IYGB - MATHEMATICAL METHODS 3 - PAPER C - QUESTION 1

MANIPULATE THE FUNCTION AS FOLLOWS

$$f(z) = \frac{1}{z+4} = \frac{1}{z\left(1+\frac{4}{z}\right)} = \frac{1}{z}\left(1+\frac{4}{z}\right)^{-1}$$

EXPANDING BINOMIALLY NOTING THAT THE RADIUS OF CONVERGENCE MUST BE

$$\left|\frac{4}{z}\right| < 1$$

$$|z| > 4$$

HENCE WE OBTAIN

$$f(z) = \frac{1}{z}\left[1 - \frac{4}{z} + \frac{16}{z^2} - \frac{64}{z^3} + \frac{256}{z^4} - \dots\right]$$

$$f(z) = \frac{1}{z} - \frac{4}{z^2} + \frac{16}{z^3} - \frac{64}{z^4} + \frac{256}{z^5} - \dots$$

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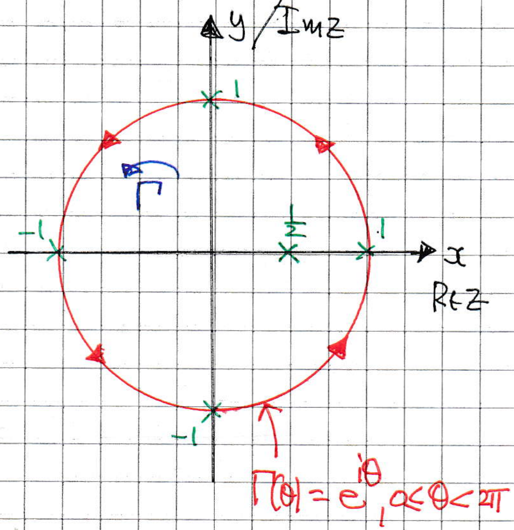
START BY USING THE CONTOUR $|z|=1$ OR $z=e^{i\theta}$ $0 \le \theta < 2\pi$

- $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \frac{1}{2}\left(z + \frac{1}{z}\right)$
- $$\begin{aligned} 4\cos\theta - 5 &= 4 \times \frac{1}{2}\left(z + \frac{1}{z}\right) - 5 \\ &= 2z + \frac{2}{z} - 5 \\ &= \frac{1}{z}\left(2z^2 - 5z + 2\right) \\ &= \frac{1}{z}(2z-1)(z-2) \end{aligned}$$



TRANSFORMING THE INTEGRAL

$$\int_0^{2\pi} \frac{1}{4\cos\theta - 5}\,d\theta = \int_\Gamma \frac{1}{\frac{1}{z}(2z-1)(z-2)}\left(-\frac{i}{z}\,dz\right)$$

$$= \int_\Gamma \frac{-i}{(2z-1)(z-2)}\,dz$$

$$\boxed{\begin{aligned} z &= e^{i\theta} \\ dz &= ie^{i\theta}\,d\theta \\ dz &= iz\,d\theta \\ d\theta &= \frac{dz}{iz} \\ d\theta &= -\frac{i}{z}\,dz \end{aligned}}$$

NOW THE INTEGRAND HAS SIMPLE POLES AT $z=2$ & $z=\frac{1}{2}$ OF WHICH ONLY THE ONE AT $z=\frac{1}{2}$ IS INSIDE Γ - FIND RESIDUE

$$\lim_{z \to \frac{1}{2}}\left[\left(z-\tfrac{1}{2}\right)f(z)\right] = \lim_{z \to \frac{1}{2}}\left[\left(z-\tfrac{1}{2}\right) \times \frac{-i}{(2z-1)(z-2)}\right]$$

$$= \lim_{z \to \frac{1}{2}}\left[\cancel{\left(z-\tfrac{1}{2}\right)} \times \frac{-i}{2\cancel{\left(z-\tfrac{1}{2}\right)}(z-2)}\right]$$

$$= \lim_{z \to \frac{1}{2}}\left[\frac{-i}{2(z-2)}\right]$$

$$= \frac{1}{3}i$$

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BY THE RESIDUE THEOREM WE HAVE

$$\int_{\Gamma} f(z)\,dz = 2\pi i \times \sum(\text{RESIDUES INSIDE } \Gamma)$$

$$\int_{\Gamma} \frac{-i}{(2z-1)(z-2)}\,dz = 2\pi i \times \frac{1}{3}i$$

$$\int_{0}^{2\pi} \frac{1}{4\cos\theta - 5}\,d\theta = -\frac{2\pi}{3}$$

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ASSUME A SOLUTION OF THE FORM $\quad y = \sum_{r=0}^{\infty} a_r x^{r+k}, \quad a_0 \neq 0, \quad k \in \mathbb{R}$

$$\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (r+k) x^{r+k-1}$$

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (r+k)(r+k-1) x^{r+k-2}$$

TIDY THE O.D.E. AND SUBSTITUTE IN

$$2x^2 \frac{d^2y}{dx^2} + (2x^2 - x)\frac{dy}{dx} + y = 0$$

$$2x^2 \frac{d^2y}{dx^2} + 2x^2 \frac{dy}{dx} - x\frac{dy}{dx} + y = 0$$

$$\sum_{r=0}^{\infty} 2a_r (r+k)(r+k-1) x^{r+k} + \sum_{r=0}^{\infty} 2a_r (r+k) x^{r+k+1} - \sum_{r=0}^{\infty} a_r (r+k) x^{r+k} + \sum_{r=0}^{\infty} a_r x^{r+k} = 0$$

WHEN $r=0$ THE LOWEST POWER OF x IS x^r AND THE HIGHEST IS x^{k+1}

PULL OUT THE LOWEST POWER OF x OUT OF THE SUMMATIONS

$$2a_0 k(k-1)x^k + \sum_{r=1}^{\infty} 2a_r (r+k)(r+k-1) x^{r+k} + \sum_{r=0}^{\infty} 2a_r (r+k) x^{r+k+1} - a_0 k x^k$$

$$- \sum_{r=1}^{\infty} a_r (r+k) x^{r+k} + a_0 x^k + \sum_{r=1}^{\infty} a_r x^{r+k} = 0$$

FORM AN INDICIAL EQUATION FROM THE EXPRESSIONS "PULLED OUT" (LOWEST POWER)

$$\Rightarrow 2a_0 k(k-1)x^k - a_0 k x^k + a_0 x^k = 0$$

$$\Rightarrow [2k(k-1) - k + 1] a_0 x^k = 0$$

$$\Rightarrow [2k(k-1) - (k-1)] = 0 \qquad a_0 \neq 0$$

$$\Rightarrow (k-1)(2k-1) = 0$$

$$\Rightarrow k = \begin{cases} 1 \\ \frac{1}{2} \end{cases}$$

DISTINCT ROOTS AND NOT DIFFERING BY AN INTEGER

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THE REST OF THE POWERS IN THE SUMMATIONS MUST ALSO EQUAL ZERO.

ADJUST THE SUMMATIONS SO THEY ALL START FROM $r=0$

$$\Rightarrow \sum_{r=1}^{\infty} 2a_r(r+k)(r+k-1)x^{r+k} + \sum_{r=0}^{\infty} 2a_r(r+k)x^{r+k+1} - \sum_{r=1}^{\infty} a_r(r+k)x^{r+k} + \sum_{r=1}^{\infty} a_r x^{r+k} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} 2a_{r+1}(r+k+1)(r+k)x^{r+k+1} + \sum_{r=0}^{\infty} 2a_r(r+k)x^{r+k+1} - \sum_{r=0}^{\infty} a_{r+1}(r+k+1)x^{r+k+1} + \sum_{r=0}^{\infty} a_{r+1}x^{r+k+1} = 0$$

$$\Rightarrow \left[2a_{r+1}(r+k+1)(r+k) + 2a_r(r+k) - a_{r+1}(r+k+1) + a_{r+1}\right]x^{r+k+1} = 0$$

$$\Rightarrow 2a_{r+1}(r+k+1)(r+k) - a_{r+1}(r+k+1) + a_{r+1} = -2a_r(r+k)$$

$$\Rightarrow \left[2(r+k+1)(r+k) - (r+k+1) + 1\right]a_{r+1} = -2a_r(r+k)$$

LET $A = r+k$

$$\begin{aligned} 2(A+1)A - (A+1) + 1 &= 2A^2 + 2A - A - 1 + 1 \\ &= 2A^2 + A \\ &= A(2A+1) \end{aligned}$$

$$\Rightarrow (r+k)(2r+2k+1)a_{r+1} = -2a_r(r+k)$$

$$\Rightarrow (2r+2k+1)a_{r+1} = -2a_r$$

$$\Rightarrow \boxed{a_{r+1} = -\frac{2}{2r+2k+1}a_r}$$

NOW IF $k=1$ THIS RECURSIVE FORMULA BECOMES

$$\boxed{a_{r+1} = -\frac{2}{2r+3}a_r}$$

- IF $r=0$ $\quad a_1 = -\frac{2}{3}a_0$
- IF $r=1$ $\quad a_2 = -\frac{2}{5}a_1 = \frac{2\times2}{3\times5}a_0$
- IF $r=2$ $\quad a_3 = -\frac{2}{7}a_2 = -\frac{2\times2\times2}{3\times5\times7}a_0$
- IF $r=3$ $\quad a_4 = -\frac{2}{9}a_3 = \frac{2\times2\times2\times2}{3\times5\times7\times9}a_0$ ETC

THUS WE NOW HAVE IF $k=1$

$$\Rightarrow y_1 = a_0x + a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5 + \dots$$

$$\Rightarrow y_1 = x\left[a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots\right]$$

$$\Rightarrow y_1 = x\left[a_0 - \frac{2}{3}a_0x + \frac{2\times2}{3\times5}a_0x^2 - \frac{2\times2\times2}{3\times5\times7}a_0x^3 + \frac{2\times2\times2\times2}{3\times5\times7\times9}a_0x^4 + \dots\right]$$

$$\Rightarrow y_1 = a_0x\left[1 - \frac{2}{3}x + \frac{2^2}{3\times5}x^2 - \frac{2^3}{3\times5\times7}x^3 + \frac{2^4}{3\times5\times7\times9}x^4 + \dots\right]$$

$$\Rightarrow y_1 = a_0x\left[1 - \frac{2\times2}{3\times2}x + \frac{2^2\times(4\times2)x^2}{5\times4\times3\times2} - \frac{2^3\times(6\times4\times2)x^3}{7\times6\times5\times4\times3\times2} + \frac{2^4\times(8\times6\times4\times2)x^4}{9\times8\times7\times6\times5\times4\times3\times2} + \dots\right]$$

$$\Rightarrow y_1 = a_0x\left[1 - \frac{2\times2\times1!}{3!}x + \frac{2^2\times2^2\times2!}{5!}x^2 - \frac{2^3\times2^3\times3!\,x^3}{7!} + \frac{2^4\times2^4\times4!\,x^4}{9!} + \dots\right]$$

$$\Rightarrow y_1 = a_0x\left[1 - \frac{2^2\times1!}{3!}x^1 + \frac{2^4\times2!}{5!}x^2 - \frac{2^6\times3!}{7!}x^3 + \frac{2^8\times4!}{9!}x^4 - \dots\right]$$

$$\Rightarrow y_1 = a_0x\sum_{r=0}^{\infty}\left[\frac{2^{2r}\times r!}{(2r+1)!}(-1)^r x^r\right]$$

$$\Rightarrow y_1 = a_0x\sum_{r=0}^{\infty}\frac{4^r(-1)^r x^r\, r!}{(2r+1)!}$$

$$\Rightarrow y_1 = Ax\sum_{r=0}^{\infty}\frac{r!}{(2r+1)!}(-4x)^r$$

NOW IF $k=\frac{1}{2}$ THE RECURRENCE RELATION YIELDS

$$\boxed{a_{r+1} = -\frac{a_r}{r+1}}$$

- $r=0$ $\quad a_1 = -a_0$
- $r=1$ $\quad a_2 = -\frac{1}{2}a_1 = \frac{1}{2}a_0$
- $r=2$ $\quad a_3 = -\frac{1}{3}a_2 = -\frac{1}{2\times3}a_0$
- $r=3$ $\quad a_4 = -\frac{1}{4}a_3 = \frac{1}{2\times3\times4}a_0$ ETC

HENCE WE NOW OBTAIN

$$y_2 = a_0x^{\frac{1}{2}} + a_1x^{\frac{3}{2}} + a_2x^{\frac{5}{2}} + a_3x^{\frac{7}{2}} + \dots$$

$$y_2 = x^{\frac{1}{2}}\left[a_0 + a_1x + a_2x^2 + a_3x^3 + \dots\right]$$

$$y_2 = x^{\frac{1}{2}}\left[a_0 - a_0x + \frac{1}{2}a_0x^2 - \frac{1}{2\times3}a_0x^3 + \frac{1}{2\times3\times4}a_0x^4 - \dots\right]$$

$$y_2 = a_0x^{\frac{1}{2}}\left[1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^5 - \dots\right]$$

$$y_2 = Bx^{\frac{1}{2}}e^{-x}$$

THUS THE GENERAL SOLUTION IS

$$\boxed{y = y_1 + y_2 = Ax\sum_{r=0}^{\infty}\frac{r!}{(2r+1)!}(-4x)^r + Bx^{\frac{1}{2}}e^{-x}}$$

IYGB - MATHEMATICAL METHODS 3 - PAPER C - QUESTION 4

LOOKING AT $\mathcal{L}^{-1}\left[\ln\left(1+\frac{1}{s^2}\right)\right]$ WE SUSPECT THAT THIS IS THE LAPLACE TRANSFORM OF $\frac{f(t)}{t}$ FOR SOME $f(t)$

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^{\infty} \bar{f}(x)\,dx = \bar{g}(s)$$

HERE $\bar{g}(s)$ IS $\ln\left(1+\frac{1}{s^2}\right)$

$$\frac{d}{ds}\left(\bar{g}(s)\right) = \frac{d}{ds}\left(\ln\left(1+\frac{1}{s^2}\right)\right) = \frac{d}{ds}\left[\ln\left(\frac{s^2+1}{s^2}\right)\right]$$

$$= \frac{d}{ds}\left[\ln(s^2+1) - \ln s^2\right] = \frac{d}{ds}\left[\ln(s^2+1) - 2\ln s\right]$$

$$= \frac{2s}{s^2+1} - \frac{2}{s} = 2\left(\frac{s}{s^2+1}\right) - 2\left(\frac{1}{s}\right)$$

WE RECOGNIZE THIS AS STANDARD RESULTS

$$2\mathcal{L}[\cos t] = 2\left(\frac{s}{s^2+1}\right)$$

$$2\mathcal{L}[1] = 2\times\frac{1}{s}$$

HENCE WE HAVE NOTING THE REVERSAL OF THE SIGNS DUE TO THE INTEGRATION LIMITS

$$\mathcal{L}^{-1}\left[\ln\left(1+\frac{1}{s^2}\right)\right] = \frac{2(1-\cos t)}{t}$$

QUICK CHECK

$$\mathcal{L}\left[\frac{2(1-\cos t)}{t}\right] = \int_s^{\infty} \mathcal{L}\left[2(1-\cos t)\right]ds = \int_s^{\infty} \mathcal{L}\left[2-2\cos t\right]ds = \int_s^{\infty} \frac{2}{s} - \frac{2s}{s^2+1}\,ds$$

$$= \left[2\ln s - \ln(s^2+1)\right]_s^{\infty} = \left[\ln s^2 - \ln(s^2+1)\right]_s^{\infty} = \left[\ln\left(\frac{s^2}{s^2+1}\right)\right]_s^{\infty}$$

$$= \cancel{\ln 1} - \ln\left(\frac{s^2}{s^2+1}\right) = -\ln\left(\frac{s^2}{s^2+1}\right) = \ln\left(\frac{s^2+1}{s^2}\right) = \ln\left(1+\frac{1}{s^2}\right)$$

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START BY THE RECURSIVE PROPERTY OF THE GAMMA FUNCTION

$$\Gamma(n+\tfrac{1}{2}) = (n-\tfrac{1}{2})(n-\tfrac{3}{2})(n-\tfrac{5}{2})\dots \tfrac{7}{2}\times\tfrac{5}{2}\times\tfrac{3}{2}\times\tfrac{1}{2}\,\Gamma(\tfrac{1}{2})$$

$$= \tfrac{1}{2}(2n-1)\times\tfrac{1}{2}(2n-3)\times\tfrac{1}{2}(2n-5)\dots(\tfrac{1}{2}\times 7)(\tfrac{1}{2}\times 5)(\tfrac{1}{2}\times 3)(\tfrac{1}{2}\times 1)\times\sqrt{\pi}$$

$$= \left(\tfrac{1}{2}\right)^n(2n-1)(2n-3)(2n-5)\dots 7\times 5\times 3\times 1\times\sqrt{\pi}$$

$$= \frac{1}{2^n}\,\frac{(2n-1)(2n-2)(2n-3)(2n-4)(2n-5)\dots 7\times 6\times 5\times 4\times 3\times 2\times 1\times\sqrt{\pi}}{(2n-2)(2n-4)(2n-6)\dots 6\times 4\times 2}$$

$$= \frac{1}{2^n}\times\frac{(2n-1)!\,\sqrt{\pi}}{2(n-1)\times 2(n-2)\times 2(n-3)\times\dots\times(2\times 3)\times(2\times 2)\times(2\times 1)}$$

$$= \frac{1}{2^n}\times\frac{\Gamma(2n)\sqrt{\pi}}{2^{n-1}\times(n-1)(n-2)(n-3)\dots\times 3\times 2\times 1}$$

$$= \frac{1}{2^n}\times\frac{\Gamma(2n)\sqrt{\pi}}{2^{n-1}(n-1)!}$$

$$= \frac{\Gamma(2n)\sqrt{\pi}}{2^{2n-1}(n-1)!}$$

AS REQUIRED

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STARTING WITH THE GENERATING FUNCTION

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right]$$

DIFFERENTIATE WITH RESPECT TO t

$$\Rightarrow -\frac{1}{2}(1-2xt+t^2)^{-\frac{3}{2}}(-2x+2t) = \sum_{n=0}^{\infty}\left[nt^{n-1}P_n(x)\right]$$

$$\Rightarrow (x-t)(1-2xt+t^2)^{-\frac{3}{2}} = \sum_{n=0}^{\infty}\left[nt^{n-1}P_n(x)\right]$$

$$\Rightarrow (x-t)(1-2xt+t^2)^{-\frac{3}{2}}(1-2xt+t^2) = (1-2xt+t^2)\sum_{n=0}^{\infty}\left[nt^{n-1}P_n(x)\right]$$

$$\Rightarrow (x-t)(1-2xt+t^2)^{-\frac{1}{2}} = (1-2xt+t^2)\sum_{n=0}^{\infty}\left[nt^{n-1}P_n(x)\right]$$

$$\Rightarrow (x-t)\sum_{n=0}^{\infty}\left[t^n P_n(x)\right] = (1-2xt+t^2)\sum_{n=0}^{\infty}\left[nt^{n-1}P_n(x)\right]$$

$$\Rightarrow \sum_{n=0}^{\infty}\left[t^n xP_n(x) - t^{n+1}P_n(x)\right] = \sum_{n=0}^{\infty}\left[nt^{n-1}P_n(x) - 2xnt^nP_n(x) + nt^{n+1}P_n(x)\right]$$

EQUATING COEFFICIENTS OF t, SAY $[t^n]$

$$\Rightarrow xP_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x)$$

$$\Rightarrow 0 = (n+1)P_{n+1}(x) - 2xnP_n(x) - xP_n(x) + (n-1)P_{n-1}(x) + P_{n-1}(x)$$

$$\Rightarrow 0 = (n+1)P_{n+1}(x) - (2xn+x)P_n(x) + nP_{n-1}(x)$$

$$\Rightarrow (n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$$

AS REQUIRED

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START WITH A SUBSTITUTION FIRST

$$u = \frac{1}{x+1} \iff x+1 = \frac{1}{u} \qquad \left(x = \frac{1}{u} - 1\right)$$

$$\implies dx = -\frac{1}{u^2}\,du$$

$$x = 0 \longmapsto u = 1$$

$$x = \infty \longmapsto u = 0$$

TRANSFORMING THE INTEGRAL INCLUDING LIMITS

$$\int_1^0 \frac{1}{\underbrace{\sqrt{\frac{1}{u}-1}}_{\sqrt{x}}\ \underbrace{\frac{1}{u}}_{x+1}} \underbrace{\left(-\frac{1}{u^2}\,du\right)}_{dx} = \int_1^0 \frac{-1}{\sqrt{\frac{1-u}{u}}\ \frac{1}{u}\ u^2}\,du$$

$$= \int_0^1 \frac{+1}{\frac{\sqrt{1-u}}{\sqrt{u}} \times u}\,du$$

$$= \int_0^1 \frac{1}{(1-u)^{\frac{1}{2}}(u^{\frac{1}{2}})}\,du$$

$$= \int_0^1 (1-u)^{-\frac{1}{2}}u^{-\frac{1}{2}}\,du$$

NOW USING THE BETA FUNCTION DEFINITION

$$\dots = \int_0^1 (1-u)^{\frac{1}{2}-1}u^{\frac{1}{2}-1}\,du = B\left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

SWITCHING INTO GAMMA FUNCTIONS FOR THE FINAL EVALUATION

$$= \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(1)} = \frac{\sqrt{\pi}\,\sqrt{\pi}}{0!} = \frac{\pi}{1} = \pi$$

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IT IS BEST TO WORK WITH EXPONENTIALS IN THIS QUESTION

$$f(z) = \frac{ze^{kz}}{z^4+1} \quad \text{HAS SIMPLE POLES AT:}$$

$$z^4 = -1 = e^{i(\pi + 2n\pi)} = e^{i\pi(2n+1)}, \quad n=0,1,2,3$$

$$z = e^{i\frac{\pi}{4}(2n+1)}, \quad n=0,1,2,3 \quad (\text{OR } -2,-1,0,1)$$

$$z = e^{i\frac{\pi}{4}},\ e^{i\frac{3\pi}{4}},\ e^{-i\frac{\pi}{4}},\ e^{-i\frac{3\pi}{4}}$$

CALCULATE THESE RESIDUES USING A GENERAL METHOD — LET A POLE BE AT $z = z_0$

$$\text{Res}(f; z_0) = \lim_{z \to z_0}\left[\frac{(z-z_0)ze^{kz}}{z^4+1}\right]$$

THIS MUST PRODUCE "ZERO OVER ZERO" AS $z-z_0$ MUST BE A FACTOR OF THE DENOMINATOR SO WE PROCEED BY L'HOSPITAL RULE

$$\text{Res}(f; z_0) = \lim_{z \to z_0}\left[\frac{\frac{d}{dz}\left[(z-z_0)ze^{kz}\right]}{\frac{d}{dz}(z^4+1)}\right] = \lim_{z \to z_0}\left[\frac{ze^{kz} + (z-z_0)e^{kz} + (z-z_0)kze^{kz}}{4z^3}\right]$$

$$\text{Res}(f; z_0) = \frac{z_0 e^{kz_0}}{4z_0^3} = \boxed{\frac{e^{kz_0}}{4z_0^2}}$$

NEXT OBTAIN THE RESIDUE AT EACH OF THE FOUR POLES

$$z_0 = e^{i\frac{\pi}{4}} \quad \text{GIVES} \quad \frac{e^{ke^{i\frac{\pi}{4}}}}{4e^{i\pi/2}} = \frac{e^{k\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)}}{4i}$$

$$z_0 = e^{-i\frac{\pi}{4}} \quad \text{GIVES} \quad \frac{e^{ke^{-i\frac{\pi}{4}}}}{4e^{-i\pi/2}} = \frac{e^{k\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)}}{-4i}$$

$$z_0 = e^{\frac{3\pi}{4}i} \quad \text{GIVES} \quad \frac{e^{ke^{i\frac{3\pi}{4}}}}{4e^{i\frac{3\pi}{2}}} = \frac{e^{k\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)}}{-4i}$$

$$z_0 = e^{-\frac{3\pi}{4}i} \quad \text{GIVES} \quad \frac{e^{ke^{-i\frac{3\pi}{4}}}}{4e^{-i\frac{3\pi}{2}}} = \frac{e^{k\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)}}{4i}$$

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ADDING THE 4 RESIDUES - LET $a=\frac{\sqrt{2}}{2}=\frac{1}{\sqrt{2}}$ FOR SIMPLICITY

$$\text{SUM OF 4 RESIDUES} = \frac{e^{k\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)}}{4i}+\frac{e^{k\left(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)}}{-4i}+\frac{e^{k\left(-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)}}{-4i}+\frac{e^{k\left(-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)}}{4i}$$

$$=\frac{1}{4i}\left[e^{k(a+ai)}-e^{k(a-ai)}-e^{k(-a+ai)}+e^{k(-a-ai)}\right]$$

$$=\frac{1}{4i}\left[e^{ak}e^{kai}-e^{ak}e^{-kai}+e^{-ak}e^{-iak}-e^{-ak}e^{aki}\right]$$

$$=\frac{1}{4i}\left[e^{ak}\left(e^{aki}-e^{-aki}\right)-e^{-ak}\left(e^{aki}-e^{-aki}\right)\right]$$

$$=\frac{1}{4i}\left[\left(e^{aki}-e^{-aki}\right)\left(e^{ak}-e^{-ak}\right)\right]$$

$$=\frac{1}{4i}\times 2\sinh(aki)\times 2\sinh(ak)$$

$$=\cancel{\frac{1}{4i}}\times \cancel{2i}\sin(ak)\times \cancel{2}\sinh(ak)$$

$$=\sin(ak)\sinh(ak)$$

$$=\sin\left(\frac{k}{\sqrt{2}}\right)\sinh\left(\frac{k}{\sqrt{2}}\right)$$

// AS REQUIRED

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STARTING WITH THE GENERATING FUNCTION FOR THE BESSEL FUNCTIONS OF THE FIRST KIND

$$\Rightarrow \quad e^{\frac{1}{2}x\left(t+\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right]$$

$$\Rightarrow \quad e^{\frac{1}{2}x\left(t+\frac{1}{t}\right)} e^{\frac{1}{2}y\left(t+\frac{1}{t}\right)} = \left[\sum_{k=-\infty}^{\infty}\left[t^k J_k(x)\right]\right]\left[\sum_{m=-\infty}^{\infty}\left[t^m J_m(y)\right]\right]$$

$$\Rightarrow \quad e^{\frac{1}{2}(x+y)\left(t+\frac{1}{t}\right)} = \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\left[J_k(x)\,J_m(y)\,t^{k+m}\right]$$

LET $n=k+m$ AND NOTE THAT SUMMATION LIMITS ARE UNCHANGED

$$\Rightarrow \quad e^{\frac{1}{2}(x+y)\left(t+\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\left[t^n J_{n-m}(x)\,J_m(y)\right]$$

$$\Rightarrow \quad e^{\frac{1}{2}(x+y)\left(t+\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty}\left[t^n \sum_{m=-\infty}^{\infty}\left[J_m(y)\,J_{n-m}(x)\right]\right]$$

$$\Rightarrow \quad \sum_{n=-\infty}^{\infty}\left[t^n J_n(x+y)\right] = \sum_{n=-\infty}^{\infty}\left[t^n\left[\sum_{m=-\infty}^{\infty}\left[J_m(y)\,J_{n-m}(x)\right]\right]\right]$$

$$\Rightarrow \quad J_n(x+y) = \sum_{m=-\infty}^{\infty}\left[J_m(y)\,J_{n-m}(x)\right]$$

AS REQUIRED

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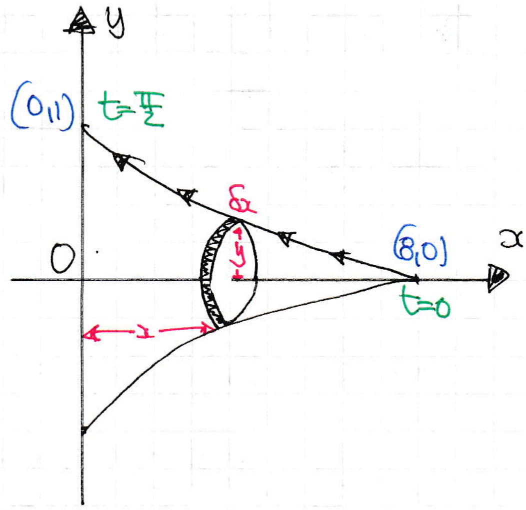
- START WITH THE DIAGRAM OPPOSITE



$$x = 8\cos^3 t, \quad y = \sin^3 t, \quad 0 \le t \le \frac{\pi}{2}$$

- THE MASS OF THE INFINITESIMAL DISC OF RADIUS y & THICKNESS δx IS GIVEN BY

$$\delta m = \pi \rho y^2 \delta x$$

$(\rho = \text{DENSITY})$

- THE "MOMENT" OF THIS INFINITESIMAL MASS, ABOUT THE y AXIS IS GIVEN BY

$$x\,\delta m = x(\pi \rho y^2 \delta x) = \pi \rho y^2 x\,\delta x$$

- SUMMING UP, TAKING LIMITS, WE OBTAIN

$$\Rightarrow M\bar{x} = \int_{x=0}^{8} \pi \rho y^2 x\,dx$$

$$\Rightarrow \bar{x}\int_{x=0}^{8} \cancel{\pi\rho}\, y^2\,dx = \int_{x=0}^{8} \cancel{\pi\rho}\, y^2 x\,dx$$

$$\Rightarrow \bar{x}\int_{t=\frac{\pi}{2}}^{0} \underbrace{(\sin^3 t)^2}_{y^2}\underbrace{(-24\cos^2 t\sin t\,dt)}_{dx} = \int_{t=\pi/2}^{0} \underbrace{(\sin^3 t)^2}_{y^2}\underbrace{(-24\cos^2 t\sin t\,dt)}_{dx}\underbrace{(8\cos^3 t)}_{x}$$

$$\Rightarrow \bar{x}\int_{0}^{\frac{\pi}{2}} \sin^7 t\cos^2 t\,dt = \int_{0}^{\pi/2} 8\sin^7 t\cos^5 t\,dt$$

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- EVALUATE USING BETA FUNCTIONS

$$\Longrightarrow \bar{x}\int_{0}^{\frac{\pi}{2}} 2(\sin t)^{2\times 4-1}(\cos t)^{2\times\frac{3}{2}-1}\,dt = 8\int_{0}^{\frac{\pi}{2}} 2(\sin t)^{2\times 4-1}(\cos t)^{2\times 3-1}\,dt$$

$$\Longrightarrow \bar{x}\, B\left(4,\tfrac{3}{2}\right) = 8\,B(4,3)$$

$$\Longrightarrow \bar{x}\,\frac{\cancel{\Gamma(4)}\,\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{11}{2}\right)} = \frac{8\,\cancel{\Gamma(4)}\,\Gamma(3)}{\Gamma(7)}$$

$$\Longrightarrow \bar{x} = \frac{8\,\Gamma(3)\,\Gamma\left(\frac{11}{2}\right)}{\Gamma(7)\,\Gamma\left(\frac{3}{2}\right)}$$

$$\Longrightarrow \bar{x} = \frac{8\times 2!}{6!}\times\frac{\frac{9}{2}\times\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\cancel{\Gamma\left(\frac{3}{2}\right)}}{\cancel{\Gamma\left(\frac{3}{2}\right)}}$$

$$\Longrightarrow \bar{x} = \frac{\cancel{16}}{720}\times\frac{9\times 7\times 5\times 3}{\cancel{16}}$$

$$\Longrightarrow \bar{x} = \frac{21}{16}$$

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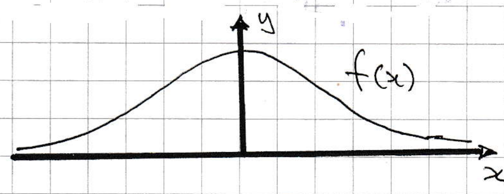
BY THE INTEGRAL DEFINITION OF FOURIER TRANSFORM

$$f(x) = \frac{1}{(a^2+x^2)^2}, \quad a>0$$



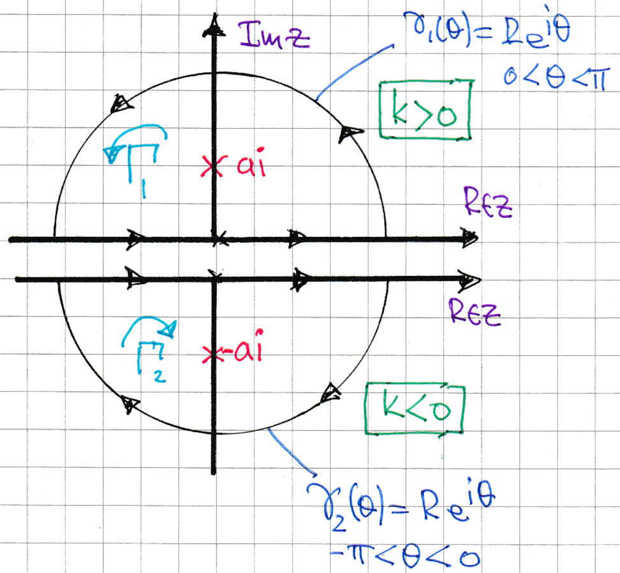
$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx$$

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{\cos kx - i \sin kx}{(a^2+x^2)^2}\, dx$$

ONLY EVEN PART SURVIVES

$$\hat{f}(k) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \frac{\cos kx}{(a^2+x^2)^2}\, dx$$

PROCEED BY CONTOUR INTEGRATION - USE DIFFERENT CONTOUR DEPENDING ON THE SIGN OF k



$f(z) = \dfrac{e^{ikz}}{(z^2+a^2)^2}$ HAS DOUBLE POLES AT $\pm ai$, ONE IN EACH OF THE CONTOURS SHOWN OPPOSITE

RESIDUE AT $z = ai$

$$\lim_{z \to ai} \left[\frac{d}{dz} \left[f(z)(z-ai)^2 \right] \right] = \lim_{z \to ai} \left[\frac{d}{dz} \left[\frac{e^{ikz}\cancel{(z-ai)^2}}{\cancel{(z-ai)^2}(z+ai)^2} \right] \right]$$

$$= \lim_{z \to ai} \left[\frac{d}{dz} \left(\frac{e^{ikz}}{(z+ai)^2} \right) \right]$$

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$$= \lim_{z \to ai} \left[\frac{ike^{ikz}(z+ai)^2 - 2(z+ai)e^{ikz}}{(z+ai)^4} \right]$$

$$= \lim_{z \to ai} \left[\frac{ike^{ikz}(z+ai) - 2e^{ikz}}{(z+ai)^3} \right] = \lim_{z \to ai} \left[\frac{e^{ikz}(ikz - ak - 2)}{(z+ai)^3} \right]$$

$$= \frac{e^{-ak}(-2ak-2)}{(2ai)^3} = \frac{-2e^{-ak}(ak+1)}{-8a^3 i} = \frac{(ak+1)e^{-ak}}{4a^3 i}$$

IN ALMOST ANALOGOUS FASHION THE RESIDUE AT $-ai$ FOR USE IN THE BOTTOM CONTOUR

$$\dots = \lim_{z \to -ai} \left[\frac{ike^{ikz}(z-ai) - 2e^{ikz}}{(z-ai)^3} \right]$$

$$= \lim_{z \to -ai} \left[\frac{e^{ikz}(ikz + ak - 2)}{(z-ai)^3} \right] = \frac{e^{ak}(2ak-2)}{(-2ai)^3}$$

$$= \frac{2(ak-1)e^{ak}}{8a^3 i} = \frac{(ak-1)e^{ak}}{4a^3 i}$$

- IF $k > 0$ USING THE "TOP" CONTOUR

$$\int_{\Gamma_1} f(z)\,dz = \int_{-R}^{R} \frac{e^{ikx}}{(x^2+a^2)^2}\,dx + \int_{\gamma_1} \frac{e^{ikz}}{(z^2+a^2)^2}\,dz = 2\pi i \left(\frac{(ak+1)e^{-ak}}{4a^3 i} \right)$$

- IF $k < 0$ USING THE "BOTTOM" CONTOUR

$$\int_{\Gamma_2} f(z)\,dz = \int_{-R}^{R} \frac{e^{ikx}}{(x^2+a^2)^2}\,dx + \int_{\gamma_2} \frac{e^{ikz}}{(z^2+a^2)^2}\,dz = -2\pi i \left(\frac{(ak-1)e^{ak}}{4a^3 i} \right)$$

CLOCKWISE PATH

THE INTEGRALS OVER THE ARCS γ_1 & γ_2 VANISH AS $R \to \infty$, AS BOTH SATISFY THE CONDITIONS OF JORDAN'S LEMMA, FOR THE CORRECT SIGN OF k IN EACH CONTOUR

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DEALING WITH EACH CASE SEPARATELY AS $R \to \infty$

- IF $k>0$

$$\int_{-\infty}^{\infty} \frac{e^{ikx}}{(x^2+a^2)^2}\,dx = \frac{\pi}{2a^3}(ak+1)e^{-ak}$$

$$\int_{0}^{\infty} \frac{2\cos kx}{(x^2+a^2)^2}\,dx = \frac{\pi e^{-ak}(ak+1)}{2a^3}$$

$$\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} \frac{2\cos kx}{(x^2+a^2)^2}\,dx = \sqrt{\frac{\pi}{2}}\,\frac{e^{-ak}(ak+1)}{a^3}$$

$$\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} \frac{\cos kx}{(x^2+a^2)^2}\,dx = \sqrt{\frac{\pi}{8}}\,\frac{e^{-ak}(ak+1)}{a^3}$$

- IF $k<0$

$$\int_{-\infty}^{\infty} \frac{e^{ikx}}{(x^2+a^2)^2}\,dx = \frac{\pi}{2a^3}(-ak+1)e^{ak}$$

$$\vdots$$

$$\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} \frac{\cos kx}{(x^2+a^2)^2}\,dx = \sqrt{\frac{\pi}{8}}\,\frac{e^{ak}(-ak+1)}{a^3}$$

COLLECTING THE RESULTS FOR ALL k

$$\hat{f}(k) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} \frac{\cos kx}{(x^2+a^2)^2}\,dx = \begin{cases} \sqrt{\frac{\pi}{8}}\,\frac{e^{-ak}(ak+1)}{a^3} & k>0 \\ \sqrt{\frac{\pi}{8}}\,\frac{e^{ak}(-ak+1)}{a^3} & k<0 \end{cases}$$

$$\hat{f}(k) = \sqrt{\frac{\pi}{8}}\,\frac{(1+a|k|)e^{-a|k|}}{a^3}$$