D OPERATOR

I.Y.G.B. madasmaths.com

Summary of D Operator

Basic Definitions

- $D\{\ \} \equiv \dfrac{d}{dx}(\ )$,

  or depending on the independent variable

  $$D\{\ \} \equiv \frac{d}{dt}(\ ),\ D\{\ \} \equiv \frac{d}{dy}(\ ),\ D\{\ \} \equiv \frac{d}{du}(\ ) \text{ etc}$$

- $D^2\{\ \} \equiv \dfrac{d^2}{dx^2}(\ ),\ D^3 \equiv \dfrac{d^3}{dx^3}\{\ \},\ D^4\{\ \} \equiv \dfrac{d^4}{dx^4}(\ )$ etc

- $\left(aD^2 + bD + c\right)\{f(x)\} \equiv aD^2\{f(x)\} + bD\{f(x)\} + c\{f(x)\}$

- $\left(\dfrac{1}{aD^2 + bD + c}\right)\{f(x)\} \equiv \left(aD^2 + bD + c\right)^{-1}\{f(x)\}$

- $\dfrac{1}{D}\{\ \} \equiv \displaystyle\int (\ )\,dx$

D Operations on Simple Functions

- $\left[f(D)\right]\left\{e^{kx}\right\} \equiv e^{kx}\left[f(k)\right]$

- $\left[f(D)\right]\left\{e^{kx}V(x)\right\} \equiv e^{kx}\left[f(D+k)\right]\left\{V(x)\right\}$

- $\left[f\left(D^2\right)\right]\{\sin kx\} \equiv \left[f\left(-k^2\right)\right]\sin kx$

- $\left[f\left(D^2\right)\right]\{\cos kx\} \equiv \left[f\left(-k^2\right)\right]\cos kx$

GENERAL PRACTICE

Question 1

Simplify each of the following expressions.

a) $D^2\{\sin 2x\}$.

b) $(D+3)\{e^{2x}\}$.

c) $D^5\{e^{-2x}\}$.

d) $\frac{1}{D}\{e^{3x}\}$.

$\boxed{-4\sin 2x}$, $\boxed{5e^{2x}}$, $\boxed{-32e^{-2x}}$, $\boxed{\frac{1}{3}e^{3x}+C}$

a) $D^2\{\sin 2x\} = \frac{d^2}{dx^2}(\sin 2x) = \frac{d}{dx}(2\cos 2x) = -4\sin 2x$

b) $(D+3)\{e^{2x}\} = D\{e^{2x}\} + 3\times e^{2x} = \frac{d}{dx}(e^{2x}) + 3e^{2x}$
$= 2e^{2x} + 3e^{2x} = 5e^{2x}$

c) $D^5\{e^{-2x}\} = (-2)^5 e^{-2x} = -32e^{-2x}$

d) $\frac{1}{D}\{e^{3x}\} = \int e^{3x}\,dx = \frac{1}{3}e^{3x} + C$

Question 2

Simplify each of the following expressions.

a) $\left(D^2+2D+2\right)\left\{e^{3x}\right\}$.

b) $\left(D^2+2D+2\right)\left\{x^2 e^{3x}\right\}$.

c) $\left(D^2+2D+2\right)\left\{\sin 2x\right\}$.

$\boxed{17e^{3x}}$, $\boxed{\left(17x^2+16x+2\right)e^{3x}}$, $\boxed{4\cos 2x - 2\sin 2x}$

a) $(D^2+2D+2)\{e^{3x}\} = (3^2+2\times3+2)e^{3x} = 17e^{3x}$

b) $(D^2+2D+2)\{x^2e^{3x}\} = e^{3x} f(D+3)\{x^2\}$

$= e^{3x}\left[(D+3)^2+2(D+3)+2\right]\{x^2\}$

$= e^{3x}\left[D^2+6D+9+2D+6+2\right]\{x^2\}$

$= e^{3x}\left[D^2+8D+17\right]\{x^2\}$

$= e^{3x}\left[2+8(2x)+17x^2\right]$

$= (17x^2+16x+2)e^{3x}$

c) $(D^2+2D+2)\{\sin 2x\} = (-2^2+2D+2)\{\sin 2x\}$

$= 2D-2\{\sin 2x\}$

$= 2(2\cos 2x) - 2\sin 2x$

$= 4\cos 2x - 2\sin 2x$

Question 3

$$f(D) = D^2 - 4D + 3.$$

Show directly, by the definition of the D operator as $D \equiv \frac{d}{dx}(\ )$, that

$$f(D)\left\{e^{kx}\right\} = f(k)e^{kx}.$$

$\boxed{\text{proof}}$

$$f(D) = D^2 - 4D + 3$$

$$\begin{aligned} f(D)\{e^{kx}\} &= (D^2 - 4D + 3)\{e^{kx}\} \\ &= D^2\{e^{kx}\} - 4D\{e^{kx}\} + 3\{e^{kx}\} \\ &= k^2e^{kx} - 4(ke^{kx}) + 3e^{kx} \\ &= (k^2 - 4k + 3)e^{kx} \\ &= f(k)e^{kx} \end{aligned}$$

AS REQUIRED

Question 4

$$g(D) = \frac{1}{D^2 - 4D + 3}.$$

Show directly, by the definition of the D operator as $D \equiv \frac{d}{dx}(\ )$, that

$$f(D)\left\{e^{kx}\right\} = \frac{1}{f(k)}e^{kx},$$

where k is a constant.

$\boxed{\text{proof}}$

$$g(D) = \frac{1}{D^2 - 4D + 3}$$

$$\begin{aligned} g(D)\{e^{kx}\} &= \frac{1}{D^2 - 4D + 3}\{e^{kx}\} = (D^2 - 4D + 3)^{-1}\{e^{kx}\} \\ &= (k^2 - 4k + 3)^{-1}e^{kx} \\ &= \frac{1}{k^2 - 4k + 3}e^{kx} \\ &= \frac{1}{f(k)}e^{kx} \end{aligned}$$

Question 5

$$f(D) = 2D^2 - D + 1.$$

Show directly, by the definition of the D operator as $D \equiv \frac{d}{dx}(\ )$, that

$$f(D)\{e^{kx}V(x)\} = e^{kx} f(D+k)\{V(x)\},$$

where $V(x)$ is a smooth function of x and k is a constant.

$\boxed{\quad}$, $\boxed{\text{proof}}$

$\boxed{f(D) = 2D^2 - D + 1 \quad \text{, HERE } \{\ \} \text{ INDICATE } D \text{ OPERATOR ARGUMENT}}$

DIRECTLY FROM THE DEFINITIONS INC LINEARITY PROPERTY

$f(D)\{e^{kx}V(x)\} = (2D^2 - D + 1)\{e^{kx}V(x)\}$

$= 2D^2\{e^{kx}V(x)\} - D\{e^{kx}V(x)\} + 1\{e^{kx}V(x)\}$

DIFFERENTIATING BY THE PRODUCT RULE

$= 2D\{ke^{kx}V(x) + e^{kx}V'(x)\} - ke^{kx}V(x) - e^{kx}V'(x) + e^{kx}V(x)$

$= 2D\{e^{kx}(kV(x) + V'(x))\} + e^{kx}[V(x) - kV(x) - V'(x)]$

BY THE PRODUCT RULE AGAIN

$= 2[ke^{kx}(kV(x) + V'(x)) + e^{kx}(kV'(x) + V''(x))] + e^{kx}[V(x) - kV(x) - V'(x)]$

$= e^{kx}[2k^2V(x) + 2kV'(x) + 2kV'(x) + 2V''(x) + V(x) - kV(x) - V'(x)]$

$= e^{kx}[2V''(x) + (4k-1)V'(x) + (2k^2 - k + 1)V(x)]$

APPLY THE DEFINITION OF D OPERATOR

$= e^{kx}[2D^2\{V(x)\} + (4k-1)D\{V(x)\} + (2k^2 - k + 1)V(x)]$

$= e^{kx}[2D^2 + (4k-1)D + (2k^2 - k + 1)]\{V(x)\}$

$= e^{kx}[2D^2 + 4Dk + 2k^2 - D - k + 1]\{V(x)\}$

$= e^{kx}[2(D^2 + 2Dk + k^2) - (D+k) + 1]\{V(x)\}$

$= e^{kx}[2(D+k)^2 - (D+k) + 1]\{V(x)\}$

$= e^{kx} f(D+k)\{V(x)\}$

AS REQUIRED

Question 6

Simplify each of the following expressions.

a) $\dfrac{1}{D^2+4D+3}\left\{30\mathrm{e}^{-2x}\right\}$.

b) $\dfrac{1}{D^2+4D+3}\{30\sin 2x\}$.

c) $\dfrac{1}{D^2+4D+4}\left\{30x^2\,\mathrm{e}^{-2x}\right\}$.

d) $\dfrac{1}{D^2+4D+4}\{30\}$.

e) $\dfrac{1}{D^2+2D}\{30\}$.

$\boxed{}$, $\boxed{-30\mathrm{e}^{-2x}}$, $\boxed{-\frac{6}{13}[8\cos 2x+\sin 2x]}$, $\boxed{\frac{5}{2}x^4\,\mathrm{e}^{-2x}}$, $\boxed{\frac{15}{2}}$, $\boxed{15x}$

a) $\dfrac{1}{D^2+4D+3}\left\{30e^{-2x}\right\} = \dfrac{30e^{-2x}}{(-2)^2+4(-2)+3} = \dfrac{30e^{-2x}}{4-8+3} = -30e^{-2x}$

b) $\dfrac{1}{D^2+4D+3}\left\{30\sin 2x\right\} = \dfrac{30}{-2^2+4D+3}\left\{\sin 2x\right\} = \dfrac{30}{4D-1}\left\{\sin 2x\right\}$

$= \dfrac{30(4D+1)}{(4D-1)(4D+1)}\left\{\sin 2x\right\} = \dfrac{30(4D+1)}{16D^2-1}\left\{\sin 2x\right\}$

$= \dfrac{30(4D+1)}{16(-2^2)-1}\left\{\sin 2x\right\} = \dfrac{30(4D+1)}{-65}\left\{\sin 2x\right\}$

$= -\dfrac{6}{13}\left[4D\left\{\sin 2x\right\} + \sin 2x\right]$

$= -\dfrac{6}{13}\left[8\cos 2x + \sin 2x\right]$

c) $\dfrac{1}{D^2+4D+4}\left\{\underbrace{30x^2}_{V(x)}e^{-2x}\right\} = \dfrac{30e^{-2x}}{(D-2)^2+4(D-2)+4}\left\{x^2\right\} = \dfrac{30e^{-2x}}{D^2}\left\{x^2\right\}$

$= \dfrac{30e^{-2x}}{D}\times\dfrac{1}{D}\left\{x^2\right\} = \dfrac{30e^{-2x}}{D}\left\{\dfrac{1}{3}x^3\right\}$

$= 10e^{-2x}\dfrac{1}{D}\left\{x^3\right\} = 10e^{-2x}\times\dfrac{1}{4}x^4$

$= \dfrac{5}{2}x^4e^{-2x}$

d) $\dfrac{1}{D^2+4D+4}\left\{30\right\} = \dfrac{1}{D^2+4D+4}\left\{30e^{0x}\right\} = \dfrac{30e^{0x}}{0+0+4}$

$= \dfrac{1}{4}\times 30 = \dfrac{15}{2}$

e) $\dfrac{1}{D^2+D}\left\{30\right\} = \dfrac{1}{D(D+2)}\left\{30\right\} = \dfrac{1}{D}\left[\dfrac{1}{D+2}\left\{30\right\}\right]$

$= \dfrac{1}{D}\left[\dfrac{1}{D+2}\left\{30e^{0x}\right\}\right] = \dfrac{1}{D}\left[\dfrac{30e^{0x}}{0+2}\right]$

$= \dfrac{1}{D}\left\{15\right\} = 15x$

Question 7

Simplify each of the following expressions.

a) $\dfrac{1}{D^2-4D+3}\left\{e^{2x}\right\}$.

b) $\dfrac{1}{D^2-4D+3}\left\{e^{3x}\right\}$.

c) $\dfrac{1}{D^2-4D+3}\left\{\sin 2x\right\}$.

d) $\dfrac{1}{D^2+1}\left\{\cos x\right\}$.

$\boxed{}$, $\boxed{-e^{2x}}$, $\boxed{\frac{1}{2}xe^{3x}}$, $\boxed{\frac{1}{65}[8\cos 2x-\sin 2x]}$, $\boxed{\frac{1}{2}x\sin x}$

a) $\frac{1}{D^2-4D+3}\{e^{2x}\} = \frac{1}{2^2-4\times2+3}\times e^{2x} = -e^{2x}$

b) $\frac{1}{D^2-4D+3}\{e^{3x}\} = \frac{1}{3^2-4\times3+3}\times e^{3x}$ .... THERE IS A PROBLEM

INTRODUCE A FUNCTION $V(x)=1=e^{0x}$

$= \frac{1}{D^2-4D+3}\{1\times e^{3x}\} = \frac{1\times e^{3x}}{(D+3)^2-4(D+3)+3}\{1\} = \frac{e^{3x}}{D^2+2D}\{1\}$

$= \frac{e^{3x}}{D^2+2D}\{e^{0x}\} = \frac{e^{3x}}{D(D+2)}\{e^{0x}\}$

$= \frac{e^{3x}}{D}\cdot\frac{1}{D+2}\{e^{0x}\} = \frac{e^{3x}}{D}\left\{\frac{1}{2}e^{0x}\right\}$

$= e^{3x}\frac{1}{D}\left\{\frac{1}{2}\right\} = \frac{1}{2}xe^{3x}$

c) $\frac{1}{D^2-4D+3}\{\sin 2x\} = \frac{1}{-2^2-4D+3}\{\sin 2x\} = \frac{1}{-1-4D}\{\sin 2x\}$

$= \frac{-1}{4D+1}\{\sin 2x\} = \frac{-1(4D-1)}{(4D+1)(4D-1)}\{\sin 2x\}$

$= \frac{1-4D}{16D^2-1}\{\sin 2x\} = \frac{1-4D}{16(-2^2)-1}\{\sin 2x\}$

$= \frac{1-4D}{-65}\{\sin 2x\} = \frac{4D-1}{65}\{\sin 2x\}$

$= \frac{1}{65}(4D-1)\{\sin 2x\} = \frac{1}{65}[4\times2\cos 2x-\sin 2x]$

$= \frac{1}{65}(8\cos 2x-\sin 2x)$

d) $\frac{1}{D^2+1}\{\cos x\} = \frac{1}{-1^2+1}\{\cos x\} = \dots$ FAILS

PROCEED BY COMPLEX NUMBERS AS FOLLOWS

$\frac{1}{D^2+1}\{\cos x\} = \frac{1}{D^2+1}\{\underset{V(x)}{1}\times \mathrm{Re}(e^{ix})\} = \mathrm{Re}\left[\frac{1}{D^2+1}\{\underset{V(x)}{1}\times e^{ix}\}\right]$

$= \mathrm{Re}\left[\frac{e^{ix}}{(D+i)^2+1}\{1\}\right]$

$= \mathrm{Re}\left[\frac{e^{ix}}{D^2+2Di}\{1\}\right] = \mathrm{Re}\left[\frac{e^{ix}}{D(D+2i)}\{1\}\right]$

$= \mathrm{Re}\left[\frac{e^{ix}}{D}\cdot\frac{1}{D+2i}\{1\}\right]$

$= \mathrm{Re}\left[\frac{e^{ix}}{D}\cdot\frac{1}{D+2i}\{e^{0x}\}\right]$

$= \mathrm{Re}\left[\frac{e^{ix}}{D}\left\{\frac{1}{0+2i}e^{0x}\right\}\right]$

$= \mathrm{Re}\left[\frac{e^{ix}}{2i}\frac{1}{D}\{1\}\right] = \mathrm{Re}\left[\frac{e^{ix}}{2i}x\right]$

$= \frac{1}{2}x\,\mathrm{Re}\left[\frac{e^{ix}}{i}\right] = \frac{1}{2}x\,\mathrm{Re}\left[-ie^{ix}\right]$

$= \frac{1}{2}x\,\mathrm{Re}\left[-i\cos x - i(i\sin x)\right]$

$= \frac{1}{2}x\sin x$

Question 8

Simplify each of the following expressions.

a) $\dfrac{1}{D^2-4D+4}\left\{e^{2x}\right\}$.

b) $\dfrac{1}{D^2-4D+4}\left\{e^{x}\sin 2x\right\}$.

c) $\dfrac{1}{D^2+2D+2}\left\{x^2 e^{-x}\right\}$.

d) $\dfrac{1}{D^2+16}\left\{\sin 4x\right\}$.

$\boxed{\frac{1}{2}x^2 e^{2x}}$, $\boxed{-e^{x}\sin 2x}$, $\boxed{e^{-x}\left(x^2-2\right)}$, $\boxed{-\frac{1}{8}x\cos 4x}$

a) $\dfrac{1}{D^2-4D+4}\left\{e^{2x}\right\} = \dfrac{1}{2^2-4\times2+4}e^{2x}$ FAILS SO WE INTRODUCE 1

$= \dfrac{1}{D^2-4D+4}\left\{e^{2x}\times 1\right\} = \dfrac{e^{2x}}{(D+2)^2-4(D+2)+4}\{1\}$

$= \dfrac{e^{2x}}{D^2}\{1\} = e^{2x}\dfrac{1}{D}\{x\} = \dfrac{1}{2}x^2e^{2x}$

b) $\dfrac{1}{D^2-2D+4}\left\{e^{x}\sin 2x\right\} = \dfrac{e^{x}}{(D+1)^2-2(D+1)+4}\{\sin 2x\} = \dfrac{e^{x}}{D^2+3}\{\sin 2x\}$

$= \dfrac{e^{x}}{-2^2+3}\{\sin 2x\} = -e^{x}\sin 2x$

c) $\dfrac{1}{D^2+2D+2}\left\{x^2e^{-x}\right\} = \dfrac{e^{-x}}{(D-1)^2+2(D-1)+2}\{x^2\} = \dfrac{e^{-x}}{D^2+1}$

$= e^{-x}(1+D^2)^{-1}\{x^2\}$

$= e^{-x}\left[1-D^2+D^4-D^6+O(D^8)\right]\{x^2\}$

$= e^{-x}\left[x^2-2\right]$

d) $\dfrac{1}{D^2+16}\{\sin 4x\} = \mathrm{Im}\left[\dfrac{1}{D^2+16}\left\{e^{0x}e^{i4x}\right\}\right]$

$= \mathrm{Im}\left[\dfrac{e^{4ix}}{(D+4i)^2+16}\left\{e^{0x}\right\}\right] = \mathrm{Im}\left[\dfrac{e^{4xi}}{D^2+8Di}\left\{e^{0x}\right\}\right]$

$= \mathrm{Im}\left[\dfrac{e^{4xi}}{D}\dfrac{1}{D+8i}\left\{e^{0x}\right\}\right] = \mathrm{Im}\left[\dfrac{e^{4xi}}{D}\dfrac{1}{8i}\left\{e^{0x}\right\}\right]$

$= \mathrm{Im}\left[\dfrac{e^{4xi}}{8i}\dfrac{1}{D}\{1\}\right] = \mathrm{Im}\left[\dfrac{e^{4xi}}{8i}x\right]$

$= \mathrm{Im}\left[\dfrac{x\cos 4x + ix\sin 4x}{8i}\right]$

$= -\dfrac{1}{8}x\cos 4x$

Question 9

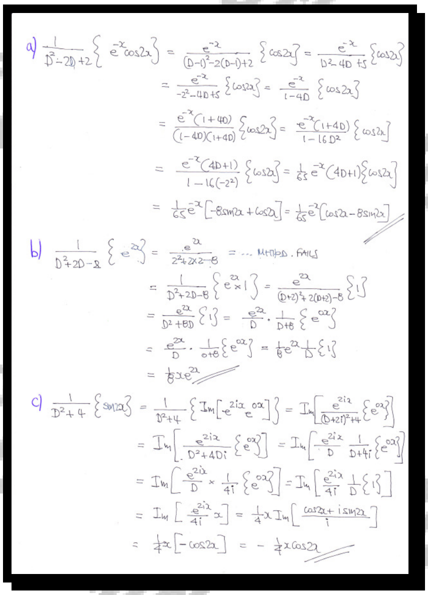
Simplify each of the following expressions.

a) $\dfrac{1}{D^2-2D+2}\left\{e^{-x}\cos 2x\right\}$.

b) $\dfrac{1}{D^2+2D-8}\left\{e^{2x}\right\}$.

c) $\dfrac{1}{D^2+4}\left\{\sin 2x\right\}$.

$\boxed{\frac{1}{65}e^{-x}\left[\cos 2x - 8\sin 2x\right]}$, $\boxed{\frac{1}{8}xe^{2x}}$, $\boxed{-\frac{1}{4}x\cos 2x}$



a) $\frac{1}{D^2-2D+2}\left\{e^{-x}\cos 2x\right\} = \frac{e^{-x}}{(D-1)^2-2(D-1)+2}\left\{\cos 2x\right\} = \frac{e^{-x}}{D^2-4D+5}\left\{\cos 2x\right\}$

$= \frac{e^{-x}}{-2^2-4D+5}\left\{\cos 2x\right\} = \frac{e^{-x}}{1-4D}\left\{\cos 2x\right\}$

$= \frac{e^{-x}(1+4D)}{(1-4D)(1+4D)}\left\{\cos 2x\right\} = \frac{e^{-x}(1+4D)}{1-16D^2}\left\{\cos 2x\right\}$

$= \frac{e^{-x}(4D+1)}{1-16(-2^2)}\left\{\cos 2x\right\} = \frac{1}{65}e^{-x}(4D+1)\left\{\cos 2x\right\}$

$= \frac{1}{65}e^{-x}\left[-8\sin 2x + \cos 2x\right] = \frac{1}{65}e^{-x}\left[\cos 2x - 8\sin 2x\right]$

b) $\frac{1}{D^2+2D-8}\left\{e^{2x}\right\} = \frac{e^{2x}}{2^2+2\times 2-8} = \ldots$ METHOD. FAILS

$= \frac{1}{D^2+2D-8}\left\{e^{2x}\times 1\right\} = \frac{e^{2x}}{(D+2)^2+2(D+2)-8}\left\{1\right\}$

$= \frac{e^{2x}}{D^2+8D}\left\{1\right\} = \frac{e^{2x}}{D}\cdot\frac{1}{D+8}\left\{e^{0x}\right\}$

$= \frac{e^{2x}}{D}\cdot\frac{1}{0+8}\left\{e^{0x}\right\} = \frac{1}{8}e^{2x}\frac{1}{D}\left\{1\right\}$

$= \frac{1}{8}xe^{2x}$

c) $\frac{1}{D^2+4}\left\{\sin 2x\right\} = \frac{1}{D^2+4}\left\{\mathrm{Im}\left[e^{2ix}e^{0x}\right]\right\} = \mathrm{Im}\left[\frac{e^{2ix}}{(D+2i)^2+4}\left\{e^{0x}\right\}\right]$

$= \mathrm{Im}\left[\frac{e^{2ix}}{D^2+4Di}\left\{e^{0x}\right\}\right] = \mathrm{Im}\left[\frac{e^{2ix}}{D}\frac{1}{D+4i}\left\{e^{0x}\right\}\right]$

$= \mathrm{Im}\left[\frac{e^{2ix}}{D}\times\frac{1}{4i}\left\{e^{0x}\right\}\right] = \mathrm{Im}\left[\frac{e^{2ix}}{4i}\frac{1}{D}\left\{1\right\}\right]$

$= \mathrm{Im}\left[\frac{e^{2ix}}{4i}x\right] = \frac{1}{4}x\,\mathrm{Im}\left[\frac{\cos 2x + i\sin 2x}{i}\right]$

$= \frac{1}{4}x\left[-\cos 2x\right] = -\frac{1}{4}x\cos 2x$

Question 10

Simplify each of the following expressions.

a) $\dfrac{1}{D^2-2D+2}\left\{x^2\right\}$.

b) $\dfrac{1}{D^2-2D+2}\left\{x^2\,\mathrm{e}^{3x}\right\}$.

$\boxed{\frac{1}{2}x^2+x+\frac{1}{2}}$, $\boxed{\frac{1}{125}\mathrm{e}^{3x}\left[25x^2-40x+22\right]}$

a) $\dfrac{1}{D^2-2D+2}\{x^2\} = \dfrac{1}{2(1-D+\frac{1}{2}D^2)}\{x^2\} = \frac{1}{2}\left(1-(D-\frac{1}{2}D^2)\right)^{-1}\{x^2\}$

$= \frac{1}{2}\left[1+(D-\frac{1}{2}D^2)^1+(D-\frac{1}{2}D^2)^2+O(D^3)\right]\{x^2\}$

$= \frac{1}{2}\left[1+D-\frac{1}{2}D^2+D^2+O(D^3)\right]\{x^2\}$

$= \frac{1}{2}\left[1+D+\frac{1}{2}D^2\right]\{x^2\} = \frac{1}{2}\left[x^2+2x+1\right]$

$= \frac{1}{2}x^2+x+\frac{1}{2}$

b) $\dfrac{1}{D^2-2D+2}\{x^2e^{3x}\} = \dfrac{e^{3x}}{(D+3)^2-2(D+3)+2}\{x^2\} = \dfrac{e^{3x}}{D^2+4D+5}\{x^2\}$

$= \dfrac{e^{3x}}{5(1+\frac{4}{5}D+\frac{1}{5}D^2)}\{x^2\} = \frac{1}{5}e^{3x}\left(1+\frac{4}{5}D+\frac{1}{5}D^2\right)^{-1}\{x^2\}$

$= \frac{1}{5}e^{3x}\left[1-(\frac{4}{5}D+\frac{1}{5}D^2)^1+(\frac{4}{5}D+\frac{1}{5}D^2)^2+O(D^3)\right]\{x^2\}$

$= \frac{1}{5}e^{3x}\left[1-\frac{4}{5}D-\frac{1}{5}D^2+\frac{16}{25}D^2+O(D^3)\right]\{x^2\}$

$= \frac{1}{5}e^{3x}\left[1-\frac{4}{5}D+\frac{11}{25}D^2+O(D^3)\right]\{x^2\}$

$= \frac{1}{5}e^{3x}\left[x^2-\frac{4}{5}(2x)+\frac{11}{25}(2)\right]$

$= \frac{1}{5}e^{3x}\left[x^2-\frac{8}{5}x+\frac{22}{25}\right]$

$= \frac{1}{125}e^{3x}\left[25x^2-40x+22\right]$

O.D.E.s APPLICATIONS

Question 1

Use D operator techniques, to find the general solution of the following differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = e^{4x}.$$

$$\boxed{y = Ae^{3x} + Be^{x} + \frac{1}{3}e^{4x}}$$

REWRITE THE O.D.E. IN D OPERATOR FORM

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = e^{4x}$$

$$(D^2 - 4D + 3)y = e^{4x}$$

$$(D-1)(D-3)y = e^{4x}$$

COMPLEMENTARY FUNCTION

$$y = Ae^{x} + Be^{3x}$$

PARTICULAR INTEGRAL

$$y = \frac{1}{D^2-4D+3}\left\{e^{4x}\right\}$$

$$y = \frac{1}{4^2-4\times4+3}\left(e^{4x}\right)$$

$$y = \frac{1}{3}e^{4x}$$

GENERAL SOLUTION

$$y = Ae^{x} + Be^{3x} + \frac{1}{3}e^{4x}$$

Question 2

Use D operator techniques, to find the general solution of the following differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-3x}.$$

$$\boxed{y = (A + Bx)e^{-x} + e^{-3x}}$$

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-3x}$$

$$\Rightarrow D^2y + 4Dy + 4y = e^{-3x}$$

$$\Rightarrow (D^2+4D+4)y = e^{-3x}$$

$$\Rightarrow (D+2)^2y = e^{-3x}$$

$D = 2$ (REPEATED)

C.F YIELDS

$$\boxed{y = (A+Bx)e^{-x}}$$

P.I. GIVES

$$y = \frac{1}{(D+2)^2}\left\{e^{-3x}\right\}$$

$$y = \frac{1}{(-3+2)^2}e^{-3x}$$

$$\boxed{y = e^{-3x}}$$

$\therefore$ GEN SOLUTION

$$y = (A+Bx)e^{-x} + e^{-3x}$$

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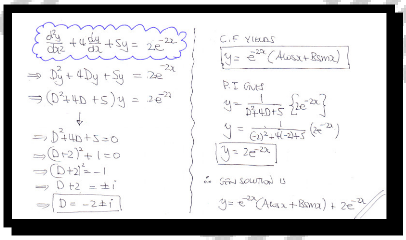
Question 3

Use D operator techniques, to find the general solution of the following differential equation

$$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+5y=2e^{-2x}.$$

$$\boxed{y=e^{-2x}\left[A\cos x+B\sin x\right]+2e^{-2x}}$$

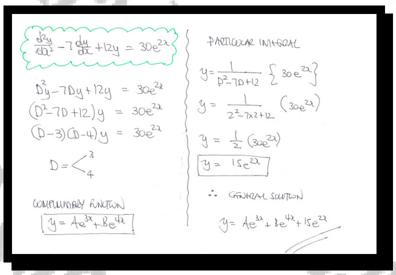


Question 4

Use D operator techniques, to find the general solution of the following differential equation

$$\frac{d^2y}{dx^2}-7\frac{dy}{dx}+12y=30e^{2x}.$$

$$\boxed{y=Ae^{3x}+Be^{4x}+15e^{2x}}$$

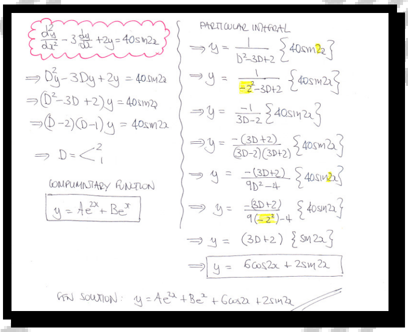


Question 5

Use D operator techniques, to find the general solution of the following differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 40\sin 2x .$$

$$\boxed{y = A\mathrm{e}^{2x} + B\mathrm{e}^{x} + 6\cos 2x + \sin 2x}$$

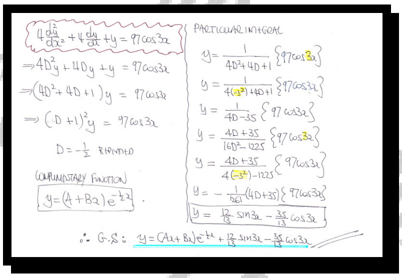


Question 6

Use D operator techniques, to find the general solution of the following differential equation

$$4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 97\cos 3x .$$

$$\boxed{y = (Ax+B)\mathrm{e}^{-\frac{1}{2}x} + \frac{12}{13}\sin 3x - \frac{35}{13}\cos 3x}$$

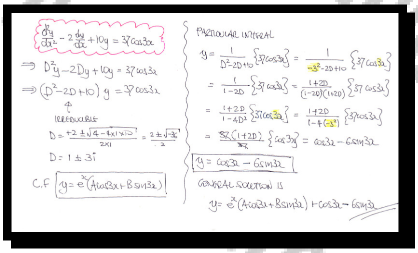


Question 7

Use D operator techniques, to find the general solution of the following differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = 37\cos 3x .$$

$$\boxed{y = \mathrm{e}^{x}\left(A\cos 3x + B\sin 3x\right) + \cos 3x - 6\sin 3x}$$

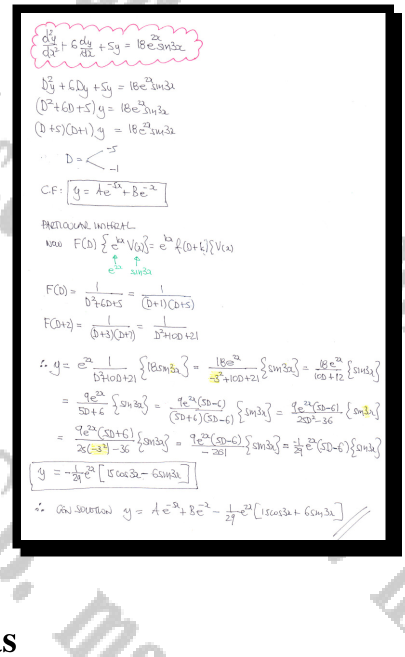


Question 8

Use D operator techniques, to find the general solution of the following differential equation

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 5y = 18\mathrm{e}^{2x}\sin 3x .$$

$$\boxed{y = A\mathrm{e}^{-5x} + B\mathrm{e}^{-x} - \frac{1}{29}\mathrm{e}^{x}\left[15\cos 3x + 6\sin 3x\right]}$$

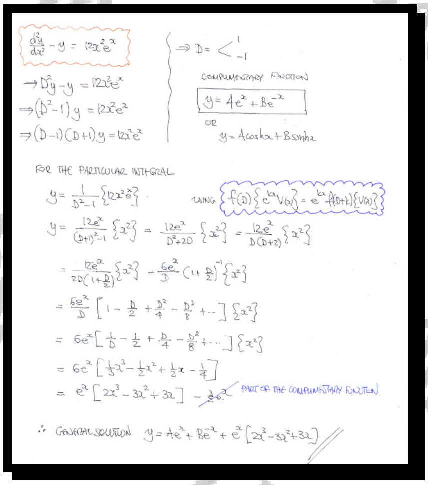


Question 9

Use D operator techniques, to find the general solution of the following differential equation

$$\frac{d^2y}{dx^2} - y = 12x^2 \mathrm{e}^x .$$

$$\boxed{y = A\mathrm{e}^x + B\mathrm{e}^{-x} + \mathrm{e}^x\left[2x^3 - 3x^2 + 3x\right]}$$

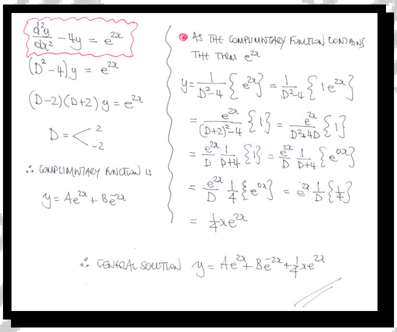


Question 10

Use D operator techniques, to find the general solution of the following differential equation

$$\frac{d^2y}{dx^2} - 4y = \mathrm{e}^{2x} .$$

$$\boxed{y = A\mathrm{e}^{2x} + B\mathrm{e}^{-2x} + \frac{1}{4}x\mathrm{e}^{2x}}$$



Question 11

Use D operator techniques, to find the general solution of the following differential equation

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 30x^4\,e^{4x}.$$

$$\boxed{y = Ae^{4x} + Bxe^{4x} + x^6\,e^{4x}}$$

$$\boxed{\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 30x^4e^{4x}}$$

- $D^2y - 8Dy + 16y = 30x^4e^{4x}$

$(D^2 - 8D + 16)y = 30x^4e^{4x}$

$(D-4)^2y = 30x^4e^{4x}$

$D = 4$ (REPEATED) $\quad \therefore$ COMPLEMENTARY FUNCTION

$$\boxed{y = Ae^{4x} + Bxe^{4x}}$$

- PARTICULAR INTEGRAL BY D OPERATOR

$y = \frac{1}{(D-4)^2}\left\{30x^4e^{4x}\right\}$

NOW

$$\boxed{f(D)\{e^{kx}V(x)\} = e^{kx}f(D+k)\{V(x)\}}$$

$y = \frac{e^{4x}}{D^2}\left[30x^4\right] = \frac{e^{4x}}{D}\left[6x^5\right]$

$$\boxed{y = x^6e^{4x}}$$

- GENERAL SOLUTION IS $\quad y = Ae^{4x} + Bxe^{4x} + x^6e^{4x}$

Question 12

Find the general solution of the following differential equation.

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 10e^{-2x},$$

a) ... by using D-operator techniques only.

b) ... by using the substitution $Y = (D+2)y$, in a method involving D-operator techniques only.

$\boxed{}$, $\boxed{y = Ae^{3x} + Be^{-2x} - 2xe^{-2x}}$

- WRITE THE O.D.E IN D OPERATOR FORM

$\Rightarrow \frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 10e^{-2x}$

$\Rightarrow D^2y - Dy - 6y = 10e^{-2x}$

$\Rightarrow (D^2 - D - 6)y = 10e^{-2x}$

$\Rightarrow (D+2)(D-3)y = 10e^{-2x}$

- HENCE WE OBTAIN

$\Rightarrow y = Ae^{-2x} + Be^{3x} + \frac{1}{D^2-D-6}\{10e^{-2x}\}$

−2 MAKES THIS ZERO

$\Rightarrow y = Ae^{-2x} + Be^{3x} + \frac{1}{D^2-D-6}\{10 \times e^{-2x}\}$

$f(x)$

- USING THE THEOREM

$$\boxed{[f(D)]\{e^{kx}V(x)\} = e^{kx}[f(D+k)]\{V(x)\}}$$

$\Rightarrow y = Ae^{-2x} + Be^{3x} + \frac{e^{-2x}}{(D-2)^2-(D-2)-6}\{10\}$

$\Rightarrow y = Ae^{-2x} + Be^{3x} + \frac{e^{-2x}}{D^2-5D}\{10\}$

$\Rightarrow y = Ae^{-2x} + Be^{3x} + \frac{1}{D} \times \frac{e^{-2x}}{D-5}\{10e^{0x}\}$

$\Rightarrow y = Ae^{-2x} + Be^{3x} + \frac{e^{-2x}}{D} \times \left[\frac{10}{0-5}e^{0x}\right]$

$\Rightarrow y = Ae^{-2x} + Be^{3x} + \frac{e^{-2x}}{D}[-2]$

$\Rightarrow y = Ae^{-2x} + Be^{3x} + e^{-2x}(-2x)$

$\Rightarrow y = Ae^{-2x} + Be^{3x} - 2xe^{-2x}$

b)

- AFTER MANIPULATING THE O.D.E INTO D OPERATOR FORM

$(D+2)(D-3)y = 10e^{-2x}$

- LET $Y = (D+2)y$ SO THE O.D.E BECOMES

$\Rightarrow (D-3)Y = 10e^{-2x}$

$\Rightarrow Y = Ae^{3x} + \frac{1}{D-3}\{10e^{-2x}\}$

$\Rightarrow Y = Ae^{3x} + \frac{1}{-2-3} \times (10e^{-2x})$

$\Rightarrow Y = Ae^{3x} - 2e^{-2x}$

$\Rightarrow (D+2)y = Ae^{3x} - 2e^{-2x}$

FROM C.F.

$\Rightarrow y = \frac{1}{D+2}\{Ae^{3x}\} - \frac{1}{D+2}\{2e^{-2x}\} + Be^{-2x}$

$\Rightarrow y = \frac{1}{-3+2}(Ae^{3x}) - \frac{e^{-2x}}{(D-2)+2}\{2\} + Be^{-2x}$

$\Rightarrow y = Ae^{3x} - e^{-2x}\frac{1}{D}\{2\} + Be^{-2x}$

$\Rightarrow y = Ae^{3x} + Be^{-2x} - 2xe^{-2x}$ AS BEFORE