C4, IYGB, PAPER X

1. $$e^y = x^x$$

$$\Rightarrow \ln(ye^y) = \ln(x^x)$$

$$\Rightarrow \ln y + \ln e^y = x\ln x$$

$$\Rightarrow \ln y + y = x\ln x$$

Diff w.r.t x

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} + 1\frac{dy}{dx} = 1\times\ln x + x\times\frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{1}{y}+1\right) = \ln x + 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1+\ln x}{\frac{1}{y}+1}$$ MULTIPLY TOP & BOTTOM BY y

$$\Rightarrow \frac{dy}{dx} = \frac{y(1+\ln x)}{1+y}$$ // AS REQUIRED

2. a) $$(3+2x)^n = 3^n\left(1+\frac{2}{3}x\right)^n$$

$$= 3^n\left[1 + \frac{n}{1}\left(\frac{2}{3}x\right) + \frac{n(n-1)}{1\times2}\left(\frac{2}{3}x\right)^2 + \frac{n(n-1)(n-2)}{1\times2\times3}\left(\frac{2}{3}x\right)^3 + O(x^4)\right]$$

$$= 3^n\left[1 + \frac{2}{3}nx + \frac{2}{9}n(n-1)x^2 + \frac{4}{81}n(n-1)(n-2)x^3 + O(x^4)\right]$$

THUS $$\frac{\frac{4}{81}\cancel{n}\cancel{(n-1)}(n-2)}{\frac{2}{9}\cancel{n}\cancel{(n-1)}} = \frac{2(n-2)}{9}$$ IF $2(n-2) : 9$ //

b) $2\left(\frac{7}{2}-2\right) : 9$

$3 : 9$

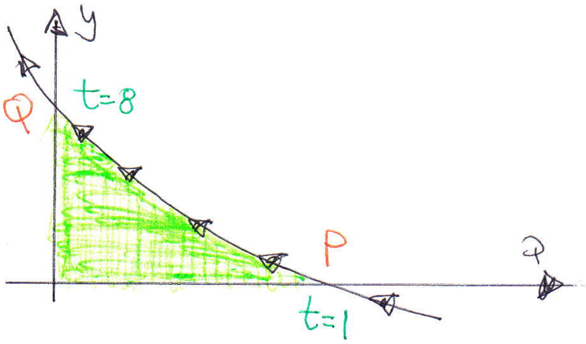
$1 : 3$ //

c) $$\frac{\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\left(-\frac{1}{2}\right)}{1\times2\times3\times4\times5.}\left(\frac{2}{3}x\right)^5$$

(FIRST NEGATIVE COEFFICIENT)

$\therefore r=5$

3. a)



$x=2-\frac{1}{4}t$

$y=2^t-2$

• P: $y=0$
$2^t-2=0$
$2^t=2$
$t=1$
$\therefore x=2-\frac{1}{4}\times1=\frac{7}{4}$
$P\left(\frac{7}{4},0\right)$

• Q: $x=0$
$2-\frac{1}{4}t=0$
$2=\frac{1}{4}t$
$t=8$
$y=2^8-2=254$
$Q(0,254)$

b) CURVE IS TRACED "BACKWARDS" (SEE ARROWS & VALUES OF t)

$$\text{AREA}=\int_{x_1}^{x_2}y(x)\,dx=\int_{t_1}^{t_2}y(t)\frac{dx}{dt}\,dt$$

$$=\int_{t=8}^{t=1}(2^t-2)\left(-\frac{1}{4}dt\right)=\int_1^8\frac{1}{4}(2^t-2)\,dt$$

$$=\int_1^8\frac{1}{4}\times2^t-\frac{1}{2}\,dt=\int_1^8 2^{-2}\times2^t-\frac{1}{2}\,dt$$

$$= \int_1^8 2^{t-2} - \frac{1}{2} \, dt$$

AS REQUIRED

c) INTEGRATING

$$\dots = \left[\frac{1}{\ln 2} 2^{t-2} - \frac{1}{2}t\right]_1^8$$

$$= \left(\frac{1}{\ln 2} \times 64 - 4\right) - \left(\frac{1}{\ln 2} \times \frac{1}{2} - \frac{1}{2}\right)$$

$$= \frac{127}{2\ln 2} - \frac{7}{2}$$

NOTE

$$\frac{d}{dx}(a^x) = a^x \ln a$$

THUS

$$\int a^x \, dx = \frac{1}{\ln a} a^x + C$$

4. a) $f(x) = \frac{1}{8}(4x + \sin 4x)$

$$f'(x) = \frac{1}{8}(4 + 4\cos 4x) = \frac{1}{2}(1 + \cos 4x)$$

$$\cos 2A = 2\cos^2 A - 1$$
$$\cos 4A = 2\cos^2 2A - 1$$

$$= \frac{1}{2}\left[\cancel{1} + (2\cos^2 2x - \cancel{1})\right] = \frac{1}{2} \times 2\cos^2 2x = \cos^2 2x$$

AS REQUIRED

b) $V = \pi \int_{x_1}^{x_2} (y(x))^2 \, dx = \pi \int_0^{\frac{\pi}{4}} (\sqrt{x} \cos 2x)^2 \, dx$

$$V = \pi \int_0^{\frac{\pi}{4}} x \cos^2 2x \, dx$$

BY PARTS, IGNORING π & LIMITS

HENCE $\int x\cos^2 2x \, dx$

x	1
$\frac{1}{8}(4x+\sin 4x)$	$\cos^2 2x$

$$= \frac{1}{8}x(4x+\sin 4x) - \int \frac{1}{8}(4x+\sin 4x)\,dx$$

$$= \frac{1}{2}x^2 + \frac{1}{8}x\sin 4x - \int \frac{1}{2}x + \frac{1}{8}\sin 4x \, dx$$

$$= \frac{1}{2}x^2 + \frac{1}{8}x\sin 4x - \left[\frac{1}{4}x^2 - \frac{1}{32}\cos 4x\right] + C$$

$$= \frac{1}{4}x^2 + \frac{1}{8}x\sin 4x + \frac{1}{32}\cos 4x + C$$

$$\therefore V = \pi\left[\frac{1}{4}x^2 + \frac{1}{8}x\sin 4x + \frac{1}{32}\cos 4x\right]_0^{\frac{\pi}{4}}$$

$$V = \pi\left[\left(\frac{\pi^2}{64} + 0 - \frac{1}{32}\right) - \left(0 + 0 + \frac{1}{32}\right)\right]$$

$$V = \pi\left[\frac{\pi^2}{64} - \frac{1}{16}\right]$$

$$V = \frac{\pi}{64}(\pi^2 - 4)$$

(P.T.O)

5. a) $\underline{r}_1 = (2,1,5) + \lambda(1,0,-1) = (\lambda+2,\ 1,\ 5-\lambda)$

$\underline{r}_2 = (2,1,5) + \mu(1,4,-1) = (\mu+2,\ 4\mu+1,\ 5-\mu)$

- $\underline{b} = (b,1,-1) = (\lambda+2,1,5-\lambda)$

$5-\lambda = -1$

$\boxed{6 = \lambda}$

$\therefore\ b = \lambda+2$

$b = 8$

- $\underline{d} = (4,d,3) = (\mu+2,4\mu+1,5-\mu)$

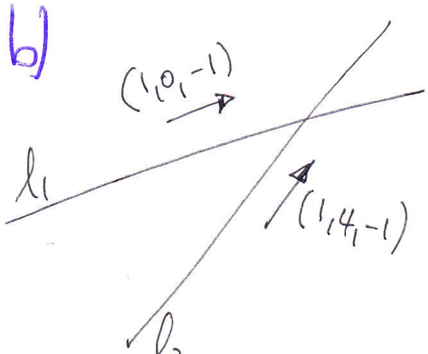
$\mu+2 = 4$

$\boxed{\mu = 2}$

$\therefore\ d = 4\mu+1$

$d = 9$

b)



DOTTING THE DIRECTION VECTORS OF THE TWO LINES

$(1,0,-1)\cdot(1,4,-1) = |1,0,-1||1,4,-1|\cos\theta$

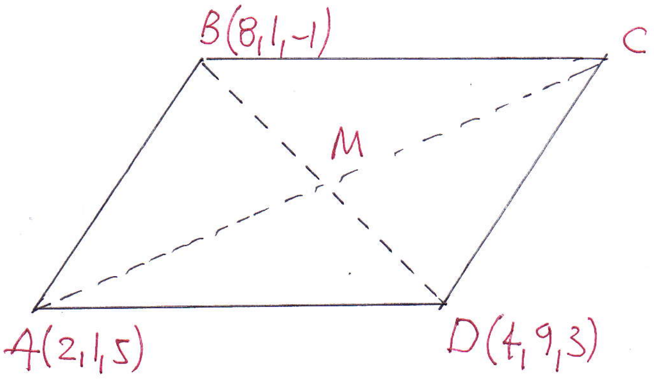
$1+0+1 = \sqrt{1+0+1}\sqrt{1+16+1}\cos\theta$

$2 = \sqrt{2}\sqrt{18}\cos\theta$

$2 = 6\cos\theta$

$\cos\theta = \frac{1}{3}$

c)



(BY INSPECTION)

- MIDPOINT OF BD IS $M(6,5,1)$

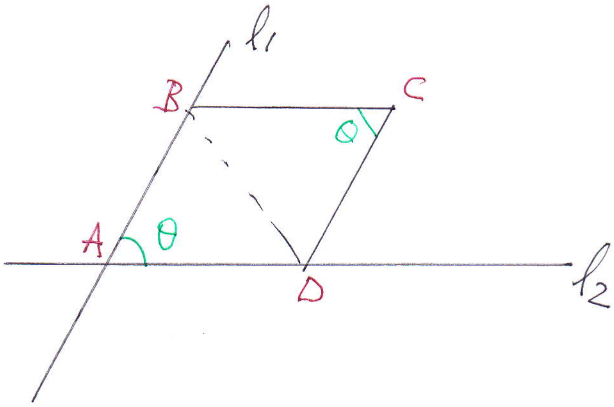
- M IS ALSO THE MIDPOINT OF AC

$A(2,1,5) \quad M(6,5,1) \quad C(10,9,-3)$

$+4 \quad +4 \quad -4$

$\therefore\ C(10,9,-3)$

d)



$\cos\theta = \frac{1}{3}$

- $|\overrightarrow{AB}| = |\underline{b} - \underline{a}| = |(8,1,-1) - (2,1,5)| = |6,0,-6| = \sqrt{36+36}$

$= \sqrt{72} = 6\sqrt{2}$

- $|\overrightarrow{AD}| = |\underline{d} - \underline{a}| = |(4,9,3) - (2,1,5)| = |2,8,-2| = \sqrt{4+64+4}$

$= \sqrt{72} = 6\sqrt{2}$

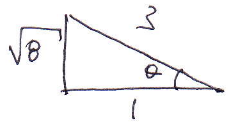
$\cos\theta = \frac{1}{3} \implies \sin\theta = +\sqrt{1-\cos^2\theta}$

$\sin\theta = +\sqrt{1-\left(\frac{1}{3}\right)^2}$

$\sin\theta = \sqrt{\frac{8}{9}}$

$\boxed{\sin\theta = \frac{2}{3}\sqrt{2}}$

OR



$\therefore \sin\theta = \frac{\sqrt{8}}{3}$

$\therefore$ AREA OF PARALLELOGRAM = (TRIANGLE ABD) × 2

$= \frac{1}{2}|\overrightarrow{AB}||\overrightarrow{AD}|\sin\theta \times 2$

$= \cancel{\frac{1}{2}}\sqrt{72}\sqrt{72} \times \frac{2}{3}\sqrt{2} \times \cancel{2}$

$= 48\sqrt{2}$

6.

$$\left.\frac{dA}{dt}\right|_{r=12} = -6 \quad \text{(GIVEN)}$$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$A = 4\pi r^2$

$\frac{dA}{dr} = 8\pi r$

$\frac{dr}{dA} = \frac{1}{8\pi r}$

$$\Rightarrow \frac{dV}{dt} = \frac{dV}{dA} \times \frac{dA}{dt}$$

$$\Rightarrow \frac{dV}{dt} = \left[\frac{dV}{dr} \times \frac{dr}{dA}\right] \times \frac{dA}{dt}$$

$$\Rightarrow \frac{dV}{dt} = 4\pi r^2 \times \frac{1}{8\pi r} \times \frac{dA}{dt}$$

$$\Rightarrow \frac{dV}{dt} = \frac{1}{2}r \times \frac{dA}{dt}$$

$$\Rightarrow \left.\frac{dV}{dt}\right|_{r=12} = \frac{1}{2} \times 12 \times \left.\frac{dA}{dt}\right|_{r=12}$$

$$\Rightarrow \left.\frac{dV}{dt}\right|_{r=12} = \frac{1}{2} \times 12 \times (-6) = -36 \text{ cm}^3\text{s}^{-1}$$

7. a)

x = infected chickens (1000s)

t = time (hours)

$t = 0,\ x = 4$

$,\ \frac{dx}{dt} = 0.032$

$$\frac{dx}{dt} = kx(20-x)$$

infected not infected

b) APPLY CONDITION $x = 4,\ \frac{dx}{dt} = 0.032$

$0.032 = k \times 4 \times 16$

$64k = 0.032$

$k = \frac{1}{2000}$

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THUS $\frac{dx}{dt} = \frac{1}{2000}x(20-x)$, SEPARATE VARIABLES

$$\Longrightarrow \frac{2000}{x(20-x)}\,dx = 1\,dt$$

$$\Longrightarrow \int \frac{2000}{x(20-x)}\,dx = \int 1\,dt$$

BY PARTIAL FRACTIONS

$$\frac{2000}{x(20-x)} \equiv \frac{A}{x} + \frac{B}{20-x}$$

$$2000 \equiv A(20-x) + Bx$$

If $x=0$, $2000 = 20A$, $\boxed{A = 100}$

If $x=20$, $2000 = 20B$, $\boxed{B = 100}$

$$\Longrightarrow \int \frac{100}{x} + \frac{100}{20-x}\,dx = \int 1\,dt$$

$$\Longrightarrow 100\ln|x| - 100\ln|20-x| = t + C$$

$$\Longrightarrow \boxed{100\ln\left|\frac{x}{20-x}\right| + C = t}$$

$t=0$, $x=4 \Longrightarrow 100\ln\frac{1}{4} + C = 0$

$C = -100\ln\frac{1}{4}$

$C = 100\ln 4$

$$\Longrightarrow 100\ln\left|\frac{x}{20-x}\right| + 100\ln 4 = t$$

$$\Longrightarrow 100\left[\ln\left|\frac{x}{20-x}\right| + \ln 4\right] = t$$

$$\Longrightarrow t = 100\ln\left|\frac{4x}{20-x}\right|$$

AS REQUIRED

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c) $t = 100\ln\left|\frac{4x}{20-x}\right|$

$\Rightarrow \frac{1}{100}t = \ln\left|\frac{4x}{20-x}\right|$

$\Rightarrow e^{0.01t} = \frac{4x}{20-x}$

$\Rightarrow 20e^{0.01t} - xe^{0.01t} = 4x$

$\Rightarrow 20e^{0.01t} = xe^{0.01t} + 4x$

$\Rightarrow x(e^{0.01t} + 4) = 20e^{0.01t}$

$\Rightarrow x = \frac{20e^{0.01t}}{e^{0.01t} + 4}$

$\Rightarrow x = \frac{20e^{0.01t}e^{-0.01t}}{e^{0.01t}e^{-0.01t} + 4e^{-0.01t}}$

$\Rightarrow x = \frac{20}{1 + 4e^{-0.01t}}$

AS REQUIRED

d) WHEN $t = 24$

$x = \frac{20}{1 + 4e^{-0.24}}$

$x = 4.82333...$

$\therefore$ 4823 CHICKENS

$\therefore$ AN EXTRA 823 CHICKENS