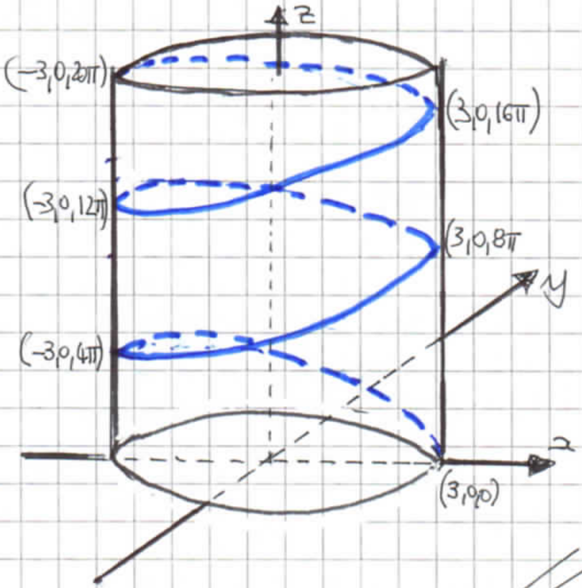
IYGB - MATHEMATICAL METHODS 1 - PAPER D - QUESTION 1

- THIS IS A HELIX
- IT "ADVANCES" ON THE z AXIS
- IT WRAPS AROUND THE CYLINDER WITH EQUATION

$$x^2 + y^2 = 9$$

- IT STARTS AT $(3,0,0)$ & ENDS AT $(-3,0,20\pi)$
- IT PERFORMS $2\frac{1}{2}$ TURNS

$$\boxed{\begin{gathered}(x,y,z) = (3\cos t, 3\sin t, 4t) \\ 0 \leq t \leq 5\pi\end{gathered}}$$



IYGB – MATHEMATICAL METHODS 1 – PAPER D – QUESTION 2

- THIS CAN BE DONE BY THE RATIO TEST

THE nth TERM OF THE SERIES IS GIVEN BY $u_n = \frac{(5x)^n}{4n^2}$

- BY THE RATIO TEST (IGNORING MODULI AS THE TERMS ARE POSITIVE)

$$\frac{u_{n+1}}{u_n} = \frac{(5x)^{n+1}}{4(n+1)^2} \times \frac{4n^2}{(5x)^n} = \frac{5xn^2}{(n+1)^2}$$

$$= \frac{5x}{1} \times \frac{n^2}{n^2+2n+1} = \frac{5x}{1+\frac{2}{n}+\frac{1}{n^2}}$$

- THIS WILL CONVERGE IF

$$\Rightarrow \frac{u_{n+1}}{u_n} \to L, \quad 0 \leqslant L < 1, \text{ AS } n \to \infty$$

$$\Rightarrow 5x \to L \quad 0 \leqslant L < 1$$

(SINCE $1+\frac{2}{n}+\frac{1}{n^2} \to 1$, AS $n \to \infty$)

$$\Rightarrow 0 \leqslant 5x < 1$$

$$0 < x < \frac{1}{5}$$

IGNORING THE TRIVIAL CASE

IYGB - MATHEMATICAL METHODS 1 - PAPER D - QUESTION 3

$$G(x,y) = F(u(x,y), v(x,y)) \qquad u = x\cos y \qquad v = x\sin y$$

START BY OBTAINING SOME BASIC PARTIAL DERIVATIVES

- $\frac{\partial u}{\partial x} = \cos y$
- $\frac{\partial v}{\partial x} = \sin y$
- $\frac{\partial u}{\partial y} = -x\sin y$
- $\frac{\partial v}{\partial y} = x\cos y$

BY THE CHAIN RULE WE HAVE

$$\frac{\partial G}{\partial x} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial F}{\partial u}\cos y + \frac{\partial F}{\partial v}\sin y$$

$$\frac{\partial G}{\partial y} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} = -\frac{\partial F}{\partial u}x\sin y + \frac{\partial F}{\partial v}x\cos y$$

FINALLY WE OBTAIN

$$\left(\frac{\partial G}{\partial x}\right)^2 + \left(\frac{1}{x}\frac{\partial G}{\partial y}\right)^2 =$$

$$= \left[\frac{\partial F}{\partial u}\cos y + \frac{\partial F}{\partial v}\sin y\right]^2 + \left[\frac{1}{x}\left(-\frac{\partial F}{\partial u}x\sin y + \frac{\partial F}{\partial v}x\cos y\right)\right]^2$$

$$= \left[\frac{\partial F}{\partial u}\cos y + \frac{\partial F}{\partial v}\sin y\right]^2 + \left[\frac{\partial F}{\partial v}\cos y - \frac{\partial F}{\partial u}\sin y\right]^2$$

$$= \left(\frac{\partial F}{\partial u}\right)^2\cos^2 y + \cancel{2\frac{\partial F}{\partial u}\frac{\partial F}{\partial v}\cos y\sin y} + \left(\frac{\partial F}{\partial v}\right)\sin^2 y$$
$$\left(\frac{\partial F}{\partial v}\right)^2\cos^2 y \cancel{- 2\frac{\partial F}{\partial u}\frac{\partial F}{\partial v}\cos y\sin y} + \left(\frac{\partial F}{\partial u}\right)^2\sin^2 y$$

$$= \left(\frac{\partial F}{\partial u}\right)^2(\cos^2 y + \sin^2 y) + \left(\frac{\partial F}{\partial v}\right)^2(\cos^2 y + \sin^2 y)$$

$$= \left(\frac{\partial F}{\partial u}\right)^2 + \left(\frac{\partial F}{\partial v}\right)^2$$

IYGB - MATHEMATICAL METHODS 1 - PAPER D - QUESTION 4

a) BY STANDARD RESULTS

$$\mathcal{L}\left[t^3+2e^{-2t}\right] = \frac{3!}{s^{3+1}} + 2\times\frac{1}{s+2} = \frac{6}{s^4}+\frac{2}{s+2}$$

b) OBTAIN THE TRANSFORM OF cosh3t FIRST

$$\mathcal{L}\left[\cosh 3t\right] = \frac{s}{s^2-3^2} = \frac{s}{s^2-9}$$

NOW USING A "SHIFT" THEOREM

$$\mathcal{L}\left[e^{-2t}\cosh 3t\right] = \frac{(s+2)}{(s+2)^2-9} = \frac{s+2}{s^2+4s-5}$$

ALTERNATIVE IN EXPONENTIALS

$$\mathcal{L}\left[e^{-2t}\cosh 3t\right] = \mathcal{L}\left[e^{-2t}\times\tfrac{1}{2}(e^{3t}+e^{-3t})\right] = \tfrac{1}{2}\mathcal{L}\left[e^{t}+e^{-5t}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-1}+\frac{1}{s+5}\right] = \frac{1}{2}\left[\frac{s+5+s-1}{(s-1)(s+5)}\right]$$

$$= \frac{1}{2}\times\frac{2s+4}{s^2+4s-5} = \frac{s+2}{s^2+4s-5}\text{, AS ABOVE}$$

c) START WITH THE TRANSFORM OF sint

$$\mathcal{L}\left[\sin t\right] = \frac{1}{s^2+1^2} = \frac{1}{s^2+1}$$

USING THE RESULT OF MULTIPLYING BY t^2, OR BY t TWICE

$$\mathcal{L}\left[t\sin t\right] = -\frac{d}{ds}\left[\mathcal{L}[\sin t]\right] = -\frac{d}{ds}\left[\frac{1}{s^2+1}\right] = -\frac{d}{ds}\left[(s^2+1)^{-1}\right]$$

$$= -\left[-(s^2+1)^{-2}\times(2s)\right] = \frac{2s}{(s^2+1)^2}$$

$$\mathcal{L}\left[t^2\sin t\right] = -\frac{d}{ds}\left[\mathcal{L}[t\sin t]\right] = -\frac{d}{ds}\left[\frac{2s}{(s^2+1)^2}\right] \leftarrow \text{QUOTIENT RULE}$$

$$= -\frac{(s^2+1)^2\times 2 - 4s(s^2+1)\times 2s}{(s^2+1)^4} = \frac{8s^2(s^2+1)-2(s^2+1)^2}{(s^2+1)^4}$$

$$= \frac{8s^2-2(s^2+1)}{(s^2+1)^3} = \frac{6s^2-2}{(s^2+1)^3}$$

IYGB - MATHEMATICAL METHODS 1 - PAPER D - QUESTION 4

d) FIRSTLY WE CHECK THE EXISTANCE OF THIS LIMIT

$$\lim_{t \to 0}\left[\frac{e^t-1}{t}\right] = \dots \text{ L'HOSPITAL } \dots \lim_{t\to 0}\left[\frac{e^t-0}{1}\right] = 1$$

AS THE LIMIT EXISTS, WE USE THE THEOREM OF DIVISION BY t

$$\mathcal{L}\left[\frac{e^t-1}{t}\right] = \int_s^\infty \mathcal{L}\left[e^t-1\right]\,ds = \int_s^\infty \frac{1}{s-1} - \frac{1}{s}\,ds$$

$$= \Big[\ln|s-1| - \ln|s|\Big]_s^\infty = \Big[\ln\frac{s-1}{s}\Big]_s^\infty$$

$$= \cancel{\ln 1} - \ln\left|\frac{s-1}{s}\right| = -\ln\left(\frac{s-1}{s}\right) = \ln\left(\frac{s}{s-1}\right)$$

e) STANDARD RESULT ON INVERSION

$$\mathcal{L}^{-1}\left[\frac{2}{2s-3}\right] = \mathcal{L}^{-1}\left[\frac{1}{s-\frac{3}{2}}\right] = e^{\frac{3}{2}t}$$

f) GETTING IT TO A DESIRABLE FORM TO BE RECOGNIZED

$$\mathcal{L}^{-1}\left[\frac{6s-17}{s^2-6s+9}\right] = \mathcal{L}^{-1}\left[\frac{6s-17}{(s-3)^2}\right] = \mathcal{L}^{-1}\left[\frac{6(s-3)+1}{(s-3)^2}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{6}{s-3} + \frac{1}{(s-3)^2}\right] = 6e^{3t} + te^{3t}$$

NOTE FOR $\mathcal{L}^{-1}\left[\frac{1}{(s-3)^2}\right]$

EITHER

$$\mathcal{L}[t] = \frac{1!}{s^2} = \frac{1}{s^2}$$

$$\mathcal{L}[te^{3t}] = \frac{1}{(s-3)^2}$$

OR

$$\mathcal{L}[e^{3t}] = \frac{1}{s-3}$$

$$\mathcal{L}[t \times e^{3t}] = -\frac{d}{ds}\left(\frac{1}{s-3}\right) = \frac{1}{(s-3)^2}$$

IYGB - MATHEMATICAL METHODS 1 - PAPER D - QUESTION 5

LET L BE THE REQUIRED LIMITING VALUE

$$\lim_{x\to\infty}\left[\left(1+\frac{1}{x^{3/2}}+\frac{1}{x^2}\right)^x\right]=L$$

AS x IS CONTAINED IN THE EXPONENT, PROCEED WITH LOGARITHMS

$$\Rightarrow \lim_{x\to\infty}\left[\ln\left(1+\frac{1}{x^{3/2}}+\frac{1}{x^2}\right)^x\right]=\ln L$$

$$\Rightarrow \lim_{x\to\infty}\left[x\ln\left(1+x^{-\frac{3}{2}}+x^{-2}\right)\right]=\ln L$$

$$\Rightarrow \lim_{x\to\infty}\left[\frac{\ln\left(1+x^{-\frac{3}{2}}+x^{-2}\right)}{\frac{1}{x}}\right]=\ln L$$

NOW THE LIMIT YIELDS $\frac{0}{0}$ AS $x\to\infty$, SO WE MAY USE L'HOSPITAL'S RULE

$$\Rightarrow \lim_{x\to\infty}\left[\frac{\frac{1}{1+x^{-3/2}+x^{-2}}\times\left[-\frac{3}{2}x^{-\frac{5}{2}}-2x^{-3}\right]}{-\frac{1}{x^2}}\right]=\ln L$$

$$\Rightarrow \lim_{x\to\infty}\left[\frac{\frac{1}{1+\frac{1}{x^{3/2}}+\frac{1}{x^2}}\times\left[\frac{-3}{2x^{5/2}}-\frac{2}{x^3}\right]}{-\frac{1}{x^2}}\right]=\ln L$$

MULTIPLY "TOP & BOTTOM" OF THE FRACTION BY $-x^2$, IN ORDER TO SIMPLIFY

$$\Rightarrow \lim_{x\to\infty}\left[\frac{1}{1+\frac{1}{x^{3/2}}+\frac{1}{x^2}}\times\left[\frac{3}{2x^{\frac{1}{2}}}-\frac{2}{x}\right]\right]=\ln L$$

IYGB - MATHEMATICAL METHODS 1 - PAPER D - QUESTION 5

TAKING THE LIMIT NOW YIELDS ZERO, SINCE

- $\lim_{x \to \infty} \left(1 + \frac{1}{x^{\frac{3}{2}}} + \frac{1}{x^2}\right) = 1$
- $\lim_{x \to \infty} \left(\frac{3}{2x^{\frac{1}{2}}} - \frac{2}{x}\right) = 0$

& $\lim_{x \to a} \left[f(x)g(x)\right] = \left[\lim_{x \to a} f(x)\right]\left[\lim_{x \to a} g(x)\right]$

HENCE $\ln L = 0$

$L = 1$

$\therefore \lim_{x \to \infty} \left[\left(1 + \frac{1}{x^{3/2}} + \frac{1}{x^2}\right)^x\right] = 1$

IYGB - MATHEMATICAL METHODS 1 - PAPER D - QUESTION 6

MANIPULATE THE PRODUCT OPERATOR AS FOLLOWS

$$\prod_{n=2}^{\infty}\left[1-\frac{1}{2-2^n}\right]=\prod_{n=2}^{\infty}\left[\frac{(2-2^n)-1}{2-2^n}\right]$$

$$=\prod_{n=2}^{\infty}\left[\frac{1-2^n}{2-2^n}\right]$$

$$=\prod_{n=2}^{\infty}\left[\frac{2^n-1}{2^n-2}\right]$$

$$=\prod_{n=2}^{\infty}\left[\frac{2^n-1}{2(2^{n-1}-1)}\right]$$

TAKING LIMITS TO INFINITY

$$=\lim_{k\to\infty}\left[\prod_{n=2}^{k}\left[\frac{2^n-1}{2(2^{n-1}-1)}\right]\right]$$

FACTORIZE $\frac{1}{2}$ OUT OF THE PRODUCT $(k-1)$ TIMES, AS n RUNS FROM 2 TO k

$$=\lim_{k\to\infty}\left[\frac{1}{2^{k-1}}\prod_{n=2}^{k}\left[\frac{2^n-1}{2^{n-1}-1}\right]\right]$$

NEXT WRITE THE PRODUCT EXPLICITLY & LOOK FOR A PATTERN

$$=\lim_{k\to\infty}\left[\frac{1}{2^{k-1}}\left[\frac{\cancel{3}}{1}\times\frac{\cancel{7}}{\cancel{3}}\times\frac{\cancel{15}}{\cancel{7}}\times\frac{\cancel{31}}{\cancel{15}}\times\cdots\times\frac{2^k-1}{\cancel{2^{k-1}-1}}\right]\right]$$

$$=\lim_{k\to\infty}\left[\frac{1}{2^{k-1}}\times(2^k-1)\right]$$

IYGB - MATHEMATICAL METHODS 1 - PAPER D - QUESTION 6

$$= \lim_{k\to\infty}\left[\frac{2^k - 1}{2^{k-1}}\right]$$

$$= \lim_{k\to\infty}\left[\frac{\frac{2^k}{2^{k-1}} - \frac{1}{2^{k-1}}}{\frac{2^{k-1}}{2^{k-1}}}\right]$$

$$= \lim_{k\to\infty}\left[\frac{2 - \frac{1}{2^{k-1}}}{1}\right]$$

$$= 2$$

ALTERNATIVE

$$\lim_{k\to\infty}\left[\frac{2^k - 1}{2^{k-1}}\right]$$

$$= \lim_{k\to\infty}\left[2 \times \frac{2^k - 1}{2^k}\right]$$

$$= \lim_{k\to\infty}\left[2\left(1 - \frac{1}{2^k}\right)\right]$$

$$= 2$$

IYGB - MATHEMATICAL METHODS 1 - PAPER D - QUESTION 7

$$\boxed{f(x,y,z) = x^2+y^2+z^2+xy-x+y}$$

- OBTAIN THE FIRST ORDER PARTIAL DERIVATIVES OF f AND SET THEM EQUAL TO ZERO

$$\left.\begin{array}{l}\dfrac{\partial f}{\partial x} = 2x+y-1 \\[2ex] \dfrac{\partial f}{\partial y} = 2y+x+1\end{array}\right\} \quad \left.\begin{array}{l}2x+y=1 \\ 2y+x=-1\end{array}\right\} \Rightarrow y = 1-2x$$

$$\frac{\partial f}{\partial z} = 2z$$

$$\Rightarrow 2(1-2x)+x=-1$$
$$\Rightarrow -3x=-3$$
$$\Rightarrow x=1$$
$$\Rightarrow y=-1$$
$$\Rightarrow z=0$$

$$\therefore f(1,-1,0) = 1+1+0-1-1-1 = -1$$

- TO CLASSIFY THE "POINT" WE REQUIRE ALL THE 2ND ORDER DERIVATIVES

$$\frac{\partial^2 f}{\partial x^2} = 2 \qquad \frac{\partial^2 f}{\partial x \partial y} = 1 \qquad \frac{\partial^2 f}{\partial x \partial z} = 0$$

$$\frac{\partial^2 f}{\partial y \partial x} = 1 \qquad \frac{\partial^2 f}{\partial y^2} = 2 \qquad \frac{\partial^2 f}{\partial y \partial z} = 0$$

$$\frac{\partial^2 f}{\partial z \partial x} = 0 \qquad \frac{\partial^2 f}{\partial z \partial y} = 0 \qquad \frac{\partial^2 f}{\partial z^2} = 2$$

- THESE NEED TO BE EVALUATED AT $(1,-1,0)$, BUT THEY ARE ALL CONSTANT

IYGB - MATHEMATICAL METHODS 1 - PAPER D - QUESTION 7

- PROCEED TO FIND THE EIGENVALUES OF THE MATRIX

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

IF ALL 3 ARE POSITIVE $\Rightarrow$ MIN
IF ALL 3 ARE NEGATIVE $\Rightarrow$ MAX
IF MIX OF POSITIVE/NEGATIVE $\Rightarrow$ "SADDLE"

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

- EXPAND BY THE THIRD COLUMN

$$\Rightarrow (2-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -(\lambda-2)\left[(2-\lambda)^2-1\right] = 0$$

$$\Rightarrow (\lambda-2)\left[(\lambda-2)^2-1\right] = 0$$

$$\Rightarrow (\lambda-2)(\lambda-2-1)(\lambda-2+1) = 0$$

$$\Rightarrow (\lambda-2)(\lambda-3)(\lambda-1) = 0$$

$$\Rightarrow \lambda = \begin{cases} 1 \\ 2 \\ 3 \end{cases}$$

- AS ALL THE EIGENVALUES ARE POSITIVE $(1,-1,0)$ YIELDS A LOCAL MINIMUM OF -1

IYGB – MATHEMATICAL METHODS 1 – PAPER D – QUESTION 8

- **WRITE THE SYSTEM IN MATRIX FORM**

$$\underbrace{\begin{pmatrix} 7 & 2 & -3 \\ 3 & 4 & -5 \\ 5 & -3 & 4 \end{pmatrix}}_{\underline{A}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 30 \\ 14 \\ 18 \end{pmatrix}$$

- **CALCULATE ALL THE RELEVANT DETERMINANTS**

- $\det \underline{A} = \begin{vmatrix} 7 & 2 & -3 \\ 3 & 4 & -5 \\ 5 & -3 & 4 \end{vmatrix} = 7\begin{vmatrix} 4 & -5 \\ -3 & 4 \end{vmatrix} - 2\begin{vmatrix} 3 & -5 \\ 5 & 4 \end{vmatrix} - 3\begin{vmatrix} 3 & 4 \\ 5 & -3 \end{vmatrix}$

$$= 7 \times 1 - 2 \times 37 - 3(-29) = \underline{20}$$

- $\det \underline{A}_x = \begin{vmatrix} 30 & 2 & -3 \\ 14 & 4 & -5 \\ 18 & -3 & 4 \end{vmatrix} = 30\begin{vmatrix} 4 & -5 \\ -3 & 4 \end{vmatrix} - 2\begin{vmatrix} 14 & -5 \\ 18 & 4 \end{vmatrix} - 3\begin{vmatrix} 14 & 4 \\ 18 & -3 \end{vmatrix}$

$$= 30 \times 1 - 2 \times 146 - 3(-114) = \underline{80}$$

- $\det \underline{A}_y = \begin{vmatrix} 7 & 30 & -3 \\ 3 & 14 & -5 \\ 5 & 18 & 4 \end{vmatrix} = 7\begin{vmatrix} 14 & -5 \\ 18 & 4 \end{vmatrix} - 30\begin{vmatrix} 3 & -5 \\ 5 & 4 \end{vmatrix} - 3\begin{vmatrix} 3 & 14 \\ 5 & 18 \end{vmatrix}$

$$= 7 \times 146 - 30 \times 37 - 3(-16) = \underline{-40}$$

- $\det \underline{A}_z \begin{vmatrix} 7 & 2 & 30 \\ 3 & 4 & 14 \\ 5 & -3 & 18 \end{vmatrix} = 7\begin{vmatrix} 4 & 14 \\ -3 & 18 \end{vmatrix} - 2\begin{vmatrix} 3 & 14 \\ 5 & 18 \end{vmatrix} + 30\begin{vmatrix} 3 & 4 \\ 5 & -3 \end{vmatrix}$

$$= 7 \times 114 - 2(-16) + 30(-29) = \underline{-40}$$

HENCE WE HAVE

- $x = \dfrac{\det \underline{A}_x}{\det \underline{A}} = \dfrac{80}{20} = 4$

- $y = \dfrac{\det \underline{A}_y}{\det \underline{A}} = \dfrac{-40}{20} = -2$

- $z = \dfrac{\det \underline{A}_z}{\det \underline{A}} = \dfrac{-40}{20} = -2$

IYGB - MATHEMATICAL METHODS 1 - PAPER D - QUESTION 9

$$\boxed{u_{n+2} = 5u_{n+1} - 6u_n + 4n \quad , \quad u_1 = 1 \quad , \quad u_2 = 3}$$

REWRITE THE EQUATION IN ORDER TO SOLVE AN AUXILARY EQUATION

$$\Longrightarrow u_{n+2} - 5u_{n+1} + 6u_n = 4n$$

AUXILARY EQUATION

$$\lambda^2 - 5\lambda + 6 = 0$$
$$(\lambda - 2)(\lambda - 3) = 0$$
$$\lambda = \begin{cases} 2 \\ 3 \end{cases}$$

"COMPLIMENTARY FUNCTION"

$$u_n = A(2^n) + B(3^n)$$

FOR "PARTICULAR INTEGRAL" TRY

$$u_n = Pn + Q$$
$$u_{n+1} = P(n+1) + Q$$
$$u_{n+2} = P(n+2) + Q$$

SUBSTITUTE INTO THE RELATION

$$P(n+2) + Q - 5\left[P(n+1) + Q\right] + 6\left[Pn + Q\right] \equiv 4n$$
$$Pn + 2P + Q - 5Pn - 5P - 5Q + 6Pn + 6Q \equiv 4n$$
$$2Pn + (2Q - 3P) \equiv 4n$$

$$\therefore P = 2 \quad \& \quad 2Q - 3P = 0$$
$$2Q = 6$$
$$Q = 3$$

GENERAL SOLUTION

$$u_n = A(2^n) + B(3^n) + 2n + 3$$

IYGB - MATHEMATICAL METHODS 1 - PAPER D - QUESTION 9

APPLYING THE CONDITIONS GIVEN

$$u_1 = 1 \implies 1 = 2A + 3B + 5$$
$$u_2 = 3 \implies 3 = 4A + 9B + 7$$

$$\implies \left.\begin{array}{l} 2A + 3B = -4 \\ 4A + 9B = -4 \end{array}\right\} \xrightarrow{\times(-2)}$$

$$\left.\begin{array}{l} -4A - 6B = 8 \\ 4A + 9B = -4 \end{array}\right\} \implies \text{ADDING EQUATIONS}$$

$$\implies 3B = 4$$
$$\implies B = \frac{4}{3}$$

$$\implies 2A + 3\left(\frac{4}{3}\right) = -4 \qquad \leftarrow \text{"}2A + 3B = -4\text{"}$$
$$\implies 2A + 4 = -4$$
$$\implies 2A = -8$$
$$\implies A = -4$$

FINALLY WE OBTAIN

$$u_n = -4(2^n) + \frac{4}{3}(3^n) + 2n + 3$$
$$u_n = -2^{n+2} + 4(3^{n-1}) + 2n + 3$$
$$u_n = 4(3^{n-1}) - 2^{n+2} + 2n + 3$$

IYGB - MATHEMATICAL METHODS 1 - PAPER D - QUESTION 10

$$\boxed{x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2e^x \quad , x \neq 0}$$

$a(x) = x^2$

OBTAIN A COMPLEMENTARY FUNCTION BY TRIAL

- $y = x^n$
- $y' = nx^{n-1}$
- $y'' = n(n-1)x^{n-2}$

SUB INTO THE O.D.E $x^2y'' + xy' - y = 0$

$$\Rightarrow x^2\left(n(n-1)x^{n-2}\right) + x\left(nx^{n-1}\right) - x^n \equiv 0$$

$$\Rightarrow n(n-1)x^n + nx^n - x^n \equiv 0$$

$$\Rightarrow \left[n(n-1) + n - 1\right]x^n \equiv 0$$

$$\Rightarrow n^2 - n + n - 1 = 0$$

$$\Rightarrow n^2 = 1$$

$$\Rightarrow n = \pm 1$$

$$\therefore \quad y = Ax^1 + Bx^{-1} = Ax + \frac{B}{x}$$

PARTICULAR INTEGRAL BY VARIATION OF PARAMETERS

$e_1 = x$

$e_2 = \frac{1}{x}$

$$\text{WRONSKIAN} = W(x) = \begin{vmatrix} e_1 & e_2 \\ e_1' & e_2' \end{vmatrix} = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix}$$

$$= -\frac{1}{x} - \frac{1}{x} = -\frac{2}{x}$$

IYGB - MATHEMATICAL METHODS 1 - PAPER D - QUESTION 10

NOW THE PARTICULAR INTEGRAL, y_P, MUST SATISFY

$$\Longrightarrow y_P = -e_1\int \frac{e_2 f}{aW}\,dx + e_2\int \frac{e_1 f}{aW}\,dx$$

WHERE
$$f = f(x) = x^2e^x$$
$$W = W(x) = -\frac{2}{x}$$
$$a = a(x) = x^2$$

$$\Longrightarrow y_P = -x\int \frac{\frac{1}{x}(x^2e^x)}{x^2\left(-\frac{2}{x}\right)}\,dx + \frac{1}{x}\int \frac{x(x^2e^x)}{x^2\left(-\frac{2}{x}\right)}\,dx$$

$$\Longrightarrow y_P = x\int \frac{1}{2}e^x - \frac{1}{2x}\underbrace{\int x^2e^x\,dx}_{\text{BY PARTS TWICE}}$$

$$x^2e^x - \int 2xe^x\,dx$$
$$= x^2e^x - \left[2xe^x - \int 2e^x\,dx\right]$$
$$= x^2e^x - 2xe^x + \int 2e^x\,dx$$
$$= x^2e^x - 2xe^x + 2e^x + C$$

x^2	$2x$
e^x	e^x

$2x$	2
e^x	e^x

COLLECTING ALL THE RESULTS

$$\Rightarrow y_P = x\left(\frac{1}{2}e^x\right) - \frac{1}{2x}\left[x^2e^x - 2xe^x + 2e^x\right] + C$$

$$\Rightarrow y_P = \cancel{\frac{1}{2}xe^x} - \cancel{\frac{1}{2}xe^x} + e^x - \frac{1}{x}e^x + C$$

$$\therefore\ y = Ax + \frac{B}{x} + e^x - \frac{1}{x}e^x$$

IYGB - MATHEMATICAL METHODS 1 - PAPER D - QUESTION 11

$$\boxed{f(D) = 2D^2 - D + 1 \quad , \text{ HERE } \{\ \} \text{ INDICATE D OPERATOR ARGUMENT}}$$

DIRECTLY FROM THE DEFINITIONS INC LINEARITY PROPERTY

$$f(D)\left\{e^{kx}V(x)\right\} = (2D^2 - D + 1)\left\{e^{kx}V(x)\right\}$$

$$= 2D^2\left\{e^{kx}V(x)\right\} - D\left\{e^{kx}V(x)\right\} + 1\left\{e^{kx}V(x)\right\}$$

DIFFERENTIATING BY THE PRODUCT RULE

$$= 2D\left\{ke^{kx}V(x) + e^{kx}V'(x)\right\} - ke^{kx}V(x) - e^{kx}V'(x) + e^{kx}V(x)$$

$$= 2D\left\{e^{kx}(kV(x) + V'(x))\right\} + e^{kx}\left[V(x) - kV(x) - V'(x)\right]$$

BY THE PRODUCT RULE AGAIN

$$= 2\left[ke^{kx}(kV(x) + V'(x)) + e^{kx}(kV'(x) + V''(x))\right] + e^{kx}\left[V(x) - kV(x) - V'(x)\right]$$

$$= e^{kx}\left[2k^2V(x) + 2kV'(x) + 2kV'(x) + 2V''(x) + V(x) - kV(x) - V'(x)\right]$$

$$= e^{kx}\left[2V''(x) + (4k-1)V'(x) + (2k^2 - k + 1)V(x)\right]$$

APPLY THE DEFINITION OF D OPERATOR

$$= e^{kx}\left[2D^2\{V(x)\} + (4k-1)D\{V(x)\} + (2k^2 - k + 1)V(x)\right]$$

$$= e^{kx}\left[2D^2 + (4k-1)D + (2k^2 - k + 1)\right]\{V(x)\}$$

$$= e^{kx}\left[2D^2 + 4Dk + 2k^2 - D - k + 1\right]\{V(x)\}$$

IYGB - MATHEMATICAL METHODS I - PAPER D - QUESTION 11

$$= e^{kx}\left[2(D^2+2Dk+k^2) - (D+k) + 1\right]\{V(x)\}$$

$$= e^{kx}\left[2(D+k)^2 - (D+k) + 1\right]\{V(x)\}$$

$$= e^{kx} f(D+k)\{V(x)\}$$

AS REQUIRED

IYGB - MATHEMATICAL METHODS 1 - PAPER D - QUESTION 12

START BY OBTAINING A RELATIONSHIP BETWEEN ψ & x

$$\frac{dy}{dx} = \tan\psi$$

(BY DEFINITION)

$$\frac{dy}{dx} = \frac{d}{dx}\left(\ln(\sin x)\right) = \frac{1}{\sin x} \times \cos x = \cot x = \tan\left(\frac{\pi}{2} - x\right)$$

$$\therefore \quad \psi = \frac{\pi}{2} - x$$

NEXT WE LINK s & ψ VIA ARCLENGTH

$$s = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int \sqrt{1 + \cot^2 x}\, dx = \int \sqrt{\operatorname{cosec}^2 x}\, dx$$

$$s = \int \operatorname{cosec} x \, dx = -\ln\left|\operatorname{cosec} x + \cot x\right| + C$$

NEED SOME FINAL CONVERSIONS

$$\cot x = \tan\psi \quad \& \quad \operatorname{cosec} x = \frac{1}{\sin x} = \frac{1}{\sin\left(\frac{\pi}{2} - \psi\right)} = \frac{1}{\cos\psi} = \sec\psi$$

I.E $\operatorname{cosec} x = \sec\psi$

$$\Rightarrow s = -\ln\left|\tan\psi + \sec\psi\right| + C$$

$$\Rightarrow s = \ln\left|\frac{1}{\tan\psi + \sec\psi}\right| + C$$

APPLY CONDITION

when $\psi = \arctan\frac{3}{4} = \arccos\frac{4}{5} = \operatorname{arcsec}\frac{5}{4}$, $s = 0$

$$\Rightarrow 0 = \ln\left(\frac{1}{\frac{3}{4} + \frac{5}{4}}\right) + C$$

$$\Rightarrow C = -\ln\frac{1}{2} = \ln 2$$

$$\therefore \quad s = \ln\left|\frac{1}{\tan\psi + \sec\psi}\right| + \ln 2 = \ln\left|\frac{2}{\tan\psi + \sec\psi}\right|$$

AS REQUIRED

IYGB - MATHEMATICAL METHODS 1 - PAPER D - QUESTION 13

a) $\frac{d}{dx}\left[\int_{x}^{x^2} e^{\sqrt{u}}\,du\right] = \ldots$ BY SUBSTITUTION

$t = \sqrt{u}$
$t^2 = u$
$2t\,dt = du$
$u = x \mapsto t = x^{\frac{1}{2}}$
$u = x^2 \mapsto t = x$

$= \frac{d}{dx}\left[\int_{x^{\frac{1}{2}}}^{x} e^{t}(2t\,dt)\right]$

$= 2\frac{d}{dx}\left[\int_{x^{\frac{1}{2}}}^{x} te^{t}\,dt\right]$

INTEGRATION BY PARTS OR INSPECTION NOW YIELDS

$= 2\frac{d}{dx}\left[\left[te^{t} - e^{t}\right]_{x^{\frac{1}{2}}}^{x}\right]$

$= 2\frac{d}{dx}\left[(xe^{x} - e^{x}) - (x^{\frac{1}{2}}e^{x^{\frac{1}{2}}} - e^{x^{\frac{1}{2}}})\right]$

$= 2\frac{d}{dx}\left[xe^{x} - e^{x} - x^{\frac{1}{2}}e^{x^{\frac{1}{2}}} + e^{x^{\frac{1}{2}}}\right]$

$= 2\times\left[\cancel{e^{x}} + xe^{x} \cancel{- e^{x}} - \frac{1}{2}x^{-\frac{1}{2}}e^{x^{\frac{1}{2}}} - x^{\frac{1}{2}}\left(\frac{1}{2}x^{-\frac{1}{2}}e^{x^{\frac{1}{2}}}\right) + \frac{1}{2}x^{-\frac{1}{2}}e^{x^{\frac{1}{2}}}\right]$

$= 2\left[xe^{x} \cancel{- \frac{1}{2}x^{-\frac{1}{2}}e^{x^{\frac{1}{2}}}} - \frac{1}{2}e^{x^{\frac{1}{2}}} \cancel{+ \frac{1}{2}x^{-\frac{1}{2}}e^{x^{\frac{1}{2}}}}\right]$

$= 2xe^{x} - e^{x^{\frac{1}{2}}}$

IYGB - MATHEMATICAL METHODS 1 - PAPER D - QUESTION 13

b) $$\frac{d}{dx}\left[\int_{x}^{x^2} e^{\sqrt{u}}\,du\right] = \frac{d}{dx}\left[\int_{0}^{x^2} e^{\sqrt{u}}\,du - \int_{0}^{x} e^{\sqrt{u}}\,du\right]$$

$$= e^{\sqrt{x^2}} \times \frac{d}{dx}(x^2) - e^{\sqrt{x}}$$

$$= e^{x} \times 2x - e^{\sqrt{x}}$$

$$= 2xe^{x} - e^{\sqrt{x}}$$

AS BEFORE.

IYGB - MATHEMATICAL METHODS 1 - PAPER D - QUESTION 14

- LOOKING AT THE LHS OF THE O.D.E WE TRY A SOLUTION OF THE FORM

$$y = x^{\lambda}$$
$$\frac{dy}{dx} = \lambda x^{\lambda-1}$$
$$\frac{d^2y}{dx^2} = \lambda(\lambda-1)x^{\lambda-2}$$
$$\frac{d^3y}{dx^3} = \lambda(\lambda-1)(\lambda-2)x^{\lambda-3}$$

- SUBSTITUTE INTO THE O.D.E (R.H.S = 0)

$$\Rightarrow \lambda(\lambda-1)(\lambda-2)x^{\lambda} + 2\lambda(\lambda-1)x^{\lambda} + \lambda x^{\lambda} - x^{\lambda} = 0$$
$$\Rightarrow x^{\lambda}\left[\lambda(\lambda-1)(\lambda-2) + 2\lambda(\lambda-1) + \lambda - 1\right] = 0$$
$$\Rightarrow (\lambda-1)\left[\lambda(\lambda-2) + 2\lambda + 1\right] = 0$$
$$\Rightarrow (\lambda-1)(\lambda^2+1) = 0$$

$$\lambda = \begin{cases} 1 \\ i \\ -i \end{cases} \qquad y = Ax^{1} + Bx^{i} + Cx^{-i}$$

- NOW NOTE THAT

$$Bx^{i} + Cx^{-i} = Be^{\ln x^{i}} + Ce^{\ln x^{-i}} = Be^{i\ln x} + Ce^{-i\ln x}$$
$$= B\left[\cos(\ln x) + i\sin(\ln x)\right] + \left[C\cos(\ln x) - i\sin(\ln x)\right]$$
$$= (B+C)\cos(\ln x) + i(B-C)\sin(\ln x)$$
$$= D\cos(\ln x) + E\sin(\ln x)$$

IYGB - MATHEMATICAL METHODS 1 - PAPER D - QUESTION 14

- FOR PARTICULAR INTEGRAL, BY INSPECTION, WE TRY $y = Px\ln x$.

$$y = Px\ln x$$
$$\frac{dy}{dx} = P + P\ln x$$
$$\frac{d^2y}{dx^2} = \frac{P}{x}$$
$$\frac{d^3y}{dx^3} = -\frac{P}{x^2}$$

- SUB INTO THE O.D.E GIVES

$$\cancel{-Px} + 2Px + \cancel{Px} + \cancel{Px\ln x} - \cancel{Px\ln x} \equiv 2x$$

$$\therefore P = 1$$

- HENCE THE GENERAL SOLUTION IS

$$y = \alpha x + \beta\cos(\ln x) + \gamma\sin(\ln x) + x\ln x$$

IYGB - MATHEMATICAL METHODS 1 - PAPER D - QUESTION 15

WE SHALL FIND THE AREA BY DIAGONALIZING THE CONIC, INTO A STANDARD CONIC

$$\Longrightarrow \quad 6x^2 + 4xy + 9y^2 - 12x - 4y = 4$$

$$\Longrightarrow \quad (x \;\; y)\begin{pmatrix} 6 & 2 \\ 2 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + (-12 \;\; -4)\begin{pmatrix} x \\ y \end{pmatrix} = 4$$

DIAGONIZE THE SYMMETRIC MATRIX — START BY THE CHARACTERISTIC EQUATION

$$\begin{vmatrix} 6-\lambda & 2 \\ 2 & 9-\lambda \end{vmatrix} = 0 \Longrightarrow (6-\lambda)(9-\lambda) - 4 = 0$$

$$\Longrightarrow (\lambda-6)(\lambda-9) - 4 = 0$$

$$\Longrightarrow \lambda^2 - 15\lambda + 50 = 0$$

$$\Longrightarrow (\lambda-10)(\lambda-5) = 0$$

$$\Longrightarrow \lambda = \begin{cases} 5 \\ 10 \end{cases}$$

FIND THE TWO (NORMALIZED) EIGENVECTORS FOR THE ABOVE EIGENVALUES

- IF $\lambda = 5$

$$\begin{Bmatrix} 6x + 2y = 5x \\ 2x + 9y = 5y \end{Bmatrix} \Longrightarrow \begin{Bmatrix} x = -2y \\ 2x = -4y \end{Bmatrix} \Longrightarrow y = -\frac{1}{2}x, \quad \alpha\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

NORMALIZED EIGENVECTORS

- IF $\lambda = 10$

$$\begin{Bmatrix} 6x + 2y = 10x \\ 2x + 9y = 10y \end{Bmatrix} \Longrightarrow \begin{Bmatrix} 2y = 4x \\ 2x = y \end{Bmatrix} \Longrightarrow y = 2x, \quad \beta\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\therefore \quad \underline{P} = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \quad \& \quad \underline{D} = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$$

$\lambda = 5$ $\lambda = 10$

IYGB - MATHEMATICAL METHODS 1 - PAPER D - QUESTION 15

SO THE CONIC CAN NOW BE WRITTEN AS

$$\Rightarrow (X \quad Y)\begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} + (-12 \quad -4)\begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = 4$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}$$

$$\Rightarrow 5X^2 + 10Y^2 + (-12 \quad -4)\begin{pmatrix} \frac{2}{\sqrt{5}}X + \frac{1}{\sqrt{5}}Y \\ -\frac{1}{\sqrt{5}}X + \frac{2}{\sqrt{5}}Y \end{pmatrix} = 4$$

$$\Rightarrow 5X^2 + 10Y^2 - \frac{20}{\sqrt{5}}X - \frac{20}{\sqrt{5}}Y = 4$$

$$\Rightarrow 5X^2 + 10Y^2 - 4\sqrt{5}X - 4\sqrt{5}Y = 4$$

MANIPULATE INTO "STANDARD" FORM

$$\Rightarrow \frac{5}{4}X^2 + \frac{5}{2}Y^2 - \sqrt{5}Y - \sqrt{5}X = 1$$

$$\Rightarrow \frac{5}{4}\left(X^2 - \frac{4\sqrt{5}}{5}X\right) + \frac{5}{2}\left(Y^2 - \frac{2\sqrt{5}}{5}Y\right) = 1$$

$$\Rightarrow \frac{5}{4}\left[\left(X - \tfrac{2}{5}\sqrt{5}\right)^2 - \frac{4}{5}\right] + \frac{5}{2}\left[\left(Y - \frac{\sqrt{5}}{5}\right)^2 - \frac{1}{5}\right] = 1$$

$$\Rightarrow \frac{5}{4}\left(X - \tfrac{2}{5}\sqrt{5}\right)^2 - 1 + \frac{5}{2}\left(Y - \frac{\sqrt{5}}{5}\right)^2 - \frac{1}{2} = 1$$

$$\Rightarrow \frac{5}{4}\left(X - \tfrac{2}{5}\sqrt{5}\right)^2 + \frac{5}{2}\left(Y - \frac{\sqrt{5}}{5}\right)^2 = \frac{5}{2}$$

$$\Rightarrow \frac{1}{2}\left(X - \tfrac{2}{5}\sqrt{5}\right)^2 + \left(Y - \frac{\sqrt{5}}{5}\right)^2 = 1$$

$$\Rightarrow \frac{\left(X - \frac{2}{5}\sqrt{5}\right)^2}{\underset{\leftarrow a^2}{2}} + \frac{\left(Y - \frac{\sqrt{5}}{5}\right)^2}{\underset{\leftarrow b^2}{1}} = 1$$

FINALLY AS TRANSLATIONS DO NOT AFFECT AREA

$$\text{AREA} = \text{"}\pi ab\text{"} = \pi \times \sqrt{2} \times 1 = \pi\sqrt{2}$$

IYGB - MATHEMATICAL METHODS 1 - PAPER D - QUESTION 16

SET UP A VOLUME INTEGRAL FOR REVOLUTION ABOUT THE y AXIS

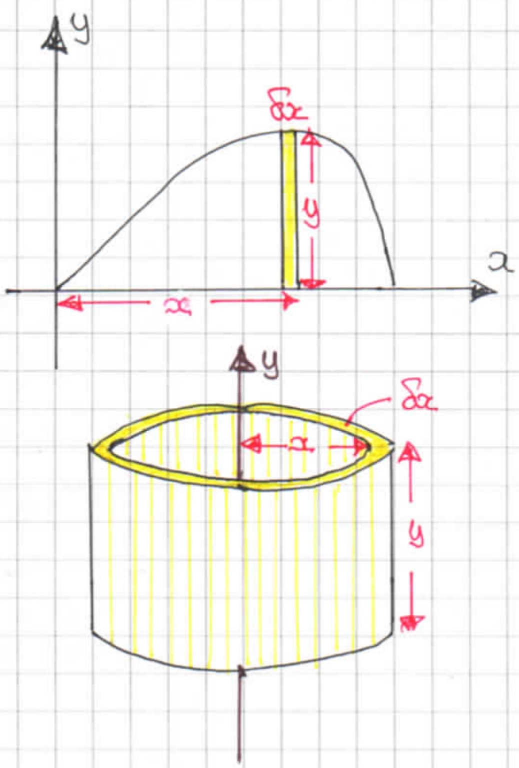
$$V = \int_{x_1}^{x_2} 2\pi x y \, dx$$

IN PARAMETRIC WE HAVE

$$V = \int_{t_1}^{t_2} 2\pi x(t)\, y(t) \frac{dx}{dt} \, dt$$

IN THIS CASE WE HAVE

$$x = 3t + \sin t$$
$$y = 2\sin t \qquad (0 \le t \le \pi)$$
$$\frac{dx}{dt} = 3 + \cos t$$



INFINITESIMAL VOLUME

$$\delta V = \pi\left[(x+\delta x)^2 - x^2\right] y$$
$$\delta V = \pi y\left[x^2 + 2x\delta x + \delta x^2 - x^2\right]$$
$$\delta V = \pi y\left[2x\delta x + \delta x^2\right]$$
$$dV = 2\pi y x \, dx$$

THE REQUIRED VOLUME IS

$$V = \int_0^{\pi} 2\pi (3t + \sin t)(2\sin t)(3 + \cos t) \, dt$$

$$V = 4\pi \int_0^{\pi} (3t\sin t + \sin^2 t)(3 + \cos t) \, dt$$

$$V = 4\pi \int_0^{\pi} 9t\sin t + 3t\sin t\cos t + 3\sin^2 t + \sin^2 t\cos t \, dt$$

IYGB - MATHEMATICAL METHODS 1 - PAPER D - QUESTION 16

$$\Rightarrow V = 4\pi\int_0^{\pi} 9t\sin t + \frac{3}{2}t\sin 2t + 3\left(\frac{1}{2} - \frac{1}{2}\cos 2t\right) + 4\sin^2 t\cos t \, dt$$

$$\Rightarrow V = \pi\int_0^{\pi} 36t\sin t + 6t\sin 2t + 6 - 6\cos 2t + 4\sin^2 t\cos t \, dt$$

$$\Rightarrow V = \pi\int_0^{\pi} \underbrace{3t\left[12\sin t + 2\sin 2t\right]}_{\text{BY PARTS}} + 6 - 6\cos 2t + 4\sin^2 t\cos t \, dt$$

$3t$	3
$-12\cos t - \cos 2t$	$2\sin 2t + 12\sin t$

$$\Rightarrow V = \pi\left\{\left[3t(-12\cos t - \cos 2t)\right]_0^{\pi} + \int_0^{\pi} \cancel{36\cos t + 3\cos 2t} \, dt + \left[6t - \cancel{3\sin 2t} + \cancel{\frac{4}{3}\sin^3 t}\right]_0^{\pi}\right\}$$

$$\Rightarrow V = \pi\left\{\left[3t\left[12\cos t + \cos 2t\right]\right]_{\pi}^{0} + \left[6t\right]_0^{\pi}\right\}$$

$$\Rightarrow V = \pi\left\{0 - 3\pi\left[-12 + 1\right] + 6\pi\right\}$$

$$\Rightarrow V = \pi(39\pi)$$

$$\Rightarrow V = 39\pi^2$$

IYGB - MATHEMATICAL METHODS 1 - PAPER D - QUESTION 17

START WITH THE DEFINITION OF A FOURIER SERIES IN t, $-L<t<L$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)\right]$$

- $a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)\,dt \qquad n=0,1,2,3,4,\ldots$
- $b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)\,dt \qquad n=1,2,3,4,\ldots$

BY MANIPULATING EULER'S FORMULA & SUBSTITUTING INTO THE ABOVE

- $\cos\frac{n\pi t}{L} = \frac{1}{2}\left[e^{i\frac{n\pi t}{L}} + e^{-i\frac{n\pi t}{L}}\right]$
- $\sin\frac{n\pi t}{L} = \frac{1}{2i}\left[e^{i\frac{n\pi t}{L}} - e^{-i\frac{n\pi t}{L}}\right]$

$$\Rightarrow f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\frac{a_n}{2}\left[e^{i\frac{n\pi t}{L}} + e^{-i\frac{n\pi t}{L}}\right] + \frac{b_n}{2i}\left[e^{i\frac{n\pi t}{L}} - e^{-i\frac{n\pi t}{L}}\right]\right]$$

$$\Rightarrow f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\left(\frac{a_n}{2} - \frac{ib_n}{2}\right)e^{i\frac{n\pi t}{L}} + \left[\frac{a_n}{2} + \frac{ib_n}{2}\right]e^{-i\frac{n\pi t}{L}}\right]$$

- LET $C_0 = \frac{1}{2}a_0 = \frac{1}{2}(a_0 + ib_0)$ WITH $b_0 = 0$
- LET $C_n = \frac{1}{2}(a_n - ib_n)$
- LET $\bar{C}_n = \frac{1}{2}(a_n + ib_n)$, AS C_n & $\bar{C}_n$ ARE CONJUGATES

$$\Rightarrow f(t) = C_0 + \sum_{n=1}^{\infty}\left[C_n e^{i\frac{n\pi t}{L}} + \bar{C}_n e^{-i\frac{n\pi t}{L}}\right]$$

NOW FOR NOTATIONAL CONVENIENCE WE WRITE THE CONJUGATES AS FOLLOWS

$$C_n \equiv \frac{1}{2}(a_n - ib_n)$$

$$\Rightarrow C_{-n} \equiv \frac{1}{2}(a_{-n} + ib_{-n}) \Rightarrow \bar{C}_n = \frac{1}{2}(a_n + ib_n)$$

IYGB - MATHEMATICAL METHODS 1 - PAPER D - QUESTION 17

$$\Rightarrow f(t) = c_0 + \sum_{n=1}^{\infty}\left[c_n e^{i\frac{n\pi t}{L}}\right] + \sum_{n=1}^{\infty}\left[\bar{c}_n e^{-i\frac{n\pi t}{L}}\right]$$

$$\Rightarrow f(t) = c_0 + \sum_{n=1}^{\infty}\left[c_n e^{i\frac{n\pi t}{L}}\right] + \sum_{n=1}^{\infty}\left[c_{-n} e^{-i\frac{n\pi t}{L}}\right]$$

$$\Rightarrow f(t) = c_0 + \sum_{n=1}^{\infty}\left[c_n e^{i\frac{n\pi t}{L}}\right] + \sum_{n=-1}^{-\infty}\left[c_n e^{+i\frac{n\pi t}{L}}\right]$$

$$\Rightarrow f(t) = \sum_{n=1}^{\infty}\left[c_n e^{i\frac{n\pi t}{L}}\right] + c_0 + \sum_{n=-1}^{-\infty}\left[c_n e^{i\frac{n\pi t}{L}}\right]$$

$$\Rightarrow f(t) = \sum_{n=-\infty}^{\infty}\left[c_n e^{i\frac{n\pi t}{L}}\right]$$

with $c_n = \frac{1}{2}(a_n - ib_n)$

$$\Rightarrow c_n = \frac{1}{2}\left[\frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt - i\frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\,dt\right]$$

$$\Rightarrow c_n = \frac{1}{2L}\int_{-L}^{L} f(t)\left[\cos\left(\frac{n\pi t}{L}\right) - i\sin\left(\frac{n\pi t}{L}\right)\right]dt$$

$$\Rightarrow c_n = \frac{1}{2L}\int_{-L}^{L} f(t)\, e^{-i\frac{n\pi t}{L}}\,dt \quad , n\in\mathbb{Z}$$