DIFFERENCE EQUATIONS

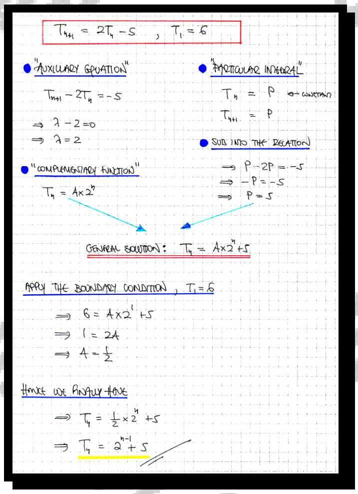
Question 1 (**)

A sequence of numbers T_1, T_2, T_3, T_4, T_5, … , T_n, … is generated by the recurrence relation

$$T_{n+1} = 2T_n - 5, \quad T_1 = 6.$$

Determine an expression for the n^{th} term of this sequence.

$\boxed{}$, $\boxed{T_n = 2^{n-1} + 5}$



$\boxed{T_{n+1} = 2T_n - 5 \quad , \quad T_1 = 6}$

• "AUXILIARY EQUATION"

$T_{n+1} - 2T_n = -5$

$\Rightarrow \lambda - 2 = 0$

$\Rightarrow \lambda = 2$

• "COMPLEMENTARY FUNCTION"

$T_n = A \times 2^n$

• "PARTICULAR INTEGRAL"

$T_n = P$ ← CONSTANT

$T_{n+1} = P$

• SUB INTO THE RELATION

$\Rightarrow P - 2P = -5$

$\Rightarrow -P = -5$

$\Rightarrow P = 5$

GENERAL SOLUTION: $T_n = A \times 2^n + 5$

APPLY THE BOUNDARY CONDITION, $T_1 = 6$

$\Rightarrow 6 = A \times 2^1 + 5$

$\Rightarrow 1 = 2A$

$\Rightarrow A = \frac{1}{2}$

HENCE WE FINALLY HAVE

$\Rightarrow T_n = \frac{1}{2} \times 2^n + 5$

$\Rightarrow T_n = 2^{n-1} + 5$

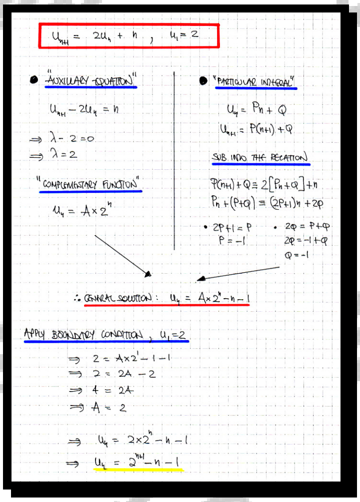
Question 2 **(**)**

A sequence of numbers u_1, u_2, u_3, u_4, $u_5 \ldots$, u_n, $\ldots$ is generated by the recurrence relation

$$u_{n+1} = 2u_n + n, \quad u_1 = 2.$$

Determine an expression for the n^{th} term of this sequence.

$\boxed{}$, $\boxed{u_n = 2^{n+1} - n - 1}$



Question 3 (+)**

A sequence of numbers b_1, b_2, b_3, b_4, b_5 … , b_n, … is generated by

$$b_{n+2} = 2b_n + n, \quad b_1 = 1, \quad b_2 = 5.$$

Determine an expression for the n^{th} term of this sequence.

$$\boxed{b_n = 2^n + (-1)^n}$$

$b_{n+2} = b_{n+1} + 2b_n$ $\quad \boxed{\begin{matrix} b_1 = 1 \\ b_2 = 5 \end{matrix}}$

$b_{n+2} - b_{n+1} - 2b_n = 0$

"AUXILIARY EQUATION"

$\lambda^2 - \lambda - 2 = 0$

$(\lambda + 1)(\lambda - 2) = 0$

$\lambda = \begin{cases} 2 \\ -1 \end{cases}$

$\therefore \boxed{b_n = A \times 2^n + B \times (-1)^n}$

$\left.\begin{matrix} b_1 = 1 \Rightarrow & 1 = 2A - B \\ b_2 = 5 \Rightarrow & 5 = 4A + B \end{matrix}\right\}$ ADD $\quad 6A = 6 \quad \boxed{A = 1}$ & $\boxed{B = 1}$

$\therefore b_n = 2^n + (-1)^n$

Question 4 (***)

A sequence of numbers u_1, u_2, u_3, u_4, $u_5 \ldots$, u_n, $\ldots$ is generated by the recurrence relation

$$u_{n+1} = 2u_n - n^2 + 3, \quad u_1 = 2.$$

Determine an expression for the n^{th} term of this sequence.

$$\boxed{u_n = n^2 + 2n - 2^{n-1}}$$

$u_{n+1} = 2u_n - n^2 + 3, \quad u_1 = 2$

$u_{n+1} - 2u_n = -n^2 + 3$

- "AUXILIARY EQUATION"

$\lambda - 2 = 0$

$\lambda = 2$ $\quad \therefore \boxed{u_n = A \times 2^n}$

- "PARTICULAR INTEGRAL"

LET $u_n = Pn^2 + Qn + R$

$u_{n+1} = P(n+1)^2 + Q(n+1) + R = P(n^2+2n+1) + Q(n+1) + R$

$= Pn^2 + 2Pn + P + Qn + Q + R$

SUB INTO THE EQUATION

$(Pn^2 + 2Pn + P + Qn + Q + R) - 2(Pn^2 + Qn + R) \equiv -n^2 + 3$

$-Pn^2 + (2P-Q)n + (P+Q-R) \equiv -n^2 + 3$

$\therefore P = 1 \quad 2P - Q = 0 \quad P + Q - R = 3$

$2 - Q = 0 \quad 1 + 2 - R = 3$

$Q = 2 \quad R = 0$

$\therefore \boxed{u_n = n^2 + 2n}$

$\therefore$ GENERAL SOLUTION $u_n = A \times 2^n + n^2 + 2n$ WITH $n=1 \quad u_1 = 2$

$2 = A \times 2 + 1 + 2$

$A = -\frac{1}{2}$

$\therefore u_n = -\frac{1}{2} \times 2^n + n^2 + 2n$

$u_n = n^2 + 2n - 2^{n-1}$

Question 5 (***)

A sequence of numbers t_1, t_2, t_3, t_4, $t_5 \dots$, t_n, $\dots$ is generated by the recurrence relation

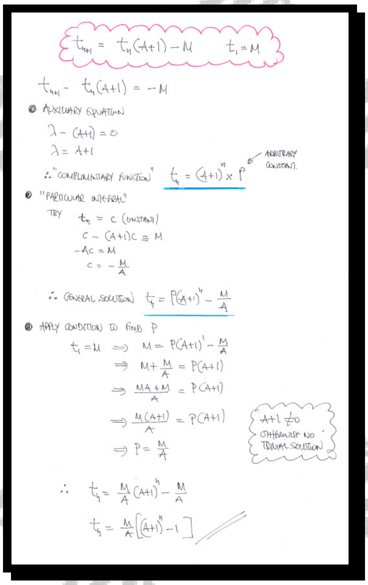
$$t_{n+1} = (A+1)t_n - M \ , \quad t_1 = M \ ,$$

where A and M are positive constants.

Determine an expression for the n^{th} term of this sequence is given by

$$t_n = \frac{M}{A}\left[(A+1)^n - 1\right].$$

proof



$$t_{n+1} = t_n(A+1) - M \qquad t_1 = M$$

$$t_{n+1} - t_n(A+1) = -M$$

- AUXILIARY EQUATION

$$\lambda - (A+1) = 0$$
$$\lambda = A+1$$

∴ "COMPLIMENTARY FUNCTION" $t_n = (A+1)^n \times P$ ← ARBITRARY CONSTANT.

- "PARTICULAR INTEGRAL"

TRY $t_n = c$ (CONSTANT)

$$c - (A+1)c = M$$
$$-Ac = M$$
$$c = -\frac{M}{A}$$

∴ GENERAL SOLUTION $t_n = P(A+1)^n - \frac{M}{A}$

- APPLY CONDITION TO FIND P

$$t_1 = M \implies M = P(A+1)^1 - \frac{M}{A}$$
$$\implies M + \frac{M}{A} = P(A+1)$$
$$\implies \frac{MA+M}{A} = P(A+1)$$
$$\implies \frac{M(A+1)}{A} = P(A+1)$$
$$\implies P = \frac{M}{A}$$

$A+1 \neq 0$ OTHERWISE NO TRIVIAL SOLUTION

$$\therefore \quad t_n = \frac{M}{A}(A+1)^n - \frac{M}{A}$$
$$t_n = \frac{M}{A}\left[(A+1)^n - 1\right]$$

Question 6 (***)

A sequence of numbers $u_1, u_2, u_3, u_4, u_5 \dots, u_n, \dots$ is generated by the following recurrence relation

$$u_{n+2} = u_{n+1} + 6u_n, \quad u_1 = 1, \quad u_2 = 1.$$

Determine an expression for the n^{th} term of this sequence.

$\boxed{}$, $\boxed{u_n = \frac{1}{5}\left[3^n - (-2)^n\right]}$

$\boxed{u_{n+2} = u_{n+1} + 6u_n \text{ WITH } u_1 = 1,\ u_2 = 1}$

REARRANGE THE RELATION

$u_{n+2} - u_{n+1} - 6u_n = 0$

"AUXILIARY EQUATION"

$\lambda^2 - \lambda - 6 = 0$

$(\lambda + 2)(\lambda - 3) = 0$

$\lambda = \begin{cases} -2 \\ 3 \end{cases}$

$\therefore$ GENERAL SOLUTION

$u_n = A(-2)^n + B(3)^n$

APPLY CONDITIONS

$u_1 = 1 \Rightarrow 1 = -2A + 3B \quad (n=1)$

$u_2 = 1 \Rightarrow 1 = 4A + 9B \quad (n=2)$

SOLVING

$\left.\begin{array}{l} 2 = -4A + 6B \\ 1 = 4A + 9B \end{array}\right\} \Rightarrow 15B = 3$

$B = \frac{1}{5}$

$\Rightarrow 1 = -2A + \frac{3}{5}$

$\Rightarrow 2A = -\frac{2}{5}$

$\Rightarrow A = -\frac{1}{5}$

FINALLY WE OBTAIN

$u_n = -\frac{1}{5}(-2)^n + \frac{1}{5}(3^n)$

$u_n = \frac{1}{5}\left[3^n - (-2)^n\right]$

Question 7 (***)

A sequence of numbers u_1, u_2, u_3, u_4, u_5 … , u_n, … is generated by the recurrence relation

$$u_{n+2} = 5u_{n+1} - 6u_n + 4n\,, \quad u_1 = 1\,,\ u_2 = 3\,.$$

Determine an expression for the n^{th} term of this sequence.

$\boxed{}$, $\boxed{u_n = 4\times 3^{n-1} - 2^{n+2} + 2n + 3}$

$\boxed{u_{n+2} = 5u_{n+1} - 6u_n + 4n\,, \quad u_1 = 1\,, \quad u_2 = 3}$

REWRITE THE EQUATION IN ORDER TO SOLVE AN AUXILIARY EQUATION

$\Rightarrow u_{n+2} - 5u_{n+1} + 6u_n = 4n$

AUXILIARY EQUATION

$\lambda^2 - 5\lambda + 6 = 0$

$(\lambda - 2)(\lambda - 3) = 0$

$\lambda = \begin{cases} 2 \\ 3 \end{cases}$

"COMPLIMENTARY FUNCTION"

$u_n = A(2^n) + B(3^n)$

FOR "PARTICULAR INTEGRAL" TRY

$u_n = Pn + Q$

$u_{n+1} = P(n+1) + Q$

$u_{n+2} = P(n+2) + Q$

SUBSTITUTE INTO THE RELATION

$P(n+2) + Q - 5[P(n+1) + Q] + 6[Pn + Q] \equiv 4n$

$Pn + 2P + Q - 5Pn - 5P - 5Q + 6Pn + 6Q \equiv 4n$

$2Pn + (2Q - 3P) \equiv 4n$

$\therefore P = 2 \quad \& \quad 2Q - 3P = 0$

$2Q = 6$

$Q = 3$

GENERAL SOLUTION

$u_n = A(2^n) + B(3^n) + 2n + 3$

APPLYING THE CONDITIONS GIVEN

$\left.\begin{array}{l} u_1 = 1 \Rightarrow 1 = 2A + 3B + 5 \\ u_2 = 3 \Rightarrow 3 = 4A + 9B + 7 \end{array}\right\} \Rightarrow$

$\left.\begin{array}{l} 2A + 3B = -4 \\ 4A + 9B = -4 \end{array}\right\} \overset{\times(-2)}{\Longrightarrow}$

$\left.\begin{array}{l} -4A - 6B = 8 \\ 4A + 9B = -4 \end{array}\right\} \Rightarrow$ ADDING EQUATIONS

$\Rightarrow 3B = 4$

$\Rightarrow B = \frac{4}{3}$

$\Rightarrow 2A + 3\left(\frac{4}{3}\right) = -4 \quad \leftarrow$ "$2A + 3B = -4$"

$\Rightarrow 2A + 4 = -4$

$\Rightarrow 2A = -8$

$\Rightarrow A = -4$

FINALLY WE OBTAIN

$u_n = -4(2^n) + \frac{4}{3}(3^n) + 2n + 3$

$u_n = -2^{n+2} + 4(3^{n-1}) + 2n + 3$

$u_n = 4(3^{n-1}) - 2^{n+2} + 2n + 3$

Question 8 (***)

A sequence of numbers u_1, u_2, u_3, u_4, $u_5 \ldots$, u_n, $\ldots$ is generated by the recurrence relation

$$u_{n+1} = u_n + 3n - 2, \quad u_1 = -1.$$

Determine an expression for the n^{th} term of this sequence.

Give the answer in the form $u_n = \frac{1}{2}(an+b)(n+c)$, where a, b and c are integers.

$\boxed{}$, $\boxed{u_n = \frac{1}{2}(3n-1)(n-2)}$

REWRITE THE DIFFERENCE EQUATION IN THE USUAL FORM

$$u_{n+1} = u_n + 3n - 2$$
$$u_{n+1} - u_n = 3n - 2$$

"AUXILIARY EQUATION" IS

$$\lambda - 1 = 0$$
$$\lambda = 1 \qquad \therefore \ u_n = A(1)^n = A$$

FOR "PARTICULAR INTEGRAL" WE NEED TO TRY $Pn^2 + Qn$ AS THE CONSTANT IS PART OF THE GENERAL SOLUTION ALREADY

$$u_n = Pn^2 + Qn$$
$$u_{n+1} = P(n+1)^2 + Q(n+1) = Pn^2 + 2Pn + P + Qn + Q$$

SUB INTO THE DIFFERENCE EQUATION

$$\Rightarrow u_{n+1} - u_n = 3n - 2$$
$$\Rightarrow \cancel{Pn^2} + 2Pn + P + \cancel{Qn} + Q - (\cancel{Pn^2} + \cancel{Qn}) \equiv 3n - 2$$
$$\Rightarrow 2Pn + (P+Q) \equiv 3n - 2$$

$$\therefore \ 2P = 3 \qquad \& \qquad P + Q = -2$$
$$P = \tfrac{3}{2} \qquad \qquad \tfrac{3}{2} + Q = -2$$
$$Q = -\tfrac{7}{2}$$

HENCE THE GENERAL SOLUTION IS

$$u_n = \tfrac{3}{2}n^2 - \tfrac{7}{2}n + A$$

FINALLY USING THE INITIAL TERM OF −1

$$u_1 = 2 \Rightarrow -1 = \tfrac{3}{2} \times 1^2 - \tfrac{7}{2} \times 1 + A$$
$$\Rightarrow -1 = \tfrac{3}{2} - \tfrac{7}{2} + A$$
$$\Rightarrow A = 1.$$

$$\therefore \ u_n = \tfrac{3}{2}n^2 - \tfrac{7}{2}n + 1$$
$$u_n = \tfrac{1}{2}(3n^2 - 7n + 2)$$
$$u_n = \tfrac{1}{2}(3n-1)(n-2)$$

// AS REQUIRED

Question 9 (***)

A sequence of numbers, u_1, u_2, u_3, u_4, ... , is defined by

$$u_{n+1} = 3u_n - 1, \quad u_1 = 2.$$

Determine, in terms of n, a simplified expression for

$$\sum_{r=1}^{n} u_r .$$

$\boxed{S_n = \frac{1}{4}\left[3^{n+1} + 2n - 3\right]}$

USING STANDARD TECHNIQUES

$u_{n+1} = 3u_n - 1 \quad , \quad u_1 = 2$

"AUXILIARY EQUATION"

$\Rightarrow u_{n+1} - 3u_n = -1$

$\Rightarrow \lambda - 3 = 0$

$\Rightarrow \lambda = 3$

GENERAL SOLUTION OF $u_{n+1} = 3u_n$ IS GIVEN BY

$u_n = A \times 3^n$

"PARTICULAR INTEGRAL" — TRY $u_n = C$ (CONSTANT)

$\Rightarrow C - 3C = -1$

$\Rightarrow -2C = -1$

$\Rightarrow C = \frac{1}{2}$

GENERAL SOLUTION OF $u_{n+1} = 3u_n - 1$ IS GIVEN BY

$\Rightarrow u_{n+1} = \frac{1}{2} + A \times 3^n$

USING THE CONDITION, $u_1 = 2$, IE $n=1$, $u_n = 2$

$\Rightarrow u_1 = \frac{1}{2} + A \times 3^1$

$\Rightarrow 2 = \frac{1}{2} + 3A$

$\Rightarrow 4 = 1 + 6A$

$\Rightarrow A = \frac{1}{2}$

$\therefore u_n = \frac{1}{2} + \frac{1}{2}(3^n)$

SUMMING UP, RECALLING THAT FOR GEOMETRIC PROGRESSIONS THE SUM OF THE FIRST n TERMS IS GIVEN BY "$\frac{a(r^n-1)}{r-1}$"

$\Rightarrow S_n = \sum_{r=1}^{n}\left(\frac{1}{2} + \frac{1}{2}(3^r)\right) = \frac{1}{2}\sum_{r=1}^{n} 1 + \frac{1}{2}\sum_{r=1}^{n} 3^r$

$\Rightarrow S_n = \frac{1}{2} \times n + \frac{1}{2}\left(3 + 3^2 + 3^3 + 3^4 + \dots + 3^n\right)$

$\Rightarrow S_n = \frac{n}{2} + \frac{1}{2}\left(\frac{3(3^n - 1)}{3-1}\right) = \frac{n}{2} + \frac{1}{4}\left(3^{n+1} - 3\right)$

$\Rightarrow S_n = \frac{n}{2} + \frac{3^{n+1} - 3}{4}$

$\Rightarrow S_n = \frac{3^{n+1} + 2n - 3}{4}$

Question 10 (***)

A sequence of numbers u_1, u_2, u_3, u_4, $u_5 \ldots$, u_n, $\ldots$ is generated by the recurrence relation

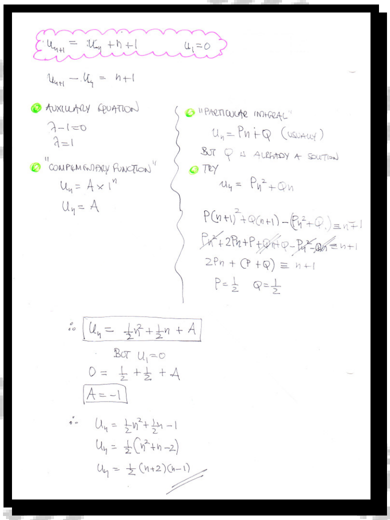
$$u_{n+1} = u_n + n + 1, \quad u_1 = 0.$$

Determine an expression for the n^{th} term of this sequence is given by

$$u_n = \frac{1}{2}(n+A)(n+B),$$

where A and B are integers to be found.

$$\boxed{u_n = \frac{1}{2}(n+2)(n-1)}$$



$u_{n+1} = u_n + n + 1 \qquad u_1 = 0$

$u_{n+1} - u_n = n + 1$

AUXILIARY EQUATION

$\lambda - 1 = 0$

$\lambda = 1$

"COMPLEMENTARY FUNCTION"

$u_n = A \times 1^n$

$u_n = A$

"PARTICULAR INTEGRAL"

$u_n = Pn + Q$ (USUALLY)

BUT Q IS ALREADY A SOLUTION

TRY

$u_n = Pn^2 + Qn$

$P(n+1)^2 + Q(n+1) - (Pn^2 + Qn) \equiv n+1$

$\cancel{Pn^2} + 2Pn + P + \cancel{Qn} + Q - \cancel{Pn^2} - \cancel{Qn} \equiv n+1$

$2Pn + (P+Q) \equiv n+1$

$P = \frac{1}{2} \quad Q = \frac{1}{2}$

$\therefore \boxed{u_n = \frac{1}{2}n^2 + \frac{1}{2}n + A}$

BUT $u_1 = 0$

$0 = \frac{1}{2} + \frac{1}{2} + A$

$\boxed{A = -1}$

$\therefore u_n = \frac{1}{2}n^2 + \frac{1}{2}n - 1$

$u_n = \frac{1}{2}(n^2 + n - 2)$

$u_n = \frac{1}{2}(n+2)(n-1)$

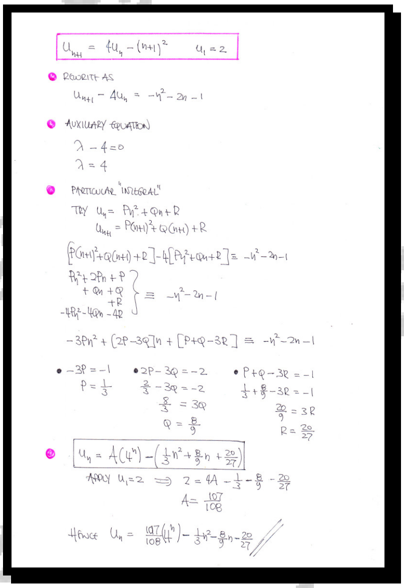
Question 11 **(***)**

A sequence of numbers u_1, u_2, u_3, u_4, $u_5 \ldots$, u_n, $\ldots$ is generated by the recurrence relation

$$u_{n+1} = 4u_n - (n+1)^2, \quad u_1 = 2.$$

Determine an expression for the n^{th} term of this sequence.

$$\boxed{u_n = \frac{107}{108}\left(4^n\right) - \frac{1}{3}n^2 - \frac{8}{9}n - \frac{20}{27} = \frac{1}{108}\left[107\left(4^n\right) - \left(36n^2 + 96n + 80\right)\right]}$$



$\boxed{u_{n+1} = 4u_n - (n+1)^2 \qquad u_1 = 2}$

- REWRITE AS

$u_{n+1} - 4u_n = -n^2 - 2n - 1$

- AUXILIARY EQUATION

$\lambda - 4 = 0$

$\lambda = 4$

- PARTICULAR "INTEGRAL"

TRY $u_n = Pn^2 + Qn + R$

$u_{n+1} = P(n+1)^2 + Q(n+1) + R$

$\left[P(n+1)^2 + Q(n+1) + R\right] - 4\left[Pn^2 + Qn + R\right] \equiv -n^2 - 2n - 1$

$$\left.\begin{array}{r} Pn^2 + 2Pn + P \\ + Qn + Q \\ + R \\ -4Pn^2 - 4Qn - 4R \end{array}\right\} \equiv -n^2 - 2n - 1$$

$-3Pn^2 + \left[2P - 3Q\right]n + \left[P + Q - 3R\right] \equiv -n^2 - 2n - 1$

- $-3P = -1$, $P = \frac{1}{3}$
- $2P - 3Q = -2$, $\frac{2}{3} - 3Q = -2$, $\frac{8}{3} = 3Q$, $Q = \frac{8}{9}$
- $P + Q - 3R = -1$, $\frac{1}{3} + \frac{8}{9} - 3R = -1$, $\frac{20}{9} = 3R$, $R = \frac{20}{27}$

- $\boxed{u_n = A(4^n) - \left(\frac{1}{3}n^2 + \frac{8}{9}n + \frac{20}{27}\right)}$

APPLY $u_1 = 2 \implies 2 = 4A - \frac{1}{3} - \frac{8}{9} - \frac{20}{27}$

$A = \frac{107}{108}$

HENCE $u_n = \frac{107}{108}(4^n) - \frac{1}{3}n^2 - \frac{8}{9}n - \frac{20}{27}$

Question 12 (***+)

A recurrence relation is defined as

$$u_{n+1} = (\sin x)u_n + \cos 2x \, , \; u_1 = k \, , \; 0 < x < \frac{\pi}{2} .$$

a) Explain why this recurrence relation will converge to a limit L, for all real values of the constant k.

b) Given that $L = \frac{1}{2}\sin x$, write the recurrence relation in the form

$$u_{n+1} = Au_n + B \, ,$$

where A and B are rational constants to be found.

c) Given further that $k = \frac{305}{24}$, show that $u_4 = 4$.

$$\boxed{u_{n+1} = \frac{1}{3}u_n + \frac{1}{9}}$$

a) $u_{n+1} = (\sin x)u_n + \cos 2x \, , \; u_1 = k \quad 0 < x < \frac{\pi}{2}$

- RECURRENCE RELATIONS OF THE FORM $u_{n+1} = Au_n + B$ WILL CONVERGE IF $-1 < A < 1$
- AS $\sin x$, WITH $0 < x < \frac{\pi}{2}$, SATISFIES THIS CONDITION, THE SEQUENCE CONVERGES. (FOR ALL STARTING VALUES)

b) NEXT, LET THE LIMIT BE L (AS $n \to \infty$)

$\Rightarrow L = L\sin x + \cos 2x$

$\Rightarrow \frac{1}{2}\sin x = \frac{1}{2}\sin^2 x + \cos 2x$

$\Rightarrow \sin x = \sin^2 x + 2\cos 2x$

$\Rightarrow \sin x = \sin^2 x + 2(1 - 2\sin^2 x)$

$\Rightarrow \sin x = \sin^2 x + 2 - 4\sin^2 x$

$\Rightarrow 3\sin^2 x + \sin x - 2 = 0$

$\Rightarrow (3\sin x - 2)(\sin x + 1) = 0$

$\Rightarrow \sin x = \begin{cases} 2/3 \\ \cancel{-1} \quad 0 < x < \frac{\pi}{2} \end{cases}$

$\therefore \cos 2x = 1 - 2\sin^2 x = 1 - 2 \times \frac{4}{9} = 1 - \frac{8}{9} = \frac{1}{9}$

$\therefore u_{n+1} = \frac{2}{3}u_n + \frac{1}{9}$

c) NOW $u_{n+1} = \frac{2}{3}u_n + \frac{1}{9} \, , \; u_1 = \frac{305}{24}$

$u_2 = \frac{2}{3}u_1 + \frac{1}{9} = \frac{2}{3} \times \frac{305}{24} + \frac{1}{9} = \frac{305}{36} + \frac{1}{9}$

$= \frac{305}{36} + \frac{4}{36} = \frac{309}{36} = \boxed{\frac{103}{12}}$

$u_3 = \frac{2}{3}u_2 + \frac{1}{9} = \frac{2}{3} \times \frac{103}{12} + \frac{1}{9} = \frac{103}{18} + \frac{1}{9}$

$= \frac{103}{18} + \frac{2}{18} = \frac{105}{18} = \boxed{\frac{35}{6}}$

$u_4 = \frac{2}{3}u_3 + \frac{1}{9} = \frac{2}{3} \times \frac{35}{6} + \frac{1}{9} = \frac{35}{9} + \frac{1}{9}$

$= \frac{36}{9}$

$= \boxed{4}$

AS REQUIRED

Question 13 (****)

The Fibonacci sequence of numbers is generated by the recurrence relation

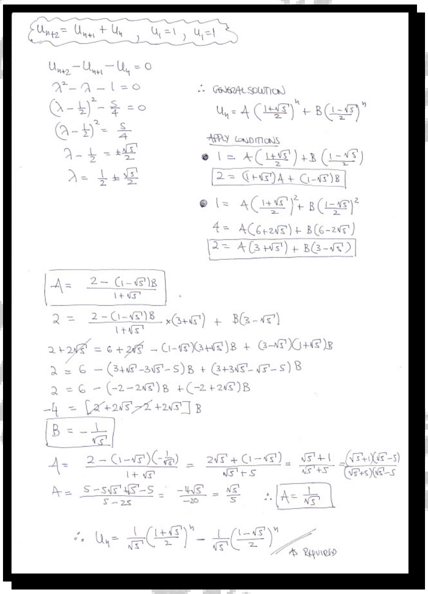
$$u_{n+2} = u_{n+1} + u_n, \quad n \in \mathbb{N},$$

with $u_1 = 1$, $u_2 = 1$.

Solve the above difference equation to show that the n^{th} term of Fibonacci sequence is given by

$$u_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

proof



$u_{n+2} = u_{n+1} + u_n, \quad u_1 = 1, \quad u_1 = 1$

$u_{n+2} - u_{n+1} - u_n = 0$

$\lambda^2 - \lambda - 1 = 0$

$\left(\lambda - \frac{1}{2}\right)^2 - \frac{5}{4} = 0$

$\left(\lambda - \frac{1}{2}\right)^2 = \frac{5}{4}$

$\lambda - \frac{1}{2} = \pm\frac{\sqrt{5}}{2}$

$\lambda = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$

∴ GENERAL SOLUTION

$u_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$

APPLY CONDITIONS

- $1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right)$

  $\boxed{2 = (1+\sqrt{5})A + (1-\sqrt{5})B}$

- $1 = A\left(\frac{1+\sqrt{5}}{2}\right)^2 + B\left(\frac{1-\sqrt{5}}{2}\right)^2$

  $4 = A(6+2\sqrt{5}) + B(6-2\sqrt{5})$

  $\boxed{2 = A(3+\sqrt{5}) + B(3-\sqrt{5})}$

$\boxed{A = \frac{2-(1-\sqrt{5})B}{1+\sqrt{5}}}$

$2 = \frac{2-(1-\sqrt{5})B}{1+\sqrt{5}} \times (3+\sqrt{5}) + B(3-\sqrt{5})$

$2 + \cancel{2\sqrt{5}} = 6 + \cancel{2\sqrt{5}} - (1-\sqrt{5})(3+\sqrt{5})B + (3-\sqrt{5})(1+\sqrt{5})B$

$2 = 6 - (3+\sqrt{5}-3\sqrt{5}-5)B + (3+3\sqrt{5}-\sqrt{5}-5)B$

$2 = 6 - (-2-2\sqrt{5})B + (-2+2\sqrt{5})B$

$-4 = [\cancel{2}+2\sqrt{5}\cancel{-2}+2\sqrt{5}]B$

$\boxed{B = -\frac{1}{\sqrt{5}}}$

$A = \frac{2-(1-\sqrt{5})\left(-\frac{1}{\sqrt{5}}\right)}{1+\sqrt{5}} = \frac{2\sqrt{5}+(1-\sqrt{5})}{\sqrt{5}+5} = \frac{\sqrt{5}+1}{\sqrt{5}+5} = \frac{(\sqrt{5}+1)(\sqrt{5}-5)}{(\sqrt{5}+5)(\sqrt{5}-5)}$

$A = \frac{5-5\sqrt{5}+\sqrt{5}-5}{5-25} = \frac{-4\sqrt{5}}{-20} = \frac{\sqrt{5}}{5} \quad \therefore \boxed{A = \frac{1}{\sqrt{5}}}$

$\therefore u_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$ AS REQUIRED

Question 14 (****)

A sequence of numbers is generated by the recurrence relation

$$u_{n+2} = 2u_{n+1} - 2u_n , \quad n \in \mathbb{N},$$

with $u_1 = 1$, $u_2 = 6$.

Determine a simplified expression for the n^{th} term of this sequence.

The final answer may not contain complex numbers

$$\boxed{u_n = \frac{1}{2}(-2+3\mathrm{i})(1-\mathrm{i})^n - \frac{1}{2}(2+3\mathrm{i})(1+\mathrm{i})^n \;=\; 2^{\frac{1}{2}n}\left[3\sin\left(\frac{1}{4}n\pi\right) - 2\cos\left(\frac{1}{4}n\pi\right)\right]},$$

$$u_{n+2} = 2u_{n+1} - 2u_n, \quad u_1 = 1, \; u_2 = 6$$

START WITH THE AUXILIARY EQUATION OF THE DIFFERENCE EQUATION

$\Rightarrow \lambda^2 = 2\lambda - 2$

$\Rightarrow \lambda^2 - 2\lambda = -2$

$\Rightarrow \lambda^2 - 2\lambda + 1 = -1$

$\Rightarrow (\lambda - 1)^2 = -1$

$\Rightarrow \lambda - 1 = \pm i$

$\Rightarrow \lambda = 1 \pm i$

THE GENERAL SOLUTION IS

$u_n = A(1+i)^n + B(1-i)^n$

WHERE A, B MAY BE COMPLEX

APPLYING CONDITIONS

- $u_1 = 1 \Rightarrow A(1+i) + B(1-i) = 1$
- $u_2 = 6 \Rightarrow A(1+i)^2 + B(1-i)^2 = 6$

$\Rightarrow \begin{cases} A(1+i) + B(1-i) = 1 \\ A(2i) + B(-2i) = 6 \end{cases} \Rightarrow \begin{cases} A(1+i) + B(1-i) = 1 \\ Ai - Bi = 3 \end{cases} \Rightarrow$

REARRANGE THE SECOND EQUATION & SUB INTO THE FIRST

$\Rightarrow Ai - Bi = 3$

$\Rightarrow A - B = \frac{3}{i}$

$\Rightarrow A - B = -3i$

$\Rightarrow A = B - 3i$

$\Rightarrow A(1+i) + B(1-i) = 1$

$\Rightarrow (B - 3i)(1+i) + B(1-i) = 1$

$\Rightarrow B(1+i) - 3i(1+i) + B(1-i) = 1$

$\Rightarrow 2B = 1 + 3i(1+i)$

$\Rightarrow 2B = 1 + 3i - 3$

$\Rightarrow B = -1 + \frac{3}{2}i$ OR $\frac{1}{2}(-2+3i)$

AND HENCE

$A = -1 + \frac{3}{2}i - 3i$

$A = -1 - \frac{3}{2}i$ OR $-\frac{1}{2}(2+3i)$

THE SOLUTION CAN NOW BE WRITTEN AS

$$u_n = \frac{1}{2}(-2+3i)(1-i)^n - \frac{1}{2}(2+3i)(1+i)^n$$

PROCEED TO ELIMINATE THE COMPLEX NUMBERS AS FOLLOWS

$\Rightarrow u_n = \frac{1}{2}\left[(-2+3i)\left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)^n - (2+3i)\left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^n\right]$

$\Rightarrow u_n = \frac{1}{2}\left[(-2+3i)\left(2^{\frac{n}{2}}e^{-i\frac{\pi n}{4}}\right) - (2+3i)\left(2^{\frac{n}{2}}e^{i\frac{\pi n}{4}}\right)\right]$

$\Rightarrow u_n = \frac{1}{2} \times 2^{\frac{n}{2}}\left[-2e^{-i\frac{\pi n}{4}} + 3ie^{-i\frac{\pi n}{4}} - 2e^{i\frac{\pi n}{4}} - 3ie^{i\frac{\pi n}{4}}\right]$

$\Rightarrow u_n = \frac{1}{2} \times 2^{\frac{n}{2}}\left[-3i\left(e^{i\frac{\pi n}{4}} - e^{-i\frac{\pi n}{4}}\right) - 2\left(e^{i\frac{\pi n}{4}} + e^{-i\frac{\pi n}{4}}\right)\right]$

$\Rightarrow u_n = \frac{1}{2} \times 2^{\frac{n}{2}}\left[-3i \times 2\sinh\left(\frac{i\pi n}{4}\right) - 2 \times 2\cosh\left(\frac{i\pi n}{4}\right)\right]$

$\Rightarrow u_n = \frac{1}{2} \times 2^{\frac{n}{2}}\left[-3i \times 2i\sin\left(\frac{\pi n}{4}\right) - 2 \times 2\cos\frac{\pi n}{4}\right]$

$\Rightarrow u_n = 2^{\frac{n}{2}}\left[3\sin\frac{\pi n}{4} - 2\cos\frac{\pi n}{4}\right]$

ALTERNATIVE SOLUTION – QUICKER & MORE DIRECT

STARTING WITH THE GENERAL SOLUTION FROM EARLIER

$u_n = A(1+i)^n + B(1-i)^n$

MANIPULATE FURTHER BEFORE USING THE CONDITIONS

$u_n = A\left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^n + B\left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)^n$

$u_n = A\left(2^{\frac{n}{2}}e^{i\frac{\pi n}{4}}\right) + B\left(2^{\frac{n}{2}}e^{-i\frac{\pi n}{4}}\right)$

$u_n = 2^{\frac{n}{2}}\left[A\cos\frac{\pi n}{4} + Ai\sin\frac{\pi n}{4} + B\cos\frac{\pi n}{4} - Bi\sin\frac{\pi n}{4}\right]$

$u_n = 2^{\frac{n}{2}}\left[(A+B)\cos\frac{\pi n}{4} + i(A-B)\sin\frac{n\pi}{4}\right]$

$u_n = 2^{\frac{n}{2}}\left[P\cos\frac{\pi n}{4} + Q\sin\frac{n\pi}{4}\right]$

APPLY CONDITIONS

- $u_1 = 1 \Rightarrow 2^{\frac{1}{2}}\left[P\cos\frac{\pi}{4} + Q\sin\frac{\pi}{4}\right] = 1$
- $u_2 = 6 \Rightarrow 2^1\left[P\cos\frac{\pi}{2} + Q\sin\frac{\pi}{2}\right] = 6$

$\Rightarrow \begin{cases} \sqrt{2}\left(P \times \frac{1}{\sqrt{2}} + Q \times \frac{1}{\sqrt{2}}\right) = 1 \\ 2(P \times 0 + Q \times 1) = 6 \end{cases}$

$\therefore P + Q = 1$

$2Q = 6$

$\therefore Q = 3$

$P = -2$

$\therefore u_n = 2^{\frac{n}{2}}\left[3\sin\frac{n\pi}{4} - 2\cos\frac{n\pi}{4}\right]$

AS BEFORE

Question 15 (****)

A sequence of numbers is generated by the recurrence relation

$$u_n = 5u_{n-1} - 4u_{n-2} - 12n + 31, \quad n \in \mathbb{N}, \quad n \geq 2$$

with $u_0 = 7$, $u_1 = 9$.

Determine a simplified expression for the n^{th} term of this sequence.

$\boxed{\text{EP4-N}}$, $\boxed{u_n = 4^n + 2n^2 - 3n + 6}$

REWRITE THE EQUATION

$u_n = 5u_{n-1} - 4u_{n-2} - 12n + 31$

$u_n - 5u_{n-1} + 4u_{n-2} = -12n + 31$

"AUXILIARY" EQUATION

$\lambda^2 - 5\lambda + 4 = 0$

$(\lambda - 1)(\lambda - 4) = 0$

$\lambda = \begin{cases} 1 \\ 4 \end{cases}$

"COMPLIMENTARY" FUNCTION

$u_n = A \times 1^n + B \times 4^n$

$u_n = A + B \times 4^n$

NEXT LOOK FOR "THE PARTICULAR INTEGRAL" — TRY A POLYNOMIAL

$u_n = Pn + Qn^2$ (AS THE CONSTANT "A" IS ALREADY THERE)

$u_{n-1} = P(n-1) + Q(n-1)^2$

$u_{n-2} = P(n-2) + Q(n-2)^2$

SUBSTITUTE BACK INTO THE LHS OF THE DIFFERENCE EQUATION

$\Rightarrow Pn + Qn^2 - 5\left[P(n-1) + Q(n-1)^2\right] + 4\left[P(n-2) + Q(n-2)^2\right] \equiv -12n + 31$

$\Rightarrow Pn + Qn^2 - 5\left[Pn - P + Qn^2 - 2Qn + Q\right] + 4\left[Pn - 2P + Qn^2 - 4Qn + 4Q\right] \equiv -12n + 31$

$\Rightarrow \cancel{Pn} + \cancel{Qn^2} - \cancel{5Pn} + 5P - \cancel{5Qn^2} + 10Qn - 5Q + \cancel{4Pn} - 8P + \cancel{4Qn^2} - 16Qn + 16Q \equiv -12n + 31$

$\Rightarrow -3P - 6Qn + 11Q \equiv -12n + 31$

$\Rightarrow -6Qn + (11Q - 3P) \equiv -12n + 31$

COMPARING COEFFICIENTS

- $-6Q = -12$
  $Q = 2$
- $11Q - 3P = 31$
  $22 - 3P = 31$
  $-3P = 9$
  $P = -3$

$\therefore \; u_n = A + B \times 4^n - 3n + 2n^2$

FINALLY APPLY CONDITIONS

- $u_0 = 7$
  $A + B = 7$
- $u_1 = 9$
  $A + 4B - 3 + 2 = 9$
  $A + 4B = 10$

$A + B = 7$

$A + 4B = 10$

$3B = 3$

$B = 1$ & $A = 6$

$\therefore \; u_n = 6 + 4^n - 3n + 2n^2$

Question 16 (****)

A sequence of numbers is generated by the recurrence relation

$$a_{n+2} = a_{n+1} - a_n + 21\left(2^n\right), \quad n \in \mathbb{N},$$

with $a_0 = 10$, $a_1 = \frac{1}{2}\left[31 + 5\sqrt{3}\right]$.

Determine a simplified expression for the n^{th} term of this sequence.

$$\boxed{a_n = 7 \times 2^n + 3\cos\left(\tfrac{1}{3}\pi n\right) + 5\sin\left(\tfrac{1}{3}\pi n\right)}$$

$a_{n+2} = a_{n+1} - a_n + 21(2^n) \quad n \geqslant 0 \quad a_0 = 10$
$a_1 = \frac{1}{2}(31 + 5\sqrt{3})$

"AUXILIARY EQUATION"

$\lambda^2 = \lambda - 1$
$\lambda^2 - \lambda + 1 = 0$
$4\lambda^2 - 4\lambda + 4 = 0$
$4\lambda^2 - 4\lambda + 1 = -3$
$(2\lambda - 1)^2 = -3$
$2\lambda - 1 = \pm\sqrt{3}\,i$
$\lambda = \frac{1 \pm \sqrt{3}\,i}{2}$

$\therefore \; a_n = A\left(\frac{1+\sqrt{3}\,i}{2}\right)^n + B\left(\frac{1-\sqrt{3}\,i}{2}\right)^n$

TRY "PARTICULAR" SOLUTION

$a_n = P \times 2^n$
$a_{n+1} = P \times 2^{n+1} = 2P \times 2^n$
$a_{n+2} = P \times 2^{n+2} = 4P \times 2^n$

SUBSTITUTING INTO THE RELATION

$4P \times 2^n = 2P \times 2^n - P \times 2^n + 21 \times 2^n$
$4P = 2P - P + 21$
$3P = 21$
$P = 7$

$\therefore \; a_n = A\left(\frac{1+\sqrt{3}\,i}{2}\right)^n + B\left(\frac{1-\sqrt{3}\,i}{2}\right)^n + 7 \times 2^n$
$a_n = A\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^n + B\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)^n + 7 \times 2^n$

$a_n = A\left(\cos\frac{\pi n}{3} + i\sin\frac{\pi n}{3}\right) + B\left(\cos\frac{\pi n}{3} - i\sin\frac{\pi n}{3}\right) + 7 \times 2^n$
$a_n = (A+B)\cos\frac{\pi n}{3} + i(A-B)\sin\frac{\pi n}{3} + 7 \times 2^n$
$a_n = D\cos\frac{\pi n}{3} + E\sin\frac{\pi n}{3} + 7 \times 2^n$

FINALLY APPLY CONDITIONS

$a_0 = 10 \implies 10 = D + 7 \implies \boxed{D = 3}$
$a_1 = \frac{1}{2}(31 + 5\sqrt{3}) \implies \frac{1}{2}(31 + 5\sqrt{3}) = D \times \frac{1}{2} + \frac{\sqrt{3}}{2}E + 14$
$\implies \frac{1}{2}(31 + 5\sqrt{3}) = \frac{3}{2} + \frac{\sqrt{3}}{2}E + 14$
$\implies \cancel{31} + 5\sqrt{3} = \cancel{3} + \sqrt{3}E + \cancel{28}$
$\implies \boxed{E = 5}$

$\therefore \; a_n = 3\cos\frac{\pi n}{3} + 5\sin\frac{\pi n}{3} + 7 \times 2^n$

Question 17 (****)

A sequence is defined by the recurrence relation

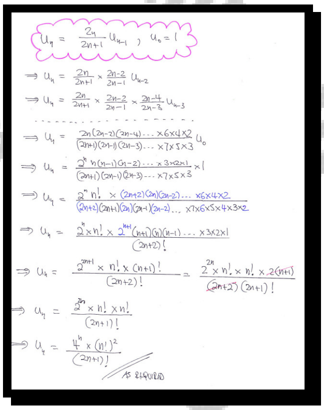
$$u_n = \frac{2n}{2n+1} u_{n-1}, \quad n \in \mathbb{N} \quad u_0 = 1.$$

Show, by direct manipulation, that

$$u_n = \frac{4^n \times (n!)^2}{(2n+1)!}.$$

[*you may not use proof by induction*]

$\boxed{\text{proof}}$



$$u_n = \frac{2n}{2n+1} u_{n-1}, \quad u_0 = 1$$

$$\Rightarrow u_n = \frac{2n}{2n+1} \times \frac{2n-2}{2n-1} u_{n-2}$$

$$\Rightarrow u_n = \frac{2n}{2n+1} \times \frac{2n-2}{2n-1} \times \frac{2n-4}{2n-3} u_{n-3}$$

$$\Rightarrow u_n = \frac{2n(2n-2)(2n-4)\ldots \times 6 \times 4 \times 2}{(2n+1)(2n-1)(2n-3)\ldots \times 7 \times 5 \times 3} u_0$$

$$\Rightarrow u_n = \frac{2^n n(n-1)(n-2)\ldots \times 3 \times 2 \times 1}{(2n+1)(2n-1)(2n-3)\ldots \times 7 \times 5 \times 3} \times 1$$

$$\Rightarrow u_n = \frac{2^n n! \times (2n+2)(2n)(2n-2)\ldots \times 6 \times 4 \times 2}{(2n+2)(2n+1)(2n)(2n-1)(2n-2)\ldots \times 7 \times 6 \times 5 \times 4 \times 3 \times 2}$$

$$\Rightarrow u_n = \frac{2^n \times n! \times 2^{n+1}(n+1)(n)(n-1)\ldots \times 3 \times 2 \times 1}{(2n+2)!}$$

$$\Rightarrow u_n = \frac{2^{2n+1} \times n! \times (n+1)!}{(2n+2)!} = \frac{2^{2n} \times n! \times n! \times \cancel{2(n+1)}}{\cancel{(2n+2)}(2n+1)!}$$

$$\Rightarrow u_n = \frac{2^{2n} \times n! \times n!}{(2n+1)!}$$

$$\Rightarrow u_n = \frac{4^n \times (n!)^2}{(2n+1)!}$$

AS REQUIRED

Question 18 (****)

A sequence is defined by the recurrence relation

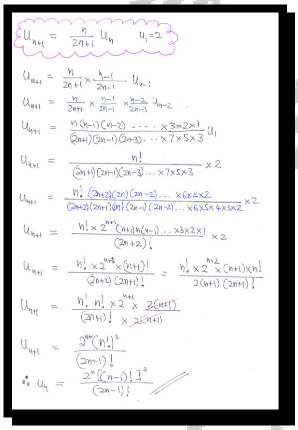
$$u_{n+1} = \frac{n}{2n+1} u_n , \ n \in \mathbb{N} \quad u_1 = 2 .$$

Show, by direct manipulation, that

$$u_n = \frac{2^n \times \left[(n-1)! \right]^2}{(2n-1)!} .$$

[*you may not use proof by induction*]

$\boxed{\text{proof}}$



$$u_{n+1} = \frac{n}{2n+1} u_n \qquad u_1 = 2$$

$$u_{n+1} = \frac{n}{2n+1} \times \frac{n-1}{2n-1} u_{n-1}$$

$$u_{n+1} = \frac{n}{2n+1} \times \frac{n-1}{2n-1} \times \frac{n-2}{2n-3} u_{n-2}$$

$$u_{n+1} = \frac{n(n-1)(n-2) \dots \times 3 \times 2 \times 1}{(2n+1)(2n-1)(2n-3) \dots \times 7 \times 5 \times 3} u_1$$

$$u_{n+1} = \frac{n!}{(2n+1)(2n-1)(2n-3) \dots \times 7 \times 5 \times 3} \times 2$$

$$u_{n+1} = \frac{n! (2n+2)(2n)(2n-2) \dots \times 6 \times 4 \times 2}{(2n+2)(2n+1)(2n)(2n-1)(2n-2) \dots \times 6 \times 5 \times 4 \times 3 \times 2} \times 2$$

$$u_{n+1} = \frac{n! \times 2^{n+1} (n+1) n (n-1) \dots \times 3 \times 2 \times 1}{(2n+2)!} \times 2$$

$$u_{n+1} = \frac{n! \times 2^{n+2} \times (n+1)!}{(2n+2)(2n+1)!} = \frac{n! \times 2^{n+2} \times (n+1) \times n!}{2(n+1)(2n+1)!}$$

$$u_{n+1} = \frac{n! \, n! \times 2^{n+1} \times \cancel{2(n+1)}}{(2n+1)! \times \cancel{2(n+1)}}$$

$$u_{n+1} = \frac{2^{n+1} (n!)^2}{(2n+1)!}$$

$$\therefore u_n = \frac{2^n [(n-1)!]^2}{(2n-1)!}$$

Question 19 (****+)

The n^{th} term of a series is given recursively by

$$A_n = \frac{a(2n+1)}{2n+4} A_{n-1}, \quad n \in \mathbb{N}, \quad n \geq 1,$$

where a is a positive constant.

Given further that $A_0 = 1$, show that

$$A_n = \left(\frac{a}{4}\right)^n \binom{2n+2}{n} \frac{1}{n+1}.$$

proof