IYGB – MATHEMATICAL METHODS 2 – PAPER D – QUESTION 1

CONSIDER THE k^{TH} COMPONENT OF $\underline{\nabla} \wedge (\phi \underline{A})$

$$\left[\underline{\nabla} \wedge (\phi \underline{A})\right]_k = \varepsilon_{ijk} \frac{\partial}{\partial x_i}(\phi A_j)$$

$$\boxed{(\underline{A} \wedge \underline{B})_k = \varepsilon_{ijk} A_i B_j}$$

APPLYING THE PRODUCT RULE WITH SCALARS

$$\dots = \varepsilon_{ijk}\left[\frac{\partial \phi}{\partial x_i} A_j + \phi \frac{\partial A_j}{\partial x_i}\right]$$

$$= \varepsilon_{ijk} \frac{\partial \phi}{\partial x_i} A_j + \phi\, \varepsilon_{ijk} \frac{\partial A_j}{\partial x_i}$$

$$= \varepsilon_{ijk} \frac{\partial \phi}{\partial x_i} A_j + \phi\, \varepsilon_{ijk} \frac{\partial}{\partial x_i} A_j$$

BACK INTO "CROSS PRODUCTS"

$$= \left[\underline{\nabla}\phi \wedge \underline{A}\right]_k + \phi\left[\underline{\nabla} \wedge \underline{A}\right]_k$$

$$= \underline{\nabla}\phi \wedge \underline{A} + \phi\, \underline{\nabla} \wedge \underline{A}$$

AS REQUIRED

IYGB - MATHEMATICAL METHODS 2 - PAPER D - QUESTION 2

PROCEED BY PARAMETERIZING THE CIRCULAR PATH

UNIT CIRCLE

$x = \cos\theta$

$y = \sin\theta$

$0 \leqslant \theta < 2\pi$

$dx = -\sin\theta \, d\theta$

$dy = \cos\theta \, d\theta$

$$\oint_C \left[y^3 \, dx + xy \, dy \right]$$

$$= \int_0^{2\pi} \sin^3\theta(-\sin\theta \, d\theta) + (\cos\theta\sin\theta)(\cos\theta \, d\theta)$$

$$= \int_0^{2\pi} \left(-\sin^4\theta + \cancel{\cos^2\theta\sin\theta}\right) d\theta = \int -(\sin^2\theta)^2$$

NO CONTRIBUTION OVER THESE LIMITS

$$= \int_0^{2\pi} -\left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta\right)^2 d\theta = \int_0^{2\pi} -\left(\tfrac{1}{4}\cos^2 2\theta + \tfrac{1}{4} - \cancel{\tfrac{1}{2}\cos 2\theta}\right) d\theta$$

NO CONTRIBUTION OVER THESE LIMITS

$$= \int_0^{2\pi} \left(-\tfrac{1}{4} - \tfrac{1}{4}\cos^2 2\theta\right) d\theta = \int_0^{2\pi} \left[-\tfrac{1}{4} - \tfrac{1}{4}\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 4\theta\right)\right] d\theta$$

$$= \int_0^{2\pi} \left(-\tfrac{3}{8} - \cancel{\tfrac{1}{8}\cos 4\theta}\right) d\theta = \int_0^{2\pi} -\tfrac{3}{8} \, d\theta$$

NO CONTRIBUTION OVER THESE LIMITS

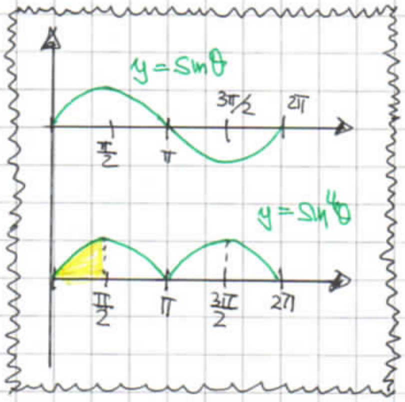
$$= -\tfrac{3}{8} \times 2\pi = -\frac{3\pi}{4}$$

ALTERNATIVE EVALUATION FROM $\int_0^{2\pi} -\sin^4\theta \, d\theta$

$$\int_0^{2\pi} -\sin^4\theta \, d\theta = -\int_0^{\pi/2} 4\sin^4\theta \, d\theta$$

$$= -2\int_0^{\frac{\pi}{2}} 2(\sin\theta)^{2\times\frac{5}{2}-1}(\cos\theta)^{2\times\frac{1}{2}-1} d\theta$$

$$= -2 \, B\left(\tfrac{5}{2}, \tfrac{1}{2}\right)$$



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SWITCH INTO GAMMA FUNCTIONS

$$= -2 \times \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(3)}$$

$$= -2 \times \frac{\left[\frac{3}{2} \times \frac{1}{2} \times \Gamma\left(\frac{1}{2}\right)\right]\Gamma\left(\frac{1}{2}\right)}{2!}$$

$$= -2 \times \frac{\frac{3}{4} \times \sqrt{\pi} \times \sqrt{\pi}}{2}$$

$$= -\frac{3\pi}{4}$$

// AS BEFORE

IYGB - MATHEMATICAL METHODS 2 - QUESTION 3 - PAPER D

AUXILIARIES FIRST INCLUDING A REGION OF INTEGRATION DIAGRAM

- $x = \sqrt{2y - y^2}$

$x^2 = 2y - y^2$

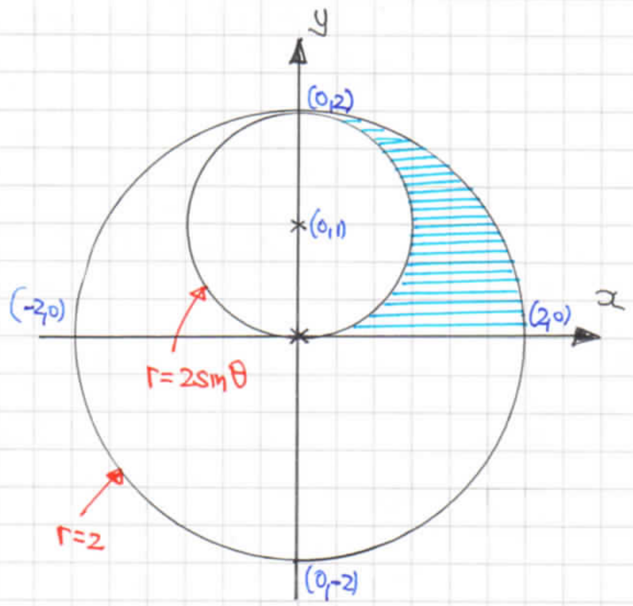
$x^2 + y^2 - 2y = 0$

$x^2 + (y-1)^2 = 1$

- $x = \sqrt{4 - y^2}$

$x^2 = 4 - y^2$

$x^2 + y^2 = 4$



SWITCH INTO PLANE POLARS

- $x^2 + y^2 - 2y = 0$

$r^2 - 2r\sin\theta = 0$

$r - 2\sin\theta = 0$

$\boxed{r = 2\sin\theta}$

- $x^2 + y^2 = 4$

$r^2 = 4$

$\boxed{r = 2}$

FINISHING OFF THE INTEGRATION

$$I = \int_0^2 \int_{\sqrt{2y-y^2}}^{\sqrt{4-y^2}} \frac{2y}{x^2+y^2}\, dx\, dy = \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=2\sin\theta}^{r=2} \frac{2(r\sin\theta)}{r^2} \underbrace{(r\, dr\, d\theta)}_{dx\,dy}$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=2\sin\theta}^{r=2} \frac{2r^2\sin\theta}{r^2}\, dr\, d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=2\sin\theta}^{r=2} 2\sin\theta\, dr\, d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \Big[2r\sin\theta \Big]_{r=2\sin\theta}^{r=2}\, d\theta$$

IYGB - MATHEMATICAL METHODS 2 - PAPER D - QUESTION 3

$$= \int_{\theta=0}^{\frac{\pi}{2}} 4\sin\theta - 2(2\sin\theta)\sin\theta \; d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} 4\sin\theta - 4\sin^2\theta \; d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} 4\sin\theta - 4\left(\tfrac{1}{2} - \tfrac{1}{2}\cos2\theta\right) d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} 4\sin\theta - 2 + 2\cos2\theta \; d\theta$$

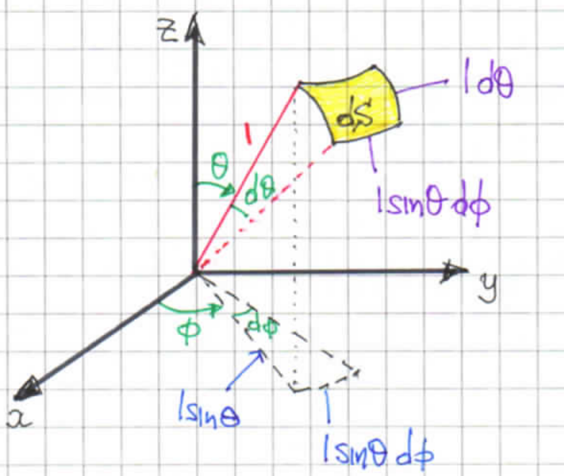
$$= \Big[-4\cos\theta - 2\theta + \sin2\theta\Big]_0^{\frac{\pi}{2}}$$

$$= (0 - \pi + 0) - (-4 - 0 + 0)$$

$$= 4 - \pi$$

IYGB - MATHEMATICAL METHODS 2 - PAPER D - QUESTION 4

USING SPHERICAL POLARS AS SUGGESTED



$x = 1\sin\theta\cos\phi$
$y = 1\sin\theta\sin\phi$
$z = 1\cos\theta$
$\Big\} \; x^2+y^2+z^2 = 1$

$$dS = \sin\theta \, d\theta \, d\phi$$

WITH $0 \le \theta \le \pi$
$0 \le \phi \le 2\pi$

PROCEED WITH THE INTEGRATION

$$\int_S x^2+y+z \; dS = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \left[\sin^2\theta\cos^2\phi + \sin\theta\sin\phi + \cos\theta\right]\left[\sin\theta \, d\theta \, d\phi\right]$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \sin^3\theta\cos^2\phi + \cancel{\sin^2\theta\sin\phi} + \cancel{\cos\theta\sin\theta} \; d\theta \, d\phi$$

NO CONTRIBUTION FROM THE INTEGRATION IN ϕ

NO CONTRIBUTION FROM THE INTEGRATION IN θ

SPLIT THE INTEGRALS, AS THE LIMITS ARE INDEPENDENT

$$\ldots = \left[\int_0^{2\pi} \cos^2\phi \, d\phi\right]\left[\int_0^{\pi} \sin^3\theta \, d\theta\right] =$$

$$= \left[\int_0^{2\pi} \tfrac{1}{2} + \cancel{\tfrac{1}{2}\cos 2\phi} \; d\phi\right]\left[\int_0^{\pi} \sin\theta(1-\cos^2\theta) \, d\theta\right]$$

NO CONTRIBUTION OVER THESE LIMITS

$$= \left[\int_0^{2\pi} \tfrac{1}{2} \, d\phi\right]\left[\int_0^{\pi} \sin\theta - \sin\theta\cos^2\theta \; d\theta\right]$$

$$= \tfrac{1}{2} \times 2\pi \times \left[-\cos\theta + \tfrac{1}{3}\cos^3\theta\right]_0^{\pi}$$

$$= \pi\left[\left(1-\tfrac{1}{3}\right) - \left(-1+\tfrac{1}{3}\right)\right] = \tfrac{4}{3}\pi$$

IYGB - MATHEMATICAL METHODS 2 - PAPER D - QUESTION 4

ALTERNATIVE BY THE DIVERGENCE THEOREM

CLEVERLY SWITCH THE SURFACE INTEGRAL INTO A FLUX INTEGRAL THROUGH THE SURFACE OF THE UNIT SPHERE

$$\int_S x^2+y+z \; dS = \int_S (x,1,1)\cdot(x,y,z) \; dS$$

NOW NOTE THAT

$$S: x^2+y^2+z^2=1 \implies f(x,y,z) = x^2+y^2+z^2-1$$
$$\implies \nabla f = \underline{n} = (2x,2y,2z) \sim (x,y,z)$$
$$\implies \underline{n} = (x,y,z)$$
$$\implies |\underline{n}| = \sqrt{x^2+y^2+z^2} = 1$$
$$\implies \hat{\underline{n}} = \frac{\underline{n}}{|\underline{n}|} = (x,y,z)$$

HENCE WE CAN NOW USE THE DIVERGENCE THEOREM

$$\dots = \int_S (x,1,1)\cdot\hat{\underline{n}} \; dS = \int_S \underline{F}\cdot\hat{\underline{n}} \; dS \qquad \boxed{\underline{F} = (x,1,1)}$$

$$= \int_V \nabla\cdot\underline{F} \; dV = \int_V \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\cdot(x,1,1) \; dV$$

$$= \int_V 1+0+0 \; dV = \int_V 1 \; dV =$$

$= $ VOLUME OF THE UNIT SPHERE

$$= \frac{4}{3}\times\pi\times 1^3$$

$$= \frac{4}{3}\pi$$

AS BEFORE

IYGB - MATHEMATICAL METHODS 2 - PAPER D - QUESTION 5

FIRSTLY DEFINE THE COMPONENTS OF SOME VECTORS

- $\underline{m} = (m_1, m_2, m_3)$ CONSTANT VECTOR
- $\underline{r} = (x, y, z)$ VARIABLE POSITION VECTOR
- $\underline{m} \wedge \underline{r} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ m_1 & m_2 & m_3 \\ x & y & z \end{vmatrix} = (m_2 z - m_3 y,\ m_3 x - m_1 z,\ m_1 y - m_2 x)$

PUTTING ALL THE RESULTS TOGETHER

$$\underline{\nabla} \cdot \left(\frac{\underline{m} \wedge \underline{r}}{r^3} \right) = \underline{\nabla} \cdot \left[\frac{m_2 z - m_3 y}{(x^2+y^2+z^2)^{\frac{3}{2}}},\ \frac{m_3 x - m_1 z}{(x^2+y^2+z^2)^{\frac{3}{2}}},\ \frac{m_1 y - m_2 x}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right]$$

APPLY THE DIVERGENCE OPERATOR

$$\dots = \frac{\partial}{\partial x}\left[\frac{m_2 z - m_3 y}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right] + \frac{\partial}{\partial y}\left[\frac{m_3 x - m_1 z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right] + \frac{\partial}{\partial z}\left[\frac{m_1 y - m_2 x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right]$$

$$= (m_2 z - m_3 y)\left(-\tfrac{3}{2}\right)(2x)(x^2+y^2+z^2)^{-\frac{5}{2}}$$
$$(m_3 x - m_1 z)\left(-\tfrac{3}{2}\right)(2y)(x^2+y^2+z^2)^{-\frac{5}{2}}$$
$$(m_1 y - m_2 x)\left(-\tfrac{3}{2}\right)(2z)(x^2+y^2+z^2)^{-\frac{5}{2}}$$

$$= 3(x^2+y^2+z^2)^{-\frac{5}{2}}\left[x(m_3 y - m_2 z) + y(m_1 z - m_3 x) + z(m_2 x - m_1 y)\right]$$

$$= 3(x^2+y^2+z^2)^{-\frac{5}{2}}\left[\cancel{m_3 xy} - \cancel{m_2 xz} + \cancel{m_1 yz} - \cancel{xy m_3} + \cancel{m_2 xz} - \cancel{m_1 yz}\right]$$

$$= 0$$

AS REQUIRED

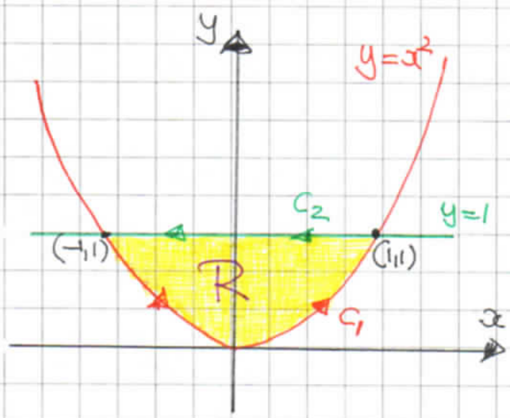
IYGB - MATHEMATICAL METHODS 2 - PAPER D - QUESTION 6

a) IF $P(x,y)$ & $Q(x,y)$ HAVE CONTINUOUS FIRST ORDER PARTIAL DERIVATIVES IN A REGION R IN THE x-y PLANE AND IN THE CLOSED BOUNDARY WHICH CONTAINS R, THEN

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

WHERE C IS TRACED ANTICLOCKWISE

b) START WITH A SKETCH SHOWING THE REGION OF INTEGRATION



$$\iint_R x^2-7y^2\,dy\,dx = \int_{x=-1}^{1}\int_{y=x^2}^{y=1} x^2-7y^2\,dy\,dx$$

$$= \int_{x=-1}^{1}\left[x^2y - \frac{7}{3}y^3\right]_{y=x^2}^{y=1} dx$$

$$= \int_{-1}^{1} x^2 - \frac{7}{3} - \left(x^4 - \frac{7}{3}x^6\right) dx$$

$$= \int_{-1}^{1} \frac{7}{3}x^6 - x^4 + x^2 - \frac{7}{3}\,dx = 2\int_0^1 \frac{14}{3}x^6 - 2x^4 + 2x^2 - \frac{14}{3}\,dx$$

$$= \left[\frac{2}{3}x^7 - \frac{2}{5}x^5 + \frac{2}{3}x^3 - \frac{14}{3}\right]_0^1 = \frac{2}{3} - \frac{2}{5} + \frac{2}{3} - \frac{14}{3} = -\frac{2}{5} - \frac{10}{3} = -\frac{56}{15}$$

c) NOW WE NEED TO CHANGE THE INTEGRAND IN A "CURL FORM"

LET $-\dfrac{\partial P}{\partial y} = x^2 - 7y^2$

$$\frac{\partial P}{\partial y} = 7y^2 - x^2$$

$$P(x,y) = \frac{7}{3}y^3 - x^2y + F(x)$$

IYGB - MATHEMATICAL METHODS 2 - PAPER D - QUESTION 6

$\therefore$ PICK $G(x)$ SUCH THAT $\frac{d}{dx}\left[G(x)\right] = F(x)$

FORMING A LINE INTEGRAL USING GREEN'S THEOREM (TAKE $Q(x,y)=0$)

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \oint P\,dx + Q\,dy$$

$$\iint_R x^2 - 7y^2 \; dx\,dy = \oint_C \tfrac{7}{3}y^3 - x^2y + F(x)\; dx$$

- ON C_1
  $y = x^2$
  $dy = 2x\,dx$
  $-1 \leqslant x \leqslant 1$
- ON C_2
  $y = 1$
  $dy = 0$
  BUT x RUNS FROM 1 TO -1

SPLIT INTO TWO PARTS C_1 & C_2

$$\dots = \int_{C_1} \tfrac{7}{3}y^3 - x^2y + F(x)\; dx + \int_{C_2} \tfrac{7}{3}y^3 - x^2y + F(x)\; dx$$

$$\dots = \int_{x=-1}^{1} \tfrac{7}{3}x^6 - x^4 + F(x)\; dx + \int_{x=1}^{x=-1} \tfrac{7}{3} - x^2 + F(x)\; dx$$

$$= \left[\tfrac{1}{3}x^7 - \tfrac{1}{5}x^5 + G(x)\right]_{-1}^{1} + \left[\tfrac{7}{3}x - \tfrac{1}{3}x^3 + G(x)\right]_{1}^{-1}$$

$$= \left[\tfrac{1}{3} - \tfrac{1}{5} + \cancel{G(1)}\right] - \left[-\tfrac{1}{3} + \tfrac{1}{5} + \cancel{G(-1)}\right] + \left[-\tfrac{7}{3} + \tfrac{1}{3} + \cancel{G(-1)}\right] - \left[\tfrac{7}{3} - \tfrac{1}{3} + \cancel{G(1)}\right]$$

$$= \frac{2}{15} + \frac{2}{15} - 2 - 2 = -\frac{56}{15}$$

AS BEFORE

IYGB - MATHEMATICAL METHODS 2 - PAPER D - QUESTION 7

a) ELIMINATE INTO CARTESIAN AS FOLLOWS

$$\left.\begin{array}{l} x(\theta,\phi) = (R + r\cos\theta)\cos\phi \\ y(\theta,\phi) = (R + r\cos\theta)\sin\phi \\ z(\theta,\phi) = r\sin\theta \end{array}\right\} \qquad \begin{array}{l} 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq 2\pi \end{array}$$

- $x^2 + y^2 = (R + r\cos\theta)^2\cos^2\phi + (R + r\cos\theta)^2\sin^2\phi$
  $= (R + r\cos\theta)^2\left[\cos^2\phi + \sin^2\phi\right]$
  $= (R + r\cos\theta)^2$

- $+\sqrt{x^2+y^2} = R + r\cos\theta$
  $-r\cos\theta = R - \sqrt{x^2+y^2}$
  $r^2\cos^2\theta = \left(R - \sqrt{x^2+y^2}\right)^2$

- $r\sin\theta = z$
  $r^2\sin^2\theta = z^2$

$$r^2\cos^2\theta + r^2\sin^2\theta = \left(R - \sqrt{x^2+y^2}\right)^2 + z^2$$

$$r^2(\cos^2\theta + \sin^2\theta) = \left(R - \sqrt{x^2+y^2}\right)^2 + z^2$$

$$z^2 + \left(R - \sqrt{x^2+y^2}\right)^2 = r^2$$

b) REARRANGING THE ABOVE EQUATION TO

$$\left(R - \sqrt{x^2+y^2}\right)^2 = r^2 - z^2$$

HERE $R = 4$ & $r = 1$

SO THE PARAMETRICS BECOME, USING PART (a)

$$x(\theta,\phi) = (4 + \cos\theta)\cos\phi$$

$$y(\theta,\phi) = (4 + \cos\theta)\sin\phi$$

$$z(\theta,\phi) = \sin\theta \qquad \text{WITH} \quad 0 \leq \theta, \phi \leq 2\pi$$

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- $\underline{r}(\theta,\phi) = [(4+\cos\theta)\cos\phi, (4+\cos\theta)\sin\phi, \sin\theta)$

  $\underline{r}(\theta,\phi) = [4\cos\phi + \cos\theta\cos\phi, 4\sin\phi + \cos\theta\sin\phi, \sin\theta]$

- $\frac{\partial \underline{r}}{\partial \theta} = [-\sin\theta\cos\phi, \cos\theta\sin\phi, \cos\theta]$

  $\frac{\partial \underline{r}}{\partial \phi} = [-4\sin\phi - \cos\theta\sin\phi, 4\cos\phi + \cos\theta\cos\phi, 0]$

- $$\left|\frac{\partial \underline{r}}{\partial \theta} \wedge \frac{\partial \underline{r}}{\partial \phi}\right| = \left\|\begin{matrix} \underline{i} & \underline{j} & \underline{k} \\ +\sin\theta\cos\phi & +\sin\theta\sin\phi & -\cos\theta \\ -4\sin\phi - \cos\theta\sin\phi & 4\cos\phi + \cos\theta\cos\phi & 0 \end{matrix}\right\|$$

(REVERSED THE SIGNS IN THE SECOND ROW FOR SIMPLICITY)

$$= |0 + 4\cos\theta\cos\phi + \cos^2\theta\cos\phi, 4\cos\theta\sin\phi + \cos^2\theta\sin\phi - 0,$$
$$, 4\sin\theta\cos^2\phi + \sin\theta\cos\theta\cos^2\phi + 4\sin\theta\sin^2\phi + \cos\theta\sin\theta\sin^2\phi|$$

$$= |\cos\theta\cos\phi(4+\cos\theta), \cos\theta\sin\phi(4+\cos\theta),$$
$$, 4\sin\theta\cos^2\phi + 4\sin\theta\sin^2\phi + \sin\theta\cos\theta\cos^2\phi + \cos\theta\sin\theta\sin^2\phi|$$

$$= |\cos\theta\cos\phi(4+\cos\theta), \cos\theta\sin\phi(4+\cos\theta),$$
$$, 4\sin\theta(\cos^2\phi + \sin^2\phi) + \sin\theta\cos\theta(\cos^2\phi + \sin^2\phi)|$$

$$= |\cos\theta\cos\phi(4+\cos\theta), \cos\theta\sin\phi(4+\cos\theta), 4\sin\theta + \sin\theta\cos\theta|$$

$$= |\cos\theta\cos\phi(4+\cos\theta), \cos\theta\sin\phi(4+\cos\theta), \sin\theta(4+\cos\theta)|$$

$$= (4+\cos\theta)|\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta|$$

$$= (4+\cos\theta)\sqrt{\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta}$$

$$= (4+\cos\theta)\sqrt{\cos^2\theta(\cos^2\phi + \sin^2\phi) + \sin^2\theta}$$

$$= (4+\cos\theta)\sqrt{\cos^2\theta + \sin^2\theta}$$

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$= (4+\cos\theta)$

HENCE THE SURFACE ELEMENT dS IN PARAMETRIC IS

$$dS = \left| \frac{\partial r}{\partial \theta} \wedge \frac{\partial r}{\partial \phi} \right| d\theta\, d\phi$$

$$\boxed{dS = (4+\cos\theta)\, d\theta\, d\phi}$$

FINALLY THE AREA CAN BE FOUND

$$\text{AREA} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{2\pi} 1 \; dS = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{2\pi} (4+\cos\theta)\; d\theta\, d\phi$$

NO CONTRIBUTION OVER THESE LIMITS

$$= \left[\int_{\phi=0}^{2\pi} 1 \; d\phi\right]\left[\int_{\theta=0}^{2\pi} (4+\cancel{\cos\theta})\; d\theta\right]$$

$$= 2\pi \times 4 \times 2\pi$$

$$= 16\pi^2$$

QUICK NOTE THE THE "STANDARD" FORMULA IS $(2\pi r)(2\pi R)$, WHICH FOR THIS TORUS FOR $r=1$, $R=4$, YIELDS $(2\pi\times1)(2\pi\times4) = 16\pi^2$

IYGB - MATHEMATICAL METHODS 2 - PAPER D - QUESTION 8

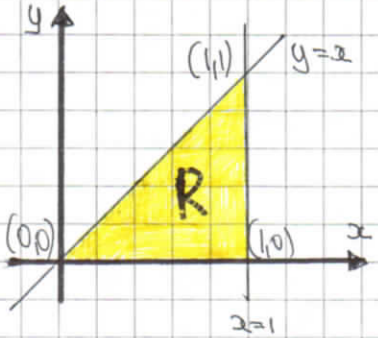
START BY OBTAINING THE JACOBIAN FROM THE GIVEN TRANSFORMATION EQUATIONS

$$du\,dv = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| dx\,dy = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} dx\,dy = \begin{vmatrix} 1 & -\frac{y}{x^2} \\ 1 & \frac{1}{x} \end{vmatrix} dx\,dy$$

$$= \left|\frac{1}{x} + \frac{y}{x^2}\right| dx\,dy = \left|\frac{x+y}{x^2}\right| dx\,dy$$

$$\therefore \boxed{dx\,dy = \frac{x^2}{x+y}\,du\,dv}$$

NEXT DRAW THE INTEGRATION REGION IN THE x-y PLANE & TRANSFORM IT INTO THE u-v PLANE



- $v = \dfrac{y}{x}$ $y = vx$
- $x + y = u$ $y = u - x$

$$vx = u - x$$
$$vx + x = u$$
$$x(v+1) = u$$
$$\underline{x = \frac{u}{v+1}} \quad \& \quad \underline{y = \frac{uv}{v+1}}$$

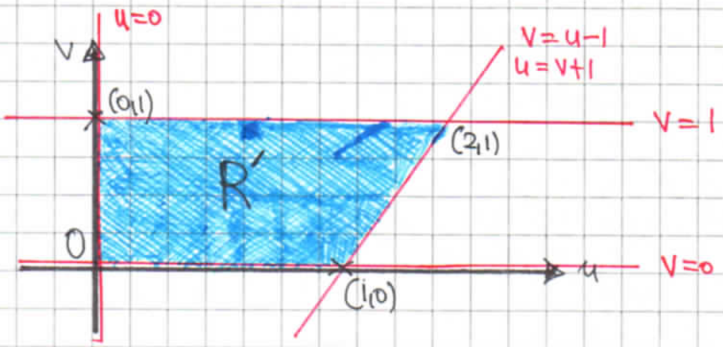
$y = vx$

NEXT OBTAIN SOME LIMITS

- $y = x$
  $\dfrac{uv}{v+1} = \dfrac{u}{v+1}$
  $v = 1$
- $x = 1$
  $\dfrac{u}{v+1} = 1$
  $v+1 = u$
  $v = u - 1$
  (OR $u = v+1$)
- $y = 0$
  $\dfrac{uv}{v+1} = 0$
  $u = 0$ OR $v = 0$

IYGB - MATHEMATICAL METHODS 2 - PAPER D - QUESTION 8

DRAW THE INTEGRATION REGION IN THE u-v PLANE



AS A LITTLE CHECK...
THE POINT $\left(\frac{1}{2},\frac{1}{4}\right)$ LIES INSIDE R IN THE x-y PLANE

ONCE TRANSFORMED IT BECOMES THE POINT $\left(\frac{3}{4},\frac{1}{2}\right)$ WHICH LIES INSIDE R' IN THE u-v PLANE

TRANSFORMING THE INTEGRAL GIVES

$$\iint_R \frac{x+y}{x^2} e^{x+y} \, dx\,dy = \iint_{R'} \cancel{\frac{x+y}{x^2}} e^{x+y} \left(\cancel{\frac{x^2}{x+y}} \, du\,dv \right)$$

$$= \int_{v=0}^{1} \int_{u=0}^{u=v+1} e^{x+y} \, du\,dv = \int_{v=0}^{1} \int_{u=0}^{u=v+1} e^{u} \, du\,dv$$

$$= \int_{v=0}^{1} \left[e^{u} \right]_{u=0}^{u=v+1} dv = \int_0^1 e^{v+1} - e^{0} \, dv$$

$$= \int_0^1 e^{v+1} - 1 \, dv = \left[e^{v+1} - v \right]_0^1$$

$$= \left(e^2 - 1\right) - \left(e^1 - 0\right) = e^2 - e - 1$$

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a) STOKES' THEOREM ASSERTS THAT

$$\iint_S \nabla \wedge \underline{F} \cdot \hat{\underline{n}} \, dS = \oint_C \underline{F} \cdot d\underline{r}$$

WHERE

- S IS AN OPEN TWO SIDED SURFACE WITH A CLOSED BOUNDARY C
- $\underline{F}$ IS A SMOOTH VECTOR FIELD
- $\hat{\underline{n}}$ IS A UNIT NORMAL TO S, SO THAT THE DIRECTION OF C & $\hat{\underline{n}}$ FORM A RIGHT HAND SET
- $d\underline{r} = (dx, dy, dz)$

b) USING STOKES THEOREM WITH $\underline{F} = \nabla\phi$

$$\Longrightarrow \iint_S \nabla \wedge \underline{F} \cdot \hat{\underline{n}} \, dS = \oint_C \underline{F} \cdot d\underline{r}$$

$$\Longrightarrow \iint_S \left[\nabla \wedge \nabla\phi\right] \cdot \hat{\underline{n}} \, dS = \oint_C \nabla\phi \cdot d\underline{r}$$

ZERO AS THIS IS A STANDARD VECTOR CALCULUS IDENTITY

$$\Longrightarrow 0 = \oint_C \nabla\phi \cdot d\underline{r}$$

$$\Longrightarrow \int_{\substack{C_1 \\ (A \text{ to } B)}} \nabla\phi \cdot d\underline{r} + \int_{\substack{C_2 \\ (B \text{ to } A)}} \nabla\phi \cdot d\underline{r} = 0 \qquad C = C_1 + C_2$$

$$\Longrightarrow \int_{\substack{C_1 \\ (A \text{ to } B)}} \nabla\phi \cdot d\underline{r} = -\int_{\substack{C_2 \\ (A \text{ to } B)}} \nabla\phi \cdot d\underline{r}$$

I.E. INDEPENDENT OF THE PATH FROM A TO B

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c) PROCEED AS FOLLOWS, LOOKING AT PARTS OF THE INTEGRAND

- $\frac{\underline{r}}{|\underline{r}|^3} = \frac{(x,y,z)}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \left[x(x^2+y^2+z^2)^{-\frac{3}{2}}, y(x^2+y^2+z^2)^{-\frac{3}{2}}, z(x^2+y^2+z^2)^{-\frac{3}{2}}\right]$

$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\left[-(x^2+y^2+z^2)^{-\frac{1}{2}}\right] = \underline{\nabla}\left(-(x^2+y^2+z^2)^{-\frac{1}{2}}\right)$

- $x\underline{i} = (x,0,0) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\left(\frac{1}{2}x^2\right) = \underline{\nabla}\left(\frac{1}{2}x^2\right)$

THUS WE NOW HAVE

$$\int_{(2,1,2)}^{(6,3,2)} \left[\frac{\underline{r}}{|\underline{r}^3|} + x\underline{i}\right]\cdot d\underline{r} = \int \underline{\nabla}\Big[\underbrace{\tfrac{1}{2}x^2 - (x^2+y^2+z^2)^{-\frac{1}{2}}}_{\phi(x,y,z)}\Big]\cdot d\underline{r}$$

$$\boxed{\begin{aligned}\underline{\nabla}\phi \cdot d\underline{r} &= \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)\cdot(dx,dy,dz) \\ &= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi\end{aligned}}$$

$$\ldots = \left[\frac{1}{2}x^2 - \frac{1}{\sqrt{x^2+y^2+z^2}}\right]_{(2,1,2)}^{(6,3,2)} \longleftarrow \int_{(2,1,2)}^{(6,3,2)} 1\, d\phi = \left[\phi\right]_{(2,1,2)}^{(6,3,2)}$$

$$= \left(18 - \frac{1}{7}\right) - \left(2 - \frac{1}{3}\right)$$

$$= 16 - \frac{1}{7} + \frac{1}{3}$$

$$= 16\frac{4}{21} \quad // \quad \left(\frac{340}{21}\right)$$