DIFFERENTIATION UNDER THE INTEGRAL SIGN

LEIBNIZ INTEGRAL RULE

Question 1

The function f satisfies the following relationship.

$$f(x) = \int_1^x \left[f(t) \right]^2 \, dt \,, \quad f(2) = \tfrac{1}{2} \,.$$

Determine the value of $f\left(\tfrac{1}{2}\right)$.

$\boxed{}$, $\boxed{f\left(\frac{1}{2}\right) = \frac{2}{7}}$

DIFFERENTIATE WITH RESPECT TO x

$$f(x) = \int_1^x (f(t))^2 \, dt.$$

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}\left[\int_1^x [f(t)]^2 \, dt\right]$$

BY LEIBNIZ INTEGRAL THEOREM

$$\frac{df}{dx} = [f(x)]^2 \times 1 - [f(1)]^2 \times 0$$

$$\frac{df}{dx} = f^2$$

SEPARATE VARIABLES AND INTEGRATE OVER THE LIMITS GIVEN

$$\Rightarrow \frac{1}{f^2} \, df = 1 \, dx$$

$$\Rightarrow \int_{\frac{1}{2}}^{f} \frac{1}{f^2} \, df = \int_2^{\frac{1}{2}} 1 \, dx$$

$$\Rightarrow \left[-\frac{1}{f}\right]_{\frac{1}{2}}^{f} = \left[x\right]_2^{\frac{1}{2}}$$

$$\Rightarrow -\frac{1}{f} - \left(-\frac{1}{\frac{1}{2}}\right) = \frac{1}{2} - 2$$

$$\Rightarrow -\frac{1}{f} + 2 = -\frac{3}{2}$$

$$\Rightarrow \frac{7}{2} = \frac{1}{f}$$

$$\Rightarrow f = \frac{2}{7}$$

$$\therefore f\left(\tfrac{1}{2}\right) = \frac{2}{7}$$

Question 2

Find the value of

$$\lim_{p \to 0}\left[\frac{d}{dp}\left[\int_{2p-1}^{3p+2} \left(\frac{x+6}{4x}\right)^{x}\ dx\right]\right].$$

$\boxed{}$, $\boxed{\frac{23}{5}}$

USING LEIBNIZ INTEGRAL FORMULA AND IGNORING THE LIMITS

$$\frac{d}{dp}\int_{2p-1}^{3p+2}\left(\frac{x+6}{4x}\right)^{x} dx = \left(\frac{3p+2+6}{4(3p+2)}\right)^{3p+2} \times 3 - \left(\frac{2p-1+6}{4(2p-1)}\right)^{2p-1} \times 2$$

$$= 3\left(\frac{3p+8}{12p+8}\right)^{3p+2} - 2\left(\frac{2p+5}{8p-4}\right)^{2p-1}$$

TAKING THE LIMIT AS $p \to 0$

$$\lim_{p \to 0}\left[\frac{d}{dp}\int_{2p-1}^{3p+2}\left(\frac{x+6}{4x}\right)^{x} dx\right] = 3(1)^{2} - 2\left(-\frac{5}{4}\right)^{-1}$$

$$= 3 - 2\left(-\frac{4}{5}\right)^{1}$$

$$= 3 + \frac{8}{5}$$

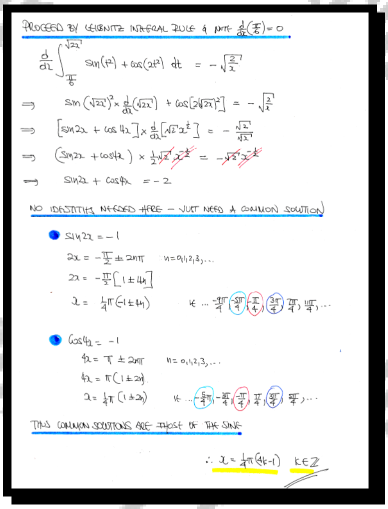
$$= \frac{23}{5}$$

Question 3

Find the general solution of the following equation

$$\frac{d}{dx}\left[\int_{\frac{1}{6}\pi}^{\sqrt{2x}} \sin\left(t^2\right)+\cos\left(2t^2\right)\ dt\right] = -\sqrt{\frac{2}{x}},\quad x\in\mathbb{R}.$$

$\boxed{}$, $\boxed{x=\frac{1}{4}\pi(4k-1)\quad k\in\mathbb{Z}}$



PROCEED BY LEIBNITZ INTEGRAL RULE & NOTE $\frac{d}{dx}\left(\frac{\pi}{6}\right)=0$

$$\frac{d}{dx}\int_{\frac{\pi}{6}}^{\sqrt{2x}} \sin(t^2)+\cos(2t^2)\ dt = -\sqrt{\frac{2}{x}}$$

$$\Longrightarrow \sin\left(\sqrt{2x}\right)^2\times\frac{d}{dx}\left(\sqrt{2x}\right)+\cos\left[2\left(\sqrt{2x}\right)^2\right] = -\sqrt{\frac{2}{x}}$$

$$\Longrightarrow \left[\sin 2x+\cos 4x\right]\times\frac{d}{dx}\left[\sqrt{2}x^{\frac{1}{2}}\right] = -\frac{\sqrt{2}}{\sqrt{x}}$$

$$\Longrightarrow \left(\sin 2x+\cos 4x\right)\times\frac{1}{2}\cancel{\sqrt{2}}\cancel{x^{-\frac{1}{2}}} = -\cancel{\sqrt{2}}\cancel{x^{-\frac{1}{2}}}$$

$$\Longrightarrow \sin 2x+\cos 4x = -2$$

NO IDENTITIES NEEDED HERE – JUST NEED A COMMON SOLUTION

- $\sin 2x = -1$

$2x = -\frac{\pi}{2}\pm 2n\pi \qquad n=0,1,2,3,\dots$

$2x = -\frac{\pi}{2}\left[1\pm 4n\right]$

$x = \frac{1}{4}\pi\left(-1\pm 4n\right) \qquad x \dots -\frac{9\pi}{4}, -\frac{5\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \dots$

- $\cos 4x = -1$

$4x = \pi\pm 2n\pi \qquad n=0,1,2,3,\dots$

$4x = \pi\left(1\pm 2n\right)$

$x = \frac{1}{4}\pi\left(1\pm 2n\right) \qquad x \dots -\frac{5}{4}\pi, -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \dots$

THUS COMMON SOLUTIONS ARE THOSE OF THE SINE

$\therefore x = \frac{1}{4}\pi(4k-1) \quad k\in\mathbb{Z}$

Question 4

The function g is defined as

$$g(x) = \int_{a(x)}^{b(x)} f(x,t)\ dt .$$

a) State Leibniz integral theorem for $g'(x)$.

b) Find a simplified expression for $\dfrac{d}{dx}\left[\displaystyle\int_{x^{-1}}^{x} \frac{\sqrt{1+x^2t^2}}{t}\ dt\right]$.

$$\boxed{\dfrac{d}{dx}\left[\displaystyle\int_{x^{-1}}^{x} \frac{\sqrt{1+x^2t^2}}{t}\ dt\right] = \frac{2\sqrt{1+x^4}}{x}}$$

a) LEIBNIZ INTEGRAL RULE STATE THAT IF $g(x) = \displaystyle\int_{a(x)}^{b(x)} f(x,t)\ dt$

$$g'(x) = f(x,b) \times \frac{db}{dx} - f(x,a) \times \frac{da}{dx} + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x}\left[f(x,t)\right]\ dt$$

b) APPLY THE RULE TO $f(x) = \dfrac{(1+x^2t^2)^{\frac{1}{2}}}{t}$

$$g(x) = \int_{\frac{1}{x}}^{x} \frac{(1+t^2x^2)^{\frac{1}{2}}}{t}\ dt$$

$$g'(x) = \frac{(1+x^4)^{\frac{1}{2}}}{x} \times 1 - \frac{(1+1)^{2}}{\frac{1}{x}}\left(-\frac{1}{x^2}\right) + \int_{\frac{1}{x}}^{x} \frac{\partial}{\partial x}\left[\frac{(1+x^2t^2)^{\frac{1}{2}}}{t}\right]\ dt$$

$$g'(x) = \frac{(1+x^4)^{\frac{1}{2}}}{x} + \frac{\sqrt{2}}{x} + \int_{\frac{1}{x}}^{x} \frac{1}{t} \times \frac{1}{2}(1+x^2t^2)^{-\frac{1}{2}} \times 2t^2x\ dt$$

$$g'(x) = \frac{(1+x^4)^{\frac{1}{2}}}{x} + \frac{\sqrt{2}}{x} + \int_{\frac{1}{x}}^{x} tx\,(1+x^2t^2)^{-\frac{1}{2}}\ dt$$

BY INSPECTION

$$\frac{d}{dt}\left[(1+x^2t^2)^{\frac{1}{2}}\right] = x^2t(1+x^2t^2)^{-\frac{1}{2}}$$

$$\frac{1}{x}\frac{d}{dt}\left[(1+x^2t^2)^{\frac{1}{2}}\right] = xt(1+x^2t^2)^{-\frac{1}{2}}$$

$$g'(x) = \frac{(1+x^4)^{\frac{1}{2}}}{x} + \frac{\sqrt{2}}{x} + \left[\frac{1}{x}\sqrt{1+x^2t^2}\right]_{t=\frac{1}{x}}^{t=x}$$

$$g'(x) = \frac{\sqrt{1+x^4}}{x} + \cancel{\frac{\sqrt{2}}{x}} + \left[\frac{\sqrt{1+x^4}}{x} - \cancel{\frac{\sqrt{2}}{x}}\right]$$

$$g'(x) = \frac{2\sqrt{1+x^4}}{x}$$

INTEGRATION APPLICATIONS INTRODUCTION

Question 1

It is given that the following integral converges.

$$\int_0^1 x^{\frac{4}{3}} \ln x \ dx .$$

a) Evaluate the above integral by introducing a parameter and carrying out a suitable differentiation under the integral sign.

b) Verify the answer obtained in part **(a)** by evaluating the integral by standard integration by parts.

V, $\boxed{\quad}$, $\boxed{-\frac{9}{49}}$

a) WE NOTE THAT $\frac{d}{d\alpha}\left[x^{\alpha}\right] = x^{\alpha}\ln x$

HENCE WE PROCEED AS FOLLOWS

$$\int_0^1 x^{\alpha}\ln x \, dx = \int_0^1 \frac{\partial}{\partial \alpha}\left(x^{\alpha}\right) dx \qquad \alpha > -1.$$

$$= \frac{\partial}{\partial \alpha}\int_0^1 x^{\alpha} \, dx$$

$$= \frac{\partial}{\partial \alpha}\left[\frac{1}{\alpha+1}x^{\alpha+1}\right]_0^1$$

$$= \frac{\partial}{\partial \alpha}\left[\frac{1}{\alpha+1} - 0\right]$$

$$= -\frac{1}{(\alpha+1)^2}$$

THUS BY LETTING $\alpha = \frac{4}{3}$ WE OBTAIN

$$\int_0^1 x^{\frac{4}{3}}\ln x \, dx = -\frac{1}{\left(\frac{4}{3}+1\right)^2} = -\frac{1}{\left(\frac{7}{3}\right)^2} = -\frac{9}{49}$$

b) VERIFICATION BY PARTS

$$\int_0^1 x^{\frac{4}{3}}\ln x \, dx = \left[\frac{3}{7}x^{\frac{7}{3}}\ln x\right]_0^1 - \frac{3}{7}\int_0^1 x^{\frac{4}{3}} dx$$

$\ln x$	$\frac{1}{x}$
$\frac{3}{7}x^{\frac{7}{3}}$	$x^{\frac{4}{3}}$

$$= \left[\frac{3}{7}x^{\frac{7}{3}}\ln x - \frac{9}{49}x^{\frac{7}{3}}\right]_0^1$$

$$= \left(\cancel{\frac{3}{7}\ln 1} - \frac{9}{49}\right) - \left(\frac{3}{7}\lim_{x\to 0}\left(x^{\frac{7}{3}}\ln x\right) - 0\right)$$

$$= -\frac{9}{49}$$

x TENDS TO ZERO FASTER THAN $\ln x$ TENDS TO $-\infty$

Question 2

$$\int_0^1 \frac{8}{\left(1+x^2\right)^2}\, dx .$$

Evaluate the above integral by introducing a parameter k and carrying out a suitable differentiation under the integral sign.

You may not use standard integration techniques in this question.

V, $\boxed{\pi+2}$

- THE INTEGRAL $\int \frac{dx}{(1+x^2)^2}$ RESEMBLES THAT OF $\int \frac{dx}{1+x^2} = \arctan x + C$
- THUS INTRODUCE A PARAMETER k

$$\frac{d}{dk}\left[\frac{1}{k^2+x^2}\right] = \frac{d}{dk}\left[(k^2+x^2)^{-1}\right] = \frac{-2k}{(k^2+x^2)^2}$$

$$-\frac{1}{2k}\frac{d}{dk}\left[\frac{1}{k^2+x^2}\right] = \dots \quad \frac{1}{(k^2+x^2)^2}$$

$$-\frac{4}{k}\frac{d}{dk}\left[\frac{1}{k^2+x^2}\right] = \dots \quad \frac{8}{(k^2+x^2)^2}$$

- THUS INTEGRATING BOTH SIDES WITH RESPECT TO x

$$\int_0^1 \frac{8}{k^2+x^2}\, dx = \int_0^1 -\frac{4}{k}\frac{d}{dk}\left(\frac{1}{k^2+x^2}\right) dx$$

$$\int_0^1 \frac{8}{k^2+x^2}\, dx = -\frac{4}{k}\frac{d}{dk}\int_0^1 \frac{1}{k^2+x^2}\, dx$$

$$\int_0^1 \frac{8}{k^2+x^2}\, dx = -\frac{4}{k}\frac{d}{dk}\left[\frac{1}{k}\arctan\frac{x}{k}\right]_0^1$$

$$\int_0^1 \frac{8}{k^2+x^2}\, dx = -\frac{4}{k}\left[-\frac{1}{k^2}\arctan\frac{x}{k} + \frac{1}{k}\underbrace{\left(-\frac{x}{k^2}\right)\times\frac{1}{1+\frac{x^2}{k^2}}}_{\frac{d}{dk}\left(\arctan\frac{x}{k}\right)}\right]_0^1$$

- SUB $k=1$

$$\int_0^1 \frac{8}{1+x^2}\, dx = -4\left[-\arctan x - \frac{x}{1+x^2}\right]_0^1$$

$$\int_0^1 \frac{8}{1+x^2}\, dx = \left[4\arctan x + \frac{4x}{1+x^2}\right]_0^1$$

$$\int_0^1 \frac{8}{1+x^2} = \left(4\times\frac{\pi}{4} + 2\right) - (0) = \pi + 2$$

Question 3

$$\int \frac{4}{\left(1-4x^2\right)^2}\,dx\,.$$

Find a simplified expression for the above integral by introducing a parameter a and carrying out a suitable differentiation under the integral sign.

You may assume

- $\displaystyle\int \frac{1}{a^2-x^2}\,dx = \frac{1}{a}\text{artanh}\left(\frac{x}{a}\right) + \text{constant}\,,\ |x| < a\,.$
- $\displaystyle\frac{d}{du}\left(\text{artanh}\,u\right) = \frac{1}{1-u^2}$

You may not use standard integration techniques in this question.

$$\boxed{\text{artanh}\,2x + \frac{2x}{1-4x^2} + C}$$

$$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{a}\text{artanh}\frac{x}{a} + C \qquad |x|<a$$

TREATING a AS A PARAMETER

$$\frac{\partial}{\partial a}\int \frac{1}{a^2-x^2}\,dx = \int \frac{\partial}{\partial a}\left[(a^2-x^2)^{-1}\right]dx = \int -2a(a^2-x^2)^{-2}\,dx$$

THUS

$$-\frac{1}{2a}\frac{\partial}{\partial a}\int \frac{1}{a^2-x^2}\,dx = \int \frac{1}{(a^2-x^2)^2}\,dx$$

$$-\frac{1}{2a}\frac{\partial}{\partial a}\left[\frac{1}{a}\text{artanh}\frac{x}{a} + C\right] = \int \frac{1}{(a^2-x^2)^2}\,dx$$

$$-\frac{1}{2a}\left[-\frac{1}{a^2}\text{artanh}\frac{x}{a} + \frac{1}{a}\times\frac{x}{-a^2}\times\underbrace{\frac{1}{1-\frac{x^2}{a^2}}}_{\boxed{\frac{d}{du}\left[\text{artanh}\,u\right] = \frac{1}{1-u^2}}}\right] = \int \frac{1}{(a^2-x^2)^2}\,dx$$

$$\int \frac{1}{(a^2-x^2)^2}\,dx = -\frac{1}{2a}\left[-\frac{1}{a^2}\text{artanh}\frac{x}{a} - \frac{x}{a^3}\,\frac{a^2}{a^2-x^2}\right] + C$$

$$\int \frac{1}{(a^2-x^2)^2}\,dx = \frac{1}{2a^3}\text{artanh}\frac{x}{a} + \frac{x}{2a^2(a^2-x^2)} + C$$

SET $a = \frac{1}{2}$

$$\int \frac{1}{(\frac{1}{4}-x^2)^2}\,dx = 4\text{artanh}\,2x + \frac{x}{\frac{1}{2}(\frac{1}{4}-x^2)} + C$$

$$\int \frac{1}{\frac{1}{16}(1-4x^2)^2}\,dx = 4\text{artanh}\,2x + \frac{x}{\frac{1}{8}(1-4x^2)} + C$$

$$\int \frac{16}{(1-4x^2)^2}\,dx = 4\text{artanh}\,2x + \frac{8x}{(1-4x^2)} + C$$

$$\int \frac{4}{(1-4x^2)^2}\,dx = \text{artanh}\,2x + \frac{2x}{1-4x^2} + C$$

Question 4

$$\int x^3 \mathrm{e}^{2x} \, dx .$$

Find a simplified expression for the above integral by introducing a parameter α and carrying out a suitable differentiation under the integral sign.

You may not use integration by parts or a reduction formula in this question.

$$\boxed{\frac{1}{8}\mathrm{e}^{2x}\left[4x^3 - 6x^2 + 6x - 3\right] + C}$$

$$\int x^3 e^{2x}\,dx = \int x^3 e^{\alpha x}\,dx \quad \text{WHERE } \alpha \text{ IS A PARAMETER}$$

THUS

$$\int x^3 e^{\alpha x}\,dx = \int \frac{\partial^3}{\partial \alpha^3}\left[e^{\alpha x}\right] dx = \frac{\partial^3}{\partial \alpha^3}\int e^{\alpha x}\,dx$$

$$= \frac{\partial^3}{\partial \alpha^3}\left[\frac{1}{\alpha}e^{\alpha x} + C\right]$$

$$= \frac{\partial^2}{\partial \alpha^2}\left[-\frac{1}{\alpha^2}e^{\alpha x} + \frac{1}{\alpha}x e^{\alpha x}\right]$$

$$= \frac{\partial^2}{\partial \alpha^2}\left[\left(\frac{1}{\alpha}x - \frac{1}{\alpha^2}\right)e^{\alpha x}\right]$$

$$= \frac{\partial}{\partial \alpha}\left[\left(-\frac{1}{\alpha^2}x + \frac{2}{\alpha^3}\right)e^{\alpha x} + \left(\frac{1}{\alpha}x - \frac{1}{\alpha^2}\right)x e^{\alpha x}\right]$$

$$= \frac{\partial}{\partial \alpha}\left[\left[-\frac{1}{\alpha^2}x + \frac{2}{\alpha^3} + \frac{1}{\alpha}x^2 - \frac{1}{\alpha^2}x\right]e^{\alpha x}\right]$$

$$= \frac{\partial}{\partial \alpha}\left[\left[\frac{1}{\alpha}x^2 - \frac{2}{\alpha^2}x + \frac{2}{\alpha^3}\right]e^{\alpha x}\right]$$

$$= \left(-\frac{1}{\alpha^2}x^2 + \frac{4}{\alpha^3}x - \frac{6}{\alpha^4}\right)e^{\alpha x} + \left(\frac{1}{\alpha}x^2 - \frac{2}{\alpha^2}x + \frac{2}{\alpha^3}\right)x e^{\alpha x}$$

$$= e^{\alpha x}\left[-\frac{1}{\alpha^2}x^2 + \frac{4}{\alpha^3}x - \frac{6}{\alpha^4} + \frac{1}{\alpha}x^3 - \frac{2}{\alpha^2}x^2 + \frac{2}{\alpha^3}x\right]$$

$$= e^{\alpha x}\left[\frac{1}{\alpha}x^3 - \frac{3}{\alpha^2}x^2 + \frac{6}{\alpha^3}x - \frac{6}{\alpha^4}\right]$$

SET $\alpha = 2$

$$\int x^3 e^{2x}\,dx = e^{2x}\left[\frac{1}{2}x^3 - \frac{3}{4}x^2 + \frac{3}{4}x - \frac{3}{8}\right] + C$$

$$= \frac{1}{8}e^{2x}\left[4x^3 - 6x^2 + 6x - 3\right] + C$$

Question 5

$$\int \frac{1}{\left(5+4x-x^2\right)^{\frac{3}{2}}}\, dx .$$

Find a simplified expression for the above integral by introducing a parameter a and carrying out a suitable differentiation under the integral sign.

You may assume

$$\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \arcsin\left(\frac{x}{a}\right) + \text{constant} , \ |x| \le a .$$

You may not use standard integration techniques in this question.

$$\boxed{\frac{x-2}{9\sqrt{5+4x-x^2}}+C}$$

$$\int \frac{1}{[5+4x-x^2]^{\frac{3}{2}}}\,dx = \int \frac{1}{(5-(x^2-4x))^{\frac{3}{2}}}\,dx$$

$$= \int \frac{1}{[9-(x^2-4x+4)]^{\frac{3}{2}}}\,dx = \int \frac{1}{[9-(x-2)^2]^{\frac{3}{2}}}\,dx$$

$$\dots \boxed{\text{SUBSTITUTION} \quad u = x-2, \ du = dx \quad \dots = \int \frac{1}{[9-u^2]^{\frac{3}{2}}}\,du}$$

$$= \int \frac{1}{(a^2-u^2)^{\frac{3}{2}}}\,du \qquad \text{WHERE } a=3$$

NOW CONSIDER $\frac{\partial}{\partial a}\left[(a^2-u^2)^{-\frac{1}{2}}\right] = -a(a^2-u^2)^{-\frac{3}{2}}$

$$\therefore -\frac{1}{a}\frac{\partial}{\partial a}\left[(a^2-u^2)^{-\frac{1}{2}}\right] = (a^2-u^2)^{-\frac{3}{2}} = \frac{1}{(a^2-u^2)^{\frac{3}{2}}}$$

THUS

$$\dots = \int -\frac{1}{a}\frac{\partial}{\partial a}\left[(a^2-u^2)^{-\frac{1}{2}}\right]du = -\frac{1}{a}\frac{\partial}{\partial a}\int (a^2-u^2)^{-\frac{1}{2}}\,du$$

$$= -\frac{1}{a}\frac{\partial}{\partial a}\int \frac{1}{\sqrt{a^2-u^2}}\,du = -\frac{1}{a}\frac{\partial}{\partial a}\left[\arcsin\frac{u}{a}+C\right]$$

$$= -\frac{1}{a} \times \frac{-u}{a^2} \times \frac{1}{\sqrt{1-\frac{u^2}{a^2}}} + k$$

SET $a=3$

$$= \frac{u}{27} \times \frac{1}{\sqrt{1-\frac{u^2}{9}}} + k = \frac{u}{27} \times \frac{1}{\sqrt{\frac{9-u^2}{9}}} + k$$

$$= \frac{u}{27} \times \frac{1}{\frac{\sqrt{9-u^2}}{3}} + k = \frac{u}{9\sqrt{9-u^2}} + k$$

$$= \frac{x-2}{9\sqrt{5+4x-x^2}} + k$$

Question 6

It is given that the following integral converges

$$\int_0^{\infty} x^n \, e^{-\alpha x} \; dx \, ,$$

where α is a positive parameter and n is a positive integer.

By carrying out a suitable differentiation under the integral sign, show that

$$\Gamma(n+1) = n! \, .$$

You may not use integration by parts or a reduction formula in this question.

proof

$$\Gamma(n) = (n-1)!$$
$$\Gamma(n+1) = n!$$

- $\Gamma(n+1) = \int_0^{\infty} x^n e^{-x} \, dx$

WRITE AS A PARAMETRIC INTEGRAL

- $\Gamma(n+1) = \int_0^{\infty} x^n e^{-\alpha x} \, dx$ $\boxed{\text{WHERE } \alpha = 1}$

$$= \int_0^{\infty} \frac{d^n}{d\alpha^n}\left[e^{-\alpha x}\right] \times (-1)^n \, dx$$

$$= \frac{d^n}{d\alpha^n} \int_0^{\infty} e^{-\alpha x} (-1)^n \, dx$$

$$= (-1)^n \times \frac{d^n}{d\alpha^n}\left[-\frac{1}{\alpha} e^{-\alpha x}\right]_0^{\infty}$$

$$= \left(-\frac{d}{d\alpha}\right)^n \left[\frac{1}{\alpha} e^{-\alpha x}\right]_{\infty}^{0}$$

$$= \left(-\frac{d}{d\alpha}\right)^n \left[\frac{1}{\alpha}\right] \qquad n+1$$

$$= \left(-\frac{d}{d\alpha}\right)^{n-1} \left[\frac{1}{\alpha^2}\right] \qquad n-1+2 = n+1$$

$$= \left[-\frac{d}{d\alpha}\right]^{n-2} \left[\frac{2\times1}{\alpha^3}\right] \qquad n-2+3 = n+1$$

$$= \left[-\frac{d}{d\alpha}\right]^{n-3} \left[\frac{3\times2\times1}{\alpha^4}\right] \qquad n-3+4 = n+1$$

$$\vdots$$

$$= \left[-\frac{d}{d\alpha}\right]^{1} \left[\frac{(n-1)(n-2)\dots3\times2\times1}{\alpha^n}\right] \qquad 1+n = n+1$$

$$= \frac{n(n-1)(n-2)\dots3\times2\times1}{\alpha^{n+1}} = \frac{n!}{\alpha^{n+1}} = n! \qquad \text{SINCE } \alpha = 1$$

Question 7

It is given that the following integral converges

$$\int_0^1 x^m [\ln x]^n \ dx ,$$

where n is a positive integer and m is a positive constant.

By carrying out a suitable differentiation under the integral sign, show that

$$\int_0^1 x^m [\ln x]^n \ dx = \frac{(-1)^n n!}{(m+1)^{n+1}} .$$

You may not use standard integration techniques in this question.

[ ], [proof]

- START FROM THE INTEGRAL

$$I(m) = \int_0^1 x^m \, dx = \left[\frac{1}{m+1} x^{m+1}\right]_0^1 = \frac{1}{m+1}$$

- DIFFERENTIATE THE ABOVE EQUATION WITH RESPECT TO m (ONCE)

$$\Longrightarrow \frac{\partial I}{\partial m} = \frac{\partial}{\partial m}\left(\frac{1}{m+1}\right)$$

$$\Longrightarrow \frac{\partial}{\partial m}\left[\int_0^1 x^m \, dx\right] = -\frac{1}{(m+1)^2}$$

$$\Longrightarrow \int_0^1 \frac{\partial}{\partial m}\left[x^m\right] dx = -\frac{1}{(m+1)^2}$$

$$\Longrightarrow \int_0^1 x^m \ln x \, dx = \frac{-1}{(m+1)^2}$$

NOTE THAT $\frac{d}{dy}(a^y) = a^y \ln a$

- DIFFERENTIATE THE ABOVE WITH RESPECT TO m AGAIN (TWICE SO FAR)

$$\Longrightarrow \int_0^1 x^m (\ln x)^2 \, dx = \frac{(-1)(-2)}{(m+1)^3} = \frac{(-1)^2 \times 2!}{(m+1)^3}$$

- DIFFERENTIATE THE ABOVE WITH RESPECT TO m AGAIN (THREE TIMES SO FAR)

$$\Longrightarrow \int_0^1 x^m (\ln x)^3 \, dx = \frac{(-1)(-2)(-3)}{(m+1)^4} = \frac{(-1)^3 \times 3!}{(m+1)^4}$$

- DIFFERENTIATING n TIMES IN TOTAL WE OBTAIN

$$\int_0^1 x^m (\ln x)^n \, dx = \frac{(-1)^n \times n!}{(m+1)^{n+1}}$$

AS REQUIRED

Question 8

$$I(\alpha)=\int_{0}^{\pi} \frac{1}{\alpha-\cos x}\,dx\,,\ |\alpha|>1.$$

a) Use an appropriate method to show that

$$I(\alpha)=\frac{\pi}{\sqrt{\alpha^{2}-1}}.$$

b) By carrying out a suitable differentiation under the integral sign, evaluate

$$\int_{0}^{\pi} \frac{1}{\left(\sqrt{2}-\cos x\right)^{2}}\,dx\,.$$

You may not use standard integration techniques in this part of the question.

$\boxed{\pi\sqrt{2}}$

a) $\displaystyle\int_{0}^{\pi} \frac{dx}{\alpha-\cos x}$ BY LITTLE t SUBSTITUTION

$$= \int_{0}^{\infty} \frac{1}{\alpha-\frac{1-t^2}{1+t^2}} \times \frac{2\,dt}{1+t^2} = \int_{0}^{\infty} \frac{2}{\alpha(1+t^2)-(1-t^2)}\,dt$$

$$= \int_{0}^{\infty} \frac{2}{(\alpha+1)t^2+(\alpha-1)}\,dt = \frac{1}{\alpha+1}\int_{0}^{\infty} \frac{2}{t^2+\frac{\alpha-1}{\alpha+1}}\,dt$$

$$= \frac{2}{\alpha+1}\int_{0}^{\infty} \frac{1}{t^2+\left[\sqrt{\frac{\alpha-1}{\alpha+1}}\right]^2}\,dt$$

STANDARD INTEGRAL TO ARCTAN

$$= \frac{2}{\alpha+1} \times \frac{1}{\sqrt{\frac{\alpha-1}{\alpha+1}}}\left[\arctan\left[\frac{t}{\sqrt{\frac{\alpha-1}{\alpha+1}}}\right]\right]_{0}^{\infty}$$

$$= \frac{2}{\alpha+1} \times \frac{\sqrt{\alpha+1}}{\sqrt{\alpha-1}}\left[\frac{\pi}{2}-0\right]$$

$$= \frac{2}{\sqrt{\alpha+1}\sqrt{\alpha-1}} \times \frac{\pi}{2}$$

$$= \frac{\pi}{\sqrt{\alpha^2-1}}$$

AS REQUIRED

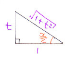
- LET $t=\tan\frac{x}{2}$

$$dt = \tfrac{1}{2}\sec^2\tfrac{x}{2}\,dx$$

$$dx = \frac{2}{\sec^2\frac{x}{2}}\,dt$$

$$dx = \frac{2}{1+\tan^2\frac{x}{2}}\,dt$$

$$\boxed{dx = \frac{2}{1+t^2}\,dt}$$

- 

$$\tan\tfrac{x}{2} = t = \tfrac{t}{1}$$

$$\sin\tfrac{x}{2} = \frac{t}{\sqrt{1+t^2}}$$

$$\cos\tfrac{x}{2} = \frac{1}{\sqrt{1+t^2}}$$

THUS

$$\cos x = \cos^2\tfrac{x}{2}-\sin^2\tfrac{x}{2}$$

$$\cos x = \frac{1}{1+t^2}-\frac{t^2}{1+t^2}$$

$$\boxed{\cos x = \frac{1-t^2}{1+t^2}}$$

- LIMITS

$x=0,\ t=\infty$

$x=0,\ t=0$

b) LET $\boxed{I = \displaystyle\int_{0}^{\pi} \frac{1}{\alpha-\cos x}\,dx = \frac{\pi}{\sqrt{\alpha^2-1}}}$

$$\Rightarrow \frac{\partial I}{\partial \alpha} = \frac{\partial}{\partial \alpha}\int_{0}^{\pi} \frac{1}{\alpha-\cos x}\,dx = \int_{0}^{\pi} \frac{\partial}{\partial \alpha}(\alpha-\cos x)^{-1}\,dx$$

$$\Rightarrow \frac{\partial I}{\partial \alpha} = \int_{0}^{\pi} -\frac{1}{(\alpha-\cos x)^2}\,dx$$

$$\Rightarrow \frac{\partial}{\partial \alpha}\left[\frac{\pi}{\sqrt{\alpha^2-1}}\right] = \int_{0}^{\pi} -\frac{1}{(\alpha-\cos x)^2}\,dx$$

$$\Rightarrow \int_{0}^{\pi} -\frac{1}{(\alpha-\cos x)^2}\,dx = \pi\frac{\partial}{\partial \alpha}\left[(\alpha^2-1)^{-\frac{1}{2}}\right]$$

$$\Rightarrow \int_{0}^{\pi} -\frac{1}{(\alpha-\cos x)^2}\,dx = \pi\left[-\tfrac{1}{2}(\alpha^2-1)^{-\frac{3}{2}}(2\alpha)\right]$$

$$\Rightarrow \int_{0}^{\pi} -\frac{1}{(\alpha-\cos x)^2}\,dx = -\frac{\pi\alpha}{(\alpha^2-1)^{3/2}}$$

THUS

$$\int_{0}^{\pi} \frac{1}{(\alpha-\cos x)^2}\,dx = \frac{\pi\alpha}{(\alpha^2-1)^{3/2}}$$

SO

$$\int_{0}^{\pi} \frac{1}{(\sqrt{2}-\cos x)^2}\,dx = \pi\sqrt{2}$$

FURTHER INTEGRATION APPLICATIONS

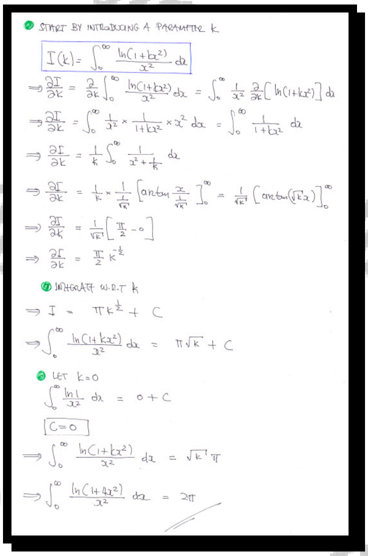
Question 1

$$I = \int_0^{\infty} \frac{\ln\left(1+4x^2\right)}{x^2} \, dx \ .$$

By introducing a parameter in the integrand and carrying a suitable differentiation under the integral sign show that

$$I = 2\pi .$$

V, $\boxed{\text{proof}}$



START BY INTRODUCING A PARAMETER k

$$\boxed{I(k) = \int_0^{\infty} \frac{\ln(1+kx^2)}{x^2}\,dx}$$

$$\Rightarrow \frac{\partial I}{\partial k} = \frac{\partial}{\partial k}\int_0^{\infty} \frac{\ln(1+kx^2)}{x^2}\,dx = \int_0^{\infty} \frac{1}{x^2}\frac{\partial}{\partial k}\left[\ln(1+kx^2)\right]dx$$

$$\Rightarrow \frac{\partial I}{\partial k} = \int_0^{\infty} \frac{1}{x^2} \times \frac{1}{1+kx^2} \times x^2\,dx = \int_0^{\infty} \frac{1}{1+kx^2}\,dx$$

$$\Rightarrow \frac{\partial I}{\partial k} = \frac{1}{k}\int_0^{\infty} \frac{1}{x^2+\frac{1}{k}}\,dx$$

$$\Rightarrow \frac{\partial I}{\partial k} = \frac{1}{k} \times \frac{1}{\frac{1}{\sqrt{k}}}\left[\arctan\frac{x}{\frac{1}{\sqrt{k}}}\right]_0^{\infty} = \frac{1}{\sqrt{k}}\left[\arctan(\sqrt{k}x)\right]_0^{\infty}$$

$$\Rightarrow \frac{\partial I}{\partial k} = \frac{1}{\sqrt{k}}\left[\frac{\pi}{2} - 0\right]$$

$$\Rightarrow \frac{\partial I}{\partial k} = \frac{\pi}{2}k^{-\frac{1}{2}}$$

INTEGRATE W.R.T k

$$\Rightarrow I = \pi k^{\frac{1}{2}} + C$$

$$\Rightarrow \int_0^{\infty} \frac{\ln(1+kx^2)}{x^2}\,dx = \pi\sqrt{k} + C$$

LET $k=0$

$$\int_0^{\infty} \frac{\ln 1}{x^2}\,dx = 0 + C$$

$$\boxed{C=0}$$

$$\Longrightarrow \int_0^{\infty} \frac{\ln(1+kx^2)}{x^2}\,dx = \sqrt{k}\,\pi$$

$$\Longrightarrow \int_0^{\infty} \frac{\ln(1+4x^2)}{x^2}\,dx = 2\pi$$

Question 2

It is given that the following integral converges.

$$I = \int_0^1 \frac{x-1}{\ln x}\, dx .$$

Evaluate I by carrying out a suitable differentiation under the integral sign.

You may not use standard integration techniques in this question.

V, , $\boxed{\ln 2}$

- START BY INTRODUCING A PARAMETER K AS FOLLOWS

$$I(k) = \int_0^1 \frac{x^k - 1}{\ln x}\, dx$$

- DIFFERENTIATING W.R.T k

$$\frac{\partial I}{\partial k} = \frac{\partial}{\partial k}\left[\int_0^1 \frac{x^k-1}{\ln x}\, dx\right] = \int_0^1 \frac{1}{\ln x}\frac{\partial}{\partial k}(x^k - 1)\, dx$$

$$\frac{\partial I}{\partial k} = \int_0^1 \frac{1}{\cancel{\ln x}}\left[x^k \cancel{\ln x}\right] dx = \int_0^1 x^k\, dx$$

$$\frac{\partial I}{\partial k} = \left[\frac{1}{k+1}x^{k+1}\right]_0^1 = \frac{1}{k+1}\left[1-0\right]$$

$$\frac{\partial I}{\partial k} = \frac{1}{k+1}$$

- INTEGRATING W.R.T k

$$I(k) = \ln|k+1| + C$$

$$\int_0^1 \frac{x^k-1}{\ln x}\, dx = \ln|k+1| + C$$

- LET $k=0$

$$0 = \cancel{\ln 1} + C \qquad \therefore C = 0$$

- THUS WE HAVE

$$\int_0^1 \frac{x^k-1}{\ln x}\, dx = \ln|k+1|$$

$$\int_0^1 \frac{x-1}{\ln x}\, dx = \ln 2$$

Question 3

$$I = \int_0^{\infty} \frac{e^{-2x} - e^{-8x}}{x} \, dx \, .$$

By introducing a parameter in the integrand and carrying a suitable differentiation under the integral sign show that

$$I = \ln 4 \, .$$

V, $\boxed{\text{proof}}$

$$\boxed{\int_0^{\infty} \frac{e^{-2x} - e^{-8x}}{x} \, dx = \ln 4}$$

METHOD A
TREAT "2" AS A PARAMETER

- $I(a) = \int_0^{\infty} \frac{e^{-ax} - e^{-8x}}{x} \, dx$

$\Rightarrow \frac{\partial I}{\partial a} = \frac{\partial}{\partial a} \int_0^{\infty} \frac{e^{-ax} - e^{-8x}}{x} \, dx$

$\Rightarrow \frac{\partial I}{\partial a} = \int_0^{\infty} \frac{\partial}{\partial a}\left[\frac{e^{-ax}}{x} - \frac{e^{-8x}}{x}\right] dx$

$\Rightarrow \frac{\partial I}{\partial a} = \int_0^{\infty} -e^{-ax} \, dx$

$\Rightarrow \frac{\partial I}{\partial a} = \left[\frac{1}{a} e^{-ax}\right]_0^{\infty}$

$\Rightarrow \frac{\partial I}{\partial a} = \frac{1}{a}\left[0 - 1\right]$

$\Rightarrow \frac{\partial I}{\partial a} = -\frac{1}{a}$

- INTEGRATING

$\Rightarrow I = -\ln a + C$

$\Rightarrow \int_0^{\infty} \frac{e^{-ax} - e^{-8x}}{x} \, dx = -\ln a + C$

LET $a = 8$

$\Rightarrow \int \frac{e^{-8x} - e^{-8x}}{x} \, dx = -\ln 8 + C$

$\Rightarrow 0 = -\ln 8 + C$

$\Rightarrow \boxed{C = \ln 8}$

$\int_0^{\infty} \frac{e^{-ax} - e^{-8x}}{x} \, dx = -\ln a + \ln 8$

$\int_0^{\infty} \frac{e^{-ax} - e^{-8x}}{x} \, dx = \ln \frac{8}{a}$

$\boxed{\text{LET } a = 2}$

$\int_0^{\infty} \frac{e^{-2x} - e^{-8x}}{x} \, dx = \ln 4$

METHOD B
TREAT "8" AS A PARAMETER

- $I(b) = \int_0^{\infty} \frac{e^{-2x} - e^{-bx}}{x} \, dx$

$\Rightarrow \frac{\partial I}{\partial b} = \frac{\partial}{\partial b} \int_0^{\infty} \frac{e^{-2x} - e^{-bx}}{x} \, dx$

$\Rightarrow \frac{\partial I}{\partial b} = \int_0^{\infty} \frac{\partial}{\partial b}\left[\frac{e^{-2x}}{x} - \frac{e^{-bx}}{x}\right] dx$

$\Rightarrow \frac{\partial I}{\partial b} = \int_0^{\infty} e^{-bx} \, dx$

$\Rightarrow \frac{\partial I}{\partial b} = \left[-\frac{1}{b} e^{-bx}\right]_0^{\infty}$

$\Rightarrow \frac{\partial I}{\partial b} = -\frac{1}{b}(0 - 1)$

$\Rightarrow \frac{\partial I}{\partial b} = \frac{1}{b}$

- INTEGRATING

$\Rightarrow I = \ln b + C$

$\Rightarrow \int_0^{\infty} \frac{e^{-2x} - e^{-bx}}{x} \, dx = \ln b + C$

LET $b = 2$

$\Rightarrow \int_0^{\infty} \frac{e^{-2x} - e^{-2x}}{x} \, dx = \ln 2 + C$

$\Rightarrow 0 = \ln 2 + C$

$\Rightarrow C = -\ln 2$

$\int_0^{\infty} \frac{e^{-2x} - e^{-bx}}{x} \, dx = \ln b - \ln 2$

$\int_0^{\infty} \frac{e^{-2x} - e^{-bx}}{x} \, dx = \ln \frac{b}{2}$

$\boxed{\text{LET } b = 8}$

$\int_0^{\infty} \frac{e^{-2x} - e^{-8x}}{x} \, dx = \ln 4$

Question 4

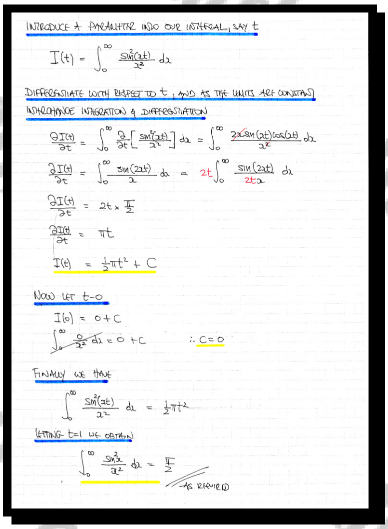
It is given that

$$\int_0^{\infty} \frac{\sin(kx)}{kx}\, dx = \frac{\pi}{2}.$$

Use Leibniz's integral rule to show that

$$\int_0^{\infty} \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2}.$$

$\boxed{}$, $\boxed{\text{proof}}$



INTRODUCE A PARAMETER INTO OUR INTEGRAL, SAY t

$$I(t) = \int_0^{\infty} \frac{\sin^2(xt)}{x^2}\, dx$$

DIFFERENTIATE WITH RESPECT TO t, AND AS THE LIMITS ARE CONSTANT INTERCHANGE INTEGRATION & DIFFERENTIATION

$$\frac{\partial I(t)}{\partial t} = \int_0^{\infty} \frac{\partial}{\partial t}\left[\frac{\sin^2(xt)}{x^2}\right] dx = \int_0^{\infty} \frac{\cancel{2x}\sin(xt)\cos(xt)}{x^{\cancel{2}}}\, dx$$

$$\frac{\partial I(t)}{\partial t} = \int_0^{\infty} \frac{\sin(2xt)}{x}\, dx = 2t\int_0^{\infty} \frac{\sin(2xt)}{2tx}\, dx$$

$$\frac{\partial I(t)}{\partial t} = 2t \times \frac{\pi}{2}$$

$$\frac{\partial I(t)}{\partial t} = \pi t$$

$$I(t) = \frac{1}{2}\pi t^2 + C$$

NOW LET $t=0$

$$I(0) = 0 + C$$

$$\int_0^{\infty} \frac{0}{x^2}\, dx = 0 + C \qquad \therefore C = 0$$

FINALLY WE HAVE

$$\int_0^{\infty} \frac{\sin^2(xt)}{x^2}\, dx = \frac{1}{2}\pi t^2$$

LETTING $t=1$ WE OBTAIN

$$\int_0^{\infty} \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2}$$

AS REQUIRED

Question 5

It is given that the following integral converges.

$$\int_0^1 \frac{x^5 - 1}{\ln x}\ dx\,.$$

Evaluate the above integral by introducing a parameter and carrying out a suitable differentiation under the integral sign.

You may not use standard integration techniques in this question.

$\boxed{\ln 6}$

$$\int_0^1 \frac{x^5-1}{\ln x}\,dx$$

INTRODUCE A PARAMETER SAY α & CONSIDER THE INTEGRAL

$$\longrightarrow \int_0^1 \frac{x^\alpha - 1}{\ln x}\,dx = F(\alpha)$$

$$\Longrightarrow \frac{d}{d\alpha}\int_0^1 \frac{x^\alpha - 1}{\ln x}\,dx = \frac{d}{d\alpha}\left[F(\alpha)\right]$$

$$\Longrightarrow \int_0^1 \frac{d}{d\alpha}\left(\frac{x^\alpha - 1}{\ln x}\right)dx = \frac{d}{d\alpha}\left[F(\alpha)\right]$$

$$\Longrightarrow \int_0^1 \frac{x^\alpha \ln x - 0}{\ln x}\,dx = \frac{d}{d\alpha}\left[F(\alpha)\right]$$

$$\Longrightarrow \int_0^1 x^\alpha\,dx = \frac{d}{d\alpha}\left[F(\alpha)\right]$$

$$\Longrightarrow \left[\frac{1}{\alpha+1}x^{\alpha+1}\right]_0^1 = \frac{d}{d\alpha}\left[F(\alpha)\right]$$

$$\Longrightarrow \frac{1}{\alpha+1} = \frac{d}{d\alpha}\left[F(\alpha)\right]$$

$$\Longrightarrow F(\alpha) = \int \frac{1}{\alpha+1}\,d\alpha$$

$$\Longrightarrow F(\alpha) = \ln|\alpha+1| + C$$

$$\Longrightarrow \int_0^1 \frac{x^\alpha - 1}{\ln x}\,dx = \ln|\alpha+1| + C$$

TO EVALUATE THE CONSTANT C, PICK A CONVENIENT VALUE FOR α, SAY $\alpha = 0$

$0 = \ln 1 + C \quad \therefore C = 0$

$$\Longrightarrow \int_1^0 \frac{x^\alpha - 1}{\ln x}\,dx = \ln|\alpha+1|$$

$$\therefore \int_0^1 \frac{x^5-1}{\ln x}\,dx = \ln 6$$

Question 6

$$I = \int_0^{\infty} \frac{e^{-2x} \sin x}{x} \, dx .$$

By introducing in the integrand a parameter k and carrying a suitable differentiation under the integral sign show that

$$I = \operatorname{arccot} 2 .$$

V, , proof

LET $I = \int_0^{\infty} \frac{e^{-kx} \sin x}{x} \, dx$, k A REAL PARAMETER

$$\Rightarrow \frac{\partial I}{\partial k} = \frac{\partial}{\partial k}\left[\int_0^{\infty} \frac{e^{-kx}\sin x}{x}\, dx\right] = \int_0^{\infty} \frac{\partial}{\partial k}\left(e^{-kx}\right)\frac{\sin x}{x}\, dx$$

$$\Rightarrow \frac{\partial I}{\partial k} = \int_0^{\infty} -\cancel{x}e^{-kx}\frac{\sin x}{\cancel{x}}\, dx = \int_0^{\infty} -e^{-kx}\sin x \, dx$$

PROCEED TO EVALUATE THE INTEGRAL BY COMPLEX NUMBERS (OR LAPLACE TRANSFORMS IF THE TECHNIQUES ARE KNOWN)

$$\frac{\partial I}{\partial k} = -\operatorname{Im}\int_0^{\infty} e^{-kx} e^{ix}\, dx$$

$$= -\operatorname{Im}\int_0^{\infty} e^{(-k+i)x}\, dx$$

$$= -\operatorname{Im}\left[\frac{1}{-k+i} e^{(-k+i)x}\right]_0^{\infty}$$

$$= -\operatorname{Im}\left[\frac{-k-i}{k^2+1} e^{-kx} e^{ix}\right]_0^{\infty} \qquad \boxed{\left|e^{ix}\right| = 1}$$

$$= -\operatorname{Im}\left[0 - \frac{-k-i}{k^2+1}\right]$$

$$= \operatorname{Im}\left[-\frac{k+i}{k^2+1}\right]$$

$$= -\frac{1}{k^2+1}$$

FINALLY WE HAVE

$$\Rightarrow \frac{\partial I}{\partial k} = -\frac{1}{k^2+1}$$

$$\Rightarrow I = -\arctan k + C$$

$$\Rightarrow \int_0^{\infty} \frac{e^{-kx}\sin x}{x}\, dx = C - \arctan k$$

LET $k=0$ IN THE ABOVE EQUATION

$$\int_0^{\infty} \frac{\sin x}{x}\, dx = C$$

$$\int_0^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2} = C$$

OR LET $k \to \infty$

$$0 = C - \arctan(\infty)$$

$$0 = C - \frac{\pi}{2}$$

$$C = \frac{\pi}{2}$$

$$\Rightarrow \int_0^{\infty} \frac{e^{-kx}\sin x}{x}\, dx = \frac{\pi}{2} - \arctan k$$

$$\Rightarrow \int_0^{\infty} \frac{e^{-kx}\sin x}{x}\, dx = \operatorname{arccot} k$$

LET $k=2$ IN THE ABOVE EQUATION, YIELDS THE REQUIRED RESULT

$$\Rightarrow \int_0^{\infty} \frac{e^{-2x}\sin x}{x}\, dx = \operatorname{arccot} 2$$

Question 7

$$I = \int_0^{\infty} \frac{e^{-x} - e^{-7x}}{x \sec x} \, dx \; .$$

By introducing in the integrand a parameter α and carrying a suitable differentiation under the integral sign show that

$$I = \ln 5 \, .$$

V, proof

$$\boxed{\int_0^{\infty} \frac{e^{-x} - e^{-7x}}{x \sec x} \, dx = \ln 5}$$

- LET $I(\alpha) = \int_0^{\infty} \frac{e^{-x} - e^{-\alpha x}}{x \sec x} \, dx$

$$\Rightarrow \frac{\partial I}{\partial \alpha} = \frac{\partial}{\partial \alpha} \int_0^{\infty} \frac{e^{-x} - e^{-\alpha x}}{x \sec x} \, dx = \int_0^{\infty} \frac{\partial}{\partial \alpha} \left[\frac{e^{-x} - e^{-\alpha x}}{x \sec x} \right] dx$$

$$\Rightarrow \frac{\partial I}{\partial \alpha} = \int_0^{\infty} \frac{\partial}{\partial \alpha} \left[\frac{e^{-x}}{x \sec x} - \frac{e^{-\alpha x}}{x \sec x} \right] dx$$

$$\Rightarrow \frac{\partial I}{\partial \alpha} = \int_0^{\infty} 0 + \frac{x e^{-\alpha x}}{x \sec x} \, dx = \int_0^{\infty} e^{-\alpha x} \cos x \, dx$$

- DOUBLE INTEGRATION BY PARTS OR COMPLEX NUMBERS

$$\dots \int_0^{\infty} e^{-\alpha x} \cos x \, dx = \mathrm{Re} \int_0^{\infty} e^{-\alpha x} e^{ix} \, dx = \mathrm{Re} \int_0^{\infty} e^{x(-\alpha + i)} \, dx$$

$$= \mathrm{Re} \left[\frac{1}{-\alpha + i} e^{x(-\alpha + i)} \right]_0^{\infty} = \mathrm{Re} \left[\frac{-\alpha - i}{\alpha^2 + 1} e^{-\alpha x} (\cos x + i \sin x) \right]_0^{\infty}$$

$$= \mathrm{Re} \left[\frac{-\alpha - i}{\alpha^2 + 1} (0 - 1) \right] = \mathrm{Re} \left[\frac{\alpha + i}{\alpha^2 + 1} \right] = \frac{\alpha}{\alpha^2 + 1}$$

- RETURNING TO THE MAIN WORK

$$\Rightarrow \frac{\partial I}{\partial \alpha} = \frac{\alpha}{\alpha^2 + 1}$$

$$\Rightarrow I = \frac{1}{2} \ln(\alpha^2 + 1) + C$$

$$\Rightarrow \int_0^{\infty} \frac{e^{-x} - e^{-\alpha x}}{x \sec x} \, dx = \frac{1}{2} \ln(\alpha^2 + 1) + C$$

- TO EVALUATE THE CONSTANT CHOOSE A SUITABLE VALUE FOR THE PARAMETER, HERE $\alpha = 1$ AS IT MAKES THE INTEGRAND ZERO

THUS $\quad I(1) = \int_0^{\infty} \frac{e^{-x} - e^{-x}}{x \sec x} \, dx = \frac{1}{2} \ln 2 + C$

$$\boxed{C = -\frac{1}{2} \ln 2}$$

- SO $\int_0^{\infty} \frac{e^{-x} - e^{-\alpha x}}{x \sec x} \, dx = \frac{1}{2} \ln(\alpha^2 + 1) - \frac{1}{2} \ln(2)$

$$\int_0^{\infty} \frac{e^{-x} - e^{-\alpha x}}{x \sec x} \, dx = \frac{1}{2} \ln \left(\frac{\alpha^2 + 1}{2} \right)$$

- IF $\alpha = 7$

$$\int_0^{\infty} \frac{e^{-x} - e^{-7x}}{x \sec x} \, dx = \frac{1}{2} \ln 25$$

$$\int_0^{\infty} \frac{e^{-x} - e^{-7x}}{x \sec x} \, dx = \ln 5$$

AS REQUIRED

Question 8

$$I = \int_0^{\infty} \frac{\cos x}{x}\left[e^{-4x} - e^{-6x}\right]\,dx\ .$$

By introducing in the integrand a parameter λ and carrying a suitable differentiation under the integral sign show that

$$I = \frac{1}{2}\ln 2 .$$

proof

$$\boxed{\int_0^{\infty} \frac{\cos 2x}{x}\left(e^{-4x} - e^{-6x}\right)\,dx = \frac{1}{2}\ln 2}$$

METHOD A – TREAT "4" AS A PARAMETER λ

$$\boxed{I(\lambda) = \int_0^{\infty} \frac{\cos 2x}{x}\left(e^{-\lambda x} - e^{-6x}\right)dx}$$

$$\Longrightarrow \frac{\partial I}{\partial \lambda} = \frac{\partial}{\partial \lambda}\int_0^{\infty} \frac{\cos 2x}{x}\left(e^{-\lambda x} - e^{-6x}\right)dx = \int_0^{\infty} \frac{\cos 2x}{x}\frac{\partial}{\partial \lambda}\left(e^{-\lambda x} - e^{-6x}\right)dx$$

$$\Longrightarrow \frac{\partial I}{\partial \lambda} = \int_0^{\infty} \frac{\cos 2x}{x}\left(-xe^{-\lambda x}\right)dx = \int_0^{\infty} -e^{-\lambda x}\cos 2x\,dx$$

$$\Longrightarrow \frac{\partial I}{\partial \lambda} = -\text{Re}\int_0^{\infty} e^{-\lambda x}e^{i2x}\,dx = -\int_0^{\infty} e^{x(-\lambda+2i)}\,dx$$

$$\Longrightarrow \frac{\partial I}{\partial \lambda} = -\text{Re}\left[\frac{1}{-\lambda+2i}e^{x(-\lambda+2i)}\right]_0^{\infty} = -\text{Re}\left[\frac{-\lambda-2i}{\lambda^2+4}e^{-\lambda x}(\cos 2x + i\sin 2x)\right]_0^{\infty}$$

$$\Longrightarrow \frac{\partial I}{\partial \lambda} = \text{Re}\left[\frac{\lambda+2i}{\lambda^2+4}(0-1)\right] = \text{Re}\left[\frac{-\lambda-2i}{\lambda^2+4}\right] = -\frac{\lambda}{\lambda^2+4}$$

INTEGRATE W.R.T λ

$$\Longrightarrow I = \int -\frac{\lambda}{\lambda^2+4}\,d\lambda = -\frac{1}{2}\ln(\lambda^2+4) + C$$

$$\Longrightarrow \int_0^{\infty} \frac{\cos 2x}{x}\left(e^{-\lambda x} - e^{-6x}\right)dx = C - \frac{1}{2}\ln(\lambda^2+4)$$

LET $\lambda = 6$

$$\Longrightarrow \int_0^{\infty} \frac{\cos 2x}{x}\left(e^{-6x} - e^{-6x}\right)dx = C - \frac{1}{2}\ln 40$$

$$0 = C - \frac{1}{2}\ln 40$$

$$\boxed{C = \frac{1}{2}\ln 40}$$

NOW LET $\lambda = 4$

$$\Longrightarrow \int_0^{\infty} \frac{\cos 2x}{x}\left(e^{-4x} - e^{-6x}\right)dx = \frac{1}{2}\ln 40 - \frac{1}{2}\ln 20$$

$$= \frac{1}{2}\ln 2$$

AS REQUIRED

METHOD B – TREAT "6" AS A PARAMETER λ

$$\boxed{I(\lambda) = \int_0^{\infty} \frac{\cos 2x}{x}\left(e^{-4x} - e^{-\lambda x}\right)dx}$$

$$\Longrightarrow \frac{\partial I}{\partial \lambda} = \frac{\partial}{\partial \lambda}\int_0^{\infty} \frac{\cos 2x}{x}\left(e^{-4x} - e^{-\lambda x}\right)dx = \int_0^{\infty} \frac{\cos 2x}{x}\frac{\partial}{\partial \lambda}\left(e^{-4x} - e^{-\lambda x}\right)dx$$

$$\Longrightarrow \frac{\partial I}{\partial \lambda} = \int_0^{\infty} \frac{\cos 2x}{x}\left(xe^{-\lambda x}\right)dx = \int_0^{\infty} e^{-\lambda x}\cos 2x\,dx$$

$$\Longrightarrow \frac{\partial I}{\partial \lambda} = \text{Re}\int_0^{\infty} e^{-\lambda x}e^{2ix}\,dx = \text{Re}\int_0^{\infty} e^{x(-\lambda+2i)}\,dx$$

$$\Longrightarrow \frac{\partial I}{\partial \lambda} = \text{Re}\left[\frac{1}{-\lambda+2i}e^{x(-\lambda+2i)}\right]_0^{\infty} = \text{Re}\left[\frac{-\lambda-2i}{\lambda^2+4}e^{-\lambda x}(\cos 2x + i\sin 2x)\right]_0^{\infty}$$

$$\Longrightarrow \frac{\partial I}{\partial \lambda} = \text{Re}\left[\frac{-\lambda-2i}{\lambda^2+4}(0-1)\right] = \text{Re}\left[\frac{\lambda+2i}{\lambda^2+4}\right] = \frac{\lambda}{\lambda^2+4}$$

INTEGRATING W.R.T λ

$$\Longrightarrow I = \int \frac{\lambda}{\lambda^2+4}\,d\lambda = \frac{1}{2}\ln(\lambda^2+4) + C$$

$$\Longrightarrow \int_0^{\infty} \frac{\cos 2x}{x}\left(e^{-4x} - e^{-\lambda x}\right)dx = \frac{1}{2}\ln(\lambda^2+4) + C$$

LET $\lambda = 4$

$$\int_0^{\infty} \frac{\cos 2x}{x}\left(e^{-4x} - e^{-4x}\right)dx = \frac{1}{2}\ln 20 + C$$

$$0 = \frac{1}{2}\ln 20 + C$$

$$\boxed{C = -\frac{1}{2}\ln 20}$$

LET $\lambda = 6$

$$\int_0^{\infty} \frac{\cos 2x}{x}\left(e^{-4x} - e^{-6x}\right)dx = \frac{1}{2}\ln 40 - \frac{1}{2}\ln 20$$

$$= \frac{1}{2}\ln 2$$

AS BEFORE.

Question 9

$$I = \int_0^{\infty} \frac{e^{-x}}{x}\left[1-\cos\left(\tfrac{3}{4}x\right)\right]\,dx \,.$$

By introducing in the integrand a parameter λ and carrying a suitable differentiation under the integral sign show that

$$I = \ln 5 - \ln 4 \,.$$

$\boxed{\text{proof}}$

$$\boxed{\int_0^{\infty} \frac{e^{-x}}{x}\left(1-\cos\tfrac{3}{4}x\right)dx = \ln 5 - \ln 4}$$

- INTRODUCE A PARAMETER λ INSTEAD OF $\frac{3}{4}$

$$I(\lambda) = \int_0^{\infty} \frac{e^{-x}}{x}(1-\cos\lambda x)\,dx$$

$$\frac{\partial I}{\partial \lambda} = \frac{\partial}{\partial \lambda}\int_0^{\infty} \frac{e^{-x}}{x}(1-\cos\lambda x)\,dx = \int_0^{\infty} \frac{e^{-x}}{x}\frac{\partial}{\partial \lambda}(1-\cos\lambda x)\,dx$$

$$\frac{\partial I}{\partial \lambda} = \int_0^{\infty} \frac{e^{-x}}{x}\,x\sin\lambda x\,dx = \int_0^{\infty} e^{-x}\sin\lambda x\,dx$$

$$\frac{\partial I}{\partial \lambda} = \mathrm{Im}\int_0^{\infty} e^{-x}e^{i\lambda x}\,dx = \mathrm{Im}\int_0^{\infty} e^{x(-1+i\lambda)}\,dx.$$

$$\frac{\partial I}{\partial \lambda} = \mathrm{Im}\left[\frac{1}{-1+i\lambda}e^{x(-1+i\lambda)}\right]_0^{\infty}$$

$$\frac{\partial I}{\partial \lambda} = \mathrm{Im}\left[\frac{-1-i\lambda}{1+\lambda^2}e^{-x}(\cos\lambda x + i\sin\lambda x)\right]_0^{\infty}$$

$$\frac{\partial I}{\partial \lambda} = \mathrm{Im}\left[\frac{-1-i\lambda}{1+\lambda^2}(0-1)\right] = \mathrm{Im}\left[\frac{1+i\lambda}{1+\lambda^2}\right]$$

$$\frac{\partial I}{\partial \lambda} = \frac{\lambda}{1+\lambda^2}$$

- INTEGRATE WITH RESPECT TO λ

$$I = \tfrac{1}{2}\ln(\lambda^2+1) + C$$

$$\int_0^{\infty} \frac{e^{-x}}{x}(1-\cos\lambda x)\,dx = \tfrac{1}{2}\ln(\lambda^2+1) + C$$

- PICK A SUITABLE VALUE FOR λ TO EVALUATE C, SAY $\lambda = 0$

$$\int_0^{\infty} \frac{e^{-x}}{x}(1-1)\,dx = \tfrac{1}{2}\ln 1 + C.$$

$$0 = 0 + C$$

$$\boxed{C = 0}$$

- HENCE

$$\int_0^{\infty} \frac{e^{-x}}{x}(1-\cos\lambda x)\,dx = \tfrac{1}{2}\ln(\lambda^2+1)$$

- LET $\lambda = \frac{3}{4}$

$$\int_0^{\infty} \frac{e^{-x}}{x}\left(1-\cos\tfrac{3}{4}x\right)dx = \tfrac{1}{2}\ln\left(\tfrac{9}{16}+1\right)$$

$$= \tfrac{1}{2}\ln\tfrac{25}{16}$$

$$= \ln\sqrt{\tfrac{25}{16}}$$

$$= \ln\tfrac{5}{4}$$

$$= \ln 5 - \ln 4$$

// AS REQUIRED

Question 10

It is given that the following integral converges

$$\int_0^{\infty} \frac{\sin t}{t}\ dt\ .$$

Evaluate the above integral by introducing the term $e^{-\alpha t}$, where α is a positive parameter and carrying out a suitable differentiation under the integral sign.

You may not use contour integration techniques in this question.

$\boxed{\text{V}}$, $\boxed{\frac{\pi}{2}}$

- CONSIDER THE FOLLOWING INTEGRAL

$$\frac{d}{d\alpha}\left[\int_0^{\infty} \frac{e^{-\alpha t}\sin t}{t}\,dt\right] = \int_0^{\infty} \frac{d}{d\alpha}\left[\frac{e^{-\alpha t}\sin t}{t}\right] dt$$

$$= \int_0^{\infty} \frac{-te^{-\alpha t}\sin t}{t}\,dt = \int_0^{\infty} -e^{-\alpha t}\sin t\,dt = \text{Im}\left[\int_0^{\infty} -e^{-\alpha t}e^{it}\,dt\right]$$

$$= \text{Im}\left[\int_0^{\infty} -e^{t(-\alpha+i)}\,dt\right] = \text{Im}\left[\left[\frac{-1}{-\alpha+i}e^{t(-\alpha+i)}\right]_0^{\infty}\right]$$

$$= \text{Im}\left[\left[\frac{1}{\alpha-i}e^{-\alpha t}[\cos t + i\sin t]\right]_0^{\infty}\right]$$

$$= \text{Im}\left[0 - \left[\frac{1}{\alpha-i}\times 1\times 1\right]\right] = -\text{Im}\left[\frac{1}{\alpha-i}\right]$$

$$= -\text{Im}\left[\frac{\alpha+i}{\alpha^2+1}\right] = -\frac{1}{\alpha^2+1}$$

- THUS $\frac{d}{d\alpha}\left[\int_0^{\infty} \frac{e^{-\alpha t}\sin t}{t}\,dt\right] = -\frac{1}{\alpha^2+1}$

$$\int_0^{\infty} \frac{e^{-\alpha t}\sin t}{t}\,dt = -\arctan\alpha + C.$$

- TO EVALUATE THE CONSTANT LET $\alpha \to +\infty$ SO $e^{-\alpha t} \to 0$, $\arctan\alpha \to \frac{\pi}{2}$

$$\therefore\ 0 = -\frac{\pi}{2} + C$$
$$C = \frac{\pi}{2}$$

$$\therefore\ \int_0^{\infty} \frac{e^{-\alpha t}\sin t}{t}\,dt = \frac{\pi}{2} - \arctan\alpha$$

- LET $\alpha = 0$

$$\int_0^{\infty} \frac{\sin t}{t}\,dt = \frac{\pi}{2}$$

Question 11

Show, by carrying out a suitable differentiation under the integral sign, that

$$\int_0^\infty \frac{e^{-ax}\sin bx}{x}\,dx = \arctan\left(\frac{b}{a}\right),$$

where a and b are positive constants.

You may assume

$$\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}.$$

V, proof

$$\boxed{\int_0^\infty \frac{e^{-ax}\sin bx}{x}\,dx = \arctan\frac{b}{a}}$$

TREAT a AS A PARAMETER & b AS A CONSTANT

$\Rightarrow I(a) = \int_0^\infty \frac{e^{-ax}\sin bx}{x}\,dx$

$\Rightarrow \frac{\partial I}{\partial a} = \frac{\partial}{\partial a}\int_0^\infty \frac{e^{-ax}\sin bx}{x}\,dx = \int_0^\infty \frac{\sin bx}{x}\frac{\partial}{\partial a}\left(e^{-ax}\right)dx$

$\Rightarrow \frac{\partial I}{\partial a} = \int_0^\infty \frac{\sin bx}{x}\left(-xe^{-ax}\right)dx = \int_0^\infty -e^{-ax}\sin bx\,dx$

$\Rightarrow \frac{\partial I}{\partial a} = -\text{Im}\int_0^\infty e^{-ax}e^{ibx}\,dx = -\text{Im}\int_0^\infty e^{x(-a+ib)}\,dx$

$\Rightarrow \frac{\partial I}{\partial a} = -\text{Im}\left[\frac{1}{-a+ib}e^{x(-a+ib)}\right]_0^\infty$

$\Rightarrow \frac{\partial I}{\partial a} = -\text{Im}\left[\frac{-a-ib}{a^2+b^2}e^{-ax}(\cos bx + i\sin bx)\right]_0^\infty$

$\Rightarrow \frac{\partial I}{\partial a} = \text{Im}\left[\frac{a+ib}{a^2+b^2}(0-1)\right] = -\frac{b}{a^2+b^2}$

INTEGRATE WITH RESPECT TO a

$\Rightarrow I = -\frac{\cancel{b}}{\cancel{b}}\arctan\frac{a}{b} + C$

$\Rightarrow \int_0^\infty \frac{e^{-ax}\sin bx}{x}\,dx = C - \arctan\frac{a}{b}$

LET $a=0$

$\int_0^\infty \frac{\sin bx}{x}\,dx = C - 0$

$\boxed{\begin{array}{ll} \text{LET } u = bx & \int_0^\infty \frac{\sin u}{\frac{u}{b}}\,\frac{du}{b} = C \\ \frac{du}{dx} = b & \\ dx = \frac{du}{b} & \int_0^\infty \frac{\sin u}{u}\,du = C \\ \text{LIMITS UNCHANGED} & \\ & C = \frac{\pi}{2} \end{array}}$

$\therefore \int_0^\infty \frac{e^{-ax}\sin bx}{x}\,dx = \frac{\pi}{2} - \arctan\frac{a}{b}$

$= \text{arccot}\,\frac{a}{b}$

$\int_0^\infty \frac{e^{-ax}\sin bx}{x}\,dx = \arctan\frac{b}{a}$ //

ALTERNATIVE
TREAT b AS A PARAMETER & a AS A CONSTANT

$\Rightarrow I(b) = \int_0^\infty \frac{e^{-ax}\sin bx}{x}\,dx$

$\Rightarrow \frac{\partial I}{\partial b} = \frac{\partial}{\partial b}\int_0^\infty \frac{e^{-ax}\sin bx}{x}\,dx = \int_0^\infty \frac{e^{-ax}}{x}\frac{\partial}{\partial b}(\sin bx)\,dx$

$\Rightarrow \frac{\partial I}{\partial b} = \int_0^\infty \frac{e^{-ax}}{x}(x\cos bx)\,dx = \int_0^\infty e^{-ax}\cos bx\,dx$

$\Rightarrow \frac{\partial I}{\partial b} = \text{Re}\int_0^\infty e^{-ax}e^{ibx}\,dx = \text{Re}\int_0^\infty e^{x(-a+ib)}\,dx$

$\Rightarrow \frac{\partial I}{\partial b} = \text{Re}\left[\frac{1}{-a+ib}e^{x(-a+ib)}\right]_0^\infty$

$\Rightarrow \frac{\partial I}{\partial b} = \text{Re}\left[\frac{-a-ib}{a^2+b^2}e^{-ax}(\cos bx + i\sin bx)\right]_0^\infty$

$\Rightarrow \frac{\partial I}{\partial b} = \text{Re}\left[\frac{-a-ib}{a^2+b^2}(0-1)\right]$

$\Rightarrow \frac{\partial I}{\partial b} = \text{Re}\left[\frac{a+ib}{a^2+b^2}\right] = \frac{a}{a^2+b^2}$

INTEGRATE WITH RESPECT TO b

$\Rightarrow I = \frac{a}{a}\arctan\left(\frac{b}{a}\right) + C$

$\Rightarrow \int_0^\infty \frac{e^{-ax}\sin bx}{x}\,dx = C + \arctan\left(\frac{b}{a}\right)$

LET $b=0$

$\int_0^\infty 0\,dx = C + \arctan(0)$

$C = 0$

$\Rightarrow \int_0^\infty \frac{e^{-ax}\sin bx}{x} = \arctan\frac{b}{a}$ //

Question 12

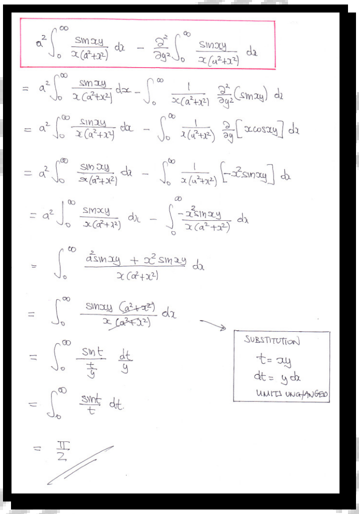
Given that a is a positive constant, find an exact simplified value for

$$a^2 \int_0^{\infty} \frac{\sin xy}{x\left(a^2+x^2\right)}\,dx - \frac{\partial^2}{\partial y^2}\left[\int_0^{\infty} \frac{\sin xy}{x\left(a^2+x^2\right)}\,dx\right].$$

You may assume

$$\int_0^{\infty} \frac{\sin t}{t}\,dt = \frac{\pi}{2}.$$

$$\boxed{\frac{\pi}{2}}$$



$$\boxed{a^2\int_0^{\infty}\frac{\sin xy}{x(a^2+x^2)}\,dx - \frac{\partial^2}{\partial y^2}\int_0^{\infty}\frac{\sin xy}{x(a^2+x^2)}\,dx}$$

$$= a^2\int_0^{\infty}\frac{\sin xy}{x(a^2+x^2)}\,dx - \int_0^{\infty}\frac{1}{x(a^2+x^2)}\frac{\partial^2}{\partial y^2}(\sin xy)\,dx$$

$$= a^2\int_0^{\infty}\frac{\sin xy}{x(a^2+x^2)}\,dx - \int_0^{\infty}\frac{1}{x(a^2+x^2)}\frac{\partial}{\partial y}\left[x\cos xy\right]dx$$

$$= a^2\int_0^{\infty}\frac{\sin xy}{x(a^2+x^2)}\,dx - \int_0^{\infty}\frac{1}{x(a^2+x^2)}\left[-x^2\sin xy\right]dx$$

$$= a^2\int_0^{\infty}\frac{\sin xy}{x(a^2+x^2)}\,dx - \int_0^{\infty}\frac{-x^2\sin xy}{x(a^2+x^2)}\,dx$$

$$= \int_0^{\infty}\frac{a^2\sin xy + x^2\sin xy}{x(a^2+x^2)}\,dx$$

$$= \int_0^{\infty}\frac{\sin xy\,\cancel{(a^2+x^2)}}{x\,\cancel{(a^2+x^2)}}\,dx$$

SUBSTITUTION
$t = xy$
$dt = y\,dx$
LIMITS UNCHANGED

$$= \int_0^{\infty}\frac{\sin t}{\frac{t}{y}}\,\frac{dt}{y}$$

$$= \int_0^{\infty}\frac{\sin t}{t}\,dt$$

$$= \frac{\pi}{2}$$

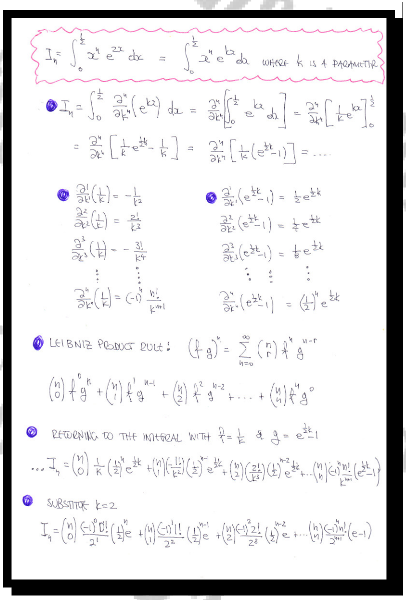
Question 13

$$I_n = \int_0^{\frac{1}{2}} x^n \, e^{2x} \, dx \, , \quad n = 0, 1, 2, 3, \ldots$$

By introducing in the integrand a parameter k and carrying a suitable differentiation under the integral sign show that

$$I_n = \frac{e}{2^{n+1}} \sum_{r=0}^{n} \left[\binom{n}{r} (-1)^n \, r! \right] - \frac{(-1)^n \, n!}{2^{n+1}} \, .$$

$\boxed{\text{proof}}$



$$I_n = \int_0^{\frac{1}{2}} x^n e^{2x}\,dx = \int_0^{\frac{1}{2}} x^n e^{kx}\,dx \quad \text{WHERE } k \text{ IS A PARAMETER}$$

- $I_n = \int_0^{\frac{1}{2}} \frac{\partial^n}{\partial k^n}\left(e^{kx}\right) dx = \frac{\partial^n}{\partial k^n}\left[\int_0^{\frac{1}{2}} e^{kx}\,dx\right] = \frac{\partial^n}{\partial k^n}\left[\frac{1}{k}e^{kx}\right]_0^{\frac{1}{2}}$

$= \frac{\partial^n}{\partial k^n}\left[\frac{1}{k}e^{\frac{1}{2}k} - \frac{1}{k}\right] = \frac{\partial^n}{\partial k^n}\left[\frac{1}{k}\left(e^{\frac{1}{2}k} - 1\right)\right] = \ldots$

- $\frac{\partial^1}{\partial k^1}\left(\frac{1}{k}\right) = -\frac{1}{k^2}$, $\frac{\partial^2}{\partial k^2}\left(\frac{1}{k}\right) = \frac{2!}{k^3}$, $\frac{\partial^3}{\partial k^3}\left(\frac{1}{k}\right) = -\frac{3!}{k^4}$, $\vdots$, $\frac{\partial^n}{\partial k^n}\left(\frac{1}{k}\right) = (-1)^n \frac{n!}{k^{n+1}}$

- $\frac{\partial^1}{\partial k^1}\left(e^{\frac{1}{2}k} - 1\right) = \frac{1}{2}e^{\frac{1}{2}k}$, $\frac{\partial^2}{\partial k^2}\left(e^{\frac{1}{2}k} - 1\right) = \frac{1}{4}e^{\frac{1}{2}k}$, $\frac{\partial^3}{\partial k^3}\left(e^{\frac{1}{2}k} - 1\right) = \frac{1}{8}e^{\frac{1}{2}k}$, $\vdots$, $\frac{\partial^n}{\partial k^n}\left(e^{\frac{1}{2}k} - 1\right) = \left(\frac{1}{2}\right)^n e^{\frac{1}{2}k}$

- LEIBNIZ PRODUCT RULE: $(fg)^n = \sum_{n=0}^{\infty} \binom{n}{r} f^n g^{n-r}$

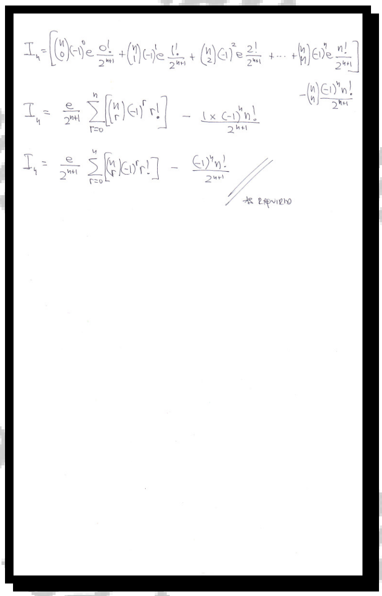
$\binom{n}{0} f^0 g^n + \binom{n}{1} f^1 g^{n-1} + \binom{n}{2} f^2 g^{n-2} + \ldots + \binom{n}{n} f^n g^0$

- RETURNING TO THE INTEGRAL WITH $f = \frac{1}{k}$ & $g = e^{\frac{1}{2}k} - 1$

$\ldots I_n = \binom{n}{0}\frac{1}{k}\left(\frac{1}{2}\right)^n e^{\frac{1}{2}k} + \binom{n}{1}\left(-\frac{1!}{k^2}\right)\left(\frac{1}{2}\right)^{n-1} e^{\frac{1}{2}k} + \binom{n}{2}\left(\frac{2!}{k^3}\right)\left(\frac{1}{2}\right)^{n-2} e^{\frac{1}{2}k} + \ldots \binom{n}{n}(-1)^n \frac{n!}{k^{n+1}}\left(e^{\frac{1}{2}k} - 1\right)$

- SUBSTITUTE $k = 2$

$I_n = \binom{n}{0}\frac{(-1)^0 0!}{2^1}\left(\frac{1}{2}\right)^n e + \binom{n}{1}\frac{(-1)^1 1!}{2^2}\left(\frac{1}{2}\right)^{n-1} e + \binom{n}{2}\frac{(-1)^2 2!}{2^3}\left(\frac{1}{2}\right)^{n-2} e + \ldots \binom{n}{n}\frac{(-1)^n n!}{2^{n+1}}(e-1)$



$I_n = \left[\binom{n}{0}(-1)^0 e\frac{0!}{2^{n+1}} + \binom{n}{1}(-1)^1 e\frac{1!}{2^{n+1}} + \binom{n}{2}(-1)^2 e\frac{2!}{2^{n+1}} + \ldots + \binom{n}{n}(-1)^n e\frac{n!}{2^{n+1}}\right] - \binom{n}{n}\frac{(-1)^n n!}{2^{n+1}}$

$I_n = \frac{e}{2^{n+1}}\sum_{r=0}^{n}\left[\binom{n}{r}(-1)^r r!\right] - \frac{1 \times (-1)^n n!}{2^{n+1}}$

$I_n = \frac{e}{2^{n+1}}\sum_{r=0}^{n}\left[\binom{n}{r}(-1)^r r!\right] - \frac{(-1)^n n!}{2^{n+1}}$ // AS REQUIRED

Question 14

$$I_n = \int_0^1 x^{2n+1} e^{x^2} \, dx \, , \qquad n = 0, 1, 2, 3, \ldots$$

By introducing in the integrand a parameter k and carrying a suitable differentiation under the integral sign show that

$$I_n = \frac{e}{2} \sum_{r=0}^{n} \left[\binom{n}{r} (-1)^n \, r! \right] - \frac{1}{2} (-1)^n \, n! \, .$$

proof

$$\boxed{\int_0^1 x^{2n+1} e^{x^2} \, dx = \int x^{2n} \cdot x e^{kx^2} \, dx \quad \text{WHERE } k=1}$$

NEXT CONSIDER THE DIFFERENTIATION

$$\frac{\partial}{\partial k}\left[x e^{kx^2}\right] = x^3 e^{kx^2}$$

SO CARRYING OUT $\underline{n}$ DIFFERENTIATIONS OVER THE INTEGRAL SIGN

$$\int_0^1 x^{2n} \times x e^{kx^2} \, dx = \int_0^1 \frac{\partial^n}{\partial k^n}\left(x e^{kx^2}\right) dx$$

$$= \frac{\partial^n}{\partial k^n} \int_0^1 x e^{kx^2} \, dx = \frac{\partial^n}{\partial k^n}\left[\frac{1}{2k} e^{kx^2}\right]_0^1 = \frac{\partial^n}{\partial k^n}\left[\frac{1}{2k} e^k - \frac{1}{2k}\right]$$

$$= \frac{\partial^n}{\partial k^n}\left[\frac{1}{2k}(e^k - 1)\right] = \frac{1}{2} \frac{\partial^n}{\partial k^n}\left[\frac{1}{k}(e^k - 1)\right]$$

LET US NOTE

$$\frac{\partial^1}{\partial k^1}\left[\frac{1}{k}\right] = -\frac{1}{k^2} \qquad \frac{\partial^1}{\partial k^1}(e^k - 1) = e^k$$

$$\frac{\partial^2}{\partial k^2}\left[\frac{1}{k}\right] = \frac{2}{k^3} \qquad \frac{\partial^2}{\partial k^2}(e^k - 1) = e^k$$

$$\frac{\partial^3}{\partial k^3}\left[\frac{1}{k}\right] = \frac{-6}{k^4} \qquad \frac{\partial^3}{\partial k^3}(e^k - 1) = e^k$$

$$\frac{\partial^4}{\partial k^4}\left[\frac{1}{k}\right] = \frac{24}{k^5} \qquad \vdots$$

$$\vdots$$

$$\boxed{\frac{\partial^n}{\partial k^n}\left[\frac{1}{k}\right] = \frac{(-1)^n n!}{k^{n+1}}} \qquad \boxed{\frac{\partial^n}{\partial k^n}(e^k - 1) = e^k}$$

BY LEIBNIZ PRODUCT RULE $\boxed{(fg)^n = \sum_{r=0}^{n} \binom{n}{r} f^r g^{n-r}}$

HERE DROPPING THE "PARTIAL" WE NEED TO FIND

$$\frac{1}{2} \frac{d^n}{dk^n}\left[\underbrace{\frac{1}{k}}_{f} \underbrace{(e^k - 1)}_{g}\right]$$

$$= \frac{1}{2}\left[\binom{n}{0} f^0 g^n + \binom{n}{1} f^1 g^{n-1} + \binom{n}{2} f^2 g^{n-2} + \ldots \binom{n}{n} f^n g^0\right]$$

$$= \frac{1}{2}\left[\binom{n}{0}\frac{1}{k} e^k + \binom{n}{1}\frac{-1}{k^2} e^k + \binom{n}{2}\frac{2}{k^3} e^k + \ldots \binom{n}{n-1}\frac{(n-1)!(-1)^{n-1}}{k^n} e^k\right]$$

$$+ \frac{1}{2}\binom{n}{n}\frac{n!(-1)^n}{k^{n+1}}(e^k - 1)$$

$$= \frac{1}{2}\left[\binom{n}{0}\frac{1}{k} e^k + \binom{n}{1}\frac{-1}{k^2} e^k + \binom{n}{2}\frac{2}{k^3} e^k + \ldots + \binom{n}{n}\frac{n!(-1)^n}{k^{n+1}} e^k\right]$$

$$- \frac{1}{2}\binom{n}{n}\frac{n!(-1)^n}{k^{n+1}}$$

SET $k=1$

$$= \frac{1}{2}\left[\binom{n}{0} e + \binom{n}{1}(-1)^1 e + \binom{n}{2} 2(-1)^2 e + \binom{n}{3} 6(-1)^3 e + \ldots \binom{n}{n} n! \, e \, (-1)^n\right]$$

$$- \frac{1}{2} n! (-1)^n$$

$$= \frac{e}{2}\left[\binom{n}{0} + \binom{n}{1}(-1)^1 1! + \binom{n}{2}(-1)^2 2! + \binom{n}{3}(-1)^3 3! + \ldots + \binom{n}{n}(-1)^n n!\right]$$

$$- \frac{1}{2}(-1)^n n!$$

$$= \frac{1}{2} e \sum_{r=0}^{n}\left[\binom{n}{r}(-1)^n r!\right] - \frac{1}{2}(-1)^n n!$$

AS REQUIRED

Question 15

$$I = \int_0^\infty \frac{\arctan 8x - \arctan 2x}{x} \, dx .$$

By introducing a parameter in the integrand and carrying a suitable differentiation under the integral sign show that

$$I = \pi \ln 2 .$$

V, $\boxed{\text{proof}}$

$$\boxed{\int_0^\infty \frac{\arctan 8x - \arctan 2x}{x} = \pi \ln 2}$$

- METHOD A – INTRODUCE A PARAMETER INSTEAD OF 8 IN THE FIRST ARCTAN

$$\Rightarrow I(\lambda) = \int_0^\infty \frac{\arctan \lambda x - \arctan 2x}{x} \, dx$$

$$\Rightarrow \frac{\partial I}{\partial \lambda} = \frac{\partial}{\partial \lambda} \int_0^\infty \frac{\arctan \lambda x - \arctan 2x}{x} \, dx = \int_0^\infty \frac{\partial}{\partial \lambda}\left[\frac{\arctan \lambda x}{x} - \frac{\arctan x}{x}\right] dx$$

$$\Rightarrow \frac{\partial I}{\partial \lambda} = \int_0^\infty x \times \frac{1}{x} \times \frac{1}{1+(\lambda x)^2} \, dx = \int_0^\infty \frac{1}{1+\lambda^2 x^2} \, dx = \frac{1}{\lambda^2}\int_0^\infty \frac{1}{\frac{1}{\lambda^2}+x^2} \, dx$$

$$\Rightarrow \frac{\partial I}{\partial \lambda} = \frac{1}{\lambda^2} \times \frac{1}{\frac{1}{\lambda}}\left[\arctan\left(\frac{x}{\frac{1}{\lambda}}\right)\right]_0^\infty = \frac{1}{\lambda}\left[\arctan \lambda x\right]_0^\infty = \frac{1}{\lambda}\left[\frac{\pi}{2} - 0\right]$$

$$\Rightarrow \frac{\partial I}{\partial \lambda} = \frac{\pi}{2}\frac{1}{\lambda}$$

- INTEGRATE WITH RESPECT TO λ

$$\Rightarrow I = \frac{\pi}{2}\ln \lambda + C$$

$$\Rightarrow \int_0^\infty \frac{\arctan \lambda x - \arctan 2x}{x} \, dx = \frac{\pi}{2}\ln \lambda + C$$

- LET $\lambda = 2 \Rightarrow 0 = \frac{\pi}{2}\ln 2 + C$

$$C = -\frac{\pi}{2}\ln 2$$

$$\Rightarrow \int_0^\infty \frac{\arctan \lambda x - \arctan 8x}{x} \, dx = \frac{\pi}{2}\ln \lambda - \frac{\pi}{2}\ln 2$$

- LET $\lambda = 8$

$$\Rightarrow \int_0^\infty \frac{\arctan 8x - \arctan 2x}{x} \, dx = \frac{\pi}{2}\ln 8 - \frac{\pi}{2}\ln 2 = \frac{\pi}{2}\ln 4 = \pi \ln 2$$

METHOD B

- INTRODUCE THE PARAMETER IN THE SECOND ARCTAN INSTEAD OF 2.

$$I(\lambda) = \int_0^\infty \frac{\arctan 8x - \arctan \lambda x}{x} \, dx$$

$$\Rightarrow \frac{\partial I}{\partial \lambda} = \frac{\partial}{\partial \lambda} \int_0^\infty \frac{\arctan 8x - \arctan \lambda x}{x} \, dx = \int_0^\infty \frac{\partial}{\partial \lambda}\left[\frac{\arctan 8x}{x} - \frac{\arctan \lambda x}{x}\right] dx$$

$$\Rightarrow \frac{\partial I}{\partial \lambda} = \int_0^\infty -\frac{x}{x}\frac{1}{1+(\lambda x)^2} \, dx = \int_0^\infty -\frac{1}{1+\lambda^2 x^2} \, dx = -\frac{1}{\lambda^2}\int_0^\infty \frac{1}{\frac{1}{\lambda^2}+x^2} \, dx$$

$$\Rightarrow \frac{\partial I}{\partial \lambda} = -\frac{1}{\lambda^2} \times \frac{1}{\frac{1}{\lambda}}\left[\arctan \frac{x}{\frac{1}{\lambda}}\right]_0^\infty = -\frac{1}{\lambda}\left[\arctan \lambda x\right]_0^\infty = -\frac{1}{\lambda}\left(\frac{\pi}{2} - 0\right)$$

$$\Rightarrow \frac{\partial I}{\partial \lambda} = -\frac{\pi}{2\lambda}$$

- INTEGRATE WITH RESPECT TO λ

$$\Rightarrow I = -\frac{\pi}{2}\ln \lambda + C$$

$$\Rightarrow \int_0^\infty \frac{\arctan 8x - \arctan \lambda x}{x} \, dx = -\frac{\pi}{2}\ln \lambda + C$$

- TO EVALUATE THE CONSTANT LET $\lambda = 8 \Longrightarrow 0 = -\frac{\pi}{2}\ln 8 + C$

$$C = \frac{\pi}{2}\ln 8$$

$$\Rightarrow \int_0^\infty \frac{\arctan 8x - \arctan \lambda x}{x} \, dx = -\frac{\pi}{2}\ln \lambda + \frac{\pi}{2}\ln 8$$

- LET $\lambda = 2$

$$\Rightarrow \int_0^\infty \frac{\arctan 8x - \arctan 2x}{x} \, dx = -\frac{\pi}{2}\ln 2 + \frac{\pi}{2}\ln 8 = \frac{\pi}{2}\ln 4$$

$$= \frac{\pi}{2}\ln 2^2 = \pi \ln 2$$

AS BEFORE.

Question 16

It is given that the following integral converges

$$\int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} \, dx \, ,$$

where a and b are positive constants.

By carrying out a suitable differentiation under the integral sign, show that

$$\int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} \, dx = \ln\left[\frac{b}{a}\right] .$$

V, proof

$$\boxed{\int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} \, dx = \ln \frac{b}{a} \qquad a, b > 0}$$

- LET $I(a) = \int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} \, dx$ TREAT a AS A PARAMETER, b AS A CONSTANT

$$\Rightarrow \frac{\partial I}{\partial a} = \frac{\partial}{\partial a} \int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} \, dx = \int_0^{\infty} \frac{\partial}{\partial a}\left[\frac{e^{-ax} - e^{-bx}}{x}\right] dx$$

$$\Rightarrow \frac{\partial I}{\partial a} = \int_0^{\infty} \frac{\partial}{\partial a}\left[\frac{e^{-ax}}{x} - \frac{e^{-bx}}{x}\right] dx = \int_0^{\infty} -e^{-ax} \, dx$$

$$\Rightarrow \frac{\partial I}{\partial a} = \left[\frac{1}{a} e^{-ax}\right]_0^{\infty} = \frac{1}{a}\left[0 - 1\right] = -\frac{1}{a}$$

$$\boxed{I = -\ln a + C}$$

- APPLY CONDITION TO FIND C

$$\Rightarrow \int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} \, dx = -\ln a + C$$

- LET $a = b$

$$\Rightarrow \int_0^{\infty} \frac{e^{-bx} - e^{-bx}}{x} \, dx = -\ln b + C$$

$$0 = -\ln b + C$$

$$\boxed{C = \ln b}$$

$$\Rightarrow \int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} \, dx = -\ln a + \ln b$$

$$\Rightarrow \int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} \, dx = \ln \frac{b}{a}$$

AS REQUIRED

ALTERNATIVE – TREAT b AS A PARAMETER & a AS A CONSTANT

- LET $I(b) = \int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} \, dx$

$$\Rightarrow \frac{\partial I}{\partial b} = \frac{\partial}{\partial b} \int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} \, dx = \int_0^{\infty} \frac{\partial}{\partial b}\left[\frac{e^{-ax}}{x} - \frac{e^{-bx}}{x}\right] dx$$

$$\Rightarrow \frac{\partial I}{\partial b} = \int_0^{\infty} e^{-bx} \, dx$$

$$\Rightarrow \frac{\partial I}{\partial b} = -\frac{1}{b}\left[e^{-bx}\right]_0^{\infty} = -\frac{1}{b}\left[0 - 1\right] = \frac{1}{b}$$

- INTEGRATING

$$\Rightarrow I = \ln b + C$$

$$\Rightarrow \int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} \, dx = \ln b + C$$

- TO FIND THE CONSTANT LET $b = a$

$$\int_0^{\infty} \frac{e^{-ax} - e^{-ax}}{x} \, dx = \ln a + C$$

$$0 = \ln a + C$$

$$\boxed{C = -\ln a}$$

$$\therefore \int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} \, dx = \ln b - \ln a$$

$$\int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} \, dx = \ln \frac{b}{a}$$

AS BEFORE

Question 17

It is given that the following integral converges

$$\int_0^\infty \frac{\cos kx}{x}\left[e^{-ax}-e^{-bx}\right]\,dx\,,$$

where k, a and b are constants with $a>0$ and $b>0$.

By carrying out a suitable differentiation under the integral sign, show that

$$\int_0^\infty \frac{\cos kx}{x}\left[e^{-ax}-e^{-bx}\right]\,dx = \frac{1}{2}\ln\left[\frac{b^2+k^2}{a^2+k^2}\right].$$

proof

$$\boxed{\int_0^\infty \frac{\cos kx}{x}\left(e^{-ax}-e^{-bx}\right)dx = \frac{1}{2}\ln\left(\frac{b^2+k^2}{a^2+k^2}\right) \quad a,b>0}$$

METHOD A — TREAT a AS A PARAMETER, b & k ARE CONSTANTS

$$\Rightarrow I(a) = \int_0^\infty \frac{\cos kx}{x}\left(e^{-ax}-e^{-bx}\right)dx$$

$$\Rightarrow \frac{\partial I}{\partial a} = \frac{\partial}{\partial a}\int_0^\infty \frac{\cos kx}{x}\left(e^{-ax}-e^{-bx}\right)dx = \int_0^\infty \frac{\cos kx}{x}\frac{\partial}{\partial a}\left[e^{-ax}-e^{-bx}\right]dx$$

$$\Rightarrow \frac{\partial I}{\partial a} = \int_0^\infty \frac{\cos kx}{x}\left(-xe^{-ax}\right)dx = -\int_0^\infty e^{-ax}\cos kx\,dx$$

$$\Rightarrow \frac{\partial I}{\partial a} = -\text{Re}\int_0^\infty e^{-ax}e^{ikx}\,dx = -\text{Re}\int_0^\infty e^{x(a+ik)}\,dx$$

$$\Rightarrow \frac{\partial I}{\partial a} = -\text{Re}\left[\frac{1}{-a+ik}e^{x(-a+ik)}\right]_0^\infty = -\text{Re}\left[\frac{-a-ik}{a^2+k^2}e^{-ax}(\cos kx+i\sin kx)\right]_0^\infty$$

$$\Rightarrow \frac{\partial I}{\partial a} = \text{Re}\left[\frac{a+ik}{a^2+k^2}[0-1]\right] = -\frac{a}{a^2+k^2}$$

INTEGRATING W.R.T a

$$\Rightarrow I = -\frac{1}{2}\ln(a^2+k^2)+C$$

$$\Rightarrow \int_0^\infty \frac{\cos kx}{x}\left(e^{-ax}-e^{-bx}\right)dx = C-\frac{1}{2}\ln(a^2+k^2)$$

LET $a=b$

$$\int \frac{\cos kx}{x}\left(e^{-bx}-e^{-bx}\right)dx = C-\frac{1}{2}\ln(b^2+k^2)$$

$$0 = C-\frac{1}{2}\ln(b^2+k^2)$$

$$C = \frac{1}{2}\ln(b^2+k^2)$$

$$\Rightarrow \int_0^\infty \frac{\cos kx}{x}\left(e^{-ax}-e^{-bx}\right)dx = \frac{1}{2}\ln(b^2+k^2)-\frac{1}{2}\ln(a^2+k^2)$$

$$= \frac{1}{2}\ln\left(\frac{b^2+k^2}{a^2+k^2}\right)$$

AS REQUIRED

METHOD B — TREAT b AS A PARAMETER & a & k AS CONSTANTS

$$\Rightarrow I(b) = \int_0^\infty \frac{\cos kx}{x}\left(e^{-ax}-e^{-bx}\right)dx$$

$$\Rightarrow \frac{\partial I}{\partial b} = \frac{\partial}{\partial b}\int_0^\infty \frac{\cos kx}{x}\left(e^{-ax}-e^{-bx}\right)dx = \int_0^\infty \frac{\cos kx}{x}\frac{\partial}{\partial b}\left[e^{-ax}-e^{-bx}\right]dx$$

$$\Rightarrow \frac{\partial I}{\partial b} = \int_0^\infty \frac{\cos kx}{x}\left(xe^{-bx}\right)dx = \int_0^\infty e^{-bx}\cos kx\,dx$$

$$\Rightarrow \frac{\partial I}{\partial b} = \text{Re}\int_0^\infty e^{-bx}e^{ikx}\,dx = \text{Re}\int_0^\infty e^{x(-b+ik)}\,dx$$

$$\Rightarrow \frac{\partial I}{\partial b} = \text{Re}\left[\frac{1}{-b+ik}e^{x(-b+ik)}\right]_0^\infty = \text{Re}\left[\frac{-b-ik}{b^2+k^2}e^{-bx}[\cos kx+i\sin kx]\right]_0^\infty$$

$$\Rightarrow \frac{\partial I}{\partial b} = \text{Re}\left[\frac{-b-ik}{b^2+k^2}(0-1)\right] = \text{Re}\left[\frac{b+ik}{b^2+k^2}\right] = \frac{b}{b^2+k^2}$$

INTEGRATING W.R.T b

$$\Rightarrow I = \int \frac{b}{b^2+k^2}\,db$$

$$\Rightarrow I = \frac{1}{2}\ln(b^2+k^2)+C$$

$$\Rightarrow \int_0^\infty \frac{\cos kx}{x}\left(e^{-ax}-e^{-bx}\right)dx = \frac{1}{2}\ln(b^2+k^2)+C$$

LET $b=a$

$$\int_0^\infty \frac{\cos kx}{x}\left(e^{-ax}-e^{-ax}\right)dx = \frac{1}{2}\ln(a^2+k^2)+C$$

$$0 = \frac{1}{2}\ln(a^2+k^2)+C$$

$$C = -\frac{1}{2}\ln(a^2+k^2)$$

$$\Rightarrow \int_0^\infty \frac{\cos kx}{x}\left(e^{-ax}-e^{-bx}\right)dx = \frac{1}{2}\ln(b^2+k^2)-\frac{1}{2}\ln(a^2+k^2)$$

$$= \frac{1}{2}\ln\left(\frac{b^2+k^2}{a^2+k^2}\right)$$

AS BEFORE

Question 18

It is given that the following integral converges

$$\int_0^1 \frac{x^a - x^b}{\ln x}\,dx\,,$$

where a and b are constants greater than -1.

By carrying out a suitable differentiation under the integral sign, show that

$$\int_0^1 \frac{x^a - x^b}{\ln x}\,dx = \ln\left[\frac{a+1}{b+1}\right].$$

proof

$$\boxed{\int_0^1 \frac{x^a - x^b}{\ln x}\,dx = \log\left(\frac{a+1}{b+1}\right) \qquad \begin{matrix} a>-1 \\ b>-1 \end{matrix}}$$

- TREAT a AS A PARAMETER & b AS A CONSTANT.

$$\Rightarrow I(a) = \int_0^1 \frac{x^a - x^b}{\ln x}\,dx$$

$$\Rightarrow \frac{\partial I}{\partial a} = \frac{\partial}{\partial a}\int_0^1 \frac{x^a - x^b}{\ln x}\,dx = \int_0^1 \frac{\partial}{\partial a}\left[\frac{x^a}{\ln x} - \frac{x^b}{\ln x}\right]dx$$

$$\Rightarrow \frac{\partial I}{\partial a} = \int_0^1 \frac{x^a \ln x}{\ln x}\,dx = \int_0^1 x^a\,dx = \left[\frac{1}{a+1}x^{a+1}\right]_0^1$$

$$\Rightarrow \frac{\partial I}{\partial a} = \frac{1}{a+1}$$

NOTE $\frac{d}{dx}(a^x) = a^x \ln a$

- INTEGRATE WITH RESPECT TO a

$$\Rightarrow I = \ln(a+1) + C$$

$$\Rightarrow \int_0^1 \frac{x^a - x^b}{\ln x}\,dx = \ln(a+1) + C$$

- USE A SUITABLE VALUE FOR a, TO EVALUATE C, SAY $a=b$

$$\int_0^1 \frac{x^b - x^b}{\ln x}\,dx = \ln(b+1) + C$$

$$0 = \ln(b+1) + C$$

$$C = -\ln(b+1)$$

$$\Rightarrow \int_0^1 \frac{x^a - x^b}{\ln x}\,dx = \ln(a+1) - \ln(b+1)$$

$$\Rightarrow \int_0^1 \frac{x^a - x^b}{\ln x}\,dx = \ln\left(\frac{a+1}{b+1}\right)$$

AS REQUIRED

- ALTERNATIVE (ALMOST IDENTICAL), TREATING b AS A PARAMETER AND a AS A KNOWN CONSTANT.

$$\Rightarrow I(b) = \int_0^1 \frac{x^a - x^b}{\ln x}\,dx$$

$$\Rightarrow \frac{\partial I}{\partial b} = \frac{\partial}{\partial b}\int_0^1 \frac{x^a - x^b}{\ln x}\,dx = \int_0^1 \frac{\partial}{\partial b}\left[\frac{x^a}{\ln x} - \frac{x^b}{\ln x}\right]dx$$

$$\Rightarrow \frac{\partial I}{\partial b} = \int_0^1 \frac{-x^b \ln x}{\ln x}\,dx = -\int_0^1 x^b\,dx = -\left[\frac{1}{b+1}x^{b+1}\right]_0^1$$

$$\Rightarrow \frac{\partial I}{\partial b} = -\frac{1}{b+1}$$

- INTEGRATE WITH RESPECT TO b

$$\Rightarrow I = -\ln(b+1) + C$$

$$\Rightarrow \int_0^1 \frac{x^a - x^b}{\ln x}\,dx = C - \ln(b+1)$$

- TO EVALUATE THE CONSTANT C, PICK A SUITABLE VALUE FOR THE PARAMETER SAY $b=a$

$$\Rightarrow \int_0^1 \frac{x^a - x^a}{\ln x}\,dx = C - \ln(a+1)$$

$$0 = C - \ln(a+1)$$

$$C = \ln(a+1)$$

- HENCE $\int_0^1 \frac{x^a - x^b}{\ln x}\,dx = \ln(a+1) - \ln(b+1)$

$$\int_0^1 \frac{x^a - x^b}{\ln x}\,dx = \ln\left(\frac{a+1}{b+1}\right)$$

Question 19

It is given that the following integral converges

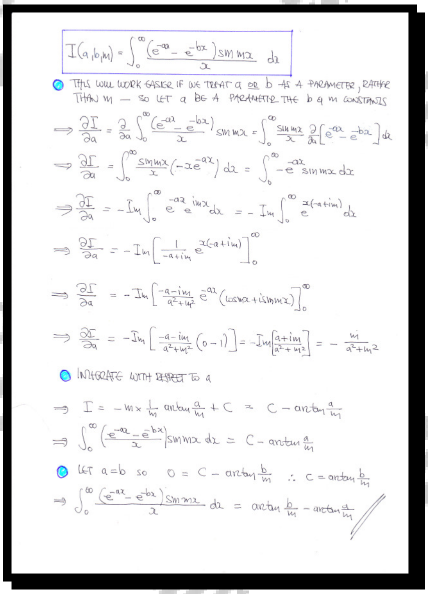
$$\int_0^{\infty} \frac{\sin mx}{x}\left[\mathrm{e}^{-ax}-\mathrm{e}^{-bx}\right]\, dx\ ,$$

where a, b and m are constants, with $m \neq 0$, $a > 0$, $b > 0$.

By carrying out a suitable differentiation under the integral sign, show that

$$\int_0^{\infty} \frac{\sin mx}{x}\left[\mathrm{e}^{-ax}-\mathrm{e}^{-bx}\right]\, dx = \arctan\left(\frac{b}{m}\right) - \arctan\left(\frac{a}{m}\right).$$

$\boxed{\text{proof}}$



$$\boxed{I(a,b,m) = \int_0^{\infty} \frac{\left(e^{-ax}-e^{-bx}\right)\sin mx}{x}\, dx}$$

THIS WILL WORK EASIER IF WE TREAT a OR b AS A PARAMETER, RATHER THAN m — SO LET a BE A PARAMETER THE b & m CONSTANTS

$$\Rightarrow \frac{\partial I}{\partial a} = \frac{\partial}{\partial a}\int_0^{\infty} \frac{\left(e^{-ax}-e^{-bx}\right)}{x}\sin mx = \int_0^{\infty} \frac{\sin mx}{x}\frac{\partial}{\partial a}\left[e^{-ax}-e^{-bx}\right] dx$$

$$\Rightarrow \frac{\partial I}{\partial a} = \int_0^{\infty} \frac{\sin mx}{x}\left(-xe^{-ax}\right) dx = \int_0^{\infty} -e^{-ax}\sin mx\, dx$$

$$\Rightarrow \frac{\partial I}{\partial a} = -\mathrm{Im}\int_0^{\infty} e^{-ax}e^{imx}\, dx = -\mathrm{Im}\int_0^{\infty} e^{x(-a+im)}\, dx$$

$$\Rightarrow \frac{\partial I}{\partial a} = -\mathrm{Im}\left[\frac{1}{-a+im}e^{x(-a+im)}\right]_0^{\infty}$$

$$\Rightarrow \frac{\partial I}{\partial a} = -\mathrm{Im}\left[\frac{-a-im}{a^2+m^2}e^{-ax}\left(\cos mx + i\sin mx\right)\right]_0^{\infty}$$

$$\Rightarrow \frac{\partial I}{\partial a} = -\mathrm{Im}\left[\frac{-a-im}{a^2+m^2}(0-1)\right] = -\mathrm{Im}\left[\frac{a+im}{a^2+m^2}\right] = -\frac{m}{a^2+m^2}$$

INTEGRATE WITH RESPECT TO a

$$\Rightarrow I = -m \times \frac{1}{m}\arctan\frac{a}{m} + C = C - \arctan\frac{a}{m}$$

$$\Rightarrow \int_0^{\infty}\left(\frac{e^{-ax}-e^{-bx}}{x}\right)\sin mx\, dx = C - \arctan\frac{a}{m}$$

LET $a=b$ SO $0 = C - \arctan\frac{b}{m}$ $\therefore C = \arctan\frac{b}{m}$

$$\Rightarrow \int_0^{\infty} \frac{\left(e^{-ax}-e^{-bx}\right)\sin mx}{x}\, dx = \arctan\frac{b}{m} - \arctan\frac{a}{m}$$

Question 20

It is given that the following integral converges

$$\int_0^{\infty} \frac{\arctan ax}{x\left(1+x^2\right)}\,dx\,,\ a>-1\,.$$

By carrying out a suitable differentiation under the integral sign, show that

$$\int_0^{\infty} \frac{\arctan ax}{x\left(1+x^2\right)}\,dx = \frac{\pi}{2}\ln\left(a+1\right)\,.$$

proof

$$\boxed{\int_0^{\infty} \frac{\arctan ax}{x(1+x^2)}\,dx = \frac{\pi}{2}\ln(a+1)}$$

- LET $I(a) = \int_0^{\infty} \frac{\arctan ax}{x(1+x^2)}\,dx$

$$\Rightarrow \frac{\partial I}{\partial a} = \frac{\partial}{\partial a}\int_0^{\infty} \frac{\arctan ax}{x(1+x^2)}\,dx = \int_0^{\infty} \frac{\partial}{\partial a}\left[\frac{\arctan ax}{x(1+x^2)}\right]dx$$

$$\Rightarrow \frac{\partial I}{\partial a} = \int_0^{\infty} \frac{1}{x(1+x^2)}\ \frac{x}{1+a^2x^2}\,dx = \int_0^{\infty} \frac{1}{(1+x^2)(1+a^2x^2)}\,dx$$

$$\Rightarrow \frac{\partial I}{\partial a} = \frac{1}{a^2}\int_0^{\infty} \frac{1}{(1+x^2)\left(x^2+\frac{1}{a^2}\right)}\,dx$$

- BY PARTIAL FRACTIONS

$$\frac{1}{(1+x^2)\left(x^2+\frac{1}{a^2}\right)} \equiv \frac{Ax+B}{1+x^2} + \frac{Cx+D}{x^2+\frac{1}{a^2}}$$

$$1 \equiv (Ax+B)\left(x^2+\tfrac{1}{a^2}\right) + (Cx+D)(1+x^2)$$

$$1 \equiv Ax^3 + Bx^2 + \frac{A}{a^2}x + \frac{B}{a^2}$$
$$Cx^3 + Dx^2 + Cx + D$$

$A+C=0 \qquad \frac{A}{a^2}+C=0 \qquad B+D=0 \qquad \frac{B}{a^2}+D=1$

$A=C=0$

$\frac{B}{a^2} - B = 1$

$B - Ba^2 = a^2$

$B(1-a^2) = a^2$

$B = \frac{a^2}{1-a^2} \qquad D = -\frac{a^2}{1-a^2}$

- HENCE

$$\Rightarrow \frac{\partial I}{\partial a} = \frac{1}{a^2}\times\frac{a^2}{1-a^2}\int_0^{\infty} \frac{1}{1+x^2} - \frac{1}{x^2+\frac{1}{a^2}}\,dx$$

$$\Rightarrow \frac{\partial I}{\partial a} = \frac{1}{1-a^2}\Big[\arctan x - a\arctan ax\Big]_0^{\infty}$$

$$\Rightarrow \frac{\partial I}{\partial a} = \frac{1}{1-a^2}\left[\left(\frac{\pi}{2} - a\frac{\pi}{2}\right) - (0)\right]$$

$$\Rightarrow \frac{\partial I}{\partial a} = \frac{1}{1-a^2}\left[\frac{\pi}{2}(1-a)\right]$$

$$\Rightarrow \frac{\partial I}{\partial a} = \frac{1}{\cancel{(1-a)}(1+a)}\ \frac{\pi}{2}\cancel{(1-a)}$$

$$\Rightarrow \frac{\partial I}{\partial a} = \frac{\pi}{2}\left(\frac{1}{a+1}\right)$$

$$\Rightarrow I = \frac{\pi}{2}\ln(a+1) + C$$

$$\Rightarrow \int_0^{\infty} \frac{\arctan(ax)}{x(1+x^2)}\,dx = \frac{\pi}{2}\ln(a+1) + C$$

LET $a=0$

$0 = \cancel{\frac{\pi}{2}\ln 1} + C$

$C = 0$

$$\Rightarrow \int_0^{\infty} \frac{\arctan(ax)}{x(1+x^2)}\,dx = \frac{\pi}{2}\ln(a+1)$$

Question 21

It is given that the following integral converges

$$\int_0^{\infty} \frac{\ln\left(1+a^2x^2\right)}{1+b^2x^2}\,dx\,,$$

where a and b are constants.

By carrying out a suitable differentiation under the integral sign, show that the exact value of the above integral is

$$\frac{\pi}{b}\ln\left|\frac{a+b}{b}\right|\,.$$

proof

LET $I(a) = \int_0^{\infty} \frac{\ln(1+a^2x^2)}{1+b^2x^2}\,dx$ & DIFFERENTIATE W.R.T a

$$\frac{\partial I(a)}{\partial a} = \frac{\partial}{\partial a}\left[\int_0^{\infty} \frac{\ln(1+a^2x^2)}{1+b^2x^2}\,dx\right] = \int_0^{\infty} \frac{\partial}{\partial a}\left[\frac{\ln(1+a^2x^2)}{1+b^2x^2}\right]dx$$

$$= \int_0^{\infty} \frac{1}{1+b^2x^2} \times \frac{1}{1+a^2x^2} \times 2ax^2\,dx$$

$$\frac{\partial I}{\partial a} = \int_0^{\infty} \frac{2ax^2}{(1+b^2x^2)(1+a^2x^2)}\,dx$$

PROCEED BY PARTIAL FRACTIONS THAT ARE "LINEAR IN x^2"

$$\frac{2ax^2}{(1+b^2x^2)(1+a^2x^2)} \equiv \frac{A}{1+b^2x^2} + \frac{B}{1+a^2x^2}$$

$$2ax^2 \equiv A(1+a^2x^2) + B(1+b^2x^2)$$

$$2ax^2 \equiv (Aa^2+Bb^2)x^2 + (A+B)$$

$A+B=0$

$A=-B$

&

$Aa^2+Bb^2 = 2a$

$-Ba^2+Bb^2 = 2a$

$B(b^2-a^2) = 2a$

$B = \frac{2a}{b^2-a^2}$

& $A = \frac{-2a}{b^2-a^2}$

RETURNING TO THE "MAIN LINE"

$$\frac{\partial I}{\partial a} = \int_0^{\infty} \frac{2a}{b^2-a^2}\left[\frac{1}{1+a^2x^2} - \frac{1}{1+b^2x^2}\right]dx$$

$$\frac{\partial I}{\partial a} = \frac{2a}{b^2-a^2}\int_0^{\infty} \frac{1}{1+a^2x^2} - \frac{1}{1+b^2x^2}\,dx$$

INTEGRATING TO ARCTANS

$$\frac{\partial I}{\partial a} = \frac{2a}{b^2-a^2}\left[\frac{1}{a}\arctan(ax) - \frac{1}{b}\arctan bx\right]_0^{\infty}$$

$$\frac{\partial I}{\partial a} = \frac{2a}{b^2-a^2}\left[\left(\frac{1}{a}\times\frac{\pi}{2} - \frac{1}{b}\times\frac{\pi}{2}\right) - 0\right]$$

$$\frac{\partial I}{\partial a} = \frac{\pi a}{b^2-a^2}\left[\frac{1}{a}-\frac{1}{b}\right] = \frac{\pi a}{b^2-a^2}\left(\frac{b-a}{ab}\right)$$

$$\frac{\partial I}{\partial a} = \frac{\pi a}{(b-a)(b+a)} \times \frac{b-a}{ab}$$

$$\frac{\partial I}{\partial a} = \frac{\pi}{b(b+a)}$$

SOLVE THE O.D.E BY INTEGRATING W.R.T a

$$I(a) = \frac{\pi}{b}\ln|b+a| + C$$

$$\int_0^{\infty} \frac{\ln(1+a^2x^2)}{1+b^2x^2}\,dx = \frac{\pi}{b}\ln|b+a| + C$$

If $a=0$

$$I(0) = 0 = \frac{\pi}{b}\ln b + C$$

$$\therefore\ C = -\frac{\pi}{b}\ln b$$

$$\therefore\ I(a) = \frac{\pi}{b}\ln|b+a| - \frac{\pi}{b}\ln b = \frac{\pi}{b}\ln\left|\frac{b+a}{b}\right|$$

$$\therefore\ \int_0^{\infty} \frac{\ln(1+a^2x^2)}{1+b^2x^2}\,dx = \frac{\pi}{b}\ln\left|\frac{b+a}{b}\right|$$

Question 22

It is given that the following integral converges

$$\int_0^\infty \frac{e^{-x}-e^{-2x}}{x}\,dx\,.$$

a) By introducing a parameter k and carrying out a suitable differentiation under the integral sign, show that

$$\int_0^\infty \frac{e^{-x}-e^{-2x}}{x}\,dx = \ln 2\,.$$

b) Use the result of part **(a)** and differentiation under the integral sign to show further that

$$\int_0^\infty \frac{e^{-x}}{x}\left[2-\frac{1}{x}+\frac{1}{x}e^{-2x}\right]dx = -2+\ln 27\,.$$

proof

a) $\boxed{\int_0^\infty \frac{e^{-x}-e^{-2x}}{x}\,dx = \ln 2}$

INTRODUCE A PARAMETER k IN THE INTEGRAL

$$\Rightarrow I(k) = \int_0^\infty \frac{e^{-x}-e^{-kx}}{x}\,dx = \int_0^\infty \frac{e^{-x}}{x}-\frac{e^{-kx}}{x}\,dx$$

$$\Rightarrow \frac{\partial I}{\partial k} = \frac{\partial}{\partial k}\int_0^\infty \frac{e^{-x}-e^{-kx}}{x}\,dx = \int_0^\infty \frac{\partial}{\partial k}\left[\frac{e^{-x}}{x}-\frac{e^{-kx}}{x}\right]dx$$

$$\Rightarrow \frac{\partial I}{\partial k} = \int_0^\infty e^{-kx}\,dx$$

$$\Rightarrow \frac{\partial I}{\partial k} = \left[-\frac{1}{k}e^{-kx}\,dx\right]_0^\infty = \frac{1}{k}\left[1-0\right] = \frac{1}{k}$$

$$I = \ln k + C$$

$$\boxed{\int_0^\infty \frac{e^{-x}-e^{-kx}}{x}\,dx = \ln k + C}$$

USE A SUITABLE VALUE FOR k, SAY $k=1$

$$\int_0^\infty 0\,dx = 0 + C \quad \therefore \boxed{C=0}$$

$$\Rightarrow \int_0^\infty \frac{e^{-x}-e^{-kx}}{x}\,dx = \ln k$$

$$\Rightarrow \int_0^\infty \frac{e^{-x}-e^{-2x}}{x}\,dx = \ln 2$$

b) NOW LET $\boxed{J = \int_0^\infty \frac{e^{-x}}{x}\left[\lambda-\frac{1}{x}+\frac{1}{x}e^{-\lambda x}\right]dx \quad (\lambda=2)}$

$$\Rightarrow \frac{\partial J}{\partial \lambda} = \frac{\partial}{\partial \lambda}\int_0^\infty \frac{e^{-x}}{x}\left[\lambda-\frac{1}{x}+\frac{1}{x}e^{-\lambda x}\right]dx$$

$$\Rightarrow \frac{\partial J}{\partial \lambda} = \int_0^\infty \frac{e^{-x}}{x}\frac{\partial}{\partial \lambda}\left[\lambda-\frac{1}{x}+\frac{1}{x}e^{-\lambda x}\right]dx$$

$$\Rightarrow \frac{\partial J}{\partial \lambda} = \int_0^\infty \frac{e^{-x}}{x}\left[1-e^{-\lambda x}\right]dx$$

$$\Rightarrow \frac{\partial J}{\partial \lambda} = \int_0^\infty \frac{e^{-x}-e^{-x(\lambda+1)}}{x}\,dx$$

FROM PART (a) $\boxed{\int_0^\infty \frac{e^{-x}-e^{-kx}}{x}\,dx = \ln k}$

$$\Rightarrow \frac{\partial J}{\partial \lambda} = \ln(\lambda+1)$$

$$\Rightarrow J = (\lambda+1)\ln(\lambda+1)-\lambda+B$$

$$\Rightarrow \int_0^\infty \frac{e^{-x}}{x}\left[\lambda-\frac{1}{x}+\frac{1}{x}e^{-\lambda x}\right]dx = (\lambda+1)\ln(\lambda+1)-\lambda+B$$

BY PARTS

$\ln(\lambda+1)$	$\frac{1}{\lambda+1}$
λ	1

$$\ldots = \lambda\ln(\lambda+1)-\int\frac{\lambda}{\lambda+1}\,d\lambda$$

$$= \lambda\ln(\lambda+1)-\int\frac{\lambda+1-1}{\lambda+1}\,d\lambda$$

$$= \lambda\ln(\lambda+1)-\int 1-\frac{1}{\lambda+1}\,d\lambda$$

$$= \lambda\ln(\lambda+1)-\lambda+\ln(\lambda+1)+B$$

$$= (\lambda+1)\ln(\lambda+1)-\lambda+B$$

LET $\lambda=0$ TO EVALUATE THE CONSTANT

$$\int_0^\infty \frac{e^{-x}}{x}\left(0-\frac{1}{x}+\frac{1}{x}\right)dx = \ln 1 - 0 + B$$

$$0 = 0 + B$$

$$\boxed{B=0}$$

$$\Rightarrow \int_0^\infty \frac{e^{-x}}{x}\left[\lambda-\frac{1}{x}+\frac{1}{x}e^{-\lambda x}\right]dx = (\lambda+1)\ln(\lambda+1)-\lambda$$

SO IF $\lambda=2$

$$\Rightarrow \int_0^\infty \frac{e^{-x}}{x}\left[2-\frac{1}{x}+\frac{1}{x}e^{-2x}\right]dx = 3\ln 3 - 2 = -2+\ln 27$$

Question 23

The integral function $y = y(x)$ is defined as

$$y(x) = \int_{\frac{1}{16}\pi^2}^{x^2} \frac{\cos x \cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} \, d\theta .$$

Evaluate $y'(\pi)$.

$\boxed{2\pi}$

$$\boxed{y(x) = \int_{\frac{\pi^2}{16}}^{x^2} \frac{\cos x \cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} \, d\theta}$$

$$y(x) = \cos x \int_{\frac{\pi^2}{16}}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} \, d\theta$$

BY THE PRODUCT RULE

$$\frac{dy}{dx} = -\sin x \int_{\frac{\pi^2}{16}}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} \, d\theta + \cos x \times \frac{d}{dx}\left[\int_{\frac{\pi^2}{16}}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} \, d\theta\right]$$

$$\frac{dy}{dx} = -\sin x \int_{\frac{\pi^2}{16}}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} \, d\theta + \cos x \left[\frac{2x\cos\sqrt{x^2}}{1+\sin\sqrt{x^2}}\right]$$

$$\frac{dy}{dx} = -\sin x \int_{\frac{\pi^2}{16}}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} \, d\theta + \frac{2x\cos^2 x}{1+\sin x}$$

$$\left.\frac{dy}{dx}\right|_{x=\pi} = 0 + \frac{2\pi(-1)^2}{1} = 2\pi$$

Question 24

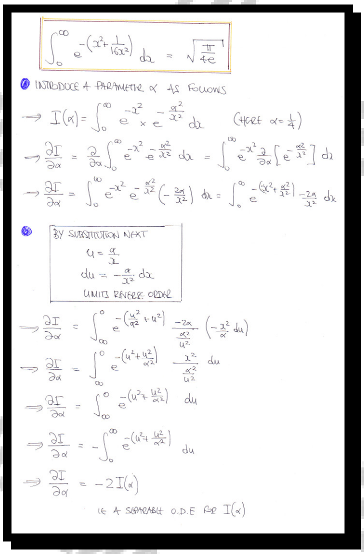
An integral I is defined in terms of a parameter α as

$$I(\alpha) = \int_0^{\infty} \exp\left[-x^2 - \frac{\alpha^2}{x^2}\right] dx .$$

By carrying out a suitable differentiation on I under the integral sign, show that

$$\int_0^{\infty} \exp\left[-x^2 - \frac{1}{16x^2}\right] dx = \sqrt{\frac{\pi}{4\mathrm{e}}} .$$

proof



$$\boxed{\int_0^{\infty} e^{-\left(x^2+\frac{1}{16x^2}\right)}\,dx = \sqrt{\frac{\pi}{4e}}}$$

INTRODUCE A PARAMETER α AS FOLLOWS

$$\Rightarrow I(\alpha) = \int_0^{\infty} e^{-x^2} \times e^{-\frac{\alpha^2}{x^2}}\,dx \qquad \left(\text{HERE } \alpha = \tfrac{1}{4}\right)$$

$$\Rightarrow \frac{\partial I}{\partial \alpha} = \frac{\partial}{\partial \alpha}\int_0^{\infty} e^{-x^2} e^{-\frac{\alpha^2}{x^2}}\,dx = \int_0^{\infty} e^{-x^2}\frac{\partial}{\partial \alpha}\left[e^{-\frac{\alpha^2}{x^2}}\right] dx$$

$$\Rightarrow \frac{\partial I}{\partial \alpha} = \int_0^{\infty} e^{-x^2} e^{-\frac{\alpha^2}{x^2}}\left(-\frac{2\alpha}{x^2}\right) dx = \int_0^{\infty} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)}\,\frac{-2\alpha}{x^2}\,dx$$

$$\boxed{\begin{array}{c}\text{BY SUBSTITUTION NEXT} \\ u = \frac{\alpha}{x} \\ du = -\frac{\alpha}{x^2}\,dx \\ \text{LIMITS REVERSE ORDER}\end{array}}$$

$$\Rightarrow \frac{\partial I}{\partial \alpha} = \int_{\infty}^{0} e^{-\left(\frac{u^2}{\alpha^2}+u^2\right)}\,\frac{-2\alpha}{\frac{\alpha^2}{u^2}}\left(-\frac{x^2}{\alpha}\,du\right)$$

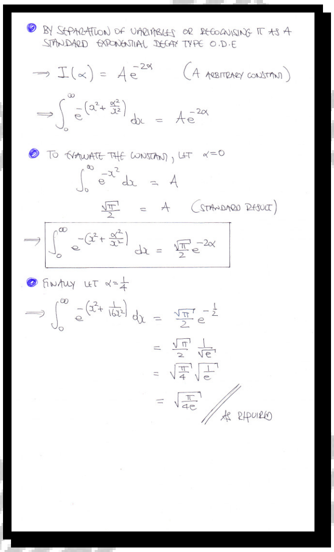
$$\Rightarrow \frac{\partial I}{\partial \alpha} = \int_{\infty}^{0} e^{-\left(u^2+\frac{u^2}{\alpha^2}\right)}\,\frac{x^2}{\frac{\alpha^2}{u^2}}\,du$$

$$\Rightarrow \frac{\partial I}{\partial \alpha} = \int_{\infty}^{0} e^{-\left(u^2+\frac{u^2}{\alpha^2}\right)}\,du$$

$$\Rightarrow \frac{\partial I}{\partial \alpha} = -\int_0^{\infty} e^{-\left(u^2+\frac{u^2}{\alpha^2}\right)}\,du$$

$$\Rightarrow \frac{\partial I}{\partial \alpha} = -2I(\alpha)$$

IE A SEPARABLE O.D.E FOR $I(\alpha)$



BY SEPARATION OF VARIABLES OR RECOGNISING IT AS A STANDARD EXPONENTIAL DECAY TYPE O.D.E

$$\Rightarrow I(\alpha) = Ae^{-2\alpha} \qquad (\text{A ARBITRARY CONSTANT})$$

$$\Rightarrow \int_0^{\infty} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)}\,dx = Ae^{-2\alpha}$$

TO EVALUATE THE CONSTANT, LET $\alpha = 0$

$$\int_0^{\infty} e^{-x^2}\,dx = A$$

$$\frac{\sqrt{\pi}}{2} = A \qquad (\text{STANDARD RESULT})$$

$$\Rightarrow \boxed{\int_0^{\infty} e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)}\,dx = \frac{\sqrt{\pi}}{2}e^{-2\alpha}}$$

FINALLY LET $\alpha = \frac{1}{4}$

$$\Rightarrow \int_0^{\infty} e^{-\left(x^2+\frac{1}{16x^2}\right)}\,dx = \frac{\sqrt{\pi}}{2}e^{-\frac{1}{2}}$$

$$= \frac{\sqrt{\pi}}{2}\,\frac{1}{\sqrt{e}}$$

$$= \sqrt{\frac{\pi}{4}}\sqrt{\frac{1}{e}}$$

$$= \sqrt{\frac{\pi}{4e}}$$

AS REQUIRED

Question 25

An integral I with variable limits is defined as

$$I(x) = \int_{x}^{x^2} e^{\sqrt{u}}\, du .$$

a) Use a suitable substitution followed by integration by parts to find a simplified expression for

$$\frac{d}{dx}\big[I(x)\big].$$

b) Verify the answer obtained in part **(a)** by carrying the differentiation over the integral sign.

$$\boxed{\frac{d}{dx}\big[I(x)\big] = 2x e^{x} - e^{\sqrt{x}}}$$

a) $\frac{d}{dx}\left[\int_{x}^{x^2} e^{\sqrt{u}}\, du\right] = \dots$ BY SUBSTITUTION

$t = \sqrt{u}$
$t^2 = u$
$2t\,dt = du$
$u = x \mapsto t = x^{\frac{1}{2}}$
$u = x^2 \mapsto t = x$

$= \frac{d}{dx}\left[\int_{x^{\frac{1}{2}}}^{x} e^{t}(2t\,dt)\right]$

$= 2\frac{d}{dx}\left[\int_{x^{\frac{1}{2}}}^{x} te^{t}\, dt\right]$

INTEGRATION BY PARTS OR INSPECTION NOW YIELDS

$= 2\frac{d}{dx}\left[\left[te^{t} - e^{t}\right]_{x^{\frac{1}{2}}}^{x}\right]$

$= 2\frac{d}{dx}\left[(xe^{x} - e^{x}) - (x^{\frac{1}{2}}e^{x^{\frac{1}{2}}} - e^{x^{\frac{1}{2}}})\right]$

$= 2\frac{d}{dx}\left[xe^{x} - e^{x} - x^{\frac{1}{2}}e^{x^{\frac{1}{2}}} + e^{x^{\frac{1}{2}}}\right]$

$= 2\times\left[\cancel{e^{x}} + xe^{x} - \cancel{e^{x}} - \frac{1}{2}x^{-\frac{1}{2}}e^{x^{\frac{1}{2}}} - x^{\frac{1}{2}}\left(\frac{1}{2}x^{-\frac{1}{2}}e^{x^{\frac{1}{2}}}\right) + \frac{1}{2}x^{-\frac{1}{2}}e^{x^{\frac{1}{2}}}\right]$

$= 2\left[xe^{x} - \cancel{\frac{1}{2}x^{-\frac{1}{2}}e^{x^{\frac{1}{2}}}} - \frac{1}{2}e^{x^{\frac{1}{2}}} + \cancel{\frac{1}{2}x^{-\frac{1}{2}}e^{x^{\frac{1}{2}}}}\right]$

$= 2xe^{x} - e^{x^{\frac{1}{2}}}$

b) $\frac{d}{dx}\left[\int_{x}^{x^2} e^{\sqrt{u}}\, du\right] = \frac{d}{dx}\left[\int_{0}^{x^2} e^{\sqrt{u}}\, du - \int_{0}^{x} e^{\sqrt{u}}\, du\right]$

$= e^{\sqrt{x^2}} \times \frac{d}{dx}(x^2) - e^{\sqrt{x}}$

$= e^{x} \times 2x - e^{\sqrt{x}}$

$= 2xe^{x} - e^{\sqrt{x}}$

AS BEFORE

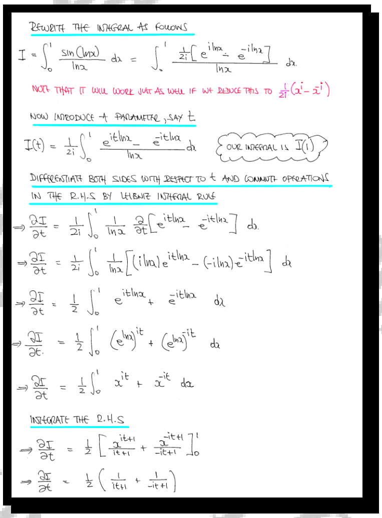
Question 26

Use complex variables and the Leibniz integral rule to evaluate

$$\int_0^1 \frac{\sin(\ln x)}{\ln x}\, dx .$$

You may assume that the integral converges.

$\boxed{}$, $\boxed{\frac{1}{4}\pi}$



REWRITE THE INTEGRAL AS FOLLOWS

$$I = \int_0^1 \frac{\sin(\ln x)}{\ln x}\, dx = \int_0^1 \frac{\frac{1}{2i}\left[e^{i\ln x} - e^{-i\ln x}\right]}{\ln x}\, dx.$$

NOTE THAT IT WILL WORK JUST AS WELL IF WE REDUCE THIS TO $\frac{1}{2i}(x^i - x^{-i})$

NOW INTRODUCE A PARAMETER, SAY t

$$I(t) = \frac{1}{2i}\int_0^1 \frac{e^{it\ln x} - e^{-it\ln x}}{\ln x}\, dx$$

OUR INTEGRAL IS $I(1)$

DIFFERENTIATE BOTH SIDES WITH RESPECT TO t AND COMMUTE OPERATIONS IN THE R.H.S BY LEIBNIZ INTEGRAL RULE

$$\Rightarrow \frac{\partial I}{\partial t} = \frac{1}{2i}\int_0^1 \frac{1}{\ln x}\frac{\partial}{\partial t}\left[e^{it\ln x} - e^{-it\ln x}\right] dx.$$

$$\Rightarrow \frac{\partial I}{\partial t} = \frac{1}{2i}\int_0^1 \frac{1}{\ln x}\left[(i\ln x)e^{it\ln x} - (-i\ln x)e^{-it\ln x}\right] dx$$

$$\Rightarrow \frac{\partial I}{\partial t} = \frac{1}{2}\int_0^1 e^{it\ln x} + e^{-it\ln x}\, dx$$

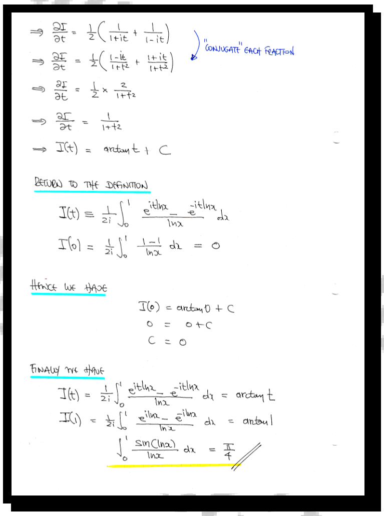
$$\Rightarrow \frac{\partial I}{\partial t} = \frac{1}{2}\int_0^1 (e^{\ln x})^{it} + (e^{\ln x})^{-it}\, dx$$

$$\Rightarrow \frac{\partial I}{\partial t} = \frac{1}{2}\int_0^1 x^{it} + x^{-it}\, dx$$

INTEGRATE THE R.H.S

$$\Rightarrow \frac{\partial I}{\partial t} = \frac{1}{2}\left[\frac{x^{it+1}}{it+1} + \frac{x^{-it+1}}{-it+1}\right]_0^1$$

$$\Rightarrow \frac{\partial I}{\partial t} = \frac{1}{2}\left(\frac{1}{it+1} + \frac{1}{-it+1}\right)$$



$$\Rightarrow \frac{\partial I}{\partial t} = \frac{1}{2}\left(\frac{1}{1+it} + \frac{1}{1-it}\right)$$

"CONJUGATE" EACH FRACTION

$$\Rightarrow \frac{\partial I}{\partial t} = \frac{1}{2}\left(\frac{1-it}{1+t^2} + \frac{1+it}{1+t^2}\right)$$

$$\Rightarrow \frac{\partial I}{\partial t} = \frac{1}{2} \times \frac{2}{1+t^2}$$

$$\Rightarrow \frac{\partial I}{\partial t} = \frac{1}{1+t^2}$$

$$\Rightarrow I(t) = \arctan t + C$$

RETURN TO THE DEFINITION

$$I(t) \equiv \frac{1}{2i}\int_0^1 \frac{e^{it\ln x} - e^{-it\ln x}}{\ln x}\, dx$$

$$I(0) = \frac{1}{2i}\int_0^1 \frac{1-1}{\ln x}\, dx = 0$$

HENCE WE HAVE

$$I(0) = \arctan 0 + C$$
$$0 = 0 + C$$
$$C = 0$$

FINALLY WE HAVE

$$I(t) = \frac{1}{2i}\int_0^1 \frac{e^{it\ln x} - e^{-it\ln x}}{\ln x}\, dx = \arctan t$$

$$I(1) = \frac{1}{2i}\int_0^1 \frac{e^{i\ln x} - e^{-i\ln x}}{\ln x}\, dx = \arctan 1$$

$$\int_0^1 \frac{\sin(\ln x)}{\ln x}\, dx = \frac{\pi}{4}$$

Question 27

$$I = \int_0^{\infty} e^{-x^2} \cos x \; dx$$

Assuming that the above integral converges, use the Leibniz integral rule to evaluate it.

Give the answer in the form $\sqrt[4]{k}$, where k is an exact constant.

You may use without proof $\int_0^{\infty} e^{-x^2} \; dx = \frac{1}{2}\sqrt{\pi}$.

$$I = \sqrt[4]{\frac{\pi^2}{16e}}$$

DEFINE SOME INTEGRAL QUANTITIES/FUNCTIONS

$$I = \int_0^{\infty} e^{-x^2}\cos x \, dx \qquad I(t) = \int_0^{\infty} e^{-x^2}\cos(tx)\, dx$$

$$I(1) = I$$

$$I(0) = \int_0^{\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$$

NOW DIFFERENTIATE $I(t)$, WITH RESPECT TO t, USING LEIBNIZ'S INTEGRAL THEOREM WHICH ALLOWS COMMUTING OF DIFFERENTIATION/INTEGRATION

$$\Rightarrow \frac{\partial I(t)}{\partial t} = \int_0^{\infty} e^{-x^2}\frac{\partial}{\partial t}\left[\cos(tx)\right] dx$$

$$\Rightarrow \frac{\partial I}{\partial t} = \int_0^{\infty} -xe^{-x^2}\sin(tx)\, dx$$

PROCEED BY STANDARD INTEGRATION BY PARTS (W.R.T x)

$$\Rightarrow \frac{\partial I}{\partial t} = \underbrace{\cancel{\left[\tfrac{1}{2}e^{-x^2}\sin(tx)\right]_0^{\infty}}}_{\text{ZERO FROM "BOTH ENDS"}} - \frac{1}{2}t\int_0^{\infty} e^{-x^2}\cos(tx)\, dx$$

$\sin(tx)$	$t\cos(tx)$
$\frac{1}{2}e^{-x^2}$	$-xe^{-x^2}$

$$\Rightarrow \frac{\partial I}{\partial t} = -\frac{1}{2}tI$$

$$\Rightarrow \frac{dI}{dt} = -\frac{t}{2}I \qquad \text{AS } I = f(t)$$

SOLVING THIS SIMPLE O.D.E. BY SEPARATING VARIABLES

$$\Rightarrow \frac{1}{I}\, dI = -\frac{t}{2}\, dt$$

$$\ln|I| = -\frac{1}{4}t^2 + C$$

$$I = Ae^{-\frac{1}{4}t^2}$$

NOW $I(0) = \frac{\sqrt{\pi}}{2}$ (GIVEN)

$$\Rightarrow \frac{\sqrt{\pi}}{2} = A$$

$$\therefore I(t) = \int_0^{\infty} e^{-x^2}\cos(tx)\, dx = \frac{\sqrt{\pi}}{2}e^{-\frac{1}{4}t^2}$$

$$I(1) = \int_0^{\infty} e^{-x^2}\cos x\, dx = \frac{\sqrt{\pi}}{2}e^{-\frac{1}{4}}$$

$$\int_0^{\infty} e^{-x^2}\cos x\, dx = \frac{\pi^{\frac{1}{2}}}{2e^{\frac{1}{4}}}$$

$$\int_0^{\infty} e^{-x^2}\cos x\, dx = \left[\left(\frac{\pi^{\frac{1}{2}}}{2e^{\frac{1}{4}}}\right)^4\right]^{\frac{1}{4}}$$

$$\int_0^{\infty} e^{-x^2}\cos x\, dx = \sqrt[4]{\frac{\pi^2}{16e}}$$

Question 28

It is given that the following integral converges

$$\int_0^\infty \frac{1-\cos\left(\frac{1}{6}x\right)}{x^2}\,dx\,.$$

By introducing a parameter in the integrand and carrying out a suitable differentiation under the integral sign, show that

$$\int_0^\infty \frac{1-\cos\left(\frac{1}{6}x\right)}{x^2}\,dx = \frac{\pi}{12}\,.$$

proof

INTRODUCE A PARAMETER a

$$\Rightarrow I = \int_0^\infty \frac{1-\cos ax}{x^2}\,dx$$

$$\Rightarrow \frac{\partial I}{\partial a} = \frac{\partial}{\partial a}\int_0^\infty \frac{1-\cos ax}{x^2}\,dx = \int_0^\infty \frac{\partial}{\partial a}\left(\frac{1}{x^2}-\frac{\cos ax}{x^2}\right)dx$$

$$\Rightarrow \frac{\partial I}{\partial a} = \int_0^\infty \frac{\sin ax}{x}\,dx$$

NOW CONSIDER ANOTHER INTEGRAL, b A PARAMETER

$$\Rightarrow J = \int_0^\infty \left(\frac{\sin ax}{x}\right)e^{-bx}\,dx$$

$$\Rightarrow \frac{\partial J}{\partial b} = \frac{\partial}{\partial b}\int_0^\infty \left(\frac{\sin ax}{x}\right)e^{-bx}\,dx = \int_0^\infty \frac{\sin ax}{x}\frac{\partial}{\partial b}\left(e^{-bx}\right)dx$$

$$\Rightarrow \frac{\partial J}{\partial b} = \int_0^\infty -e^{-bx}\sin ax\,dx$$

$$\Rightarrow \frac{\partial J}{\partial b} = -\mathrm{Im}\int_0^\infty e^{-bx}e^{iax}\,dx = -\mathrm{Im}\int_0^\infty e^{x(-b+ia)}\,dx$$

$$\Rightarrow \frac{\partial J}{\partial b} = -\mathrm{Im}\left[\frac{1}{-b+ai}e^{x(-b+ai)}\right]_0^\infty$$

$$\Rightarrow \frac{\partial J}{\partial b} = -\mathrm{Im}\left[\frac{-b-ai}{b^2+a^2}e^{-bx}(\cos ax+i\sin ax)\right]_0^\infty$$

$$\Rightarrow \frac{\partial J}{\partial b} = \mathrm{Im}\left[\frac{b+ai}{a^2+b^2}e^{-bx}(\cos ax+i\sin ax)\right]_0^\infty$$

$$\Rightarrow \frac{\partial J}{\partial b} = \mathrm{Im}\left[\frac{b+ai}{a^2+b^2}(0-1)\right] = \mathrm{Im}\left[\frac{-b-ai}{a^2+b^2}\right]$$

$$\Rightarrow \frac{\partial J}{\partial b} = -\frac{a}{a^2+b^2}$$

$$\Rightarrow J = -a\times\frac{1}{a}\arctan\frac{b}{a} + C_1$$

$$\Rightarrow J = C_1 - \arctan\left(\frac{b}{a}\right)$$

$$\Rightarrow \int_0^\infty \left(\frac{\sin ax}{x}\right)e^{-bx}\,dx = C_1 - \arctan\left(\frac{b}{a}\right)$$

LET $b \to +\infty$ $\quad \int_0^\infty 0\,dx = C_1 - \arctan(\infty)$

$$0 = C_1 - \frac{\pi}{2}$$

$$\boxed{C_1 = \frac{\pi}{2}}$$

$$\Rightarrow \int_0^\infty \frac{\sin ax}{x}e^{-bx}\,dx = \frac{\pi}{2} - \arctan\frac{b}{a}$$

LET $b=0$

$$\Rightarrow \int_0^\infty \frac{\sin ax}{x}\,dx = \frac{\pi}{2}$$

THUS RETURNING TO THE ORIGINAL INTEGRAL

$$\Rightarrow \frac{\partial I}{\partial a} = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi a}{2} + C_2$$

$$\Rightarrow \int_0^\infty \frac{1-\cos ax}{x^2}\,dx = \frac{\pi a}{2} + C_2$$

LET $a=0 \Rightarrow 0 = 0 + C_2$ $\quad \boxed{C_2 = 0}$

$$\Rightarrow \int_0^\infty \frac{1-\cos ax}{x^2}\,dx = \frac{\pi a}{2}$$

$$\Rightarrow \int_0^\infty \frac{1-\cos\frac{1}{6}x}{x^2}\,dx = \frac{\pi}{12}$$

Question 29

It is given that the following integral converges

$$I = \int_0^1 \left[\frac{\sqrt{x} - 1}{\ln x} \right]^2 \, dx \, .$$

By carrying out a suitable differentiation under the integral sign, show that

$$I = 5\ln 3 - 3\ln 3 \, .$$

V, [ ], proof

INTRODUCE A PARAMETER t AFTER SOME INITIAL MANIPULATIONS

$$\int_0^1 \left(\frac{\sqrt{x}-1}{\ln x}\right)^2 dx = \int_0^1 \frac{(x^{\frac{1}{2}}-1)^2}{(\ln x)^2} dx$$

LET $I(t) = \int_0^1 \frac{(x^t-1)^2}{(\ln x)^2} dx$ AND DIFFERENTIATE WITH RESPECT TO t

$$\Rightarrow \frac{dI}{dt} = \frac{d}{dt}\left[\int_0^1 \frac{(x^t-1)^2}{(\ln x)^2} dx\right] = \int_0^1 \frac{\partial}{\partial t}\left[\frac{(x^t-1)^2}{(\ln x)^2}\right] dx$$

$$\Rightarrow \frac{dI}{dt} = \int_0^1 \frac{2x^t \ln x \,(x^t-1)}{(\ln x)^2} dx = \int_0^1 \frac{2x^t(x^t-1)}{\ln x} dx$$

$$\Rightarrow \boxed{\frac{dI}{dt} = 2\int_0^1 \frac{x^{2t}-x^t}{\ln x} dx}$$

DIFFERENTIATE ONCE MORE WITH RESPECT TO t

$$\Rightarrow \frac{d^2I}{dt^2} = 2\frac{d}{dt}\left[\int_0^1 \frac{x^{2t}-x^t}{\ln x} dx\right] = 2\int_0^1 \frac{\partial}{\partial t}\left(\frac{x^{2t}-x^t}{\ln x}\right) dx$$

$$\Rightarrow \frac{d^2I}{dt^2} = 2\int_0^1 \frac{2x^{2t}\ln x - x^t \ln x}{\ln x} dx = \int_0^1 4x^{2t} - 2x^t \, dx$$

INTEGRATING THE R.H.S WITH RESPECT TO x

$$\Rightarrow \frac{d^2I}{dt^2} = \left[\frac{4}{2t+1}x^{2t+1} - \frac{2}{t+1}x^{t+1}\right]_0^1 = \frac{4}{2t+1} - \frac{2}{t+1}$$

NEXT INTEGRATE WITH RESPECT TO t

$$\Rightarrow \frac{dI}{dt} = 2\ln(2t+1) - 2\ln(t+1) + C$$

EVALUATE THE CONSTANT AS FOLLOWS

$$\frac{dI}{dt} = 2\int_0^1 \frac{x^{2t}-x^t}{\ln x} = 2\ln(2t+1) - 2\ln(t+1) + C$$

LET $t=0$

$$\Rightarrow 0 = 2\ln 1 - 2\ln 1 + C$$

$$\Rightarrow C = 0$$

$$\Rightarrow \frac{dI}{dt} = 2\ln(2t+1) - 2\ln(t+1)$$

INTEGRATE WITH RESPECT TO t ONCE MORE

$$\Rightarrow I = \int 2\ln(2t+1) - 2\ln(t+1) \, dt$$

$$\Rightarrow I = \int 2\ln(2t+1) \, dt - \int 2\ln(t+1) \, dt$$

$u = 2t+1$, $\frac{du}{dt} = 2$, $dt = \frac{du}{2}$

$v = t+1$, $\frac{dv}{dt} = 1$, $dv = dt$

$$\Rightarrow I = \int 2\ln u \left(\frac{du}{2}\right) - \int 2\ln v \, dv$$

$$\Rightarrow I = \int \ln u \, du - \int 2\ln v \, dv$$

USING STANDARD INTEGRATION RESULT OR SHOW INTEGRATION BY PARTS FROM SCRATCH FOR ~~EACH~~ INTEGRAL

$$\boxed{\int \ln x \, dx = x\ln x - x + C}$$

$$\Rightarrow I = u\ln u - u - 2(v\ln v - v) + D$$

$$\Rightarrow I = (2t+1)\ln(2t+1) - (2t+1) - 2(t+1)\ln(t+1) + 2(t+1) + D$$

$$\Rightarrow \int_0^1 \left(\frac{x^t-1}{\ln x}\right)^2 dx = (2t+1)\ln(2t+1) - 2(t+1)\ln(t+1) + E$$

LET $t=0$

$$0 = \ln 1 - 2\ln 1 + E$$

$$E = 0$$

$$\Rightarrow \int_0^1 \left(\frac{x^t-1}{\ln x}\right)^2 dx = (2t+1)\ln(2t+1) - 2(t+1)\ln(t+1)$$

FINALLY IF $t = \frac{1}{2}$

$$\int_0^1 \left(\frac{\sqrt{x}-1}{\ln x}\right)^2 dx = 2\ln 2 - 3\ln\frac{3}{2}$$

$$= 2\ln 2 - 3(\ln 3 - \ln 2)$$

$$= 5\ln 2 - 3\ln 3$$

AS REQUIRED

Question 30

It is given that the following integral converges

$$\int_0^{\infty} e^{-\frac{1}{2}t} \ln t \; dt .$$

Evaluate the above integral by introducing a new parametric term in the integrand and carrying out a suitable differentiation under the integral sign.

You may assume that

$$\Gamma'(x) = \Gamma(x)\left[-\gamma + \frac{1}{x} + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{x+k}\right)\right].$$

$\boxed{2(-\gamma + \ln 2)}$

$$\boxed{I = \int_0^{\infty} e^{-\frac{1}{2}t} \ln t \, dt}$$

- THIS HAS THE SIGNATURE OF A GAMMA FUNCTION

$$\Gamma(x) = \int_0^{\infty} e^{-t}\, t^{x-1}\, dt$$

- INTRODUCE A PARAMETER x, AS IN THE GAMMA FUNCTION

$\Rightarrow$ LET $J = \int_0^{\infty} t^{x-1} e^{-\frac{1}{2}t} \ln t \, dt$ $\boxed{\text{WHERE } I \text{ IS } J \text{ WITH } x=1}$

- WE OBSERVE THAT $\frac{d}{dx}\left(t^{x-1}\right) = t^{x-1} \times \ln t \times 1 = t^{x-1}\ln t$
- THUS WE MAY WRITE J AS

$$\Rightarrow J = \int_0^{\infty} \frac{\partial}{\partial x}\left[t^{x-1} e^{-\frac{1}{2}t}\right] dt = \frac{\partial}{\partial x}\left[\int_0^{\infty} t^{x-1} e^{-\frac{1}{2}t}\, dt\right]$$

WHICH IS ALMOST A GAMMA FUNCTION

LET $u = \frac{1}{2}t \Leftrightarrow t = 2u$
$du = \frac{1}{2}dt$
$dt = 2du$
LIMITS UNCHANGED

$$\Rightarrow J = \frac{\partial}{\partial x}\left[\int_0^{\infty} (2u)^{x-1} e^{-u} (2\,du)\right] = \frac{\partial}{\partial x}\left[\int_0^{\infty} 2^x u^{x-1} e^{-u}\, du\right]$$

$$\Rightarrow J = \frac{\partial}{\partial x}\left[2^x \int_0^{\infty} e^{-u} u^{x-1}\, du\right] = \frac{\partial}{\partial x}\left[2^x\, \Gamma(x)\right]$$

- DIFFERENTIATING THE PRODUCT

$$\Rightarrow J = 2^x \ln 2\, \Gamma(x) + 2^x\, \Gamma'(x)$$

$$\Rightarrow I = J(1) = 2\ln 2\, \Gamma(1) + 2\Gamma'(1)$$

$$\Rightarrow I = 2\ln 2 + 2\Gamma'(1) \qquad (\Gamma(1) = 0! = 1)$$

- NOW $\Gamma'(x) = \Gamma(x)\left[-\gamma - \frac{1}{x} + \sum_{k=1}^{\infty}\left[\frac{1}{k} - \frac{1}{x+k}\right]\right]$

$$\Rightarrow \Gamma'(1) = \Gamma(1)\left[-\gamma - 1 + \sum_{k=1}^{\infty}\left[\frac{1}{k} - \frac{1}{k+1}\right]\right]$$

$$\Rightarrow \Gamma'(1) = -\gamma - 1 + \left[\begin{array}{l} 1 + \cancel{\frac{1}{2}} + \cancel{\frac{1}{3}} + \cancel{\frac{1}{4}} + \cancel{\frac{1}{5}} + \cdots \\ \quad - \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} - \cancel{\frac{1}{5}} - \cdots \end{array}\right]$$

$$\Rightarrow \Gamma'(1) = -\gamma - 1 + 1$$

$$\Rightarrow \boxed{\Gamma'(1) = -\gamma}$$

$$\Rightarrow I = 2\ln 2 + 2(-\gamma)$$

$$\Rightarrow \int_0^{\infty} e^{-\frac{1}{2}t} \ln t \, dt = 2\left[-\gamma + \ln 2\right]$$

Question 31

$$I = \int_0^{\frac{\pi}{2}} \frac{\ln\left(1+\cos\alpha\cos x\right)}{\cos x}\, dx .$$

By carrying out a suitable differentiation on I under the integral sign, show that

$$I = \tfrac{1}{8}\pi^2 - \tfrac{1}{2}\alpha^2 .$$

proof

$$\boxed{\int_0^{\frac{\pi}{2}} \frac{\ln(1+\cos\alpha\cos x)}{\cos x}\,dx = \frac{\pi^2}{8} - \frac{\alpha^2}{2}}$$

LET $I(\alpha) = \int_0^{\frac{\pi}{2}} \frac{\ln(1+\cos\alpha\cos x)}{\cos x}\,dx$

$$\frac{\partial I}{\partial \alpha} = \frac{\partial}{\partial \alpha}\int_0^{\frac{\pi}{2}} \frac{\ln(1+\cos\alpha\cos x)}{\cos x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial \alpha}\left[\frac{\ln(1+\cos\alpha\cos x)}{\cos x}\right] dx$$

$$\frac{\partial I}{\partial \alpha} = \int_0^{\frac{\pi}{2}} \frac{1}{\cancel{\cos x}} \times \frac{1}{1+\cos\alpha\cos x} \times \cancel{\cos x}(-\sin\alpha)\,dx$$

$$\frac{\partial I}{\partial \alpha} = \int_0^{\frac{\pi}{2}} \frac{-\sin\alpha}{1+\cos\alpha\cos x}\,dx = -\sin\alpha\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos\alpha\cos x}\,dx$$

BY THE LITTLE t IDENTITIES (STANDARD PROCEDURE)

IF $t = \tan\frac{x}{2}$

$$\cos x = \frac{1-t^2}{1+t^2} \quad \& \quad dx = \frac{2}{1+t^2}\,dt$$

$$x = 0 \mapsto t = 0$$
$$x = \tfrac{\pi}{2} \mapsto t = 1$$

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos\alpha\cos x}\,dx = \int_0^1 \frac{1}{1+\cos\alpha\,\frac{1-t^2}{1+t^2}}\;\frac{2}{1+t^2}\,dt$$

$$= \int_0^1 \frac{2}{(1+t^2)+(1-t^2)\cos\alpha}\,dt = \int_0^1 \frac{2}{t^2(1-\cos\alpha)+(1+\cos\alpha)}\,dt$$

$$= \frac{2}{1-\cos\alpha}\int_0^1 \frac{1}{t^2+\frac{1+\cos\alpha}{1-\cos\alpha}}\,dt$$

$$= \frac{2}{1-\cos\alpha} \times \frac{1}{\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}}\left[\arctan\left[\frac{t}{\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}}\right]\right]_0^1$$

$$= \frac{2}{1-\cos\alpha} \times \frac{\sqrt{1-\cos\alpha}}{\sqrt{1+\cos\alpha}}\left[\arctan\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} - 0\right]$$

$$= \frac{2}{\sqrt{1-\cos\alpha}\sqrt{1+\cos\alpha}}\arctan\sqrt{\frac{1-(1-2\sin^2\frac{\alpha}{2})}{1+(2\cos^2\frac{\alpha}{2}-1)}} = \frac{2}{\sqrt{1-\cos^2\alpha}}\arctan\sqrt{\frac{\sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}}}$$

$$= \frac{2}{\sin\alpha}\arctan\left(\tan\frac{\alpha}{2}\right) = \frac{2}{\sin\alpha}\times\frac{\alpha}{2} = \frac{\alpha}{\sin\alpha}$$

$$\Rightarrow \frac{\partial I}{\partial \alpha} = -\sin\alpha \times \frac{\alpha}{\sin\alpha}$$

$$\Rightarrow \frac{\partial I}{\partial \alpha} = -\alpha$$

SOLVING THE O.D.E

$$\Rightarrow I = -\tfrac{1}{2}\alpha^2 + C$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{\ln(1+\cos\alpha\cos x)}{\cos x}\,dx = -\tfrac{1}{2}\alpha^2 + C$$

LET $\alpha = \frac{\pi}{2}$

$$\int_0^{\frac{\pi}{2}} \frac{\ln 1}{\cos x}\,dx = -\tfrac{1}{2}\left(\tfrac{\pi}{2}\right)^2 + C$$

$$0 = C - \frac{\pi^2}{8}$$

$$\boxed{C = \frac{\pi^2}{8}}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{\ln(1+\cos\alpha\cos x)}{\cos x}\,dx = \frac{\pi^2}{8} - \frac{\alpha^2}{2}$$

AS REQUIRED

Question 32

$$I = \int_0^{\frac{\pi}{2}} \frac{\ln\left(1+3\sin^2 x\right)}{\sin^2 x}\, dx .$$

By introducing a parameter a in the integrand and carrying out differentiation on I under the integral sign, show that

$$I = \pi .$$

proof

$$\boxed{\int_0^{\frac{\pi}{2}} \frac{\ln(1+3\sin^2 x)}{\sin^2 x}\, dx = \pi}$$

- INTRODUCE A PARAMETER a IN THE INTEGRAND

$$I(a) = \int_0^{\frac{\pi}{2}} \frac{\ln(1+a\sin^2 x)}{\sin^2 x}\, dx$$

$$\frac{\partial I}{\partial a} = \frac{\partial}{\partial a}\left[\int_0^{\frac{\pi}{2}} \frac{\ln(1+a\sin^2 x)}{\sin^2 x}\, dx\right] = \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial a}\left[\frac{\ln(1+a\sin^2 x)}{\sin^2 x}\right] dx$$

$$\frac{\partial I}{\partial a} = \int_0^{\frac{\pi}{2}} \frac{1}{\cancel{\sin^2 x}} \times \frac{1}{1+a\sin^2 x} \times \cancel{\sin^2 x}\, dx = \int_0^{\frac{\pi}{2}} \frac{1}{1+a\sin^2 x}\, dx$$

- TO INTEGRATE THIS WE USE THE LITTLE t IDENTITIES OR BETTER PROCEED AS FOLLOWS

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+a\sin^2 x}\, dx = \int_0^{\frac{\pi}{2}} \frac{1}{1+\frac{a}{\operatorname{cosec}^2 x}}\, dx = \int_0^{\frac{\pi}{2}} \frac{\operatorname{cosec}^2 x}{\operatorname{cosec}^2 x + a}\, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\operatorname{cosec}^2 x}{1+\cot^2 x + a}\, dx = \dots \text{ SUBSTITUTION}$$

$u = \cot x$

$\frac{du}{dx} = -\operatorname{cosec}^2 x$

$dx = \frac{du}{-\operatorname{cosec}^2 x}$

$x = 0 \mapsto u = \infty$

$x = \frac{\pi}{2} \mapsto u = 0$

$$\dots = \int_\infty^0 \frac{\operatorname{cosec}^2 x}{1+u^2+a}\left(-\frac{du}{\operatorname{cosec}^2 x}\right) = \int_0^\infty \frac{1}{u^2+(a+1)}\, du$$

$$= \frac{1}{\sqrt{a+1}}\left[\arctan\left(\frac{u}{\sqrt{a+1}}\right)\right]_0^\infty = \frac{1}{\sqrt{a+1}}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{2\sqrt{a+1}}$$

- RETURNING TO THE MAIN PROBLEM

$$\Rightarrow \frac{\partial I}{\partial a} = \int_0^{\frac{\pi}{2}} \frac{1}{1+a\sin^2 x}\, dx$$

$$\Rightarrow \frac{\partial I}{\partial a} = \frac{\pi}{2}\left(\frac{1}{\sqrt{a+1}}\right) = \frac{\pi}{2}(a+1)^{-\frac{1}{2}}$$

$$\Rightarrow I = \frac{\pi}{\cancel{2}} \times \cancel{2}(a+1)^{\frac{1}{2}} + C$$

$$\Rightarrow \int_0^{\pi/2} \frac{\ln(1+a\sin^2 x)}{\sin^2 x}\, dx = \pi(a+1)^{\frac{1}{2}} + C$$

- TO EVALUATE THE CONSTANT LET $a=0$

$$\int_0^{\pi/2} \frac{\ln 1}{\sin^2 x}\, dx = \pi + C$$

$$0 = \pi + C$$

$$\boxed{C = -\pi}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{\ln(1+a\sin^2 x)}{\sin^2 x}\, dx = \pi(a+1)^{\frac{1}{2}} - \pi$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{\ln(1+a^2\sin^2 x)}{\sin^2 x}\, dx = \pi\left[\sqrt{a+1} - 1\right]$$

& FINALLY IF $a=3$

$$\int_0^{\pi/2} \frac{\ln(1+3\sin^2 x)}{\sin^2 x} = \pi\left[\sqrt{3+1} - 1\right] = \pi$$

AS REQUIRED

Question 33

$$I = \int_0^\infty \frac{e^{-x^n} - e^{-(2x)^n}}{x} \, dx, \; n \in \mathbb{N}.$$

By carrying out a suitable differentiation on I under the integral sign, show that for all $n \in \mathbb{N}$,

$$I = \ln 2 .$$

proof

$$\boxed{\int_0^\infty \frac{e^{-x^n} - e^{-(2x)^n}}{x} \, dx = \ln 2, \; n \in \mathbb{N}}$$

- START BY AN OBVIOUS SUBSTITUTION

$$\boxed{\begin{array}{l} t = x^n \\ x = t^{\frac{1}{n}} \\ dx = \frac{1}{n} t^{\frac{1}{n}-1} \, dt \\ \text{LIMITS UNCHANGED} \end{array}}$$

$$\int_0^\infty \frac{e^{-x^n} - e^{-2^n \times x^n}}{x} \, dx = \int_0^\infty \frac{e^{-t} - e^{-2^n t}}{t^{\frac{1}{n}}} \left(\frac{1}{n} t^{\frac{1}{n}-1} \, dt \right)$$

$$= \int_0^\infty \frac{e^{-t} - e^{-2^n t}}{n} \times t^{-\frac{1}{n}} \times t^{\frac{1}{n}-1} \, dt = \frac{1}{n} \int_0^\infty \frac{e^{-t} - e^{-2^n t}}{t} \, dt$$

- NEXT WE CONSIDER THE FOLLOWING INTEGRAL, SUBJECT TO 2 PARAMETERS a & b, AND DIFFERENTIATE UNDER THE INTEGRAL SIGN.

$$I = \int_0^\infty \frac{e^{-at} - e^{-bt}}{t} \, dt$$

- DIFFERENTIATE WITH RESPECT TO a (W.R.T b "WORKS" JUST AS WELL)

$$\Rightarrow \frac{\partial I}{\partial a} = \frac{\partial}{\partial a} \left[\int_0^\infty \frac{e^{-at} - e^{-bt}}{t} \, dt \right] = \int_0^\infty \frac{1}{t} \frac{\partial}{\partial a} \left[e^{-at} - e^{-bt} \right] dt$$

$$\Rightarrow \frac{\partial I}{\partial a} = \int_0^\infty \frac{1}{t} \times (-t) \left[e^{-at} \right] dt$$

$$\Rightarrow \frac{\partial I}{\partial a} = \int_0^\infty -e^{-at} \, dt$$

$$\Rightarrow \frac{\partial I}{\partial a} = \left[\frac{1}{a} e^{-at} \right]_0^\infty = 0 - \frac{1}{a} = -\frac{1}{a}$$

$$\Rightarrow I = C - \ln a$$

- APPLY CONDITION TO FIND C

$$\Rightarrow \int_0^\infty \frac{e^{-a} - e^{-b}}{t} \, dt = -\ln a + C$$

LET $a = b$

$$\int_0^\infty \frac{e^{-bt} - e^{-bt}}{t} \, dt = -\ln b + C$$

$$0 = C - \ln b$$

$$C = \ln b$$

$$\Rightarrow \int_0^\infty \frac{e^{-at} - e^{-bt}}{t} \, dt = -\ln a + \ln b$$

$$\Rightarrow \int_0^\infty \frac{e^{-at} - e^{-bt}}{t} \, dt = \ln \frac{b}{a}$$

- RETURNING TO THE REQUIRED INTEGRAL, WITH $a = 1$ & $b = 2^n$

$$\Rightarrow \frac{1}{n} \int_0^\infty \frac{e^{-t} - e^{-2^n \times t}}{t} \, dt = \frac{1}{n} \ln \left(\frac{2^n}{1} \right) = \frac{1}{n} \ln (2^n)$$

$$\Rightarrow \int_0^\infty \frac{e^{-x^n} - e^{-(2x^n)}}{x} \, dx = \ln 2$$

Question 34

$$I = \int_0^{\frac{1}{2}\pi} \frac{\exp\left(-\frac{1}{\sqrt{3}}\tan x\right) - \exp\left(-\sqrt{3}\tan x\right)}{\sin 2x}\, dx .$$

By carrying out a suitable differentiation on I under the integral sign, show that

$$I = \frac{1}{2}\ln 3 .$$

proof

$$\boxed{\int_0^{\frac{\pi}{2}} \frac{e^{-\frac{\tan x}{\sqrt{3}}} - e^{-\sqrt{3}\tan x}}{\sin 2x}\, dx = \frac{1}{2}\ln 3}$$

- START BY THE OBVIOUS SUBSTITUTION

$$\boxed{\begin{aligned} &t = \tan x \\ &dt = \sec^2 x\, dx \\ &dx = \cos^2 x\, dt \\ &x = 0 \mapsto t = 0 \\ &x = \tfrac{\pi}{2} \mapsto t = \infty \end{aligned}}$$

- THUS WE NOW HAVE

$$\int_0^{\infty} \frac{e^{-\frac{t}{\sqrt{3}}} - e^{-\sqrt{3}t}}{\sin 2x} \times (\cos^2 x\, dt)$$

$$= \int_0^{\infty} \left(e^{-\frac{t}{\sqrt{3}}} - e^{-\sqrt{3}t}\right) \times \left(\frac{\cos^2 x}{2\sin x\cos x}\right) dt$$

$$= \int_0^{\infty} \left(e^{-\frac{t}{\sqrt{3}}} - e^{-\sqrt{3}t}\right) \times \left(\frac{1}{2}\cot x\right) dt$$

$$= \int_0^{\infty} \left(e^{-\frac{t}{\sqrt{3}}} - e^{-\sqrt{3}t}\right) \times \frac{1}{2t}\, dt$$

$$= \frac{1}{2}\int_0^{\infty} \frac{e^{-\frac{t}{\sqrt{3}}} - e^{-\sqrt{3}t}}{t}\, dt$$

- NOW DEFINE $I(a,b)$ WHERE a, b ARE POSITIVE PARAMETERS

$$I(a,b) = \int_0^{\infty} \frac{e^{-at} - e^{-bt}}{t}\, dt$$

- DIFFERENTIATE W.R.T TO a (OR b)

$$\frac{\partial I}{\partial a} = \frac{\partial}{\partial a}\int_0^{\infty} \frac{e^{-at} - e^{-bt}}{t}\, dt$$

$$\frac{\partial I}{\partial a} = \int_0^{\infty} \frac{1}{t}\frac{\partial}{\partial a}\left(e^{-at} - e^{-bt}\right) dt$$

$$\frac{\partial I}{\partial a} = \int_0^{\infty} \frac{1}{t} \times (-t)e^{-at}\, dt$$

$$\frac{\partial I}{\partial a} = \int_0^{\infty} -e^{-at}\, dt$$

$$\frac{\partial I}{\partial a} = \left[\frac{1}{a}e^{-at}\right]_0^{\infty} = (0) - \frac{1}{a}$$

- INTEGRATING W.R.T a

$$I = C - \ln a$$

$$\int_0^{\infty} \frac{e^{-at} - e^{-bt}}{t}\, dt = C - \ln a$$

- TO FIND THE CONSTANT LET THE PARAMETER $a = b$

$$\int_0^{\infty} \frac{e^{-bt} - e^{-bt}}{t}\, dt = C - \ln b$$

$$0 = C - \ln b$$

$$C = \ln b$$

- HENCE $\int_0^{\infty} \frac{e^{-at} - e^{-bt}}{t}\, dt = \ln b - \ln a$

$$\Rightarrow \frac{1}{2}\int_0^{\infty} \frac{e^{-\frac{1}{\sqrt{3}}t} - e^{-\sqrt{3}t}}{t}\, dt = \frac{1}{2}\left(\ln\sqrt{3} - \ln\frac{1}{\sqrt{3}}\right)$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{e^{-\frac{1}{\sqrt{3}}\tan x} - e^{-\sqrt{3}\tan x}}{\sin 2x}\, dx = \frac{1}{2}\ln 3$$

ALTERNATIVE VARIATION BY LAPLACE TRANSFORMS AFTER THE SUBSTITUTION

- CONSIDER THE LAPLACE TRANSFORM OF $\frac{e^{-\frac{1}{\sqrt{3}}t} - e^{-\sqrt{3}t}}{t}$ (USING THE DIVISION BY t RULE)

$$\Rightarrow \mathcal{L}\left[\frac{e^{-\frac{1}{\sqrt{3}}t} - e^{-\sqrt{3}t}}{t}\right] = \int_s^{\infty} \mathcal{L}\left[e^{-\frac{1}{\sqrt{3}}t} - e^{-\sqrt{3}t}\right] d\sigma$$

- CHECK THAT THE LIMIT EXISTS

$$\lim_{t \to 0}\left[\frac{e^{-\frac{1}{\sqrt{3}}t} - e^{-\sqrt{3}t}}{t}\right] = \dots \text{ BY L'HOSPITAL}$$

$$= \lim_{t \to 0}\left[\frac{-\frac{1}{\sqrt{3}}e^{-\frac{1}{\sqrt{3}}t} + \sqrt{3}e^{-\sqrt{3}t}}{1}\right]$$

$$= -\frac{1}{\sqrt{3}} + \sqrt{3} = \frac{2}{3}\sqrt{3} \quad \text{ie IT EXISTS}$$

- CONTINUING THE TRANSFORM

$$\Rightarrow \mathcal{L}\left[\frac{e^{-\frac{1}{\sqrt{3}}t} - e^{-\sqrt{3}t}}{t}\right] = \int_s^{\infty} \frac{1}{\sigma + \frac{1}{\sqrt{3}}} - \frac{1}{\sigma + \sqrt{3}}\, d\sigma$$

$$\Rightarrow \int_0^{\infty} e^{-st}\left[\frac{e^{-\frac{1}{\sqrt{3}}t} - e^{-\sqrt{3}t}}{t}\right] dt = \left[\ln\left[\frac{\sigma + \frac{1}{\sqrt{3}}}{\sigma + \sqrt{3}}\right]\right]_s^{\infty}$$

$$\Rightarrow \int_0^{\infty} e^{-st}\left[\frac{e^{-\frac{1}{\sqrt{3}}t} - e^{-\sqrt{3}t}}{t}\right] dt = \cancel{\ln 1} - \ln\left[\frac{s + \frac{1}{\sqrt{3}}}{s + \sqrt{3}}\right]$$

- FINALLY LET $s = 0$

$$\Rightarrow \int_0^{\infty} \frac{e^{-\frac{1}{\sqrt{3}}t} - e^{-\sqrt{3}t}}{t}\, dt = -\ln\left(\frac{\frac{1}{\sqrt{3}}}{\sqrt{3}}\right) = -\ln\frac{1}{3} = \ln 3$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{e^{-\frac{1}{\sqrt{3}}\tan x} - e^{-\sqrt{3}\tan x}}{\sin 2x}\, dx = \frac{1}{2}\int_0^{\infty} \frac{e^{-\frac{1}{\sqrt{3}}t} - e^{-\sqrt{3}t}}{t}\, dt = \frac{1}{2}\ln 3$$

Question 35

$$J = \int_0^{\frac{\pi}{2}} \frac{1}{1+k^2\tan^2 x}\, dx, \quad |k| \neq 1.$$

a) Use appropriate methods to find, in terms of k, a simplified expression for J.

$$I(k) = \int_0^{\frac{\pi}{2}} \frac{\arctan(k\tan x)}{\tan x}\, dx, \quad |k| \neq 1.$$

b) By carrying out a suitable differentiation on I under the integral sign, show that

$$\int_0^{\frac{\pi}{2}} x\cot x \; dx = \tfrac{1}{2}\pi \ln 2 .$$

c) Deduce the value of

$$\int_0^{\frac{\pi}{2}} \ln(\sin x)\; dx .$$

$$\boxed{J = \frac{\pi}{2(k+1)}}, \quad \boxed{-\tfrac{1}{2}\pi \ln 2}$$

a) $\int_0^{\frac{\pi}{2}} \frac{1}{k^2\tan^2 x+1}\, dx = \dots$ BY SUBSTITUTION ...

$u = k\tan x$

$\frac{du}{dx} = k\sec^2 x$

$dx = \frac{du}{k\sec^2 x}$

$dx = \frac{du}{k(\tan^2 x+1)}$

$dx = \frac{du}{k\left(\frac{u^2}{k^2}+1\right)}$

$x = 0 \mapsto u = 0$

$x = \frac{\pi}{2} \mapsto u = \infty$

$= \int_0^{\infty} \frac{1}{u^2+1} \frac{du}{k\left(\frac{u^2}{k^2}+1\right)} = \int_0^{\infty} \frac{1}{u^2+1} \frac{k}{u^2+k^2}\, du$

PARTIAL FRACTIONS

$\frac{k}{(u^2+1)(u^2+k^2)} \equiv \frac{Au+B}{u^2+1} + \frac{Cu+D}{u^2+k^2}$

$k \equiv (Au+B)(u^2+k^2) + (Cu+D)(u^2+1)$

$k \equiv Au^3 + Bu^2 + Ak^2u + Bk^2$
$\quad Cu^3 + Du^2 + Cu + D$

• $A+C=0$ • $B+D=0$ • $Ak^2+C=0$ • $Bk^2+D=k$

THUS $\boxed{D = k - Bk^2}$ & $\boxed{C = -Ak^2}$

$B + (k - Bk^2) = 0 \qquad A + (Ak^2) = 0$

$B - Bk^2 = -k \qquad A(1+k^2) = 0$

$B(1-k^2) = -k \qquad \boxed{A=0}$

$\boxed{B = \frac{k}{k^2-1}} \qquad \boxed{C=0}$

$\boxed{D = -\frac{k}{k^2-1}}$

$\dots = \int_0^{\infty} \frac{\frac{k}{k^2-1}}{u^2+1} - \frac{\frac{k}{k^2-1}}{u^2+k^2}\, du = \frac{k}{k^2-1}\int_0^{\infty} \frac{1}{u^2+1} - \frac{1}{u^2+k^2}\, du$

$= \frac{k}{k^2-1}\left[\arctan u - \frac{1}{k}\arctan\frac{u}{k}\right]_0^{\infty} = \frac{k}{k^2-1}\left[\left(\frac{\pi}{2} - \frac{1}{k}\frac{\pi}{2}\right) - 0\right]$

$= \frac{k}{k^2-1} \times \frac{\pi}{2}\left(1 - \frac{1}{k}\right) = \frac{\pi}{2}\,\frac{k}{k^2-1} \times \frac{k-1}{k} = \frac{\pi k(k-1)}{2k(k+1)(k-1)}$

$= \frac{\pi}{2(k+1)}$

b) • CONSIDER THE INTEGRAL

$\boxed{I(k) = \int_0^{\frac{\pi}{2}} \frac{\arctan(k\tan x)}{\tan x}\, dx}$

• DIFFERENTIATE WITH RESPECT TO k

$\Rightarrow \frac{\partial I}{\partial k} = \frac{\partial}{\partial k}\int_0^{\frac{\pi}{2}} \frac{\arctan(k\tan x)}{\tan x}\, dx$

$\Rightarrow \frac{\partial I}{\partial k} = \int_0^{\frac{\pi}{2}} \frac{1}{\tan x}\frac{\partial}{\partial k}\left[\arctan(k\tan x)\right] dx$

$\Rightarrow \frac{\partial I}{\partial k} = \int_0^{\frac{\pi}{2}} \frac{1}{\tan x}\left[\tan x \times \frac{1}{1+(k\tan x)^2}\right] dx$

$\Rightarrow \frac{\partial I}{\partial k} = \int_0^{\frac{\pi}{2}} \frac{1}{k^2\tan^2 x+1}\, dx$

$\Rightarrow \frac{\partial I}{\partial k} = \frac{\pi}{2(k+1)}$

$\Rightarrow I = \frac{\pi}{2}\ln(k+1) + C$

$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{\arctan(k\tan x)}{\tan x}\, dx = \frac{\pi}{2}\ln(k+1) + C$

• LET $k=0 \Rightarrow 0 = 0 + C$

$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{\arctan(k\tan x)}{\tan x}\, dx = \frac{\pi}{2}\ln(k+1)$

• FINALLY LET $k=1$

$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{x}{\tan x}\, dx = \frac{\pi}{2}\ln 2$

$\Rightarrow \int_0^{\frac{\pi}{2}} x\cot x\, dx = \frac{\pi}{2}\ln 2$

c) $\int_0^{\frac{\pi}{2}} \ln(\sin x)\, dx = \dots$ by parts

$\ln(\sin x)$	$\cot x$
x	1

$= \left[\cancel{x\ln(\sin x)}\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} x\cot x\, dx$

$= 0 - \int_0^{\frac{\pi}{2}} x\cot x\, dx$

$= -\frac{\pi}{2}\ln 2$

Question 36

The integral function $I(k)$ is defined as

$$I(k) = \int_0^{\pi} e^{k\cos x} \cos(k \sin x) \, dx \,, \quad k \in \mathbb{R} \,.$$

By carrying out a suitable differentiation on I under the integral sign, show that

$$\int_0^{\pi} e^{\cos x} \cos(\sin x) \, dx = \pi \,.$$

$\boxed{\text{proof}}$

CONSIDER THE INTEGRAL

$$\boxed{I(k) = \int_0^{\pi} e^{k\cos x} \cos(k \sin x) \, dx}$$

AS INTEGRAND IS EVEN WE MAY REWRITE IT AS

$$I(k) = \frac{1}{2}\int_{-\pi}^{\pi} e^{k\cos x} \cos(k \sin x) \, dx$$

AND USING PERIODICITY PROPERTIES

$$\Rightarrow I(k) = \frac{1}{2}\int_0^{2\pi} e^{k\cos x} \cos(k \sin x) \, dx$$

$$\Rightarrow I(k) = \frac{1}{2}\text{Re}\left[\int_0^{2\pi} e^{k\cos x} \times e^{ik\sin x} \, dx\right]$$

$$\Rightarrow I(k) = \frac{1}{2}\text{Re}\int_0^{2\pi} e^{k\cos x + ik\sin x} \, dx$$

$$\Rightarrow I(k) = \frac{1}{2}\text{Re}\int_0^{2\pi} e^{k(\cos x + i\sin x)} \, dx$$

$$\Rightarrow I(k) = \frac{1}{2}\text{Re}\int_0^{2\pi} e^{ke^{ix}} \, dx$$

NOW DIFFERENTIATING W.R.T k

$$\Rightarrow \frac{\partial I}{\partial k} = \frac{1}{2}\text{Re}\int_0^{2\pi} \frac{\partial}{\partial k}\left[e^{ke^{ix}}\right] dx$$

$$\Rightarrow \frac{\partial I}{\partial k} = \frac{1}{2}\text{Re}\int_0^{2\pi} e^{ke^{ix}} \times e^{ix} \, dx$$

$$\Rightarrow \frac{\partial I}{\partial k} = \frac{1}{2}\text{Re}\left[\frac{1}{ki} e^{ke^{ix}}\right]_0^{2\pi}$$

$$\Rightarrow \frac{\partial I}{\partial k} = \frac{1}{2k}\text{Re}\left[\frac{1}{i} e^{ke^{2\pi i}} - \frac{1}{i} e^{k}\right]$$

$$\Rightarrow \frac{\partial I}{\partial k} = \frac{1}{2k}\text{Re}\left[\frac{1}{i} e^{k} - \frac{1}{i} e^{k}\right]$$

$$\Rightarrow \frac{\partial I}{\partial k} = 0$$

$$\Rightarrow I(k) = \text{CONSTANT}$$

$$\Rightarrow \int_0^{\pi} e^{k\cos x} \cos(k \sin x) \, dx = \text{CONSTANT}.$$

TAKE $k=0$

$\int_0^{\pi} 1 \, dx = \text{CONSTANT}$

CONSTANT $= \pi$

$$\Rightarrow \int_0^{\pi} e^{k\cos x} \cos(k \sin x) \, dx = \pi \quad (\text{FOR ALL } k)$$

$$\Rightarrow \int_0^{\pi} e^{\cos x} \cos(\sin x) \, dx = \pi$$

Question 37

$$I = \int_0^{\pi} \frac{\ln(1+\cos\alpha\cos\theta)}{\cos\theta}\, d\theta .$$

By carrying out a suitable differentiation on I under the integral sign, show that

$$\int_0^{\pi} \frac{\ln(1+\cos\theta)}{\cos\theta}\, d\theta = \frac{\pi^2}{2} .$$

$\boxed{\text{proof}}$

CONSIDER THE INTEGRAL

$$I = \int_0^{\pi} \frac{\ln(1+\cos\alpha\cos\theta)}{\cos\theta}\, d\theta \quad \text{WHERE } \alpha = 0$$

$$\frac{\partial I}{\partial \alpha} = \frac{\partial}{\partial \alpha}\int_0^{\pi} \frac{\ln(1+\cos\alpha\cos\theta)}{\cos\theta}\, d\theta = \int_0^{\pi} \frac{1}{\cos\theta}\frac{\partial}{\partial \alpha}\left[\ln(1+\cos\alpha\cos\theta)\right] d\theta$$

$$\frac{\partial I}{\partial \alpha} = \int_0^{\pi} \frac{1}{\cos\theta}\,\frac{1}{1+\cos\alpha\cos\theta} \times \cos\theta(-\sin\alpha)\, d\theta$$

$$\frac{\partial I}{\partial \alpha} = \int_0^{\pi} \frac{-\sin\alpha}{1+\cos\alpha\cos\theta}\, d\theta$$

$$\frac{\partial I}{\partial \alpha} = -\sin\alpha \int_0^{\pi} \frac{1}{1+\cos\alpha\cos\theta}\, d\theta$$

BY LITTLE t IDENTITIES $t = \tan\frac{\theta}{2}$

$$\cos\theta = \frac{1-t^2}{1+t^2} \quad \& \quad d\theta = \frac{2}{1+t^2}\, dt$$

$$\theta = 0 \quad t = 0$$
$$\theta = \pi \quad t = \infty$$

$$\int_0^{\pi} \frac{1}{1+\cos\alpha\cos\theta}\, d\theta = \int_0^{\infty} \frac{1}{1+\cos\alpha\left(\frac{1-t^2}{1+t^2}\right)}\,\frac{2\,dt}{1+t^2}$$

$$= \int_0^{\infty} \frac{2}{(1+t^2)+(1-t^2)\cos\alpha}\, dt = \int_0^{\infty} \frac{2}{t^2(1-\cos\alpha)+(1+\cos\alpha)}\, dt$$

$$= \frac{2}{1-\cos\alpha}\int_0^{\infty} \frac{1}{t^2+\frac{1+\cos\alpha}{1-\cos\alpha}}\, dt$$

$$= \frac{2}{1-\cos\alpha} \times \frac{1}{\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}}\left[\arctan\left[\frac{t}{\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}}\right]\right]_0^{\infty}$$

$$= \frac{2}{1-\cos\alpha} \times \frac{\sqrt{1-\cos\alpha}}{\sqrt{1+\cos\alpha}}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{\sqrt{1-\cos^2\alpha}} = \frac{\pi}{\sin\alpha}$$

RETURNING TO OUR EQUATION

$$\Longrightarrow \frac{\partial I}{\partial \alpha} = -\sin\alpha \int_0^{\pi} \frac{1}{1+\cos\alpha\cos\theta}\, d\theta$$

$$\Longrightarrow \frac{\partial I}{\partial \alpha} = -\sin\alpha\left(\frac{\pi}{\sin\alpha}\right)$$

$$\Longrightarrow \frac{\partial I}{\partial \alpha} = -\pi$$

$$\boxed{I = -\pi\alpha + C}$$

$$\Longrightarrow \int_0^{\pi} \frac{\ln(1+\cos\alpha\cos\theta)}{\cos\theta}\, d\theta = -\pi\alpha + C.$$

- LET $\alpha = \frac{\pi}{2}$

$$\Longrightarrow \int_0^{\pi} \frac{\ln 1}{\cos\theta}\, d\theta = -\pi\left(\frac{\pi}{2}\right) + C$$

$$\boxed{C = \pi\left(\frac{\pi}{2}\right)}$$

$$\therefore \int_0^{\pi} \frac{\ln(1+\cos\alpha\cos\theta)}{\cos\theta}\, d\theta = -\pi\alpha + \pi\left(\frac{\pi}{2}\right)$$

$$\Longrightarrow \int_0^{\pi} \frac{\ln(1+\cos\alpha\cos\theta)}{\cos\theta}\, d\theta = \pi\left[\frac{\pi}{2} - \alpha\right]$$

& IF $\alpha = 0$

$$\Longrightarrow \int_0^{\pi} \frac{\ln(1+\cos\theta)}{\cos\theta}\, d\theta = \frac{\pi^2}{2}$$

Question 38

$$I(k) \equiv \int_0^{\pi} \ln(1 - k\cos x)\ dx\,, \quad |k| < 1\,.$$

By differentiating both sides of the above equation with respect to k, show that

$$I(k) = \pi \ln\left[\frac{1}{2}\left(1 - \sqrt{1-k^2}\right)\right].$$

proof

$$\boxed{\int_0^{\pi} \ln(1-k\cos x)\, dx = \pi \ln\left[\frac{1+\sqrt{1-k^2}}{2}\right] \qquad |k|<1}$$

LET $I(k) = \int_0^{\pi} \ln(1-k\cos x)\, dx$

$$\Rightarrow \frac{\partial I}{\partial k} = \frac{\partial}{\partial k}\left[\int_0^{\pi} \ln(1-k\cos x)\, dx\right] = \int_0^{\pi} \frac{\partial}{\partial k}\left[\ln(1-k\cos x)\right] dx$$

$$\Rightarrow \frac{\partial I}{\partial k} = \int_0^{\pi} \frac{1}{1-k\cos x} \times (-\cos x)\, dx = \int_0^{\pi} \frac{-\cos x}{1-k\cos x}\, dx$$

$$\Rightarrow \frac{\partial I}{\partial k} = \frac{1}{k}\int_0^{\pi} \frac{-k\cos x}{1-k\cos x}\, dx = \frac{1}{k}\int_0^{\pi} \frac{(1-k\cos x)-1}{1-k\cos x}\, dx$$

$$\Rightarrow \frac{\partial I}{\partial k} = \frac{1}{k}\int_0^{\pi} 1 - \frac{1}{1-k\cos x}\, dx$$

SPLIT THE INTEGRAL & EMPLOY THE LITTLE t IDENTITIES FOR THE 2ND PART

LET $t = \tan\frac{x}{2}$

$\cos x = \frac{1-t^2}{1+t^2}$ & $dx = \frac{2}{1+t^2}\, dt$

$x = 0 \mapsto t = 0$

$x = \pi \mapsto t = 1$

$$\int_0^{\pi} \frac{1}{1-k\cos x}\, dx = \int_0^{\infty} \frac{1}{1-k\left(\frac{1-t^2}{1+t^2}\right)}\left(\frac{2}{1+t^2}\, dt\right) = \int_0^{\infty} \frac{2}{(1+t^2)-k(1-t^2)}\, dt$$

$$= \int_0^{\infty} \frac{2}{(1+k)t^2+(1-k)}\, dt = \frac{2}{1+k}\int_0^{\infty} \frac{1}{t^2+\frac{1-k}{1+k}}\, dt$$

$$= \frac{2}{1+k} \times \frac{1}{\sqrt{\frac{1-k}{1+k}}}\left[\arctan\left(\frac{t}{\sqrt{\frac{1-k}{1+k}}}\right)\right]_0^{\infty} = \frac{2}{1+k}\frac{\sqrt{1+k}}{\sqrt{1-k}}\left[\frac{\pi}{2} - 0\right]$$

$$= \frac{2}{\sqrt{1+k}\sqrt{1-k}} \times \frac{\pi}{2} = \frac{\pi}{\sqrt{1-k^2}}$$

RETURNING TO THE INTEGRAL

$$\Rightarrow \frac{\partial I}{\partial k} = \frac{1}{k}\int_0^{\pi} 1 - \frac{1}{1-k\cos x}\, dx$$

$$\Rightarrow \frac{\partial I}{\partial k} = \frac{1}{k}\int_0^{\pi} 1\, dx - \frac{1}{k}\int_0^{\pi} \frac{1}{1-k\cos x}\, dx$$

$$\Rightarrow \frac{\partial I}{\partial k} = \frac{\pi}{k} - \frac{1}{k} \times \frac{\pi}{\sqrt{1-k^2}}$$

$$\Rightarrow \frac{\partial I}{\partial k} = \frac{\pi}{k} - \frac{\pi}{k\sqrt{1-k^2}}$$

$$\Rightarrow I = \pi \ln k - \pi \int \frac{1}{k\sqrt{1-k^2}}\, dk$$

SUBSTITUTION

$k = \sin\theta \Rightarrow \theta = \arcsin k$

$dk = \cos\theta\, d\theta$

$$\text{So } \int \frac{1}{\sin\theta\sqrt{1-\sin^2\theta}}(\cos\theta\, d\theta) = \int \operatorname{cosec}\theta\, d\theta$$

$$= -\ln|\operatorname{cosec}\theta + \cot\theta| = -\ln\left|\frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}\right|$$

$$= -\ln\left|\frac{1+\cos\theta}{\sin\theta}\right| = -\ln\left|\frac{1+\sqrt{1-\sin^2\theta}}{\sin\theta}\right|$$

$$= -\ln\left|\frac{1+\sqrt{1-k^2}}{k}\right|$$

$$\Rightarrow I = \pi \ln k - \pi\left[-\ln\frac{1+\sqrt{1-k^2}}{k}\right] + C$$

$$\Rightarrow \int_0^{\pi} \ln(1-k\cos x)\, dx = \pi \ln k + \pi \ln\left[\frac{1+\sqrt{1-k^2}}{k}\right] + C$$

$$\Rightarrow \int_0^{\pi} \ln(1-k\cos x)\, dx = \pi \ln\left[k \times \frac{1+\sqrt{1-k^2}}{k}\right] + C$$

$$\Rightarrow \int_0^{\pi} \ln(1-k\cos x)\, dx = \pi \ln\left[1+\sqrt{1-k^2}\right] + C$$

LET $k = 0$

$$\int_0^{\pi} \ln 1\, dx = \pi \ln 2 + C$$

$$0 = \pi \ln 2 + C$$

$$C = -\pi \ln 2$$

$$\Rightarrow \int_0^{\pi} \ln(1-k\cos x)\, dx = \pi \ln\left[1+\sqrt{1-k^2}\right] - \pi \ln 2$$

$$\therefore \int_0^{\pi} \ln(1-k\cos x)\, dx = \pi \ln\left[\frac{1+\sqrt{1-k^2}}{2}\right]$$

Question 39

Find the following inverse Laplace transform, by using differentiation under the integral sign.

$$\mathcal{L}^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right],\ a>0.$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right]=\frac{t\sin at}{2a}}$$

- STARTING WITH $\frac{s}{s^2+a^2} = s\left(s^2+a^2\right)^{-1}$

$$\frac{d}{da}\left[\frac{s}{s^2+a^2}\right] = -\frac{2as}{\left(s^2+a^2\right)^2}$$

- TAKING INVERSE LAPLACE TRANSFORMS ON BOTH SIDES

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{d}{da}\left(\frac{s}{s^2+a^2}\right)\right] = -2a\,\mathcal{L}^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right]$$

$$\Rightarrow \frac{d}{da}\left[\mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right]\right] = -2a\,\mathcal{L}^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right]$$

$$\Rightarrow \frac{d}{da}\left[\cos(at)\right] = -2a\,\mathcal{L}^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right]$$

$$\Rightarrow -t\sin(at) = -2a\,\mathcal{L}^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right]$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right] = \frac{1}{2a}t\sin(at)$$

Question 40

The integral I is defined in terms of the constants α and k, by

$$I(\alpha,k) \equiv \int_0^\infty e^{-\alpha x^2}\cos(kx)\ dx\,, \quad \alpha > 0\,.$$

By differentiating both sides of the above equation with respect to k, followed by integration by parts, show that

$$\int_0^\infty e^{-\alpha x^2}\cos(kx)\ dx \ = \ \sqrt{\frac{\pi}{4\alpha}}\exp\left(-\frac{k^2}{4\alpha}\right).$$

You may assume without proof that

$$\int_0^\infty e^{-x^2}\ dx \ = \ \frac{1}{2}\sqrt{\pi}\,.$$

You may not use contour integration techniques in this question.

$\boxed{\text{proof}}$

$$\boxed{I = \int_0^\infty e^{-\alpha x^2}\cos(kx)\,dx \qquad \begin{matrix}\alpha > 0 \\ k = \text{CONSTANT}\end{matrix}}$$

$$\Rightarrow \frac{\partial I}{\partial k} = \frac{\partial}{\partial k}\int_0^\infty e^{-\alpha x^2}\cos(kx)\,dx$$

$$\Rightarrow \frac{\partial I}{\partial k} = \int_0^\infty \frac{\partial}{\partial k}\left[e^{-\alpha x^2}\cos(kx)\right]dx$$

$$\Rightarrow \frac{\partial I}{\partial k} = \int_0^\infty e^{-\alpha x^2}(-x\sin kx)\,dx = \int_0^\infty \left(-xe^{-\alpha x^2}\right)(\sin kx)\,dx$$

- BY PARTS

$\sin kx$	$k\cos kx$
$\frac{1}{2\alpha}e^{-\alpha x^2}$	$-xe^{-\alpha x^2}$

$$\Rightarrow \frac{\partial I}{\partial k} = \cancel{\left[\frac{1}{2\alpha}e^{-\alpha x^2}\sin kx\right]_0^\infty} - \frac{k}{2\alpha}\int_0^\infty e^{-\alpha x^2}\cos kx\,dx$$

$$\Rightarrow \frac{\partial I}{\partial k} = -\frac{k}{2\alpha}I$$

- IE A SEPARABLE O.D.E FOR I

$$\Rightarrow \frac{1}{I}\partial I = -\frac{k}{2\alpha}\partial k$$

$$\Rightarrow \ln I = -\frac{k^2}{4\alpha} + C$$

$$\Rightarrow I = Be^{-\frac{k^2}{4\alpha}} \qquad (B \text{ ARBITRARY})$$

- THUS SO FAR

$$\boxed{\int_0^\infty e^{-\alpha x^2}\cos(kx)\,dx = Be^{-\frac{k^2}{4\alpha}}}$$

- TO EVALUATE THE CONSTANT PICK A CONVENIENT VALUE, SAY $k=0$

$$\Rightarrow \int_0^\infty e^{-\alpha x^2}\,dx = B$$

$$\Rightarrow \int_0^\infty e^{-y^2}\frac{dy}{\sqrt{\alpha}} = B$$

$$\Rightarrow \frac{1}{\sqrt{\alpha}}\underbrace{\int_0^\infty e^{-y^2}\,dy} = B$$

$$\Rightarrow B = \left(\frac{\sqrt{\pi}}{2}\right)\times\frac{1}{\sqrt{\alpha}}$$

BY SUBSTITUTION

$y = \sqrt{\alpha}\,x$

$y^2 = \alpha x^2$

$dy = \sqrt{\alpha}\,dx$

LIMITS UNCHANGED

HENCE

$$\int_0^\infty e^{-\alpha x^2}\cos kx\,dx = Be^{-\frac{k^2}{4\alpha}}$$

$$\int_0^\infty e^{-\alpha x^2}\cos kx\,dx = \sqrt{\frac{\pi}{4\alpha}}\,e^{-\frac{k^2}{4\alpha}}$$

Question 41

It is given that the following integral converges

$$\int_0^{\infty} \frac{e^{-x}}{x}\left[3-\frac{1}{x}+\frac{1}{x}e^{-3x}\right]\,dx\,.$$

By introducing a parameter λ and carrying out a suitable differentiation under the integral sign, show that

$$\int_0^{\infty} \frac{e^{-x}}{x}\left[3-\frac{1}{x}+\frac{1}{x}e^{-3x}\right]\,dx=-3+\ln 256\ .$$

proof

$$\boxed{\int_0^{\infty} \frac{e^{-x}}{x}\left(3-\frac{1}{x}+\frac{1}{x}e^{-3x}\right)dx = -3+\ln 256}$$

- INTRODUCE A PARAMETER λ IN THE INTEGRAND, INSTEAD OF 3

$$\Rightarrow I(\lambda)=\int_0^{\infty}\frac{e^{-x}}{x}\left[\lambda-\frac{1}{x}+\frac{1}{x}e^{-\lambda x}\right]dx$$

$$\Rightarrow \frac{\partial I}{\partial \lambda}=\frac{\partial}{\partial \lambda}\int_0^{\infty}\frac{e^{-x}}{x}\left[\lambda-\frac{1}{x}+\frac{1}{x}e^{-\lambda x}\right]dx$$

$$\Rightarrow \frac{\partial I}{\partial \lambda}=\int_0^{\infty}\frac{e^{-x}}{x}\frac{\partial}{\partial \lambda}\left[\lambda-\frac{1}{x}+\frac{1}{x}e^{-\lambda x}\right]dx$$

$$\Rightarrow \frac{\partial I}{\partial \lambda}=\int_0^{\infty}\frac{e^{-x}}{x}\left[1-0-e^{-\lambda x}\right]dx$$

$$\Rightarrow \frac{\partial I}{\partial \lambda}=\int_0^{\infty}\frac{e^{-x}-e^{-x(\lambda+1)}}{x}\,dx$$

- LET $J=\frac{\partial I}{\partial \lambda}=\int_0^{\infty}\frac{e^{-x}-e^{-x(\lambda+1)}}{x}\,dx$

$$\Rightarrow \frac{\partial J}{\partial \lambda}=\frac{\partial}{\partial \lambda}\int_0^{\infty}\frac{e^{-x}-e^{-x(\lambda+1)}}{x}\,dx=\int_0^{\infty}\frac{\partial}{\partial \lambda}\left[\frac{e^{-x}}{x}-\frac{e^{-x(\lambda+1)}}{x}\right]dx$$

$$\Rightarrow \frac{\partial J}{\partial \lambda}=\int_0^{\infty}0+e^{-x(\lambda+1)}\,dx$$

$$\Rightarrow \frac{\partial J}{\partial \lambda}=\left[-\frac{1}{\lambda+1}e^{-x(\lambda+1)}\right]_0^{\infty}=\frac{1}{\lambda+1}\left[e^{-x(\lambda+1)}\right]_{\infty}^{0}=\frac{1}{\lambda+1}$$

$$\Rightarrow J=\ln|\lambda+1|+A$$

$$\boxed{\int_0^{\infty}\frac{e^{-x}-e^{-x(\lambda+1)}}{x}\,dx=\ln|\lambda+1|+A}$$

- TO EVALUATE THE CONSTANT, LET $\lambda=0$, SO

$$\int_0^{\infty}\frac{e^{-x}-e^{-x}}{x}\,dx=\ln 1+A$$

$$0=0+A$$

$$\boxed{A=0}$$

$$\boxed{\int_0^{\infty}\frac{e^{-x}-e^{-x(\lambda+1)}}{x}\,dx=\ln|\lambda+1|}$$

$$\Rightarrow \frac{\partial I}{\partial \lambda}=\ln|\lambda+1|$$

BY PARTS (OR STANDARD RESULTS)

$\ln(\lambda+1)$	$\frac{1}{\lambda+1}$
λ	1

$$\int \lambda+1\,d\lambda = \lambda\ln(\lambda+1)-\int\frac{\lambda}{\lambda+1}\,d\lambda$$

$$=\lambda\ln(\lambda+1)-\int\frac{\lambda+1-1}{\lambda+1}\,d\lambda$$

$$=\lambda\ln(\lambda+1)-\int 1-\frac{1}{\lambda+1}\,d\lambda$$

$$=\lambda\ln(\lambda+1)-\left(\lambda-\ln(\lambda+1)\right)$$

$$=\lambda\ln(\lambda+1)-\lambda+\ln(\lambda+1)+C$$

$$=(\lambda+1)\ln(\lambda+1)-\lambda+C$$

$$\Rightarrow I=(\lambda+1)\ln(\lambda+1)-\lambda+B$$

$$\Rightarrow \int_0^{\infty}\frac{e^{-x}}{x}\left[\lambda-\frac{1}{x}+\frac{1}{x}e^{-\lambda x}\right]dx=(\lambda+1)\ln(\lambda+1)-\lambda+B$$

- LET $\lambda=0$ (TO EVALUATE THE CONSTANT)

$$\int\frac{e^{-x}}{x}\left[0-\frac{1}{x}+\frac{1}{x}\right]dx=\ln 1+B$$

$$0=0+B$$

$$\Rightarrow \int_0^{\infty}\frac{e^{-x}}{x}\left[\lambda-\frac{1}{x}+\frac{1}{x}e^{-\lambda x}\right]dx=(\lambda+1)\ln(\lambda+1)-\lambda$$

$$\Rightarrow \int_0^{\infty}\frac{e^{-x}}{x}\left[3-\frac{1}{x}+\frac{1}{x}e^{-3x}\right]dx=4\ln 4-3=-3+\ln 256$$

Question 42

The following integral is to be evaluated

$$\int_0^{\frac{\pi}{2}} \ln\left[a^2\cos^2\theta + b^2\sin^2\theta\right]\, d\theta\,,$$

where a and b are distinct constants such that $a+b>0$.

By carrying out a suitable differentiation under the integral sign, show that

$$\int_0^{\frac{\pi}{2}} \ln\left[a^2\cos^2\theta + b^2\sin^2\theta\right]\, d\theta = \pi\ln\left[\frac{a+b}{2}\right].$$

proof

$$\boxed{\int_0^{\frac{\pi}{2}} \ln\left(a^2\cos^2\theta+b^2\sin^2\theta\right) d\theta = \pi\ln\left(\frac{a+b}{2}\right) \quad \begin{matrix} a\neq b \\ a+b>0 \end{matrix}}$$

- TREAT a AS A PARAMETER & b AS A CONSTANT

$$\Rightarrow I(a) = \int_0^{\frac{\pi}{2}} \ln\left(a^2\cos^2\theta+b^2\sin^2\theta\right) d\theta$$

$$\Rightarrow \frac{\partial I}{\partial a} = \frac{\partial}{\partial a}\int_0^{\frac{\pi}{2}} \ln\left(a^2\cos^2\theta+b^2\sin^2\theta\right) d\theta = \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial a}\left[\ln\left(a^2\cos^2\theta+b^2\sin^2\theta\right)\right] d\theta$$

$$\Rightarrow \frac{\partial I}{\partial a} = \int_0^{\frac{\pi}{2}} \frac{1}{a^2\cos^2\theta+b^2\sin^2\theta} \times 2a\cos^2\theta \; d\theta$$

$$\Rightarrow \frac{\partial I}{\partial a} = \int_0^{\frac{\pi}{2}} \frac{2a\cos^2\theta}{a^2\cos^2\theta+b^2\sin^2\theta}\, d\theta = \int_0^{\frac{\pi}{2}} \frac{2a\cos^2\theta}{a^2\cos^2\theta+b^2-b^2\cos^2\theta}\, d\theta$$

$$\Rightarrow \frac{\partial I}{\partial a} = \int_0^{\frac{\pi}{2}} \frac{2a\cos^2\theta}{(a^2-b^2)\cos^2\theta+b^2}\, d\theta$$

$$\Rightarrow \frac{\partial I}{\partial a} = \frac{2a}{a^2-b^2}\int_0^{\frac{\pi}{2}} \frac{(a^2-b^2)\cos^2\theta}{(a^2-b^2)\cos^2\theta+b^2}\, d\theta$$

$$\Rightarrow \frac{\partial I}{\partial a} = \frac{2a}{a^2-b^2}\int_0^{\frac{\pi}{2}} \frac{(a^2-b^2)\cos^2\theta+b^2-b^2}{(a^2-b^2)\cos^2\theta+b^2}\, d\theta$$

$$\Rightarrow \frac{\partial I}{\partial a} = \frac{2a}{a^2-b^2}\int_0^{\frac{\pi}{2}} 1 - \frac{b^2}{(a^2-b^2)\cos^2\theta+b^2}\, d\theta$$

$$\Rightarrow \frac{\partial I}{\partial a} = \frac{2a}{a^2-b^2}\int_0^{\frac{\pi}{2}} 1 - \frac{b^2\sec^2\theta}{(a^2-b^2)+b^2\sec^2\theta}\, d\theta$$

$$\Rightarrow \frac{\partial I}{\partial a} = \frac{2a}{a^2-b^2}\int_0^{\frac{\pi}{2}} 1 - \frac{b^2\sec^2\theta}{(a^2-b^2)+b^2(1+\tan^2\theta)}\, d\theta$$

$$\Rightarrow \frac{\partial I}{\partial a} = \frac{2a}{a^2-b^2}\int_0^{\frac{\pi}{2}} 1 - \frac{b^2\sec^2\theta}{a^2+b^2\tan^2\theta}\, d\theta$$

$$\Rightarrow \frac{\partial I}{\partial a} = \frac{2a}{a^2-b^2}\left(\frac{\pi}{2}\right) - \frac{2ab^2}{a^2-b^2}\int_0^{\frac{\pi}{2}} \frac{\sec^2\theta}{a^2+b^2\tan^2\theta}\, d\theta$$

SUBSTITUTION OR RECOGNITION

$u=\tan\theta$

$\frac{du}{d\theta}=\sec^2\theta$

$d\theta = \frac{du}{\sec^2\theta}$

$\theta=0 \longmapsto u=0$

$\theta=\frac{\pi}{2} \longmapsto u=\infty$

- APPLYING THE SUBSTITUTION & TIDYING UP

$$\Rightarrow \frac{\partial I}{\partial a} = \frac{\pi a}{a^2-b^2} - \frac{2ab^2}{a^2-b^2}\int_0^{\infty} \frac{\cancel{\sec^2\theta}}{a^2+b^2u^2} \times \frac{du}{\cancel{\sec^2\theta}}$$

$$\Rightarrow \frac{\partial I}{\partial a} = \frac{\pi a}{a^2-b^2} - \frac{2ab^2}{a^2-b^2}\int_0^{\infty} \frac{1}{a^2+b^2u^2}\, du$$

$$\Rightarrow \frac{\partial I}{\partial a} = \frac{\pi a}{a^2-b^2} - \frac{2a\cancel{b^2}}{a^2-b^2}\times\frac{1}{\cancel{b^2}}\int_0^{\infty} \frac{1}{u^2+\left(\frac{a}{b}\right)^2}\, du$$

$$\Rightarrow \frac{\partial I}{\partial a} = \frac{\pi a}{a^2-b^2} - \frac{2a}{a^2-b^2}\times\frac{1}{\frac{a}{b}}\left[\arctan\left(\frac{u}{\frac{a}{b}}\right)\right]_0^{\infty}$$

$$\Rightarrow \frac{\partial I}{\partial a} = \frac{\pi a}{a^2-b^2} - \frac{2b}{a^2-b^2}\left[\frac{\pi}{2}-0\right]$$

$$\Rightarrow \frac{\partial I}{\partial a} = \frac{\pi a}{a^2-b^2} - \frac{\pi b}{a^2-b^2} = \frac{\pi(a-b)}{a^2-b^2}$$

$$\Rightarrow \frac{\partial I}{\partial a} = \frac{\pi\cancel{(a-b)}}{\cancel{(a-b)}(a+b)} \quad (a\neq b)$$

$$\Rightarrow \boxed{\frac{\partial I}{\partial a} = \frac{\pi}{a+b}}$$

- INTEGRATING WITH RESPECT TO a

$$\Rightarrow I = \int \frac{\pi}{a+b}\, da$$

$$\Rightarrow I = \pi\ln(a+b) + C$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \ln\left(a^2\cos^2\theta+b^2\sin^2\theta\right) d\theta = C + \pi\ln(a+b)$$

- TO EVALUATE THE CONSTANT LET a (PARAMETER) $= b$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \ln\left(b^2\cos^2\theta+b\sin^2\theta\right) d\theta = C+\pi\ln(2b)$$

$$\Rightarrow \int_0^{\pi/2} \ln b^2\, d\theta = C+\pi\ln(2b)$$

$$\Rightarrow 2\ln b \times \frac{\pi}{2} = C+\pi\ln(2b)$$

$$\Rightarrow \pi\ln b = C+\pi\ln 2b$$

$$\Rightarrow \boxed{C = -\pi\ln b}$$

- FINALLY PUTTING EVERYTHING TOGETHER

$$\int_0^{\frac{\pi}{2}} \ln\left(a^2\cos^2\theta+b^2\sin^2\theta\right) d\theta = \pi\ln(a+b) - \pi\ln 2$$

$$\int_0^{\frac{\pi}{2}} \ln\left(a^2\cos^2\theta+b^2\sin^2\theta\right) d\theta = \pi\ln\left(\frac{a+b}{2}\right)$$

AS REQUIRED

Question 43

It is given that

$$y = \arcsin\left[\frac{\alpha + \cos x}{1 + \alpha \cos x}\right],$$

where α is a constant.

a) Show that

$$\frac{dy}{dx} = -\frac{\sqrt{1-\alpha^2}}{1 + \alpha \cos x}.$$

The integral function $I(\alpha, x)$ is defined as

$$I(\alpha) = \int_0^{\pi} \ln(1 + \alpha \cos x) \, dx .$$

b) By differentiating both sides of the above relationship with respect to α, show further that

$$I(1) = -\pi \ln 2 .$$

$\boxed{\text{proof}}$

[solution overleaf]

a)

$$\frac{d}{dx}\left[\arcsin\left[\frac{\alpha+\cos x}{1+\alpha\cos x}\right]\right] = \frac{1}{\sqrt{1-\left(\frac{\alpha+\cos x}{1+\alpha\cos x}\right)^2}} \times \frac{d}{dx}\left[\frac{\alpha+\cos x}{1+\alpha\cos x}\right]$$

$$= \frac{1}{\sqrt{\frac{(1+\alpha\cos x)^2-(\alpha+\cos x)^2}{(1+\alpha\cos x)^2}}} \times \frac{(1+\alpha\cos x)(-\sin x)-(\alpha+\cos x)(-\alpha\sin x)}{(1+\alpha\cos x)^2}$$

$$= \frac{1}{\frac{\sqrt{1+\cancel{2\alpha\cos x}+\alpha^2\cos^2 x-\alpha^2-\cancel{2\alpha\cos x}-\cos^2 x}}{1+\alpha\cos x}} \times \frac{-\sin x-\cancel{\alpha\cos x\sin x}+\alpha^2\sin x+\cancel{\alpha\sin x\cos x}}{(1+\alpha\cos x)^2}$$

$$= \frac{\cancel{1+\alpha\cos x}}{\sqrt{1-\cos^2 x-\alpha^2+\alpha^2\cos^2 x}} \times \frac{-\sin x(1-\alpha^2)}{(1+\alpha\cos x)^{\cancel{2}}}$$

$$= \frac{1}{\sqrt{(1-\cos^2 x)-\alpha^2(1-\cos^2 x)}} \times \frac{-\sin x(1-\alpha^2)}{1+\alpha\cos x}$$

$$= \frac{1}{\sqrt{(1-\cos^2 x)(1-\alpha^2)}} \times \frac{-\sin x(1-\alpha^2)}{1+\alpha\cos x}$$

$$= \frac{-\cancel{\sin x}(1-\alpha^2)}{\cancel{\sin x}\sqrt{1-\alpha^2}} \times \frac{1}{1+\alpha\cos x}$$

$$= -\frac{\sqrt{1-\alpha^2}}{1+\alpha\cos x}$$

b)

$$\boxed{I = \int_0^{\pi} \ln(1+\cos x)\, dx}$$

- INTRODUCE A PARAMETER α

$$\Rightarrow I(\alpha, x) = \int_0^{\pi} \ln(1+\alpha\cos x)\, dx$$

$$\Rightarrow \frac{\partial I}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left[\int_0^{\pi} \ln(1+\alpha\cos x)\, dx\right] = \int_0^{\pi} \frac{\partial}{\partial \alpha}\left[\ln(1+\alpha\cos x)\right] dx$$

$$\Rightarrow \frac{\partial I}{\partial \alpha} = \int_0^{\pi} \frac{\cos x}{1+\alpha\cos x}\, dx$$

$$\Rightarrow \frac{\partial I}{\partial \alpha} = \frac{1}{\alpha}\int_0^{\pi} \frac{\alpha\cos x}{1+\alpha\cos x}\, dx$$

$$\Rightarrow \frac{\partial I}{\partial \alpha} = \frac{1}{\alpha}\int_0^{\pi} \frac{(1+\alpha\cos x)-1}{1+\alpha\cos x}\, dx = \frac{1}{\alpha}\int_0^{\pi} 1 - \frac{1}{1+\alpha\cos x}\, dx$$

FROM PART (a)

$$\boxed{\frac{\partial}{\partial x}\left[\arcsin\left(\frac{\alpha+\cos x}{1+\alpha\cos x}\right)\right] = -\frac{\sqrt{1-\alpha^2}}{1+\alpha\cos x}}$$

$$\Rightarrow \frac{\partial I}{\partial \alpha} = \frac{1}{\alpha}\left[x + \frac{1}{\sqrt{1-\alpha^2}}\arcsin\left(\frac{\alpha+\cos x}{1+\alpha\cos x}\right)\right]_0^{\pi}$$

$$\Rightarrow \frac{\partial I}{\partial \alpha} = \frac{1}{\alpha}\left[\pi + \frac{1}{\sqrt{1-\alpha^2}}\arcsin\left(\frac{\alpha-1}{1-\alpha}\right) - \frac{1}{\sqrt{1-\alpha^2}}\arcsin\left(\frac{\alpha+1}{1+\alpha}\right)\right]$$

$$\Rightarrow \frac{\partial I}{\partial \alpha} = \frac{1}{\alpha}\left[\pi + \frac{1}{\sqrt{1-\alpha^2}}\left[\arcsin(-1) - \arcsin 1\right]\right]$$

$$\Rightarrow \frac{\partial I}{\partial \alpha} = \frac{1}{\alpha}\left[\pi + \frac{1}{\sqrt{1-\alpha^2}}\left[-\frac{\pi}{2}-\frac{\pi}{2}\right]\right]$$

$$\Rightarrow \frac{\partial I}{\partial \alpha} = \frac{1}{\alpha}\left[\pi - \frac{\pi}{\sqrt{1-\alpha^2}}\right]$$

$$\Rightarrow \frac{\partial I}{\partial \alpha} = \frac{\pi}{\alpha}\left[1 - \frac{1}{\sqrt{1-\alpha^2}}\right]$$

- INTEGRATE WITH RESPECT TO α

$$\Rightarrow I = \pi\int \frac{1}{\alpha} - \frac{1}{\alpha\sqrt{1-\alpha^2}}\, d\alpha$$

$$\Rightarrow I = \pi\left[\ln\alpha\right] - \pi\int \frac{1}{\alpha\sqrt{1-\alpha^2}}\, d\alpha \leftarrow$$

SUBSTITUTION

$$u = \sqrt{1-\alpha^2}$$
$$u^2 = 1-\alpha^2$$
$$2u\,du = -2\alpha\, d\alpha$$
$$d\alpha = -\frac{u}{\alpha}\, du$$

$$\Rightarrow I = \pi\ln\alpha - \pi\int \frac{1}{\alpha u}\left(-\frac{u}{\alpha}\, du\right)$$

$$\Rightarrow I = \pi\ln\alpha + \pi\int \frac{1}{\alpha^2}\, du$$

$$\Rightarrow I = \pi\ln\alpha + \pi\int \frac{1}{1-u^2}\, du$$

$$\Rightarrow I = \pi\ln\alpha + \pi\int \frac{1}{(1-u)(1+u)}\, du \leftarrow$$

STANDARD PARTIAL FRACTIONS

$$\Rightarrow I = \pi\ln\alpha + \pi\int \frac{\frac{1}{2}}{1+u} + \frac{\frac{1}{2}}{1-u}\, du$$

$$\Rightarrow I = \pi\ln\alpha + \frac{\pi}{2}\ln\left|\frac{1+u}{1-u}\right| + C$$

$$\Rightarrow I = \pi\ln\alpha + \frac{\pi}{2}\ln\left|\frac{(1+u)^2}{1-u^2}\right| + C$$

$$\Rightarrow I = \pi\ln\alpha + \frac{\pi}{2}\ln\left|\frac{(1+u)^2}{\alpha^2}\right| + C$$

$$\Rightarrow I = \pi\ln\alpha + \frac{\pi}{2}\ln\left|\frac{1+u}{\alpha}\right|^2 + C$$

$$\Rightarrow I = \pi\ln\alpha + \pi\ln\left|\frac{1+u}{\alpha}\right| + C$$

$$\Rightarrow I = \pi\ln\alpha + \pi\ln\left|\frac{1+\sqrt{1-\alpha^2}}{\alpha}\right| + C$$

$$\Rightarrow I = \pi\ln\left[1+\sqrt{1-\alpha^2}\right] + C$$

- NEED TO EVALUATE THE CONSTANT C

$$\int_0^{\pi} \ln(1+\alpha\cos x)\, dx = \pi\ln\left[1+\sqrt{1-\alpha^2}\right] + C$$

LET $\alpha = 0$

$$\cancel{\int_0^{\pi} \ln 1\, dx} = \pi\ln 2 + C$$

$$\boxed{C = -\pi\ln 2}$$

$$\Rightarrow \int_0^{\pi} \ln(1+\alpha\cos x)\, dx = \pi\ln\left[1+\sqrt{1-\alpha^2}\right]$$

$$\Rightarrow \int_0^{\pi} \ln(1+\alpha\cos x)\, dx = \pi\ln\left[\frac{1+\sqrt{1-\alpha^2}}{2}\right]$$

$$\therefore \int_0^{\pi} \ln(1+\cos x)\, dx = \pi\ln\frac{1}{2} = -\pi\ln 2$$

Question 44

By carrying out suitable differentiations on I under the integral sign, show that

$$I = \int_0^{\infty} \operatorname{arccot}(2x)\operatorname{arccot}(4x)\ dx \ = \ \tfrac{1}{8}\pi \ln\left(\tfrac{27}{4}\right).$$

V, $\boxed{\text{proof}}$

$$\boxed{\int_0^{\infty} \operatorname{arccot}(2x)\operatorname{arccot}(4x)\ dx = \frac{\pi}{8}\ln\frac{27}{4}}$$

- INTRODUCE PARAMETERS a & b IN THE INTEGRAND

$$I = \int_0^{\infty} \operatorname{arccot}(ax)\operatorname{arccot}(bx)\ dx$$

$$\boxed{\begin{aligned} \text{NOW}\quad \frac{d}{da}\left[\operatorname{arccot}(ax)\right] &= \frac{d}{da}\left[\arctan\frac{1}{ax}\right] \\ &= \frac{1}{1+\left(\frac{1}{ax}\right)^2}\times -\frac{1}{xa^2} = -\frac{1}{a^2x}\times\frac{1}{1+\frac{1}{a^2x^2}} \\ &= -\frac{1}{a^2x}\times\frac{a^2x^2}{a^2x^2+1} = -\frac{x}{a^2x^2+1} \end{aligned}}$$

$$\Rightarrow \frac{\partial I}{\partial a} = \int_0^{\infty}\left(-\frac{x}{a^2x^2+1}\right)\operatorname{arccot} bx\ dx$$

$$\Rightarrow \frac{\partial^2 I}{\partial a\partial b} = \int_0^{\infty}\left(-\frac{x}{a^2x^2+1}\right)\left(-\frac{x}{b^2x^2+1}\right)\ dx$$

$$\Rightarrow \frac{\partial^2 I}{\partial a\partial b} = \int_0^{\infty}\frac{x^2}{(a^2x^2+1)(b^2x^2+1)}\ dx$$

- THE "DENOMINATORS" OF THE INTEGRAND WRITTEN AS PARTIAL FRACTIONS ARE IRREDUCIBLE — HOWEVER THE CURRENT NUMERATOR SUGGESTS THAT THE INDIVIDUAL NUMERATORS OF THE PARTIAL FRACTIONS ARE AT MOST CONSTANTS — IN OTHER WORDS

$$\Rightarrow \frac{x^2}{(a^2x^2+1)(b^2x^2+1)} \equiv \frac{A}{a^2x^2+1}+\frac{B}{b^2x^2+1}$$

$$\Rightarrow \boxed{x^2 \equiv A(b^2x^2+1)+B(a^2x^2+1)}$$

$$\Rightarrow x^2 \equiv (Ab^2+Ba^2)x^2+(A+B)$$

$A+B=0$

$\boxed{A=-B} \quad \Rightarrow \quad -Bb^2+Ba^2=1$

$B(a^2-b^2)=1$

$\boxed{B=\frac{1}{a^2-b^2}}$ & $\boxed{A=\frac{1}{b^2-a^2}}$

- HENCE THE INTEGRAL BECOMES

$$\Rightarrow \frac{\partial^2 I}{\partial a\partial b} = \int_0^{\infty}\frac{\frac{1}{b^2-a^2}}{a^2x^2+1}+\frac{\frac{1}{a^2-b^2}}{b^2x^2+1}\ dx$$

$$\Rightarrow \frac{\partial^2 I}{\partial a\partial b} = \frac{1}{b^2-a^2}\int_0^{\infty}\frac{1}{a^2x^2+1}-\frac{1}{b^2x^2+1}\ dx$$

$$\Rightarrow \frac{\partial^2 I}{\partial a\partial b} = \frac{1}{b^2-a^2}\int_0^{\infty}\frac{1}{a^2}\left(\frac{1}{x^2+\frac{1}{a^2}}\right)-\frac{1}{b^2}\left(\frac{1}{x^2+\frac{1}{b^2}}\right)\ dx$$

$$\Rightarrow \frac{\partial^2 I}{\partial a\partial b} = \frac{1}{b^2-a^2}\left[\frac{1}{a^2}\times\frac{1}{\frac{1}{a}}\times\arctan\frac{x}{\frac{1}{a}}-\frac{1}{b^2}\times\frac{1}{\frac{1}{b}}\times\arctan\frac{x}{\frac{1}{b}}\right]_0^{\infty}$$

$$\Rightarrow \frac{\partial^2 I}{\partial a\partial b} = \frac{1}{b^2-a^2}\left[\frac{1}{a}\arctan(ax)-\frac{1}{b}\arctan(bx)\right]_0^{\infty}$$

$$\Rightarrow \frac{\partial^2 I}{\partial a\partial b} = \frac{1}{b^2-a^2}\left[\frac{\pi}{2a}-\frac{\pi}{2b}\right] = \frac{\pi}{2(b^2-a^2)}\left[\frac{1}{a}-\frac{1}{b}\right]$$

$$\Rightarrow \frac{\partial^2 I}{\partial a\partial b} = \frac{\pi}{2\cancel{(b-a)}(b+a)}\ \frac{\cancel{b-a}}{ab} = \frac{\pi}{2ab(a+b)}$$

- REGARDLESS OF INTEGRATION W.R.T a <u>OR</u> b WE NEED PARTIAL FRACTIONS

- ATTEMPTING INTEGRATION WRT b FIRST

$$\Rightarrow \frac{\partial^2 I}{\partial a\partial b} = \frac{\pi}{2a}\left[\frac{1}{b(b+a)}\right]$$

by cover up

$$\Rightarrow \frac{\partial^2 I}{\partial a\partial b} = \frac{\pi}{2a}\left[\frac{\frac{1}{a}}{b}-\frac{\frac{1}{a}}{b+a}\right]$$

$$\Rightarrow \frac{\partial^2 I}{\partial a\partial b} = \frac{\pi}{2a^2}\left[\frac{1}{b}-\frac{1}{b+a}\right]$$

$$\Rightarrow \frac{\partial I}{\partial a} = \frac{\pi}{2a^2}\left[\ln b-\ln(b+a)\right]+C$$

$$\Rightarrow \frac{\partial I}{\partial a} = \frac{\pi}{2a^2}\left[\ln\frac{b}{b+a}\right]+C$$

- TO EVALUATE THE CONSTANT WE PROCEED AS FOLLOWS

$$-\int_0^{\infty}\left(\frac{x}{a^2x^2+1}\right)\operatorname{arccot}(bx) = \frac{\pi}{2a^2}\ln\left(\frac{b}{b+a}\right)+C$$

AS $b\to\infty$ FOR A FIXED VALUE OF a

$$0 = \cancel{\frac{\pi}{2a^2}\ln 1}+C \quad\Rightarrow\quad \boxed{C=0}$$

- HENCE WE OBTAIN

$$\frac{\partial I}{\partial a} = \frac{\pi}{2a^2}\ln\left(\frac{b}{a+b}\right) \text{, WHERE } b \text{ IS NOW FIXED}$$

$$\frac{\partial I}{\partial a} = \frac{\pi}{2a^2}\left[\ln b-\ln(a+b)\right]$$

$$\frac{\partial I}{\partial a} = \frac{\pi}{2}\left[\frac{\ln b}{a^2}-\frac{1}{a^2}\ln(a+b)\right]$$

- WE NEED INTEGRATION BY PARTS FOR $\int\frac{1}{a^2}\ln(a+b)\ da$

$\ln(a+b)$	$\frac{1}{a+b}$
$-\frac{1}{a}$	$\frac{1}{a^2}$

$$\int\frac{1}{a^2}\ln(a+b)\ da = -\frac{1}{a}\ln(a+b)+\int\frac{1}{a(a+b)}\ da$$

$$= -\frac{1}{a}\ln(a+b)+\int\frac{\frac{1}{b}}{a}-\frac{\frac{1}{b}}{a+b}\ da$$

(BY COVER UP)

$$= -\frac{1}{a}\ln(a+b)+\frac{1}{b}\int\frac{1}{a}-\frac{1}{a+b}\ da$$

$$= -\frac{1}{a}\ln(a+b)+\frac{1}{b}\ln\left(\frac{a}{a+b}\right)+\text{CONSTANT}$$

- RETURNING TO THE "MAIN LINE" & INTEGRATING W.R.T a

$$\Rightarrow I = \frac{\pi}{2}\left[-\frac{\ln b}{a}-\left[-\frac{1}{a}\ln(a+b)+\frac{1}{b}\ln\left(\frac{a}{a+b}\right)\right]\right]+k$$

$$\Rightarrow I = \frac{\pi}{2}\left[\frac{1}{a}\ln\left(\frac{a+b}{b}\right)-\frac{1}{b}\ln\left(\frac{a}{a+b}\right)\right]+k$$

- TO EVALUATE THE CONSTANT FOR A FIXED b LET $a\to\infty$

$$\int_0^{\infty}\operatorname{arccot}(ax)\operatorname{arccot}(bx)\ dx = \frac{\pi}{2}\left[\frac{1}{a}\ln\left(\frac{a+b}{b}\right)-\frac{1}{b}\ln\left(\frac{a}{a+b}\right)\right]+k$$

$$0 = \frac{\pi}{2}\left[0+\frac{1}{b}\ln 1\right]+k$$

$\frac{1}{a}\to 0$ "FASTER" THAN $\ln a\to\infty$

$\therefore\ k=0$

- FINALLY WE HAVE

$$\int_0^{\infty}\operatorname{arccot}(ax)\operatorname{arccot}(bx)\ dx = \frac{\pi}{2}\left[\frac{1}{a}\ln\frac{a+b}{b}-\frac{1}{b}\ln\frac{a}{a+b}\right]$$

$$\int_0^{\infty}\operatorname{arccot}(2x)\operatorname{arccot}(4x)\ dx = \frac{\pi}{2}\left[\frac{1}{2}\ln\frac{6}{4}-\frac{1}{4}\ln\frac{2}{6}\right]$$

$$= \frac{\pi}{2}\left[\frac{1}{2}\ln\frac{3}{2}-\frac{1}{4}\ln\frac{1}{3}\right]$$

$$= \frac{\pi}{8}\left[2\ln\frac{3}{2}-\ln\frac{1}{3}\right]$$

$$= \frac{\pi}{8}\left[\ln\frac{9}{4}-\ln\frac{1}{3}\right]$$

$$= \frac{\pi}{8}\ln\frac{27}{4}$$

Question 45

$$A(t) \equiv \left[\int_0^t e^{-x^2} \, dx \right]^2 .$$

By differentiating both sides of the above equation with respect to t, followed by the substitution $x = ty$, show that

$$\int_0^\infty e^{-x^2} \, dx = \tfrac{1}{2}\sqrt{\pi} .$$

proof

CONSIDER $A(t) = \left[\int_0^t e^{-x^2}\,dx\right]^2$

DIFFERENTIATE WITH RESPECT TO t

$$\Rightarrow \frac{dA}{dt} = 2\left[\int_0^t e^{-x^2}\,dx\right]^1 \times \frac{\partial}{\partial t}\left[\int_0^t e^{-x^2}\,dx\right]$$

$$\Rightarrow \frac{dA}{dt} = 2\int_0^t e^{-x^2}\,dx \times e^{-t^2}$$

$$\boxed{\frac{\partial}{\partial t}\left[\int_0^t f(x)\,dx\right] = f(t)}$$

$$\Rightarrow \frac{dA}{dt} = 2e^{-t^2}\int_0^t e^{-x^2}\,dx$$

LET $x = ty$ t IS A CONSTANT (PARAMETER) AS FAR AS THIS SUBSTITUTION IS CONCERNED

$dx = t\,dy$

$x = 0,\ y = 0$

$x = t,\ y = 1$

$$\Rightarrow \frac{dA}{dt} = 2e^{-t^2}\int_0^1 e^{-t^2y^2}\,t\,dy$$

$$\Rightarrow \frac{dA}{dt} = \int_0^1 2te^{-t^2y^2}e^{-t^2}\,dy = \int_0^1 2te^{-t^2(1+y^2)}\,dy$$

$$\Rightarrow \frac{dA}{dt} = \int_0^1 \frac{\partial}{\partial t}\left[-\frac{e^{-t^2(1+y^2)}}{1+y^2}\right]dy$$

$$\Rightarrow \frac{dA}{dt} = -\frac{\partial}{\partial t}\int_0^1 \frac{e^{-t^2(1+y^2)}}{1+y^2}\,dy$$

$$\Rightarrow A(t) = -\int_0^1 \frac{e^{-t^2(1+y^2)}}{1+y^2}\,dy + k$$

$$\Rightarrow \boxed{\left[\int_0^t e^{-x^2}\,dx\right]^2 = -\int_0^1 \frac{e^{-t^2(1+y^2)}}{1+y^2}\,dy + k}$$

NOW LET $t \to 0^+$

LHS $= 0$ RHS $= k - \int_0^1 \frac{1}{1+y^2}\,dy$

$0 = k - \left[\arctan y\right]_0^1$

$0 = k - \frac{\pi}{4}$

$\boxed{k = \frac{\pi}{4}}$

$$\Rightarrow \left[\int_0^t e^{-x^2}\,dx\right]^2 = -\int_0^1 \frac{e^{-t^2(1+y)}}{1+y^2}\,dy + \frac{\pi}{4}$$

NOW LET $t \to \infty$

$$\Rightarrow \left[\int_0^\infty e^{-x^2}\,dx\right]^2 = 0 + \frac{\pi}{4}$$

$$\Rightarrow \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

Question 46

$$I(t) \equiv \left[\int_0^t e^{-i x^2} \, dx \right]^2 .$$

By differentiating both sides of the above equation with respect to t, followed by the substitution $x = ty$, show that

$$\int_0^\infty \cos\left(x^2\right) \, dx = \int_0^\infty \sin\left(x^2\right) \, dx = \frac{1}{4}\sqrt{2\pi} .$$

proof

CONSIDER

$$\cos(x^2) - i\sin(x^2) = e^{-ix^2}$$

DEFINE A "FUNCTION/INTEGRAL" BY

$$\boxed{I(t) = \left[\int_0^t e^{-ix^2}\,dx\right]^2}$$

DIFFERENTIATE W.R.T.

$$\Rightarrow \frac{\partial I}{\partial t} = 2\left[\int_0^t e^{-ix^2}\,dx\right] \times \frac{\partial}{\partial t}\left[\int_0^t e^{-ix^2}\,dx\right]$$

$$\boxed{\frac{\partial}{\partial t}\left[\int_0^t f(x)\,dx\right] = f(t)}$$

SIMPLIFYING

$$\Rightarrow \frac{\partial I}{\partial t} = 2\int_0^t e^{-ix^2}\,dx \times e^{-it^2}$$

$$\Rightarrow \frac{\partial I}{\partial t} = 2e^{-it^2}\int_0^t e^{-ix^2}\,dx$$

NEXT A SUBSTITUTION LET $\boxed{x = ty}$, WHERE t IS A CONSTANT (PARAMETER)

$$dx = t\,dy$$

$$x = 0 \mapsto y = 0$$

$$x = t \mapsto y = 1$$

$$\Rightarrow \frac{\partial I}{\partial t} = 2e^{-it^2}\int_0^1 e^{-it^2y^2}\,t\,dy$$

$$\Rightarrow \frac{\partial I}{\partial t} = \int_0^1 2te^{-it^2}e^{-it^2y^2}\,dy$$

$$\Rightarrow \frac{\partial I}{\partial t} = \int_0^1 2te^{-it^2(1+y^2)}\,dy$$

$$\Rightarrow \frac{\partial I}{\partial t} = \int_0^1 \frac{\partial}{\partial t}\left[-\frac{e^{-it^2(1+y^2)}}{i(1+y^2)}\right]dy$$

$$\Rightarrow \frac{\partial I}{\partial t} = \frac{\partial}{\partial t}\int_0^1 -\frac{e^{-it^2(1+y^2)}}{i(1+y^2)}\,dy$$

$$\Rightarrow I = -\int_0^1 \frac{e^{-it^2(1+y^2)}}{i(1+y^2)} + C$$

$$\Rightarrow \left[\int_0^t e^{-ix^2}\,dx\right]^2 = -\int_0^1 \frac{e^{-it^2(1+y^2)}}{i(1+y^2)}\,dy + C$$

AS $t \to 0^+$
- LHS $\to 0$
- RHS $\to -\int_0^1 \frac{1}{i(1+y^2)}\,dy + C$

RHS $\to -\frac{1}{i}\int_0^1 \frac{1}{1+y^2}\,dy + C$

RHS $\to i\left[\arctan y\right]_0^1 + C$

RHS $\to i\left[\frac{\pi}{4} - 0\right] + C$

RHS $\to C + \frac{\pi}{4}i$

THUS $0 = C + \frac{\pi}{4}i$

$$\boxed{C = -\frac{\pi}{4}i}$$

SO AS $t \to \infty$

$$\Rightarrow \left[\int_0^\infty e^{-ix^2}\,dx\right]^2 = 0 - \frac{\pi}{4}i$$

$$\Rightarrow \int_0^\infty e^{-ix^2}\,dx = \pm\sqrt{-\frac{\pi}{4}i}$$

$$\Rightarrow \int_0^\infty \cos x^2 - i\sin x^2\,dx = \pm\sqrt{-\frac{\pi}{4}i}$$

NOW $-i = 1e^{i\frac{3\pi}{2}}$

$$(-i)^{\frac{1}{2}} = \left(e^{i\frac{3\pi}{2}}\right)^{\frac{1}{2}} = e^{i\frac{3\pi}{4}} = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$$

(& OF COURSE THE MINUS $-\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}$)

$$\text{SO } \pm\sqrt{-\frac{\pi}{4}i} = \pm\sqrt{\frac{\pi}{4}}\sqrt{-i} = \pm\sqrt{\frac{\pi}{4}}\left[-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right]$$

$$= \pm\left(-\frac{\sqrt{2\pi}}{4} + i\frac{\sqrt{2\pi}}{4}\right)$$

$$\Rightarrow \int_0^\infty \cos x^2 - i\sin x^2\,dx = \cancel{-\frac{\sqrt{2\pi}}{4} + i\frac{\sqrt{2\pi}}{4}}$$

$$= \frac{\sqrt{2\pi}}{4} - i\frac{\sqrt{2\pi}}{4}$$

SEPARATING REAL & IMAGINARY TO OBTAIN

$$\therefore \int_0^\infty \sin(x^2)\,dx = \frac{\sqrt{2\pi}}{4}$$

$$\int_0^\infty \cos(x^2)\,dx = \frac{\sqrt{2\pi}}{4}$$

INTEGRATION UNDER THE INTEGRAL SIGN

Question 1

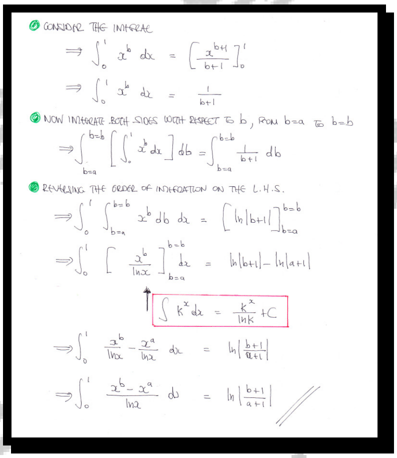
By integrating both sides of an appropriate integral relationship, with suitable limits, show that

$$\int_0^1 \frac{x^b - x^a}{\ln x}\ dx = \ln\left[\frac{b+1}{a+1}\right],$$

where $b > a > 0$.

You may assume that for $k > 0$, $\displaystyle\int k^x\ dx = \frac{k^x}{\ln k} + \text{constant}$.

$\boxed{\text{proof}}$



CONSIDER THE INTEGRAL

$$\Rightarrow \int_0^1 x^b\ dx = \left[\frac{x^{b+1}}{b+1}\right]_0^1$$

$$\Rightarrow \int_0^1 x^b\ dx = \frac{1}{b+1}$$

NOW INTEGRATE BOTH SIDES WITH RESPECT TO b, FROM $b=a$ TO $b=b$

$$\Rightarrow \int_{b=a}^{b=b} \left[\int_0^1 x^b\,dx\right] db = \int_{b=a}^{b=b} \frac{1}{b+1}\ db$$

REVERSING THE ORDER OF INTEGRATION ON THE L.H.S.

$$\Rightarrow \int_0^1 \int_{b=a}^{b=b} x^b\,db\ dx = \Big[\ln|b+1|\Big]_{b=a}^{b=b}$$

$$\Rightarrow \int_0^1 \left[\frac{x^b}{\ln x}\right]_{b=a}^{b=b} dx = \ln|b+1| - \ln|a+1|$$

$$\boxed{\int k^x\,dx = \frac{k^x}{\ln k} + C}$$

$$\Rightarrow \int_0^1 \frac{x^b}{\ln x} - \frac{x^a}{\ln x}\ dx = \ln\left|\frac{b+1}{a+1}\right|$$

$$\Rightarrow \int_0^1 \frac{x^b - x^a}{\ln x}\ dx = \ln\left|\frac{b+1}{a+1}\right|$$

Question 2

By integrating both sides of an appropriate integral relationship, with suitable limits, show that

$$\int_0^{\infty} \frac{e^{-ax^2} - e^{-bx^2}}{x^2} \, dx = \sqrt{\pi b} - \sqrt{\pi a},$$

where $b > a > 0$.

You may assume that $\int_0^{\infty} e^{-t^2} \, dt = \frac{1}{2}\sqrt{\pi}$.

$\boxed{\text{proof}}$

- FIRSTLY CONSIDER THE INTEGRAL

$$\int_0^{\infty} e^{-ax^2} dx = \dots \text{by substitution} \dots$$

$$= \int_0^{\infty} e^{-t^2} \frac{dt}{\sqrt{a}} = \frac{1}{\sqrt{a}} \int_0^{\infty} e^{-t^2} dt = \frac{1}{\sqrt{a}} \frac{\sqrt{\pi}}{2}$$

SUBSTITUTION
$t = \sqrt{a}\,x$
$t^2 = ax^2$
$dt = \sqrt{a}\,dx$
LIMITS UNCHANGED

- HENCE

$$\int_0^{\infty} e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}} = \frac{1}{2}\sqrt{\pi}\,a^{-\frac{1}{2}}$$

- INTEGRATE BOTH SIDES WITH RESPECT TO a, FROM $a=b$ TO $a=a$

$$\Longrightarrow \int_{a=b}^{a=a} \left[\int_0^{\infty} e^{-ax^2} dx \right] da = \int_{a=b}^{a=a} \frac{1}{2}\sqrt{\pi}\,a^{-\frac{1}{2}} da$$

- REVERSING THE ORDER OF INTEGRATION

$$\Longrightarrow \int_0^{\infty} \left[\int_{a=b}^{a=a} e^{-ax^2} da \right] dx = \left[\sqrt{\pi}\,a^{\frac{1}{2}} \right]_{a=b}^{a=a}$$

$$\Longrightarrow \int_0^{\infty} \left[-\frac{1}{x^2} e^{-ax^2} \right]_{a=b}^{a=a} dx = \sqrt{\pi a} - \sqrt{\pi b}$$

$$\Longrightarrow \int_0^{\infty} \left[\frac{1}{x^2} e^{-ax^2} \right]_{a=a}^{a=b} dx = \sqrt{\pi a} - \sqrt{\pi b}$$

$$\Longrightarrow \int_0^{\infty} \frac{1}{x^2} e^{-bx^2} - \frac{1}{x^2} e^{-ax^2} dx = \sqrt{\pi a} - \sqrt{\pi b}$$

$$\Longrightarrow \int_0^{\infty} \frac{e^{-bx^2} - e^{-ax^2}}{x^2} dx = \sqrt{\pi a} - \sqrt{\pi b}$$

$$\Longrightarrow \int_0^{\infty} \frac{e^{-ax^2} - e^{-bx^2}}{x^2} dx = \sqrt{\pi b} - \sqrt{\pi a}$$

Question 3

The integral I is defined as

$$I = \int_0^{\infty} e^{kx} \sin x \ dx .$$

where k is a constant.

a) Use a suitable method to show that

$$I = \frac{1}{k^2+1} .$$

b) By integrating both sides of an appropriate integral relationship with respect to k, with suitable limits, show further that

$$\int_0^{\infty} \frac{e^{-2x} \sin x}{x} \ dx \ = \ \operatorname{arccot} 2 .$$

You may assume that $\displaystyle\int_0^{\infty} \frac{\sin x}{x} \ dx = \frac{\pi}{2}$.

$\boxed{\text{proof}}$

a) CONSIDER $I = \int_0^{\infty} e^{-kx} \sin x \, dx$

$$I = \operatorname{Im} \int_0^{\infty} e^{-kx} e^{ix} \, dx = \operatorname{Im} \int_0^{\infty} e^{x(-k+i)} \, dx = \operatorname{Im} \left[\frac{1}{-k+i} e^{x(-k+i)} \right]_0^{\infty}$$

$$= \operatorname{Im} \left[\frac{1}{-k+i} e^{-kx} e^{ix} \right]_{x=0}^{\infty} = \operatorname{Im} \left[0 - \frac{1}{-k+i} \right] = \operatorname{Im} \left[\frac{1}{k-i} \right]$$

$$= \operatorname{Im} \left[\frac{k+i}{k^2+1} \right] = \frac{1}{k^2+1}$$

b) NOW INTEGRATE BOTH SIDES OF THE ABOVE EQUATION W.R.T k, FROM 0 TO "k"

$$\int_0^{\infty} e^{-kx} \sin x \, dx = \frac{1}{k^2+1}$$

$$\int_0^{k} \left[\int_0^{\infty} e^{-kx} \sin x \, dx \right] dk = \int_0^{k} \frac{1}{k^2+1} \, dk$$

REVERSING THE ORDER OF INTEGRATION

$$\int_{x=0}^{\infty} \int_{k=0}^{k} e^{-kx} \sin x \, dk \, dx = \left[\arctan k \right]_0^{k}$$

$$\int_0^{\infty} \left[-\frac{1}{x} e^{-kx} \sin x \right]_{k=0}^{k=k} dx = \arctan k - 0$$

$$\int_0^{\infty} -\frac{1}{x} e^{-kx} \sin x + \frac{1}{x} \sin x \, dx = \arctan k$$

$$-\int_0^{\infty} \frac{1}{x} e^{-kx} \sin x \, dx + \underbrace{\int_0^{\infty} \frac{\sin x}{x} \, dx}_{\text{GIVEN}} = \arctan k$$

LET $k = 2$

$$-\int_0^{\infty} \frac{e^{-2x} \sin x}{x} \, dx + \frac{\pi}{2} = \arctan 2$$

$$\int_0^{\infty} \frac{e^{-2x} \sin x}{x} \, dx = \frac{\pi}{2} - \arctan 2 = \operatorname{arccot} 2$$

Question 4

By integrating both sides of an appropriate integral relationship with respect to b, with suitable limits, show that

$$\int_0^{\infty} \frac{e^{-x}\sinh bx}{x}\, dx = \frac{1}{2}\ln\left[\frac{1+b}{1-b}\right].$$

proof

CONSIDER THE INTEGRAL

$$\boxed{\int_0^{\infty} e^{-x}\cosh bx \, dx \qquad |b|<1}$$

INTEGRATING IT BY PARTS OR EXPONENTIALS

$$\int_0^{\infty} e^{-x}\left(\tfrac{1}{2}e^{bx}+\tfrac{1}{2}e^{-bx}\right) dx = \int_0^{\infty} \tfrac{1}{2}e^{(b-1)x} + \tfrac{1}{2}e^{-(b+1)x}\, dx$$

$$= \left[\frac{1}{2(b-1)}e^{(b-1)x} - \frac{1}{2(b+1)}e^{-(b+1)x}\right]_0^{\infty} = 0 - \left[\frac{1}{2(b-1)} - \frac{1}{2(b+1)}\right]$$

$$= -\frac{1}{2}\left[\frac{1}{b-1} - \frac{1}{b+1}\right] = -\frac{1}{2}\left[\frac{b+1-b+1}{b^2-1}\right] = -\frac{1}{2}\,\frac{2}{b^2-1} = \frac{1}{1-b^2}$$

NOW $\displaystyle\int_0^{\infty} e^{-x}\cosh bx \, dx = \frac{1}{1-b^2}$

INTEGRATE BOTH SIDES WITH RESPECT TO b FROM $b=0$ TO $b=$ "b"

$$\Rightarrow \int_0^{b}\left[\int_0^{\infty} e^{-x}\cosh bx \, dx\right] db = \int_0^{b} \frac{1}{1-b^2}\, db$$

REVERSING THE ORDER OF INTEGRATION GIVES

$$\Rightarrow \int_0^{\infty}\left[\int_0^{b} e^{-x}\cosh bx \, db\right] dx = \int_0^{b} \frac{\frac{1}{2}}{1+b} + \frac{\frac{1}{2}}{1-b}\, db$$

$$\Rightarrow \int_0^{\infty}\left[e^{-x}\times\frac{1}{x}\sinh bx\right]_{b=0}^{b} dx = \frac{1}{2}\Big[\ln|1+b| - \ln|1-b|\Big]_0^{b}$$

$$\Rightarrow \int_0^{\infty} \frac{e^{-x}}{x}\sinh bx - 0 \, dx = \frac{1}{2}\ln\left|\frac{1+b}{1-b}\right| - \cancel{\frac{1}{2}\ln 1}$$

$$\Rightarrow \int_0^{\infty} \frac{e^{-x}\sinh bx}{x}\, dx = \frac{1}{2}\ln\left|\frac{1+b}{1-b}\right|$$