IYGB

Mathematical Methods 1

Practice Paper E

Time: 3 hours

Candidates may use any non programmable, non graphical calculator which does not have the capability of storing data or manipulating algebraic expressions.

Information for Candidates

This practice paper follows the most common syllabi of Mathematical Methods used the United Kingdom Universities for Mathematics, Physics and Engineering degrees.

Booklets of *Mathematical formulae and statistical tables* may NOT be used.

Full marks may be obtained for answers to MOST of the questions.

The marks for the parts of questions are shown in round brackets, e.g. **(2)**.

There are 17 questions in this question paper.

The total mark for this paper is 139.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.

You must show sufficient working to make your methods clear to the Examiner.

Answers without working may not gain full credit.

Non exact answers should be given to an appropriate degree of accuracy.

The examiner may refuse to mark any parts of questions if deemed not to be legible.

Scoring

The maximum number of marks on this paper is 100, which can be obtained without attempting all the questions on the paper.

If the number of marks exceeds 100 then the paper will score maximum marks.

Topics Examined Under Mathematical Methods 1

- Revision and extensions of Pre-University Algebra and Calculus, including intrinsic coordinates.
- Partial Differentiation
  [includes applications and the use of the gradient operator]
- Curve Sketching in 2 and 3 Dimensions
- Limits
- Ordinary Differential Equations
  [First and second order including substitutions, variation of parameters, D operator and simple Laplace transforms]
- Sequences and Series
- Recurrence Relations
- Product Operator
- Fourier Series
- Conics and Quadrics
- Differentiation and Integration under the Integral Sign

Question 1

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}[u_i] = u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k .$$

Simplify, showing a clear method

$$\prod_{r=1}^{n}\left[\frac{r+1}{r}\right].$$ **(2)**

Question 2

The surface S has Cartesian equation

$$x = y^2 + z^2 .$$

Sketch the graph of S. **(3)**

Question 3

A sequence of numbers T_1, T_2, T_3, T_4, T_5, … , T_n, … is generated by the recurrence relation

$$T_{n+1} = 2T_n - 5, \quad T_1 = 6 .$$

Determine an expression for the n^{th} term of this sequence. **(7)**

Question 4

$$x+2y+z=1$$
$$x+y+3z=2$$
$$3x+5y+5z=4$$

Show that the solution of the above simultaneous equations is

$$x=3-5t, \quad y=2t-1, \quad z=t$$

where t is a parameter. **(6)**

Question 5

Simplify each of the following expressions.

a) $\dfrac{1}{D^2-4D+3}\left\{e^{2x}\right\}$. **(1)**

b) $\dfrac{1}{D^2-4D+3}\left\{e^{3x}\right\}$. **(3)**

c) $\dfrac{1}{D^2-4D+3}\{\sin 2x\}$. **(3)**

d) $\dfrac{1}{D^2+1}\{\cos x\}$. **(3)**

Question 6

The function of two variables f is defined as

$$f(x,y) \equiv 2x^3+6xy^2-3y^3-150x, \quad x \in \mathbb{R}, \; y \in \mathbb{R}.$$

Find the coordinates of each of the stationary points of f, where $z=f(x,y)$, and further determine their nature. **(11)**

Question 7

Use the method of variation of parameters to find the general solution of the following differential equation.

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 6x e^{2x} .$$ **(7)**

Question 8

By using a suitable test and justifying every step in the workings, determine the convergence or divergence of the following series.

a) $$\sum_{n=1}^{\infty} \frac{10^n}{n!} .$$ **(3)**

b) $$\sum_{k=1}^{\infty} \frac{k^4}{(k+1)^6} .$$ **(2)**

c) $$\sum_{r=1}^{\infty} \frac{(r+1)(2r+1)(3r+1)}{r^4} .$$ **(2)**

Question 9

A solid's base is bounded by the circle with equation

$$x^2 + y^2 = 1 .$$

Every vertical cross-section of the solid, perpendicular to the x axis, is a right angled isosceles triangle, with one of its non hypotenuse sides on the base of the solid.

Determine the volume of the solid. **(6)**

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Question 10

Use two distinct methods to evaluate the following limit.

$$\lim_{x \to 1}\left[\frac{\sqrt{x+3}-2\sqrt{x}}{\sqrt{x}-1}\right].$$ **(8)**

Question 11

A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

a) State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series. **(2)**

b) Find the Fourier series of

$$f(x) = x^2, \; -1 \le x \le 1.$$ **(6)**

c) Hence determine the exact value of

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \dots.$$ **(3)**

Question 12

$$I = \int_0^{\infty} \frac{e^{-2x}\sin x}{x} \, dx .$$

By introducing in the integrand a parameter k and carrying a suitable differentiation under the integral sign show that

$$I = \operatorname{arccot} 2 .$$ **(8)**

Question 13

The function $w = \varphi\left[u(x,y), v(x,y)\right]$ satisfies

$$x = e^{u}\cos v \quad \text{and} \quad y = e^{-u}\sin v .$$

Determine simplified expressions for $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$, in terms of u and v. **(9)**

Question 14

Use Laplace transforms to solve the differential equation

$$\frac{d^2 y}{dx^2} - 4y = 24\cos 2x, \;\; x \ge 0 ,$$

subject to the boundary conditions $y = 3$, $\frac{dy}{dx} = 4$ at $x = 0$. **(9)**

Question 15

A curve C has Cartesian equation

$$\sin y = e^{x}, \;\; x \le 0 .$$

Find an intrinsic equation for C, in the form $s = f(\psi)$, where s is measured from the point with Cartesian coordinates $\left(0, \frac{\pi}{2}\right)$, and ψ is the angle the tangent to C makes with the positive x axis. **(8)**

Question 16

$$x^2 + 2y^2 + z^2 + 2xy + 2yz = 9.$$

The quadric surface, with the above Cartesian equation, is rotated out of its standard position.

By suitably removing the cross terms, give the Cartesian equation of the quadric surface in the rotated system.

You are expected to name the conic and give the direction of any axes of symmetry in the original coordinate system. **(12)**

Question 17

The curve with equation $y = f(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} + e^{-y} = 0, \quad \frac{dy}{dx} \geq 0.$$

If $y = 0$, $\frac{dy}{dx} = -1$ at $x = \frac{1}{2}\pi$, solve the differential equation to show that

$$y = \ln(1 - \cos x).$$ **(15)**

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