IYGB - MATHEMATICAL METHODS 3 - PAPER A - QUESTION 1

- f(z) HAS A SIMPLE POLE AT $z=0$, WHICH IS VERY EASY TO FIND FROM ITS LAURENT EXPANSION

$$f(z) = \frac{\sin z}{z^2} = \frac{1}{z^2}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} + O(z^7)\right]$$

$$= \frac{1}{z} - \frac{1}{6}z + \frac{1}{120}z^3 + O(z^5)$$

$\therefore$ RESIDUE IS 1

- ALTERNATIVE IS TO USE THE STANDARD METHOD FOR A SIMPLE POLE AT $z=a$

$$\lim_{z \to a}\left[\frac{d}{dz}\left[(z-a)^2 f(z)\right]\right] = \lim_{z \to 0}\left[\frac{d}{dz}\left[z \, \frac{\sin z}{z^2}\right]\right]$$

$$= \lim_{z \to 0}\left[\frac{d}{dz}\left(\frac{\sin z}{z}\right)\right]$$

BY L'HOSPITAL RULE

$$= \lim_{z \to 0}\left(\frac{\cos z}{1}\right)$$

$$= 1$$

AS BEFORE

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- STARTING WITH A SUBSTITUTION

$$\Rightarrow t = \ln\left(\frac{1}{x}\right)$$
$$\Rightarrow t = -\ln x$$
$$\Rightarrow -t = \ln x$$
$$\Rightarrow x = e^{-t}$$
$$\Rightarrow dx = -e^{-t}\, dt$$

LIMITS

$x = 0 \longmapsto t = \infty$

$x = 1 \longmapsto t = 0$

- THE INTEGRAL TRANSFORMS TO

$$\int_0^1 \left[\ln\left(\frac{1}{x}\right)\right]^{a-1} dx = \int_\infty^0 t^{a-1}(-e^{-t})\, dt$$

$$= \int_0^\infty t^{a-1} e^{-t}\, dt$$

WHICH IS THE EXACT DEFINITION OF $\Gamma(a)$ //

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- WE PROCEED AS FOLLOWS

$$\frac{1}{z} = \frac{1}{1+(z-1)} = \frac{1}{z-1}\left(\frac{1}{1+\frac{1}{z-1}}\right)$$

- EXPAND USING $\frac{1}{1+x} = 1 - x + x^2 + x^3 - \dots \quad |x| < 1$

$$\Longrightarrow \frac{1}{z} = \frac{1}{z-1}\left[1 - \frac{1}{(z-1)} + \frac{1}{(z-1)^2} - \frac{1}{(z-1)^3} + \frac{1}{(z-1)^4} - \dots\right]$$

$$\Longrightarrow \frac{1}{z} = \frac{1}{z-1} - \frac{1}{(z-1)^2} + \frac{1}{(z-1)^2} - \frac{1}{(z-1)^3} + \frac{1}{(z-1)^4} \dots$$

VALID FOR $\left|\frac{1}{z-1}\right| < 1$

$|z-1| > 1$

$$\therefore \frac{1}{z} = \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{(z-1)^r} \quad \text{FOR ANNULUS } |z-1| > 1$$

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- START BY A SELF SUGGESTING SUBSTITUTION

$$\Rightarrow u = 8x^3$$
$$\Rightarrow x^3 = \frac{1}{8}u$$
$$\Rightarrow x = \frac{1}{2}u^{\frac{1}{3}}$$
$$\Rightarrow dx = \frac{1}{6}u^{-\frac{2}{3}}\,du$$

& LIMITS ARE UNCHANGED

- THE INTEGRAL TRANSFORMS TO

$$\int_0^\infty \frac{x^3}{(1+8x^3)^2}\,dx = \int_0^\infty \frac{\frac{1}{8}u}{(1+u)^2}\left(\frac{1}{6}u^{-\frac{2}{3}}\,du\right) = \frac{1}{48}\int_0^\infty \frac{u^{\frac{1}{3}}}{(1+u)^2}\,du$$

- NOW USING THE ALTERNATIVE FORM OF A BETA FUNCTION

$$\boxed{B(m,n) \equiv \int_0^\infty \frac{u^{m-1}}{(1+u)^{m+n}}\,du}$$

- THE INTEGRAL BECOMES

$$\dots = \frac{1}{48}\int_0^\infty \frac{u^{\frac{4}{3}-1}}{(1+u)^{\frac{4}{3}+\frac{2}{3}}}\,du = \frac{1}{48}B\left(\frac{4}{3},\frac{2}{3}\right) = \frac{1}{48}\,\frac{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{4}{3}+\frac{2}{3}\right)}$$

$$= \frac{1}{48}\,\frac{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma(2)} = \frac{1}{48}\,\frac{\frac{1}{3}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{1!} = \frac{1}{144}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)$$

- FINALLY USING $\boxed{\Gamma(x)\Gamma(1-x) \equiv \frac{\pi}{\sin \pi x}}$

$$= \frac{1}{144}\Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right) = \frac{1}{144}\,\frac{\pi}{\sin\frac{\pi}{3}} = \frac{1}{144}\,\frac{\pi}{\frac{\sqrt{3}}{2}} = \frac{\pi}{72\sqrt{3}}$$

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a) START BY DIFFERENTIATING THE GENERATING FUNCTION FOR $J_n(x)$, WITH RESPECT TO x

$$\Rightarrow e^{\frac{1}{2}x(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right]$$

$$\Rightarrow \frac{1}{2}(t-\tfrac{1}{t})e^{\frac{1}{2}x(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty}\left[t^n J_n'(x)\right]$$

$$\Rightarrow \frac{1}{2}te^{\frac{1}{2}x(t-\frac{1}{t})} - \frac{1}{2t}e^{\frac{1}{2}x(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty}\left[t^n J_n'(x)\right]$$

$$\Rightarrow \frac{1}{2}t\sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right] - \frac{1}{2t}\sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right] = \sum_{n=-\infty}^{\infty}\left[t^n J_n'(x)\right]$$

$$\Rightarrow \frac{1}{2}\sum_{n=-\infty}^{\infty}\left[t^{n+1} J_n(x)\right] - \frac{1}{2}\sum_{n=-\infty}^{\infty}\left[t^{n-1} J_n(x)\right] = \sum_{n=-\infty}^{\infty}\left[t^n J_n'(x)\right]$$

EQUATING POWERS OF t, SAY $[t^n]$, GIVES

$$\Rightarrow \frac{1}{2}t^n J_{n-1}(x) - \frac{1}{2}t^n J_{n+1}(x) = t^n J_n'(x)$$

$$\Rightarrow \frac{1}{2}J_{n-1}(x) - \frac{1}{2}J_{n+1}(x) = J_n'(x)$$

// AS REQUIRED

b) NEXT DIFFERENTIATE WITH RESPECT TO t

$$\Rightarrow e^{\frac{1}{2}x(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right]$$

$$\Rightarrow \frac{1}{2}x(1+\tfrac{1}{t^2})e^{\frac{1}{2}x(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty}\left[nt^{n-1} J_n(x)\right]$$

$$\Rightarrow \frac{1}{2}xe^{\frac{1}{2}x(t-\frac{1}{t})} + \frac{x}{2t^2}e^{\frac{1}{2}x(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty}\left[nt^{n-1} J_n(x)\right]$$

IYGB - MATHEMATICAL METHODS 3 - PAPER A - QUESTION 5

$$\Rightarrow \frac{1}{2}x\sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right] + \frac{x}{2t^2}\sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right] = \sum_{n=-\infty}^{\infty}\left[nt^{n-1}J_n(x)\right]$$

$$\Rightarrow \frac{1}{2}x\sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right] + \frac{1}{2}x\sum_{n=-\infty}^{\infty}\left[t^{n-2} J_n(x)\right] = \sum_{n=-\infty}^{\infty}\left[nt^{n-1}J_n(x)\right]$$

EQUATING POWERS OF t, SAY $\left[t^{n-1}\right]$ GIVES

$$\Rightarrow \frac{1}{2}x\left[t^{n-1}J_{n-1}(x)\right] + \frac{1}{2}x\left[t^{n-1}J_{n+1}(x)\right] = nt^{n-1}J_n(x)$$

$$\Rightarrow \frac{1}{2}xJ_{n-1}(x) + \frac{1}{2}xJ_{n+1}(x) = nJ_n(x)$$

$$\Rightarrow J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$$

AS REQUIRED

c) DIFFERENTIATING, USING THE PRODUCT RULE, WITH RESPECT TO x

$$\frac{d}{dx}\left[(x^n + x^{-n})J_n(x)\right] = (nx^{n-1} - nx^{-n-1})J_n(x) + (x^n + x^{-n})J_n'(x)$$

USING PART (b) AND PART (a) IN THE RESPECTIVE TERMS GIVES

$$= (nx^{n-1} - nx^{-n-1})\frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right] + (x^n + x^{-n})\left[\frac{1}{2}J_{n-1}(x) - \frac{1}{2}J_{n+1}(x)\right]$$

$$= \frac{1}{2}(x^n - x^{-n})\left[J_{n-1}(x) + J_{n+1}(x)\right] + \frac{1}{2}(x^n + x^{-n})\left[J_{n-1}(x) - J_{n+1}(x)\right]$$

$$= \frac{1}{2}\left[\begin{array}{l} x^nJ_{n-1}(x) + \cancel{x^nJ_{n+1}(x)} - \cancel{x^{-n}J_{n-1}(x)} - x^{-n}J_{n+1}(x) \\ x^nJ_{n-1}(x) - \cancel{x^nJ_{n+1}(x)} + \cancel{x^{-n}J_{n-1}(x)} - x^{-n}J_{n+1}(x)\end{array}\right]$$

$$= x^nJ_{n-1}(x) - x^{-n}J_{n+1}(x)$$

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$$\boxed{f(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\left(2 - 2e^{-4s} - e^{-6s}\right)\right]}$$

- INVERTING INTO "HEAVISIDES"

$$f(t) = \mathcal{L}^{-1}\left[\frac{2}{s^2} - \frac{2e^{-4s}}{s^2} - \frac{e^{-6s}}{s^2}\right]$$

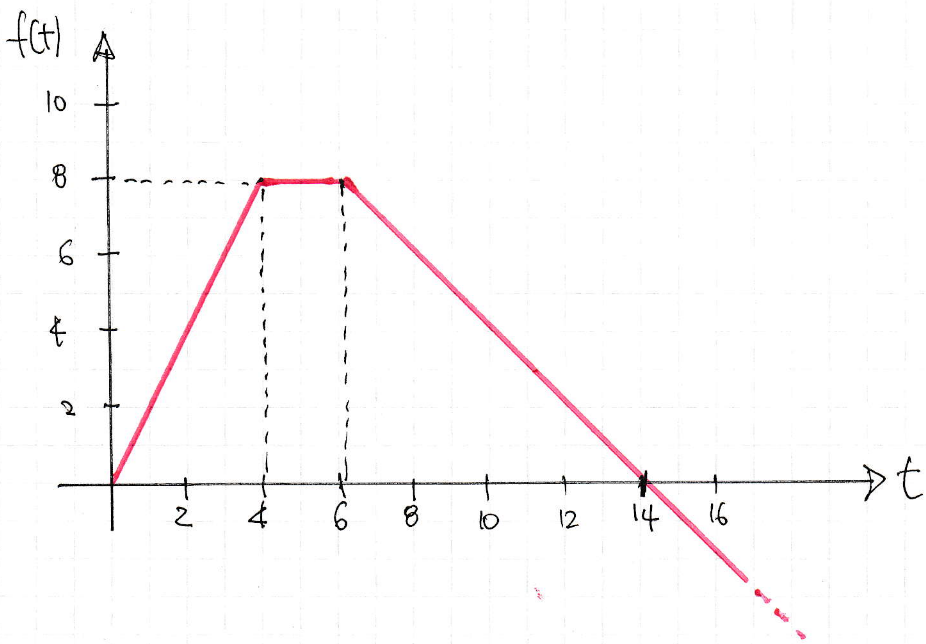
$$f(t) = 2t - 2(t-4)H(t-4) - (t-6)H(t-6)$$

$$f(t) = 2tH(t) - 2(t-4)H(t-4) - (t-6)H(t-6)$$

- USE A TABLE TO EXTRACT THE FUNCTION

INTERVAL	$2tH(t)$	$-2(t-4)H(t-4)$	$-(t-6)H(t-6)$	$f(t)$
$0 \le t \le 4$	$2t$	0	0	$2t$
$4 < t \le 6$	$2t$	$8-2t$	0	8
$t > 6$	$2t$	$8-2t$	$6-t$	$14-t$

- FINALLY BEFORE SKETCHING CHECK FOR CONTINUITY AT $x=4$, $x=6$



IYGB - MATHEMATICAL METHODS 3 - PAPER A - QUESTION 7

$$\boxed{(x^2-1)\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = 0}$$

- AS THE O.D.E IS ANALYTIC AT $x=0$ (WHEN WE DIVIDE BY x^2-1, THE POLES AT ± 1), WE MAY SEEK FOR A SOLUTION OF THE FORM

$$y = \sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

- DIFFERENTIATE WITH RESPECT TO x

$$\frac{dy}{dx} = \sum_{r=1}^{\infty} a_r r x^{r-1} \quad , \quad \frac{d^2y}{dx^2} = \sum_{r=2}^{\infty} a_r r(r-1)x^{r-2}$$

- SUBSTITUTE INTO THE O.D.E.

$$\Rightarrow (x^2-1)\sum_{r=2}^{\infty} a_r r(r-1)x^{r-2} + 4x\sum_{r=1}^{\infty} a_r r x^{r-1} + 2\sum_{r=0}^{\infty} a_r x^r = 0$$

$$\Rightarrow \sum_{r=2}^{\infty} a_r r(r-1)x^r - \sum_{r=2}^{\infty} a_r r(r-1)x^{r-2} + \sum_{r=1}^{\infty} 4a_r r x^r + \sum_{r=0}^{\infty} 2a_r x^r = 0$$

LOWEST POWER IS x^2 — LOWEST POWER IS x^0 — LOWEST POWER IS x^1 — LOWEST POWER OF x^0

- PULL OUT x^0 AND x^1 OUT OF THE SUMMATIONS

$$\Rightarrow \left.\begin{array}{l} \displaystyle\sum_{r=2}^{\infty} a_r r(r-1)x^r - a_2 \times 2 \times 1 \times x^0 + 4a_1 \times 1 \times x^1 + 2a_0 x^0 \\ \displaystyle\qquad - a_3 \times 3 \times 2 \times x^1 + \sum_{r=2}^{\infty} 4a_r r x^r + 2a_1 x^1 \\ \displaystyle\qquad - \sum_{r=4}^{\infty} a_r r(r-1)x^{r-2} + \sum_{r=2}^{\infty} 2a_r x^r \end{array}\right\} = 0$$

- TIDYING UP

$$\Longrightarrow (2a_0 - 2a_2) + (6a_1 - 6a_3)$$

$$+ \sum_{r=2}^{\infty} a_r r(r-1)x^r - \sum_{r=4}^{\infty} a_r r(r-1)x^{r-2} + \sum_{r=2}^{\infty} 4a_r r x^r + \sum_{r=2}^{\infty} 2a_r x^r = 0$$

- ADJUST THE SUMMATIONS SO THEY ALL START FROM $r=2$, BY MAPPING "$r \mapsto r+2$" IN THE 2ND SUMMATION (IGNORE THE LOOSE TERMS)

$$\Longrightarrow \sum_{r=2}^{\infty} a_r r(r-1)x^r - \sum_{r=2}^{\infty} a_{r+2}(r+2)(r+1)x^r + \sum_{r=2}^{\infty} 4a_r r x^r + \sum_{r=2}^{\infty} 2a_r x^r = 0$$

$$\Longrightarrow \sum_{r=2}^{\infty} \Big[\big[a_r [r(r-1) + 4r + 2] - a_{r+2}(r+2)(r+1) \big] x^r \Big] = 0$$

- EQUATING POWERS IN x IN THE SUMMATION

$$\Longrightarrow a_{r+2}(r+2)(r+1) = a_r [r(r-1) + 4r + 2]$$

$$\Longrightarrow a_{r+2}(r+2)(r+1) = a_r (r^2 + 3r + 2)$$

$$\Longrightarrow a_{r+2} = \frac{(r+1)(r+2)}{(r+2)(r+1)} a_r$$

$$\Longrightarrow \underline{\underline{a_{r+2} = a_r}}$$

- THIS IS A TRIVIAL RECURRENCE YIELDING

$$a_0 = a_2 = a_4 = a_6 = \dots$$

$$a_1 = a_3 = a_5 = a_7 = \dots$$

- WE FINALLY HAVE

$$\Rightarrow y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots$$

$$\Rightarrow y = a_0 + a_1x + a_0x^2 + a_1x^3 + a_0x^4 + a_1x^5 + a_0x^6 + \dots$$

$$\Rightarrow y = a_0\left[1 + x^2 + x^4 + x^6 + \dots\right] + a_1\left[x + x^3 + x^5 + x^7 + \dots\right]$$

$$\Rightarrow y = A\left(1 + x^2 + x^4 + x^6 + \dots\right) + Bx\left(1 + x^2 + x^4 + x^6 + \dots\right)$$

$$\Rightarrow y = \frac{A}{1-x^2} + \frac{Bx}{1-x^2}$$

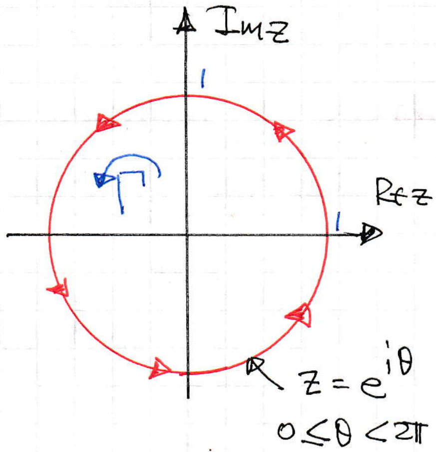
$$\Rightarrow y = \frac{A + Bx}{1-x^2}$$

IYGB - MATHEMATICAL METHODS 3 - PAPER A - QUESTION 8

- USING THE CONTOUR $|z|=1$ OR $z=e^{i\theta}$, $0 \leq \theta < 2\pi$

THUS $z = e^{i\theta}$

$dz = ie^{i\theta}\,d\theta$



- HENCE THE INTEGRAL BECOMES

$$\int_0^{2\pi} \cos^6\theta \sin^6\theta \, d\theta$$

$$= \int_0^{2\pi} (\cos\theta \sin\theta)^6 \, d\theta = \int_0^{2\pi} \left(\tfrac{1}{2}\sin 2\theta\right)^6 d\theta$$

$$= \frac{1}{64}\int_0^{2\pi} \sin^6 2\theta \, d\theta = \frac{1}{64}\int_0^{2\pi} \left[\frac{1}{2i}\left(e^{2i\theta} - e^{-2i\theta}\right)\right]^6 d\theta$$

$$= \frac{1}{64}\left(\frac{1}{2i}\right)^6 \int_0^{2\pi} \left[(e^{i\theta})^2 - (e^{-i\theta})^2\right]^6 d\theta$$

$$= \frac{1}{64} \times \frac{1}{-64} \int_\Gamma \left(z^2 - \frac{1}{z^2}\right)^6 \left(\frac{dz}{iz}\right) \longleftarrow \boxed{d\theta = \frac{dz}{ie^{i\theta}}}$$

$$= \frac{i}{64^2}\int_\Gamma \frac{1}{z}\left(z^2 - \frac{1}{z^2}\right)^6 dz$$

- EXPAND BINOMIALLY

$$= \frac{i}{2^{12}}\int_\Gamma \frac{1}{z}\left[z^{12} - 6z^8 + 15z^4 - 20 + \frac{15}{z^4} - \frac{6}{z^8} + \frac{1}{z^{12}}\right] dz$$

$$= \frac{i}{2^{12}} \int_{\Gamma} z^{11} - 6z^{7} + 15z^{3} - \frac{20}{z} + \frac{15}{z^{5}} - \frac{6}{z^{9}} + \frac{1}{z^{13}} \, dz$$

- AS THE INTEGRAND IS IN SERIES FORM THE ONLY CONTRIBUTION IS FROM THE $\frac{1}{z}$ TERM IE THE RESIDUE IS -20

$$\Rightarrow \int_{0}^{2\pi} \cos^6\theta \sin^6\theta \, d\theta = 2\pi i \times \sum(\text{RESIDUES INSIDE } \Gamma)$$

$$\Rightarrow \int_{0}^{2\pi} \cos^6\theta \sin^6\theta \, d\theta = 2\pi i \times \frac{i}{2^{12}} \times (-20)$$

$$\Rightarrow \int_{0}^{2\pi} \cos^6\theta \sin^6\theta \, d\theta = \frac{5\pi \times 2^3}{2^{12}} = \frac{5\pi}{2^9}$$

$$\Rightarrow \int_{0}^{2\pi} \cos^6\theta \sin^6\theta \, d\theta = \frac{5\pi}{512}$$

IYGB - MATHEMATICAL METHODS 3 - PAPER A - QUESTION 9

a) • STARTING FROM THE GENERATING FUNCTION FOR LEGENDRE'S POLYNOMIALS

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right]$$

• LETTING $x=1$, IN THE ABOVE RELATIONSHIP

$$\Rightarrow (1-2t+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(1)\right]$$

$$\Rightarrow \left[(1-t)^2\right]^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(1)\right]$$

$$\Rightarrow (1-t)^{-1} = \sum_{n=0}^{\infty}\left[t^n P_n(1)\right]$$

$$\Rightarrow 1+t+t^2+t^3+\dots = P_0(1)+tP_1(1)+t^2P_2(1)+t^3P_3(1)+\dots$$

• HENCE THE RESULT FOLLOWS BY COMPARISON $P_n(1)=1$

b) • STARTING WITH LEGENDRE'S EQUATION, WHOSE SOLUTION IS $y=P_n(x)$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

$$\Rightarrow (1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$$\Rightarrow (1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0$$

• LET $x=1$ & NOTE FROM PART (a), $P_n(1)=1$

$$\Rightarrow -2P_n'(1) + n(n+1) = 0$$

$$\Rightarrow P_n'(1) = \frac{1}{2}n(n+1)$$

AS REQUIRED

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- STARTING BY THE DIFFERENTIAL EQUATION

$$\frac{d^2x}{dt^2} = a^2x$$

WITH GENERAL SOLUTION

$$x = A\cosh at + B\sinh at$$
$$\dot{x} = Aa\sinh at + Ba\cosh at$$

- PICK INITIAL CONDITIONS FOR EACH CASE.

$t=0,\ x=1,\ \dot{x}=0$

$\Longrightarrow x = \cosh at$

$\Longrightarrow \dot{x} = a\sinh at$

$t=0,\ x=0,\ \dot{x}=a$

$\Longrightarrow x = \sinh at$

$\Longrightarrow \dot{x} = a\cosh at$

- TAKING THE LAPLACE TRANSFORM OF THE O.D.E

$$\Longrightarrow \ddot{x} = a^2x$$
$$\Longrightarrow s^2\bar{x} - sx_0 - \dot{x}_0 = a^2\bar{x}$$
$$\Longrightarrow (s^2-a^2)\bar{x} = sx_0 + \dot{x}_0$$
$$\Longrightarrow \bar{x} = \frac{sx_0+\dot{x}_0}{s^2-a^2}$$

$$\Longrightarrow \bar{x} = \frac{s}{s^2-a^2}$$

$$\Longrightarrow \mathcal{L}[\cosh at] = \frac{s}{s^2-a^2}$$

$$\Longrightarrow \bar{x} = \frac{a}{s^2-a^2}$$

$$\Longrightarrow \mathcal{L}[\sinh at] = \frac{a}{s^2-a^2}$$

IYGB − MATHEMATICAL METHODS 3 − PAPER A − QUESTION 11

- START BY THE DEFINITION OF THE FOURIER TRANSFORM

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\left(Ae^{-\alpha x^2}\right)dx$$

$$= \frac{A}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}e^{-\alpha x^2}\,dx = \frac{2A}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-\alpha x^2}\cos kx\,dx$$

(EVEN INTEGRAND)

- THIS INTEGRAL CAN BE DONE BY DIFFERENTIATION UNDER THE INTEGRAL SIGN, WHICH WILL THEN ALLOW INTEGRATION BY PARTS

$$\Rightarrow I = \int_0^{\infty} e^{-\alpha x^2}\cos kx\,dx$$

$$\Rightarrow \frac{\partial I}{\partial k} = \frac{\partial}{\partial k}\int_0^{\infty} e^{-\alpha x^2}\cos kx\,dx = \int_0^{\infty} e^{-\alpha x^2}\frac{\partial}{\partial k}(\cos kx)\,dx$$

$$\Rightarrow \frac{\partial I}{\partial k} = \int_0^{\infty}(-x\sin kx)e^{-\alpha x^2}\,dx = \int_0^{\infty} -xe^{-\alpha x^2}\sin kx\,dx$$

- FOLLOWING BY INTEGRATION BY PARTS

$\sin kx$	$k\cos kx$
$\frac{1}{2\alpha}e^{-\alpha x^2}$	$-xe^{-\alpha x^2}$

$$\Rightarrow \frac{\partial I}{\partial k} = \left[\cancel{\frac{1}{2\alpha}e^{-\alpha x^2}\sin kx}\right]_0^{\infty} - \frac{k}{2\alpha}\int_0^{\infty} e^{-\alpha x^2}\cos kx\,dx$$

$$\Rightarrow \frac{\partial I}{\partial k} = -\frac{k}{2\alpha}I$$

IYGB - MATHEMATICAL METHODS 3 - PAPER A - QUESTION 11

- SOLVING THE O.D.E FOR I

$$\Rightarrow \frac{dI}{dk} = \frac{-k}{2\alpha} I$$

$$\Rightarrow \int \frac{dI}{I} = \int -\frac{k}{2\alpha}\, dk$$

$$\Rightarrow \ln I = -\frac{k^2}{4\alpha} + C$$

$$\Rightarrow I = Be^{-\frac{k^2}{4\alpha}} \qquad (B \text{ ARBITRARY})$$

- HENCE SO FAR WE HAVE

$$I = \int_0^\infty e^{-\alpha x^2} \cos kx \, dx = Be^{-\frac{k^2}{4\alpha}}$$

LET $k=0$

$$\int_0^\infty e^{-\alpha x^2}\, dx = B$$

USING A SUBSTITUTION

$$y^2 = \alpha x^2$$
$$y = \sqrt{\alpha}\, x$$
$$dy = \sqrt{\alpha}\, dx$$

LIMITS UNCHANGED

$$B = \int_0^\infty e^{-y^2} \frac{dy}{\sqrt{\alpha}}$$

$$B = \frac{1}{\sqrt{\alpha}} \int_0^\infty e^{-y^2}\, dy$$

$$B = \frac{1}{\sqrt{\alpha}} \left(\frac{\sqrt{\pi}}{2}\right)$$

- FINALLY WE HAVE

$$\hat{f}(k) = \frac{2A}{\sqrt{2\pi}} I = \frac{\cancel{2}A}{\sqrt{2\pi}} \left(\frac{1}{\sqrt{\alpha}} \frac{\sqrt{\pi}}{\cancel{2}}\right) e^{-\frac{k^2}{4\alpha}} = \frac{A}{\sqrt{2\alpha}} e^{-\frac{k^2}{4\alpha}}$$

ALTERNATIVE APPROACH BY CONTOUR INTEGRATION

- STARTING BY THE DEFINITION OF THE FOURIER TRANSFORM

$$\hat{f}(k) = \frac{A}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha x^2} e^{-ikx} \, dx = \frac{A}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha\left[x^2 + \frac{ik}{\alpha}x\right]} \, dx$$

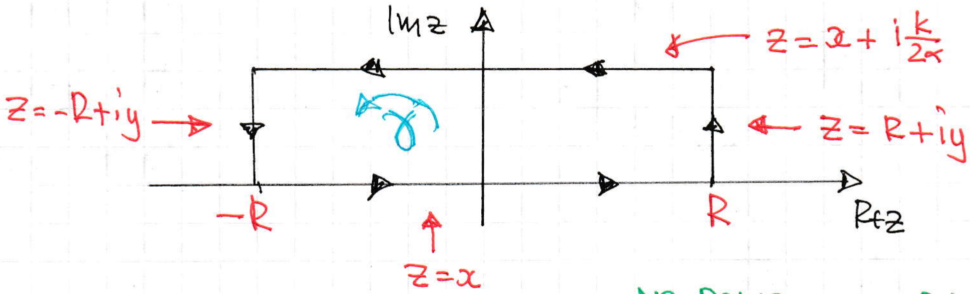
- COMPLETING THE SQUARE IN THE EXPONENT, IN x

$$x^2 + \frac{ik}{\alpha}x = \left(x + \frac{ik}{2\alpha}\right)^2 - \left(\frac{ik}{2\alpha}\right)^2 = \left(x + \frac{ik}{2\alpha}\right)^2 + \frac{k^2}{4\alpha^2}$$

- RETURNING TO THE TRANSFORM

$$\hat{f}(k) = \frac{A}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha\left(x + \frac{ik}{2\alpha}\right)^2} \times e^{-\frac{k^2}{4\alpha}} \, dx = \frac{A e^{-\frac{k^2}{4\alpha}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha\left(x + \frac{k}{2\alpha}i\right)^2} \, dx$$

- NOW CONSIDER $f(z) = e^{-\alpha z^2}$ OVER THE CONTOUR SHOWN BELOW



NO POLES INSIDE γ

$$\Rightarrow \int_{\gamma} e^{-\alpha z^2} \, dz = 0$$

$$\Rightarrow \int_{-R}^{R} e^{-\alpha x^2} \, dx + \int_{y=0}^{y=\frac{k}{2\alpha}} e^{-\alpha(R+iy)^2} (i\,dy) + \int_{R}^{-R} e^{-\alpha\left(x + \frac{ik}{2\alpha}\right)^2} \, dx + \int_{y=\frac{k}{2\alpha}}^{y=0} e^{-\alpha(-R+iy)^2} (i\,dy) = 0$$

$z = x$	$z = R + iy$	$z = x + i\frac{k}{2\alpha}$	$z = -R + iy$
$dz = dx$	$dz = i\,dy$	$dz = dx$	$dz = i\,dy$

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- NOW AS $R \to \infty$, THE 2ND & 4TH INTEGRALS VANISH BECAUSE OF THE TERM $-\alpha R^2$ IN THE EXPONENT

$$\Longrightarrow \int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx + \int_{\infty}^{-\infty} e^{-\alpha\left(x+i\frac{k}{2\alpha}\right)^2}\,dx = 0$$

$$\Longrightarrow \int_{-\infty}^{\infty} e^{-\alpha\left(x+i\frac{k}{2\alpha}\right)^2}\,dx = \int_{-\infty}^{\infty} e^{-\alpha x^2}$$

- RETURNING TO THE TRANSFORM

$$\hat{f}(k) = \frac{Ae^{-\frac{k^2}{4\alpha}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha\left(x+i\frac{k}{2\alpha}\right)^2}\,dx = \frac{A\,e^{-\frac{k^2}{4\alpha}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx$$

$$\hat{f}(k) = \frac{Ae^{-\frac{k^2}{4\alpha}}}{\sqrt{2\pi}} \times \frac{1}{\sqrt{\alpha}} \int_{-\infty}^{\infty} e^{-y^2}\,dy$$

LET $y = \sqrt{\alpha}\,x$

$dy = \sqrt{\alpha}\,dx$

$dx = \frac{1}{\sqrt{\alpha}}\,dy$

LIMITS UNCHANGED

$$\hat{f}(k) = \frac{Ae^{-\frac{k^2}{4\alpha}}}{\sqrt{2\cancel{\pi}}} \times \frac{1}{\sqrt{\alpha}} \times \cancel{\sqrt{\pi}}$$

$$\hat{f}(k) = \frac{A}{\sqrt{2\alpha}} e^{-\frac{k^2}{4\alpha}}$$

AS BEFORE