

# D OPERATOR

## Summary of D Operator

### Basic Definitions

- $D\{ \} \equiv \frac{d}{dx}( \ ),$

or depending on the independent variable

$$D\{ \} \equiv \frac{d}{dt}( \ ), \quad D\{ \} \equiv \frac{d}{dy}( \ ), \quad D\{ \} \equiv \frac{d}{du}( \ ) \text{ etc}$$

- $D^2\{ \} \equiv \frac{d^2}{dx^2}( \ ), \quad D^3 \equiv \frac{d^3}{dx^3}\{ \}, \quad D^4\{ \} \equiv \frac{d^4}{dx^4}( \ ) \text{ etc}$

- $(aD^2 + bD + c)\{f(x)\} \equiv aD^2\{f(x)\} + bD\{f(x)\} + c\{f(x)\}$

- $\left(\frac{1}{aD^2 + bD + c}\right)\{f(x)\} \equiv (aD^2 + bD + c)^{-1}\{f(x)\}$

- $\frac{1}{D}\{ \} \equiv \int ( \ ) dx$

### D Operations on Simple Functions

- $[f(D)]\{e^{kx}\} \equiv e^{kx}[f(k)]$

- $[f(D)]\{e^{kx}V(x)\} \equiv e^{kx}[f(D+k)]\{V(x)\}$

- $[f(D^2)]\{\sin kx\} \equiv [f(-k^2)]\sin kx$

- $[f(D^2)]\{\cos kx\} \equiv [f(-k^2)]\cos kx$

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# GENERAL PRACTICE

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## Question 1

Simplify each of the following expressions.

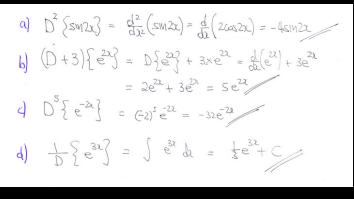
a)  $D^2 \{\sin 2x\}.$

b)  $(D+3)\{e^{2x}\}.$

c)  $D^5 \{e^{-2x}\}.$

d)  $\frac{1}{D}\{e^{3x}\}.$

$$\boxed{-4 \sin 2x}, \quad \boxed{5e^{2x}}, \quad \boxed{-32e^{-2x}}, \quad \boxed{\frac{1}{3}e^{3x} + C}$$



a)  $D^2 \{\sin 2x\} = \frac{d^2}{dx^2}(\sin 2x) = \frac{d}{dx}(2\cos 2x) = -4\sin 2x$   
 b)  $(D+3)\{e^{2x}\} = D\{e^{2x}\} + 3e^{2x} = \frac{d}{dx}(e^{2x}) + 3e^{2x} = 2e^{2x} + 3e^{2x} = 5e^{2x}$   
 c)  $D^5 \{e^{-2x}\} = (D^2)^2 \{e^{-2x}\} = (-2)^2 e^{-2x} = -32e^{-2x}$   
 d)  $\frac{1}{D}\{e^{3x}\} = \int e^{3x} dx = \frac{1}{3}e^{3x} + C$

## Question 2

Simplify each of the following expressions.

a)  $(D^2 + 2D + 2)\{e^{3x}\}.$

b)  $(D^2 + 2D + 2)\{x^2 e^{3x}\}.$

c)  $(D^2 + 2D + 2)\{\sin 2x\}.$

$$\boxed{17e^{3x}}, \quad \boxed{(17x^2 + 16x + 2)e^{3x}}, \quad \boxed{4\cos 2x - 2\sin 2x}$$

Handwritten solutions for Question 2:

a)  $(D^2 + 2D + 2)\{e^{3x}\} = (3^2 + 2 \cdot 3 + 2)e^{3x} = 17e^{3x}$

b)  $(D^2 + 2D + 2)\{x^2 e^{3x}\} = e^{3x} \{(D+3)\{x^2\}\}$   
 $= e^{3x} \{(0 \cdot 3^2 + 2 \cdot 0 \cdot 3 + 2)\{x^2\}\}$   
 $= e^{3x} \{0^2 + 6 \cdot 0 + 9 + 2D + 4 + 2\}\{x^2\}$   
 $= e^{3x} \{0^2 + 0 + 17\}\{x^2\}$   
 $= e^{3x} \{2 + 8(2x) + 17x^2\}$   
 $= (17x^2 + 16x + 2)e^{3x}$

c)  $(D^2 + 2D + 2)\{\sin 2x\} = (-2^2 + 2D + 2)\{\sin 2x\}$   
 $= 2D - 2\{\sin 2x\}$   
 $= 2(2\cos 2x) - 2\sin 2x$   
 $= 4\cos 2x - 2\sin 2x$

## Question 3

$$f(D) = D^2 - 4D + 3.$$

Show directly, by the definition of the  $D$  operator as  $D \equiv \frac{d}{dx}(\ )$ , that

$$f(D)\{e^{kx}\} = f(k)e^{kx}.$$

proof

$$\begin{aligned} f(D) &= D^2 - 4D + 3 \\ f(D)\{e^{kx}\} &= (D^2 - 4D + 3)\{e^{kx}\} \\ &= D^2\{e^{kx}\} - 4D\{e^{kx}\} + 3\{e^{kx}\} \\ &= k^2 e^{kx} - 4(ke^{kx}) + 3e^{kx} \\ &= (k^2 - 4k + 3)e^{kx} \\ &= f(k)e^{kx} \end{aligned}$$

## Question 4

$$g(D) = \frac{1}{D^2 - 4D + 3}.$$

Show directly, by the definition of the  $D$  operator as  $D \equiv \frac{d}{dx}(\ )$ , that

$$g(D)\{e^{kx}\} = \frac{1}{f(k)}e^{kx},$$

where  $k$  is a constant.

proof

$$\begin{aligned} g(D) &= \frac{1}{D^2 - 4D + 3} \\ g(D)\{e^{kx}\} &= \frac{1}{D^2 - 4D + 3}\{e^{kx}\} = (D^2 - 4D + 3)^{-1}\{e^{kx}\} \\ &= (k^2 - 4k + 3)^{-1}e^{kx} \\ &= \frac{1}{k^2 - 4k + 3}e^{kx} \\ &= \frac{1}{f(k)}e^{kx} \end{aligned}$$

## Question 5

$$f(D) = 2D^2 - D + 1.$$

Show directly, by the definition of the  $D$  operator as  $D \equiv \frac{d}{dx}(\ )$ , that

$$f(D)\{e^{kx}V(x)\} = e^{kx}f(D+k)\{V(x)\},$$

where  $V(x)$  is a smooth function of  $x$  and  $k$  is a constant.

 , proof

$f(D) = 2D^2 - D + 1$ , here  $\frac{d}{dx}$  indicates D operator. (Handwritten)

DIRECTLY FROM THE DEFINITIONS, INC. UNUSUAL PROBABLY

$$f(D)\{e^{kx}V(x)\} = (2D^2 - D + 1)\{e^{kx}V(x)\}$$

$$= 2D^2\{e^{kx}V(x)\} - D\{e^{kx}V(x)\} + 1\{e^{kx}V(x)\}$$

DIFFERENTIATING BY THE PRODUCT RULE

$$= 2D\{e^{kx}V(x) + e^{kx}V(x)\} - k e^{kx}V(x) - e^{kx}V(x) + e^{kx}V(x)$$

$$= 2D\{e^{kx}(kV(x) + V(x))\} + e^{kx}[V(x) - kV(x) - V(x)]$$

BY THE PRODUCT RULE AGAIN

$$= 2[e^{kx}(kV(x) + V(x)) + e^{kx}(kV(x) + V(x))] + e^{kx}[V(x) - kV(x) - V(x)]$$

$$= e^{kx}[2k^2V(x) + 2kV(x) + 2kV(x) + 2V(x) + V(x) - kV(x) - V(x)]$$

$$= e^{kx}[2k^2V(x) + (4k-1)V(x) + (2k^2-k+1)V(x)]$$

APPLY THE DEFINITION OF D OPERATOR

$$= e^{kx}[2k^2\{V(x)\} + (4k-1)D\{V(x)\} + (2k^2-k+1)V(x)]$$

$$= e^{kx}[2k^2 + (4k-1)D + (2k^2-k+1)]\{V(x)\}$$

$$= e^{kx}[2k^2 + 4k + 2k^2 - D - k + 1]\{V(x)\}$$

$$= e^{kx}[2(k^2 + 2k + k^2) - (D+k) + 1]\{V(x)\}$$

$$= e^{kx}[2(2k+k^2) - (D+k) + 1]\{V(x)\}$$

$$= e^{kx}f(D+k)\{V(x)\}$$

As Required

## Question 6

Simplify each of the following expressions.

a)  $\frac{1}{D^2 + 4D + 3} \{30e^{-2x}\}.$

b)  $\frac{1}{D^2 + 4D + 3} \{30\sin 2x\}.$

c)  $\frac{1}{D^2 + 4D + 4} \{30x^2 e^{-2x}\}.$

d)  $\frac{1}{D^2 + 4D + 4} \{30\}.$

e)  $\frac{1}{D^2 + 2D} \{30\}.$

$$\boxed{\phantom{000}}, \boxed{-30e^{-2x}}, \boxed{-\frac{6}{13}[8\cos 2x + \sin 2x]}, \boxed{\frac{5}{2}x^4 e^{-2x}}, \boxed{\frac{15}{2}}, \boxed{15x}$$

a)  $\frac{1}{D^2 + 4D + 3} \{30e^{-2x}\} = \frac{30e^{-2x}}{(D+1)(D+3)} = \frac{30e^{-2x}}{4-4+3} = \frac{30e^{-2x}}{3} = 10e^{-2x}$

b)  $\frac{1}{D^2 + 4D + 3} \{30\sin 2x\} = \frac{30}{D^2 + 4D + 3} \{\sin 2x\} = \frac{30}{(D+1)(D+3)} \{\sin 2x\}$   
 $= \frac{30}{(D+1)(D+3)} \{\sin 2x\} = \frac{30}{(D+1)(D+3)} \{\sin 2x\}$   
 $= \frac{30}{(D+1)(D+3)} \{\sin 2x\} = \frac{30}{(D+1)(D+3)} \{\sin 2x\}$   
 $= \frac{30}{(D+1)(D+3)} \{\sin 2x\} = \frac{30}{(D+1)(D+3)} \{\sin 2x\}$   
 $= -\frac{6}{13} [8\cos 2x + \sin 2x]$

c)  $\frac{1}{D^2 + 4D + 4} \{30x^2 e^{-2x}\} = \frac{30}{(D+2)^2} \{x^2\} = \frac{30}{(D+2)^2} \{x^2\}$   
 $= \frac{30}{(D+2)^2} \{x^2\} = \frac{30}{(D+2)^2} \{x^2\}$   
 $= \frac{30}{(D+2)^2} \{x^2\} = \frac{30}{(D+2)^2} \{x^2\}$   
 $= \frac{5}{2} x^4 e^{-2x}$

d)  $\frac{1}{D^2 + 4D + 4} \{30\} = \frac{1}{(D+2)^2} \{30\} = \frac{30}{(D+2)^2}$   
 $= \frac{1}{4} \times 30 = \frac{15}{2}$

e)  $\frac{1}{D^2 + 2D} \{30\} = \frac{1}{D(D+2)} \{30\} = \frac{1}{D} \left[ \frac{1}{D+2} \{30\} \right]$   
 $= \frac{1}{D} \left[ \frac{1}{D+2} \{30e^{0x}\} \right] = \frac{1}{D} \left[ \frac{30e^{0x}}{0+2} \right]$   
 $= \frac{1}{D} \{15\} = 15x$



## Question 7

Simplify each of the following expressions.

a)  $\frac{1}{D^2 - 4D + 3} \{e^{2x}\}.$

b)  $\frac{1}{D^2 - 4D + 3} \{e^{3x}\}.$

c)  $\frac{1}{D^2 - 4D + 3} \{\sin 2x\}.$

d)  $\frac{1}{D^2 + 1} \{\cos x\}.$

$$\boxed{\phantom{000000}}, \boxed{-e^{2x}}, \boxed{\frac{1}{2}xe^{3x}}, \boxed{\frac{1}{65}[8\cos 2x - \sin 2x]}, \boxed{\frac{1}{2}x\sin x}$$

a)  $\frac{1}{D^2 - 4D + 3} \{e^{2x}\} = \frac{1}{2^2 - 4(2) + 3} \times e^{2x} = -\frac{e^{2x}}{1}$

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b)  $\frac{1}{D^2 - 4D + 3} \{e^{3x}\} = \frac{1}{3^2 - 4(3) + 3} \times e^{3x} \dots$  THERE IS A PROBLEM  
 INTERPOLATE & FIND THE VALUE  $V(x) = e^{3x}$   
 $= \frac{1}{D^2 - 4D + 3} \{1 \times e^{3x}\} = \frac{(D+3)(D-1)}{(D+3)(D-1)} \{1\} = \frac{1}{D^2 - 4D + 3} \{1\}$   
 $= \frac{1}{D^2 - 4D + 3} \{e^{0x}\} = \frac{1}{0^2 - 4(0) + 3} \{e^{0x}\} = \frac{1}{3} \{e^{0x}\} = \frac{1}{3} \{1\}$   
 $= \frac{1}{3} \times \frac{1}{D^2 - 4D + 3} \{e^{3x}\} = \frac{1}{3} \times \frac{1}{2} e^{3x}$

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c)  $\frac{1}{D^2 - 4D + 3} \{\sin 2x\} = \frac{1}{-2^2 - 4(2) + 3} \{\sin 2x\} = \frac{1}{-14} \{\sin 2x\}$   
 $= \frac{-1}{14(1)} \{\sin 2x\} = \frac{-1(40+1)}{14(1)(40+1)} \{\sin 2x\}$   
 $= \frac{1-40}{14(40+1)} \{\sin 2x\} = \frac{1-40}{14(40+1)} \{\sin 2x\}$   
 $= \frac{1-40}{14(40+1)} \{\sin 2x\} = \frac{40-1}{14(40+1)} \{\sin 2x\}$   
 $= \frac{1}{65} (40-1) \{\sin 2x\} = \frac{1}{65} [8\cos 2x - \sin 2x]$   
 $= \frac{1}{65} (8\cos 2x - \sin 2x)$

d)  $\frac{1}{D^2 + 1} \{\cos x\} = \frac{1}{-1^2 + 1} \{\cos x\} = \dots$  FAILS  
 PROCEED BY COMPLEX NUMBERS AS ABOVE  
 $\frac{1}{D^2 + 1} \{\cos x\} = \frac{1}{D^2 + 1} \{1 \times \cos x\} = \text{Re} \left[ \frac{1}{D^2 + 1} \{1 \times e^{ix}\} \right]$   
 $= \text{Re} \left[ \frac{1}{(D+1i)(D-1i)} \{1\} \right]$   
 $= \text{Re} \left[ \frac{1}{D^2 + 1} \{1\} \right] = \text{Re} \left[ \frac{1}{5(1+2i)} \{1\} \right]$   
 $= \text{Re} \left[ \frac{1}{5} \cdot \frac{1}{1+2i} \{1\} \right]$   
 $= \text{Re} \left[ \frac{1}{5} \cdot \frac{1}{D+1i} \{e^{ix}\} \right]$   
 $= \text{Re} \left[ \frac{1}{5} \cdot \frac{1}{D+1i} \{1\} \right] = \text{Re} \left[ \frac{1}{5} \cdot \frac{1}{1+2i} \{1\} \right]$   
 $= \frac{1}{5} \times \text{Re} \left[ \frac{1}{1+2i} \{1\} \right] = \frac{1}{5} \times \text{Re} \left[ -ie^{ix} \right]$   
 $= \frac{1}{5} \times \text{Re} \left[ -\cos x - i \sin x \right]$   
 $= \frac{1}{5} \times \sin x$

Simplify each of the following expressions.

**b)**  $\frac{1}{D^2 - 4D + 4} \{e^x \sin 2x\}.$

d)  $\frac{1}{D^2 + 16} \{\sin 4x\}.$

$$\left[\frac{1}{2}x^2e^{2x}\right], \left[-e^x\sin 2x\right], \left[e^{-x}(x^2-2)\right], \left[-\frac{1}{8}x\cos 4x\right]$$

$$\begin{aligned}
 \text{a)} \quad \frac{1}{15^2 - 40 + 44} \left\{ e^{1/2} \right\} &= \frac{1}{2^2 - 12 + 14} e^{1/2} \quad \text{falls so was in der Regel!} \\
 &= \frac{1}{15^2 - 40 + 44} \left\{ e^{2/1} \right\} = \frac{e^{2/1}}{(15)^2 - (40) + 44} \left\{ 1 \right\} \\
 &= \frac{e^{2/1}}{15^2} \left\{ 1 \right\} = e^{2/15} \left\{ 2 \right\} = \frac{1}{2} e^{2/15} \quad // \\
 \text{b)} \quad \frac{1}{15^2 - 20 + 44} \left\{ e^{2 \sin 2\lambda} \right\} &= \frac{e^{2 \sin 2\lambda}}{(15)^2 - (20) + 44} \left\{ \sin 2\lambda \right\} = \frac{e^{2 \sin 2\lambda}}{15^2 + 24} \left\{ \sin 2\lambda \right\} \\
 &= -\frac{e^{2 \sin 2\lambda}}{2^2 + 3} \left\{ \sin 2\lambda \right\} = -e^{2 \sin 2\lambda} \quad // \\
 \text{c)} \quad \frac{1}{15^2 + 22 + 42} \left\{ e^{2 \cdot e^2} \right\} &= \frac{e^{2 \cdot e^2}}{(15)^2 + (22) + 42} \left\{ 2 \cdot e^2 \right\} = \frac{e^{e^2}}{3^2 + 41} \\
 &= e^{e^2} (1 + 1)^{-1} \left\{ 2 \cdot e^2 \right\} \\
 &= e^{e^2} \left[ (1 - 3^2 + 3^2 - 3^4 + \dots) \right] \left\{ 2 \cdot e^2 \right\} \\
 &= e^{e^2} \left[ 2 \cdot e^2 - 2 \right] \quad // \\
 \text{d)} \quad \frac{1}{15^2 + 16} \left\{ \sin 2\lambda \right\} &= \left[ \frac{1}{15^2 + 16} \left\{ e^{e^{1/2} + i \lambda} \right\} \right] \\
 &= \frac{e^{1/2}}{(15)^2 + 16} \left\{ e^{i \lambda} \right\} = \frac{1}{15^2 + 16} \left\{ e^{i \lambda} \right\} \\
 &= \frac{1}{15} \left[ \frac{e^{i \lambda}}{1} \frac{1}{1 + 16} \left\{ e^{e^{1/2}} \right\} \right] = \frac{1}{15} \left[ \frac{e^{i \lambda}}{1} \frac{1}{16} \left\{ e^{e^{1/2}} \right\} \right] \\
 &= \frac{1}{15} \left[ \frac{e^{i \lambda}}{16} \left\{ 1 \right\} \right] = \frac{1}{15} \left[ \frac{e^{i \lambda}}{16} \cdot 1 \right] \\
 &= \frac{1}{15} \left[ \frac{2 \cos 16 + i 2 \sin 16}{16} \right] \\
 &= -\frac{1}{8} \sin 16 \quad //
 \end{aligned}$$

Simplify each of the following expressions.

**b)**  $\frac{1}{D^2 + 2D - 8} \{e^{2x}\}.$

c)  $\frac{1}{D^2 + 4} \{\sin 2x\}.$

$$\left[\frac{1}{65}e^{-x}[\cos 2x - 8\sin 2x]\right], \left[\frac{1}{8}xe^{2x}\right], \left[-\frac{1}{4}x\cos 2x\right]$$

$$\begin{aligned}
 a) \quad \frac{1}{(z-1)^2+4} \left\{ e^{-x} \cos 2x \right\} &= \frac{e^{-x}}{(z-1)^2-2(z-1)+2} \left\{ \cos 2x \right\} = \frac{e^{-x}}{(z-2-4i)(z-1+4i)} \left\{ \cos 2x \right\} \\
 &= \frac{e^{-x}}{z^2-4iz+10} \left\{ \cos 2x \right\} = \frac{e^{-x}}{z-4i} \left\{ \cos 2x \right\} \\
 &= \frac{e^{-x}(1+4i)}{(1-4i)(1+4i)} \left\{ \cos 2x \right\} = \frac{e^{-x}(1+4i)}{1-16i^2} \left\{ \cos 2x \right\} \\
 &= \frac{e^{-x}(4b+1)}{1-(4c-2^2)} \left\{ \cos 2x \right\} = \frac{1}{4} e^{-x} (4b+1) \left\{ \cos 2x \right\} \\
 &= \frac{1}{4} e^{-x} [-8 \sin 2x + \cos 2x] = \frac{1}{4} e^{-x} [\cos 2x - 8 \sin 2x]
 \end{aligned}$$

$$\begin{aligned}
 b) \quad \frac{1}{D^2+2D-8} \left\{ e^{3x} \right\} &= \frac{e^{3x}}{2^2+2 \cdot 2-8} = \dots \text{undefined, find} \\
 &= \frac{e^{3x}}{D^2+2D-8} \left\{ e^{3x} \right\} = \frac{e^{3x}}{(D+4)(D-2)} \left\{ e^{3x} \right\} \\
 &= \frac{e^{3x}}{D^2+6D} \left\{ e^{3x} \right\} = \frac{e^{3x}}{D} \cdot \frac{1}{D+6} \left\{ e^{3x} \right\} \\
 &= \frac{e^{3x}}{D} \cdot \frac{1}{16} \left\{ e^{3x} \right\} = \frac{1}{16} e^{3x} \cdot \frac{1}{D} \left\{ e^{3x} \right\} \\
 &= \frac{1}{16} e^{3x}
 \end{aligned}$$

$$\begin{aligned}
 c) \quad \frac{1}{D^2+4} \left\{ \sin 2x \right\} &= \frac{1}{D^2+4} \left\{ \operatorname{Im} \left[ e^{-i2x} \cdot e^{2x} \right] \right\} = \operatorname{Im} \left[ \frac{e^{-i2x}}{D^2+4} \left\{ e^{2x} \right\} \right] \\
 &= \operatorname{Im} \left[ \frac{-e^{i2x}}{D^2+4iD} \left\{ e^{2x} \right\} \right] = \operatorname{Im} \left[ \frac{-e^{i2x}}{D} \cdot \frac{1}{D+4i} \left\{ e^{2x} \right\} \right] \\
 &= \operatorname{Im} \left[ \frac{e^{i2x}}{D} \cdot \frac{1}{4i} \left\{ e^{2x} \right\} \right] = \operatorname{Im} \left[ \frac{e^{i2x}}{4i} \left\{ e^{2x} \right\} \right] \\
 &= \operatorname{Im} \left[ \frac{e^{i2x}}{4i} \cdot 2 \right] = \frac{1}{4} \cdot \frac{1}{i} \operatorname{Im} \left[ \frac{e^{i2x} + i e^{i2x}}{1} \right] \\
 &= \frac{1}{4} \cdot \frac{1}{i} [-\cos 2x] = -\frac{1}{4} x \cos 2x
 \end{aligned}$$

## Question 10

Simplify each of the following expressions.

a)  $\frac{1}{D^2 - 2D + 2} \{x^2\}.$

b)  $\frac{1}{D^2 - 2D + 2} \{x^2 e^{3x}\}.$

$$\boxed{\frac{1}{2}x^2 + x + \frac{1}{2}}, \quad \boxed{\frac{1}{125}e^{3x} [25x^2 - 40x + 22]}$$

a)  $\frac{1}{D^2 - 2D + 2} \{x^2\} = \frac{1}{2(-0 + \frac{1}{2}D^2)} \{x^2\} = \frac{1}{2}(-0 + \frac{1}{2}D^2) \{x^2\}$   
 $= \frac{1}{2} [1 + (-0 + \frac{1}{2}D^2) + (-0 + \frac{1}{2}D^2) + 0(D^2)] \{x^2\}$   
 $= \frac{1}{2} [1 + 0 + \frac{1}{2}D^2 + 0 + 0(D^2)] \{x^2\}$   
 $= \frac{1}{2} [1 + 0 + \frac{1}{2}D^2] \{x^2\} = \frac{1}{2} [x^2 + 2x + 1]$   
 $= \frac{1}{2}x^2 + x + \frac{1}{2}$

b)  $\frac{1}{D^2 - 2D + 2} \{x^2 e^{3x}\} = \frac{e^{3x}}{(D+3)^2 - 2(D+3) + 2} \{x^2\} = \frac{e^{3x}}{D^2 + 4D + 5} \{x^2\}$   
 $= \frac{e^{3x}}{5(1 + \frac{4}{5}D + \frac{1}{5}D^2)} \{x^2\} = \frac{1}{5} \frac{e^{3x}}{(1 + \frac{4}{5}D + \frac{1}{5}D^2)} \{x^2\}$   
 $= \frac{1}{5} \frac{e^{3x}}{1 - (\frac{4}{5}D + \frac{1}{5}D^2) + (\frac{4}{5}D + \frac{1}{5}D^2) + 0(D^2)} \{x^2\}$   
 $= \frac{1}{5} \frac{e^{3x}}{1 - \frac{4}{5}D - \frac{1}{5}D^2 + \frac{8}{5}D + \frac{1}{5}D^2 + 0(D^2)} \{x^2\}$   
 $= \frac{1}{5} \frac{e^{3x}}{1 - \frac{4}{5}D + \frac{4}{5}D + \frac{1}{5}D^2 + 0(D^2)} \{x^2\}$   
 $= \frac{1}{5} \frac{e^{3x}}{1} \{x^2 - \frac{4}{5}(2x) + \frac{1}{5}(2)\}$   
 $= \frac{1}{5} e^{3x} [x^2 - \frac{4}{5}x + \frac{2}{5}]$   
 $= \frac{1}{125} e^{3x} [25x^2 - 40x + 22]$

# O.D.E.s APPLICATIONS

## Question 1

Use  $D$  operator techniques, to find the general solution of the following differential equation

$$\frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 3y = e^{4x}.$$

$$\boxed{\phantom{000000}}, \quad \boxed{y = Ae^{3x} + Be^x + \frac{1}{3}e^{4x}}$$

Handwritten solution for Question 1:

REWRITE THE O.D.E. IN D OPERATOR FORM

$$\frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 3y = e^{4x}$$

$$(D^2 - 4D + 3)y = e^{4x}$$

$$(D-1)(D-3)y = e^{4x}$$

... ..

COMPLEMENTARY FUNCTION

$$y = Ae^{3x} + Be^x$$

PARTICULAR INTEGRAL

$$y = \frac{1}{D^2 - 4D + 3} \{e^{4x}\}$$

$$y = \frac{1}{4^2 - 4(4) + 3} (e^{4x})$$

$$y = \frac{1}{3} e^{4x}$$

GENERAL SOLUTION

$$y = Ae^{3x} + Be^x + \frac{1}{3}e^{4x}$$

## Question 2

Use  $D$  operator techniques, to find the general solution of the following differential equation

$$\frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} + 4y = e^{-3x}.$$

$$\boxed{y = (A + Bx)e^{-x} + e^{-3x}}$$

Handwritten solution for Question 2:

$$\frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} + 4y = e^{-3x}$$

$$\Rightarrow D^2 y + 4Dy + 4y = e^{-3x}$$

$$\Rightarrow (D^2 + 4D + 4)y = e^{-3x}$$

$$\Rightarrow (D+2)^2 y = e^{-3x}$$

$D = -2$  (EIGENVAL)

C.F. YIELDS

$$y = (A + Bx)e^{-2x}$$

P.I. GIVES

$$y = \frac{1}{(D+2)^2} \{e^{-3x}\}$$

$$y = \frac{1}{(-3+2)^2} e^{-3x}$$

$$y = e^{-3x}$$

GENERAL SOLUTION

$$y = (A + Bx)e^{-2x} + e^{-3x}$$

## Question 3

Use  $D$  operator techniques, to find the general solution of the following differential equation

$$\frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} + 5y = 2e^{-2x}$$

$$y = e^{-2x} [A \cos x + B \sin x] + 2e^{-2x}$$

Handwritten solution for Question 3:

$\frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} + 5y = 2e^{-2x}$   
 $\Rightarrow D^2 y + 4Dy + 5y = 2e^{-2x}$   
 $\Rightarrow (D^2 + 4D + 5)y = 2e^{-2x}$   
 $\downarrow$   
 $\Rightarrow D^2 + 4D + 5 = 0$   
 $\Rightarrow (D+2)^2 + 1 = 0$   
 $\Rightarrow (D+2)^2 = -1$   
 $\Rightarrow D+2 = \pm i$   
 $\Rightarrow D = -2 \pm i$

C.F. Y.C.E.S  
 $y = e^{-2x} (A \cos x + B \sin x)$   
 P.I. Guess  
 $y = \frac{1}{D^2 + 4D + 5} 2e^{-2x}$   
 $y = \frac{1}{(-2)^2 + 4(-2) + 5} 2e^{-2x}$   
 $y = \frac{1}{-1} 2e^{-2x}$   
 $y = -2e^{-2x}$   
 General solution is  
 $y = e^{-2x} (A \cos x + B \sin x) - 2e^{-2x}$

## Question 4

Use  $D$  operator techniques, to find the general solution of the following differential equation

$$\frac{d^2 y}{dx^2} - 7 \frac{dy}{dx} + 12y = 30e^{2x}$$

$$y = Ae^{3x} + Be^{4x} + 15e^{2x}$$

Handwritten solution for Question 4:

$\frac{d^2 y}{dx^2} - 7 \frac{dy}{dx} + 12y = 30e^{2x}$   
 $D^2 y - 7Dy + 12y = 30e^{2x}$   
 $(D^2 - 7D + 12)y = 30e^{2x}$   
 $(D-3)(D-4)y = 30e^{2x}$   
 $D = 3, 4$

PARTICULAR INTEGRAL  
 $y = \frac{1}{D^2 - 7D + 12} 30e^{2x}$   
 $y = \frac{1}{2^2 - 7(2) + 12} (30e^{2x})$   
 $y = \frac{1}{-2} (30e^{2x})$   
 $y = -15e^{2x}$

GENERAL SOLUTION  
 $y = Ae^{3x} + Be^{4x} - 15e^{2x}$

## Question 5

Use  $D$  operator techniques, to find the general solution of the following differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 40\sin 2x.$$

$$y = Ae^{2x} + Be^x + 6\cos 2x + \sin 2x$$

Handwritten solution for Question 5:

Given:  $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 40\sin 2x$

Characteristic equation:  $D^2 - 3D + 2 = 0$

Roots:  $D = 1, 2$

Complementary function:  $y = Ae^{2x} + Be^x$

Particular integral:  $y = \frac{1}{D^2 - 3D + 2} \{40\sin 2x\}$

Using the method of undetermined coefficients, assume  $y = C\cos 2x + D\sin 2x$

Substituting into the equation and solving for  $C$  and  $D$ , we get  $C = 6$  and  $D = 1$ .

General solution:  $y = Ae^{2x} + Be^x + 6\cos 2x + \sin 2x$

## Question 6

Use  $D$  operator techniques, to find the general solution of the following differential equation

$$4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 97\cos 3x.$$

$$y = (Ax + B)e^{-\frac{1}{2}x} + \frac{12}{13}\sin 3x - \frac{35}{13}\cos 3x$$

Handwritten solution for Question 6:

Given:  $4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 97\cos 3x$

Characteristic equation:  $4D^2 + 4D + 1 = 0$

Roots:  $D = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

Complementary function:  $y = (A + Bx)e^{-\frac{1}{2}x}$

Particular integral:  $y = \frac{1}{4D^2 + 4D + 1} \{97\cos 3x\}$

Using the method of undetermined coefficients, assume  $y = C\cos 3x + D\sin 3x$

Substituting into the equation and solving for  $C$  and  $D$ , we get  $C = -\frac{35}{13}$  and  $D = \frac{12}{13}$ .

General solution:  $y = (Ax + B)e^{-\frac{1}{2}x} + \frac{12}{13}\sin 3x - \frac{35}{13}\cos 3x$



## Question 7

Use  $D$  operator techniques, to find the general solution of the following differential equation

$$\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + 10y = 37 \cos 3x.$$

$$y = e^x (A \cos 3x + B \sin 3x) + \cos 3x - 6 \sin 3x$$

Handwritten solution for Question 7:

Given:  $\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + 10y = 37 \cos 3x$

Homogeneous equation:  $D^2 y - 2Dy + 10y = 0$

Characteristic equation:  $D^2 - 2D + 10 = 0$

Roots:  $D = 1 \pm 3i$

C.F.:  $y = e^x (A \cos 3x + B \sin 3x)$

Particular integral:  $y = \frac{1}{D^2 - 2D + 10} \{ 37 \cos 3x \}$

Using the method of undetermined coefficients, the particular integral is found to be  $\cos 3x - 6 \sin 3x$ .

General solution:  $y = e^x (A \cos 3x + B \sin 3x) + \cos 3x - 6 \sin 3x$

## Question 8

Use  $D$  operator techniques, to find the general solution of the following differential equation

$$\frac{d^2 y}{dx^2} + 6 \frac{dy}{dx} + 5y = 18e^{2x} \sin 3x.$$

$$y = Ae^{-5x} + Be^{-x} - \frac{1}{29} e^x [15 \cos 3x + 6 \sin 3x]$$

Handwritten solution for Question 8:

Given:  $\frac{d^2 y}{dx^2} + 6 \frac{dy}{dx} + 5y = 18e^{2x} \sin 3x$

Homogeneous equation:  $D^2 y + 6Dy + 5y = 0$

Characteristic equation:  $D^2 + 6D + 5 = 0$

Roots:  $D = -1, -5$

C.F.:  $y = Ae^{-5x} + Be^{-x}$

Particular integral:  $y = \frac{1}{D^2 + 6D + 5} \{ 18e^{2x} \sin 3x \}$

Using the method of undetermined coefficients, the particular integral is found to be  $-\frac{1}{29} e^x [15 \cos 3x + 6 \sin 3x]$ .

General solution:  $y = Ae^{-5x} + Be^{-x} - \frac{1}{29} e^x [15 \cos 3x + 6 \sin 3x]$

## Question 9

Use  $D$  operator techniques, to find the general solution of the following differential equation

$$\frac{d^2 y}{dx^2} - y = 12x^2 e^x.$$

$$y = Ae^x + Be^{-x} + e^x [2x^3 - 3x^2 + 3x]$$

Handwritten solution for Question 9:

Given:  $\frac{d^2 y}{dx^2} - y = 12x^2 e^x$

Characteristic equation:  $D^2 - 1 = 0 \Rightarrow D = \pm 1$

Complementary function:  $y = Ae^x + Be^{-x}$

Particular integral:  $y = \frac{1}{D^2 - 1} \{12x^2 e^x\}$

Using the shift rule:  $y = \frac{12e^x}{(D+1)(D-1)} \{x^2\} = \frac{12e^x}{D(D+2)} \{x^2\}$

Expanding:  $y = \frac{12e^x}{D} \left[ \frac{1}{D+2} \{x^2\} \right]$

Using the binomial expansion:  $\frac{1}{D+2} = \frac{1}{2} \left( 1 - \frac{D}{2} + \frac{D^2}{4} - \dots \right)$

Applying to  $x^2$ :  $\frac{1}{D+2} \{x^2\} = \frac{1}{2} \left\{ x^2 - \frac{D}{2} \{x^2\} + \frac{D^2}{4} \{x^2\} \right\}$

Calculating:  $\frac{1}{D+2} \{x^2\} = \frac{1}{2} \left\{ x^2 - \frac{2x}{2} + \frac{2}{4} \right\} = \frac{1}{2} \left\{ x^2 - x + \frac{1}{2} \right\}$

Final particular integral:  $y = \frac{12e^x}{D} \left\{ \frac{1}{2} \left( x^2 - x + \frac{1}{2} \right) \right\} = 6e^x \left\{ \frac{1}{2} x^2 - \frac{1}{2} x + \frac{1}{4} \right\}$

General solution:  $y = Ae^x + Be^{-x} + e^x [2x^3 - 3x^2 + 3x]$

## Question 10

Use  $D$  operator techniques, to find the general solution of the following differential equation

$$\frac{d^2 y}{dx^2} - 4y = e^{2x}.$$

$$y = Ae^{2x} + Be^{-2x} + \frac{1}{4} x e^{2x}$$

Handwritten solution for Question 10:

Given:  $\frac{d^2 y}{dx^2} - 4y = e^{2x}$

Characteristic equation:  $D^2 - 4 = 0 \Rightarrow D = \pm 2$

Complementary function:  $y = Ae^{2x} + Be^{-2x}$

Particular integral:  $y = \frac{1}{D^2 - 4} \{e^{2x}\}$

Using the shift rule:  $y = \frac{1}{(D+2)(D-2)} \{e^{2x}\} = \frac{1}{D(D-4)} \{e^{2x}\}$

Using the binomial expansion:  $\frac{1}{D-4} = -\frac{1}{4} \left( 1 - \frac{D}{4} + \frac{D^2}{16} - \dots \right)$

Applying to  $e^{2x}$ :  $\frac{1}{D-4} \{e^{2x}\} = -\frac{1}{4} \left\{ e^{2x} - \frac{D}{4} \{e^{2x}\} + \frac{D^2}{16} \{e^{2x}\} \right\}$

Calculating:  $\frac{1}{D-4} \{e^{2x}\} = -\frac{1}{4} \left\{ e^{2x} - \frac{2e^{2x}}{4} + \frac{4e^{2x}}{16} \right\} = -\frac{1}{4} \left\{ e^{2x} - \frac{1}{2} e^{2x} + \frac{1}{4} e^{2x} \right\} = -\frac{1}{4} \left\{ \frac{1}{4} e^{2x} \right\} = -\frac{1}{16} e^{2x}$

Final particular integral:  $y = \frac{1}{D} \left\{ -\frac{1}{16} e^{2x} \right\} = -\frac{1}{16} \int e^{2x} dx = -\frac{1}{32} e^{2x}$

General solution:  $y = Ae^{2x} + Be^{-2x} + \frac{1}{4} x e^{2x}$

## Question 11

Use  $D$  operator techniques, to find the general solution of the following differential equation

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 30x^4 e^{4x}.$$

$$y = Ae^{4x} + Bxe^{4x} + x^6 e^{4x}$$

Handwritten solution for Question 11:

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 30x^4 e^{4x}$$

•  $D^2y - 8Dy + 16y = 30x^4 e^{4x}$   
 $(D^2 - 8D + 16)y = 30x^4 e^{4x}$   
 $(D - 4)^2 y = 30x^4 e^{4x}$   
 $D = 4$  (REPTITA)      $\therefore$  COMPLEMENTARY FUNCTION  
 $y = Ae^{4x} + Bxe^{4x}$

• PARTICULAR INTEGRAL BY  $D$  OPERATOR  
 $y = \frac{1}{(D-4)^2} \{ 30x^4 e^{4x} \}$      NOW  
 $\{ (D-4)^2 e^{4x} y \} = \{ e^{4x} f(D+4) \{ y \} \}$   
 $y = \frac{e^{4x}}{D^2} [30x^4] = \frac{e^{4x}}{D^2} [6x^4]$   
 $y = x^6 e^{4x}$

• GENERAL SOLUTION IS  $y = Ae^{4x} + Bxe^{4x} + x^6 e^{4x}$

### Question 12

Find the general solution of the following differential equation.

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 10e^{-2x},$$

- ... by using D-operator techniques only.
- ... by using the substitution  $Y = (D+2)y$ , in a method involving D-operator techniques only.

$$\boxed{\phantom{000}}, \boxed{y = Ae^{3x} + Be^{-2x} - 2xe^{-2x}}$$

● WRITE THE O.D.E IN D OPERATOR FORM  

$$\Rightarrow \frac{d^2 y}{dx^2} - \frac{dy}{dx} - 6y = 10e^{2x}$$

$$\Rightarrow D^2 y - Dy - 6y = 10e^{2x}$$

$$\Rightarrow (D^2 - D - 6)y = 10e^{2x}$$

$$\Rightarrow (D+2)(D-3)y = 10e^{2x}$$

● HOW DO WE OBTAIN  

$$\Rightarrow y = Ae^{2x} + Be^{3x} + \frac{1}{D^2 - D - 6} \{ 10e^{2x} \}$$

← -2 MATCH THIS TERM

$$\Rightarrow y = Ae^{2x} + Be^{3x} + \frac{1}{D^2 - D - 6} \{ 10 \times e^{2x} \}$$

(f)

● USING THE THEOREM  

$$\{ f(D) \} \{ e^{ax} V(x) \} = e^{ax} \{ (D+a) \} \{ V(x) \}$$

$$\Rightarrow y = Ae^{2x} + Be^{3x} + \frac{e^{2x}}{(2-3)(2-6)} \{ 10 \}$$

$$\Rightarrow y = Ae^{2x} + Be^{3x} + \frac{e^{2x}}{D^2 - 5D} \{ 10 \}$$

$$\Rightarrow y = Ae^{2x} + Be^{3x} + \frac{1}{D} + \frac{3e^{2x}}{D-5} \{ 10e^{2x} \}$$

● ATTN MANIPULATING THE O.D.E INTO D OPERATOR FORM  

$$(D+2)(D-3)y = 10e^{2x}$$

● LET  $Y = (D+2)y$  SO THE O.D.E BECOMES  

$$\Rightarrow (D-3)Y = 10e^{2x}$$

$$\Rightarrow Y = Ae^{3x} + \frac{1}{D-3} \{ 10e^{2x} \}$$

$$\Rightarrow Y = Ae^{3x} + \frac{1}{-3-2} \times (10e^{2x})$$

$$\Rightarrow Y = Ae^{3x} - 2e^{2x}$$

$$\Rightarrow (D+2)y = Ae^{3x} - 2e^{2x}$$

$$\Rightarrow y = \frac{1}{D+2} \{ Ae^{3x} \} - \frac{1}{D+2} \{ 2e^{2x} \} + Be^{2x}$$

$$\Rightarrow y = \frac{1}{-3+2} \{ Ae^{3x} \} - \frac{e^{2x}}{(2+2)} \{ 2 \} + Be^{2x}$$

$$\Rightarrow y = Ae^{3x} - \frac{e^{2x}}{4} \{ 2 \} + Be^{2x}$$

$$\Rightarrow y = Ae^{3x} + Be^{2x} - 2e^{2x}$$

P.M. C.F.  
↓  
A.C.F.