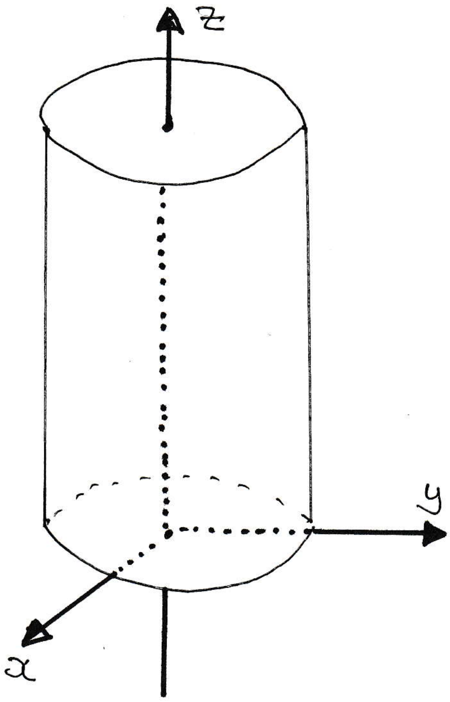
IYGB - MATHEMATICAL METHODS 1 - PAPER B - QUESTION 1



THIS REPRESENT THE CURVED SURFACE OF AN INFINITE CYLINDER, WHOSE LINE OF SYMMETRY IS THE z AXIS, AND HAS RADIUS 5

(NOTE IN THE PICTURE OPPOSITE THE CYLINDER IS SUCH SO THAT $-\infty < z < \infty$)

IYGB - MATHEMATICAL METHODS 1 - PAPER B - QUESTION 2

a)
$$\frac{1}{D^2+4D+3}\left\{30e^{-2x}\right\} = \frac{30e^{-2x}}{(-2)^2+4(-2)+3} = \frac{30e^{-2x}}{4-8+3} = -30e^{-2x}$$

b)
$$\frac{1}{D^2+4D+3}\left\{30\sin 2x\right\} = \frac{30}{-2^2+4D+3}\left\{\sin 2x\right\} = \frac{30}{4D-1}\left\{\sin 2x\right\}$$
$$= \frac{30(4D+1)}{(4D-1)(4D+1)}\left\{\sin 2x\right\} = \frac{30(4D+1)}{16D^2-1}\left\{\sin 2x\right\}$$
$$= \frac{30(4D+1)}{16(-2^2)-1}\left\{\sin 2x\right\} = \frac{30(4D+1)}{-65}\left\{\sin 2x\right\}$$
$$= -\frac{6}{13}\left[4D\left\{\sin 2x\right\} + \sin 2x\right]$$
$$= -\frac{6}{13}\left[8\cos 2x + \sin 2x\right]$$

c)
$$\frac{1}{D^2+4D+4}\left\{\underbrace{30x^2}_{V(x)}e^{-2x}\right\} = \frac{30e^{-2x}}{(D-2)^2+4(D-2)+4}\left\{x^2\right\} = \frac{30e^{-2x}}{D^2}\left\{x^2\right\}$$
$$= \frac{30e^{-2x}}{D}\times\frac{1}{D}\left\{x^2\right\} = \frac{30e^{-2x}}{D}\left\{\frac{1}{3}x^3\right\}$$
$$= 10e^{-2x}\frac{1}{D}\left\{x^3\right\} = 10e^{-2x}\times\frac{1}{4}x^4$$
$$= \frac{5}{2}x^4e^{-2x}$$

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d) $$\frac{1}{D^2+4D+4}\{30\} = \frac{1}{D^2+4D+4}\{30e^{0x}\} = \frac{30e^{0x}}{0+0+4}$$

$$= \frac{1}{4} \times 30 = \frac{15}{2}$$

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e) $$\frac{1}{D^2+D}\{30\} = \frac{1}{D(D+2)}\{30\} = \frac{1}{D}\left[\frac{1}{D+2}\{30\}\right]$$

$$= \frac{1}{D}\left[\frac{1}{D+2}\{30e^{0x}\}\right] = \frac{1}{D}\left[\frac{30e^{0x}}{0+2}\right]$$

$$= \frac{1}{D}\{15\} = 15x$$

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a) WRITE THE SYSTEM AS AN AUGMENTED MATRIX

$$\left.\begin{aligned} x+2y+3z&=5\\ 3x+y+2z&=18\\ 4x-y+z&=27 \end{aligned}\right\} \Longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 5\\ 3 & 1 & 2 & 18\\ 4 & -1 & 1 & 27 \end{array}\right]$$

$$\begin{matrix} r_{12}(-3)\\ r_{13}(-4) \end{matrix} = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 5\\ 0 & -5 & -7 & 3\\ 0 & -9 & -11 & 7 \end{array}\right] \quad r_2\left(-\tfrac{1}{5}\right) = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 5\\ 0 & 1 & \frac{7}{5} & -\frac{3}{5}\\ 0 & -9 & -11 & 7 \end{array}\right]$$

$$r_{23}(9) = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 5\\ 0 & 1 & \frac{7}{5} & -\frac{3}{5}\\ 0 & 0 & \frac{8}{5} & \frac{8}{5} \end{array}\right] \quad r_3\left(\tfrac{5}{8}\right) = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 5\\ 0 & 1 & \frac{7}{5} & -\frac{3}{5}\\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$r_{21}(-2) = \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{5} & \frac{31}{5}\\ 0 & 1 & \frac{7}{5} & -\frac{3}{5}\\ 0 & 0 & 1 & 1 \end{array}\right] \quad \begin{matrix} r_{32}\left(-\frac{7}{5}\right)\\ r_{31}\left(-\frac{1}{5}\right) \end{matrix} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6\\ 0 & 1 & 0 & -2\\ 0 & 0 & 1 & 1 \end{array}\right]$$

THUS THE UNIQUE SOLUTION IS $\begin{pmatrix} x\\ y\\ z \end{pmatrix} = \begin{pmatrix} 6\\ -2\\ 1 \end{pmatrix}$

b) PREPARE THE REQUIRED DETERMINANTS OF THE SYSTEM

$$\begin{bmatrix} 1 & 2 & 3\\ 3 & 1 & 2\\ 4 & -1 & 1 \end{bmatrix}\begin{bmatrix} x\\ y\\ z \end{bmatrix} = \begin{bmatrix} 5\\ 18\\ 27 \end{bmatrix}$$

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$$|\underline{A}| = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 4 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} + 3\begin{vmatrix} 3 & 1 \\ 4 & -1 \end{vmatrix}$$

$$= 3 - 2(-5) + 3(-7) = 3 + 10 - 21 = -8$$

$$|\underline{A}_x| = \begin{vmatrix} 5 & 2 & 3 \\ 18 & 1 & 2 \\ 27 & -1 & 1 \end{vmatrix} = 5\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} - 2\begin{vmatrix} 18 & 2 \\ 27 & 1 \end{vmatrix} + 3\begin{vmatrix} 18 & 1 \\ 27 & -1 \end{vmatrix}$$

$$= 5 \times 3 - 2(-36) + 3(-45) = 15 + 72 - 135 = -48$$

$$|\underline{A}_y| = \begin{vmatrix} 1 & 5 & 3 \\ 3 & 18 & 2 \\ 4 & 27 & 1 \end{vmatrix} = \begin{vmatrix} 18 & 2 \\ 27 & 1 \end{vmatrix} - 5\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} + 3\begin{vmatrix} 3 & 18 \\ 4 & 27 \end{vmatrix}$$

$$= -36 - 5(-5) + 3(9) = -36 + 25 + 27 = 16$$

$$|\underline{A}_z| = \begin{vmatrix} 1 & 2 & 5 \\ 3 & 1 & 18 \\ 4 & -1 & 27 \end{vmatrix} = \begin{vmatrix} 1 & 18 \\ -1 & 27 \end{vmatrix} - 2\begin{vmatrix} 3 & 18 \\ 4 & 27 \end{vmatrix} + 5\begin{vmatrix} 3 & 1 \\ 4 & -1 \end{vmatrix}$$

$$= 45 - 2(9) + 5(-7) = 45 - 18 - 35 = -8$$

HENCE BY CRAMER'S RULE

$$x = \frac{|\underline{A}_x|}{|\underline{A}|} = \frac{-48}{-8} = 6$$

$$y = \frac{|\underline{A}_y|}{|\underline{A}|} = \frac{16}{-8} = -2$$

$$z = \frac{|\underline{A}_z|}{|\underline{A}|} = \frac{-8}{-8} = 1$$

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a) ATTEMPTING TO SHOW CONVERGENCE BY COMPARISON AS THE WE HAVE $\sum_{k=1}^{\infty} O\left(\frac{1}{k^{3/2}}\right)$

$$\sum_{k=1}^{\infty}\left[\frac{\sqrt{k}}{k^2+4k+1}\right] < \sum_{k=1}^{\infty}\left[\frac{\sqrt{k}}{k^2}\right] = \sum_{k=1}^{\infty}\left[\frac{1}{k^{3/2}}\right]$$

WHICH CONVERGES BY THE "p - TEST"

$$\sum_{n=1}^{\infty}\left[\frac{1}{n^p}\right] \begin{cases} \text{CONVERGES IF } p>1 \\ \text{DIVERGES IF } p\leq 1 \end{cases}$$

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b) BY COMPARISON WE HAVE

$$\sum_{n=1}^{\infty}\left[\frac{3^n+2}{2^n+3}\right] < \sum_{n=1}^{\infty}\left[\frac{3^n+2}{2^n}\right] = \sum_{n=1}^{\infty}\left[\frac{3^n}{2^n}+\frac{2}{2^n}\right]$$

$$= \underbrace{\sum_{n=1}^{\infty}\left[\left(\frac{3}{2}\right)^n\right]}_{\text{DIVERGENT G.P.}} + \underbrace{2\sum_{n=1}^{\infty}\left[2^{-n}\right]}_{\text{CONVERGENT G.P}}$$

$$\therefore \sum_{n=1}^{\infty}\left[\frac{3^n+2}{2^n+3}\right] \text{ DIVERGES}$$

ALTERNATIVE

$$\lim_{n\to\infty}\left[\frac{3^n+2}{2^n+3}\right] = \lim_{n\to\infty}\left[\frac{\left(\frac{3}{2}\right)^n+\frac{2}{2^n}}{1+\frac{3}{2^n}}\right] \sim \left(\frac{3}{2}\right)^n \text{ AS } n\to\infty$$

IF A SERIES IS TO CONVERGE THE NECESSARY (BUT NOT SUFFICIENT) CONDITION IS THAT THE ABOVE LIMIT IS ZERO, WHICH IS NOT THE CASE HERE

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- START FROM THE INTEGRAL

$$I(m) = \int_0^1 x^m \, dx = \left[\frac{1}{m+1}x^{m+1}\right]_0^1 = \frac{1}{m+1}$$

- DIFFERENTIATE THE ABOVE EQUATION WITH RESPECT TO m (ONCE)

$$\Rightarrow \frac{\partial I}{\partial m} = \frac{\partial}{\partial m}\left(\frac{1}{m+1}\right)$$

$$\Rightarrow \frac{\partial}{\partial m}\left[\int_0^1 x^m \, dx\right] = -\frac{1}{(m+1)^2}$$

$$\Rightarrow \int_0^1 \frac{\partial}{\partial m}\left[x^m\right] dx = -\frac{1}{(m+1)^2}$$

$$\Rightarrow \int_0^1 x^m \ln x \, dx = \frac{-1}{(m+1)^2}$$

NOTE THAT
$$\frac{d}{dy}(a^y) = a^y \ln a$$

- DIFFERENTIATE THE ABOVE WITH RESPECT TO m AGAIN (TWICE SO FAR)

$$\Rightarrow \int_0^1 x^m (\ln x)^2 \, dx = \frac{(-1)(-2)}{(m+1)^3} = \frac{(-1)^2 \times 2!}{(m+1)^3}$$

- DIFFERENTIATE THE ABOVE WITH RESPECT TO m AGAIN (THREE TIMES SO FAR)

$$\Rightarrow \int_0^1 x^m (\ln x)^3 \, dx = \frac{(-1)(-2)(-3)}{(m+1)^4} = \frac{(-1)^3 \times 3!}{(m+1)^4}$$

- DIFFERENTIATING n TIMES IN TOTAL WE OBTAIN

$$\int_0^1 x^m (\ln x)^n \, dx = \frac{(-1)^n \times n!}{(m+1)^{n+1}}$$

AS REQUIRED

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- ATTEMPT THE LIMIT BY LOGARITHMS

$$\Rightarrow \lim_{x \to 0^+} \left[x^{-\sin x} \right] = L$$

$$\Rightarrow \lim_{x \to 0^+} \left[\ln\left(x^{-\sin x}\right) \right] = \ln(L)$$

$$\Rightarrow \lim_{x \to 0^+} \left[(-\sin x)(\ln x) \right] = \ln L$$

- THIS IS AN INDETERMINATE FORM OF THE TYPE "$0 \times \pm\infty$", SO WE MAY USE L'HOSPITAL'S RULE AFTER A SIMPLE MANIPULATION

$$\Rightarrow \lim_{x \to 0^+} \left[\frac{-\sin x}{\frac{1}{\ln x}} \right] = \ln L$$

THIS IS NOW OF THE FORM $\frac{0}{0}$, SO DIFFERENTIATE "TOP" AND "BOTTOM" SEPARATELY

$$\Rightarrow \lim_{x \to 0^+} \left[\frac{-\cos x}{-\frac{1}{(\ln x)^2} \times \frac{1}{x}} \right] = \ln L$$

$$\Rightarrow \lim_{x \to 0^+} \left[(x \cos x)(\ln x)^2 \right] = \ln L$$

THIS LIMIT EXISTS AS $x \to 0$, FASTER THAN $(\ln x)^2 \to \infty$

$$\Rightarrow \ln L = 0$$

$$\Rightarrow L = 1$$

$$\Rightarrow \lim_{x \to 0^+} \left[x^{-\sin x} \right] = 1$$

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- WRITE A FEW TERMS OUT SEEKING PATTERNS

$$\prod_{r=1}^{n}\left[\frac{2r}{2r+1}\right] = \frac{2}{3}\times\frac{4}{5}\times\frac{6}{7}\times \dots \times \frac{2n-4}{2n-3}\times\frac{2n-2}{2n-1}\times\frac{2n}{2n+1}$$

$$= \frac{2^n\left[1\times2\times3\times\dots\times(n-2)(n-1)n\right]}{3\times5\times7\times\dots\times(2n-3)(2n-1)(2n+1)}$$

$$= \frac{2^n\times n!}{3\times5\times7\times\dots\times(2n-3)(2n-1)(2n+1)}$$

- IN ORDER TO "COMPLETE" A FACTORIAL EXPRESSION AT THE DENOMINATOR MULTIPLY "TOP & BOTTOM" BY THE EVEN "MISSING" TERMS

$$= \frac{2^n\times n!\times\left[2\times4\times6\dots(2n-4)(2n-2)(2n)\right]}{2\times3\times4\times5\times6\times7\times\dots\times(2n-4)(2n-3)(2n-2)(2n-1)(2n)(2n+1)}$$

$$= \frac{2^n\times n!\times2^n\left[1\times2\times3\times\dots\times(n-2)(n-1)n\right]}{(2n+1)!}$$

$$= \frac{(2^n)^2\times n!\times n!}{(2n+1)!}$$

$$= \frac{4^n\times(n!)^2}{(2n+1)!}$$

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- TAKING THE LAPLACE TRANSFORM OF THE O.D.E, W.R.T t

$$\Longrightarrow \frac{dx}{dt} - 2x = 4 \qquad [t=0, x=1]$$

$$\Longrightarrow \mathcal{L}\left[\frac{dx}{dt}\right] - \mathcal{L}[2x] = \mathcal{L}[4]$$

$$\Longrightarrow s\bar{x} - x_0 - 2\bar{x} = \frac{4}{s}$$

$$\Longrightarrow s\bar{x} - 1 - 2\bar{x} = \frac{4}{s}$$

$$\Longrightarrow (s-2)\bar{x} = \frac{4}{s} + 1$$

$$\Longrightarrow (s-2)\bar{x} = \frac{4+s}{s}$$

$$\Longrightarrow \bar{x} = \frac{s+4}{s(s-2)}$$

- INVERT BY PARTIAL FRACTIONS (COVER UP)

$$\Longrightarrow \bar{x} = \frac{3}{s-2} - \frac{2}{s}$$

$$\Longrightarrow x = \mathcal{L}^{-1}\left[\frac{3}{s-2} - \frac{2}{s}\right]$$

- THESE ARE SIMPLE STANDARD RESULTS

$$\Longrightarrow x(t) = 3e^{2t} - 2$$

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- USING THE STANDARD FOURIER SERIES FORMULA

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right] \qquad L = \text{HALF PERIOD} = \frac{a+b}{2}$$

WHERE $a_0 = \frac{1}{L}\int_a^b f(x)\,dx$

$$a_n = \frac{1}{L}\int_a^b f(x)\cos\frac{n\pi x}{L}\,dx \qquad n = 1,2,3,4,\ldots$$

$$b_n = \frac{1}{L}\int_a^b f(x)\sin\frac{n\pi x}{L}\,dx \qquad n = 1,2,3,4,\ldots$$

- USING THE ABOVE RESULTS, WITH $a=-\pi$, $b=\pi$, $f(x)=x$ WE OBTAIN

- $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = 0$ (ODD INTEGRAND IN A SYMMETRICAL DOMAIN)

- $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underset{\text{ODD}}{x}\,\underset{\text{EVEN}}{\cos nx}\,dx = 0$ (ODD INTEGRAND IN A SYMMETRICAL DOMAIN)

- $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$

- PROCEED BY INTEGRATION BY PARTS

$$b_n = \frac{2}{\pi}\left\{\left[-\frac{x}{n}\cos nx\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,dx\right\}$$

x	1
$-\frac{1}{n}\cos nx$	$\sin nx$

$$b_n = \frac{2}{\pi}\left\{\left[\frac{x\cos nx}{n}\right]_{\pi}^{0} + \cancel{\left[\frac{1}{n^2}\sin nx\right]_0^{\pi}}\right\}$$

$$\Rightarrow b_n = \frac{2}{\cancel{\pi}}\left[0 - \frac{\cancel{\pi}\cos n\pi}{n}\right]$$

$$\Rightarrow b_n = -\frac{2\cos n\pi}{n}$$

$$\Rightarrow b_n = -\frac{2}{n}(-1)^n$$

- FINALLY WE HAVE THE FOURIER SERIES

$$f(x) = \sum_{n=1}^{\infty}\left[-\frac{2}{n}(-1)^n \sin nx\right]$$

$$f(x) = 2\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{n}\sin nx\right]$$

$$x = 2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \dots\right]$$

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a) USING STANDARD POLARS THE RESULT IS TRIVIAL

$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} \Rightarrow \begin{array}{l} r^2 = x^2+y^2 \\ \tan\theta = \frac{y}{x} \end{array}$$

HENCE WE HAVE

$$z = r^2\tan\theta = (x^2+y^2)\left(\frac{y}{x}\right) = xy + y^3x^{-1}$$

- $$\frac{\partial z}{\partial x} = y - xy^3x^{-2} = y - \frac{xy^3}{x^2} = \frac{x^2y - xy^3}{x^2} = \frac{y(x^2-y^2)}{x^2}$$

- $$\frac{\partial z}{\partial y} = x + 3y^2x^{-1} = x + \frac{3y^2}{x} = \frac{x^2+3y^2}{x}$$

b) FIRSTLY COMPUTE THE JACOBIAN

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$J = r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta+\sin^2\theta) = r$$

NEXT WE USE THE STANDARD RESULT

$$\begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{bmatrix} = \frac{1}{J}\begin{bmatrix} \frac{\partial y}{\partial \theta} & -\frac{\partial x}{\partial \theta} \\ -\frac{\partial y}{\partial r} & \frac{\partial x}{\partial r} \end{bmatrix}$$

NOTICE THE MATRIX INVERSE RESEMBLANCE

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DEALING WITH EACH OF THE ENTRIES OF THE MATRIX

- $\frac{\partial r}{\partial x} = \frac{1}{J}\frac{\partial y}{\partial \theta} = \frac{1}{r}(r\cos\theta) = \cos\theta$
- $\frac{\partial r}{\partial y} = -\frac{1}{J}\frac{\partial x}{\partial \theta} = -\frac{1}{r}(-r\sin\theta) = \sin\theta$
- $\frac{\partial \theta}{\partial x} = -\frac{1}{J}\frac{\partial y}{\partial r} = -\frac{1}{r}(\sin\theta) = -\frac{\sin\theta}{r}$
- $\frac{\partial \theta}{\partial y} = \frac{1}{J}\frac{\partial x}{\partial r} = \frac{1}{r}(\cos\theta) = \frac{\cos\theta}{r}$

NOW BY THE CHAIN RULE WE HAVE

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial x} = (2r\tan\theta)(\cos\theta) + (r^2\sec^2\theta)\left(-\frac{\sin\theta}{r}\right)$$

$$= 2r\sin\theta - r\tan\theta\sec\theta$$

$$= 2(r\sin\theta) - r\tan\theta\left(\frac{1}{\cos\theta}\right)$$

$$= 2y - r\left(\frac{y}{x}\right)\left(\frac{r}{x}\right)$$

$$= 2y - \frac{r^2 y}{x^2} = 2y - \frac{y(x^2+y^2)}{x^2}$$

$$= \frac{2yx^2 - x^2y - y^3}{x^2} = \frac{yx^2 - y^3}{x^2}$$

$$= \frac{y(x^2-y^2)}{x^2}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial y} = (2r\tan\theta)\sin\theta + (r^2\sec^2\theta)\left(\frac{\cos\theta}{r}\right)$$

$$= 2\tan\theta(r\sin\theta) + r\cos\theta \times \frac{1}{\cos^2\theta}$$

$$= 2\left(\frac{y}{x}\right)y + x\left(\frac{r^2}{x^2}\right) = \frac{2y^2}{x} + \frac{r^2}{x}$$

$$= \frac{2y^2+r^2}{x} = \frac{2y^2+x^2+y^2}{x} = \frac{x^2+3y^2}{x}$$

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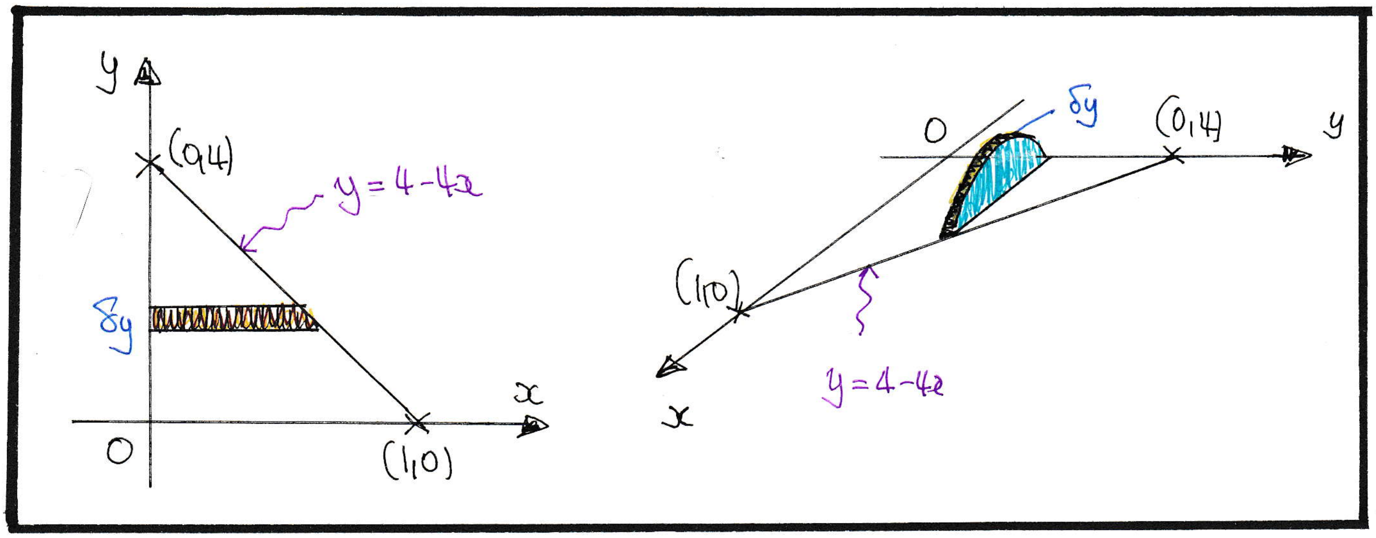
- START BY DRAWING SKETCH TO "SEE" THE INFINITESIMAL VOLUME



- REARRANGE THE EQUATION OF THE LINE FOR x

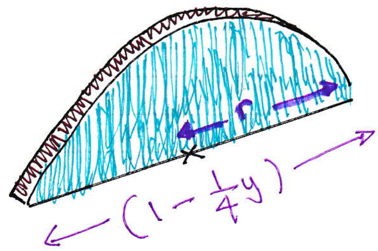
$$y = 4 - 4x$$
$$4x = 4 - y$$
$$x = 1 - \tfrac{1}{4}y$$

- HENCE THE RADIUS OF AN INFINITESIMAL ARBITRARY SEMICIRCLE WILL BE

$$r = \tfrac{1}{2}x = \tfrac{1}{2} - \tfrac{1}{8}y$$

- THE VOLUME OF THE INFINITESIMAL SLICE WILL BE

$$\delta V = \underbrace{\tfrac{1}{2}\pi\left(\tfrac{1}{2} - \tfrac{1}{8}y\right)^2}_{\text{"}\frac{1}{2}\pi r^2\text{"}}\delta y$$



- SUMMING AND TAKING LIMITS

$$\Rightarrow V = \sum \delta V = \sum \left[\tfrac{1}{2}\pi\left(\tfrac{1}{2} - \tfrac{1}{8}y\right)^2 \delta y\right]$$

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$$\Rightarrow V = \int_{y=0}^{y=4} \frac{1}{2}\pi\left(\frac{1}{2} - \frac{1}{8}y\right)^2 \, dy$$

$$\Rightarrow V = \int_{y=0}^{y=4} \frac{1}{2}\pi\left(\frac{1}{8}\right)^2 (4-y)^2 \, dy$$

$$\Rightarrow V = \int_{y=0}^{y=4} \frac{\pi}{128}(4-y)^2 \, dy$$

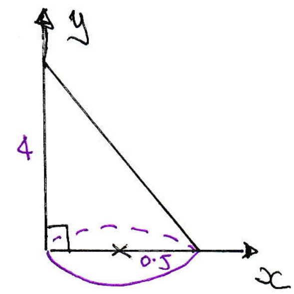
$$\Rightarrow V = \frac{\pi}{128}\left[-\frac{1}{3}(4-y)^3\right]_0^4$$

$$\Rightarrow V = \frac{\pi}{128} \times \frac{1}{3}\left[(4-y)^3\right]_4^0$$

$$\Rightarrow V = \frac{\pi}{\cancel{128}_{2}} \times \frac{1}{3} \times (\cancel{64} - 0)$$

$$\Rightarrow V = \frac{\pi}{6}$$

CONFIRMING THE ABOVE RESULT BY THE STANDARD FORMULA FOR A CONE



$$V = \frac{1}{2}\left[\frac{1}{3}\pi r^2 h\right] = \frac{1}{6}\pi \times 0.5^2 \times 4$$

$\Uparrow$

HALF A CONE

$$= \frac{1}{6}\pi \times \frac{1}{4} \times 4$$

$$= \frac{\pi}{6}$$

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$$u_{n+2} = 2u_{n+1} - 2u_n \quad , \qquad u_1 = 1 \ , \ u_2 = 6$$

START WITH THE AUXILIARY EQUATION OF THE DIFFERENCE EQUATION

$$\Rightarrow \lambda^2 = 2\lambda - 2$$

$$\Rightarrow \lambda^2 - 2\lambda = -2$$

$$\Rightarrow \lambda^2 - 2\lambda + 1 = -1$$

$$\Rightarrow (\lambda - 1)^2 = -1$$

$$\Rightarrow \lambda - 1 = \pm i$$

$$\Rightarrow \lambda = 1 \pm i$$

THE GENERAL SOLUTION IS

$$u_n = A(1+i)^n + B(1-i)^n$$

WHERE A, B MAY BE COMPLEX

APPLYING CONDITIONS

- $u_1 = 1 \Rightarrow$
- $u_2 = 6 \Rightarrow$

$$\left. \begin{array}{l} A(1+i) + B(1-i) = 1 \\ A(1+i)^2 + B(1-i)^2 = 6 \end{array} \right\} \Rightarrow$$

$$\left\{ \begin{array}{l} A(1+i) + B(1-i) = 1 \\ A(2i) + B(-2i) = 6 \end{array} \right\} \Rightarrow$$

$$\left\{ \begin{array}{l} A(1+i) + B(1-i) = 1 \\ Ai - Bi = 3 \end{array} \right\} \Rightarrow$$

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REARRANGE THE SECOND EQUATION & SUB INTO THE FIRST

$$\Rightarrow Ai - Bi = 3$$
$$\Rightarrow A - B = \frac{3}{i}$$
$$\Rightarrow A - B = -3i$$
$$\Rightarrow A = B - 3i$$

$$\Rightarrow A(1+i) + B(1-i) = 1$$
$$\Rightarrow (B-3i)(1+i) + B(1-i) = 1$$
$$\Rightarrow B(1+i) - 3i(1+i) + B(1-i) = 1$$
$$\Rightarrow 2B = 1 + 3i(1+i)$$
$$\Rightarrow 2B = 1 + 3i - 3$$
$$\Rightarrow B = -1 + \frac{3}{2}i \quad \text{OR} \quad \frac{1}{2}(-2+3i)$$

AND HENCE

$$A = -1 + \frac{3}{2}i - 3i$$
$$A = -1 - \frac{3}{2}i \quad \text{OR} \quad -\frac{1}{2}(2+3i)$$

THE SOLUTION CAN NOW BE WRITTEN AS

$$u_n = \frac{1}{2}(-2+3i)(1-i)^n - \frac{1}{2}(2+3i)(1+i)^n$$

PROCEED TO ELIMINATE THE COMPLEX NUMBERS AS FOLLOWS

$$\Rightarrow u_n = \frac{1}{2}\left[(-2+3i)\left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)^n - (2+3i)\left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^n\right]$$
$$\Rightarrow u_n = \frac{1}{2}\left[(-2+3i)\left(2^{\frac{n}{2}}e^{-i\frac{\pi n}{4}}\right) - (2+3i)\left(2^{\frac{n}{2}}e^{i\frac{\pi n}{4}}\right)\right]$$
$$\Rightarrow u_n = \frac{1}{2}\times 2^{\frac{n}{2}}\left[-2e^{-i\frac{\pi n}{4}} + 3ie^{-i\frac{\pi n}{4}} - 2e^{i\frac{\pi n}{4}} - 3ie^{i\frac{\pi n}{4}}\right]$$

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$$\Longrightarrow u_n = \frac{1}{2} \times 2^{\frac{n}{2}}\left[-3i\left(e^{i\frac{\pi n}{4}} - e^{-i\frac{\pi n}{4}}\right) - 2\left(e^{i\frac{\pi n}{4}} + e^{-i\frac{\pi n}{4}}\right)\right]$$

$$\Longrightarrow u_n = \frac{1}{2} \times 2^{\frac{n}{2}}\left[-3i \times 2\sinh\left(\frac{i\pi n}{4}\right) - 2 \times 2\cosh\left(\frac{i\pi n}{4}\right)\right]$$

$$\Longrightarrow u_n = \frac{1}{2} \times 2^{\frac{n}{2}}\left[-3i \times 2i\sin\left(\frac{\pi n}{4}\right) - 2 \times 2\cos\frac{\pi n}{4}\right]$$

$$\Longrightarrow u_n = 2^{\frac{n}{2}}\left[3\sin\frac{\pi n}{4} - 2\cos\frac{\pi n}{4}\right]$$

ALTERNATIVE SOLUTION - QUICKER & MORE DIRECT

STARTING WITH THE GENERAL SOLUTION FROM EARLIER

$$u_n = A(1+i)^n + B(1-i)^n$$

MANIPULATE FURTHER BEFORE USING THE CONDITIONS

$$u_n = A\left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^n + B\left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)^n$$

$$u_n = A\left(2^{\frac{n}{2}}e^{i\frac{\pi n}{4}}\right) + B\left(2^{\frac{n}{2}}e^{-i\frac{\pi n}{4}}\right)$$

$$u_n = 2^{\frac{n}{2}}\left[A\cos\frac{\pi n}{4} + Ai\sin\frac{\pi n}{4} + B\cos\frac{\pi n}{4} - Bi\sin\frac{\pi n}{4}\right]$$

$$u_n = 2^{\frac{n}{2}}\left[(A+B)\cos\frac{\pi n}{4} + i(A-B)\sin\frac{n\pi}{4}\right]$$

$$u_n = 2^{\frac{n}{2}}\left[P\cos\frac{\pi n}{4} + Q\sin\frac{n\pi}{4}\right]$$

APPLY CONDITIONS

- $u_1 = 1 \Longrightarrow 2^{\frac{1}{2}}\left[P\cos\frac{\pi}{4} + Q\sin\frac{\pi}{4}\right] = 1$
- $u_2 = 6 \Longrightarrow 2^{1}\left[P\cos\frac{\pi}{2} + Q\sin\frac{\pi}{2}\right] = 6$

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$$\Rightarrow \begin{cases} \sqrt{2}\left(P\times\frac{1}{\sqrt{2}} + Q\times\frac{1}{\sqrt{2}}\right) = 1 \\ 2\left(P\times 0 + Q\times 1\right) = 6 \end{cases}$$

$$\therefore \quad \begin{aligned} P + Q &= 1 \\ 2Q &= 6 \end{aligned}$$

$$\therefore \quad \begin{aligned} Q &= 3 \\ P &= -2 \end{aligned}$$

$$\therefore \quad u_n = 2^{\frac{n}{2}}\left[3\sin\frac{n\pi}{4} - 2\cos\frac{n\pi}{4}\right]$$

AS BEFORE

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$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \frac{e^x}{x}, \quad x>0$$

START BY FINDING THE COMPLIMENTARY FUNCTION

$$\Longrightarrow \lambda^2 - 2\lambda + 1 = 0$$

$$\Longrightarrow (\lambda - 1)^2 = 0$$

$$\Longrightarrow \lambda = 1 \quad \text{(REPEATED)}$$

COMPLIMENTARY FUNCTION : $y = Ae^x + Bxe^x$

BY THE METHOD OF VARIATION OF PARAMETERS

$$\underbrace{1}_{a(x)}\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \underbrace{\left\{\frac{e^x}{x}\right\}}_{f(x)}$$

- $a(x) = 1$
- $e_1 = e^x$
- $e_2 = xe^x$
- $f(x) = \frac{e^x}{x}$

NEXT WE OBTAIN THE WRONSKIAN, $W(x)$

$$W(x) = \begin{vmatrix} e_1 & e_2 \\ e_1' & e_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x} + \cancel{xe^{2x}} - \cancel{xe^{2x}} = e^{2x}$$

USING THE STANDARD FORMULA FOR THE PARTICULAR INTEGRAL, $y_p(x)$

$$\Longrightarrow y_p = -e_1 \int \frac{e_2 f(x)}{a(x)W(x)}\, dx + e_2 \int \frac{e_1 f(x)}{a(x)W(x)}\, dx$$

$$\Longrightarrow y_p = -e^x \int \frac{(xe^x)\left(\frac{e^x}{x}\right)}{1 \times e^{2x}}\, dx + xe^x \int \frac{(e^x)\left(\frac{e^x}{x}\right)}{1 \times e^{2x}}\, dx$$

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$$\Longrightarrow y_p = -e^x \int 1 \, dx + xe^x \int \frac{1}{x} \, dx$$

$$\Longrightarrow y_p = -e^x \times x + xe^x \times \ln x$$

$$\Longrightarrow y_p = -xe^x + xe^x \ln x$$

$$\Longrightarrow y_p = xe^x (\ln x - 1)$$

FINALLY WE HAVE A GENERAL SOLUTION

$$y = Ae^x + Bxe^x + xe^x(\ln x - 1)$$

or

$$y = Ae^x + xe^x[B - 1 + \ln x]$$

RELABELLING

$$y = Ae^x + xe^x(B + \ln x)$$

IYGB - MATHEMATICAL METHODS 1 - PAPER B - QUESTION 14

- START BY WRITING THE CONIC IN THE USUAL MATRIX FORM

$$\Longrightarrow 5x^2 - 4xy + 8y^2 = 36$$

$$\Longrightarrow \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 36$$

$$\Longrightarrow$$

- THE MATRIX ABOVE BEING SYMMETRIC WILL DIAGONALIZE WITH NORMALIZED EIGENVECTORS — START WITH THE CHARACTERISTIC EQUATION

$$\begin{vmatrix} 5-\lambda & -2 \\ -2 & 8-\lambda \end{vmatrix} = 0 \Longrightarrow (5-\lambda)(8-\lambda) - 4 = 0$$

$$\Longrightarrow (\lambda-5)(\lambda-8) - 4 = 0$$

$$\Longrightarrow \lambda^2 - 13\lambda + 40 - 4 = 0$$

$$\Longrightarrow \lambda^2 - 13\lambda + 36 = 0$$

$$\Longrightarrow (\lambda \quad 4)(\lambda \quad 9) = 0$$

$$\Longrightarrow \lambda = \begin{cases} 4 \\ 9 \end{cases}$$

- IF $\lambda = 4$

$$\left.\begin{aligned} 5x - 2y &= 4x \\ -2x + 8y &= 4y \end{aligned}\right\} \text{BOTH GIVE } x = 2y$$

- EIGENVECTOR: $\alpha \begin{pmatrix} 2 \\ 1 \end{pmatrix}$
- NORMALIZED EIGENVECTOR: $\begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$

- IF $\lambda = 9$

$$\left.\begin{aligned} 5x - 2y &= 9x \\ -2x + 8y &= 9y \end{aligned}\right\} \text{BOTH GIVE } y = -2x$$

- EIGENVECTOR: $\beta \begin{pmatrix} 1 \\ -2 \end{pmatrix}$
- NORMALIZED EIGENVECTOR: $\begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix}$

IYGB - MATHEMATICAL METHODS 1 - PAPER B - QUESTION 14

- HENCE THE MATRIX $\underline{P}$ WHICH DIAGONALIZES THE MATRIX INTO $\underline{D}$

$$\underline{P} = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{bmatrix} \quad \& \quad \underline{D} = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$$

- SO WE HAVE IN NEW CO-ORDINATES (X, Y)

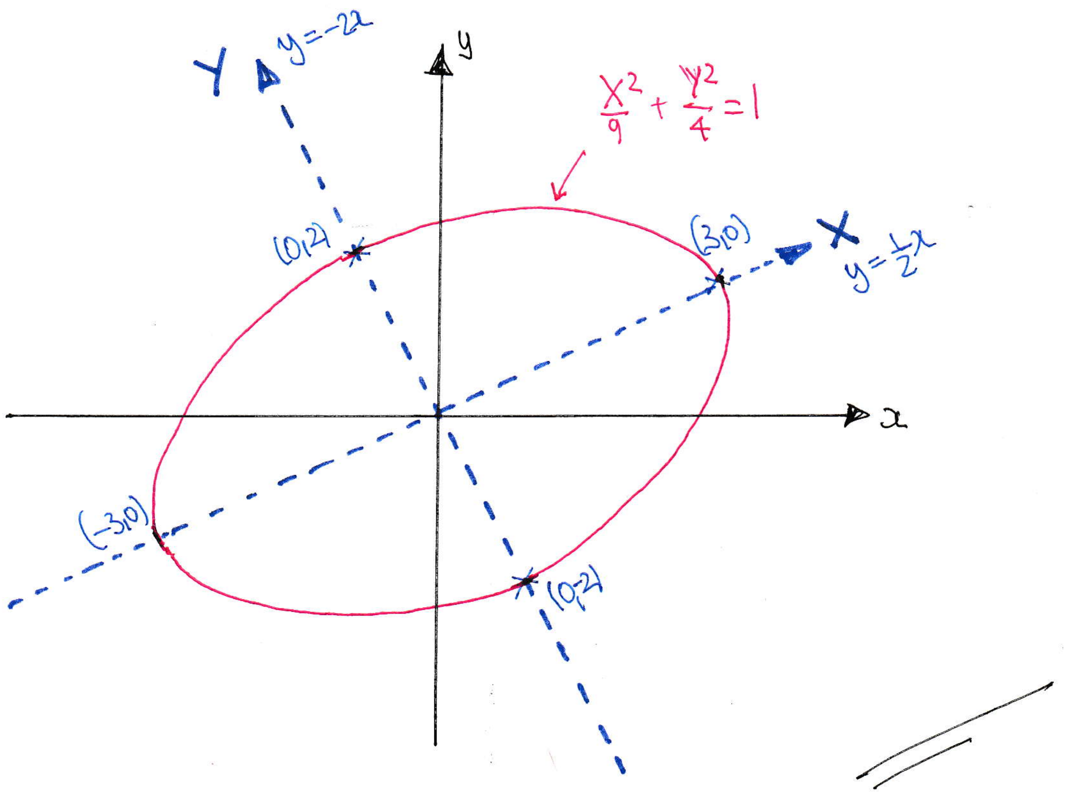
$$\Longrightarrow (X \ Y)\begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = 36$$

$$\Longrightarrow 4X^2 + 9Y^2 = 36$$

$$\Longrightarrow \frac{X^2}{9} + \frac{Y^2}{4} = 1$$

IE AN ELLIPSE

- FINALLY THE SKETCH



IYGB, MATHEMATICAL METHODS 1 – PAPER B – QUESTION 15

USING LAGRANGE'S METHOD, WE HAVE IN THE USUAL NOTATION

- OBJECTIVE FUNCTION

$$f(x,y) = x^2+y^2$$

- CONSTRAINT

$$(x+1)^2+(y-1)^2 = 1$$
$$x^2+2x+1+y^2-2y+1 = 1$$
$$x^2+y^2+2x-2y+1 = 0$$
$$\phi(x,y) = x^2+y^2+2x-2y+1$$

HENCE WE HAVE THE FOLLOWING EQUATIONS

$$\left.\begin{array}{ll} \text{(I)} & \dfrac{\partial f}{\partial x} + \lambda\dfrac{\partial \phi}{\partial x} = 0 \\ \text{(II)} & \dfrac{\partial f}{\partial y} + \lambda\dfrac{\partial \phi}{\partial y} = 0 \\ \text{(III)} & \phi(x,y) = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} 2x+\lambda(2x+2) = 0 \\ 2y+\lambda(2y-2) = 0 \\ x^2+y^2+2x-2y+1 = 0 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left\{\begin{array}{l} x = -\lambda(x+1) \\ y = -\lambda(y-1) \\ x^2+y^2+2x-2y+1 = 0 \end{array}\right\} \Rightarrow$$

DIVIDING THE FIRST TWO EQUATIONS

$$\Rightarrow \frac{x}{y} = \frac{x+1}{y-1}$$

$$\Rightarrow \cancel{xy}-x = \cancel{xy}+y$$

$$\Rightarrow y = -x$$

IYGB - MATHEMATICAL METHODS I - PAPER B - QUESTION 15

SUBSTITUTE INTO EQUATION (III)

$$\Rightarrow x^2+y^2+2x-2y+1=0$$
$$\Rightarrow x^2+(-x)^2+2x-2(-x)+1=0$$
$$\Rightarrow x^2+x^2+2x+2x+1=0$$
$$\Rightarrow 2x^2+4x+1=0$$
$$\Rightarrow x^2+2x+\tfrac{1}{2}=0$$
$$\Rightarrow x^2+2x+1=\tfrac{1}{2}$$
$$\Rightarrow (x+1)^2=\tfrac{1}{2}$$
$$\Rightarrow x+1 = \begin{cases} +\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{cases}$$
$$\Rightarrow x = \begin{cases} -1+\frac{\sqrt{2}}{2} = \frac{-2+\sqrt{2}}{2} \\ -1-\frac{\sqrt{2}}{2} = \frac{-2-\sqrt{2}}{2} \end{cases} \qquad y = \begin{cases} \frac{2-\sqrt{2}}{2} \\ \frac{2+\sqrt{2}}{2} \end{cases}$$

FINALLY WE OBTAIN BY SUBSTITUTING INTO $f(x,y)=x^2+y^2$

- $f\left[\frac{-2+\sqrt{2}}{2},\frac{2-\sqrt{2}}{2}\right] = \left(\frac{-2+\sqrt{2}}{2}\right)^2+\left(\frac{2-\sqrt{2}}{2}\right)^2 = \frac{1}{4}\left[4-4\sqrt{2}+2+4-4\sqrt{2}+2\right]$
$$= \tfrac{1}{4}\left[12-8\sqrt{2}\right] = \underline{3-2\sqrt{2}}$$

- $f\left[\frac{-2-\sqrt{2}}{2},\frac{2+\sqrt{2}}{2}\right] = \left(\frac{-2-\sqrt{2}}{2}\right)^2+\left(\frac{2+\sqrt{2}}{2}\right)^2 = \frac{1}{4}\left[4+4\sqrt{2}+2+4+4\sqrt{2}+2\right]$
$$= \tfrac{1}{4}\left[12+8\sqrt{2}\right] = \underline{3+2\sqrt{2}}$$

$$\therefore \; f(x,y)_{MIN} = 3-2\sqrt{2}$$
$$f(x,y)_{MAX} = 3+2\sqrt{2}$$

IYGB - MATHEMATICAL METHODS 1 - PAPER B - QUESTION 16

FIND THE VOLUME BY THE "SHELL METHOD"

- CONSIDER AN INFINITESIMAL STRIP OF THICKNESS δy, ROTATED ABOUT THE x AXIS, FORMING A THIN INFINITESIMAL TUBE, AS IN THE DIAGRAMS OPPOSITE

- THE VOLUME OF THE INFINITESIMAL TUBE IS

$$\Rightarrow \delta V = \pi x(y+\delta y)^2 - \pi x y^2$$
$$\Rightarrow \delta V = \pi x\left[(y+\delta\ )^2 - y^2\right]$$
$$\Rightarrow \delta V = \pi x\left[\cancel{y^2} + 2y\delta y + \delta y^2 - \cancel{y^2}\right]$$
$$\Rightarrow \delta V = \pi x\left[2y\delta y + \delta y^2\right]$$
$$\Rightarrow \delta V = 2\pi x y\,\delta y + \cancel{\pi x \delta y^2}$$
$$\Rightarrow \delta V \approx 2\pi x y\,\delta y$$

IGNORED COMPARED TO δy

- SUMMING UP ALL THESE TUBES FROM $y=2$ TO $y=6$

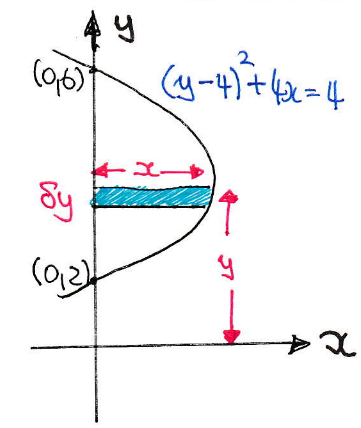
$$V = \sum \delta V = \sum(2\pi x y\,\delta y)$$

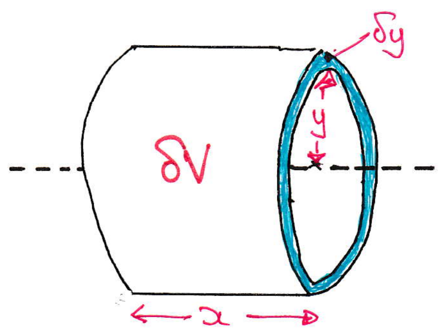
$x=0$

$(y-4)^2=4$

$y-4 = \begin{cases} 2 \\ -2 \end{cases}$

$y = \begin{cases} 6 \\ 2 \end{cases}$





TAKING LIMITS AND CARRY OUT THE RESULTING INTEGRATIONS

$$V = \int_{y=2}^{6} 2\pi x y\ dy = \int_2^6 2\pi y\left[1 - \tfrac{1}{4}(y-4)^2\right]\ dy$$

$$V = \int_2^6 2\pi y\left[1 - \tfrac{1}{4}(y^2 - 8y + 16)\right]\ dy$$

$$\Rightarrow V = \pi \int_{2}^{6} 2y \left[1 - \tfrac{1}{4}y^2 + 2y - 4 \right] dy$$

$$\Rightarrow V = \pi \int_{2}^{6} 2y \left[-\tfrac{1}{4}y^2 + 2y - 3 \right] dy$$

$$\Rightarrow V = \pi \int_{2}^{6} -\tfrac{1}{2}y^3 + 4y^2 - 6y \, dy$$

$$\Rightarrow V = \pi \left[-\tfrac{1}{8}y^4 + \tfrac{4}{3}y^3 - 3y^2 \right]_{2}^{6}$$

$$\Rightarrow V = \pi \left[(-162 + 288 - 108) - (-2 + \tfrac{32}{3} - 12) \right]$$

$$\Rightarrow V = \pi \left[18 - (-\tfrac{10}{3}) \right]$$

$$\Rightarrow V = \frac{64\pi}{3}$$

IYGB - MATHEMATICAL METHODS 1 - PAPER B - QUESTION 17

a) • START BY RE-WRITING THE O.D.E

$$\Rightarrow x^2\frac{dy}{dx} + xy = y^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2 - xy}{x^2}$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{y}{x}\right)^2 - \left(\frac{y}{x}\right)$$

• THIS IS A STANDARD FIRST ORDER HOMOGENEOUS O.D.E AS IT IS OF THE FORM $y = f\left(\frac{y}{x}\right)$

$$V = \frac{y}{x} \Rightarrow y = xV$$

$$\Rightarrow \frac{dy}{dx} = V + x\frac{dV}{dx}$$

• HENCE WE MAY TRANSFORM THE O.D.E TO

$$\Rightarrow V + x\frac{dV}{dx} = V^2 - V$$

$$\Rightarrow x\frac{dV}{dx} = V^2 - 2V$$

$$\Rightarrow \frac{1}{V^2 - 2V}\,dV = \frac{1}{x}\,dx$$

$$\Rightarrow \int \frac{1}{V(V-2)}\,dV = \int \frac{1}{x}\,dx$$

• PARTIAL FRACTIONS BY INSPECTION

$$\Rightarrow \int \frac{\frac{1}{2}}{V-2} - \frac{\frac{1}{2}}{V}\,dV = \int \frac{1}{x}\,dx$$

$$\Rightarrow \int \frac{1}{V-2} - \frac{1}{V}\,dV = \int \frac{2}{x}\,dx$$

IYGB - MATHEMATICAL METHODS 1 - PAPER B - QUESTION 17

$$\Longrightarrow \ln|v-2| - \ln|v| = 2\ln|x| + \ln A$$

$$\Longrightarrow \ln\left|\frac{v-2}{v}\right| = \ln(Ax^2)$$

$$\Longrightarrow \frac{v-2}{v} = Ax^2$$

$$\Longrightarrow \frac{\frac{y}{x}-2}{\frac{y}{x}} = Ax^2$$

$$\Longrightarrow \frac{y-2x}{y} = Ax^2$$

$$\Longrightarrow 1 - \frac{2x}{y} = Ax^2$$

$$\Longrightarrow 1 + Ax^2 = \frac{2x}{y}$$

$$\Longrightarrow y = \frac{2x}{1+Ax^2}$$

- FINALLY APPLY CONDITIONS $y(\frac{1}{2}) = 2$

$$2 = \frac{1}{1 + A \times \frac{1}{4}}$$

$$1 + \frac{1}{4}A = \frac{1}{2}$$

$$\frac{1}{4}A = -\frac{1}{2}$$

$$A = -2$$

- HENCE WE OBTAIN

$$y = \frac{2x}{1-2x^2}$$

b) • LOOKING AT THE ODE ONCE DIVIDED THROUGH BY x^2

$$\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$$

• THIS IS A BERNOULLI EQUATION ALSO, AS IT IS OF THE FORM

$$\frac{dy}{dx} + y f(x) = y^n g(x) \qquad n \neq -1$$

WHICH IS SOLVED BY THE SUBSTITUTION

$$z = \frac{1}{y^{n-1}}$$

• IN THIS EXAMPLE WE HAVE

$$z = \frac{1}{y}$$

$$\frac{dz}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$$

• MULTIPLY THE O.D.E BY $-\frac{1}{y^2}$ TO OBTAIN

$$\Longrightarrow -\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{xy} = -\frac{1}{x^2}$$

$$\Longrightarrow \frac{dz}{dx} - \frac{z}{x} = -\frac{1}{x^2}$$

• THIS IS A STANDARD FIRST ORDER WITH INTEGRATING FACTOR

$$e^{\int -\frac{1}{x}\,dx} = e^{-\ln x} = e^{\ln\left(\frac{1}{x}\right)} = \frac{1}{x}$$

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- MULTIPLYING BY THE INTEGRATING ~~FACTOR~~ MAKES THE ODE EXACT

$$\Rightarrow \frac{1}{x}\frac{dz}{dx} - \frac{z}{x^2} = -\frac{1}{x^3}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{z}{x}\right) = -\frac{1}{x^3}$$

$$\Rightarrow \frac{z}{x} = \int -\frac{1}{x^3}\,dx$$

$$\Rightarrow \frac{z}{x} = \frac{1}{2x^2} + C$$

$$\Rightarrow z = \frac{1}{2x} + Cx$$

$$\Rightarrow \frac{1}{y} = \frac{1+Cx^2}{2x}$$

$$\Rightarrow y = \frac{2x}{1+Cx^2}$$

- APPLYING CONDITION $y\left(\frac{1}{2}\right)=2$, GIVES $C=-2$ AS IN PART (a), YIELDING THE SAME SOLUTION

$$\boxed{y = \frac{2x}{1-2x^2}}$$

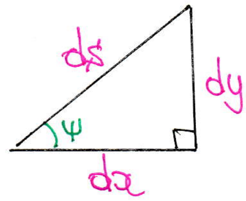
IYGB – MATHEMATICAL METHODS 1 – PAPER B – QUESTION 18

- WE ARE GIVEN THE INTRINSIC EQUATION & CONDITIONS – PROCEED WITH THE STANDARD SET UP

$$s = 4\sin\psi$$

$$s=0,\ \psi=0,\ x=0,\ y=0$$

$$\rho = \frac{ds}{d\psi} = 4\cos\psi$$



- $\dfrac{dy}{ds} = \sin\psi$
- $\dfrac{dx}{ds} = \cos\psi$

- SOLVING EACH OF THE DIFFERENTIAL EQUATIONS

$$\Rightarrow \frac{dx}{ds} = \cos\psi$$

$$\Rightarrow dx = \cos\psi \, ds$$

$$\Rightarrow dx = \cos\psi \frac{ds}{d\psi} d\psi$$

$$\Rightarrow dx = \cos\psi (4\cos\psi)\, d\psi$$

$$\Rightarrow dx = 4\cos^2\psi \, d\psi$$

$$\Rightarrow \int_{x=0}^{x} 1 \, dx = \int_{\psi=0}^{\psi} 4\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2\psi\right) d\psi$$

$$\Rightarrow \int_0^x 1 \, dx = \int_0^{\psi} 2 + 2\cos 2\psi \, d\psi$$

$$\Rightarrow \frac{dy}{ds} = \sin\psi$$

$$\Rightarrow dy = \sin\psi \, ds$$

$$\Rightarrow dy = \sin\psi \frac{ds}{d\psi} d\psi$$

$$\Rightarrow dy = \sin\psi (4\cos\psi)\, d\psi$$

$$\Rightarrow dy = 2\sin 2\psi \, d\psi$$

$$\Rightarrow \int_{y=0}^{y} 1 \, dy = \int_{\psi=0}^{\psi} 2\sin 2\psi \, d\psi$$

$$\Rightarrow \left[y \right]_0^y = \left[-\cos 2\psi \right]_0^{\psi}$$

IYGB - MATHEMATICAL METHODS 1 - PAPER B - QUESTION 18

$\Rightarrow \left[x\right]_0^x = \left[2\psi + \sin 2\psi\right]_0^\psi$

$\Rightarrow x - 0 = (2\psi + \sin 2\psi) - 0$

$\Rightarrow \underline{x = 2\psi + \sin 2\psi}$

$\Rightarrow y - 0 = -\cos 2\psi + 1$

$\Rightarrow \underline{y = 1 - \cos 2\psi}$

- FINALLY WE LET $t = 2\psi$, $0 \leqslant \psi \leqslant \pi$, $0 \leqslant t \leqslant 2\pi$

HENCE THE PARAMETRIC REPRESENTATION IS

$$\boxed{x = t + \sin t \quad \& \quad y = 1 - \cos t, \quad 0 \leqslant t \leqslant 2\pi}$$