IYGB – FP3 PAPER P – QUESTION 1

a) FORMING A TABLE OF VALUES

x	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\cos^2 x$	1	$\frac{2+\sqrt{3}}{4}$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
	FIRST	ODD	EVEN	ODD	LAST

BY SIMPSON RULE

$$\int_0^{\frac{\pi}{3}} \cos^2 x \, dx \simeq \frac{\text{"THICKNESS"}}{3}\Big[\text{FIRST} + \text{LAST} + 4 \times \text{ODDS} + 2 \times \text{EVENS}\Big]$$

$$\simeq \frac{\pi/12}{3}\left[1 + \frac{1}{4} + 4\left[\frac{2+\sqrt{3}}{4} + \frac{1}{2}\right] + 2 \times \frac{3}{4}\right]$$

$$\simeq \frac{\pi}{36}\left[1 + \frac{1}{4} + 2 + \sqrt{3} + 2 + \frac{3}{2}\right]$$

$$\simeq \frac{\pi}{36} \times \frac{27 + 4\sqrt{3}}{4}$$

$$\simeq 0.7401985696\ldots$$

$$\simeq 0.740$$

b) USING $\cos^2 x + \sin^2 x \equiv 1$

$$\int_0^{\frac{\pi}{3}} \sin^2 x \, dx = \int_0^{\frac{\pi}{3}} 1 - \cos^2 x \, dx = \int_0^{\frac{\pi}{3}} 1 \, dx - \int_0^{\frac{\pi}{3}} \cos^2 x \, dx$$

USING THE APPROXIMATION OF PART (a)

$$\simeq \Big[x\Big]_0^{\frac{\pi}{3}} - 0.740198\ldots$$

$$\simeq \frac{\pi}{3} - 0.740198\ldots$$

$$\simeq 0.30699\ldots$$

$$\simeq 0.307$$

IYGB – FP3 PAPER P – QUESTION 2

METHOD A

$$\frac{x+3}{x} \geqslant \frac{x}{2-x}$$

$$\Rightarrow \frac{x+3}{x} - \frac{x}{2-x} \geqslant 0$$

$$\Rightarrow \frac{(x+3)(2-x) - x^2}{x(2-x)} \geqslant 0$$

$$\Rightarrow \frac{2x - x^2 + 6 - 3x - x^2}{x(2-x)} \geqslant 0$$

$$\Rightarrow \frac{-2x^2 - x + 6}{x(2-x)} \geqslant 0$$

$$\Rightarrow \frac{2x^2 + x - 6}{x(2-x)} \leqslant 0$$

$$\Rightarrow \frac{(2x-3)(x+2)}{x(2-x)} \leqslant 0$$

THE CRITICAL VALUES FOR THIS INEQUALITY EXPRESSION ARE

$x = 0, \ 2$ (asymptotes)

$x = -2, \ \frac{3}{2}$ (numerator zero)

	-2		0		$\frac{3}{2}$		2	
$\frac{\ominus\ominus}{\ominus\oplus}$		$\frac{\ominus\oplus}{\ominus\oplus}$		$\frac{\ominus\oplus}{\oplus\oplus}$		$\frac{\oplus\oplus}{\oplus\oplus}$		$\frac{\oplus\oplus}{\oplus\ominus}$
NEG		POS		NEG		POS		NEG

$$x \leqslant -2 \ \cup \ 0 < x \leqslant \tfrac{3}{2} \ \cup \ x > 2$$

METHOD B

$$\frac{x+3}{x} \geqslant \frac{x}{2-x}$$

$$\Rightarrow \frac{x(x+3)}{x^2} \geqslant \frac{x(2-x)}{(2-x)^2}$$

$$\Rightarrow x(x+3)(2-x)^2 \geqslant x^3(2-x)$$

$$\Rightarrow x(x+3)(2-x)^2 - x^3(2-x) \geqslant 0$$

$$\Rightarrow x(x+3)(x-2)^2 + x^3(x-2) \geqslant 0$$

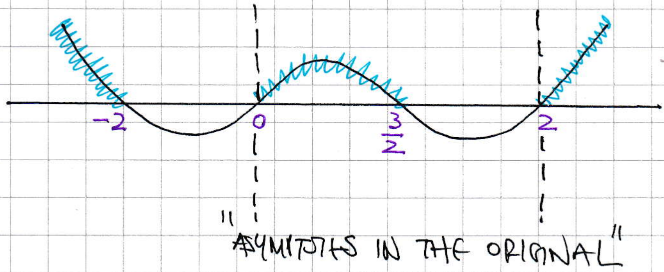
$$\Rightarrow x(x-2)\left[(x+3)(x-2) + x^2\right] \geqslant 0$$

$$\Rightarrow x(x-2)(x^2 + x - 6 + x^2) \geqslant 0$$

$$\Rightarrow x(x-2)(2x^2 + x - 6) \geqslant 0$$

$$\Rightarrow x(x-2)(2x-3)(x+2) \geqslant 0$$

LOOKING AT THE QUARTIC



$$x \leqslant -2 \ \cup \ 0 < x \leqslant \tfrac{3}{2} \ \cup \ x > 2$$

IYGB - FP3 PAPER P - QUESTION 3

a) OBTAIN THE FIRST THREE DERIVATIVES OF $y = x^{-\frac{1}{2}}$

$$y' = -\tfrac{1}{2}x^{-\frac{3}{2}}, \quad y'' = \tfrac{3}{4}x^{-\frac{5}{2}}, \quad y''' = -\tfrac{15}{8}x^{-\frac{7}{2}}$$

EVALUATE AT $x=1$

$$y_1 = 1, \quad y_1' = -\tfrac{1}{2}, \quad y_1'' = \tfrac{3}{4}, \quad y''' = -\tfrac{15}{8}$$

BY THE TAYLOR FORMULA

$$y = y_a + (x-a)y_a' + \frac{(x-a)^2}{2!}y_a'' + \frac{(x-a)^3}{3!}y_a''' + O\left[(x-a)^4\right]$$

$$\frac{1}{\sqrt{x}} = 1 - \tfrac{1}{2}(x-1) + \tfrac{1}{2}(x-1)^2\times\left(\tfrac{3}{4}\right) + \tfrac{1}{6}(x-\ )^3\times\left(-\tfrac{15}{8}\right) + O\left[(x-1)^4\right]$$

$$\frac{1}{\sqrt{x}} = 1 - \tfrac{1}{2}(x-1) + \tfrac{3}{8}(x-1)^2 - \tfrac{5}{16}(x-1)^3 + O\left[(x-1)^4\right]$$

b) NOW USING THE FIRST THREE TERMS WITH $x=\frac{8}{9}$

$$\Rightarrow \frac{1}{\sqrt{\frac{8}{9}}} = 1 - \tfrac{1}{2}\times\left(\tfrac{8}{9}-1\right) + \tfrac{3}{8}\left(\tfrac{8}{9}-1\right)^2 + \dots$$

$$\Rightarrow \frac{3}{\sqrt{8}} = 1 - \tfrac{1}{2}\left(-\tfrac{1}{9}\right) + \tfrac{3}{8}\left(\tfrac{1}{81}\right) + \dots$$

$$\Rightarrow \frac{3\sqrt{2}}{\sqrt{8}\sqrt{2}} = 1 + \frac{1}{18} + \frac{1}{216} + \dots$$

$$\Rightarrow \tfrac{3}{4}\sqrt{2} = \frac{229}{162} + \dots$$

$$\Rightarrow \sqrt{2} = \frac{229}{162} + \dots$$

$$\therefore \sqrt{2} \simeq \frac{229}{162}$$

IYGB - FP3 PAPER P - QUESTION 4

USING STANDARD EXPANSIONS

- $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + O(x^5)$

$\ln(1-x^2) = (\cancel{x^2}) - \frac{1}{2}(\cancel{x^2})^2 + \frac{1}{3}(-3x^2)^3 + O(x^8)$

$\ln(1-3x^2) = -3x^2 - \frac{9}{2}x^4 - 9x^6 + O(x^8)$

- $\sqrt{x+4} = (4+x)^{\frac{1}{2}} = 4^{\frac{1}{2}}\left(1+\frac{1}{4}x\right)^{\frac{1}{2}}$

$= 2\left[1 + \frac{\frac{1}{2}}{1}\left(\frac{1}{4}x\right)^1 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1\times 2}\left(\frac{1}{4}x\right)^2 + O(x^3)\right]$

$= 2\left[1 + \frac{1}{8}x - \frac{1}{128}x^2 + O(x^3)\right]$

$= 2 + \frac{1}{4}x - \frac{1}{64}x^2 + O(x^3)$

APPLYING THESE RESULTS TO THE LIMIT

$$\lim_{x\to 0}\left[\frac{2x - x\sqrt{x+4}}{\ln(1-3x^2)}\right]$$

$$= \lim_{x\to 0}\left[\frac{2x - x\left(2+\frac{1}{4}x - \frac{1}{64}x^2 + O(x^3)\right)}{-3x^2 - \frac{9}{2}x^4 + O(x^6)}\right]$$

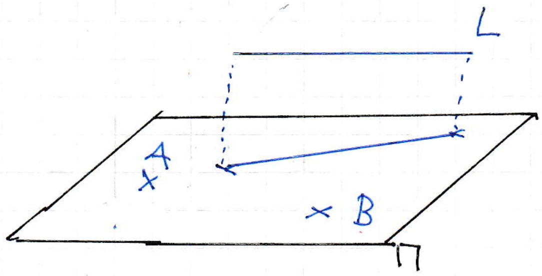
$$= \lim_{x\to 0}\left[\frac{\cancel{2x} - \cancel{2x} - \frac{1}{4}x^2 + O(x^3)}{-3x^2 - \frac{9}{2}x^4 + O(x^6)}\right]$$

$$= \lim_{x\to 0}\left[\frac{-\frac{1}{4}x^2 + O(x^3)}{-3x^2 + O(x^4)}\right]$$

$$= \lim_{x\to 0}\left[\frac{-\frac{1}{4} + O(x)}{-3 + O(x^2)}\right] = \frac{1}{12}$$

IYGB – FP3 PAPER P – QUESTION 5

a) START BY OBTAINING A NORMAL BY "CROSSING" $\vec{AB}$ & THE DIRECTION OF L



$$\vec{AB} = \underline{b} - \underline{a} = (15,12,5) - (11,13,5)$$
$$= (4,-1,0)$$

$$\underline{n} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 4 & -1 & 0 \\ -2 & 2 & -3 \end{vmatrix} = (3,12,6)$$

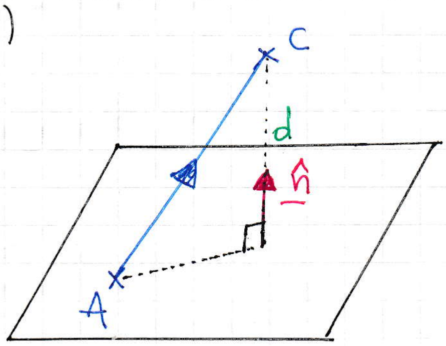
SCALING THE NORMAL TO $(1,4,2)$ & USING POINT $A(11,13,5)$

$$x + 4y + 2z = \text{CONSTANT}$$
$$11 + 4\times13 + 2\times5 = \text{CONSTANT}$$
$$\text{CONSTANT} = 11 + 52 + 10$$
$$\text{CONSTANT} = 73$$

$$\therefore\ x + 4y + 2z = 73$$

NOW TO FIND THE SHORTEST DISTANCE, TAKE ANY POINT ON THE LINE SAY $C(3,7,0)$, FIND $\vec{AC}$ AND FIND ITS PROJECTION ON A NORMAL

$$\vec{AC} = \underline{c} - \underline{a} = (3,7,0) - (15,12,5) = (-12,-5,-5)$$



ALSO $$\hat{n} = \frac{1}{\sqrt{1^2+4^2+2^2}}(1,4,2)$$

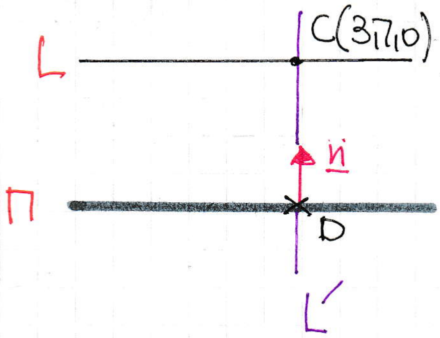
$$\hat{n} = \frac{(1,4,2)}{\sqrt{21}}$$

IYGB – FP3 – PAPER P – QUESTION 5

$$d = |\vec{AC} \cdot \hat{n}| = \left| (-12,-5,-5) \cdot \frac{(1,4,2)}{\sqrt{21}} \right| = \left| \frac{-12-20-10}{\sqrt{21}} \right|$$

$$= \left| \frac{-42}{\sqrt{21}} \right| = \frac{42\sqrt{21}}{21} = 2\sqrt{21}$$

b) FIND AN EQUATION OF L'



$\underline{r} = (x,y,z) = (3,7,0) + \lambda(1,4,2)$

$(x,y,z) = (\lambda+3,\ 4\lambda+7,\ 2\lambda)$

SOLVING SIMULTANEOUSLY WITH THE EQUATION OF THE PLANE

$$\Rightarrow \quad x + 4y + 2z = 73$$
$$\Rightarrow \quad (\lambda+3) + 4(4\lambda+7) + 2(2\lambda) = 73$$
$$\Rightarrow \quad \lambda + 3 + 16\lambda + 28 + 4\lambda = 73$$
$$\Rightarrow \quad 21\lambda = 42$$
$$\Rightarrow \quad \lambda = 2$$

THUS $D(5,15,4)$

HENCE WE HAVE THE REFLECTION OF $C(3,7,0)$ ABOUT Π AS THE POINT $E(7,23,8)$ (BY INSPECTION AS D IS THE MIDPOINT OF CE)

$\therefore$ REQUIRED LINE WILL BE

$$\underline{r} = (7,23,8) + \mu(2,-2,3)$$

IYGB – FP3 PAPER P – QUESTION 6

a) LOOKING AT THE REFLECTION OF THE PARABOLA

$$y = \frac{1}{12}x^2 \longmapsto x = \frac{1}{12}y^2$$

$$\longmapsto y^2 = 12x$$

$$\longmapsto y^2 = 4(3x)$$

FOCUS AT $(3,0)$

SWAP x AND y, FOR THE REFLECTION IN THE LINE $y=x$

SO FOCUS OF $y = \frac{1}{12}x^2$ WILL BE AT $(0,3)$

$$\therefore d = \sqrt{3^2+3^2} = \sqrt{18} = 3\sqrt{2}$$

b) POINT A, MUST LIE ON THE LINE $y=x$, SO

$$\left.\begin{array}{l} y = x \\ y = \frac{1}{12}x^2 \end{array}\right\} \Rightarrow x = \frac{1}{12}x^2$$

$$\Rightarrow 12x = x^2$$

$$\Rightarrow 12 = x \quad \text{AS } x \neq 0 \text{ AT } A$$

$$\therefore A(12,12)$$

c) THE COMMON TANGENT MUST BE AT RIGHT ANGLES TO $y=x$

$$\therefore L: y = -x + c$$

SOLVING SIMULTANEOUSLY WITH EITHER CURVE AND LOOK FOR REPEATED ROOTS

$$\left.\begin{array}{l} y = -x+c \\ y = \frac{1}{12}x^2 \end{array}\right\} \Rightarrow \frac{1}{12}x^2 = -x+c$$

$$\Rightarrow x^2 = -12x + 12c$$

$$\Rightarrow x^2 + 12x - 12c = 0$$

NOW $b^2 - 4ac = 0$

$$\Rightarrow 12^2 - 4 \times 1 \times (-12c) = 0$$

$$\Rightarrow 144 + 48c = 0$$

$$\Rightarrow 48c = -144$$

IYGB - FP3 PAPER P - QUESTION 6

$\Rightarrow c = -3$

$\therefore$ THE QUADRATIC WHICH SHOULD YIELD REPEATED ROOTS IS

$$x^2 + 12x - 12(-3) = 0$$
$$x^2 + 12x + 36 = 0$$
$$(x+6)^2 = 0$$
$$x = -6 \quad \& \quad y = \frac{1}{12}(-6)^2 = 3$$

$\therefore$ $P(-6,3)$ & ITS REFLECTION ABOUT $y=x$ $(6,-3)$

IYGB - FP3 PAPER P - QUESTION 7

USING THE SUBSTITUTION GIVEN

$$y = \frac{1}{z} \implies \text{diff w.r.t } x \qquad \frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{z}\right)$$

$$\frac{dy}{dx} = -\frac{1}{z^2}\frac{dz}{dx}$$

TRANSFORM THE O.D.E.

$$\implies x^2\frac{dy}{dx} + xy = y^2$$

$$\implies x^2\left[-\frac{1}{z^2}\frac{dz}{dx}\right] + x\left(\frac{1}{z}\right) = \left(\frac{1}{z}\right)^2$$

$$\implies -\frac{x^2}{z^2}\frac{dz}{dx} + \frac{x}{z} = \frac{1}{z^2}$$

$$\implies -x^2\frac{dz}{dx} + xz = 1$$

$$\implies \frac{dz}{dx} - \frac{z}{x} = -\frac{1}{x^2}$$

LOOK FOR AN INTEGRATING FACTOR

$$e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln\left(\frac{1}{x}\right)} = \frac{1}{x}.$$

HENCE MULTIPLYING BY $\frac{1}{x}$ WILL MAKE THE L.H.S EXACT

$$\implies \frac{1}{x}\frac{dz}{dx} - \frac{z}{x^2} = -\frac{1}{x^3}$$

$$\implies \frac{d}{dx}\left(\frac{z}{x}\right) = -\frac{1}{x^3}$$

IYGB - FP3 PAPER P - QUESTION 7

$$\Longrightarrow \frac{z}{x} = \int -\frac{1}{x^3}\,dx$$

$$\Longrightarrow \frac{z}{x} = \frac{1}{2x^2} + C$$

$$\Longrightarrow z = \frac{1}{2x} + Cx$$

$$\Longrightarrow \frac{1}{y} = \frac{1}{2x} + Cx$$

APPLY THE CONDITION $x = \frac{1}{2}$ $y = 2$

$$\frac{1}{2} = 1 + \frac{1}{2}C$$

$$\frac{1}{2}C = -\frac{1}{2}$$

$$C = -1$$

HENCE WE HAVE

$$\frac{1}{y} = \frac{1}{2x} - x$$

$$\frac{1}{y} = \frac{1-2x^2}{2x}$$

$$y = \frac{2x}{1-2x^2}$$

IYGB - FP3 QUESTION P - QUESTION 8

$$\frac{d^2y}{dx^2} = x+y+2 \quad , \quad x=0, \; y=0, \; \frac{dy}{dx}=1$$

a) BY CONSIDERING TAYLOR EXPANSIONS WE OBTAIN

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + O(h^3)$$
$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) + O(h^3)$$

ADDING THE EXPRESSIONS

$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + O(h^4)$$

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

$$y''_n \approx \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2}$$

AS REQUIRED

SUBTRACTING THE EXPRESSIONS

$$f(x+h) - f(x-h) = 2hf'(x) + O(h^3)$$

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

$$y'_n \approx \frac{y_{n+1} - y_{n-1}}{2h}$$

AS REQUIRED

b) PROCEED AS FOLLOWS

$$\left.\begin{aligned} y_{n+1} + y_{n-1} &\approx 2y_n + h^2 y''_n \\ y_{n+1} - y_{n-1} &\approx 2hy'_n \end{aligned}\right\} \text{ADDING}$$

IYGB, FP3 PAPER P — QUESTION 8

$\Longrightarrow 2y_{n+1} = 2y_n + h^2 y_n'' + 2hy_n'$

$\Longrightarrow 2y_1 = 2y_0 + h^2 y_0'' + 2hy_0'$ $\quad n=0$

$\Longrightarrow 2y_1 = 2y_0 + h^2[x_0 + y_0 + 2] + 2h y_0'$

NOW $x_0 = 0,\ y_0 = 0,\ y_0' = 1,\ h = 0.1$

$\Longrightarrow 2y_1 = 2 \times 0 + (0.1)^2[0 + 0 + 2] + 2(0.1) \times 1$

$\Longrightarrow 2y_1 = 0.22$

$\Longrightarrow y_1 = 0.11$

c) FINALLY USING

$y_{n+1} \approx 2y_n + h^2 y_n'' - y_{n-1}$

$\Longrightarrow y_{n+1} \approx 2y_n - y_{n-1} + h^2[x_n + y_n + 2]$

$x_0 = 0 \qquad y_0 = 0$
$x_1 = 0.1 \qquad y_1 = 0.11$
$x_2 = 0.2$

ETC

- $y_2 \approx 2y_1 - y_0 + h^2(x_1 + y_1 + 2)$

  $y_2 \approx 2(0.11) - 0 + (0.1)^2[0.1 + 0.11 + 2]$

  $y_2 \approx 0.2421$

- $y_3 \approx 2y_2 - y_1 + h^2(x_2 + y_2 + 2)$

  $y_3 \approx 2(0.2421) - 0.11 + (0.1)^2[0.2 + 0.2421 + 2]$

  $y_3 \approx 0.398621 \approx 0.3986$