YGB - MATHEMATICAL METHODS 2 - PAPER E - QUESTION 1

PROCEED AS FOLLOWS

$$\oint_C \underline{F} \cdot d\underline{r} = \oint_C \left(\sin x^3 - xy,\ y^3 \sin y + x\right) \cdot (dx, dy)$$

$$= \oint_C \left(\sin x^3 - xy\right) dx + \left(y^3 \sin y + x\right) dy$$

NOW GREEN'S THEOREM ON THE PLANE ASSERTS THAT

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

APPLYING IT HERE YIELDS

$$\dots = \iint_R \left[\frac{\partial}{\partial x}\left[y^3 \sin y + x\right] - \frac{\partial}{\partial y}\left[\sin x^3 - xy\right]\right] dx\,dy$$

$$= \iint_R 1 - x \quad dx\,dy$$

NOW LOOKING AT THE REGION R, WHICH IS THE ELLIPSE ANALYSED OPPOSITE WE HAVE:

$$= \iint_R 1 \quad dx\,dy$$

AS x IS AN ODD POWER IN A SYMMETRICAL DOMAIN IN x

$$= 1 \times \text{AREA OF THE ELLIPSE}$$

$$= 1 \times \pi \times \frac{1}{3} \times \frac{1}{\sqrt{6}}$$

$$= \frac{\pi}{3\sqrt{6}}$$

R: $2x^2 + 3y^2 = 2y$

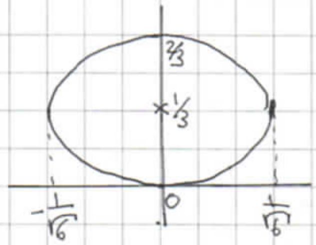
$$2x^2 + 3y^2 - 2y = 0$$

$$\frac{2}{3}x^2 + y^2 - \frac{2}{3}y = 0$$

$$\frac{2}{3}x^2 + \left(y - \frac{1}{3}\right)^2 = \frac{1}{9}$$

$$6x^2 + 9\left(y - \frac{1}{3}\right)^2 = 1$$

$$\frac{x^2}{\frac{1}{6}} + \frac{\left(y - \frac{1}{3}\right)^2}{\frac{1}{9}} = 1$$



IYGB - MATHEMATICAL METHODS 2 - PAPER E - QUESTION 2

USING THE STANDARD INDEX NOTATION DEFINITION OF A CROSS PRODUCT, FOR ITS k^{TH} COMPONENT

$$(\underline{p} \wedge \underline{q})_k = \varepsilon_{ijk} \, p_i \, q_j$$

THUS WE HAVE

$$(\underline{A} \wedge \underline{B}) \cdot (\underline{C} \wedge \underline{D}) = (\varepsilon_{ijk} A_i B_j)(\varepsilon_{lnk} C_l D_n)$$

$$= \varepsilon_{ijk} \, \varepsilon_{lnk} \, A_i B_j C_l D_n$$

"STANDARD" IDENTITY

$$= \begin{vmatrix} \delta_{il} & \delta_{in} \\ \delta_{jl} & \delta_{jn} \end{vmatrix} A_i B_j C_l D_n$$

$$= \left[\delta_{il} \delta_{jn} - \delta_{jl} \delta_{in} \right] A_i B_j C_l D_n$$

$$= \delta_{il} \delta_{jn} A_i B_j C_l D_n - \delta_{jl} \delta_{in} A_i B_j C_l D_n$$

$$= A_l B_j C_l D_j - A_i B_l C_l D_i$$

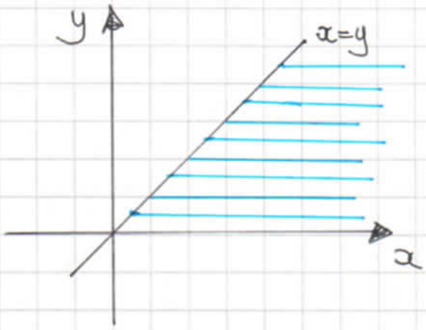
$$= A_l C_l B_j D_j - A_i D_i B_l C_l$$

$$= (\underline{A} \cdot \underline{C})(\underline{B} \cdot \underline{D}) - (\underline{A} \cdot \underline{D})(\underline{B} \cdot \underline{C})$$

// AS REQUIRED

IYGB - MATHEMATICAL METHODS 2 - PAPER E - QUESTION 3

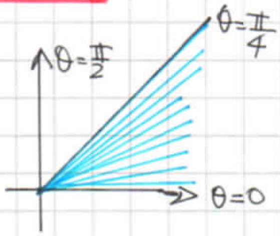
START WITH A DIAGRAM SHOWING THE REGION OF INTEGRATION



THE LIMITS NOW IN POLARS BECOME

$r=0$ TO $r=\infty$

$\theta=0$ TO $\theta=\frac{\pi}{4}$



HENCE WE NOW HAVE

$$\int_{y=0}^{\infty}\int_{x=y}^{\infty}\frac{e^{-x}}{x}\,dx\,dy=\int_{\theta=0}^{\frac{\pi}{4}}\int_{r=0}^{\infty}\frac{e^{-(r\cos\theta)}}{r\cos\theta}\,(\underbrace{r\,dr\,d\theta}_{dx\,dy})$$

$$=\int_{\theta=0}^{\frac{\pi}{4}}\int_{r=0}^{\infty}\frac{e^{-r\cos\theta}}{\cos\theta}\,dr\,d\theta=\int_{\theta=0}^{\frac{\pi}{4}}\left[\frac{1}{\cos\theta}\times\frac{1}{-\cos\theta}e^{-r\cos\theta}\right]_{r=0}^{\infty}d\theta$$

$$=\int_{\theta=0}^{\frac{\pi}{4}}\left[-\frac{e^{-r\cos\theta}}{\cos^2\theta}\right]_{r=0}^{\infty}d\theta=\int_{\theta=0}^{\frac{\pi}{4}}\left[+e^{-r\cos\theta}\sec^2\theta\right]_{r=\infty}^{r=0}d\theta$$

$$=\int_{0}^{\frac{\pi}{4}}\sec^2\theta-0\;d\theta=\int_{0}^{\frac{\pi}{4}}\sec^2\theta\;d\theta$$

$$=\left[\tan\theta\right]_{\theta=0}^{\theta=\frac{\pi}{4}}=1$$

IYGB - MATHEMATICAL METHODS 2 - PAPER E - QUESTION 4

a) TIDY BY COMPLETING THE SQUARE IN CARTESIAN

$$x^2+y^2+z^2=2x$$
$$x^2-2x+y^2+z^2=0$$
$$(x-1)^2+y^2+z^2=1$$

$\in$ A SPHERE OF RADIUS 1, CENTRE AT $(1,0,0)$

FOR PARAMETRIZATION USE SPHERICAL POLARS

$$\left.\begin{aligned} x-1 &= 1\sin\theta\cos\phi \\ y &= 1\sin\theta\sin\phi \\ z &= 1\cos\theta \end{aligned}\right\} \Rightarrow \begin{aligned} x &= 1+\sin\theta\cos\phi \\ y &= \sin\theta\sin\phi \\ z &= \cos\theta \end{aligned}$$

HENCE $\underline{r}(u,v) = \left[1+\sin u\cos v,\ \sin u\sin v,\ \cos\ \right]$ $\quad 0 \leqslant u \leqslant \pi$, $\ 0 \leqslant v \leqslant 2\pi$

b) NOW WE HAVE

$$x^2+y^2+z^2=2x, \qquad \frac{3}{5} \leqslant z \leqslant \frac{4}{5}$$

$\Rightarrow \arccos\frac{3}{5} \leqslant \cos u \leqslant \arccos\frac{4}{5}$

$\Rightarrow$ NOTE THAT SINCE COS IS DECREASING u RUNS FROM $\arccos\frac{4}{5}$ TO $\arccos\frac{3}{5}$

NOW THE dS, SINCE WE ESSENTIALLY HAVE SPHERICAL POLARS ON A UNIT SPHERE IS $\sin\theta\, d\theta\, d\phi$ OR WITH OUR VARIABLES $\sin u\, du\, dv$

OR WE CAN DERIVE AS

- $\frac{\partial \underline{r}}{\partial u} = (\cos u\cos v, \cos u\sin v, -\sin u)$
- $\frac{\partial \underline{r}}{\partial v} = (-\sin u\sin v, \sin u\cos v, 0)$

- $$\left|\frac{\partial \underline{r}}{\partial u} \wedge \frac{\partial \underline{r}}{\partial v}\right| = \left\|\begin{matrix} \underline{i} & \underline{j} & \underline{k} \\ \cos u\cos v & \cos u\sin v & -\sin u \\ -\sin u\sin v & \sin u\cos v & 0 \end{matrix}\right\|$$

IYGB - MATHEMATICAL METHODS 2 - PAPER E - QUESTION 4

$$= \left| 0 + \sin^2 u\cos v,\ + \sin^2 u \sin v - 0,\ \sin u\cos u\cos^2 v + \sin u\cos u \sin^2 v \right|$$

$\sin u\cos u\,(\cos^2 v + \sin^2 v)$

$$= \left| \sin^2 u \cos v,\ \sin^2 u \sin v,\ \sin u\cos u \right| = \sin u \left| \sin u\cos v,\ \sin u \sin v,\ \cos u \right|$$

$$= \sin u \sqrt{\sin^2 u\cos^2 v + \sin^2 u \sin^2 v + \cos^2 u}$$

$$= \sin u \sqrt{\sin^2 u\,\cancel{(\cos^2 v + \sin^2 v)} + \cos^2 u} = \sin u \sqrt{\cancel{\sin^2 u + \cos^2 u}}$$

$$= \sin u$$

$$\therefore\ dS = \left| \frac{\partial \underline{r}}{\partial u} \wedge \frac{\partial \underline{r}}{\partial v} \right| du\,dv$$

$$\Rightarrow\ dS = \sin u\ du\,dv \quad \text{(AS INDEED EXPECTED)}$$

FINALLY WE HAVE

$$\text{AREA} = \int_{v=0}^{2\pi} \int_{u=\arccos\frac{4}{5}}^{u=\arccos\frac{3}{5}} 1\ dS = \int_{v=0}^{2\pi} \int_{u=\arccos\frac{4}{5}}^{u=\arccos\frac{3}{5}} \sin u\ du\,dv$$

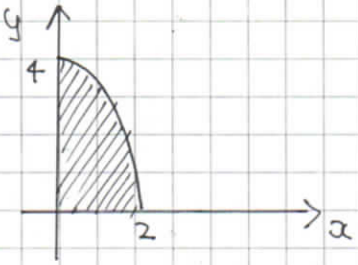
$$= \int_{v=0}^{2\pi} \Big[-\cos u\Big]_{\arccos\frac{4}{5}}^{\arccos\frac{3}{5}} dv = \int_{v=0}^{2\pi} \Big[\cos u\Big]_{\arccos\frac{3}{5}}^{\arccos\frac{4}{5}} dv$$

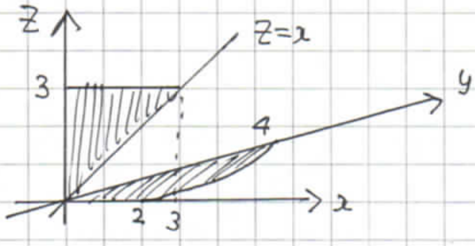
$$= \int_0^{2\pi} \left(\frac{4}{5} - \frac{3}{5}\right) dv = \int_0^{2\pi} \frac{1}{5}\ dv = \frac{1}{5} \times 2\pi$$

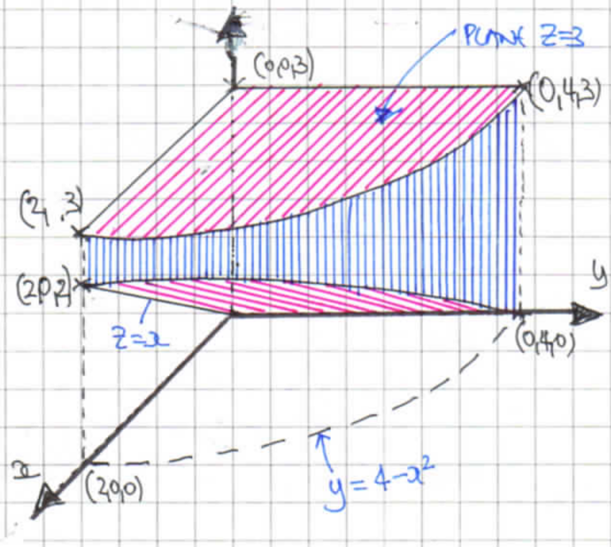
$$= \frac{2\pi}{5}$$

IYGB – MATHEMATICAL METHODS 2 – PAPER E – QUESTION 5

THIS IS DIFFICULT TO DRAW/VISUALISE SO DRAW IN SEVERAL SECTIONS







SETTING UP A VOLUME INTEGRAL IN CARTESIAN

$$V = \iiint_{\text{REGION}} 1 \, dV = \int_{z=x}^{z=3} \int_{x=0}^{2} \int_{y=0}^{y=4-x^2} 1 \quad dy\,dx\,dz$$

BECAUSE OF THE DEPENDENCE OF LIMITS WE GO FROM z TO x TO y

$$\Rightarrow V = \int_{y=0}^{y=4} \int_{x=0}^{x=+\sqrt{4-y}} \int_{z=x}^{z=3} 1 \quad dz\,dx\,dy$$

$$\Rightarrow V = \int_{y=0}^{4} \int_{x=0}^{x=\sqrt{4-y}} \Big[z \Big]_{z=x}^{z=3} \, dx\,dy$$

IYGB - MATHEMATICAL METHODS 2 - PAPER E - QUESTION 5

$$\Rightarrow V = \int_{y=0}^{4} \int_{x=0}^{x=\sqrt{4-y}} 3 - x \ dx\, dy$$

$$\Rightarrow V = \int_{y=0}^{4} \left[3x - \frac{1}{2}x^2 \right]_{x=0}^{x=\sqrt{4-y}} dy$$

$$\Rightarrow V = \int_{y=0}^{4} -3(4-y)^{\frac{1}{2}} + \frac{1}{2}(4-y) \ dy$$

$$\Rightarrow V = \left[-2(4-y)^{\frac{3}{2}} + \frac{1}{4}(4-y)^2 \right]_0^4$$

$$\Rightarrow V = (0 + 0) - \left(-2\times 8 + \frac{1}{4}\times 16\right)$$

$$\Rightarrow V = -(-16+4)$$

$$\Rightarrow V = 12$$

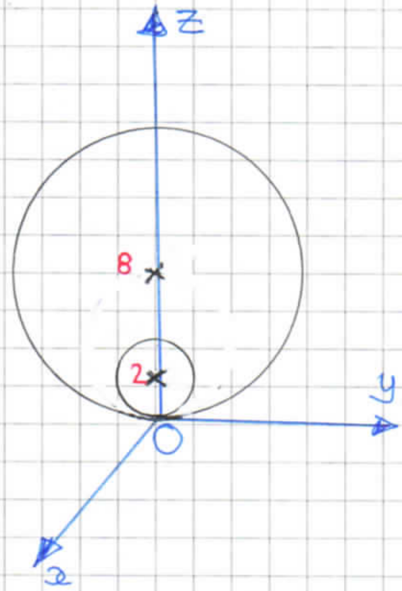
IYGB - MATHEMATICAL METHODS 2 - PAPER E - QUESTION 6

THE INTEGRATION REGION SUGGEST SPHERICAL POLARS SINCE

- $4z = x^2+y^2+z^2$
  $x^2+y^2+z^2-4z=0$
  $x^2+y^2+(z-2)^2=4$
  SPHERE OF RADIUS 2, CENTRE AT $(0,0,2)$
- $x^2+y^2+z^2=16z$
  $x^2+y^2+z^2-16z=0$
  $x^2+y^2+(z-8)^2=64$
  SPHERE OF RADIUS 8, CENTRE AT $(0,0,8)$

USING SPHERICAL POLARS

$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$ & $x^2+y^2+z^2=r^2$
$z = r\cos\theta$



TRANSFORM THE EQUATIONS OF SPHERES

$x^2+y^2+z^2=4z$ | $x^2+y^2+z^2=16z$
$r^2 = 4r\cos\theta$ | $r^2 = 16r\cos\theta$
$r = 4\cos\theta$ | $r=16\cos\theta$

TRANSFORM THE LIMITS

$z>0 \Rightarrow 0 \leq \theta \leq \frac{\pi}{2}$
$0 \leq \phi \leq 2\pi$

VOLUME ELEMENT IN SPHERICAL POLARS

$dV = r^2\sin\theta\, dr\, d\theta\, d\phi$

TRANSFORMING THE INTEGRAL YIELDS

$$\iiint_V \left(\frac{z}{8}\right)^3 dx\,dy\,dz = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=4\cos\theta}^{r=16\cos\theta} \left(\frac{r\cos\theta}{8}\right)^3 (r^2\sin\theta\, dr\, d\theta\, d\phi)$$

IYGB - MATHEMATICAL METHODS 2 - PAPER E - QUESTION 6

$$\dots = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=4\cos\theta}^{r=16\cos\theta} \frac{r^5\cos^3\theta\sin\theta}{512}\,dr\,d\theta\,d\phi$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{2}} \left[\frac{r^6\cos^3\theta\sin\theta}{512\times 6}\right]_{r=4\cos\theta}^{r=16\cos\theta} d\theta\,d\phi$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{2}} \frac{16384}{3}\cos^9\theta\sin\theta - \frac{4}{3}\cos^9\theta\sin\theta\ d\theta\,d\phi$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{2}} 5460\cos^9\theta\sin\theta\ d\theta\,d\phi$$

$$= \int_{\phi=0}^{2\pi} \left[-546\cos^{10}\theta\right]_0^{\frac{\pi}{2}} d\phi$$

$$= \int_{\phi=0}^{2\pi} \left[546\cos^{10}\theta\right]_{\frac{\pi}{2}}^{0} d\phi$$

$$= \int_{\phi=0}^{2\pi} 546 - 0\ d\phi$$

$$= \left[546\phi\right]_0^{2\pi}$$

$$= 1092\pi$$

IYGB - MATHEMATICAL METHODS 2 - PAPER E - QUESTION 7

a) DEFINE SOME QUANTITIES FIRST

$$\underline{A} = \left(A_1(x,y,z), A_2(x,y,z), A_3(x,y,z)\right) \quad \& \quad \phi = \phi(x,y,z)$$

THEN WE HAVE

$$\underline{\nabla}\cdot(\phi\underline{A}) = \underline{\nabla}\cdot(\phi A_1, \phi A_2, \phi A_3)$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(\phi A_1, \phi A_2, \phi A_3)$$

BY THE PRODUCT RULE

$$= \frac{\partial}{\partial x}(\phi A_1) + \frac{\partial}{\partial y}(\phi A_2) + \frac{\partial}{\partial z}(\phi A_3)$$

$$= \frac{\partial \phi}{\partial x}A_1 + \phi\frac{\partial A_1}{\partial x} + \frac{\partial \phi}{\partial y}A_2 + \phi\frac{\partial A_2}{\partial y} + \frac{\partial \phi}{\partial z}A_3 + \phi\frac{\partial A_3}{\partial z}$$

$$= \left[\frac{\partial \phi}{\partial x}A_1 + \frac{\partial \phi}{\partial y}A_2 + \frac{\partial \phi}{\partial z}A_3\right] + \phi\left[\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right]$$

$$= \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right)\cdot(A_1, A_2, A_3) + \phi\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(A_1, A_2, A_3)$$

$$= \underline{\nabla}\phi\cdot\underline{A} + \phi\,\underline{\nabla}\cdot\underline{A}$$

AS REQUIRED

b) PROCEED USING THE RESULT OF PART (a)

$$\underline{\nabla}\cdot[f\underline{\nabla}g - g\underline{\nabla}f] = \underline{\nabla}\cdot[\underbrace{f}_{\phi}\,\underbrace{\underline{\nabla}g}_{\underline{A}}] - \underline{\nabla}\cdot[\underbrace{g}_{\phi}\,\underbrace{\underline{\nabla}f}_{\underline{A}}]$$

$$= \underline{\nabla}f\cdot\underline{\nabla}g + f\,\underline{\nabla}\cdot\underline{\nabla}g - \underline{\nabla}g\cdot\underline{\nabla}f - g\,\underline{\nabla}\cdot\underline{\nabla}f$$

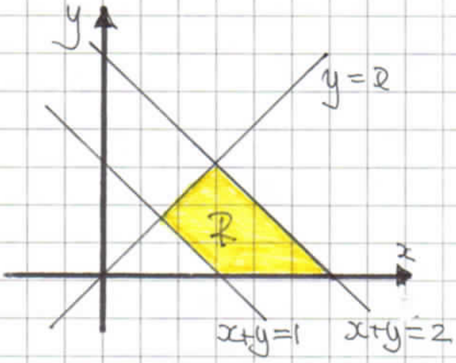
$$= \cancel{\underline{\nabla}f\cdot\underline{\nabla}g} + f\,\underline{\nabla}^2 g - \cancel{\underline{\nabla}f\cdot\underline{\nabla}g} - g\,\underline{\nabla}^2 f$$

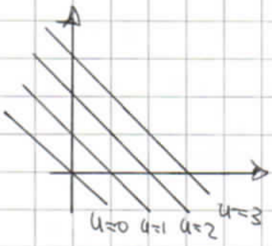
$$= f\,\underline{\nabla}^2 g - g\,\underline{\nabla}^2 f$$

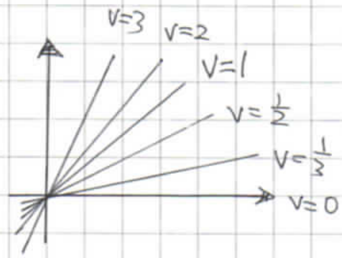
AS REQUIRED

IYGB - MATHEMATICAL METHODS 2 - PAPER E - QUESTION 8

START BY SKETCHING THE INTEGRATION REGION & SEEK A "JACOBIAN" TRANSFORMATION







LET $u = x+y$ LET $v = \frac{y}{x}$

$1 \leq u \leq 2$ $0 \leq v \leq 1$

FIND THE JACOBIAN (EASIER TO DO IT "BACKWARDS")

$$du\,dv = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| dy\,dx = \left|\begin{matrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{matrix}\right| dy\,dx = \left|\begin{matrix} 1 & 1 \\ -\frac{y}{x^2} & \frac{1}{x}\end{matrix}\right| dy\,dx$$

$$= \left|\frac{1}{x} + \frac{y}{x^2}\right| dy\,dx = \left|\frac{x+y}{x^2}\right| dy\,dx$$

$$\therefore \boxed{dy\,dx = \frac{x^2}{x+y}\,du\,dv}$$

TRANSFORM THE INTEGRAL TO OBTAIN

$$\iint_R \frac{y\ln(x+y)}{x^2}\,dx\,dy = \iint_{R'} \frac{y\ln(x+y)}{\cancel{x^2}}\left(\frac{\cancel{x^2}}{x+y}\,du\,dv\right)$$

$$= \iint_{R'} \frac{y\ln(x+y)}{x+y}\,du\,dv$$

$$= \iint_{R'} \frac{y\ln u}{u}\,du\,dv$$

ELIMINATE y TO OBTAIN $y = y(u,v)$ FROM THE TRANSFORMATION EQUATIONS

IYGB - MATHEMATICAL METHODS 2 - PAPER E - QUESTION 8

$$\left.\begin{aligned} v &= \frac{y}{x} \\ u &= x+y \end{aligned}\right\} \Rightarrow x = u-y$$

$$\Rightarrow v = \frac{y}{u-y}$$

$$\Rightarrow uv - vy = y$$

$$\Rightarrow uv = y + vy$$

$$\Rightarrow uv = y(v+1)$$

$$\Rightarrow \boxed{y = \frac{uv}{v+1}}$$

RETURNING TO THE INTEGRAL WE OBTAIN

$$\dots = \iint_{R'} \left(\frac{\cancel{u}v}{v+1}\right)\frac{\ln u}{\cancel{u}}\,du\,dv = \int_{u=1}^{2}\int_{v=0}^{1} \frac{v}{v+1}\ln u \; dv\,du$$

$$= \left[\int_1^2 \ln u \; du\right]\left[\int_0^1 \frac{v}{v+1}\,dv\right]$$

BY PARTS / STANDARD RESULT

$$= \Big[u\ln u - u\Big]_1^2 \int_0^1 \frac{v+1-1}{v+1}\,dv$$

$$= \Big[(2\ln 2 - 2) - (\cancel{\ln 1} - 1)\Big]\int_0^1 1 - \frac{1}{v+1}\,dv$$

$$= (2\ln 2 - 1)\Big[v - \ln|v+1|\Big]_0^1$$

$$= (2\ln 2 - 1)\big((1-\ln 2) - (0 - \cancel{\ln 1})\big)$$

$$= (2\ln 2 - 1)(1 - \ln 2)$$

AS REQUIRED

IYGB - MATHEMATICAL METHODS 2 - PAPER E - QUESTION 9

METHOD A

$$x = r\cos\theta \Rightarrow dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta \Rightarrow dx = \cos\theta\, dr - r\sin\theta\, d\theta$$

$$y = r\sin\theta \Rightarrow dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta \Rightarrow dy = \sin\theta\, dr + r\cos\theta\, d\theta$$

PROCEED WITH THE POLAR PARAMETRIZATION

$$\oint_C \underline{F} \cdot d\underline{r} = \oint_C (x+y, -x+y) \cdot (dx, dy)$$

$$= \oint_C (r\cos\theta + r\sin\theta, -r\cos\theta + r\sin\theta) \cdot (\cos\theta\, dr - r\sin\theta\, d\theta, \sin\theta\, dr + r\cos\theta\, d\theta)$$

$$= \oint_C \begin{matrix} (r\cos\theta + r\sin\theta)(\cos\theta\, dr) + (r\cos\theta + r\sin\theta)(-r\sin\theta\, d\theta) \\ (-r\cos\theta + r\sin\theta)(\sin\theta\, dr) + (-r\cos\theta + r\sin\theta)(r\cos\theta\, d\theta) \end{matrix}$$

$$= \oint_C \begin{matrix} r\cos^2\theta\, dr + \cancel{r\sin\theta\cos\theta\, dr} - \cancel{r^2\cos\theta\sin\theta\, d\theta} - r^2\sin^2\theta\, d\theta \\ r\sin^2\theta\, dr - \cancel{r\sin\theta\cos\theta\, dr} + \cancel{r^2\cos\theta\sin\theta\, d\theta} - r^2\cos^2\theta\, d\theta \end{matrix}$$

$$= \oint_C r(\cos^2\theta + \sin^2\theta)\, dr - r^2(\sin^2\theta + \cos^2\theta)\, d\theta$$

$$= \oint_C \left[r\, dr - r^2\, d\theta \right]$$

FINALLY WE HAVE

$$= \int_0^{-2\pi} (3+\sin\theta)(\cos\theta\, d\theta) - (3+\sin\theta)^2 (d\theta)$$

$$= \int_0^{-2\pi} 3\cos\theta + \sin\theta\cos\theta - 9 - 6\sin\theta - \sin^2\theta \; d\theta$$

$$= \int_0^{-2\pi} \cancel{3\cos\theta} + \cancel{\sin\theta\cos\theta} - 9 - \cancel{6\sin\theta} - \left(\tfrac{1}{2} - \tfrac{1}{2}\cancel{\cos 2\theta}\right) d\theta$$

NO CONTRIBUTION OVER THESE LIMITS

$$= \int_0^{-2\pi} -\frac{19}{2}\, d\theta = -\frac{19}{2}(-2\pi) = 19\pi$$

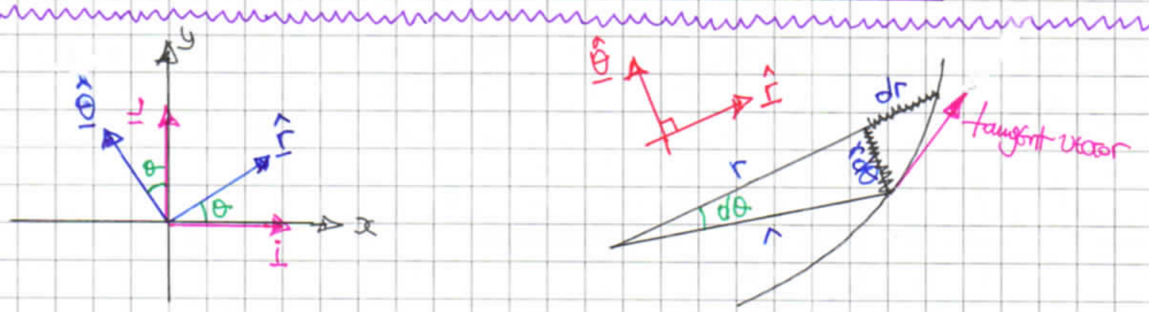
$r = 3 + \sin\theta$
$dr = \cos\theta\, d\theta$

θ RUNS FROM 0 TO -2π, AS THE PATH IS CLOCKWISE

IYGB, MATHEMATICAL METHODS 2, PAPER E, QUESTION 9

METHOD B

STARTING WITH SOME AUXILIARIES IN THE DIAGRAMS BELOW



$$\underline{i} = (1\cos\theta)\hat{\underline{r}} - (1\sin\theta)\hat{\underline{\theta}}$$
$$\underline{j} = (1\sin\theta)\hat{\underline{r}} + (1\cos\theta)\hat{\underline{\theta}}$$

IN ITS UNIT THE TANGENT VECTOR WILL BE

$$(r\,d\theta)\hat{\underline{\theta}} + (dr)\hat{\underline{r}}$$

RETURNING TO THE POLAR LINE INTEGRAL

$$\oint_C \underline{F}\cdot d\underline{r} = \oint_C (x+y, -x+y)\cdot(dx, dy)$$

$$= \oint_C \left[(x+y)\underline{i} + (-x+y)\underline{j}\right]\cdot\left[\underline{i}\,dx + \underline{j}\,dy\right]$$

$$= \oint_C \left[(r\cos\theta + r\sin\theta)\underbrace{(\cos\theta\hat{\underline{r}} - \sin\theta\hat{\underline{\theta}})}_{\underline{i}} + (-r\cos\theta + r\sin\theta)\underbrace{(\sin\theta\hat{\underline{r}} + \cos\theta\hat{\underline{\theta}})}_{\underline{j}}\right]\cdot\underbrace{\left[dr\,\hat{\underline{r}} + r\,d\theta\,\hat{\underline{\theta}}\right]}_{\text{tangent vector}}$$

$$= \oint_C \begin{bmatrix}(r\cos^2\theta + \cancel{r\sin\theta\cos\theta} - \cancel{r\cos\theta\sin\theta} + r\sin^2\theta)\hat{\underline{r}} \\ (-r\sin^2\theta - \cancel{r\sin\theta\cos\theta} + \cancel{r\cos\theta\sin\theta} - r\cos^2\theta)\hat{\underline{\theta}}\end{bmatrix}\cdot\left(dr\,\hat{\underline{r}} + r\,d\theta\,\hat{\underline{\theta}}\right)$$

$$= \oint_C \left[r(\cos^2\theta + \sin^2\theta)\hat{\underline{r}} - r(\cos^2\theta + \sin^2\theta)\hat{\underline{\theta}}\right]\cdot\left[dr\,\hat{\underline{r}} + r\,d\theta\,\hat{\underline{\theta}}\right]$$

$$= \oint_C (r\hat{\underline{r}} - r\hat{\underline{\theta}})\cdot(dr\,\hat{\underline{r}} + r\,d\theta\,\hat{\underline{\theta}})$$

$$= \oint_C r\,dr - r^2\,d\theta$$

WHICH FROM THIS POINT ONWARDS MERGES WITH METHOD A

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METHOD C

START BY PARAMETERIZING DIRECTLY FROM THE POLARS

$$\boxed{\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}} \Longrightarrow \begin{array}{l} x = (3+\sin\theta)\cos\theta \\ y = (3+\sin\theta)\sin\theta \end{array} \Rightarrow \boxed{\begin{array}{l} x = 3\cos\theta + \sin\theta\cos\theta = 3\cos\theta + \frac{1}{2}\sin 2\theta \\ y = 3\sin\theta + \sin^2\theta \end{array}}$$

$$\boxed{\begin{array}{l} dx = (-3\sin\theta + \cos 2\theta)\, d\theta \\ dy = (3\cos\theta + 2\sin\theta\cos\theta)\, d\theta = (3\cos\theta + \sin 2\theta)\, d\theta \end{array}}$$

HENCE WE NOW HAVE

$$\oint_C \underline{F} \cdot d\underline{r} = \oint_C (x+y, -x+y)\cdot(dx, dy) = \oint_C (x+y)\,dx + (-x+y)\,dy$$

$$= \oint_C \Big[(r\cos\theta + r\sin\theta)(-3\sin\theta + \cos 2\theta) + (-r\cos\theta + r\sin\theta)(3\cos\theta + \sin 2\theta)\Big]\, d\theta$$

$$= \oint_C r\Big[\cancel{-3\cos\theta\sin\theta} + \cos\theta\cos 2\theta - 3\sin^2\theta + \sin\theta\cos 2\theta - 3\cos^2\theta - \cos\theta\sin 2\theta + \cancel{3\sin\theta\cos\theta} + \sin\theta\sin 2\theta\Big]\, d\theta$$

$$= \oint_C r\Big[(\cos 2\theta\cos\theta + \sin 2\theta\sin\theta) - (\sin 2\theta\cos\theta - \cos 2\theta\sin\theta) - 3(\cos^2\theta + \sin^2\theta)\Big]\, d\theta$$

$$= \oint_C r\Big[\cos(2\theta - \theta) - \sin(2\theta - \theta) - 3\Big]\, d\theta$$

$$= \oint_0^{-2\pi} (3 + \sin\theta)(\cos\theta - \sin\theta - 3)\, d\theta$$

$$= \int_0^{-2\pi} \cancel{3\cos\theta} - \cancel{3\sin\theta} - 9 + \cancel{\sin\theta\cos\theta} - \sin^2\theta - \cancel{3\sin\theta}\, d\theta$$ (NO CONTRIBUTION OVER THESE LIMITS)

$$= \int_0^{-2\pi} -9 - \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta\right) d\theta = \int_0^{-2\pi} -\tfrac{19}{2} + \cancel{\tfrac{1}{2}\cos 2\theta}\, d\theta$$ (NO CONTRIBUTION OVER THESE LIMITS)

$$= \int_0^{-2\pi} -\tfrac{19}{2}\, d\theta = -\tfrac{19}{2}(-2\pi) = 19\pi$$

AS BEFORE