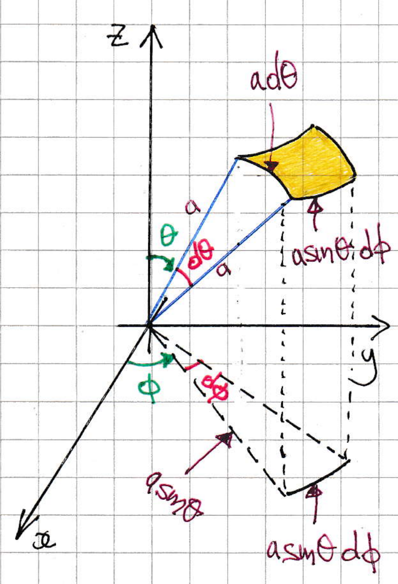
# IYGB – MATHEMATICAL METHODS 2 – PAPER C – QUESTION 1

THE TOTAL CHARGE ON THE SURFACE IS $\int_S \phi(x,y)\, dS$

HERE WE HAVE

$$\text{TOTAL CHARGE} = \int_S (2x^2+y^2)\, dS$$

SWITCH INTO SPHERICAL POLARS




IN SPHERICAL POLARS

- $x = a\sin\theta\cos\phi$
- $y = a\sin\theta\sin\phi$
- $z = a\cos\theta$

$x^2+y^2+z^2 = a^2$

$$\boxed{dS = a^2\sin\theta\, d\theta\, d\phi}$$

TOTAL CHARGE = ...

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \left[2(a\sin\theta\cos\phi)^2 + (a\sin\theta\sin\phi)^2\right](a^2\sin\theta\, d\theta\, d\phi)$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} a^4\left[2\sin^3\theta\cos^2\phi + \sin^3\theta\sin^2\phi\right] d\theta\, d\phi$$

$$= a^4\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \sin^3\theta\left[2\cos^2\phi + \sin^2\phi\right] d\theta\, d\phi$$

$$= a^4\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \sin^3\theta\,(1+\cos^2\phi)\, d\theta\, d\phi$$

$$= a^4\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \sin\theta(1-\cos^2\theta)\left(1+\tfrac{1}{2}+\tfrac{1}{2}\cos 2\phi\right) d\theta\, d\phi$$

$$= a^4\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \left(\sin\theta - \sin\theta\cos^2\theta\right)\left(\tfrac{3}{2}+\tfrac{3}{2}\cos 2\phi\right) d\theta\, d\phi$$

# IYGB – MATHEMATICAL METHODS 2 – PAPER C – QUESTION 1

SPLITTING THE INTEGRAL, AS THERE IS NO DEPENDENCE IN $\theta$ & $\phi$

$$\text{TOTAL CHARGE} = \left[a^4 \int_{\phi=0}^{2\pi} \frac{3}{2} + \cancel{\frac{1}{2}\cos 2\phi} \, d\phi\right]\left[\int_{\theta=0}^{\pi} \sin\theta - \sin\theta\cos^2\theta \, d\theta\right]$$

NO CONTRIBUTION OVER THESE LIMITS

$$= a^4 \times \frac{3}{2} \times 2\pi \times \left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]_0^{\pi}$$

$$= 3\pi a^4 \left[\left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right)\right]$$

$$= 3\pi a^4 \times \frac{4}{3}$$

$$= 4\pi a^4$$

# IYGB - MATHEMATICAL METHODS 2 - PAPER C - QUESTION 2

$$\iint_{\Omega} x^3 y \, dx\, dy = \ldots\ldots$$

$$= \int_{x=0}^{x=1} \int_{y=1-x}^{y=\sqrt{1-x^2}} x^3 y \, dy\, dx$$

$$= \int_{x=0}^{1} \left[ \tfrac{1}{2} x^3 y^2 \right]_{y=1-x}^{y=+\sqrt{1-x^2}} dx$$

$$= \int_0^1 \tfrac{1}{2} x^3 \left[ (1-x^2) - (1-x)^2 \right] dx$$

$$= \int_0^1 \tfrac{1}{2} x^3 \left[ 1 - x^2 - (1 - 2x + x^2) \right] dx$$

$$= \int_0^1 \tfrac{1}{2} x^3 (2x - 2x^2) \, dx$$

$$= \int_0^1 x^4 - x^5 \, dx$$

$$= \left[ \tfrac{1}{5} x^5 - \tfrac{1}{6} x^6 \right]_0^1$$

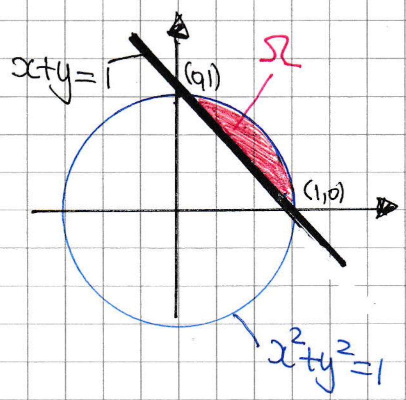
$$= \left( \tfrac{1}{5} - \tfrac{1}{6} \right) - (0)$$

$$= \frac{1}{30}$$

REGION $\Omega$

$x^2 + y^2 \leq 1$

$x + y \geq 1$




LIMITS

$y$: FROM $y = 1 - x$

TO $y = +\sqrt{1-x^2}$

$x$: FROM $x = 0$

TO $x = 1$

# IYGB - MATHEMATICAL METHODS 2 - PAPER C - QUESTION 3

START FROM THE R.H.S — AFTER REGROUPING THE TERMS CONSIDER THE $n^{\text{TH}}$ COMPONENT OF THE R.H.S

$$\left[(\underline{B}\cdot\underline{\nabla})\underline{A} + (\underline{A}\cdot\underline{\nabla})\underline{B} + \underline{B}\wedge(\underline{\nabla}\wedge\underline{A}) + \underline{A}\wedge(\underline{\nabla}\wedge\underline{B})\right]_n$$

$$= \left[\underline{B}\wedge(\underline{\nabla}\wedge\underline{A}) + \underline{A}\wedge(\underline{\nabla}\wedge\underline{B}) + (\underline{B}\cdot\underline{\nabla})\underline{A} + (\underline{A}\cdot\underline{\nabla})\underline{B}\right]_n$$

$$= \varepsilon_{\ell kn} B_\ell \left[\varepsilon_{ijk}\frac{\partial}{\partial x_i}A_j\right] + \varepsilon_{\ell kn} A_\ell\left[\varepsilon_{ijk}\frac{\partial}{\partial x_i}B_j\right] + \left[B_i\frac{\partial}{\partial x_i}\right]A_n + \left[A_i\frac{\partial}{\partial x_i}\right]B_n$$

$$= \varepsilon_{\ell kn}\varepsilon_{ijk}B_\ell\frac{\partial A_j}{\partial x_i} + \varepsilon_{\ell kn}\varepsilon_{ijk}A_\ell\frac{\partial B_j}{\partial x_i} + B_i\frac{\partial A_n}{\partial x_i} + A_i\frac{\partial B_n}{\partial x_i}$$

$$= -\varepsilon_{\ell nk}\varepsilon_{ijk}B_\ell\frac{\partial A_j}{\partial x_i} - \varepsilon_{\ell nk}\varepsilon_{ijk}A_\ell\frac{\partial B_j}{\partial x_i} + B_i\frac{\partial A_n}{\partial x_i} + A_i\frac{\partial B_n}{\partial x_i}$$

NOW USING THE IDENTITY

$$\varepsilon_{abc}\varepsilon_{dec} = \begin{vmatrix}\delta_{ad} & \delta_{ae}\\ \delta_{bd} & \delta_{be}\end{vmatrix} = \delta_{ad}\delta_{be} - \delta_{bd}\delta_{ae}$$

$$-\varepsilon_{\ell nk}\varepsilon_{ijk} = -\begin{vmatrix}\delta_{\ell i} & \delta_{\ell j}\\ \delta_{ni} & \delta_{nj}\end{vmatrix} = \delta_{ni}\delta_{\ell j} - \delta_{\ell i}\delta_{nj}$$

RETURNING TO THE "MAIN LINE"

$$= (\delta_{ni}\delta_{\ell j} - \delta_{\ell i}\delta_{nj})B_\ell\frac{\partial A_j}{\partial x_i} + (\delta_{ni}\delta_{\ell j} - \delta_{\ell i}\delta_{nj})A_\ell\frac{\partial B_j}{\partial x_i} + B_i\frac{\partial A_n}{\partial x_i} + A_i\frac{\partial B_n}{\partial x_i}$$

# IYGB - MATHEMATICAL METHODS 2 - PAPER C - QUESTION 3

EXPAND & USE "DELTA" SUBSTITUTION PROPERTY

$$= \delta_{ni}\delta_{lj}B_l\frac{\partial A_j}{\partial x_i} - \delta_{li}\delta_{nj}B_l\frac{\partial A_j}{\partial x_i} + \delta_{ni}\delta_{lj}A_l\frac{\partial B_j}{\partial x_i} - \delta_{li}\delta_{nj}A_l\frac{\partial B_j}{\partial x_i} + B_i\frac{\partial A_n}{\partial x_i} + A_i\frac{\partial B_n}{\partial x_i}$$

$$= B_j\frac{\partial A_j}{\partial x_n} - \cancel{B_l\frac{\partial A_n}{\partial x_l}} + A_j\frac{\partial B_j}{\partial x_n} - \cancel{A_l\frac{\partial B_n}{\partial x_l}} + \cancel{B_i\frac{\partial A_n}{\partial x_i}} + \cancel{A_i\frac{\partial B_n}{\partial x_i}}$$

$$= B_j\frac{\partial A_j}{\partial x_n} + A_j\frac{\partial B_j}{\partial x_n}$$

$$= \frac{\partial}{\partial x_n}\left[A_jB_j\right]$$

$$= \left[\underline{\nabla}(\underline{A}\cdot\underline{B})\right]_n$$

$$\therefore \quad \underline{\nabla}(\underline{A}\cdot\underline{B}) \equiv (\underline{B}\cdot\underline{\nabla})\underline{A} + (\underline{A}\cdot\underline{\nabla})\underline{B} + \underline{B}\wedge(\underline{\nabla}\wedge\underline{A}) + \underline{A}\wedge(\underline{\nabla}\wedge\underline{B})$$

# IYGB - MATHEMATICAL METHODS 2 - PAPER C - QUESTION 4

- START BY FINDING THE INTERSECTION OF THE TWO OBJECTS

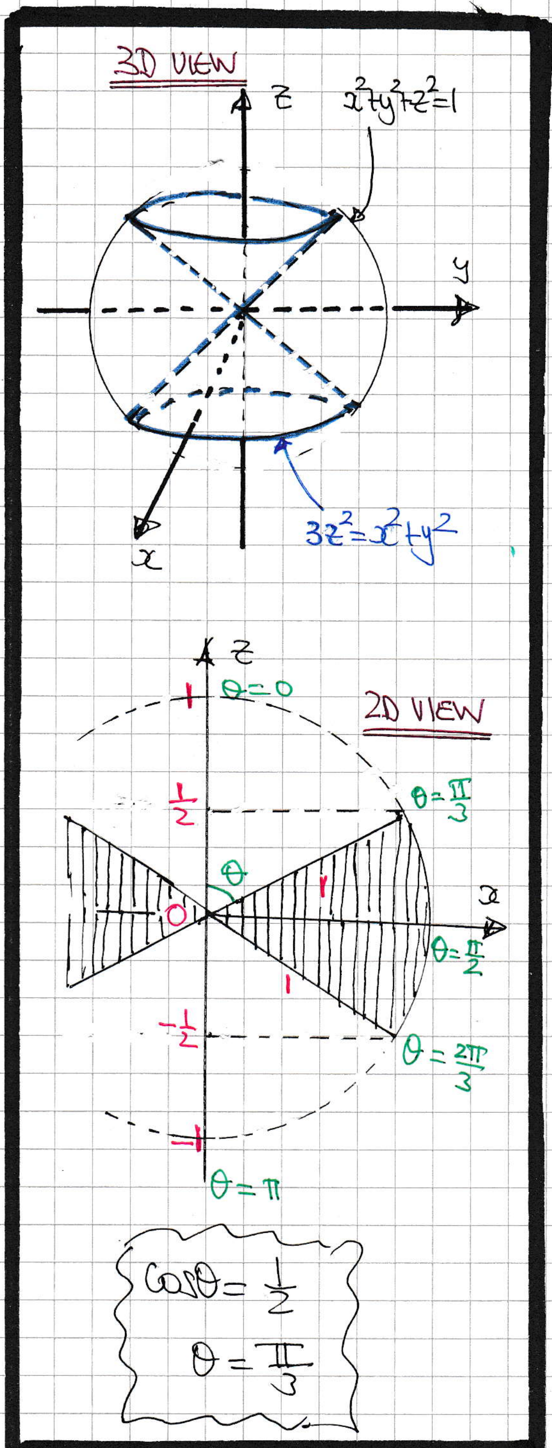
$$\left.\begin{array}{l} x^2+y^2+z^2=1 \\ 3z^2 = x^2+y^2 \end{array}\right\} \Rightarrow$$

$$\Rightarrow 3z^2+z^2=1$$

$$\Rightarrow 4z^2=1$$

$$\Rightarrow z^2=\frac{1}{4}$$

$$\Rightarrow z=\pm\frac{1}{2}$$




- SWITCHING INTO SPHERICAL POLARS, THE REQUIRED VOLUME, WHICH IS WHAT REMAINS OF THE SPHERE IS GIVEN BY

$$\Rightarrow V = \int_{\text{"REGION"}} 1 \, dV$$

$$\Rightarrow V = \int_{\phi=0}^{2\pi}\int_{\theta=\frac{\pi}{3}}^{\frac{2\pi}{3}}\int_{r=0}^{1} 1 \; \underbrace{r^2\sin\theta \, dr\, d\theta\, d\phi}_{dV \text{ IN S.P.C}}$$

$$\Rightarrow V = \left[\int_{\phi=0}^{2\pi} 1 \, d\phi\right]\left[\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sin\theta \, d\theta\right]\left[\int_{r=0}^{1} r^2 \, dr\right]$$

$$\Rightarrow V = 2\pi \times \left[-\cos\theta\right]_{\pi/3}^{2\pi/3} \times \frac{1}{3}$$

$$\Rightarrow V = 2\pi \times 1 \times \frac{1}{3} = \frac{2}{3}\pi$$

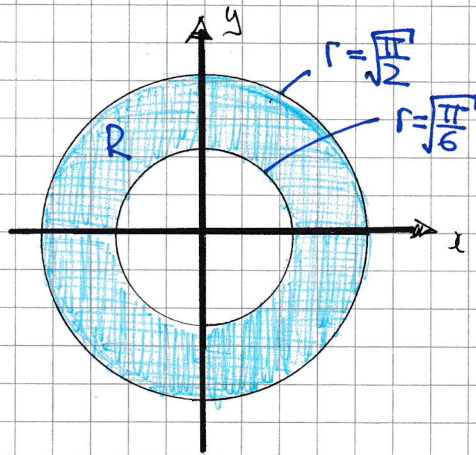
# IYGB - MATHEMATICAL METHODS 2 - PAPER C - QUESTION 5

## SWITCHING THE DOMAIN & INTEGRAND INTO PLANE POLARS

$$4x^4 + 4y^4 \leqslant \pi^2 - 8x^2y^2$$
$$4x^4 + 8x^2y^2 + 4y^4 \leqslant \pi^2$$
$$x^4 + 2x^2y^2 + y^4 \leqslant \frac{\pi^2}{4}$$
$$(x^2+y^2)^2 \leqslant \frac{\pi^2}{4}$$
$$(r^2)^2 \leqslant \frac{\pi^2}{4}$$
$$r \leqslant \sqrt{\frac{\pi}{2}}$$

$$6x^2 + 6y^2 \geqslant \pi$$
$$x^2 + y^2 \geqslant \frac{\pi}{6}$$
$$r^2 \geqslant \frac{\pi}{6}$$
$$r \geqslant \sqrt{\frac{\pi}{6}}$$

## SKETCHING THE DOMAIN & TRANSFORM THE INTEGRAND




$$\iint_R \cos(x^2+y^2)\,dx\,dy$$
$$= \iint_R \cos(r^2)\,(r\,dr\,d\theta)$$
$$= \int_{\theta=0}^{2\pi}\int_{r=\sqrt{\frac{\pi}{6}}}^{\sqrt{\frac{\pi}{2}}} r\cos r^2\,dr\,d\theta$$
$$= \left[\int_{\theta=0}^{2\pi} 1\,d\theta\right]\left[\int_{\sqrt{\frac{\pi}{6}}}^{\sqrt{\frac{\pi}{2}}} r\cos r^2\,dr\right]$$
$$= 2\pi \times \left[\tfrac{1}{2}\sin r^2\right]_{\sqrt{\frac{\pi}{6}}}^{\sqrt{\frac{\pi}{2}}}$$
$$= \pi\left(1 - \tfrac{1}{2}\right)$$
$$= \tfrac{1}{2}\pi$$

# IYGB - MATHEMATICAL METHODS 2 - PAPER C - QUESTION 6

**START BY MANIPULATING THE EXPONENT**

$$\frac{5}{4}x^2 - xy + 2y^2 = \frac{5}{4}(u+2v)^2 - (u+2v)(u-v) + 2(u-v)^2$$

$$= \frac{5}{4}(u^2+4uv+4v^2) - (u^2+uv-2v^2) + 2(u^2-2uv+v^2)$$

$$= \frac{5}{4}u^2 + \cancel{5uv} + 5v^2 - u^2 - \cancel{uv} + 2v^2 + 2u^2 - \cancel{4uv} + 2v^2$$

$$= \frac{9}{4}u^2 + 9v^2$$

**NEXT CALCULATE THE "SCALING FACTOR"**

$$dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \left\|\begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{matrix}\right\| du\,dv$$

$$= \left\|\begin{matrix} 1 & 2 \\ 1 & -1\end{matrix}\right\| du\,dv = |-3|\,du\,dv$$

$$\therefore \boxed{dx\,dy = 3\,du\,dv}$$

**HENCE WE HAVE THE FOLLOWING VOLUME INTEGRAL**

$$V = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(\frac{5}{4}x^2 - xy + 2y^2\right)}\,dx\,dy$$

CHANGE THE VARIABLES INTO THE $u-v$ PLANE, NOTING THE LIMITS ARE UNCHANGED

$$\dots = \int_{v=-\infty}^{\infty}\int_{u=-\infty}^{\infty} e^{-\left(\frac{9}{4}u^2+9v^2\right)}\,(3\,du\,dv)$$

SPLIT THE INTEGRAL, AS THERE IS NO DEPENDENCE, BETWEEN $u$ & $v$

$$\dots = \left[\int_{-\infty}^{\infty} 3e^{-\frac{9}{4}u^2}\,du\right]\left[\int_{-\infty}^{\infty} e^{-9v^2}\,dv\right]$$

BY SUBSTITUTION

$$\boxed{\begin{aligned} t &= \frac{3}{2}u \\ dt &= \frac{3}{2}\,du \\ du &= \frac{2}{3}\,dt \\ &\text{LIMITS UNCHANGED}\end{aligned}}$$

$$\boxed{\begin{aligned} s &= 3v \\ ds &= 3\,dv \\ dv &= \frac{1}{3}\,ds \\ &\text{LIMITS UNCHANGED}\end{aligned}}$$

IYGB - MATHEMATICAL METHODS 2 - PAPER C - QUESTION 6

... TRANSFORMING THE TWO INTEGRALS

$$= \left[ \int_{-\infty}^{\infty} 3e^{-t^2} \left(\tfrac{2}{3}\,dt\right) \right] \left[ \int_{-\infty}^{\infty} e^{-s^2} \left(\tfrac{1}{3}\,ds\right) \right]$$

$$= \frac{2}{3} \left[ \int_{-\infty}^{\infty} e^{-t^2}\,dt \right] \left[ \int_{-\infty}^{\infty} e^{-s^2}\,ds \right]$$

$$= \frac{2}{3} \sqrt{\pi}\,\sqrt{\pi}$$

$$= \frac{2}{3}\pi$$

# IYGB - MATHEMATICAL METHODS 2 - PAPER C - QUESTION 7

$$\underline{F}(x,y,z) = \begin{pmatrix} y \\ x^2 \\ z \end{pmatrix} \qquad \underline{r}(u,v) = \begin{pmatrix} u \\ v \\ u+v \end{pmatrix} \qquad \begin{matrix} 0 \leq u \leq 1 \\ 1 \leq v \leq 4 \end{matrix}$$

## FIND AN EXPRESSION FOR THE "AREA FLUX ELEMENT" $d\underline{S}$

- $\dfrac{\partial \underline{r}}{\partial u} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$
- $\dfrac{\partial \underline{r}}{\partial v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$
- NORMAL $= \dfrac{\partial \underline{r}}{\partial u} \wedge \dfrac{\partial \underline{r}}{\partial v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = (-1,-1,1)$
- UNIT NORMAL $\underline{\hat{n}} = \dfrac{\underline{n}}{|\underline{n}|}$

$$\underline{\hat{n}} = \frac{\dfrac{\partial \underline{r}}{\partial u} \wedge \dfrac{\partial \underline{r}}{\partial v}}{\left|\dfrac{\partial \underline{r}}{\partial u} \wedge \dfrac{\partial \underline{r}}{\partial v}\right|}$$

## COLLECTING THESE RESULTS

$$\boxed{dS = \left|\frac{\partial \underline{r}}{\partial u} \wedge \frac{\partial \underline{r}}{\partial v}\right| \, du \, dv}$$

$$\underline{\hat{n}} \, dS = \underline{\hat{n}} \left|\frac{\partial \underline{r}}{\partial u} \wedge \frac{\partial \underline{r}}{\partial v}\right| du \, dv$$

$$d\underline{S} = \frac{\dfrac{\partial \underline{r}}{\partial u} \wedge \dfrac{\partial \underline{r}}{\partial v}}{\cancel{\left|\dfrac{\partial \underline{r}}{\partial u} \wedge \dfrac{\partial \underline{r}}{\partial v}\right|}} \cancel{\left|\frac{\partial \underline{r}}{\partial u} \wedge \frac{\partial \underline{r}}{\partial v}\right|} \, du \, dv$$

$$d\underline{S} = \left(\frac{\partial \underline{r}}{\partial u} \wedge \frac{\partial \underline{r}}{\partial v}\right) du \, dv$$

$$\boxed{d\underline{S} = (-1,-1,1) \, du \, dv}$$

# IYGB - MATHEMATICAL METHODS 2 - PAPER C - QUESTION 7

FINALLY THE FLUX CAN BE CALCULATED

$$\text{flux} = \int_S \underline{F} \cdot d\underline{S} = \int \underline{F}(u,v) \cdot d\underline{S}$$

$$= \int_{v=1}^{4} \int_{u=0}^{1} (v, u^2, u+v) \cdot (-1,-1,1) \, du \, dv$$

$$= \int_{v=1}^{4} \int_{u=0}^{1} (-v - u^2 + u + v) \, du \, dv$$

$$= \int_{v=1}^{4} \int_{u=0}^{1} (u - u^2) \, du \, dv$$

$$= \int_{v=1}^{4} \left[ \frac{1}{2}u^2 - \frac{1}{3}u^3 \right]_0^1 dv$$

$$= \int_{1}^{4} \left( \frac{1}{2} - \frac{1}{3} \right) dv$$

$$= \int_{1}^{4} \frac{1}{6} \, dv$$

$$= \left[ \frac{1}{6} v \right]_1^4$$

$$= \frac{2}{3} - \frac{1}{6}$$

$$= \frac{1}{2}$$

# IYGB - MATHEMATICAL METHODS 2 - PAPER C - QUESTION 8

a) FIRSTLY DETERMINE THE CO-ORDINATES OF A & B

- $x=0$

$$1 = 16(1-y^2)$$
$$\frac{1}{16} = 1-y^2$$
$$y^2 = \frac{15}{16}$$
$$y = +\frac{1}{4}\sqrt{15}$$
$$A\left(0, \tfrac{1}{4}\sqrt{15}\right)$$

- $y=0$

$$(x-1)^2 = 16$$
$$x-1 = \begin{cases} 4 \\ -4 \end{cases}$$
$$x = 5$$
$$B(5,0)$$

NEXT WE TEST WHETHER THE INTEGRAL IS INDEPENDENT OF THE PATH.

$$\frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy = d\phi$$
$$x^2+xy \qquad\qquad y^2+\tfrac{1}{2}x^2$$

$$\frac{\partial}{\partial y}\left(x^2+xy\right) = x \qquad \frac{\partial}{\partial x}\left(y^2+\tfrac{1}{2}x^2\right) = x \qquad \text{IT EXACT}$$

- $\frac{\partial \phi}{\partial x} = x^2+xy \Rightarrow \phi(x,y) = \frac{1}{3}x^3 + \frac{1}{2}x^2y + f(y)$
- $\frac{\partial \phi}{\partial y} = y^2+\frac{1}{2}x^2 \Rightarrow \phi(x,y) = \frac{1}{3}y^3 + \frac{1}{2}x^2y + g(x)$

$$\therefore \phi(x,y) = \frac{1}{3}x^3 + \frac{1}{3}y^3 + \frac{1}{2}x^2y + C$$

HENCE WE OBTAIN

$$\int_A^B \left[(x^2+xy)\,dx + (y^2+\tfrac{1}{2}x^2)\,dy\right] = \int_A^B d\phi$$

# IYGB - MATHEMATICAL METHODS 2 - PAPER C - QUESTION 8

$$= \left[\frac{1}{3}x^3 + \frac{1}{3}y^3 + \frac{1}{2}x^2y\right]_{(0,\frac{1}{4}\sqrt{15})}^{(5,0)}$$

$$= \frac{1}{3} \times 5^3 - \frac{1}{3}\left(\frac{1}{4}\sqrt{15}\right)^3$$

$$= \frac{125}{3} - \frac{1}{3}\,\frac{15}{64}\sqrt{15} = \frac{125}{3} - \frac{5}{64}\sqrt{15}$$

b) CHECKING FOR PATH INDEPENDENCE FOR (b)

$$\frac{\partial}{\partial y}\left(y^3\right) = 3y^2 \qquad \frac{\partial}{\partial x}\left(\frac{1}{16}(x-1)^3\right) = \frac{3}{16}(x-1)^2$$

DIFFERENTIAL IS NOT EXACT
SO IT DEPENDS ON THE PATH

PARAMETRIZE THE ELLIPSE

$$\Rightarrow (x-1)^2 = 16(1-y^2)$$

$$\Rightarrow (x-1)^2 = 16 - 16y^2$$

$$\Rightarrow (x-1)^2 + 16y^2 = 16$$

$$\Rightarrow \frac{(x-1)^2}{16} + y^2 = 1$$

$$\left[\cos^2\theta + \sin^2\theta = 1\right]$$

$$\therefore \cos\theta = \frac{x-1}{4} \quad \& \quad \sin\theta = y$$

$$\boxed{x = 1 + 4\cos\theta} \quad \& \quad \boxed{y = \sin\theta}$$

$$dx = -4\sin\theta \quad \& \quad dy = \cos\theta\, d\theta$$

# IYGB - MATHEMATICAL METHODS 2 - PAPER C - QUESTION 8

## TRANSFORMING THE UNITS

$A(0, \frac{1}{4}\sqrt{15}) \longmapsto \theta = \arcsin\frac{\sqrt{15}}{4} = \arccos\left(\frac{1}{4}\right)$

$B(5,0) \longmapsto \theta = 0$

## TRANSFORMING THE INTEGRAL INTO PARAMETRIC

$$\int_C y^3\,dx + \frac{1}{16}(x-1)^3\,dy$$

$$= \int_{\substack{\arccos\left(\frac{1}{4}\right)\\ \arcsin\frac{\sqrt{15}}{4}}}^{0} \left[\sin^3\theta(-4\sin\theta) + \frac{1}{16}(1+4\cos\theta-1)^3\cos\theta\right]d\theta$$

$$= \int_{\substack{\arcsin\frac{\sqrt{15}}{4}\\ \arccos\left(\frac{1}{4}\right)}}^{0} -4\sin^4\theta + 4\cos^4\theta\,d\theta$$

$$= \int_{\substack{\arcsin\frac{\sqrt{15}}{4}\\ \arccos\left(\frac{1}{4}\right)}}^{0} 4(\cos^4\theta - \sin^4\theta)\,d\theta$$

$$= \int_{\substack{\arcsin\frac{\sqrt{15}}{4}\\ \arccos\left(\frac{1}{4}\right)}}^{0} 4(\cos^2\theta-\sin^2\theta)\cancel{(\cos^2\theta+\sin^2\theta)}\,d\theta$$

$$= \int_{\substack{\arcsin\frac{\sqrt{15}}{4}\\ \arccos\left(\frac{1}{4}\right)}}^{0} 4\cos 2\theta\,d\theta$$

IYGB - MATHEMATICAL METHODS 2 - PAPER C - QUESTION 8

$$= \Big[\ 2\sin 2\theta\ \Big]_{\theta=\arcsin\frac{\sqrt{15}}{4}\atop \theta=\arccos\left(-\frac{1}{4}\right)}^{0}$$

$$= \Big[\ 4\sin\theta\cos\theta\ \Big]_{\theta=\arcsin\frac{\sqrt{15}}{4}\atop \theta=\arccos\left(-\frac{1}{4}\right)}^{0}$$

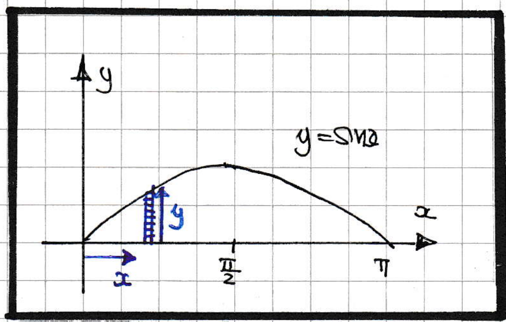
$$= 0 - 4 \times \frac{\sqrt{15}}{4} \times \left(-\frac{1}{4}\right)$$

$$= \frac{1}{4}\sqrt{15}$$

# IYGB - MATHEMATICAL METHODS 2 - PAPER C - QUESTION 9

a) LOOKING AT THE INTEGRATION REGION

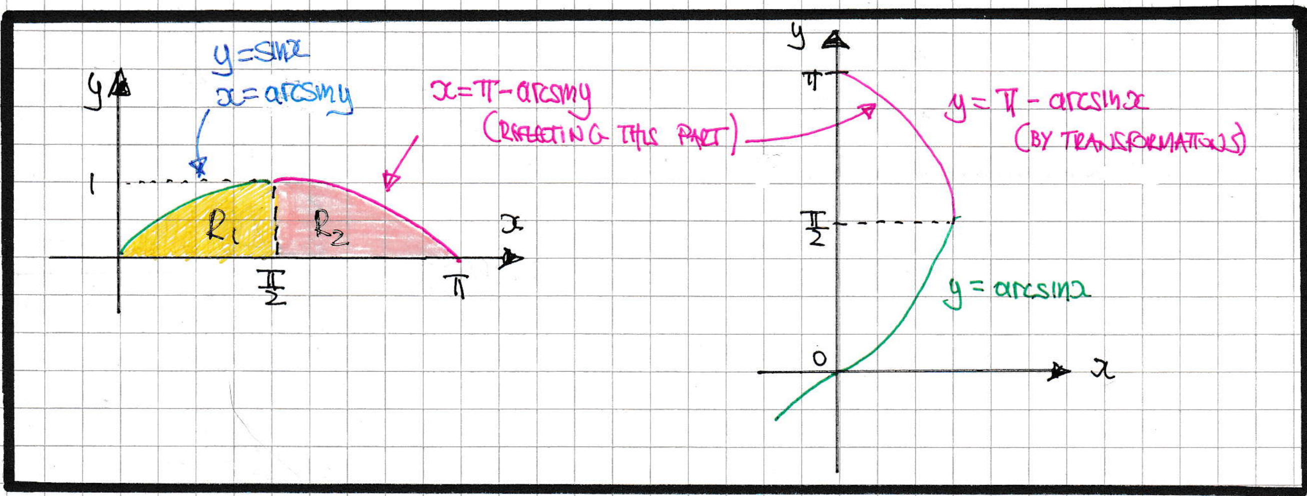
$$\int_{x=0}^{\pi}\int_{y=0}^{y=\sin x} 4y \, dy \, dx = \int_{x=0}^{\pi} \left[2y^2\right]_{y=0}^{y=\sin x} dx$$




$$= \int_0^{\pi} 2\sin^2 x \, dx = \int_0^{\pi} 1 - \cos 2x \, dx$$

$$= \left[x - \tfrac{1}{2}\sin 2x\right]_0^{\pi} = \pi$$

b) REVERSING THE ORDER REQUIRES SPLITTING THE AREA OF INTEGRATION




$$\underbrace{\int_{y=0}^{1}\int_{x=\arcsin y}^{x=\frac{\pi}{2}} 4y \, dx \, dy}_{R_1} + \underbrace{\int_{y=0}^{1}\int_{x=\frac{\pi}{2}}^{\pi-\arcsin y} 4y \, dx \, dy}_{R_2}$$

$$= \int_0^1 \left[4yx\right]_{x=\arcsin y}^{x=\frac{\pi}{2}} dy + \int_0^1 \left[4xy\right]_{x=\frac{\pi}{2}}^{x=\pi-\arcsin y} dy$$

$$= \int_0^1 2\pi y - 4y\arcsin y \, dy + \int_0^1 4y(\pi - \arcsin y) - 2\pi y \, dy$$

IYGB - MATHEMATICAL METHODS 2 - PAPER C - QUESTION 9

$$= \int_0^1 \cancel{2\pi y} - 4y\arcsin y + 4\pi y - 4y\arcsin y - \cancel{2\pi y} \, dy$$

$$= \int_0^1 4\pi y - 8y\arcsin y \, dy$$

$$= \left[2\pi y^2\right]_0^1 - \int_0^1 8y\arcsin y \, dy$$

BY SUBSTITUTION FOLLOWED BY INTEGRATION BY PARTS

$$\boxed{\begin{array}{l} \theta = \arcsin y \\ y = \sin\theta \\ dy = \cos\theta \, d\theta \\ y=0 \mapsto \theta = 0 \\ y=1 \mapsto \theta = \frac{\pi}{2} \end{array}}$$

$$\ldots = 2\pi - \int_0^{\frac{\pi}{2}} (8\sin\theta)(\cos\theta \, d\theta)$$

$$= 2\pi - \int_0^{\pi/2} 8\theta\sin\theta\cos\theta \, d\theta$$

$$- \int_0^{\frac{\pi}{2}} 4\theta\sin 2\theta \, d\theta$$

BY PARTS

| $4\theta$ | $4$ |
|---|---|
| $-\frac{1}{2}\cos 2\theta$ | $\sin 2\theta$ |

$$= 2\pi - \left\{\left[-2\theta\cos 2\theta\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -2\cos 2\theta \, d\theta\right\}$$

$$= 2\pi - \left\{\left[-2\theta\cos 2\theta + \sin 2\theta\right]_0^{\pi/2}\right\}$$

$$= 2\pi - \left\{\left[-2\times\frac{\pi}{2}\times(-1) + 0\right] - \left[0\right]\right\}$$

$$= 2\pi - \pi$$

$$= \pi$$

AS BEFORE

# IYGB – MATHEMATICAL METHODS 2 – PAPER C – QUESTION 10

(a) COMPUTING THE FLUX THROUGH $S$ DIRECTLY

$$\text{FLUX} = \int_S \underline{A} \cdot d\underline{S} = \int_S \underline{A} \cdot \underline{\hat{n}}\, dS$$

OBTAIN THE UNIT NORMAL TO $S$

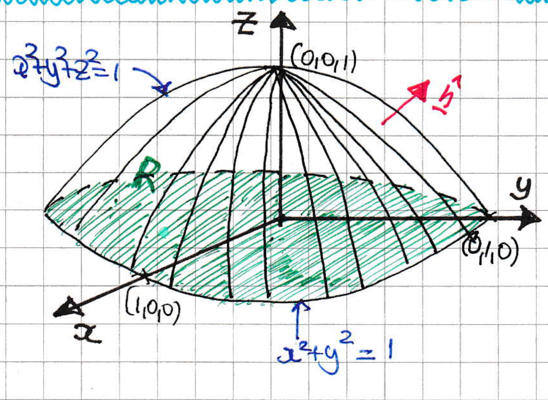
LET $f(x,y,z) = x^2+y^2+z^2-1$

$$\nabla f = (2x, 2y, 2z)$$

$$\underline{n} = (x,y,z)$$

$$|\underline{n}| = \sqrt{x^2+y^2+z^2} = 1$$

$$\underline{\hat{n}} = \frac{\underline{n}}{|\underline{n}|} = (x,y,z)$$




$$\dots = \int_S (2, -1, 4y-3) \cdot (x,y,z)\, dS = \int_S 2x - y + 4yz - 3z\, dS$$

PROJECT ONTO THE CIRCULAR REGION R (ABOVE DIAGRAM) OR SWITCH INTO SPHERICAL POLAR CO-ORDINATES

$$\dots = \int_R (2x - y + 4yz - 3z) \underbrace{\frac{dx\,dy}{\underline{\hat{n}} \cdot \underline{k}}}_{dS}$$

WHERE R SATISFIES $x^2+y^2 \leq 1$

$$= \int_R (2x - y + 4yz - 3z) \frac{dx\,dy}{(x,y,z)\cdot(0,0,1)} = \int_R (2x - y + 4yz - 3z)\frac{dx\,dy}{z}$$

$$= \int_R \left[\frac{2x}{z} - \frac{y}{z} + 4y - 3\right] dx\,dy$$

$$= \int_R \left[\underbrace{\cancel{\frac{2x}{+\sqrt{1-x^2-y^2}}}}_{\text{ODD IN } x} - \underbrace{\cancel{\frac{y}{+\sqrt{1-x^2-y^2}}}}_{\text{ODD IN } y} + \underbrace{\cancel{4y}}_{\text{ODD IN } y} - 3\right] dx\,dy$$

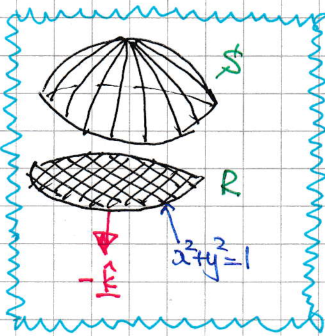
(AS R IS A SYMMETRICAL DOMAIN IN BOTH $x$ & $y$)

$$= -3\int_R dx\,dy = -3 \times (\text{AREA OF R}) = -3\pi$$

# IYGB - MATHEMATICAL METHODS 2 - PAPER C - QUESTION 10

b) IN ORDER TO USE THE DIVERGENCE THEOREM WHICH APPLIES TO CLOSED SURFACES, WE SHALL CLOSE THE HEMISPHERE AT THE BOTTOM WITH A CIRCULAR DISC WHOSE OUTWARD UNIT NORMAL IS $-\underline{\hat{k}}$

- FLUX THROUGH THE ABOVE MENTIONED DISC




$$\int_R \underline{A} \cdot d\underline{S} = \int_R \underline{A} \cdot \hat{\underline{n}} \, dS$$

$$= \int_R (2,-1,2y-3) \cdot (0,0,-1) \, dx\,dy$$

$$= \int_R (\cancel{-2y} + 3) \, dx\,dy \quad = \quad \int_R 3 \, dx\,dy$$

ODD FUNCTION IN $y$ IN A SYMMETRICAL DOMAIN IN $y$

$$= 3 \times (\text{AREA OF } R) \quad = 3\pi$$

- BY THE DIVERGENCE THEOREM ON THE "CLOSED HEMISPHERE"

$$\Rightarrow \iiint_V \underline{\nabla} \cdot \underline{A} \, dV = \iint_{S+R} \underline{A} \cdot d\underline{S}$$

$$\Rightarrow \iiint_V \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (2,-1,2y-3) \, dV = \iint_S \underline{A} \cdot d\underline{S} + \iint_R \underline{A} \cdot d\underline{S}$$

$$\Rightarrow \iiint_V 0 \, dV = \iint_S \underline{A} \cdot d\underline{S} + 3\pi \quad \leftarrow \text{FOUND ABOVE}$$

$$\Rightarrow 0 = \iint_S \underline{A} \cdot d\underline{S} + 3\pi$$

$$\Rightarrow \text{FLUX THROUGH } S = \iint_S \underline{A} \cdot d\underline{S} = -3\pi$$

AS IN PART (a)

# IYGB - MATHEMATICAL METHODS 2 - PAPER C - QUESTION 10

c) TO USE STOKES THEOREM, IE CONVERTING THE FLUX INTO A LINE INTEGRAL WE MUST FIRST FIND A VECTOR FUNCTION $\underline{F}$, SO THAT $\nabla \wedge \underline{F} = \underline{A}$

ATTEMPT TO "INVERT A CURL", NOTING THAT $\nabla \wedge (\underline{A}+\underline{B}) \equiv \nabla \wedge \underline{A} + \nabla \wedge \underline{B}$

$$\Longrightarrow \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P_i & Q_i & R_i \end{vmatrix} = (2, -1, 4y-3)$$

{WHERE $i = 1, 2, 3$}

$$\Longrightarrow \left[\frac{\partial R_i}{\partial y} - \frac{\partial Q_i}{\partial z}, \frac{\partial P_i}{\partial z} - \frac{\partial R_i}{\partial x}, \frac{\partial Q_i}{\partial x} - \frac{\partial P_i}{\partial y}\right] = (2, -1, 4y-3)$$

- LET $i=1$ & TRY TO PRODUCE $(2,0,0)$

  $\boxed{R_1 = 2y}$ $\quad \boxed{Q_1 = 0}$ $\quad \boxed{P_1 = 0}$ ALL 3 AGREE

- LET $i=2$ & TRY TO PRODUCE $(0,-1,0)$

  $\boxed{R_2 = x}$ $\quad \boxed{Q_2 = 0}$ $\quad \boxed{P_2 = 0}$ ALL 3 AGREE

- LET $i=3$ & TRY TO PRODUCE $(0,0,4y-3)$

  $\boxed{Q_3 = 4xy - 3x}$ $\quad \boxed{P_3 = 0}$ $\quad \boxed{R_3 = 0}$ ALL 3 AGREE

ADDING AS THE CURL OPERATOR IS LINEAR

$$\underline{F} = [P_1 + P_2 + P_3, Q_1 + Q_2 + Q_3, R_1 + R_2 + R_3]$$

$$\boxed{\underline{F} = [0, 4xy - 3x, 2y + x]}$$

HENCE THE FLUX OF $\underline{A}$ THROUGH THE HEMISPHERE $S$ IS GIVEN BY

$$\text{FLUX} = \int_S \underline{A} \cdot d\underline{S} = \int_S (\nabla \wedge \underline{F}) \cdot d\underline{S}$$

# IYGB - MATHEMATICAL METHODS 2 - PAPER C - QUESTION 10

## APPLYING STOKE'S THEOREM FOR OPEN SURFACES

$$\text{FLUX} = \int_S (\nabla \wedge \underline{F}) \cdot d\underline{S} = \int_{\substack{C \\ (x^2+y^2=1,\ z=0)}} \underline{F} \cdot d\underline{r}$$

$$= \int_{\substack{C \\ (x^2+y^2=1,z=0)}} (0,\ 4xy-3x,\ x+2y) \cdot (dx, dy, dz)$$

$$= \int_{\substack{C \\ (x^2+y^2=1,z=0)}} (4xy-3x)\,dy + \cancel{(x+2y)\,dz}$$

$z=0$ ON $C$
$dz=0$

## PARAMETERIZING THE CIRCLE $x^2+y^2=1$

$$\boxed{\begin{array}{lll} x = \cos\theta & (dx = -\sin\theta\, d\theta) & \\ y = \sin\theta & dy = \cos\theta\, d\theta & 0 \leq \theta < 2\pi \end{array}}$$

$$\dots = \int_{\theta=0}^{2\pi} \left[4\cos\theta\sin\theta - 3\cos\theta\right](\cos\theta\, d\theta)$$

$$= \int_0^{2\pi} \cancel{4\cos^2\theta\sin\theta} - 3\cos^2\theta\, d\theta$$

NO CONTRIBUTION FOR $0 \leq \theta < 2\pi$

$$= \int_0^{2\pi} -3\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta\right) d\theta =$$

$$= \int_0^{2\pi} -\tfrac{3}{2} - \cancel{\tfrac{3}{2}\cos 2\theta}\ d\theta$$

NO CONTRIBUTION FOR $0 \leq \theta < 2\pi$

$$= -\tfrac{3}{2} \times 2\pi$$

$$= -3\pi$$

AS BEFORE