IYGB - MATHEMATICAL METHODS 4 - PAPER A - QUESTION 1

AS THIS IS A LINEAR FIRST ORDER P.D.E WITH ONLY ONE PARTIAL DERIVATIVE PRESENT WE CAN JUST SOLVE IT AS AN O.D.E WHERE THE OTHER INDEPENDENT VARIABLE IS TREATED AS A CONSTANT (x HERE)

$$z = f(x,y)$$

$$\frac{\partial z}{\partial y} + 2yz = xy^3$$

$$\text{I.F.} = e^{\int 2y\,dy} = e^{y^2}$$

$$\Longrightarrow \frac{\partial}{\partial y}\left(ze^{y^2}\right) = xy^3e^{y^2}$$

$$\Longrightarrow ze^{y^2} = \int xy^3e^{y^2}\,dy$$

$$\Longrightarrow ze^{y^2} = x\int y^2\left(ye^{y^2}\right)dy$$

BY PARTS (W.R.T y)

y^2	$2y$
$\frac{1}{2}e^{y^2}$	ye^{y^2}

$$\Longrightarrow ze^{y^2} = x\left[\frac{1}{2}y^2e^{y^2} - \int ye^{y^2}\,dy\right]$$

$$\Longrightarrow ze^{y^2} = x\left[\frac{1}{2}y^2e^{y^2} - \frac{1}{2}e^{y^2} + A(x)\right]$$

$$\Longrightarrow ze^{y^2} = \frac{1}{2}xy^2e^{y^2} - \frac{1}{2}xe^{y^2} + B(x)$$

$$\Longrightarrow z = \frac{1}{2}xy^2 - \frac{1}{2}x + B(x)e^{-y^2}$$

$$\Longrightarrow z(x,y) = \frac{1}{2}x(y^2-1) + B(x)e^{-y^2}$$

IYGB - MATHEMATICAL METHODS 4 - PAPER A - QUESTION 2

- TAKING THE FOURIER TRANSFORM OF THE P.D.E, I.E MULTIPLY BY $\frac{1}{\sqrt{2\pi}}e^{-ikx}$ AND INTEGRATE FROM $-\infty$ TO ∞, WITH RESPECT TO x

$$\Longrightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

$$\Longrightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\partial^2 \phi}{\partial x^2} e^{-ikx}\,dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\partial^2 \phi}{\partial y} e^{-ikx}\,dx = \cancel{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 0\, e^{-ikx}\,dx}$$

$$\Longrightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\partial^2 \phi}{\partial x^2} e^{-ikx} + \frac{\partial}{\partial y}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \phi(x,y) e^{-ikx}\right] = 0$$

- NOW THE FOURIER TRANSFORM OF DERIVATIVES IS

$$\boxed{\begin{aligned} \mathcal{F}\left[f'(x)\right] &= ik\,\mathcal{F}\left[f(x)\right] = ik\hat{f}(k) \\ \mathcal{F}\left[f''(x)\right] &= (ik)^2\,\mathcal{F}\left[f(x)\right] = -k^2\hat{f}(k) \end{aligned}}$$

- HENCE WE HAVE

$$\Longrightarrow -\hat{k}\,\hat{\phi}(k,y) + \frac{\partial}{\partial y^2}\left[\hat{\phi}(k,y)\right] = 0$$

- AS k IS A CONSTANT AS FAR AS y IS CONCERNED, THIS REDUCES TO A SIMPLE O.D.E

$$\frac{d^2\hat{\phi}}{dy^2} - k^2\hat{\phi} = 0 \quad \text{FOR } \hat{\phi} = \hat{\phi}(k,y)$$

IYGB - MATHEMATICAL METHODS 4 - PAPER A - QUESTION 3

a) • START BY PREPARING THE DERIVATIVES BY THE CHAIN RULE

$$u = x+y \qquad v = x-y$$

• $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}\cdot 1 + \frac{\partial z}{\partial v}\cdot 1 = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$

• $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u}\cdot 1 + \frac{\partial z}{\partial v}(-1) = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$

• THE P.D.E NOW BECOMES

$$\Longrightarrow \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 2z(x+y)$$

$$\Longrightarrow \left(\frac{\partial z}{\partial u} + \cancel{\frac{\partial z}{\partial v}}\right) + \left(\frac{\partial z}{\partial u} - \cancel{\frac{\partial z}{\partial v}}\right) = 2zu$$

$$\Longrightarrow 2\frac{\partial z}{\partial u} = 2zu$$

$$\Longrightarrow \frac{\partial z}{\partial u} = zu$$

• SOLVE BY SEPARATING VARIABLES — V IS TREATED AS A CONSTANT

$$\Longrightarrow \frac{1}{z}\,\partial z = u\,\partial u$$

$$\Longrightarrow \ln|z| = \tfrac{1}{2}u^2 + A(v)$$

$$\Longrightarrow z = e^{\frac{1}{2}u^2 + A(v)} = e^{\frac{1}{2}u^2} \times e^{A(v)} = B(v)e^{\frac{1}{2}u^2}$$

$$\Longrightarrow z(x,y) = f(x-y)\,e^{\frac{1}{2}(x+y)^2}$$

b) APPLYING THE BOUNDARY CONDITION

WHEN $x+y=1$ $\quad z(x,y)=x^2$

$$\Rightarrow z(x,y) = f(x-y)\,e^{\frac{1}{2}(x+y)^2}$$

$$\Rightarrow x^2 = f[x-(1-x)]\,e^{\frac{1}{2}(1)^2}$$

$$\Rightarrow x^2 = f(2x-1)\,e^{\frac{1}{2}}$$

NOW LET $w=2x-1 \iff x=\frac{1}{2}(w+1)$

$$\Rightarrow \frac{1}{4}(w+1)^2 = f(w)\,e^{\frac{1}{2}}$$

$$\Rightarrow f(w) = \frac{1}{4}e^{-\frac{1}{2}}(w+1)^2$$

$$\Rightarrow f(x-y) = \frac{1}{4}e^{-\frac{1}{2}}(x-y+1)^2$$

HENCE THE SPECIFIC SOLUTION IS

$$\boxed{z(x,y) = \frac{1}{4}(x-y+1)^2 e^{-\frac{1}{2}} \times e^{\frac{1}{2}(x+y)^2}}$$

$$\therefore z(1,0) = \frac{1}{4}(1-0+1)^2 e^{-\frac{1}{2}} \times e^{\frac{1}{2}(1+0)^2}$$

$$z(1,0) = e^{-\frac{1}{2}}\,e^{\frac{1}{2}}$$

$$\underline{z(1,0) = 1}$$

IYGB - MATHEMATICAL METHODS 4 - PAPER A - QUESTION 4

a) SOLVING $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2 z}{\partial t^2}$ FOR $z = z(x,t)$

SUBJECT TO THE INITIAL CONDITIONS $z(x,0) = F(x)$

$$\frac{\partial z}{\partial t}(x,0) = G(x)$$

- AUXILIARY EQUATION FOR A SECOND ORDER P.D.E IS

$\lambda^2 = \dfrac{1}{c^2}$ (KEEPING THE LAMDA ON $\dfrac{\partial z}{\partial x}$ "BIT")

$\lambda = \pm\dfrac{1}{c}$

GENERAL SOLUTION IS

$$z(x,t) = f\left(-\tfrac{1}{c}x + t\right) + g\left(\tfrac{1}{c}x + t\right)$$

$$z(x,t) = f(x-ct) + g(x+ct)$$

- APPLYING CONDITIONS

$z(x,0) = F(x)$

$f(x) + g(x) = F(x)$

↓ DIFFERENTIATE W.R.T x

$f'(x) + g'(x) = F'(x)$

$\dfrac{\partial z}{\partial t} = -cf'(x-ct) + cg'(x+ct)$

$\dfrac{\partial z}{\partial t}(x,0) = G(x)$

$-cf'(x) + cg'(x) = G(x)$

$-f'(x) + g'(x) = \dfrac{1}{c}G(x)$

- ADDING AND SUBTRACTING

$$\left.\begin{aligned} 2f'(x) &= F'(x) - \tfrac{1}{c}G(x) \\ 2g'(x) &= F'(x) + \tfrac{1}{c}G(x) \end{aligned}\right\} \Rightarrow \begin{aligned} f'(x) &= F'(x) - \tfrac{1}{2c}G(x) \\ g'(x) &= F'(x) + \tfrac{1}{2c}G(x) \end{aligned}$$

$$\Longrightarrow \begin{pmatrix} f(x) = \frac{1}{2}F(x) - \frac{1}{2c}\int_0^x G(\xi)\,d\xi \\ g(x) = \frac{1}{2}F(x) + \frac{1}{2c}\int_0^x G(\xi)\,d\xi \end{pmatrix}$$

INTEGRATED THE EQUATION WRT x

NOTE HERE THAT

$$\frac{d}{dx}\left[\int_0^x f(t)\,dt\right] = f(x)$$

- NOW THE ABOVE RELATIONSHIPS HOLD FOR ALL x, AND IN PARTICULAR THEY WILL HOLD FOR $(x-ct)$ & $(x+ct)$

$$f(x-ct) = \frac{1}{2}F(x-ct) - \frac{1}{2c}\int_0^{x-ct} G(\xi)\,d\xi = \frac{1}{2}F(x-ct) + \frac{1}{2c}\int_{x-ct}^{0} G(\xi)\,d\xi$$

$$g(x+ct) = \frac{1}{2}F(x+ct) + \frac{1}{2c}\int_0^{x+ct} G(\xi)\,d\xi$$

- FINALLY WE HAVE WHEN COMBINING RESULTS

$$\Longrightarrow z(x,t) = f(x-ct) + g(x+ct)$$

$$\Longrightarrow z(x,t) = \frac{1}{2}F(x-ct) + \frac{1}{2c}\int_{x-ct}^{0} G(\xi)\,d\xi + \frac{1}{2}F(x+ct) + \frac{1}{2c}\int_0^{x+ct} G(\xi)\,d\xi$$

$$\Longrightarrow z(x,t) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\,d\xi$$

AS REQUIRED

b) • NOW THE INITIAL CONDITIONS ARE SPECIFIED

$$\bullet\ F(x) = z(x,0) = \begin{cases} 1 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$

$$\bullet\ G(x) = \frac{\partial z}{\partial t}(x,0) = 0$$

(WE HAVE A UNIT STEP "SQUARE" WAVE, WITH NO INITIAL VERTICAL VELOCITY TO START WITH)

$$\Longrightarrow z(x,t) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right] + \cancel{\frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\, d\xi}$$

$$\Longrightarrow z(x,t) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right]$$

• WE OBTAIN FOR THESE VALUES OF t

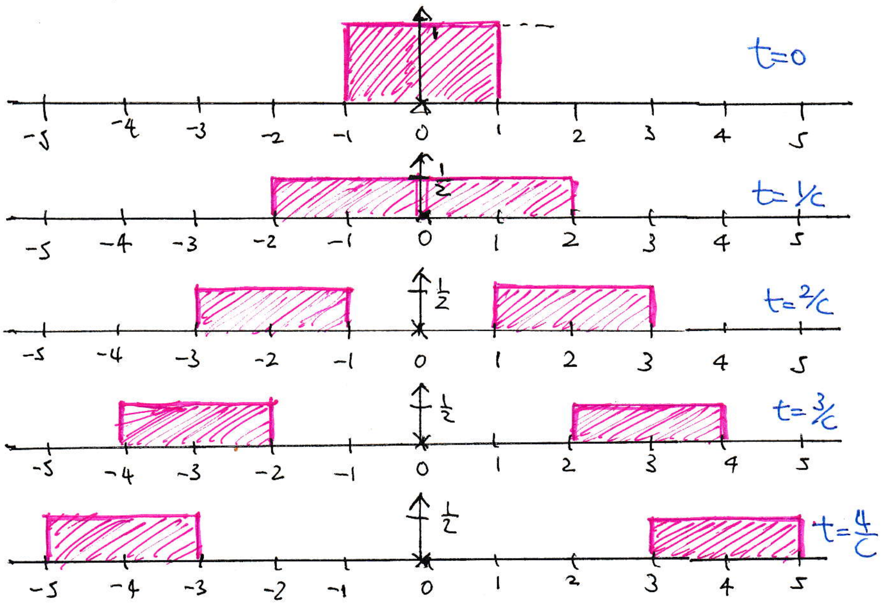
$t=0$, $z(x,0) = \frac{1}{2}F(x) + \frac{1}{2}F(x) = F(x)$

$t=\frac{1}{c}$, $z(x,\frac{1}{c}) = \frac{1}{2}F(x-1) + \frac{1}{2}F(x+1)$

$t=\frac{2}{c}$, $z(x,\frac{2}{c}) = \frac{1}{2}F(x-2) + \frac{1}{2}F(x+2)$

$t=\frac{3}{c}$, $z(x,\frac{3}{c}) = \frac{1}{2}F(x-3) + \frac{1}{2}F(x+3)$

$t=\frac{4}{c}$, $z(x,\frac{4}{c}) = \frac{1}{2}F(x-4) + \frac{1}{2}F(x+4)$



IYGB - MATHEMATICAL METHODS 4 - PAPER A - QUESTION 5

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{25}\frac{\partial^2 u}{\partial t^2}, \quad u = u(x,t) \qquad 0 \leq x \leq 0.5, \quad t \geq 0$$

SUBJECT TO THE CONDITIONS

I $u(0,t) = 0$

II $u(0.5,t) = 0$ ← FIXED AT ENDPOINTS

III $u(x,0) = \sin(20\pi x)$ ← INITIAL SHAPE

III $\frac{\partial u}{\partial t}(x,0) = 0$ ← RELEASED FROM REST (INITIALLY)

- ASSUME A SOLUTION IN VARIABLE SEPARATE FORM

$$u(x,t) = X(x)\,T(t)$$

- DIFFERENTIATE AND SUBSTITUTE INTO THE P.D.E

$$\frac{\partial^2 u}{\partial x^2} = X''(x)T(t) \quad \& \quad \frac{\partial^2 u}{\partial t^2} = X(x)T''(t)$$

$$\Rightarrow X''(x)T(t) = \frac{1}{25}X(x)T''(t)$$

$$\Rightarrow \frac{X''(x)T(t)}{X(x)T(t)} = \frac{1}{25}\frac{X(x)T''(t)}{X(x)T(t)}$$

$$\Rightarrow \frac{X''(x)}{X(x)} = \frac{1}{25}\frac{T''(t)}{T(t)} = \lambda$$

- BOTH SIDES IN THE EQUATION ABOVE ARE AT MOST A CONSTANT, SAY λ, AS THE L.H.S IS A FUNCTION OF x ONLY AND THE R.H.S IS A FUNCTION OF t ONLY — FURTHERMORE LOOKING AT THE BOUNDARY CONDITIONS (I) & (II) WE ARE LOOKING FOR A PERIODIC (OR CONSTANT) SOLUTION IN x, WHICH IS POSSIBLE IF λ IS NEGATIVE

IYGB - MATHEMATICAL METHODS 4 - PAPER A - QUESTION 5

- LET $\lambda = -p^2$

$$\frac{X''(x)}{X(x)} = -p^2 \qquad\qquad \frac{1}{25}\frac{T''(t)}{T(t)} = -p^2$$

$$X''(x) = -p^2 X(x) \qquad\qquad T''(t) = -25p^2$$

$$X(x) = A\cos px + B\sin px \qquad\qquad T(t) = C\cos 5pt + D\sin 5pt$$

$$\boxed{u(x,t) = X(x)T(t) = (A\cos px + B\sin px)(C\cos 5pt + D\sin 5pt)}$$

- APPLYING CONDITION (I), $u(0,t) = 0$ FOR ALL $t \geq 0$

$$\Rightarrow 0 = A[C\cos 5pt + D\sin 5pt]$$

$$\Rightarrow A = 0$$

ABSORBING "B" INTO C & D WE OBTAIN

$$\boxed{u(x,t) = (C\cos 5pt + D\sin 5pt)\sin px}$$

- NEXT, APPLY CONDITION (IV), $\frac{\partial u}{\partial t}(x,0) = 0$, FOR ALL x $0 \leq x \leq \frac{1}{2}$

$$\Rightarrow \frac{\partial u}{\partial t} = [-5pC\sin 5pt + 5pD\cos 5pt]\sin px$$

$$\Rightarrow 0 = 5pD\cos 5pt\sin px$$

$$\Rightarrow D = 0$$

$$\boxed{u(x,t) = C\sin px\cos 5pt}$$

- APPLY CONDITION (II), $u(\frac{1}{2},t)=0$, FOR ALL $t \geqslant 0$

$$\Rightarrow 0 = C\sin\tfrac{1}{2}p\cos 5pt$$

$$\Rightarrow \sin\tfrac{1}{2}p = 0 \qquad C \neq 0$$

$$\Rightarrow \tfrac{1}{2}p = n\pi \;,\; n=1,2,3,4,\ldots$$

$$\Rightarrow p = 2n\pi \;,\; n=1,2,3,4,\ldots$$

$$\boxed{\begin{array}{l} u_n(x,t) = C_n\sin(2n\pi x)\cos(10n\pi t) \\ \hline u(x,t) = \displaystyle\sum_{n=1}^{\infty}\left[C_n\sin(2n\pi x)\cos(10n\pi t)\right] \end{array}}$$

- FINALLY CONDITION (III), $u(x,0) = \sin(20\pi x)$

$$\sin(20\pi x) = \sum_{n=1}^{\infty}\left[C_n\sin(2n\pi x)\right]$$

$$\therefore C_{10} = 1 \;,\; C_n = 0 \text{ OTHERWISE}$$

$$\therefore u(x,t) = \sin(20\pi x)\cos(100\pi t)$$

IYGB - MATHEMATICAL METHODS 4 - PAPER A - QUESTION 6

- SOLVING THE HEAT EQUATION BY SEPARATION OF VARIABLES AND IGNORING ANY CONDITIONS AT THIS STAGE

LET $\theta(x,t) = X(x)T(t)$

$$\frac{\partial^2 \theta}{\partial x^2}(x,t) = X''(x)\,T(t)$$

$$\frac{\partial \theta}{\partial t}(x,t) = X(x)\,T'(t)$$

- SUBSTITUTE INTO THE P.D.E

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial \theta}{\partial t} \implies X''(x)T(t) = \frac{1}{\alpha^2}X(x)T'(t)$$

$$\implies \frac{X''(x)}{X(x)} = \frac{1}{\alpha^2}\frac{T'(t)}{T(t)} = \lambda$$

- BOTH SIDES OF THE ABOVE EQUATION MUST AT MOST BE A CONSTANT λ AS THE L.H.S IS A FUNCTION OF x ONLY AND THE R.H.S IS A FUNCTION OF t ONLY. THIS CONSTANT MAY BE ZERO, POSITIVE OR NEGATIVE

• IF $\lambda = 0$

$$\left.\begin{aligned} X''(x) = 0 &\implies X(x) = Ax+B \\ T'(t) = 0 &\implies T(t) = C \end{aligned}\right\} \implies \theta(x,t) = (Ax+B)\times C$$

$$\theta(x,t) = Ax+B$$

I.E STEADY FLOW WITHOUT TIME DEPENDENCY

• IF $\lambda > 0$, SAY p^2

$$\implies \frac{X''(x)}{X(x)} = p^2 \qquad \& \qquad \implies \frac{1}{\alpha^2}\frac{T'(t)}{T(t)} = p^2$$

$$\implies X''(x) = p^2 X(x) \qquad \implies T'(t) = \alpha^2 p^2 T(t)$$

$$\implies X(x) = Ae^{px} + Be^{-px} \qquad \implies T(t) = Ce^{\alpha^2 p^2 t}$$

IYGB – MATHEMATICAL METHODS 4 – PAPER A – QUESTION 6

$$\therefore \quad \theta(x,t) = Ce^{\alpha^2 p^2 t}\left(Ae^{px} + Be^{-px}\right)$$

THIS SOLUTION IS NOT APPLICABLE AS IT PRODUCES UNBOUNDED TEMPERATURE $\theta(x,t)$ AS $t \to \infty$

- IF $\lambda < 0$, SAY $\lambda = -p^2$

$$\Rightarrow \frac{X''(x)}{X(x)} = -p^2 \qquad \Rightarrow \frac{1}{\alpha^2}\frac{T'(t)}{T(t)} = -p^2$$

$$\Rightarrow X''(x) = -p^2 X(x) \qquad \Rightarrow T'(t) = -\alpha^2 p^2 T(t)$$

$$\Rightarrow X(x) = A\cos px + B\sin px \qquad \Rightarrow T(t) = Ce^{-\alpha^2 p^2 t}$$

$$\therefore \theta(x,t) = Ce^{-\alpha^2 p^2 t}\left(A\cos px + B\sin px\right)$$

$$\theta(x,t) = e^{-\alpha^2 p^2 t}\left(A\cos px + B\sin px\right)$$

THIS SOLUTION IS "ACCEPTABLE" TO THIS TYPE OF PROBLEM AS $\theta(x,t)$ IS BOUNDED AS $t \to \infty$

NOW THE INITIAL & BOUNDARY CONDITIONS NEED TO BE BUILT IN

LET $\boxed{\theta(x,t) = T_1 + \tilde{\theta}(x,t)}$ WHERE $\tilde{\theta}(x,t)$ SATISFIES

$$\frac{\partial^2 \tilde{\theta}}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial \tilde{\theta}}{\partial t}$$

SO THAT $\tilde{\theta}(x,t) \to 0$, AS $t \to \infty$

AND HENCE $\theta(x,t) \to T_1$, AS $t \to \infty$

- AT THE END $x=0$, $\theta(0,t) = T_1 \Rightarrow \tilde{\theta}(0,t) = 0$, $t > 0$ — I
- AT THE END $x=L$, $\theta(L,t) = T_1 \Rightarrow \tilde{\theta}(L,t) = 0$, $t > 0$ — II
- INITIAL TEMPERATURE IS ZERO, $\theta(x,0) = 0 \Rightarrow \tilde{\theta}(x,0) = -T_1$, $0 < x < L$ — III

IYGB - MATHEMATICAL METHODS 4 - PAPER A - QUESTION 6

- APPLYING EACH OF THESE CONDITIONS IN TURN

$$\tilde{\theta}(x,t) = e^{-p^2\alpha^2 t}\left(A\cos px + B\sin px\right)$$

(I) $0 = e^{-p^2\alpha^2 t}(A)$ FOR ALL $t>0$ $\therefore A=0$

$$\tilde{\theta}(x,t) = Be^{-p^2\alpha^2 t}\sin px$$

(II) $0 = Be^{-p^2\alpha^2 t}\sin pL$ FOR ALL $t>0$ $\therefore pL = n\pi,\ n\in\mathbb{Z}$

$\therefore p = \frac{n\pi}{L}$

$$\tilde{\theta}(x,t) = \sum_{n=1}^{\infty}\left[B_n e^{-\frac{n^2\pi^2\alpha^2}{L^2}t}\sin\frac{n\pi x}{L}\right]$$

NEGATIVE n ABSORBED INTO B_n

(III) $-T_1 = \sum_{n=1}^{\infty}\left[B_n \sin\frac{n\pi x}{L}\right]$. WHICH IS A SIMPLE FOURIER IN $(0,L)$ WITH $f(x) = -T_1$

$$\therefore B_n = \frac{1}{L/2}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L -T_1\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}\left[\frac{LT_1}{n\pi}\cos\frac{n\pi x}{L}\right]_0^L = \frac{2T_1}{n\pi}\left[\cos n\pi - 1\right] = \frac{2T_1}{n\pi}\left[(-1)^n - 1\right]$$

$$= \begin{cases} -2 & \text{IF } n \text{ IS ODD} \\ 0 & \text{IF } n \text{ IS EVEN}\end{cases}$$

$$\therefore B_{2m-1} = \frac{-4T_1}{(2m-1)\pi} \qquad m = 1,2,3,\dots$$

$$\tilde{\theta}(x,t) = \sum_{m=1}^{\infty}\left[\frac{-4T_1}{\pi(2m-1)}e^{-\frac{\alpha^2\pi^2(2m-1)^2 t}{L^2}}\sin\left[\frac{(2m-1)\pi x}{L}\right]\right]$$

$$\therefore \boxed{\theta(x,t) = T_1 - \frac{4T_1}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{2m-1}e^{-\frac{\alpha^2\pi^2(2n-1)^2 t}{L^2}}\sin\left[\frac{(2n-1)\pi x}{L}\right]\right]}$$

IYGB - MATHEMATICAL METHODS 4 - PAPER A - QUESTION 7

$$\frac{\partial z}{\partial x} = 2\frac{\partial z}{\partial t} + z \quad \underline{\text{SUBJECT TO}} \quad z(x,0) = 6e^{-3x}, \; x \geqslant 0$$

$$z = z(x,t), \; x \geqslant 0, \; t \geqslant 0 \qquad z(x,t) \text{ IS BOUNDED } x \geqslant 0, \; t \geqslant 0$$

- TAKING LAPLACE TRANSFORMS OF THE P.D.E W.R.T t

$$\Longrightarrow \mathcal{L}\left[\frac{\partial z}{\partial x}\right] = \mathcal{L}\left[2\frac{\partial z}{\partial t}\right] + \mathcal{L}[z]$$

$$\Longrightarrow \frac{\partial}{\partial x}\bar{z} = 2\left[s\bar{z} - z(x,0)\right] + \bar{z}$$

$$\Longrightarrow \frac{\partial \bar{z}}{\partial x} = 2s\bar{z} - 12e^{-3x} + \bar{z}$$

$$\Longrightarrow \frac{\partial \bar{z}}{\partial x} - (2s+1)\bar{z} = -12e^{-3x}$$

- THIS IS A FIRST ORDER O.D.E FOR $\bar{z} = \bar{z}(x,s)$, WHERE s IS TREATED AS A CONSTANT — LOOK FOR AN INTEGRATING FACTOR

$$e^{\int -(2s+1)\,dx} = e^{-(2s+1)x}$$

- HENCE WE OBTAIN

$$\Longrightarrow \frac{\partial}{\partial x}\left[\bar{z}e^{-(2s+1)x}\right] = -12e^{-3x}e^{-(2s+1)x}$$

$$\Longrightarrow \frac{\partial}{\partial x}\left[\bar{z}e^{-(2s+1)x}\right] = -12e^{-(2s+4)x}$$

$$\Longrightarrow \bar{z}e^{-2(s+1)x} = \int -12e^{-(2s+4)x}\,dx$$

$$\Longrightarrow \bar{z}e^{-2(s+1)x} = \frac{12}{2s+4}e^{-(2s+4)x} + A(s)$$

$$\Rightarrow \bar{z} = \frac{6}{s+2} e^{-(2s+4)x} \times e^{(2s+1)x} + A(s) e^{(2s+1)x}$$

$$\Rightarrow \boxed{\bar{z}(x,s) = \frac{6}{s+2} e^{-3x} + A(s) e^{(2s+1)x}}$$

- NOW $A(s) = 0$ SINCE $z(x,t)$ IS BOUNDED AS $x \to \infty$, SO MUST $\bar{z}(x,s)$ AS $x \to \infty$

$$\Rightarrow \boxed{\bar{z}(x,s) = \frac{6}{s+2} e^{-3x}}$$

- INVERTING BACK INTO t, NOTING x IS A CONSTANT WITH RESPECT TO THE TRANSFORM

$$\Rightarrow z(x,t) = 6e^{-2t} e^{-3x}$$

$$\Rightarrow \boxed{z(x,t) = 6e^{-(2t+3x)}}$$

IYGB – MATHEMATICAL METHODS 4 – QUESTION 8 – PAPER A

- ASSUME A SOLUTION IN VARIABLE SEPARABLE FORM

$$\phi(r,\theta) = R(r)\,\Theta(\theta)$$

$$\frac{\partial \phi}{\partial r} = R'(r)\,\Theta(\theta)$$

$$\frac{\partial^2 \phi}{\partial r^2} = R''(r)\,\Theta(\theta)$$

$$\frac{\partial^2 \phi}{\partial \theta^2} = R(r)\,\Theta''(\theta)$$

- SUBSTITUTE THESE EXPRESSIONS INTO THE P.D.E

$$\Rightarrow \boxed{\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0}$$

$$\Rightarrow R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0$$

$$\Rightarrow r^2R''(r)\Theta(\theta) + rR'(r)\Theta(\theta) + R(r)\Theta''(\theta) = 0$$

$$\Rightarrow r^2R''(r)\Theta(\theta) + rR'(r)\Theta(\theta) = -R(r)\Theta''(\theta)$$

$$\Rightarrow \frac{r^2R''(r)\cancel{\Theta(\theta)}}{R(r)\cancel{\Theta(\theta)}} + \frac{rR'(r)\cancel{\Theta(\theta)}}{R(r)\cancel{\Theta(\theta)}} = -\frac{\cancel{R(r)}\Theta''(\theta)}{\cancel{R(r)}\Theta(\theta)}$$

$$\Rightarrow \boxed{\frac{r^2R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)}}$$

- AS THE LHS OF THE ABOVE EXPRESSION IS A FUNCTION OF r ONLY AND THE RHS IS A FUNCTION OF θ ONLY, THEN BOTH SIDES ARE AT MOST A CONSTANT, SAY λ, WHICH MAY BE POSITIVE, NEGATIVE OR ZERO

- LOOKING AT THE R.H.S FIRST

$$\boxed{-\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda} \iff \boxed{\Theta''(\theta) = -\lambda\Theta(\theta)}$$

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- IF $\lambda=0$ $\quad \Theta''(\theta)=0 \implies \boxed{\Theta(\theta)=A\theta+B}$ — (I)

- IF $\lambda>0,\ \lambda=p^2$ $\quad \Theta''(\theta)=-p^2\Theta(\theta) \implies \boxed{\Theta(\theta)=A\cos p\theta+B\sin p\theta}$ — (II)

- IF $\lambda<0,\ \lambda=-p^2$ $\quad \Theta(\theta)=p^2(\theta) \implies \boxed{\Theta(\theta)=A\cosh p\theta+B\sinh p\theta}$ — (III)

(OR EXPONENTIALS)

- UNDER A SYSTEM OF POLAR CO-ORDINATES (r,θ), A UNIQUE CARTESIAN POINT (x,y) GETS MAPPED TO A POLAR POINT $(r,\theta+2k\pi)$, $k\in\mathbb{Z}$ HENCE WE REQUIRE Θ TO PRODUCE PERIODIC (OR CONSTANT) SOLUTIONS

  - SOLUTION (III) IS DISCARDED AS IT HAS NO PERIODICITY
  - SOLUTION (II) IS FINE AS IT IS PERIODIC
  - SOLUTION (I) IS O.K IF $A=0$, BUT THEN IT IS CONSTANT AND IT CAN BE ABSORBED INTO (II)

- LOOKING FURTHER INTO (II) WITH SINES (OR COSINES)

$$\sin\theta=\sin(\theta+2\pi)$$
$$\sin p\theta=\sin[p(\theta+2\pi)]$$

$$\therefore\ p=n,\ n=\text{INTEGER}$$

- HENCE WE CONCLUDE AT THIS STAGE

$$\Theta_n(\theta)=A_n\cos n\theta+B_n\sin n\theta\ ,\quad n=0,1,2,3,4,\dots$$

AS WE SHALL SEE LATER NEGATIVE INTEGERS WILL BE INCLUDED IN THE FINAL SOLUTION, DUE TO ITS NATURE

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RETURNING TO THE L.H.S OF THE P.D.E (IN SEPARATED FORM)

$$\boxed{\frac{r^2 R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} = \lambda = 0,1,2,3,4,\ldots}$$

IF $\lambda = n = 0$

$$\Rightarrow \frac{r^2 R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} = 0$$

$$\Rightarrow rR''(r) + R'(r) = 0$$

$$\Rightarrow rR''(r) = -R'(r)$$

$$\Rightarrow \frac{R''(r)}{R'(r)} = -\frac{1}{r}$$

INTEGRATE W.R.T r

$$\Rightarrow \ln|R'(r)| = -\ln|r| + \ln C$$

$$\Rightarrow \ln|R'(r)| = \ln\left|\frac{C}{r}\right|$$

$$\Rightarrow R'(r) = \frac{C}{r}$$

INTEGRATE W.R.T. r

$$\Rightarrow R(r) = C\ln|r| + D$$

IF $\lambda = n = 1,2,3,4,5,\ldots$

$$\Rightarrow \frac{r^2 R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} = n^2 \leftarrow \text{(II)}$$

$$\Rightarrow r^2R''(r) + rR'(r) - n^2R(r) = 0$$

CAUCHY - EULER O.D.E

LET $R(r) = r^\lambda$

$R'(r) = \lambda r^{\lambda-1}$

$R''(r) = \lambda(\lambda-1)r^{\lambda-2}$

SUB INTO ABOVE O.D.E

$\lambda(\lambda-1)\cancel{r^\lambda} + \lambda \cancel{r^\lambda} - n^2\cancel{r^\lambda} = 0$

$\lambda^2 - \lambda + \lambda - n^2 = 0$

$\lambda^2 = n^2$

$\lambda = \pm n$

HENCE

$$R_n(r) = \alpha_n r^n + \beta_n r^{-n} \qquad n = 1, 2, 3, 4, \ldots$$

AND NOTE THAT NEGATIVE INTEGERS ARE NOW INCLUDED

- FINALLY WE CAN COMBINE ALL THE SOLUTIONS INTO A GENERAL SOLUTION

- $n=0$ $\quad \Theta_0(\theta) = B$
  $R_0(r) = C\ln r + D$ ABSORB B INTO D

- $n = 1, 2, 3, 4, \ldots$ $\quad \Theta_n(\theta) = A_n \cos n\theta + B_n \sin n\theta$
  $R_n(r) = \alpha r^n + \beta r^{-n}$

$$\Phi(r,\theta) = \left[C\ln r + D\right] + \sum_{n=1}^{\infty}\left[(A_n \cos n\theta + B_n \sin n\theta)(\alpha_n r^n + \beta_n r^{-n})\right]$$

ABSORBING & RELABELLING SOME OF THESE CONSTANTS, WE OBTAIN

$$\boxed{\Phi(r,\theta) = A + B\ln r + \sum_{n=1}^{\infty}\left[C_n r^n \cos n\theta + D_n r^{-n}\cos n\theta + E_n r^n \sin n\theta + F_n r^{-n}\sin n\theta\right]}$$

b) APPLYING THE CONDITIONS

- $\nabla^2 \phi_1 = 0 \qquad r \geqslant 1$
- $\nabla^2 \phi_2 = 0 \qquad 0 < r < 1$

(1) $\lim_{r \to \infty} \left[\phi_1 - r\cos\theta \right] = 2$

(2) $\phi_1(1,\theta) = \phi_2(1,\theta)$

(3) $1 + \frac{\partial \phi_1}{\partial r}(1,\theta) = \frac{\partial \phi_2}{\partial r}(1,\theta)$

(4) $\lim_{r \to 0} \left[r\phi_2 - \cos\theta \right) = 0$

LET ϕ_1 & ϕ_2 BE

- $\phi_1(r,\theta) = A + B\ln r + \sum_{n=1}^{\infty} \left[C_n r^n \cos n\theta + D_n r^{-n} \cos n\theta + E_n r^n \sin n\theta + F_n r^{-n} \sin n\theta \right]$
- $\phi_2(r,\theta) = G + H\ln r + \sum_{n=1}^{\infty} \left[K_n r^n \cos n\theta + L_n r^{-n} \cos n\theta + M_n r^n \sin n\theta + P_n r^{-n} \sin n\theta \right]$

BY CONDITION (1)

AS $r \to \infty$, $\phi_1(r,\theta) \longrightarrow 2 + r\cos\theta$

$\Longrightarrow A = 2$

$\Longrightarrow B = 0$

$\Longrightarrow E_n = 0$

$\Longrightarrow C_1 = 1, \; C_n = 0 \; n \neq 1$

$\Longrightarrow D_n, F_n$ UNDETERMINED

$$\therefore \boxed{\phi_1(r,\theta) = 2 + r\cos\theta + \sum_{n=1}^{\infty} \left[D_n r^{-n} \cos n\theta + F_n r^{-n} \sin n\theta \right]}$$

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- BY CONDITION (4)

AS $r \to 0 \quad r\phi_2(r,\theta) \longrightarrow \cos\theta$ $\Rightarrow G, H, K_n, M_n$ UNDETERMINED

$\Rightarrow P_n = 0$

$\Rightarrow L_1 = 1, \ L_n = 0 \ n \geq 2$

$$\therefore \boxed{\phi_2(r,\theta) = G + H\ln r + \frac{1}{r}\cos\theta + \sum_{n=1}^{\infty}\left[K_n r^n \cos n\theta + M_n r^n \sin n\theta\right]}$$

- BY CONTINUITY AT $r=1$, CONDITION (2)

$\phi_1(1,\theta) = \phi_2(1,\theta)$

$$2 + \cancel{\cos\theta} = \sum_{n=1}^{\infty}\left[D_n \cos n\theta + F_n \sin n\theta\right] = G + \cancel{\cos\theta} + \sum_{n=1}^{\infty}\left[K_n \cos n\theta + M_n \sin n\theta\right]$$

$G = 2 \qquad D_n = K_n, \ \forall n$

$\qquad\qquad F_n = M_n, \ \forall n$

- DIFFERENTIATE TO APPLY (3)

$$\frac{\partial\phi_1}{\partial r} = \cos\theta + \sum_{n=1}^{\infty}\left[-nD_n r^{-n-1}\cos n\theta - nF_n r^{-n-1}\sin n\theta\right]$$

$$\frac{\partial\phi_2}{\partial r} = \frac{H}{r} - \frac{1}{r^2}\cos\theta + \sum_{n=1}^{\infty}\left[nK_n r^{n-1}\cos n\theta + nM_n r^n \sin n\theta\right]$$

$$1 + \frac{\partial\phi_1}{\partial r}(1,\theta) = \frac{\partial\phi_2}{\partial r}(1,\theta)$$

$$1 + \cos\theta + \sum_{n=1}^{\infty}\left[-nD_n\cos n\theta - nF_n \sin n\theta\right] = H - \cos\theta + \sum_{n=1}^{\infty}\left[nK_n\cos n\theta + nM_n \sin n\theta\right]$$

$\therefore\ H = 1$ & BY TIDYING UP THE REST

$$2\cos\theta = \sum_{n=1}^{\infty}\Big[n\left[D_n + k_n\right]\cos n\theta + n\left[F_n + M_n\right]\sin n\theta\Big]$$

- $n = 1, 2, 3, 4, \ldots$

  $n\left[F_n + M_n\right] = 0$

  &

  FROM EARLIER

  $F_n = M_n,\ \forall n$

  $\therefore F_n = M_n = 0$

- $n = 1$

  $2\cos\theta = (D_1 + k_1)\cos\theta$

  $D_1 + k_1 = 2$

  BUT FROM EARLIER

  $D_n = k_n,\ \forall n$

  $\therefore D_1 = k_1 = 1$

- $n \geqslant 2$

  $n\left[D_n + k_n\right] = 0$

  &

  FROM EARLIER

  $D_n = k_n,\ \forall n$

  $\therefore D_n = k_n = 0$ $\quad n \geqslant 2$

- FINALLY WE HAVE EXPRESSIONS FOR BOTH ϕ_1 & ϕ_2

$$\phi_1(r,\theta) = 2 + r\cos\theta + \frac{1}{r}\cos\theta$$

$$\phi_2(r,\theta) = 2 + \ln r + \frac{1}{r}\cos\theta + r\cos\theta$$

$$\phi_1(r,\theta) = 2 + \left(r + \frac{1}{r}\right)\cos\theta \qquad r > 1$$

$$\phi_2(r,\theta) = 2 + \ln r + \left(r + \frac{1}{r}\right)\cos\theta \qquad 0 < r < 1$$