IYGB, MATHEMATICAL METHODS 2 – PAPER A – QUESTION 1

- FIRSTLY WE HAVE IN SUMMATION NOTATION

  - THE k^{th} COMPONENT OF $\underline{A} \wedge \underline{B}$ IS $(\underline{A} \wedge \underline{B})_k = \varepsilon_{ijk} A_i B_j$
  - THE DIVERGENCE OF A VECTOR u_k IS $\underline{\nabla} \cdot \underline{u} = \frac{\partial}{\partial x_k} u_k$

- USING THESE RESULTS WE NOW HAVE

$$\underline{\nabla} \cdot (\underline{\nabla} \wedge \underline{F}) = \frac{\partial}{\partial x_k}\left[\varepsilon_{ijk} \frac{\partial}{\partial x_i} F_j\right] = \varepsilon_{ijk} \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_i} F_j$$

$$= \varepsilon_{ijk} \frac{\partial^2 F_j}{\partial x_k \partial x_i}$$

AS i & k ARE DUMMY VARIABLES WE MAY INTERCHANGE THEM

$$= \varepsilon_{kji} \frac{\partial^2 F_j}{\partial x_i \partial x_k}$$

SWAP i & k IN THE PERMUTATION SYMBOL GENERATES A MINUS

$$= -\varepsilon_{ijk} \frac{\partial^2 F_j}{\partial x_i \partial x_k}$$

REVERSE THE DIFFERENTIATION ORDER IN THE PARTIAL DERIVATIVE

$$= -\varepsilon_{ijk} \frac{\partial^2 F_j}{\partial x_k \partial x_i}$$

$$= -\underline{\nabla} \cdot (\underline{\nabla} \wedge \underline{F})$$

- CONCLUDING THE ARGUMENT

$\underline{\nabla} \cdot (\underline{\nabla} \wedge \underline{F}) = -\underline{\nabla} \cdot (\underline{\nabla} \wedge \underline{F})$ FOR ANY SMOOTH VECTOR FIELD $\underline{B}$

$$\therefore \underline{\nabla} \cdot (\underline{\nabla} \wedge \underline{F}) = \underline{0}$$

IYGB – MATHEMATICAL METHODS 2, PAPER A – QUESTION 2

- START BY PARAMETRIZING THE LINE SEGMENT USING VECTORS

$\underline{a} = (1,1,0)$

$\underline{b} = (5,3,4)$

$\overrightarrow{AB} = \underline{b} - \underline{a} = (5,3,4) - (1,1,0)$
$= (4,2,4)$

$\underline{r} = (x,y,z) = (1,1,0) + t(4,2,4)$

$\underline{r} = (x,y,z) = (4t+1,\ 2t+1,\ 4t)$

HENCE WE HAVE

$$\left.\begin{array}{l} x = 4t+1 \\ y = 2t+1 \\ z = 4t \end{array}\right\} \Longrightarrow \begin{array}{l} dx = 4\,dt \\ dy = 2\,dt \\ dz = 4\,dt \end{array}$$

$$\underset{(1,1,0)}{0} \leqslant t \leqslant \underset{(5,3,4)}{1}$$

- RETURNING TO THE LINE INTEGRAL

$$\int_{(1,1,0)}^{(5,3,4)} (3x-2y)\,dx + (y+z)\,dy + (1-z^2)\,dz$$

$$= \int_{t=0}^{t=1} \Big\{ \big[3(4t+1) - 2(2t+1)\big](4\,dt) + \big[(2t+1) + 4t\big](2\,dt) + \big[1 - 16t^2\big](4\,dt) \Big\}$$

$$= \int_{t=0}^{t=1} \Big\{ (8t+1)(4\,dt) + (6t+1)(2\,dt) + (1-16t^2)(4\,dt) \Big\}$$

$$= \int_0^1 \left(32t + 4 + 12t + 2 + 4 - 64t^2\right) dt$$

$$= \int_0^1 -64t^2 + 44t + 10\,dt$$

$$= \left[-\frac{64}{3}t^3 + 22t^2 + 10t\right]_0^1$$

$$= \left(-\frac{64}{3} + 22 + 10\right) - \cancel{(0)}$$

$$= \frac{-64 + 66 + 30}{3} = \frac{32}{3}$$

IYGB - MATHEMATICAL METHODS 2 - PAPER A - QUESTION 3

- STARTING BY THE DEFINITION OF MASS FOR VARIABLE DENSITY

$$\text{MASS} = \int_V \rho(x,y,z)\, dV = \int_{\substack{\text{VOLUME OF} \\ x^2+y^2+z^2=a^2}} \sqrt{x^2+y^2}\, dx\, dy\, dz$$

- SWITCH INTO SPHERICAL POLARS

$$\Rightarrow \text{MASS} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} \sqrt{\underbrace{(r\sin\theta\cos\phi)^2}_{x^2} + \underbrace{(r\sin\theta\sin\phi)^2}_{y^2}}\ \underbrace{r^2\sin\theta\, dr\, d\theta\, d\phi}_{dV}$$

$$\Rightarrow \text{MASS} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} \sqrt{r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi}\ r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$\Rightarrow \text{MASS} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} (r\sin\theta)\cancel{\sqrt{\cos^2\phi + \sin^2\phi}}\ r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$\Rightarrow \text{MASS} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} r^3\sin^2\theta\, dr\, d\theta\, d\phi$$

- INTEGRATING WITH RESPECT TO r FIRST

$$\Rightarrow \text{MASS} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \left[\frac{1}{4}r^4\sin^2\theta\right]_0^a d\theta\, d\phi$$

$$\Rightarrow \text{MASS} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \frac{1}{4}a^4\sin^2\theta\, d\theta\, d\phi$$

$$\Rightarrow \text{MASS} = \frac{1}{4}a^4 \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \sin^2\theta\, d\theta\, d\phi$$

IYGB - MATHEMATICAL METHODS 2 - PAPER A - QUESTION 3

- INTEGRATING WITH RESPECT TO ϕ NEXT

$$\text{MASS} = \frac{1}{4}a^4 \times 2\pi \times \int_{\theta=0}^{\pi} \frac{1}{2} - \cancel{\frac{1}{2}\cos 2\theta}\, d\theta$$

$$\text{MASS} = \frac{1}{4}a^4 \times 2\pi \times \frac{1}{2}\pi$$

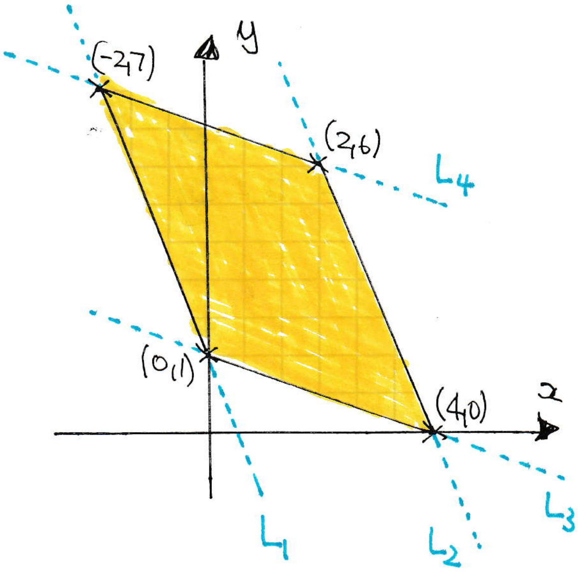
$$\text{MASS} = \frac{1}{4}a^4\pi^2$$

- NOW THE VOLUME OF A SPHERE OF RADIUS a IS $V = \frac{4}{3}\pi a^3$

$$\therefore \text{AVERAGE DENSITY} = \frac{\text{MASS}}{\text{VOLUME}} = \frac{\frac{1}{4}\pi^2 a^4}{\frac{4}{3}\pi a^3} = \frac{3}{16}\pi a$$

IYGB - MATHEMATICAL METHODS 2 - PAPER A - QUESTION 4

- START WITH A SKETCH OF THE REGION



GRADIENT OF L_1: $\frac{7-1}{-2-0} = \frac{6}{-2} = -3$

GRADIENT OF L_3: $\frac{0-1}{4-0} = -\frac{1}{4}$

- NOW WE HAVE THE EQUATIONS OF ALL 4 LINES WHICH DEFINE THE REGION

L_1: $y = -3x+1 \implies 3x+y=1$

L_2: $y = -3x+12 \implies 3x+y=12$

L_3: $y = -\frac{1}{4}x+1 \implies x+4y=4$

L_4: $y = -\frac{1}{4}x+\frac{13}{2} \implies x+4y=26$

- DEFINE A NEW CO-ORDINATE SYSTEM, PARALLEL TO THE SIDES L_1 TO L_4

$$\boxed{\begin{array}{ll} u = 3x+y & 1 \le u \le 12 \\ v = x+4y & 4 \le v \le 26 \end{array}}$$

$$\implies du\,dv = \frac{\partial(u,v)}{\partial(x,y)}\,dx\,dy$$

$$\implies du\,dv = \left\|\begin{matrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{matrix}\right\| dx\,dy$$

$$\implies du\,dv = \left\|\begin{matrix} 3 & 1 \\ 1 & 4 \end{matrix}\right\| dx\,dy$$

$$\implies du\,dv = 11\,dx\,dy$$

$$\implies \boxed{dx\,dy = \frac{1}{11}\,du\,dv}$$

IYGB - MATHEMATICAL METHODS 2 - PAPER A - QUESTION 4

- FINALLY GETTING A SIMPLIFIED EXPRESSION FOR THE INTEGRAND IN TERMS OF u & v

$$\left.\begin{array}{l} u = 3x+y \\ v = x+4y \end{array}\right\} \Longrightarrow \begin{array}{l} -4u = -12x - 4y \\ \quad v = \quad x + 4y \end{array}$$

$$\Longrightarrow v - 4u = -11x$$

$$\Longrightarrow 11x = 4u - v$$

$$\Longrightarrow 121x^2 = (4u-v)^2$$

$$\Longrightarrow \boxed{x^2 = \frac{1}{121}(4u-v)^2}$$

- RETURNING TO THE ACTUAL INTEGRAL

$$\int_R x^2 \, dx\,dy = \int_{v=4}^{26}\int_{u=1}^{12} \frac{1}{121}(4u-v)^2 \left(\frac{1}{11}\,du\,dv\right)$$

$$= \frac{1}{11^3}\int_{v=4}^{26}\int_{u=1}^{12} (4u-v)^2 \, du\,dv = \frac{1}{11^3}\int_{v=4}^{26}\left[\frac{1}{12}(4u-v)^3\right]_{u=1}^{12} dv$$

$$= \frac{1}{11^3\times 12}\int_4^{26} (48-v)^3 - (4-v)^3 \, dv = \frac{1}{11^3\times 12}\left[-\frac{1}{4}(48-v)^4 + \frac{1}{4}(4-v)^4\right]_4^{26}$$

$$= \frac{1}{11^3\times 12\times 4}\left[(4-v)^4 - (48-v)^4\right]_4^{26} = \frac{1}{11^3\times 12\times 4}\left[\cancel{\left[(-22)^4 - (22)^4\right]} - \left[0 - (44)^4\right]\right]$$

$$= \frac{1}{11^3\times 12\times 4}(44)^4 = \frac{4^4\times 11^4}{4\times 4\times 3\times 11^3} = \frac{4^2\times 11}{3} = \frac{16\times 11}{3}$$

$$= \frac{176}{3}$$

IYGB - MATHEMATICAL METHODS 2 - PAPER A - QUESTION 5

- START "EXPANDING" BY THE VECTOR CALCULUS IDENTITY

$$\boxed{\nabla\cdot(\phi \underline{A}) = \phi\, \nabla\cdot\underline{A} + \nabla\phi\cdot\underline{A}}$$

$$\Longrightarrow \nabla\cdot(r^n \underline{r}) = r^n\,\nabla\cdot\underline{r} + \nabla r^n\cdot\underline{r}$$

$$= r^n\left[\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right]\cdot(x,y,z) + \left[\frac{\partial}{\partial x}r^n, \frac{\partial}{\partial y}r^n, \frac{\partial}{\partial z}r^n\right]\cdot(x,y,z)$$

$$= r^n\left[\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z}\right] + \left[x\frac{\partial}{\partial x}(r^n) + y\frac{\partial}{\partial y}(r^n) + z\frac{\partial}{\partial z}(r^n)\right]$$

$$= r^n[1+1+1] + \left[x\frac{\partial}{\partial x}(r^n) + y\frac{\partial}{\partial y}(r^n) + z\frac{\partial}{\partial z}(r^n)\right]$$

- NOW $r = |\underline{r}| = |x,y,z| = \sqrt{x^2+y^2+z^2} = (x^2+y^2+z^2)^{\frac{1}{2}}$

$$r^n = (x^2+y^2+z^2)^{\frac{n}{2}}$$

$$\frac{\partial}{\partial x}(r^n) = \frac{\partial}{\partial x}\left((x^2+y^2+z^2)^{\frac{n}{2}}\right) = nx(x^2+y^2+z^2)^{\frac{n}{2}-1}$$

AND SIMILARLY THE REST AS THESE EXPRESSIONS ARE SYMMETRICAL

- TIDYING UP FURTHER WE GET

$$\Longrightarrow \nabla\cdot(r^n\underline{r}) = 3r^n + nx^2(x^2+y^2+z^2)^{\frac{n}{2}-1} + ny^2(x^2+y^2+z^2)^{\frac{n}{2}-1} + nz^2(x^2+y^2+z^2)^{\frac{n}{2}-1}$$

$$= 3r^n + n(x^2+y^2+z^2)^{\frac{n}{2}-1}\left[x^2+y^2+z^2\right]^1$$

$$= 3r^n + n(x^2+y^2+z^2)^{\frac{n}{2}}$$

$$= 3r^n + n\left[(x^2+y^2+z^2)^{\frac{1}{2}}\right]^n$$

$$= 3r^n + nr^n$$

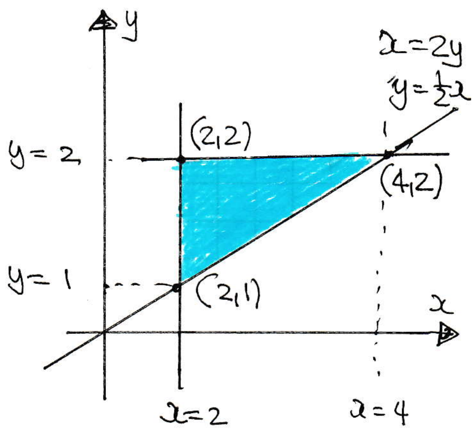
$$= (n+3)r^n$$

AS REQUIRED

IYGB - MATHEMATICAL METHODS 2 - PAPER A - QUESTION 6

- LOOKING AT THE INTEGRAND, IT LOOKS LIKE IT MIGHT BE EASIER TO INTEGRATE WITH RESPECT TO y FIRST
- SKETCH THE REGION IN ORDER TO "REVERSE THE LIMITS"

$$\Rightarrow I = \int_{y=1}^{y=2} \int_{x=2}^{x=2y} \frac{16y}{(16-x^2)^{\frac{3}{2}}}\, dx\, dy$$



$$\Rightarrow I = \int_{x=2}^{x=4} \int_{y=\frac{1}{2}x}^{y=2} \frac{16y}{(16-x^2)^{\frac{3}{2}}}\, dy\, dx$$

$$\Rightarrow I = \int_{x=2}^{x=4} \left[\frac{8y^2}{(16-x^2)^{3/2}} \right]_{y=\frac{1}{2}x}^{y=2} dx$$

$$\Rightarrow I = \int_{x=2}^{x=4} \frac{32 - 8\left(\frac{1}{2}x\right)^2}{(16-x^2)^{3/2}}\, dx = \int_2^4 \frac{32 - 2x^2}{(16-x^2)^{3/2}}\, dx$$

$$\Rightarrow I = \int_2^4 \frac{2(16-x^2)}{(16-x^2)^{3/2}}\, dx = \int_2^4 \frac{2}{\sqrt{16-x^2}}\, dx$$

- THIS IS NOW A STANDARD INTEGRAL (OR USE THE SUBSTITUTION $x=4\sin\theta$)

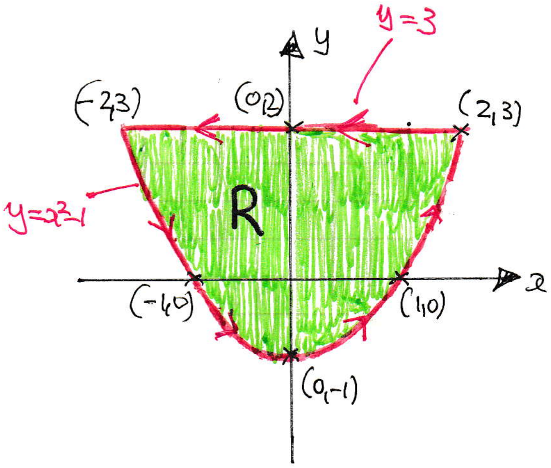
$$\Rightarrow I = \left[2\arcsin\left(\frac{x}{4}\right) \right]_2^4 = 2\arcsin 1 - 2\arcsin\frac{1}{2}$$

$$\Rightarrow I = 2\times\frac{\pi}{2} - 2\times\frac{\pi}{6} = \pi - \frac{\pi}{3}$$

$$\Rightarrow I = \frac{2\pi}{3}$$

IYGB - MATHEMATICAL METHODS 2 - PAPER A - QUESTION 7

- START BY SKETCHING THE PATH C



$$\oint_C \underline{F} \cdot d\underline{r}$$

$$= \oint_C \left[x\cos x, 15xy + \ln(1+y^3)\right] \cdot \left[dx, dy\right]$$

$$= \oint_C \Big[\underbrace{(x\cos x)}_{P}\, dx + \underbrace{\left[15xy + \ln(1+y^3)\right]}_{Q} dy\Big]$$

- AS THIS INTEGRATION LOOKS LIKE AN IMPOSSIBILITY, INVESTIGATE WHETHER GREEN'S THEOREM ON THE PLANE CAN BE USED

$$\boxed{\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy}$$

$$= \iint_R (15y - 0)\, dx\,dy = \int_{x=-2}^{x=2} \int_{y=x^2-1}^{y=3} 15y \; dy\,dx$$

$$= \int_{-2}^{2} \left[\frac{15}{2}y^2\right]_{y=x^2-1}^{y=3} dx = \frac{15}{2}\int_{-2}^{2} 3^2 - (x^2-1)^2 \, dx$$

$$= \frac{15}{2}\int_{-2}^{2} 9 - (x^2-1)^2 \, dx$$

- THE INTEGRAND IS EVEN IN x

$$= 15\int_0^2 9 - (x^2-1)^2 \, dx$$

$$= 15\int_0^2 9 - (x^4 - 2x^2 + 1)\, dx$$

$$= 15\int_0^2 8 + 2x^2 - x^4\, dx$$

$$= \int_0^2 120 + 30x^2 - 15x^4\, dx$$

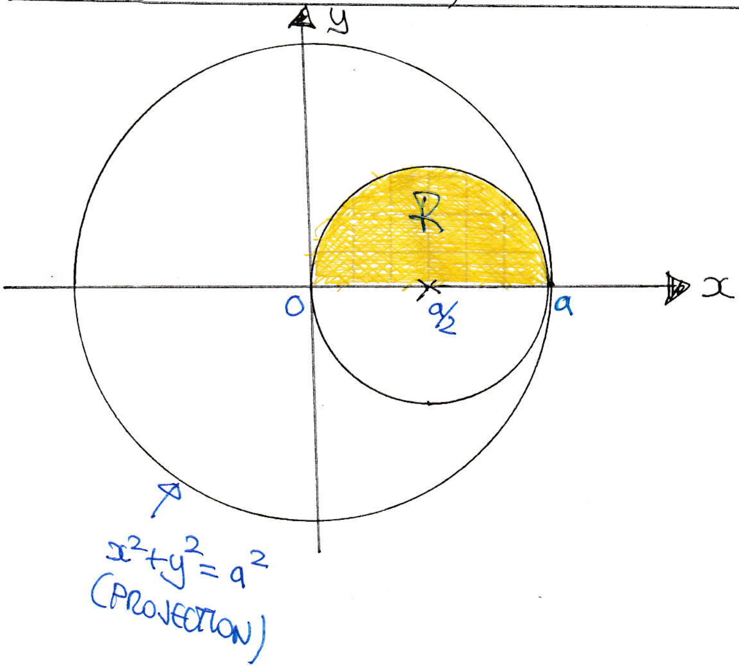
$$= \Big[120x + 10x^3 - 3x^5\Big]_0^2$$

$$= (240 + 80 - 96) - (0)$$

$$= 224$$

IYGB - MATHEMATICAL METHODS 2 - PAPER A - QUESTION 8

- START WITH A DIAGRAM OF THE PROJECTION OF THE "TOP HALF" OF THE SPHERICAL SURFACE, PROJECTED ONTO THE xy PLANE



TO "SEE" THE CYLINDER

$$x^2+y^2=ax$$

$$x^2-ax+y^2=0$$

$$\left(x-\frac{a}{2}\right)^2-\frac{a^2}{4}+y^2=0$$

$$\left(x-\frac{a}{2}\right)^2+y^2=\frac{a^2}{4}$$

(MARKED OPPOSITE)

- THE SPHERICAL SURFACE FOR WHICH $z \geqslant 0$ (TOP HALF), HAS EQUATION

$$z=(a^2-x^2-y^2)^{\frac{1}{2}}$$

$$\frac{\partial z}{\partial x}=\frac{1}{2}(-2x)(a^2-x^2-y^2)^{-\frac{1}{2}}=-\frac{x}{(a^2-x^2-y^2)^{\frac{1}{2}}}$$

$$\frac{\partial z}{\partial x}=\frac{1}{2}(-2y)(a^2-x^2-y^2)^{-\frac{1}{2}}=-\frac{y}{(a^2-x^2-y^2)^{\frac{1}{2}}}$$

NOTE THAT THE PROJECTION IS ALSO POSSIBLE USING VECTOR METHODS

$$dS=\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$

$$dS=\sqrt{1+\frac{x^2}{a^2-x^2-y^2}+\frac{y^2}{a^2-x^2-y^2}}\,dx\,dy$$

$$dS=\sqrt{\frac{a^2-x^2-y^2+x^2+y^2}{a^2-x^2-y^2}}\,dx\,dy$$

$$\boxed{dS=\frac{a}{\sqrt{a^2-x^2-y^2}}\,dx\,dy}$$

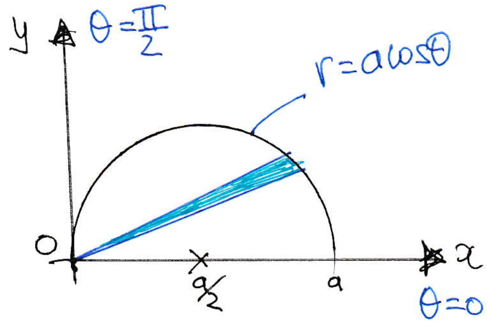
IYGB - MATHEMATICAL METHODS 2 - PAPER A - QUESTION 2

- THE REQUIRED SURFACE IS 4 TIMES THE "PROJECTION" ONTO THE REGION R, SHOWN IN YELLOW, BY SYMMETRY

$$\Rightarrow \text{AREA} = 4\int_S 1 \, dS = 4\int_R 1\left(\frac{a}{\sqrt{a^2-x^2-y^2}}\right) dx\,dy$$

$$\Rightarrow \text{AREA} = 4a\int_R \frac{1}{\sqrt{a^2-y^2-x^2}} \, dx\,dy$$

- SWITCHING THE INTEGRAL INTO PLANE POLARS



$$\Rightarrow x^2+y^2 = ax$$

$$\Rightarrow r^2 = a(r\cos\theta)$$

$$\Rightarrow r = a\cos\theta$$

WITH

$$0 \leqslant r \leqslant a\cos\theta$$

$$0 \leqslant \theta \leqslant \pi/2$$

AND

$$dx\,dy = r\,dr\,d\theta$$

- SO THE INTEGRAL BECOMES

$$\Rightarrow \text{AREA} = 4a\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{r=a\cos\theta} \frac{1}{\sqrt{a^2-r^2}}\,(r\,dr\,d\theta)$$

$$\Rightarrow \text{AREA} = 4a\int_{\theta=0}^{\pi/2}\int_{r=0}^{r=a\cos\theta} r(a^2-r^2)^{-\frac{1}{2}}\,dr\,d\theta$$

IYGB - MATHEMATICAL METHODS 2 - PAPER A - QUESTION 8

- BY INSPECTION (OR SUBSTITUTION $u = a^2 - r^2$)

$$\Rightarrow \text{AREA} = 4a \int_{\theta=0}^{\pi/2} \left[-(a^2 - r^2)^{\frac{1}{2}} \right]_{r=0}^{r=a\cos\theta} d\theta$$

$$\Rightarrow \text{AREA} = 4a \int_{\theta=0}^{\pi/2} \left[\sqrt{a^2 - r^2} \right]_{r=a\cos\theta}^{r=0} d\theta$$

$$\Rightarrow \text{AREA} = 4a \int_{\theta=0}^{\pi/2} a - \sqrt{a^2 - a^2\cos^2\theta} \, d\theta$$

$$\Rightarrow \text{AREA} = 4a \int_{\theta=0}^{\pi/2} a - a\sqrt{1 - \cos^2\theta} \, d\theta$$

$$\Rightarrow \text{AREA} = 4a \int_{\theta=0}^{\pi/2} a - a\sin\theta \, d\theta$$

$$\Rightarrow \text{AREA} = 4a^2 \int_0^{\frac{\pi}{2}} 1 - \sin\theta \, d\theta$$

$$\Rightarrow \text{AREA} = 4a^2 \left[\theta + \cos\theta \right]_0^{\pi/2}$$

$$\Rightarrow \text{AREA} = 4a^2 \left[\left(\frac{\pi}{2} + 0 \right) - (0 + 1) \right]$$

$$\Rightarrow \text{AREA} = 4a^2 \left(\frac{\pi}{2} - 1 \right)$$

$$\Rightarrow \text{AREA} = 2a^2(\pi - 2)$$

IYGB - MATHEMATICAL METHODS 2 - PAPER A - QUESTION 9

PREPARE ALL THE AUXILIARY ITEMS

- $\underline{F}(x,y,z) = (x^2, y^2, z^2)$
- $\underline{r}(u,v) = (u+v, u-v, u), \quad 0 \leq u \leq 2, \quad 0 \leq v \leq 3$

(THIS IS IN FACT A PLANE THROUGH O)

- $\frac{\partial r}{\partial u} = (1,1,1) \qquad \frac{\partial r}{\partial v} = (1,-1,0)$

- $$\frac{\partial r}{\partial u} \wedge \frac{\partial r}{\partial v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = (1,1,-2)$$

$$\therefore \boxed{\underline{n} = \frac{\partial r}{\partial u} \wedge \frac{\partial r}{\partial v} = (1,1,-2)}$$

$$\boxed{dS = \left\| \frac{\partial r}{\partial u} \wedge \frac{\partial r}{\partial v} \right\| \, du \, dv}$$

HENCE WE NOW HAVE IN PARAMETRIC

$$\int_S \underline{F} \cdot d\underline{S} = \int_S \underline{F} \cdot \hat{\underline{n}} \, dS$$

$$= \int_S \underline{F} \cdot \underbrace{\frac{\frac{\partial r}{\partial u} \wedge \frac{\partial r}{\partial v}}{\left\| \frac{\partial r}{\partial u} \wedge \frac{\partial r}{\partial v} \right\|}}_{\hat{n}} \underbrace{\left\| \frac{\partial r}{\partial u} \wedge \frac{\partial r}{\partial v} \right\| du \, dv}_{dS}$$

$$= \int_S \underline{F} \cdot \left(\frac{\partial r}{\partial u} \wedge \frac{\partial r}{\partial v} \right) du \, dv$$

IYGB - MATHEMATICAL METHODS 2 - PAPER A - QUESTION 9

- SUBSTITUTING FULLY INTO THE REQUIRED SURFACE INTEGRAL

$$\Rightarrow \int_S \underline{F} \cdot d\underline{S} = \int_{v=0}^{3} \int_{u=0}^{2} \left[(u+v)^2, (u-v)^2, u^2\right] \cdot \left[1,1,-2\right] \, du \, dv$$

$$= \int_{v=0}^{3} \int_{u=0}^{2} (u+v)^2 + (u-v)^2 - 2u^2 \, du \, dv$$

$$= \int_{v=0}^{3} \int_{u=0}^{2} \cancel{u^2} + \cancel{2uv} + v^2 + \cancel{u^2} - \cancel{2uv} + v^2 - \cancel{2u^2} \, du \, dv$$

$$= \int_{v=0}^{3} \int_{u=0}^{2} 2v^2 \, du \, dv$$

$$= \int_{v=0}^{3} \left[2v^2 u\right]_{u=0}^{2} \, dv$$

$$= \int_{v=0}^{3} 4v^2 \, dv$$

$$= \left[\frac{4}{3}v^3\right]_0^3$$

$$= \frac{4}{3} \times 27$$

$$= 36$$

IYGB - MATHEMATICAL METHODS 2 - PAPER A - QUESTION 10

a) THE SURFACE IS A CONE, TRUNCATED IN THE POSITIVE z DIRECTION BY 1 UNIT

- WHEN $x=0$

$(z-1)^2 = y^2$

$z-1 = \begin{cases} y \\ -y \end{cases}$

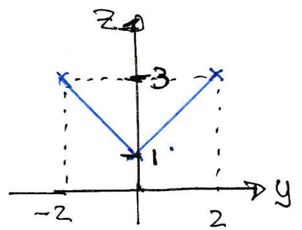
$z = \begin{cases} 1+y \\ 1-y \end{cases}$

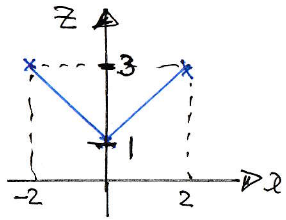
- WHEN $y=0$

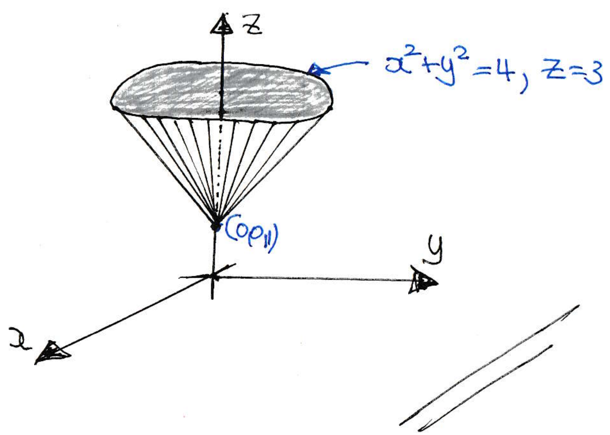
$(z-1)^2 = x^2$

$z-1 = \begin{cases} x \\ -x \end{cases}$

$z = \begin{cases} 1+x \\ 1-x \end{cases}$







b) EVALUATING THE GIVEN CURL

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ zx^2 & xy^2 & yz^2 \end{vmatrix} = (z^2, x^2, y^2) = \underline{F}$$

c) FINALLY WE HAVE

$$\int_S \underline{F} \cdot d\underline{S} = \int_S (z^2, x^2, y^2) \cdot \hat{\underline{n}}\ dS$$

BY PART (b)

$$= \int_{S} \underline{\nabla} \wedge (zx^2, xy^2, yz^2) \cdot \hat{\underline{n}}\ dS$$

BY STOKES THEOREM FOR OPEN SURFACES

$$= \oint_C \overset{\underline{A}}{(zx^2, xy^2, yz^2)} \cdot \overset{d\underline{r}}{(dx, dy, dz)}$$

WHERE C IS $x^2+y^2=4,\ z=3$

$$= \oint_C \left[zx^2\,dx + xy^2\,dy + yz^2\,dz \right]$$

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PARAMETRIZE THE LINE INTEGRAL ON C

$$\boxed{\left.\begin{array}{ll} x = 2\cos\theta\,, & dx = -2\sin\theta \\ y = 2\sin\theta\,, & dy = 2\cos\theta \\ z = 3\,, & dz = 0 \end{array}\right\} \quad 0 \le \theta < 2\pi}$$

$$\ldots = \int_{\theta=0}^{2\pi} 3(2\cos\theta)^2(-2\sin\theta\, d\theta) + (2\cos\theta)(2\sin\theta)^2(2\cos\theta\, d\theta) + 0$$

$$= \int_0^{2\pi} \left[\cancel{-24\cos^3\theta\sin\theta} + 16\cos^2\theta\sin^2\theta\right] d\theta$$

ZERO CONTRIBUTION OVER THESE LIMITS

$$= \int_0^{2\pi} 16\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta\right)\left(\tfrac{1}{2}-\tfrac{1}{2}\cos 2\theta\right) d\theta$$

$$= \int_0^{2\pi} 16\left(\tfrac{1}{4}-\tfrac{1}{4}\cos^2 2\theta\right) d\theta = \int_0^{2\pi} 4 - 4\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 4\theta\right) d\theta$$

$$= \int_0^{2\pi} 2 - \cancel{2\cos 4\theta}\, d\theta$$

ZERO CONTRIBUTION OVER THESE LIMITS

$$= 2 \times 2\pi = 4\pi$$

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ALTERNATIVE FOR PART (C) - DIRECT EVALUATION OF THE SURFACE INTEGRAL

$$\int_S \underline{F} \cdot d\underline{S} = \int_S (z^2, x^2, y^2) \cdot \hat{\underline{n}} \, dS = \int_R (z^2, x^2, y^2) \cdot \hat{\underline{n}} \frac{dx\,dy}{\hat{\underline{n}} \cdot \hat{\underline{k}}}$$

PROJECT ONTO THE xy PLANE ONTO REGION
$R: x^2+y^2 \leq 4$

$$= \int_R (z^2, x^2, y^2) \cdot \frac{\underline{n}}{|\underline{n}|} \frac{dx\,dy}{\frac{\underline{n}}{|\underline{n}|} \cdot \underline{k}} = \int_R (z^2, x^2, y^2) \cdot \underline{n} \frac{dx\,dy}{\underline{n} \cdot \underline{k}}$$

$$= \int_R (z^2, x^2, y^2) \cdot (x, y, 1-z) \frac{dx\,dy}{1-z} = \int_R \frac{xz^2}{1-z} + \frac{x^2 y}{1-z} + y^2 \, dx\,dy$$

$$= \int_R \left[\cancel{\frac{x\left[1+\sqrt{x^2+y^2}\right]^2}{1-\left[1+\sqrt{x^2+y^2}\right]}} + \cancel{\frac{x^2 y}{1-\left[1+\sqrt{x^2+y^2}\right]}} + y^2 \right] dx\,dy$$

$\dfrac{(\text{ODD IN } x)(\text{EVEN IN } x)}{(\text{EVEN IN } x)}$ $\quad$ $\dfrac{(\text{ODD IN } y)}{(\text{EVEN IN } y)}$ $\quad$ EVEN IN y

$\therefore$ ODD IN x IN A SYMMETRICAL DOMAIN, R

$\therefore$ ODD IN y IN A SYMMETRICAL DOMAIN, R

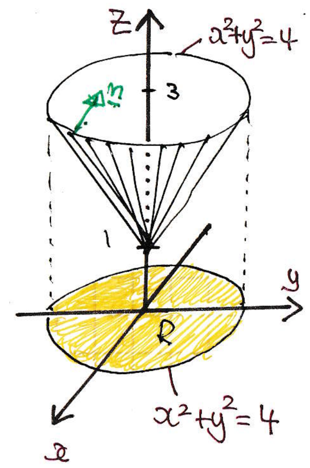
- CONE EQUATION

$$(z-1)^2 = x^2+y^2$$

- $\phi(x,y,z) = x^2+y^2-(z-1)^2$

$$\underline{\nabla}\phi = [2x, 2y, -2(z-1)]$$

$$\boxed{\underline{n} = (x, y, 1-z)}$$



$$\boxed{\underline{n} \cdot \hat{\underline{k}} = 1-z}$$

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WITH THE INTEGRAND GREATLY SIMPLIFIED, SWITCH INTO PLANE POLARS, OVER R

$$\int_S \underline{F} \cdot d\underline{S} = \dots \int_R y^2 \, dx\,dy = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \underbrace{(r\sin\theta)^2}_{y^2} \underbrace{(r\,dr\,d\theta)}_{dx\,dy}$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} r^3 \sin^2\theta \, dr\,d\theta = \int_{\theta=0}^{2\pi} \left[\tfrac{1}{4} r^4 \sin^2\theta \right]_{r=0}^{2} d\theta$$

$$= \int_0^{2\pi} 4\sin^2\theta \, d\theta = \int_0^{2\pi} 4\left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta\right) d\theta$$

$$= \int_0^{2\pi} 2 - \cancel{2\cos 2\theta} \, d\theta$$

NO CONTRIBUTION OVER THESE LIMITS

$$= 2 \times 2\pi$$

$$= 4\pi$$