C3, IYGB, PAPER V

1. $f(x) = x\ln(1+x^2)$

- BY PRODUCT RULE

$$f'(x) = 1 \times \ln(1+x^2) + x \times \frac{1}{1+x^2} \times 2x$$

$$f'(x) = \ln(1+x^2) + \frac{2x^2}{1+x^2}$$

$$f'(1) = \ln 2 + \frac{2}{2}$$

$\therefore$ GRADIENT $1+\ln 2$

- $f(1) = \ln 2 \quad \therefore (1, \ln 2)$

TANGENT EQUATION

$$y - y_0 = m(x - x_0)$$

$$y - \ln 2 = (1+\ln 2)(x-1)$$

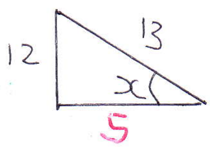
$$y - \ln 2 = (1+\ln 2)x - (1+\ln 2)$$

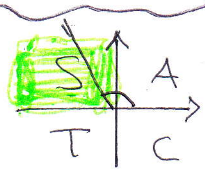
$$y - \cancel{\ln 2} = x(1+\ln 2) - 1 - \cancel{\ln 2}$$

$$y = x(1+\ln 2) - 1$$

AS REQUIRED

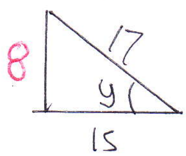
2. $\sin x = \frac{12}{13}$

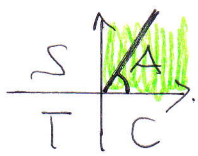




(OBTUSE)

$\cos y = \frac{15}{17}$





(ACUTE)

$$\sin(x-y) = \sin x\cos y - \cos x\sin y$$

$$= \frac{12}{13} \times \frac{15}{17} - \left(-\frac{5}{13}\right)\frac{8}{17}$$

$$= \frac{180}{221} + \frac{40}{221}$$

$$= \frac{220}{221}$$

3. $f(x) = x^2+3x-7, \quad x \in \mathbb{R}$
$g(x) = ax+b, \quad x \in \mathbb{R}$

$$f(g(x)) = f(ax+b) = (ax+b)^2+3(ax+b)-7$$
$$g(f(x)) = g(x^2+3x-7) = a(x^2+3x-7)+b$$

- $f(g(-2)) = 21$

$$(-2a+b)^2+3(-2a+b)-7=21$$

- $g(f(1)) = 0$

$$a(1+3-7)+b=0$$
$$-3a+b=0$$
$$\boxed{b=3a}$$

$$(-2a+3a)^2+3(-2a+3a)-28=0$$
$$a^2+3a-28=0$$
$$(a+7)(a-4)=0$$

$$a = 4 \text{ or } \cancel{-7} \qquad b = 12$$

4. a) $f(x) = a\ln bx \quad x>0$

- $\left(\frac{1}{3},0\right) \Rightarrow 0 = a\ln\left(\frac{1}{3}b\right)$

$$\Rightarrow \ln\left(\tfrac{1}{3}b\right) = 0 \qquad (a \neq 0)$$
$$\Rightarrow \tfrac{1}{3}b = e^0$$
$$\Rightarrow \tfrac{1}{3}b = 1$$
$$\Rightarrow b = 3$$

C3, IYGB, PAPER V

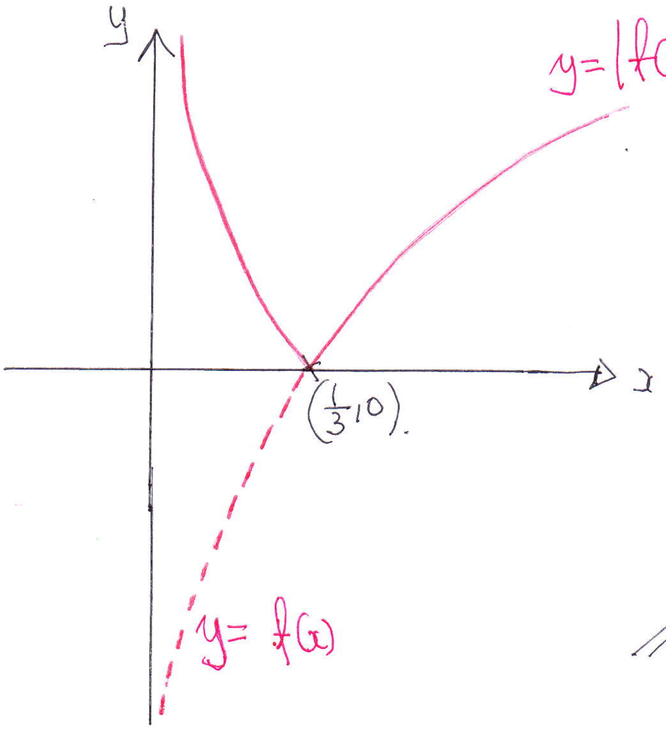
- $(3,4) \Rightarrow 4 = a\ln(3\times3)$

$\Rightarrow 4 = a\ln 9$

$\Rightarrow 4 = 2a\ln 3$

$\Rightarrow a = \dfrac{2}{\ln 3}$

b)



c) $|f(x)| = 8$ DEFINITELY HAS TWO SOLUTIONS (FROM GRAPH)

- $\dfrac{2}{\ln 3}\ln(3x) = 8$

$\Rightarrow \ln 3x = 4\ln 3$

$\Rightarrow \ln 3x = \ln 81$

$\Rightarrow 3x = 81$

$\Rightarrow x = 27$

- $\dfrac{2}{\ln 3}\ln 3x = -8$

$\ln 3x = -4\ln 3$

$\ln 3x = \ln\left(\dfrac{1}{81}\right)$

$3x = \dfrac{1}{81}$

$x = \dfrac{1}{243}$

5. a)

$$\cos 3x = \cos(2x + x)$$
$$= \cos 2x \cos x - \sin 2x \sin x$$
$$= (2\cos^2 x - 1)\cos x - (2\sin x \cos x)\sin x$$
$$= 2\cos^3 x - \cos x - 2\sin^2 x \cos x$$
$$= 2\cos^3 x - \cos x - 2(1 - \cos^2 x)\cos x$$
$$= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x$$
$$= 4\cos^3 x - 3\cos x$$

AS REQUIRED

b)

$$8\cos^3 x - 6\cos x + 1 = 0$$
$$8\cos^3 x - 6\cos x = -1 \quad) \div 2$$
$$4\cos^3 x - 3\cos x = -\frac{1}{2}$$
$$\cos 3x = -\frac{1}{2}$$

$$\arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

$$\begin{cases} 3x = \frac{2\pi}{3} \pm 2n\pi \\ 3x = \frac{4\pi}{3} \pm 2n\pi \end{cases} \qquad n = 0, 1, 2, 3, \ldots$$

$$\begin{cases} x = \frac{2\pi}{9} \pm \frac{2}{3}n\pi \\ x = \frac{4\pi}{9} \pm \frac{2}{3}n\pi \end{cases}$$

$$x = \frac{2\pi}{9}, \frac{8\pi}{9}, \frac{14\pi}{9}, \frac{4\pi}{9}, \frac{10\pi}{9}, \frac{16\pi}{9}$$

6. a) $y = 4x^{\frac{3}{2}} - \frac{7}{8}\ln 4x$

$\frac{dy}{dx} = 6x^{\frac{1}{2}} - \frac{7}{8} \times \frac{1}{x}$

$\left.\frac{dy}{dx}\right|_{x=\frac{1}{4}} = 6 \times \frac{1}{2} - \frac{7}{8} \times 4 = 3 - \frac{7}{2} = -\frac{1}{2}$

- NORMAL GRADIENT $= 2$
- $y = 4 \times \left(\frac{1}{4}\right)^{\frac{3}{2}} - \cancel{\frac{7}{8}\ln 1} = \frac{1}{2}$ i.e. $\left(\frac{1}{4}, \frac{1}{2}\right)$

$y - y_0 = m(x - x_0)$

$y - \frac{1}{2} = 2\left(x - \frac{1}{4}\right)$

$y \cancel{- \frac{1}{2}} = 2x \cancel{- \frac{1}{2}}$

$y = 2x$

b) $y = 2x$ & $y = 4x^{\frac{3}{2}} - \frac{7}{8}\ln 4x$

$2x = 4x^{\frac{3}{2}} - \frac{7}{8}\ln 4x$

$2x + \frac{7}{8}\ln 4x = 4x^{\frac{3}{2}}$

$\frac{16x + 7\ln 4x}{8} = 4x^{\frac{3}{2}}$

$x^{\frac{3}{2}} = \frac{16x + 7\ln 4x}{32}$

$x = \left(\frac{16x + 7\ln 4x}{32}\right)^{\frac{2}{3}}$

c)

$$x_{n+1} = \left[\frac{16x_n + 7\ln(4x_n)}{32}\right]^{\frac{2}{3}}$$

$x_0 = 0.7$

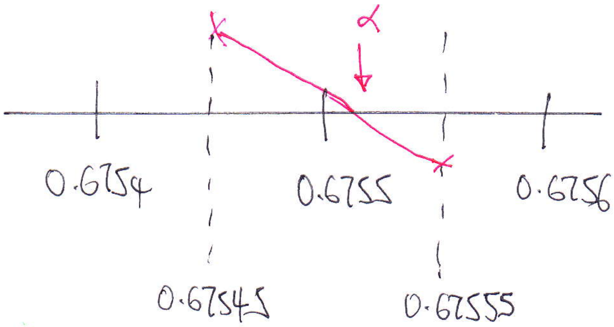
$x_1 = 0.692$

$x_2 \simeq 0.686$

$x_3 \simeq 0.683$

$x_4 \simeq 0.680$

d)



EQUATION SOLVED

$2x = 4x^{\frac{3}{2}} - \frac{7}{8}\ln 4x$

$2x - 4x^{\frac{3}{2}} + \frac{7}{8}\ln 4x = 0$

LET $f(x) = 2x - 4x^{\frac{3}{2}} + \frac{7}{8}\ln 4x$

$$\left.\begin{aligned} f(0.67545) &= 0.000083 > 0 \\ f(0.67555) &= -0.00008 < 0 \end{aligned}\right\} \Rightarrow$$

CHANGE OF SIGN (AND CONTINUITY) IMPLY

$0.67545 < \alpha < 0.6755$

$\alpha \simeq 0.6755$

CORRECT TO 4 d.p.

C3, IYGB, PAPER V

7. $y = \left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}(2x+1) \qquad 1 < x \leq 1$

$$\Rightarrow \ln y = \ln\left[\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}(2x+1)\right]$$

$$\Rightarrow \ln y = \ln(1-x)^{\frac{1}{2}} - \ln(1+x)^{\frac{1}{2}} + \ln(2x+1)$$

$$\Rightarrow \ln y = \frac{1}{2}\ln(1-x) - \frac{1}{2}\ln(1+x) + \ln(2x+1)$$

Diff w.r.t x

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = -\frac{1}{2}\times\frac{1}{1-x} - \frac{1}{2}\times\frac{1}{1+x} + \frac{1}{2x+1}\times 2$$

$$\Rightarrow \frac{dy}{dx}\frac{1}{y} = \frac{2}{2x+1} - \frac{1}{2}\left(\frac{1}{1-x} + \frac{1}{1+x}\right)$$

$$\Rightarrow \frac{dy}{dx}\frac{1}{y} = \frac{2}{2x+1} - \frac{1}{2}\left(\frac{1+\cancel{x}+1-\cancel{x}}{(1-x)(1+x)}\right)$$

$$\Rightarrow \frac{dy}{dx}\frac{1}{y} = \frac{2}{2x+1} - \frac{1}{1-x^2}$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{2}{2x+1} - \frac{1}{1-x^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}(2x+1)\left[\frac{2}{2x+1} - \frac{1}{1-x^2}\right]$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{x=\frac{1}{2}} = \left(\frac{1/2}{3/2}\right)^{\frac{1}{2}}\times 2\left[1 - \frac{1}{3/4}\right]$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{x=\frac{1}{2}} = \left(\frac{1}{3}\right)^{\frac{1}{2}}\times 2\left(-\frac{1}{3}\right) = -\frac{2}{3}\frac{1}{\sqrt{3}} = -\frac{2}{9}\sqrt{3}$$

C3, IYGB, PAPER V

8. a) $\boxed{P_1 - P_2 = 4800 \quad \text{when } t = T}$

$1600e^{\frac{1}{4}T} - 100e^{\frac{1}{2}T} = 4800$

$16e^{\frac{1}{4}T} - e^{\frac{1}{2}T} = 48$

$e^{\frac{1}{2}T} - 16e^{\frac{1}{4}T} + 48 = 0$

$(e^{\frac{1}{4}T} - 4)(e^{\frac{1}{4}T} - 12) = 0$

$e^{\frac{1}{4}T} = \begin{cases} 4 \\ 12 \end{cases}$

$\frac{1}{4}T = \begin{cases} \ln 4 \\ \ln 12 \end{cases}$

$T = \begin{cases} 4\ln 4 = 8\ln 2 \\ 4\ln 12 \end{cases}$

b) LET $\boxed{\begin{array}{l} P = P_1 - P_2 \\ P = 1600e^{\frac{1}{4}t} - 100e^{\frac{1}{2}t} \end{array}}$

$\frac{dP}{dt} = 400e^{\frac{1}{4}t} - 50e^{\frac{1}{2}t}$

- SOLVE FOR ZERO

$\Rightarrow 400e^{\frac{1}{4}t} - 50e^{\frac{1}{2}t} = 0$

$\Rightarrow 400e^{\frac{1}{4}t} = 50e^{\frac{1}{2}t}$

$\Rightarrow 8 = e^{\frac{1}{4}t}$) $\div e^{\frac{1}{4}t}$, $\div 50$

ie $\boxed{e^{\frac{1}{4}t} = 8}$

$$\therefore \quad P = 1600e^{\frac{1}{4}t} - 100e^{\frac{1}{2}t}$$

$$\Longrightarrow P = 1600\left(e^{\frac{1}{4}t}\right) - 100\left(e^{\frac{1}{4}t}\right)^2$$

$$\Longrightarrow P = 1600 \times 8 - 100 \times 8^2$$

$$\Longrightarrow P = 12800 - 6400$$

$$\Longrightarrow P = 6400$$

JUSTIFICATION NOT ACTUALLY NEEDED SINCE AT $t=T$, $P_1 - P_2 = 4800$

$$\frac{d^2P}{dt^2} = 100e^{\frac{1}{4}t} - 50e^{\frac{1}{2}t}$$

$$\left.\frac{d^2P}{dt^2}\right|_{e^{\frac{1}{4}t}=8} = 100 \times 8 - 50 \times 8^2 = 800 - 3200 < 0$$

$\therefore$ INDEED GREATEST