IYGB - SYNOPTIC PAPER J - QUESTION 1

DIFFERENTIATING IMPLICITLY W.R.T x

$$\Rightarrow 2\ln y = x\ln x$$

$$\Rightarrow \frac{d}{dx}(2\ln y) = \frac{d}{dx}(x\ln x)$$

$$\Rightarrow 2\times\frac{1}{y}\frac{dy}{dx} = 1\times\ln x + x\times\frac{1}{x}$$

$$\Rightarrow \frac{2}{y}\frac{dy}{dx} = 1+\ln x$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}y(1+\ln x)$$

NOW WHEN $x=4$

$$2\ln y = 4\ln 4$$

$$\ln y = 2\ln 4$$

$$\ln y = \ln 16$$

$$y = 16$$

FINALLY WE OBTAIN

$$\left.\frac{dy}{dx}\right|_{(4,16)} = \frac{1}{2}\times 16\times(1+\ln 4) = 8(1+\ln 4) = 8(1+2\ln 2)$$

ALTERNATIVE BY REARRANGING FIRST

$$\Rightarrow 2\ln y = x\ln x$$

$$\Rightarrow \ln y = \frac{1}{2}x\ln x$$

$$\Rightarrow e^{\ln y} = e^{\frac{1}{2}x\ln x}$$

$$\Rightarrow y = e^{\frac{1}{2}x\ln x}$$

IYGB - SYNOPTIC PAPER J - QUESTION 1

NOW DIFFERENTIATING W.R.T x

$$\Longrightarrow \frac{dy}{dx} = e^{\frac{1}{2}x\ln x} \times \left(\frac{1}{2}\ln x + \frac{1}{2}x \times \frac{1}{x}\right)$$

$$\Longrightarrow \frac{dy}{dx} = e^{\frac{1}{2}x\ln x} \times \left(\frac{1}{2}\right.$$

$$\Longrightarrow \left.\frac{dy}{dx}\right|_{x=4} = e^{2\ln 4} \times \left(\frac{1}{2}\ln 4 + \frac{1}{2}\right)$$

$$= e^{\ln 16} \times \frac{1}{2}(1+\ln 4)$$

$$= 16 \times \frac{1}{2}(1+2\ln 2)$$

$$= 8(1+2\ln 2)$$

AS BEFORE

IYGB – SYNOPTIC PAPER J – QUESTION 2

METHOD A – BY PARTIAL FRACTIONS

$$\frac{7x+2}{(x-2)(x+2)(2x+1)} \equiv \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{2x+1}$$

$$\boxed{7x+2 \equiv A(x+2)(2x+1) + B(x-2)(2x+1) + C(x+2)(x-2)}$$

- IF $x=2 \Rightarrow 16 = A \times 4 \times 5$
  $\Rightarrow 16 = 20A$
  $\Rightarrow A = \frac{4}{5}$

- IF $x=-2 \Rightarrow -12 = B(-4)(-3)$
  $\Rightarrow -12 = 12B$
  $\Rightarrow B = -1$

- IF $x=-\frac{1}{2} \Rightarrow -\frac{3}{2} = C\left(\frac{3}{2}\right)\left(-\frac{5}{2}\right)$
  $\Rightarrow -\frac{3}{2} = -\frac{15}{4}C$
  $\Rightarrow C = \frac{2}{5}$

$$\therefore \quad y = \frac{4}{5}(x-2)^{-1} - (x+2)^{-1} + \frac{2}{5}(2x+1)^{-1}$$

$$\frac{dy}{dx} = -\frac{4}{5}(x-2)^{-2} + (x+2)^{-2} - \frac{2}{5}(2x+1)^{-2} \times 2$$

$$\frac{dy}{dx} = -\frac{4}{5(x-2)^2} + \frac{1}{(x+2)^2} - \frac{4}{5(2x+1)^2}$$

$$\left.\frac{dy}{dx}\right|_{x=-1} = -\frac{4}{5\times 9} + \frac{1}{\text{[illegible]}} - \frac{4}{5\times 1}$$

$$\left.\frac{dy}{dx}\right|_{x=-1} = -\frac{4}{45} + \text{[illegible]} - \frac{4}{5}$$

$$\left.\frac{dy}{dx}\right|_{x=-1} = \frac{\text{[illegible]}}{9}$$

IYGB - SYNOPTIC PAPER J - QUESTION 2

ALTERNATIVE BY IMPLICIT DIFFERENTIATION (LOGARITHMIC)

$$y = \frac{7x+2}{(x-2)(x+2)(2x+1)}$$

$$\ln y = \ln\left[\frac{7x+2}{(x-2)(x+2)(2x+1)}\right]$$

$$\ln y = \ln(7x+2) - \ln(x-2) - \ln(x+2) - \ln(2x+1)$$

DIFF WITH RESPECT TO x

$$\frac{1}{y}\frac{dy}{dx} = \frac{7}{7x+2} - \frac{1}{x-2} - \frac{1}{x+2} - \frac{2}{2x+1}$$

$$\frac{dy}{dx} = y\left[\frac{7}{7x+2} - \frac{1}{x-2} - \frac{1}{x+2} - \frac{2}{2x+1}\right]$$

FIND FIRST y AT $x=-1$

$$y\Big|_{x=-1} = \frac{-5}{(-3)(1)(-1)} = \frac{-5}{3} = -\frac{5}{3}$$

FINALLY AT $x=-1$

$$\frac{dy}{dx}\Big|_{x=-1} = -\frac{5}{3}\left[\frac{7}{-5} - \frac{1}{-3} - 1 - \frac{2}{-1}\right]$$

$$= -\frac{5}{3} \times \left[-\frac{7}{5} + \frac{1}{3} - 1 + 2\right]$$

$$= -\frac{5}{3} \times \left(-\frac{1}{15}\right)$$

$$= \frac{1}{9}$$

AS BEFORE

IYGB - SYNOPTIC PAPER J - QUESTION 3

$$\boxed{S_n = 2n(4n-7) \qquad n \in \mathbb{N}}$$

THE FIFTH TERM SATISFIES

$$u_5 = S_5 - S_4$$

$$u_5 = 2 \times 5 \times (20-7) - 2 \times 4 \times (16-7)$$

$$u_5 = 130 - 72$$

$$u_5 = 58$$

IYGB - SYNOPTIC PAPER J - QUESTION 4

PROCEED AS FOLLOWS

$$\Rightarrow f(x) = \frac{(1-x)^2}{\sqrt{1+2x}} = (1-x)^2(1+2x)^{-\frac{1}{2}}$$

$$\Rightarrow f(x) = (1-2x+x^2)\left[1 + \frac{-\frac{1}{2}}{1}(2x)^1 + \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)}{1\times 2}(2x)^2 + \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{1\times 2\times 3}(2x)^3 + O(x^4)\right]$$

$$\Rightarrow f(x) = (1-2x+x^2)\left[1 - x + \frac{3}{2}x^2 - \frac{5}{2}x^3 + O(x^4)\right]$$

MULTIPLYING FOLLOWS

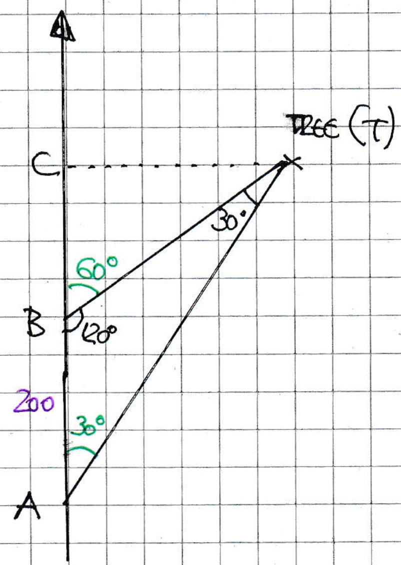
$$\begin{aligned}\Rightarrow f(x) = \; & 1 - x + \tfrac{3}{2}x^2 - \tfrac{5}{2}x^3 + O(x^4) \\ & \quad - 2x + 2x^2 - 3x^3 + O(x^4) \\ & \qquad\qquad x^2 - x^3 + O(x^4)\end{aligned}$$

$$\Rightarrow f(x) = 1 - 3x + \frac{9}{2}x^2 - \frac{13}{2}x^3 + O(x^4)$$

// AS REQUIRED

IYGB – SYNOPTIC PAPER J – QUESTION 5

a) STARTING WITH A DIAGRAM – FILL IN ALL THE MISSING ANGLES



- EVIDENTLY AS $\overset{\triangle}{ABT}$ IS ISOSCELES, $|BT| = 200$
- BY SIMPLE TRIGONOMETRY ON $\overset{\triangle}{BCT}$

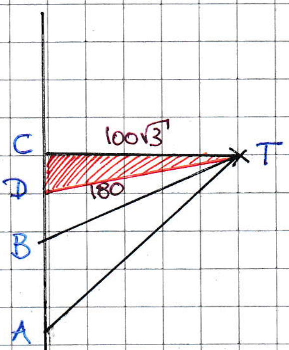
$$\frac{|CT|}{|BT|} = \sin 60°$$

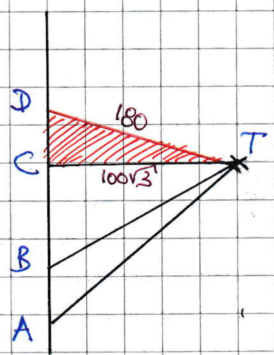
$$|CT| = |BT| \sin 60°$$

$$|CT| = 200 \times \frac{\sqrt{3}}{2}$$

$$|CT| = 100\sqrt{3} \simeq 173$$

b) THERE ARE TWO CASES TO CONSIDER





- IN EITHER CASE FROM THE ORIGINAL DIAGRAM BY TRIGONOMETRY OR PYTHAGORAS

$$|BC|^2 + |CT|^2 = |BT|^2$$

$$|BC|^2 + (100\sqrt{3})^2 = 200^2$$

$$|BC|^2 + 30000 = 40000$$

$$|BC| = 100$$

IYGB - SYNOPTIC PAPER J - QUESTION 5

THEN BY PYTHAGORAS, LOOKING AT THE ORANGE/RED TRIANGLE IN EACH CASE

$$|CD|^2 + |CT|^2 = |DT|^2$$
$$|CD|^2 + (100\sqrt{3})^2 = 180^2$$
$$|CD|^2 + 30000 = 32400$$
$$|CD|^2 = 2400$$
$$|CD| = 20\sqrt{6} \simeq 48.98$$

HENCE WE OBTAIN

$$|AD| = \begin{cases} |AB| + |BC| - |CD| = 200 + 100 - 20\sqrt{6} \\ |AB| + |BC| + |CD| = 200 + 100 + 20\sqrt{6} \end{cases}$$

$$|AD| = \begin{cases} 300 - 20\sqrt{6} \approx 251 \\ 300 + 20\sqrt{6} \approx 349 \end{cases}$$

IYGB - SYNOPTIC PAPER J - QUESTION 6

a) $f(x) = \dfrac{3x+1}{x+4}, \quad x \in \mathbb{R}, \quad x > -4$

$$\Rightarrow y = \frac{3x+1}{x+4}$$

$$\Rightarrow y(x+4) = 3x+1$$

$$\Rightarrow yx + 4y = 3x+1$$

$$\Rightarrow yx - 3x = 1 - 4y$$

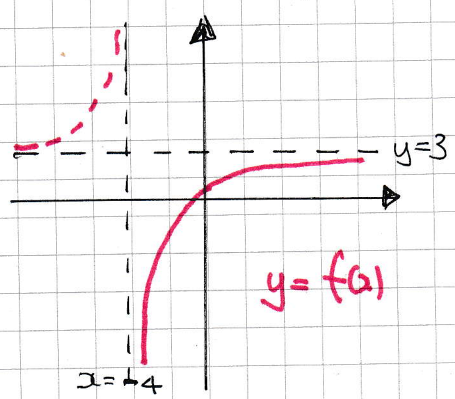
$$\Rightarrow x(y-3) = 1-4y$$

$$\Rightarrow x = \frac{1-4y}{y-3}$$

$$\therefore \; f^{-1}(x) = \frac{1-4x}{x-3} \quad \text{OR} \quad f^{-1}(x) = \frac{4x-1}{3-x}$$

b) FIRSTLY OBTAIN THE RANGE OF $f(x)$ VIA A QUICK SKETCH

- VERTICAL ASYMPTOTE : $x = -4$ (DENOMINATOR ZERO)
- HORIZONTAL ASYMPTOTE : $y = 3$ $\left[\lim\limits_{x \to \infty} \left(\dfrac{3x+1}{x+4} \right) = 3 \right]$
- $x = 0 \Rightarrow y = \frac{1}{4}$
- HENCE WE SKETCH



$\therefore$ RANGE OF $f(x)$ IS

$f(x) < 3$

IYGB - SYNOPTIC PAPER J - QUESTION 6

THUS WE HAVE

	$f(x)$	$f^{-1}(x)$
DOMAIN	$x \geqslant -4$ (GIVEN)	$x < 3$
RANGE	$f(x) < 3$	$f^{-1}(x) \geqslant -4$

c) FIRSTLY OBTAIN AN EXPRESSION FOR THE COMPOSITION

- $f(g(x)) = f(e^x - 3) = \dfrac{3(e^x-3)+1}{(e^x-3)+4} = \dfrac{3e^x - 8}{e^x+1}$

- $f(g(x)) = \dfrac{4}{5}$

$$\Rightarrow \frac{3e^x - 8}{e^x+1} = \frac{4}{5}$$

$$\Rightarrow 15e^x - 40 = 4e^x + 4$$

$$\Rightarrow 11e^x = 44$$

$$\Rightarrow e^x = 4$$

$$\Rightarrow x = \ln 4$$

$$\Rightarrow x = 2\ln 2$$

"ACCEPTED" BY $f(g(x))$

IYGB – SYNOPTIC PAPER J – QUESTION 7

a) START BY DIFFERENTIATING THE EXPRESSION W.R.T t

$$N = Ae^{kt}$$

$$\frac{dN}{dt} = Ake^{kt} \quad \text{OR} \quad \frac{dN}{dt} = kN$$

- $t = \ln 64 \quad N = 1200$

$$1200 = Ae^{k\ln 64}$$

- $t = \ln 64 \quad \frac{dN}{dt} = 200 \quad N = 1200$

$$200 = k \times 1200$$

$$k = \frac{1}{6}$$

$$1200 = Ae^{\frac{1}{6}\ln 64}$$

$$1200 = Ae^{\ln 64^{\frac{1}{6}}}$$

$$1200 = Ae^{\ln 2}$$

$$1200 = A \times 2$$

$$A = 600$$

b) WHEN $t = 0$, $N = 1200$

WHEN $t = T$, $N = 3600$ $(\times 3)$

$$\Rightarrow N = 600e^{\frac{1}{6}t}$$

$$\Rightarrow 3600 = 600e^{\frac{1}{6}t}$$

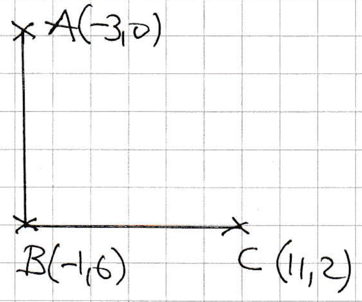
$$\Rightarrow 6 = e^{\frac{1}{6}t}$$

$$\Rightarrow \ln 6 = \frac{1}{6}t$$

$$\Rightarrow t = 6\ln 6$$

IYGB - SYNOPTIC PAPER J - QUESTION 8

a) CONSIDERING GRADIENTS

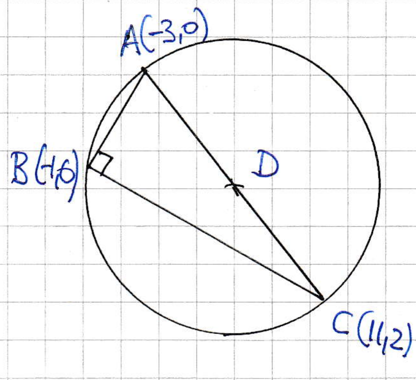


$$\text{GRADIENT } AB = \frac{0-6}{-3-(-1)} = \frac{-6}{-2} = 3$$

$$\text{GRADIENT } BC = \frac{6-2}{-1-11} = \frac{4}{-12} = -\frac{1}{3}$$

AS THESE GRADIENTS ARE NEGATIVE RECIPROCALS OF EACH OTHER $A\hat{B}C = 90°$

b) BY A CIRCLE THEOREM IF $A\hat{B}C = 90°$, & ALL 3 POINTS LIE ON THE CIRCLE, WITH AC BEING A DIAMETER



- D IS THE MIDPOINT OF AC

$$D\left(\frac{-3+11}{2}, \frac{0+2}{2}\right) = D(4,1)$$

- A(-3,0) & D(4,1)

$$r = |AD| = \sqrt{(-3-4)^2 + (0-1)^2} = \sqrt{50}$$

- EQUATION OF CIRCLE IS

$$(x-4)^2 + (y-1)^2 = \sqrt{50}^2$$

$$x^2 - 8x + 16 + y^2 - 2y + 1 = 50$$

$$x^2 + y^2 - 8x - 2y - 33 = 0$$

IYGB – SYNOPTIC PAPER J – QUESTION 9

- USING THE SUBSTITUTION GIVEN

$$x = -\frac{1}{2} + \frac{1}{2}\sin\theta$$

$$dx = \frac{1}{2}\cos\theta \, d\theta$$

- CHANGING THE LIMITS

• $x = 0 \implies 0 = -\frac{1}{2} + \frac{1}{2}\sin\theta$

$\frac{1}{2} = \frac{1}{2}\sin\theta$

$\sin\theta = 1$

$\theta = \frac{\pi}{2}$

• $x = -\frac{1}{4} \implies -\frac{1}{4} = -\frac{1}{2} + \frac{1}{2}\sin\theta$

$\implies \frac{1}{4} = \frac{1}{2}\sin\theta$

$\implies \sin\theta = \frac{1}{2}$

$\implies \theta = \frac{\pi}{6}$

- TRANSFORMING THE DENOMINATOR OF THE INTEGRAND

$$\sqrt{-x(x+1)} = \sqrt{-\left(-\tfrac{1}{2}+\tfrac{1}{2}\sin\theta\right)\left(-\tfrac{1}{2}+\tfrac{1}{2}\sin\theta+1\right)}$$

$$= \sqrt{\left(\tfrac{1}{2}-\tfrac{1}{2}\sin\theta\right)\left(\tfrac{1}{2}+\tfrac{1}{2}\sin\theta\right)}$$

$$= \sqrt{\tfrac{1}{4}-\tfrac{1}{4}\sin^2\theta}$$

$$= \sqrt{\tfrac{1}{4}(1-\sin^2\theta)}$$

$$= \sqrt{\tfrac{1}{4}\cos^2\theta}$$

$$= \tfrac{1}{2}\cos\theta$$

- HENCE THE INTEGRAL NOW BECOMES

$$\int_{-\frac{1}{4}}^{0} \frac{3}{\sqrt{-x(x+1)}}\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{3}{\cancel{\frac{1}{2}\cos\theta}}\left(\cancel{\tfrac{1}{2}\cos\theta}\,d\theta\right)$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 3\,d\theta$$

$$= \Big[3\theta\Big]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \frac{3\pi}{2} - \frac{\pi}{2}$$

$$= \pi$$

IYGB - SYNOPTIC PAPER J - QUESTION 10

USING THE TAN COMPOUND ANGLE IDENTITY & THE PYTHAGOREAN IDENTITY CONNECTING $\tan^2\theta$ & $\sec^2\theta$

$$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$1 + \tan^2 A \equiv \sec^2 A$$

$$\Rightarrow 4\tan(\theta+60)\tan(\theta-60) = \sec^2\theta - 16$$

$$\Rightarrow 4\left(\frac{\tan\theta + \tan 60}{1-\tan\theta\tan 60}\right)\times\left(\frac{\tan\theta - \tan 60}{1+\tan\theta\tan 60}\right) = (1+\tan^2\theta) - 16$$

$$\boxed{\tan 60 = \sqrt{3}}$$

$$\Rightarrow \frac{4(\tan\theta+\sqrt{3})}{1-\sqrt{3}\tan\theta} \times \frac{\tan\theta - \sqrt{3}}{1+\sqrt{3}\tan\theta} = \tan^2\theta - 15$$

$$\Rightarrow \frac{4(\tan^2\theta - 3)}{1-3\tan^2\theta} = \tan^2\theta - 15$$

$$\Rightarrow \frac{4(T-3)}{1-3T} = T - 15 \quad , \text{ WHERE } T = \tan\theta$$

$$\Rightarrow 4(T-3) = (T-15)(1-3T)$$

$$\Rightarrow 4T - 12 = T - 3T^2 - 15 + 45T$$

$$\Rightarrow 3T^2 - 42T + 3 = 0$$

$$\Rightarrow T^2 - 14T + 1 = 0$$

$$\Rightarrow (T-7)^2 - 48 = 0$$

$$\Rightarrow (T-7)^2 = 48$$

$$\Rightarrow T - 7 = \pm\sqrt{48}$$

IYGB - SYNOPTIC PAPER J - QUESTION 10

$$\Rightarrow T = 7 \pm \sqrt{48}$$

$$\Rightarrow \tan^2\theta = 7 \pm \sqrt{48}$$

$$\Rightarrow \tan\theta = \pm\sqrt{7 \pm \sqrt{48}}$$

INVERTING ALL 4 POSSIBLE ARCTANS

$$\left\{\begin{array}{l} \theta = 15 \pm 180n \\ \theta = -15 \pm 180n \\ \theta = 75 \pm 180n \\ \theta = -75 \pm 180n \end{array}\right. \qquad n = 0, 1, 2, 3, \ldots$$

$$\therefore \ \theta = 15^\circ, 75^\circ, 105^\circ, 165^\circ$$

IYGB − SYNOPTIC PAPER J − QUESTION 11

a) BY INSPECTION $P(0,6)$

$y = \frac{1}{2}x^2 - 2x + 6$

$\frac{dy}{dx} = x - 2$

$\left.\frac{dy}{dx}\right|_{x=0} = -2$

NORMAL GRADIENT $= +\frac{1}{2}$

EQUATION OF L_1 MUST BE

$y = \frac{1}{2}x + 6$

(USING $y = mx + c$)

b) $\left.\frac{dy}{dx}\right|_{x=3} = 1 \implies$ NORMAL GRADIENT MUST BE -1

$\implies y - y_0 = m(x - x_0)$

$\implies y - \frac{9}{2} = -1(x-3)$

$\implies 2y - 9 = -2(x-3)$

$\implies 2y - 9 = -2x + 6$

$\implies 2x + 2y = 15$

FIND THE INTERSECTION BETWEEN L_1 & L_2

$L_1: \quad y = \frac{1}{2}x + 6$

$L_2: \quad 2x + 2y = 15$

$\implies 2x + 2\left(\frac{1}{2}x + 6\right) = 15$

$\implies 2x + x + 12 = 15$

$\implies 3x = 3$

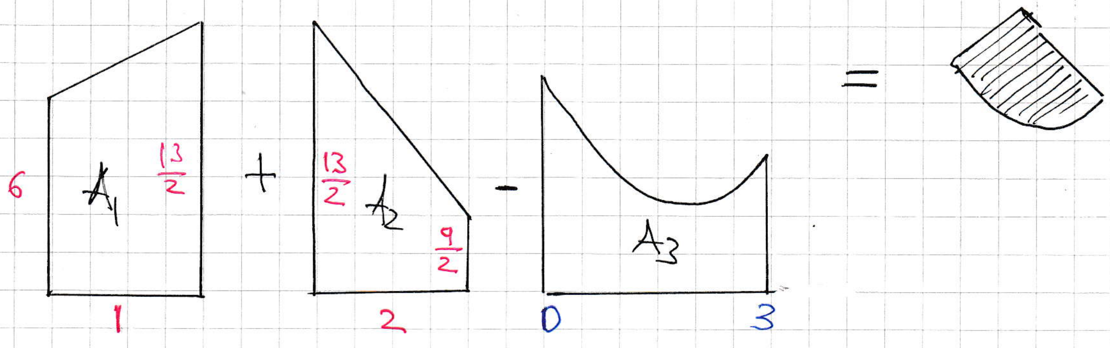
$\implies x = 1$

$\implies y = 6\frac{1}{2} = \frac{13}{2}$

$\therefore \left(1, \frac{13}{2}\right)$

IYGB - SYNOPTIC PAPER J - QUESTION 11

LOOKING AT THE DIAGRAM BELOW



- $A_1 = \dfrac{\frac{13}{2}+6}{2} \times 1 = \dfrac{25}{4}$

- $A_2 = \dfrac{\frac{13}{2}+\frac{9}{2}}{2} \times 2 = 11$

- $A_3 = \displaystyle\int_0^3 \tfrac{1}{2}x^2 - 2x + 6 \, dx = \left[\tfrac{1}{6}x^3 - x^2 + 6x\right]_0^3$

$$= \left(\tfrac{9}{2} - 9 + 18\right) - (0) = \tfrac{27}{2}$$

REQUIRED AREA $= A_1 + A_2 - A_3$

$$= \frac{25}{4} + 11 - \frac{27}{2}$$

$$= \frac{15}{4}$$

i

IYGB – SYNOPTIC PAPER J – QUESTION 12

TAKE "A" TO BE THE ORIGIN

$$\vec{AP} + 4\vec{BP} + 3\vec{PC} = \vec{0}$$

$$\vec{AP} + 4(\vec{BA}+\vec{AP}) + 3(\vec{PA}+\vec{AC}) = \vec{0}$$

$$\vec{AP} + 4\vec{BA} + 4\vec{AP} + 3\vec{PA} + 3\vec{AC} = \vec{0}$$

$$\vec{AP} + 4\vec{AP} + 3\vec{PA} = -4\vec{BA} - 3\vec{AC}$$

$$\vec{AP} + 4\vec{AP} - 3\vec{AP} = -\vec{BA} - 3\vec{BA} - 3\vec{AC}$$

$$2\vec{AP} = -\vec{BA} - 3(\vec{BA}+\vec{AC})$$

$$2\vec{AP} = \vec{AB} - 3\vec{BC}$$

$$\vec{AP} = \frac{1}{2}(\vec{AB} - 3\vec{BC})$$

// AS REQUIRED

IYGB - SYNOPTIC PAPER J - QUESTION 13

a) ADDING BY OBTAINING COMMON DENOMINATOR

$$f(x) = 2 + \frac{1}{2x-1} = \frac{2(2x-1)}{(2x-1)} + \frac{1}{2x-1} = \frac{2(2x-1)+1}{2x-1} = \frac{4x-2+1}{2x-1}$$

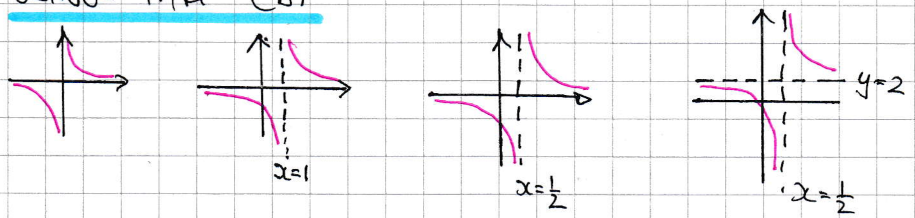
$$\therefore f(x) = \frac{4x-1}{2x-1}$$

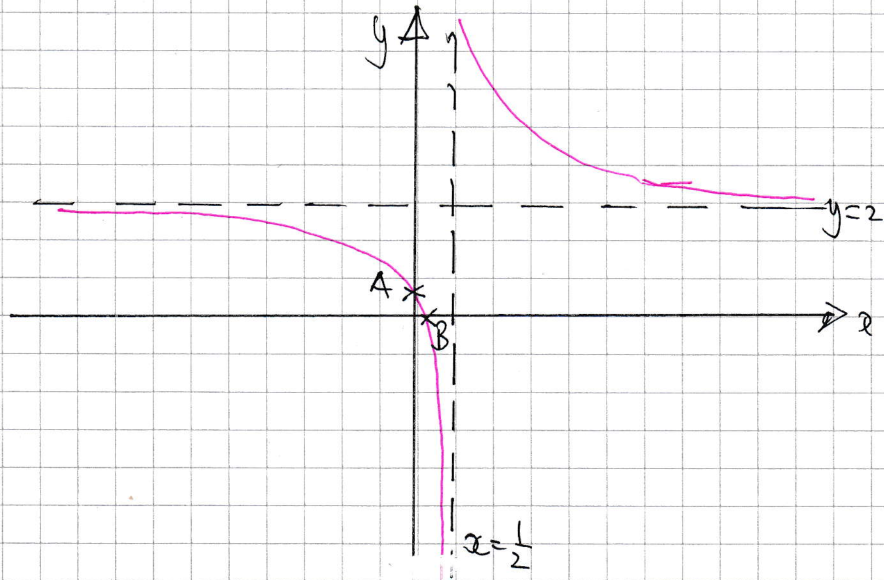
b) T_1: TRANSLATION BY THE VECTOR $\binom{1}{0}$ "$f(x-1)$"

T_2: STRETCH PARALLEL TO THE x AXIS BY SCALE FACTOR $\frac{1}{2}$ "$f(2x)$"

T_3: TRANSLATION BY THE VECTOR $\binom{0}{2}$ "$f(x)+2$"

c) USING PART (b)





$x=0$
$y=1$ $\therefore A(0,1)$

$y=0$
$4x-1=0$
$4x=1$
$x=\frac{1}{4}$ $\therefore B(\frac{1}{4},0)$

d) SOLVING $y=3$

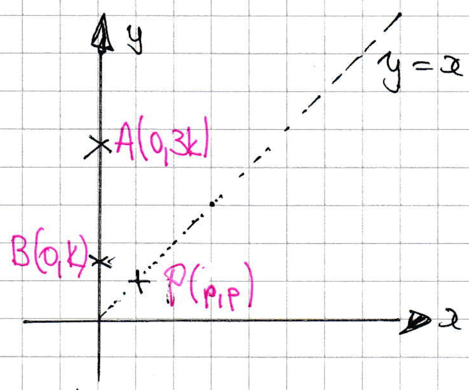
$$\frac{4x-1}{2x-1} = 3$$

$$4x-1 = 6x-3$$

$$2 = 2x$$

$\therefore (1,3)$

IYGB – SYNOPTIC PAPER J – QUESTION 14



GRADIENT AP = $\dfrac{p-3k}{p}$

GRADIENT PB = $\dfrac{p-k}{p}$

EQUATION OF AP

$$y = \frac{p-3k}{p}x + 3k$$

EQUATION OF BP

$$y = \frac{p-k}{p}x + k$$

x INTERCEPT OF AP

$$0 = \frac{p-3k}{p}x + 3k$$
$$0 = (p-3k)x + 3kp$$
$$(p-3k)x = -3kp$$
$$x = \frac{-3kp}{p-3k}$$

x INTERCEPT OF BP

$$0 = \frac{p-k}{p}x + k$$
$$0 = (p-k)x + kp$$
$$(p-k)x = -kp$$
$$x = \frac{-kp}{p-k}$$

FINALLY WE HAVE

$$\frac{1}{x_1} - \frac{1}{x_2} = \frac{p-3k}{-3kp} - \frac{p-k}{-kp}$$
$$= \frac{3k-p}{3kp} - \frac{k-p}{kp}$$
$$= \frac{3k-p}{3kp} - \frac{3k-3p}{3kp}$$
$$= \frac{3k-p-3k+3p}{3kp}$$
$$= \frac{2p}{3kp}$$
$$= \frac{2}{3k}$$

AS REQUIRED

IYGB - SYNOPTIC PAPER J - QUESTION 15

LET $f(x) = \dfrac{1}{2-x}$

$$f(x+h) = \frac{1}{2-(x+h)} = \frac{1}{2-h-x}$$

FROM THE DEFINITION OF THE DERIVATIVE

$$f'(x) = \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]$$

$$f'(x) = \lim_{h\to 0}\left[\frac{\frac{1}{2-x-h}-\frac{1}{2-x}}{h}\right]$$

$$f'(x) = \lim_{h\to 0}\left[\frac{\frac{\cancel{(2-x)}-(\cancel{2-x}-h)}{(2-x-h)(2-x)}}{h}\right]$$

$$f'(x) = \lim_{h\to 0}\left[\frac{\frac{h}{(2-x)(2-x-h)}}{h}\right]$$

$$f'(x) = \lim_{h\to 0}\left[\frac{h}{(2-x)(2-x-h)} \div h\right]$$

$$f'(x) = \lim_{h\to 0}\left[\frac{\cancel{h}}{(2-x)(2-x-h)} \times \frac{1}{\cancel{h}}\right]$$

$$f'(x) = \lim_{h\to 0}\left[\frac{1}{(2-x)(2-x-h)}\right]$$

$$f'(x) = \frac{1}{(2-x)^2}$$

IYGB - SYNOPTIC PAPER J - QUESTION 16

a) FORMING A DIFFERENTIAL EQUATION

$$\frac{dh}{dt} = -k\,h^{\frac{1}{2}}$$

RATE

LEAK

PROPORTIONAL

SQUARE ROOT OF HEIGHT

h = HEIGHT OF CHEMICAL (cm)

t = TIME IN MINUTES

$t=0$, $h=100$

$$\left.\frac{dh}{dt}\right|_{\substack{t=0\\h=100}} = -0.25$$

APPLY THE CONDITION $\left.\frac{dh}{dt}\right|_{h=100} = -0.25$

$$-0.25 = -k \times 100^{\frac{1}{2}}$$

$$10k = 0.25$$

$$k = \frac{1}{40}$$

$$\therefore \frac{dh}{dt} = -\frac{1}{40}h^{\frac{1}{2}}$$

b) SEPARATING VARIABLES TO OBTAIN

$$\Rightarrow \frac{dh}{dt} = -\frac{1}{40}h^{\frac{1}{2}}$$

$$\Rightarrow dh = -\frac{1}{40}h^{\frac{1}{2}}\,dt$$

$$\Rightarrow \frac{1}{h^{\frac{1}{2}}}\,dh = -\frac{1}{40}\,dt$$

$$\Rightarrow \int h^{-\frac{1}{2}}\,dh = \int -\frac{1}{40}\,dt$$

$$\Rightarrow 2h^{\frac{1}{2}} = -\frac{1}{40}t + C$$

IYGB - SYNOPTIC PAPER - QUESTION 16

APPLY THE CONDITION $t=0$ $h=100$ cm

$$\Longrightarrow \quad 2\times 100^{\frac{1}{2}} = -\frac{1}{40}\times 0 + C$$

$$\Longrightarrow \quad C = 20$$

$$\Longrightarrow \quad 2h^{\frac{1}{2}} = 20 - \frac{1}{40}t$$

THE LEAKING STARTED WHEN $h=160$ cm

$$\Longrightarrow \quad 2\times\sqrt{160} = 20 - \frac{1}{40}t$$

$$\Longrightarrow \quad 80\sqrt{160} = 800 - t$$

$$\Longrightarrow \quad t = 800 - 80\sqrt{160}$$

$$\Longrightarrow \quad t = -211.92\ldots \text{ minutes}$$

$$\Longrightarrow \quad t = -3 \text{ HOURS } \& \text{ 32 MINUTES, BEFORE } t=0$$

$\therefore$ DRUM HAS BEEN LEAKING FOR 3 HOURS & 32 MINUTES

IYGB - SYNOPTIC PAPER J - QUESTION 17

START COLLECTING AUXILIARY INFORMATION

- WHEN $y=0$

$\sin t = 0$

$t = \begin{cases} 0 & \leftarrow \text{ORIGIN} \\ \pi & \leftarrow A(\pi^2, 0) \end{cases}$

- WHEN $t = \frac{2\pi}{3}$

$x = \left(\frac{2\pi}{3}\right)^2 = \frac{4}{9}\pi^2$

$y = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

$P\left(\frac{4}{9}\pi^2, \frac{\sqrt{3}}{2}\right)$

FIND THE EQUATION OF THE TANGENT AT P

- $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t}{2t}$

- $\left.\frac{dy}{dx}\right|_{t=\frac{2\pi}{3}} = \frac{\cos\frac{2\pi}{3}}{2\times\frac{2\pi}{3}} = \frac{-\frac{1}{2}}{\frac{4\pi}{3}} = -\frac{3}{8\pi}$

- $y - y_0 = m(x - x_0)$

$y - \frac{\sqrt{3}}{2} = -\frac{3}{8\pi}\left(x - \frac{4}{9}\pi^2\right)$

WHEN $y=0$ (TO FIND B)

$\Rightarrow -\frac{\sqrt{3}}{2} = -\frac{3}{8\pi}\left(x - \frac{4}{9}\pi^2\right)$

$\Rightarrow \frac{8\pi\sqrt{3}}{6} = x - \frac{4}{9}\pi^2$

$\Rightarrow x = \frac{4}{9}\pi^2 + \frac{4}{3}\pi\sqrt{3}$

$\therefore B\left(\frac{4}{9}\pi^2 + \frac{4}{3}\pi\sqrt{3}, 0\right)$

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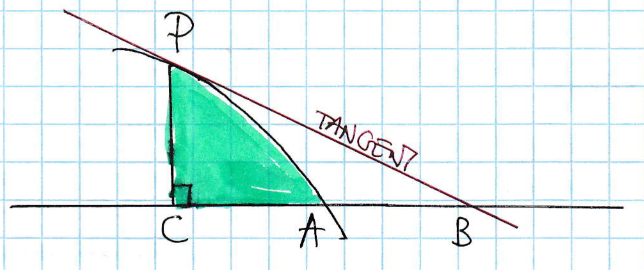
LOOKING AT THE DIAGRAM BELOW



- $|PC| = \frac{\sqrt{3}}{2}$
- $|BC| = \left(\cancel{\frac{4}{9}\pi^2} + \frac{4}{3}\pi\sqrt{3}\right) - \cancel{\frac{4}{9}\pi^2}$
$= \frac{4}{3}\sqrt{3}\pi$

- AREA OF $\overset{\Delta}{BCP} = \frac{1}{2}|PC||BC| = \frac{1}{2} \times \frac{\sqrt{3}}{2} \times \frac{4}{3}\sqrt{3}\pi = \pi$

- AREA UNDER THE PARAMETRIC CURVE (SHOWN IN GREEN)

$$\text{AREA} = \int_{x_C}^{x_A} y(x)\,dx = \int_{t_P}^{t_A} y(t)\frac{dx}{dt}\,dt$$

$$= \int_{\frac{2}{3}\pi}^{\pi} (\sin t)(2t)\,dt = \int_{\frac{2}{3}\pi}^{\pi} 2t\sin t\,dt$$

INTEGRATION BY PARTS

$2t$	2
$-\cos t$	$\sin t$

$$\ldots = \left[-2t\cos t\right]_{\frac{2\pi}{3}}^{\pi} - \int_{\frac{2\pi}{3}}^{\pi} -2\cos t\,dt$$

$$= \left[-2t\cos t\right]_{\frac{2\pi}{3}}^{\pi} + \int_{\frac{2\pi}{3}}^{\pi} 2\cos t\,dt$$

$$= \left[2\sin t - 2t\cos t\right]_{\frac{2}{3}\pi}^{\pi}$$

$$= (0 + 2\pi) - \left(\sqrt{3} + \frac{2\pi}{3}\right) = \frac{4}{3}\pi - \sqrt{3}$$

FINALLY THE REQUIRED AREA

$$\pi - \left(\frac{4}{3}\pi - \sqrt{3}\right) = -\frac{\pi}{3} + \sqrt{3} = \frac{1}{3}\left(3\sqrt{3} - \pi\right)$$

IYGB - SYNOPTIC PAPER J - QUESTION 18

START BY FACTORIZING THE EXPRESSION

$$f(k) = k^3 + 2k = k(k^2+2)$$

THE POSITIVE INTEGER k HAS ONE OF THE FOLLOWING THREE FORMS

$$k = 3n \ , \ 3n+1 \ , \ 3n+2$$

EXAMINING EACH CASE

- $f(3n) = 3n(9n^2+2) = 3\left[n(9n^2+2)\right]$
- $f(3n+1) = (3n+1)\left[(3n+1)^2+2\right] = (3n+1)(9n^2+6n+1+2)$
  $= (3n+1)(9n^2+6n+3) = 3(3n+1)(3n^2+2n+1)$
- $f(3n+2) = (3n+2)\left[(3n+2)^2+2\right] = (3n+2)(9n^2+12n+4+2)$
  $= (3n+2)(9n^2+12n+6) = 3(3n+2)(3n^2+4n+2)$

$\therefore$ BY EXHAUSTION $f(k) = k^3+2k$ IS A MULTIPLE OF 3, $k \in \mathbb{N}$

IYGB - SYNOPTIC PAPER J - QUESTION 19

SPLIT THE SUMMATION AS FOLLOWS

$$\sum_{n=1}^{\infty}\left[\frac{3^n-2}{4^{n+1}}\right] = \sum_{n=1}^{\infty}\left[\frac{3^n}{4^{n+1}} - \frac{2}{4^{n+1}}\right] = \sum_{n=1}^{\infty}\left[\frac{3^n}{4^{n+1}}\right] - \sum_{n=1}^{\infty}\left[\frac{2}{4^{n+1}}\right]$$

$$= \sum_{n=1}^{\infty}\left[\frac{3^n}{4\times 4^n}\right] - \sum_{n=1}^{\infty}\left[\frac{2}{4\times 4^n}\right]$$

$$= \frac{1}{4}\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n - \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n$$

THIS IS A G.P.

$a = \frac{3}{4}$

$r = \frac{3}{4}$

$S_{\infty} = \frac{\frac{3}{4}}{1-\frac{3}{4}}$

$S_{\infty} = 3$

THIS IS A G.P

$a = \frac{1}{4}$

$r = \frac{1}{4}$

$S_{\infty} = \frac{\frac{1}{4}}{1-\frac{1}{4}}$

$S_{\infty} = \frac{1}{3}$

PUTTING ALL THE RESULTS TOGETHER

$$\sum_{n=1}^{\infty}\left[\frac{3^n-2}{4^{n+1}}\right] = \frac{1}{4}\times 3 - \frac{1}{2}\times\frac{1}{3} = \frac{3}{4} - \frac{1}{6} = \frac{7}{12}$$

IYGB – SYNOPTIC PAPER J – QUESTION 20

a) SUPPOSE THAT

$$\sqrt{6+2\sqrt{6}} \le \sqrt{3}+\sqrt{2}$$

$$\Longrightarrow \quad 6+2\sqrt{6} \le \left(\sqrt{3}+\sqrt{2}\right)^2$$

$$\Longrightarrow \quad 6+2\sqrt{6} \le 3+2\sqrt{6}+2$$

$$\Longrightarrow \quad 6 \le 5$$

WHICH IS A CONTRADICTION

$$\therefore \quad \sqrt{6+2\sqrt{6}} > \sqrt{3}+\sqrt{2}$$

b) SUPPOSE THAT

$$\sqrt[3]{3} \le \sqrt{2}$$

$$\Longrightarrow \quad 3^{\frac{1}{3}} \le 2^{\frac{1}{2}}$$

$$\Longrightarrow \quad \left(3^{\frac{1}{3}}\right)^6 \le \left(2^{\frac{1}{2}}\right)^6$$

$$\Longrightarrow \quad 3^2 \le 2^3$$

$$\Longrightarrow \quad 9 \le 8$$

THIS IS A CONTRADICTION

$$\therefore \quad \sqrt[3]{3} > \sqrt{2}$$

c) SUPPOSE THAT

$$\sqrt{2}-1 \le \sqrt{3}-\sqrt{2}$$

$$\Longrightarrow \quad 2\sqrt{2} \le \sqrt{3}+1$$

$$\Longrightarrow \quad 8 \le \left(\sqrt{3}+1\right)^2$$

$$\Longrightarrow \quad 8 \le 3+2\sqrt{3}+1$$

$$\Longrightarrow \quad \text{[illegible]} \le 2\sqrt{3}$$

$$\Longrightarrow \quad 2 \le \sqrt{3}$$

IYGB - SYNOPTIC PAPER J - QUESTION 20

$\Rightarrow 4 \leqslant 3$

THIS IS A CONTRADICTION

$\therefore \sqrt{2}-1 > \sqrt{3}-\sqrt{2}$

ALTERNATIVE APPROACH BASED ON

IF $a>b>0 \Rightarrow a^n > b^n$ FOR $n=1,2,3,4\ldots$

a) $\sqrt{6+2\sqrt{6}}$ SQUARES TO $6+2\sqrt{6}$

$(\sqrt{3}+\sqrt{2})^2 = 3+2\sqrt{6}+2 = 5+2\sqrt{6}$

$\therefore \sqrt{6+2\sqrt{6}} > \sqrt{3}+\sqrt{2}$

b) $\sqrt[3]{3} = 3^{\frac{1}{3}}$ & $(3^{\frac{1}{3}})^6 = 3^2 = 9$

$\sqrt{2} = 2^{\frac{1}{2}}$ & $(2^{\frac{1}{2}})^6 = 2^3 = 8$

$\therefore \sqrt[3]{3} > \sqrt{2}$

IYGB – SYNOPTIC PAPER J – QUESTION 21

PROCEED AS FOLLOWS

$$\Rightarrow 2 + 4\cos^2\theta = 7\cos\theta\sin\theta$$
$$\Rightarrow 2(\cos^2\theta + \sin^2\theta) + 4\cos^2\theta = 7\cos\theta\sin\theta$$
$$\Rightarrow 6\cos^2\theta - 7\cos\theta\sin\theta + 2\sin^2\theta = 0$$

FACTORIZE THE QUADRATIC EXPRESSION

$$\Rightarrow (3\cos\theta - 2\sin\theta)(2\cos\theta - \sin\theta) = 0$$

HENCE WE OBTAIN TWO EQUATIONS

$$\Rightarrow 3\cos\theta - 2\sin\theta = 0$$
$$\Rightarrow 3\cos\theta = 2\sin\theta$$
$$\Rightarrow \frac{3\cos\theta}{\cos\theta} = \frac{2\sin\theta}{\cos\theta}$$
$$\Rightarrow 3 = 2\tan\theta$$
$$\Rightarrow \tan\theta = \frac{3}{2}$$
$$\arctan\left(\frac{3}{2}\right) = 56.3°$$
$$\Rightarrow \theta = 56.3 \pm 180n \quad n = 0, 1, 2, 3, \ldots$$

$$\Rightarrow 2\cos\theta - \sin\theta = 0$$
$$\Rightarrow 2\cos\theta = \sin\theta$$
$$\Rightarrow \frac{2\cos\theta}{\cos\theta} = \frac{\sin\theta}{\cos\theta}$$
$$\Rightarrow 2 = \tan\theta$$
$$\arctan(2) = 63.4°$$
$$\Rightarrow \theta = 63.4 \pm 180n \quad n = 0, 1, 2, 3, \ldots$$

$$\theta = 56.3°, 63.4°, 236.3°, 243.4°$$

ALTERNATIVE APPROACH USING MINOR TRIG RATIOS AS FOLLOWS

$$\Rightarrow 2 + 4\cos^2\theta = 7\cos\theta\sin\theta$$
$$\Rightarrow \frac{2}{\cos^2\theta} + \frac{4\cos^2\theta}{\cos^2\theta} = \frac{7\cos\theta\sin\theta}{\cos^2\theta}$$

IYGB - SYNOPTIC PAPER J - QUESTION 21

$$\Rightarrow 2\sec^2\theta + 4 = 7\tan\theta$$

$$\Rightarrow 2(1+\tan^2\theta) + 4 = 7\tan\theta$$

$$\Rightarrow 2 + 2\tan^2\theta + 4 = 7\tan\theta$$

$$\Rightarrow 2\tan^2\theta - 7\tan\theta + 6 = 0$$

$$\Rightarrow (2\tan\theta - 3)(\tan\theta - 2) = 0$$

$$\Rightarrow \tan\theta = \begin{cases} 3/2 \\ 2 \end{cases}$$

ETC. ETC AS WITH THE PREVIOUS METHOD