IYGB - FP3 PAPER U - QUESTION 1

$$\boxed{\frac{d^2y}{dx^2} = 1 + x\sin y \quad \text{SUBJECT TO} \quad x=1,\ y=1,\ \frac{dy}{dx}=2}$$

START BY USING THE FORMULAS

$$\left(\frac{dy}{dx}\right)_{r+1} \simeq \frac{y_{r+2} - y_r}{h} \qquad \left(\frac{d^2y}{dx^2}\right)_{r+1} \simeq \frac{y_{r+2} - 2y_{r+1} + y_r}{h^2}$$

$$h y'_{r+1} \simeq y_{r+2} - y_r \qquad h^2 y''_{r+1} \simeq y_{r+2} - 2y_{r+1} + y_r$$

ELIMINATE y_r BETWEEN THE EQUATIONS

$$\Longrightarrow h y'_{r+1} + h^2 y''_{r+1} \simeq 2y_{r+2} - 2y_{r+1}$$

$$\Longrightarrow h y'_{r+1} + h^2\left(1 + x_{r+1}\sin y_{r+1}\right) \simeq 2y_{r+2} - 2y_{r+1}$$

LET $r=0$ & $x_1 = 1$, $y_1 = 1$, $y'_1 = 2$

$$\Longrightarrow h y'_1 + h^2(1 + x_1 \sin y_1) \simeq 2y_2 - 2y_1$$

$$\Longrightarrow y_2 \simeq \frac{1}{2}\left[h y'_1 + h^2(1 + x_1\sin y_1) + 2y_1\right]$$

$$\Longrightarrow y_2 \simeq \frac{1}{2}\left[0.05 \times 2 + 0.05^2(1 + 1\times\sin 1) + 2\times 1\right]$$

$$\Longrightarrow y_2 \simeq 0.5261509194\ldots$$

NOW USING $y_{r+2} \simeq h^2 y''_{r+1} + 2y_{r+1} - y_r$

LET $r=1$ & NOTE THAT $x_1 = 1 \quad y_1 = 1$

$x_2 = 1.05 \quad y_2 \simeq 0.52615\ldots$

$$\Longrightarrow y_3 \simeq h^2 y''_2 + 2y_2 - y_1$$

IYGB - FP3 PAPER D - QUESTION 1

$\Rightarrow y_3 \approx h^2(1 + x_2 \sin y_2) + 2y_2 - y_1$

$\Rightarrow y_3 \approx (0.05)^2(1 + 1.1 \times \sin(0.52615\ldots)) + 2 \times 0.52615\ldots - 1$

$\Rightarrow y_3 \approx 0.0562$

IYGB - FP3 PAPER U - QUESTION 2

THIS IS A ZERO OVER ZERO LIMIT – DUE TO THE "NATURE" OF THE DENOMINATOR

WE PROCEED BY L'HOSPITAL'S RULE

$$\lim_{x \to 0}\left[\frac{e^{5x}-5x-1}{\sin 4x \sin 3x}\right] = \lim_{x \to 0}\left[\frac{\frac{d}{dx}\left(e^{5x}-5x-1\right)}{\frac{d}{dx}\left(\sin 4x \sin 3x\right)}\right]$$

$$= \lim_{x \to 0}\left[\frac{5e^{5x}-5}{4\cos 4x \sin 3x + \sin 4x(3\cos 3x)}\right]$$

THIS IS AGAIN A ZERO OVER ZERO WITH TWO (2) PRODUCTS IN THE DENOMINATOR

BY L'HOSPITAL'S AGAIN

$$= \lim_{x \to 0}\left[\frac{\frac{d}{dx}\left(5e^{5x}-5\right)}{\frac{d}{dx}\left(4\cos 4x \sin 3x + 3\sin 4x \cos 3x\right)}\right]$$

$$= \lim_{x \to 0}\left[\frac{25e^{5x}}{-16\sin 4x \sin 3x + 12\cos 4x \cos 3x + 12\cos 4x \cos 3x + 9\sin 4x \sin 3x}\right]$$

AND THE LIMIT NOW EXISTS

$$= \frac{25}{0+12+12+0}$$

$$= \frac{25}{24}$$

ALTERNATIVE BY POWER SERIES

$$\lim_{x \to 0}\left[\frac{e^{5x}-5x-1}{\sin 4x \sin 3x}\right] = \lim_{x \to 0}\left[\frac{\cancel{1}+\cancel{5x}+\frac{1}{2}(5x)^2+O(x^3)-\cancel{5x}-\cancel{1}}{[4x+O(x^3)][3x+O(x^3)]}\right]$$

$$= \lim_{x \to 0}\left[\frac{\frac{25}{2}x^2+O(x^3)}{12x^2+O(x^4)}\right] = \lim_{x \to 0}\left[\frac{\frac{25}{2}+O(x)}{12+O(x)}\right]$$

$$= \frac{25}{24}$$

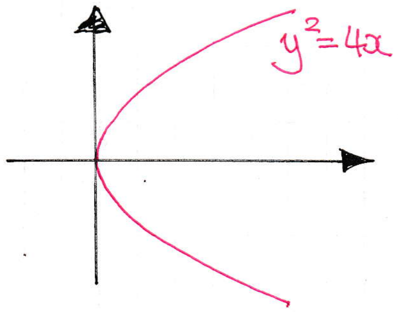
AS ABOVE

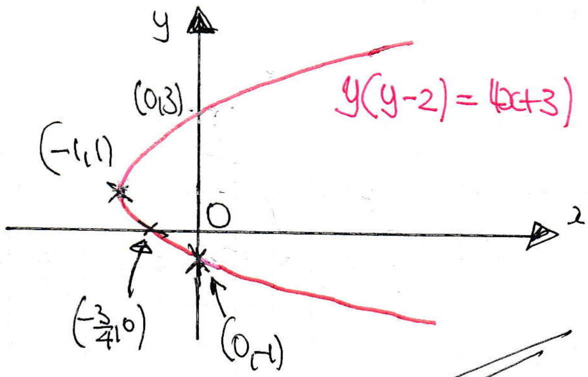
IYGB - FP3 PAPER U - QUESTION 3

- START BY COMPLETING THE SQUARE IN y

$$\Rightarrow y(y-2) = 4x+3$$
$$\Rightarrow y^2-2y = 4x+3$$
$$\Rightarrow y^2-2y+1 = 4x+4$$
$$\Rightarrow (y-1)^2 = 4(x+1)$$

- THIS IS A SIMPLE TRANSLATION OF $y^2=4x$ BY THE VECTOR $\begin{pmatrix}-1\\1\end{pmatrix}$



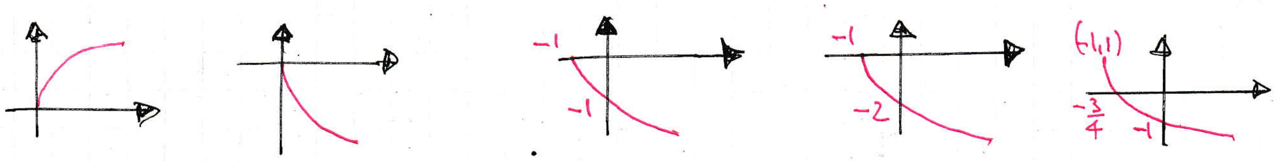


ALTERNATIVE WE HAVE

$$y-1 = \pm 2\sqrt{x+1}$$
$$y = \begin{cases} 1+2\sqrt{x+1} \\ 1-2\sqrt{x+1} \end{cases}$$

- $\sqrt{x} \longmapsto \sqrt{x+1} \longmapsto 2\sqrt{x+1} \longmapsto 2\sqrt{x+1}+1$

- $\sqrt{x} \longmapsto -\sqrt{x} \longmapsto -\sqrt{x+1} \longmapsto -2\sqrt{x+1} \longmapsto -2\sqrt{x+1}+1$



IYGB - FP3 PAPER U - QUESTION 4

PROCEED AS FOLLOWS

$$(\underline{a}+2\underline{b}-3\underline{c})_\wedge(\underline{a}+2\underline{b}+k\underline{c}) = \lambda(\underline{i}-\underline{j})$$

AS THE "CROSS PRODUCT" IS DISTRIBUTIVE OVER ADDITION/SUBTRACTION

$$\Rightarrow \Big[[\underline{a}+2\underline{b}]-3\underline{c}\Big]_\wedge\Big[[\underline{a}+2\underline{b}]+k\underline{c}\Big] = \lambda(\underline{i}-\underline{j})$$

$$\Rightarrow \cancel{(\underline{a}+2\underline{b})_\wedge(\underline{a}+2\underline{b})} + (\underline{a}+2\underline{b})_\wedge k\underline{c} - \cancel{3k\underline{c}_\wedge\underline{c}} - 3\underline{c}_\wedge(\underline{a}+2\underline{b}) = \lambda(\underline{i}-\underline{j})$$

$$\Rightarrow (\underline{a}+2\underline{b})_\wedge k\underline{c} + 3(\underline{a}+2\underline{b})_\wedge\underline{c} = \lambda(\underline{i}-\underline{j})$$

$$\Rightarrow k\underline{a}_\wedge\underline{c} + 2k\underline{b}_\wedge\underline{c} + 3\underline{a}_\wedge\underline{c} + 6\underline{b}_\wedge\underline{c} = \lambda(\underline{i}-\underline{j})$$

$$\Rightarrow (k+3)(\underline{a}_\wedge\underline{c}) + (2k+6)(\underline{b}_\wedge\underline{c}) = \lambda(\underline{i}-\underline{j})$$

BUT $\underline{b}_\wedge\underline{c} = 2\underline{i}$ & $\underline{a}_\wedge\underline{c} = \mu\underline{j}$

$$\Rightarrow (k+3)(\mu\underline{j}) + (2k+6)(2\underline{i}) = \lambda\underline{i} - \lambda\underline{j}$$

COMPARING COMPONENTS

$$\left\{\begin{array}{l} 4k+12 = \lambda \\ (k+3)\mu = -\lambda \end{array}\right\}$$

ADDING GIVES

$$4k+12+\mu(k+3) = 0$$
$$4(k+3)+\mu(k+3) = 0$$
$$(k+3)(\mu+4) = 0$$
$$k=3 \text{ OR } \mu=-4$$

FINALLY WE HAVE

$$k \neq 3 \text{ AND } \mu = -4$$

(YIELDS NO DIRECTION)

IYGB - FP3 PAPER U - QUESTION 5

a) FORMING A TABLE OF VALUES

x	0	1	2	3	4	5	6	7	8
2^x	1	2	4	8	16	32	64	128	256
	FIRST	ODD	EVEN	ODD	EVEN	ODD	EVEN	ODD	LAST

BY SIMPSON RULE

$$\int_0^8 2^x\,dx \simeq \frac{\text{"THICKNESS"}}{3}\left[\text{FIRST} + \text{LAST} + 4\times\text{ODD} + 2\times\text{EVEN}\right]$$

$$\simeq \frac{1}{3}\left[1+256+4(2+8+32+128)+2(4+16+64)\right]$$

$$\simeq \frac{1}{3}\times 1105$$

$$\simeq \frac{1105}{3}$$ // AS REQUIRED

b) INTEGRATING DIRECTLY

$$I = \int_0^8 2^x\,dx = \int_0^8 e^{x\ln 2}\,dx = \left[\frac{1}{\ln 2}e^{x\ln 2}\right]_0^8$$

$$= \frac{1}{\ln 2}\left[2^x\right]_0^8 = \frac{1}{\ln 2}\left[256-1\right] = \frac{255}{\ln 2}$$ //

c) COMBINING RESULTS WE DEDUCE

$$\Rightarrow I = \frac{255}{\ln 2} \simeq \frac{1105}{3}$$

$$\Rightarrow \ln 2 \approx \frac{3\times 255}{1105}$$

$$\Rightarrow \ln 2 \approx \frac{9}{13}$$ // AS REQUIRED

IYGB - FP3 PAPER U - QUESTION 6

USING THE SUBSTITUTION GIVEN

$$(1-x^2)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = (1-x)^2$$

$$(1-x^2)\frac{dw}{dx} + 2w = (1-x)^2$$

$$\frac{dw}{dx} + \frac{2}{1-x^2}w = \frac{(1-x)^2}{1-x^2}$$

$$\frac{dw}{dx} + \frac{2}{(1-x)(1+x)}w = \frac{(1-x)^2}{(1-x)(1+x)}$$

$$\frac{dw}{dx} + \left(\frac{1}{1+x} + \frac{1}{1-x}\right)w = \frac{1-x}{1+x}$$

PARTIAL FRACTIONS BY INSPECTION

FIND THE INTEGRATING FACTOR

$$e^{\int \frac{1}{1+x} + \frac{1}{1-x}\,dx} = e^{\ln(1+x) - \ln(1-x)} = e^{\ln\left(\frac{1+x}{1-x}\right)} = \frac{1+x}{1-x}$$

HENCE WE NOW HAVE

$$\frac{d}{dx}\left[w\left(\frac{1+x}{1-x}\right)\right] = \frac{1-x}{1+x}\left(\frac{1+x}{1-x}\right)$$

$$\frac{d}{dx}\left[w\left(\frac{1+x}{1-x}\right)\right] = 1$$

$$w\left(\frac{1+x}{1-x}\right) = \int 1\,dx$$

$$\boxed{w\left(\frac{1+x}{1-x}\right) = x + A}$$

APPLY CONDITION $x=0$, $w = \frac{dy}{dx} = -1$

$$-1 = A$$

$$\therefore\ w\left(\frac{1+x}{1-x}\right) = x - 1$$

IYGB – FP3 PAPER U – QUESTION 6

$$\Longrightarrow \quad w = \frac{(1-x)(x-1)}{1+x}$$

$$\Longrightarrow \quad \frac{dy}{dx} = -\frac{(x-1)^2}{x+1}$$

$$\Longrightarrow \quad \frac{dy}{dx} = -\frac{x^2-2x+1}{x+1}$$

LONG DIVISION OR MANIPULATION

$$\Longrightarrow \quad \frac{dy}{dx} = -\frac{x(x+1)-3(x+1)+4}{x+1}$$

$$\Longrightarrow \quad \frac{dy}{dx} = -\left[x - 3 + \frac{4}{x+1}\right]$$

$$\Longrightarrow \quad y = -\left[\tfrac{1}{2}x^2 - 3x + 4\ln(x+1)\right] + C$$

APPLY CONDITION $x=0$ $y=-4.5$ TO OBTAIN $C=-4.5$.

$$\Longrightarrow \quad y = -\tfrac{1}{2}x^2 + 3x - 4.5 - 4\ln(x+1)$$

$$\Longrightarrow \quad y = -\tfrac{1}{2}\left(x^2 - 6x + 9\right) - 4\ln(x+1)$$

$$\Longrightarrow \quad y = -\tfrac{1}{2}(x-3)^2 - 4\ln(x+1)$$

IE $\alpha = -\frac{1}{2}$

$\beta = -4$

IYGB - FP3 PAPER V - QUESTION 7

START BY DERIVING INFORMATION BASED ON THE GIVEN SUBSTITUTION

- $t = \tan\left(\frac{1}{2}x\right)$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\left(\frac{1}{2}x\right)$

$\frac{dt}{dx} = \frac{1}{2}\left(1 + \tan^2\left(\frac{1}{2}x\right)\right)$

$\frac{dt}{dx} = \frac{1}{2}(1+t^2)$

$\frac{dx}{dt} = \frac{2}{1+t^2}$

$dx = \frac{2}{1+t^2}\,dt$

- $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2\sin\frac{x}{2}}{\cos\frac{x}{2}}\cos^2\frac{x}{2}$

$= 2\tan\frac{x}{2}\cos^2\frac{x}{2} = 2\tan\frac{x}{2} \times \frac{1}{\sec^2\frac{x}{2}}$

$= 2\tan\frac{x}{2} \times \frac{1}{1+\tan^2\frac{x}{2}} = \frac{2t}{1+t^2}$

- $\cos x = 2\cos^2\frac{x}{2} - 1 = \frac{2}{\sec^2\frac{x}{2}} - 1$

$= \frac{2}{1+\tan^2\frac{x}{2}} - 1 = \frac{2}{1+t^2} - 1$

$= \frac{2-(1+t^2)}{1+t^2} = \frac{1-t^2}{1+t^2}$

FINALLY THE LIMITS

$x=0 \longmapsto t=0$

$x=\frac{\pi}{2} \longmapsto t=1$

TRANSFORMING THE INTEGRAL

$$\int_0^{\frac{\pi}{2}} \frac{2}{1+\sin x + 2\cos x}\,dx = \int_0^1 \frac{2}{1 + \frac{2t}{1+t^2} + \frac{2(1-t^2)}{1+t^2}}\left(\frac{2}{1+t^2}\,dt\right)$$

$$= \int_0^1 \frac{4}{1+t^2+2t+2(1-t^2)}\,dt$$

$$= \int_0^1 \frac{4}{1+t^2+2t+2-2t^2}\,dt$$

$$= \int_0^1 \frac{4}{-t^2+2t+3}\,dt$$

$$= \int_1^0 \frac{4}{t^2-2t-3}\,dt$$

IYGB - FP3 PAPER V - QUESTION 7

PROCEED BY PARTIAL FRACTIONS (BY INSPECTION)

$$\int_1^0 \frac{4}{t^2-2t-3}\,dt = \int_1^0 \frac{4}{(t+1)(t-3)}\,dt$$

$$= \int_1^0 \frac{1}{t-3} - \frac{1}{t+1}\,dt$$

$$= \Big[\ln|t-3| - \ln|t+1|\Big]_1^0$$

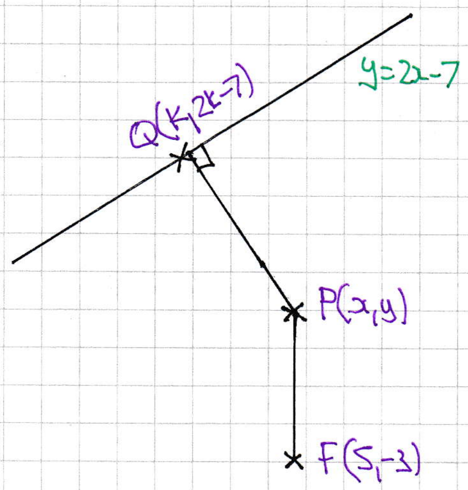
$$= \Big[\ln|-3| - \cancel{\ln 1}\Big] - \Big[\ln|-2| - \ln 2\Big]$$

$$= \ln 3 - \cancel{\ln 2} + \cancel{\ln 2}$$

$$= \ln 3$$

IYGB – FP3 PAPER U – QUESTION 8

STARTING WITH A DIAGRAM AND SOME STANDARD RESULTS



- $\frac{|PF|}{|PQ|} = e = \frac{\sqrt{5}}{10} < 1$
- GRAD PQ $= -\frac{1}{2}$

$$\frac{y-2k+7}{x-k} = -\frac{1}{2}$$

$$2y-4k+14 = -x+k$$

$$2y+x+14 = 5k$$

$$5k = 2y+x+14$$

THUS WE NOW HAVE

$$\Rightarrow |PF| = e\,|PQ|$$

$$\Rightarrow |PF|^2 = e^2|PQ|^2$$

$$\Rightarrow (x-5)^2+(y+3)^2 = \frac{1}{20}\left[(x-k)^2+(y-2k+7)^2\right]$$

MULTIPLY THE EQUATION BY 25 TO CREATE $5k = \ldots$ IN THE R.H.S

$$\Rightarrow 25\left[(x-5)^2+(y+3)^2\right] = \frac{1}{20}\left[(5x-5k)^2+(5y-2\times 5k+35)^2\right]$$

$$\Rightarrow 25\left[(x-5)^2+(y+3)^2\right] = \frac{1}{20}\left[(5x-2y-x-14)^2+(5y-4y-2x-28+35)^2\right]$$

$$\Rightarrow 25\left[(x-5)^2+(y+3)^2\right] = \frac{1}{20}\left[(4x-2y-14)^2+(y-2x+7)^2\right]$$

NOW THIS IS THE EQUATION OF THE ELLIPSE – NO NEED TO SIMPLIFY AS WE ARE ONLY INTERESTED IN INTERSECTIONS WITH $y=-3$

$$\Rightarrow 25\left[(x-5)^2+0^2\right] = \frac{1}{20}\left[(4x-8)^2+(-2x+4)^2\right]$$

$$\Rightarrow 500(x-5)^2 = (4x-8)^2+(2x-4)^2$$

$$\Rightarrow 500(x-5)^2 = 16(x-2)^2+4(x-2)^2$$

IYGB - FP3 PAPER U - QUESTION 8

$$\Rightarrow 500(x-5)^2 = 20(x-2)^2$$

$$\Rightarrow 25(x-5)^2 = (x-2)^2$$

$$\Rightarrow \frac{(x-5)^2}{(x-2)^2} = \frac{1}{25}$$

$$\Rightarrow \frac{x-5}{x-2} = \frac{1}{5} \quad \text{OR} \quad \frac{x-5}{x-2} = -\frac{1}{5}$$

$5x-25 = x-2$

$4x = 23$

$x = \frac{23}{4}$

$5x-25 = -x+2$

$6x = 27$

$2x = 9$

$x = \frac{9}{2}$

AND FINALLY WE HAVE

$$\left(\frac{23}{4}, -3\right) \text{ \& } \left(\frac{9}{2}, -3\right)$$