FOURIER SERIES

The Fourier Theorem

If $f(x)$ is a piecewise continuous function on (α, β), then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right],$$

where $a_n = \frac{1}{L}\int_{\alpha}^{\beta} f(x)\cos\left(\frac{n\pi x}{L}\right)\, dx$

$$b_n = \frac{1}{L}\int_{\alpha}^{\beta} f(x)\sin\left(\frac{n\pi x}{L}\right)\, dx$$

$$L = \frac{\beta - \alpha}{2} = \text{half period}$$

Parseval's Identity

$$\frac{1}{L}\int_{\alpha}^{\beta}\left[f(x)\right]^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left[a_n^2 + b_n^2\right]$$

FOURIER SERIES EXPANSIONS

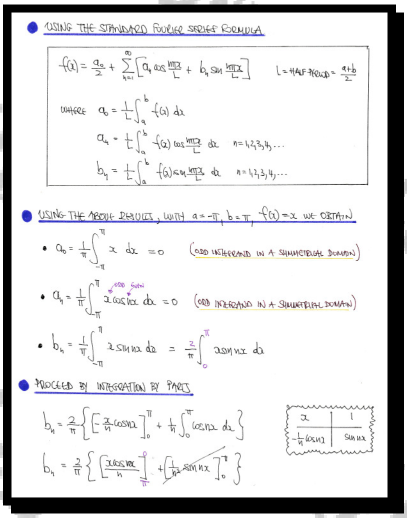
Question 1

$$f(x) = x\,,\; x \in \mathbb{R}\,,\; -\pi \le x \le \pi\,.$$

$$f(x) = f(x + 2\pi)\,.$$

Determine the Fourier series expansion of $f(x)$.

$$\boxed{}\,,\quad \boxed{f(x) = 2\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}\sin nx}{n}\right]}$$



USING THE STANDARD FOURIER SERIES FORMULA

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right] \qquad L = \text{HALF PERIOD} = \frac{a+b}{2}$$

WHERE $a_0 = \frac{1}{L}\int_a^b f(x)\,dx$

$a_n = \frac{1}{L}\int_a^b f(x)\cos\frac{n\pi x}{L}\,dx \quad n = 1,2,3,4,\ldots$

$b_n = \frac{1}{L}\int_a^b f(x)\sin\frac{n\pi x}{L}\,dx \quad n = 1,2,3,4,\ldots$

USING THE ABOVE RESULTS, WITH $a = -\pi$, $b = \pi$, $f(x) = x$ WE OBTAIN

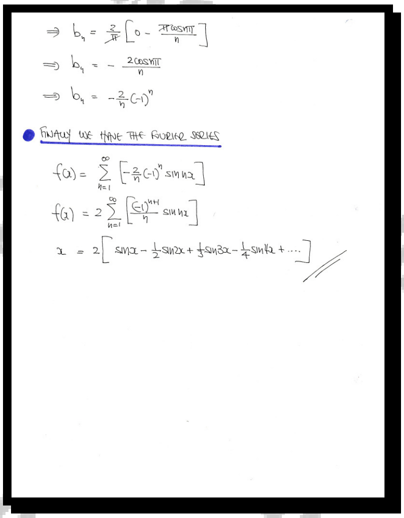
- $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = 0$ (ODD INTEGRAND IN A SYMMETRICAL DOMAIN)
- $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx = 0$ (ODD INTEGRAND IN A SYMMETRICAL DOMAIN) [ODD, EVEN]
- $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$

PROCEED BY INTEGRATION BY PARTS

$$b_n = \frac{2}{\pi}\left\{\left[-\frac{x}{n}\cos nx\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,dx\right\}$$

x	1
$-\frac{1}{n}\cos nx$	$\sin nx$

$$b_n = \frac{2}{\pi}\left\{\left[\frac{x\cos nx}{n}\right]_{\pi}^{0} + \left[\frac{1}{n^2}\sin nx\right]_0^{\pi}\right\}$$



$\Rightarrow b_n = \frac{2}{\pi}\left[0 - \frac{\pi\cos n\pi}{n}\right]$

$\Rightarrow b_n = -\frac{2\cos n\pi}{n}$

$\Rightarrow b_n = -\frac{2}{n}(-1)^n$

FINALLY WE HAVE THE FOURIER SERIES

$$f(x) = \sum_{n=1}^{\infty}\left[-\frac{2}{n}(-1)^n \sin nx\right]$$

$$f(x) = 2\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{n}\sin nx\right]$$

$$x = 2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \ldots\right]$$

Question 2

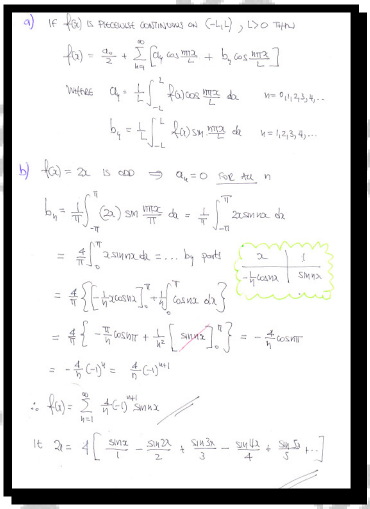
A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

a) State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.

b) Find the Fourier series of

$$f(x) = 2x, \; -\pi \le x \le \pi .$$

$$2x = \sum_{n=1}^{\infty} \left[\frac{4(-1)^{n+1}}{n} \sin(nx) \right]$$



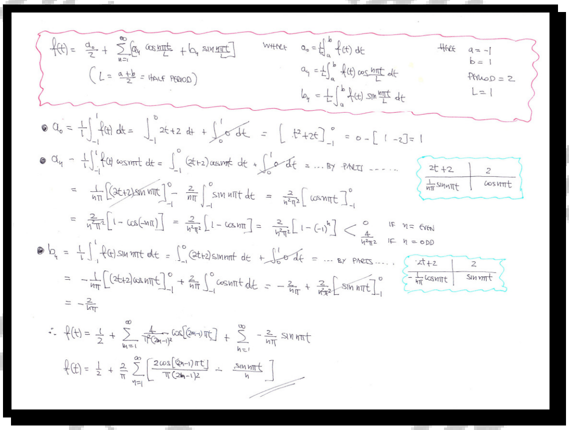
Question 3

$$f(t) = \begin{cases} 2t+2 & -1 \le t \le 0 \\ 0 & 0 \le t \le 1 \end{cases}.$$

$$f(t) = f(t+2).$$

Determine the Fourier series expansion of $f(t)$.

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{2\cos\left[(2n-1)\pi t\right]}{\pi(2n-1)^2} - \frac{\sin(n\pi t)}{n}\right]$$



$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi t}{L} + b_n \sin\frac{n\pi t}{L}\right]$ WHERE $a_0 = \frac{1}{L}\int_a^b f(t)\,dt$, $a_n = \frac{1}{L}\int_a^b f(t)\cos\frac{n\pi t}{L}\,dt$, $b_n = \frac{1}{L}\int_a^b f(t)\sin\frac{n\pi t}{L}\,dt$

$\left(L = \frac{a+b}{2} = \text{HALF PERIOD}\right)$

HERE $a=-1$, $b=1$, PERIOD $=2$, $L=1$

- $a_0 = \frac{1}{1}\int_{-1}^{1} f(t)\,dt = \int_{-1}^{0} 2t+2\,dt + \int_0^1 0\,dt = \left[t^2+2t\right]_{-1}^{0} = 0-[1-2] = 1$
- $a_n = \frac{1}{1}\int_{-1}^{1} f(t)\cos n\pi t\,dt = \int_{-1}^{0}(2t+2)\cos n\pi t\,dt + \int_0^1 0\,dt = \dots$ BY PARTS ……

$2t+2$	2
$\frac{1}{n\pi}\sin n\pi t$	$\cos n\pi t$

$= \frac{1}{n\pi}\left[(2t+2)\sin n\pi t\right]_{-1}^{0} - \frac{2}{n\pi}\int_{-1}^{0}\sin n\pi t\,dt = \frac{2}{n^2\pi^2}\left[\cos n\pi t\right]_{-1}^{0}$

$= \frac{2}{n^2\pi^2}\left[1-\cos(-n\pi)\right] = \frac{2}{n^2\pi^2}\left[1-\cos n\pi\right] = \frac{2}{n^2\pi^2}\left[1-(-1)^n\right]$: 0 IF n = EVEN, $\frac{4}{n^2\pi^2}$ IF n = ODD

- $b_n = \frac{1}{1}\int_{-1}^{1} f(t)\sin n\pi t\,dt = \int_{-1}^{0}(2t+2)\sin n\pi t\,dt + \int_0^1 0\,dt = \dots$ BY PARTS ……

$2t+2$	2
$-\frac{1}{n\pi}\cos n\pi t$	$\sin n\pi t$

$= -\frac{1}{n\pi}\left[(2t+2)\cos n\pi t\right]_{-1}^{0} + \frac{2}{n\pi}\int_{-1}^{0}\cos n\pi t\,dt = -\frac{2}{n\pi} + \frac{2}{n^2\pi^2}\left[\sin n\pi t\right]_{-1}^{0}$

$= -\frac{2}{n\pi}$

$\therefore f(t) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{4}{\pi^2(2n-1)^2}\cos\left[(2n-1)\pi t\right] + \sum_{n=1}^{\infty} -\frac{2}{n\pi}\sin n\pi t$

$f(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{2\cos\left[(2n-1)\pi t\right]}{\pi(2n-1)^2} - \frac{\sin n\pi t}{n}\right]$

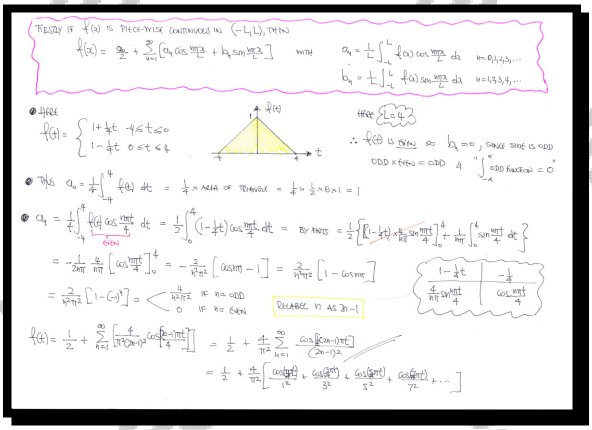
Question 4

$$f(t)=\begin{cases}1+\frac{1}{4}t & -4\le t\le 0\\ 1-\frac{1}{4}t & 0\le t\le 4\end{cases}.$$

$$f(t)=f(t+8).$$

Determine the Fourier series expansion of $f(t)$.

$$f(t)=\frac{1}{2}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{\cos\left[\frac{1}{4}(2n-1)\pi t\right]}{(2n-1)^2}\right]$$



Question 5

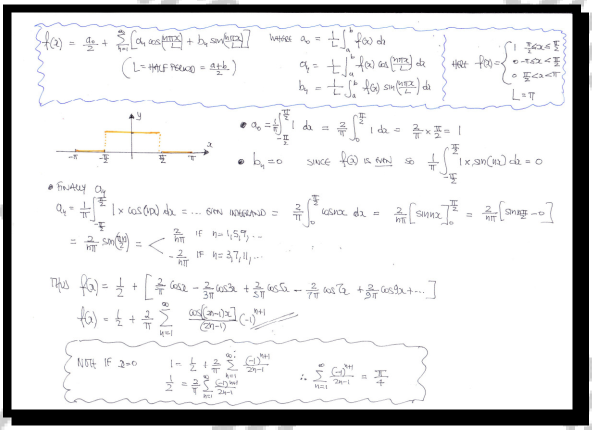
The "Top Hat" function is defined as

$$f(x) = \begin{cases} 1 & |x| \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < |x| \le \pi \end{cases}$$

for $x \in \mathbb{R}$, $f(x) = f(x + 2\pi)$.

Determine the Fourier series expansion of $f(x)$.

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}\cos[(2n-1)x]}{2n-1}\right]$$


$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$ WHERE $a_0 = \frac{1}{L}\int_a^b f(x)\,dx$

(L = HALF PERIOD = $\frac{a+b}{2}$) $a_n = \frac{1}{L}\int_a^b f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

$b_n = \frac{1}{L}\int_a^b f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

HERE $f(x) = \begin{cases} 1 & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ 0 & -\pi \le x < -\frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x < \pi \end{cases}$ $L = \pi$

- $a_0 = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\,dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} 1\,dx = \frac{2}{\pi} \times \frac{\pi}{2} = 1$
- $b_n = 0$ SINCE $f(x)$ IS EVEN SO $\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 \times \sin(nx)\,dx = 0$
- FINALLY a_n

$a_n = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 \times \cos(nx)\,dx = \ldots$ EVEN INTEGRAND $= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos nx\,dx = \frac{2}{n\pi}\left[\sin nx\right]_0^{\frac{\pi}{2}} = \frac{2}{n\pi}\left[\sin\frac{n\pi}{2} - 0\right]$

$= \frac{2}{n\pi}\sin\left(\frac{\pi n}{2}\right) = \begin{cases} \frac{2}{n\pi} & \text{IF } n = 1, 5, 9, \ldots \\ -\frac{2}{n\pi} & \text{IF } n = 3, 7, 11, \ldots \end{cases}$

THUS $f(x) = \frac{1}{2} + \left[\frac{2}{\pi}\cos x - \frac{2}{3\pi}\cos 3x + \frac{2}{5\pi}\cos 5x - \frac{2}{7\pi}\cos 7x + \frac{2}{9\pi}\cos 9x + \ldots\right]$

$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos[(2n-1)x]}{(2n-1)}(-1)^{n+1}$

NOTE IF $x = 0$ $1 = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}$

$\frac{1}{2} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}$ $\therefore \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4}$

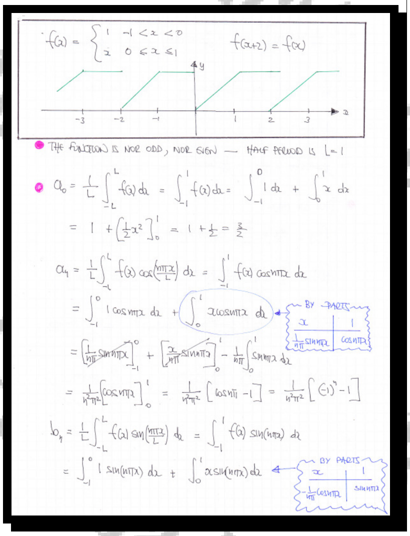
Question 6

$$f(x) = \begin{cases} 1 & -1 \le x \le 0 \\ x & 0 \le x \le 1 \end{cases}.$$

$$f(x+2) = f(x).$$

Determine the Fourier series expansion of $f(x)$.

$$f(x) = \frac{3}{4} - \frac{2}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{\cos\left[(2n-1)\pi x\right]}{(2n-1)^2}\right] - \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{\sin(n\pi x)}{n}\right]$$



$f(x) = \begin{cases} 1 & -1 < x < 0 \\ x & 0 \le x \le 1 \end{cases}$ $\quad f(x+2) = f(x)$

- THE FUNCTION IS NOR ODD, NOR EVEN — HALF PERIOD IS $L=1$
- $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} 1\,dx + \int_{0}^{1} x\,dx$

$= 1 + \left[\frac{1}{2}x^2\right]_0^1 = 1 + \frac{1}{2} = \frac{3}{2}$

$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \int_{-1}^{1} f(x)\cos n\pi x\,dx$

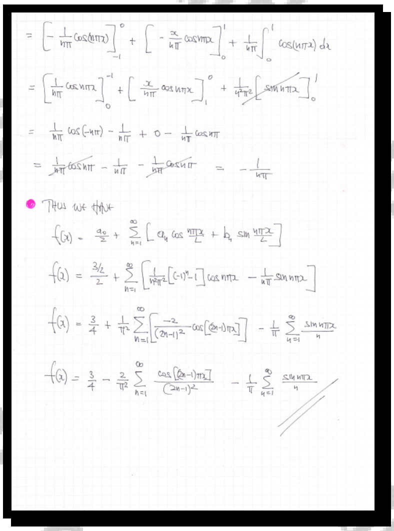
$= \int_{-1}^{0} 1\cos n\pi x\,dx + \int_{0}^{1} x\cos n\pi x\,dx$ ← BY PARTS: x | 1; $\frac{1}{n\pi}\sin n\pi x$ | $\cos n\pi x$

$= \left[\frac{1}{n\pi}\sin n\pi x\right]_{-1}^{0} + \left[\frac{x}{n\pi}\sin n\pi x\right]_0^1 - \frac{1}{n\pi}\int_0^1 \sin n\pi x\,dx$

$= \frac{1}{n^2\pi^2}\left[\cos n\pi x\right]_0^1 = \frac{1}{n^2\pi^2}\left[\cos n\pi - 1\right] = \frac{1}{n^2\pi^2}\left[(-1)^n - 1\right]$

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \int_{-1}^{1} f(x)\sin(n\pi x)\,dx$

$= \int_{-1}^{0} 1\sin(n\pi x)\,dx + \int_0^1 x\sin(n\pi x)\,dx$ ← BY PARTS: x | 1; $-\frac{1}{n\pi}\cos n\pi x$ | $\sin n\pi x$



$= \left[-\frac{1}{n\pi}\cos(n\pi x)\right]_{-1}^{0} + \left[-\frac{x}{n\pi}\cos n\pi x\right]_0^1 + \frac{1}{n\pi}\int_0^1 \cos(n\pi x)\,dx$

$= \left[\frac{1}{n\pi}\cos n\pi x\right]_0^{-1} + \left[\frac{x}{n\pi}\cos n\pi x\right]_1^0 + \frac{1}{n^2\pi^2}\left[\sin n\pi x\right]_0^1$

$= \frac{1}{n\pi}\cos(-n\pi) - \frac{1}{n\pi} + 0 - \frac{1}{n\pi}\cos n\pi$

$= \frac{1}{n\pi}\cos n\pi - \frac{1}{n\pi} - \frac{1}{n\pi}\cos n\pi = -\frac{1}{n\pi}$

- THUS WE HAVE

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right]$

$f(x) = \frac{3/2}{2} + \sum_{n=1}^{\infty}\left[\frac{1}{n^2\pi^2}\left[(-1)^n - 1\right]\cos n\pi x - \frac{1}{n\pi}\sin n\pi x\right]$

$f(x) = \frac{3}{4} + \frac{1}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{-2}{(2n-1)^2}\cos\left[(2n-1)\pi x\right]\right] - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\pi x}{n}$

$f(x) = \frac{3}{4} - \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos\left[(2n-1)\pi x\right]}{(2n-1)^2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\pi x}{n}$

Question 7

A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

a) State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.

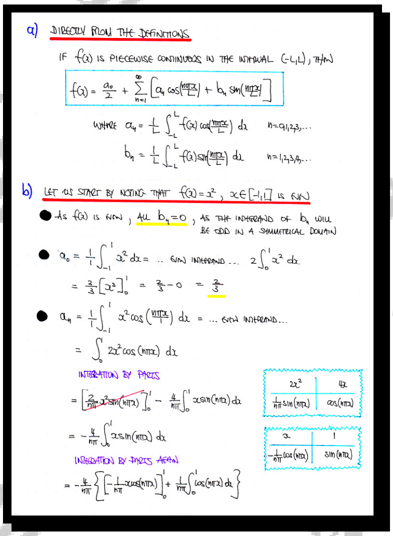
b) Find the Fourier series of

$$f(x) = x^2, \; -1 \le x \le 1.$$

c) Hence determine the exact value of

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \dots .$$

$$x^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^2}\cos(n\pi x)\right], \quad 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \dots = \frac{\pi^2}{12}$$



a) DIRECTLY FROM THE DEFINITIONS

IF $f(x)$ IS PIECEWISE CONTINUOUS IN THE INTERVAL $(-L, L)$, THEN

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

WHERE $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \quad n = 0,1,2,3,\dots$

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad n = 1,2,3,4,\dots$

b) LET US START BY NOTING THAT $f(x) = x^2$, $x \in [-1,1]$ IS EVEN

- AS $f(x)$ IS EVEN, ALL $b_n = 0$, AS THE INTEGRAND OF b_n WILL BE ODD IN A SYMMETRICAL DOMAIN
- $a_0 = \frac{1}{1}\int_{-1}^{1} x^2\,dx = \dots$ EVEN INTEGRAND $\dots$ $2\int_0^1 x^2\,dx$ $= \frac{2}{3}\left[x^3\right]_0^1 = \frac{2}{3} - 0 = \frac{2}{3}$
- $a_n = \frac{1}{1}\int_{-1}^{1} x^2\cos\left(\frac{n\pi x}{1}\right)dx = \dots$ EVEN INTEGRAND $\dots$ $= \int_0^1 2x^2\cos(n\pi x)\,dx$

INTEGRATION BY PARTS

$2x^2$	$4x$
$\frac{1}{n\pi}\sin(n\pi x)$	$\cos(n\pi x)$

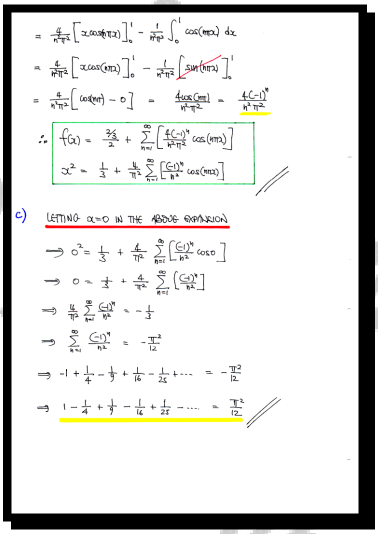
$= \left[\frac{2}{n\pi}x^2\sin(n\pi x)\right]_0^1 - \frac{4}{n\pi}\int_0^1 x\sin(n\pi x)\,dx$

$= -\frac{4}{n\pi}\int_0^1 x\sin(n\pi x)\,dx$

INTEGRATION BY PARTS AGAIN

x	1
$-\frac{1}{n\pi}\cos(n\pi x)$	$\sin(n\pi x)$

$= -\frac{4}{n\pi}\left\{\left[-\frac{1}{n\pi}x\cos(n\pi x)\right]_0^1 + \frac{1}{n\pi}\int_0^1 \cos(n\pi x)\,dx\right\}$



$= \frac{4}{n^2\pi^2}\left[x\cos(n\pi x)\right]_0^1 - \frac{1}{n^2\pi^2}\int_0^1 \cos(n\pi x)\,dx$

$= \frac{4}{n^2\pi^2}\left[x\cos(n\pi x)\right]_0^1 - \frac{1}{n^2\pi^2}\left[\sin(n\pi x)\right]_0^1$

$= \frac{4}{n^2\pi^2}\left[\cos(n\pi) - 0\right] = \frac{4\cos(n\pi)}{n^2\pi^2} = \frac{4(-1)^n}{n^2\pi^2}$

$\therefore \; f(x) = \frac{2/3}{2} + \sum_{n=1}^{\infty}\left[\frac{4(-1)^n}{n^2\pi^2}\cos(n\pi x)\right]$

$x^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^2}\cos(n\pi x)\right]$

c) LETTING $x = 0$ IN THE ABOVE EXPANSION

$\Rightarrow 0^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^2}\cos 0\right]$

$\Rightarrow 0 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^2}\right]$

$\Rightarrow \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\frac{1}{3}$

$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$

$\Rightarrow -1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} + \dots = -\frac{\pi^2}{12}$

$\Rightarrow 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \dots = \frac{\pi^2}{12}$

Question 8

A function $f(x)$ is defined in an interval $(-\pi, \pi)$.

a) State the general formula for the Fourier series of $f(x)$ in $(-\pi, \pi)$, giving general expressions for the coefficients of the series.

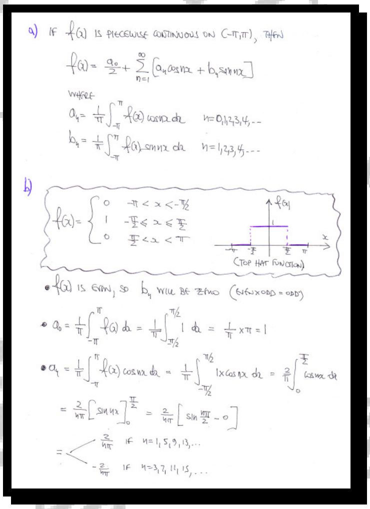
b) Find the Fourier series of

$$f(x) = \begin{cases} 0 & -\pi \le x \le -\frac{1}{2}\pi \\ 1 & -\frac{1}{2}\pi < x \le \frac{1}{2}\pi \\ 0 & \frac{1}{2}\pi \le x \le \pi \end{cases}$$

c) Hence determine the exact value of

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots.$$

$$\boxed{f(x) = \frac{1}{2} + \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}\cos(nx)}{2n-1}\right]}, \quad \boxed{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots = \frac{\pi}{4}}$$



a) IF $f(x)$ IS PIECEWISE CONTINUOUS ON $(-\pi,\pi)$, THEN

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos nx + b_n \sin nx\right]$

WHERE

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx \quad n = 0,1,2,3,4,\dots$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx \quad n = 1,2,3,4,\dots$

b) $f(x) = \begin{cases} 0 & -\pi < x < -\frac{\pi}{2} \\ 1 & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x < \pi \end{cases}$

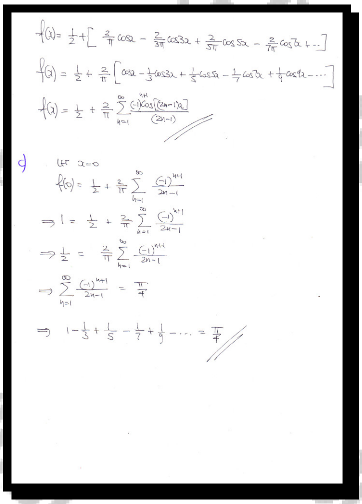
• $f(x)$ IS EVEN, SO b_n WILL BE ZERO (EVEN × ODD = ODD)

• $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 1\,dx = \frac{1}{\pi} \times \pi = 1$

• $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 1 \times \cos nx\,dx = \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} \cos nx\,dx$

$= \frac{2}{n\pi}\left[\sin nx\right]_0^{\frac{\pi}{2}} = \frac{2}{n\pi}\left[\sin\frac{n\pi}{2} - 0\right]$

$= \begin{cases} \frac{2}{n\pi} & \text{IF } n = 1,5,9,13,\dots \\ -\frac{2}{n\pi} & \text{IF } n = 3,7,11,15,\dots \end{cases}$



$f(x) = \frac{1}{2} + \left[\frac{2}{\pi}\cos x - \frac{2}{3\pi}\cos 3x + \frac{2}{5\pi}\cos 5x - \frac{2}{7\pi}\cos 7x + \dots\right]$

$f(x) = \frac{1}{2} + \frac{2}{\pi}\left[\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \frac{1}{7}\cos 7x + \frac{1}{9}\cos 9x - \dots\right]$

$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos\left[(2n-1)x\right]}{(2n-1)}$

c) LET $x = 0$

$f(0) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}$

$\Rightarrow 1 = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}$

$\Rightarrow \frac{1}{2} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}$

$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4}$

$\Rightarrow 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots = \frac{\pi}{4}$

Question 9

A function $f(x)$ is defined in an interval $(\alpha, \alpha+2L)$, $L>0$.

a) State the general formula for the Fourier series of $f(x)$ in $(\alpha, \alpha+2L)$, giving general expressions for the coefficients of the series.

$$f(x) = x,\ 0 \le x \le 4.$$

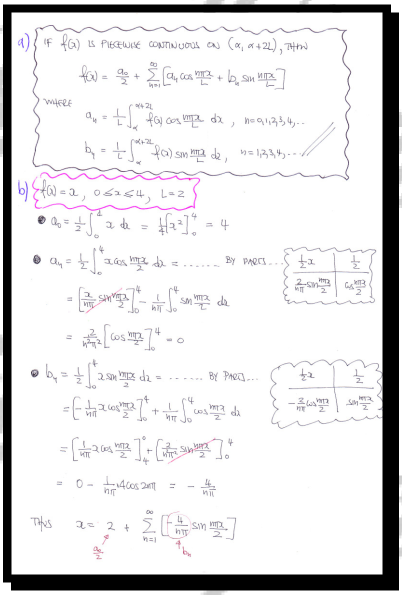
b) Find the Fourier series of $f(x)$ …

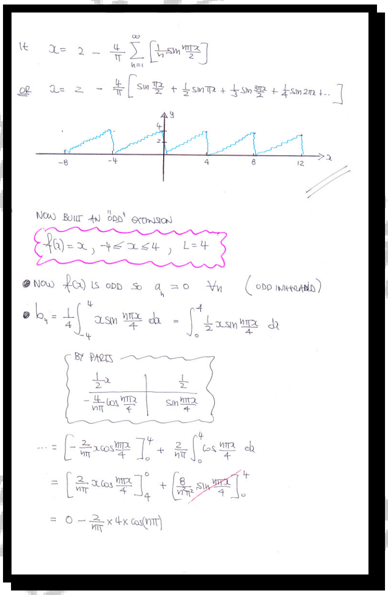
i. … in the interval $0 \le x \le 4$, with period 4.

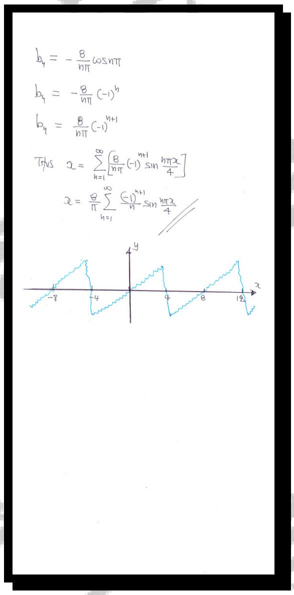
ii. … in the interval $0 \le x \le 4$, with period 8, by building a suitable "extension" to $f(x)$.

Illustrate the solution in each case with a sketch.

$$\boxed{x = 2 - \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n}\sin\left(\tfrac{1}{2}n\pi x\right)\right]},\quad \boxed{x = \frac{8}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{n}\sin\left(\tfrac{1}{4}n\pi x\right)\right]}$$







Question 10

A function $f(x)$ is defined in an interval $(\alpha, \alpha + 2L)$, $L > 0$.

a) State the general formula for the Fourier series of $f(x)$ in $(\alpha, \alpha + 2L)$, giving general expressions for the coefficients of the series.

$$f(x) = x^2, \; 0 \le x \le 1.$$

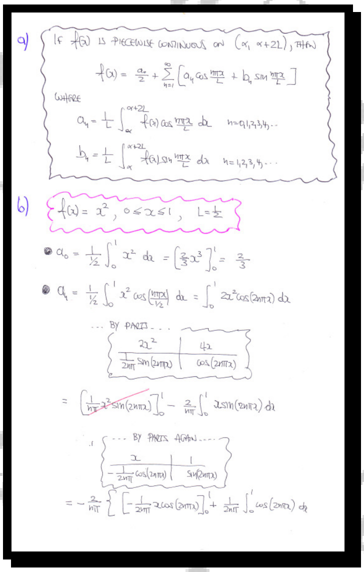
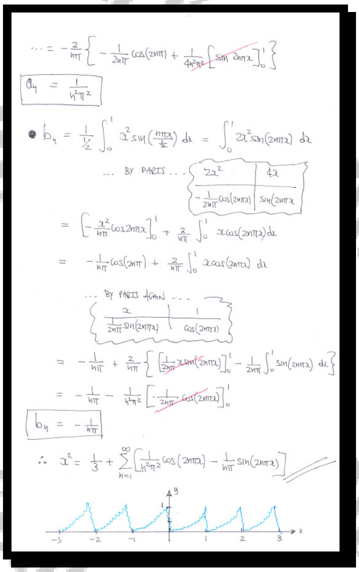
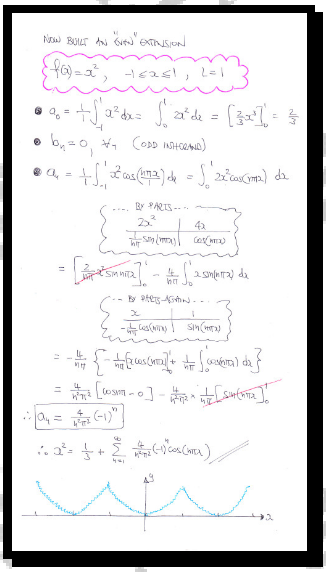
b) Find the Fourier series of $f(x)$ …

i. … in the interval $0 \le x \le 1$, with period 1.

ii. … in the interval $0 \le x \le 1$, with period 2, by building a suitable "extension" to $f(x)$.

Illustrate the solution in each case with a sketch.

$$x^2 = \frac{1}{3} + \sum_{n=1}^{\infty}\left[\frac{\cos(2n\pi x)}{n^2\pi^2} - \frac{\sin(2n\pi x)}{n\pi}\right], \quad x^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^2}\cos(n\pi x)\right]$$

Question 11

$$f(x) = \begin{cases} \pi - x & 0 \le x \le \pi \\ \pi + x & -\pi < x \le 0 \end{cases}$$

for $x \in \mathbb{R}$, $f(x) = f(x + 2\pi)$.

a) Determine the Fourier series expansion of $f(x)$.

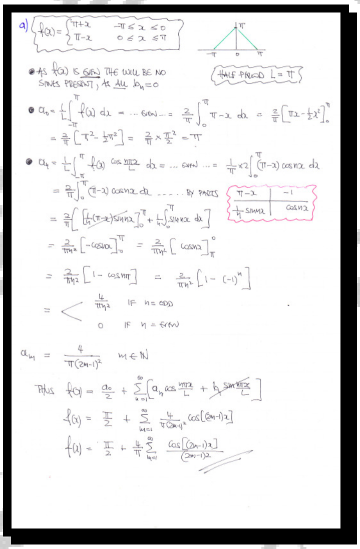
b) Hence determine the exact value of

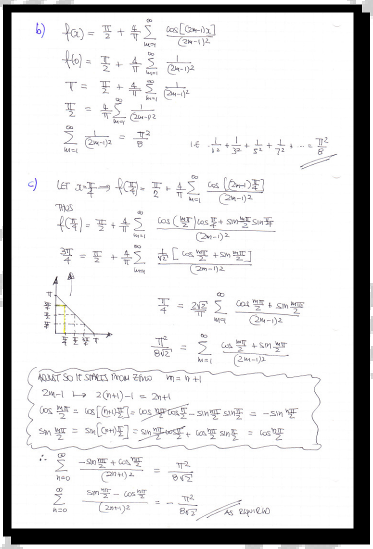
$$\sum_{r=1}^{\infty} \frac{1}{(2r-1)^2}.$$

c) Show that

$$\sum_{n=0}^{\infty} \left[\frac{\sin \frac{n\pi}{2} - \cos \frac{n\pi}{2}}{(2n+1)^2} \right] = -\frac{\pi^2}{8\sqrt{2}}$$

$$f(x) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1}^{\infty} \left[\frac{\cos[(2n-1)x]}{(2n-1)^2} \right], \quad \sum_{r=1}^{\infty} \frac{1}{(2r-1)^2} = \frac{\pi^2}{8}$$





Question 12

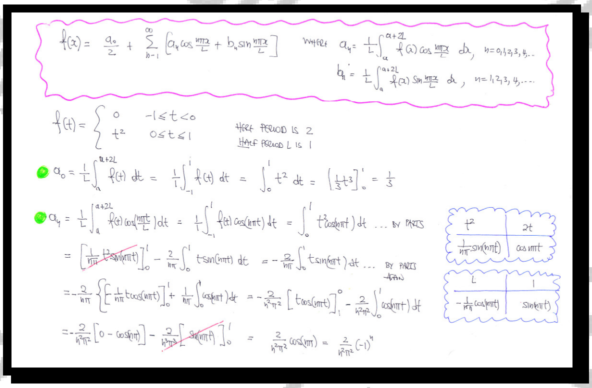
The periodic function f is defined as

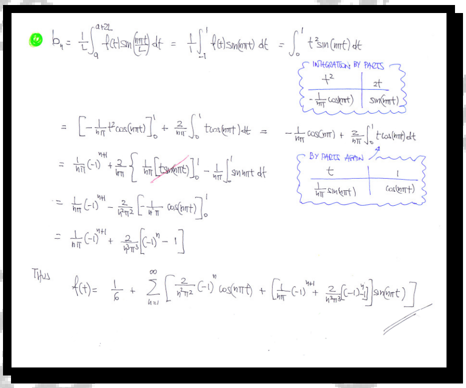
$$f(t)=\begin{cases} 0 & -1\le t<0 \\ t^2 & 0\le t\le 1 \end{cases}$$

for $t\in\mathbb{R}$, $f(t)=f(t+2)$.

Determine the Fourier series expansion of $f(t)$.

$$f(t)=\frac{1}{6}+\sum_{n=1}^{\infty}\left\{\frac{(-1)^n\times 2\cos(n\pi t)}{n^2\pi^2}+\left[\frac{(-1)^{n+1}}{n\pi}+\frac{2}{n^3\pi^3}\left[(-1)^n-1\right]\right]\sin(n\pi t)\right\}$$





Question 13

$$f(x) = x, \; x \in \mathbb{R}, \; 0 \le x \le 2\pi .$$

$$f(x) = f(x + 2\pi).$$

a) Determine the Fourier series expansion of $f(x)$.

b) Hence determine the exact value of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}.$$

$$\boxed{f(x) = \pi - 2\sum_{n=1}^{\infty} \frac{\sin nx}{n}}, \quad \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4}}$$

(a) $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{T} + b_n \sin\frac{2n\pi x}{T}$ WITH $a_0 = \frac{2}{T}\int_a^{a+T} f(x)\,dx$

$a_n = \frac{2}{T}\int_a^{a+T} f(x)\cos\frac{2n\pi x}{T}\,dx$

$b_n = \frac{2}{T}\int_a^{a+T} f(x)\sin\frac{2n\pi x}{T}\,dx$

HERE PERIOD $T = 2\pi$

$\frac{2n\pi}{T} = n$

$\frac{2}{T} = \frac{1}{\pi}$

- $a_0 = \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{2\pi}\left[x^2\right]_0^{2\pi} = \frac{1}{2\pi}\left[4\pi^2 - 0\right] = 2\pi$
- $a_n = \frac{1}{\pi}\int_0^{2\pi} x\cos nx = \dots \text{parts} \dots = \frac{1}{\pi}\left[\left[\frac{1}{n}x\sin nx\right]_0^{2\pi} - \frac{1}{n}\int_0^{2\pi}\sin nx\,dx\right] = 0$
- $b_n = \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx = \dots \text{parts} \dots = \frac{1}{\pi}\left[\left[-\frac{1}{n}x\cos nx\right]_0^{2\pi} + \frac{1}{n}\int_0^{2\pi}\cos nx\,dx\right]$

$= -\frac{1}{\pi n}\left[2\pi - 0\right] = -\frac{2}{n}$

x	1
$\frac{1}{n}\sin nx$	$\cos nx$

x	1
$-\frac{1}{n}\cos nx$	$\sin nx$

THUS $f(x) = x = \pi + \sum_{n=1}^{\infty} -\frac{2}{n}\sin nx = \pi - 2\sum_{n=1}^{\infty}\frac{\sin nx}{n}$

$= \pi - 2\left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 4x}{4} + \dots\right]$

(b) LET $x = \frac{\pi}{2}$ $\quad \frac{\pi}{2} = \pi - 2\left[1 + 0 - \frac{1}{3} + 0 + \frac{1}{5} + 0 - \frac{1}{7} + 0 + \frac{1}{9} + \dots\right]$

$\frac{\pi}{2} = \pi - 2\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9}\right]$

$-\frac{\pi}{2} = -2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}$

$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4}$

Question 14

A function $f(x)$ is defined in an interval $(-\pi,\pi)$.

a) State the general formula for the Fourier series of $f(x)$ in $(-\pi,\pi)$, giving general expressions for the coefficients of the series.

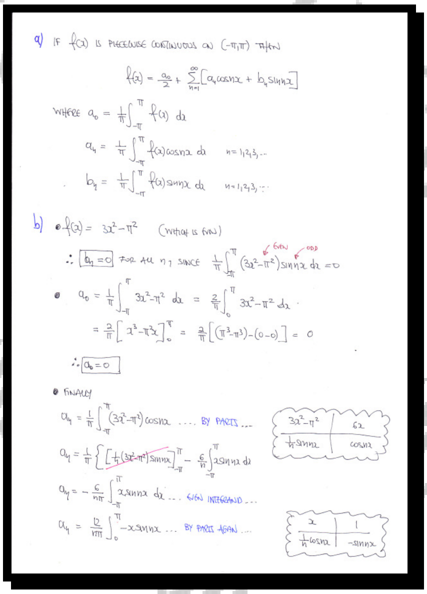
b) Find the Fourier series of

$$f(x) = 3x^2 - \pi^2, \ -\pi \le x \le \pi .$$

c) Hence determine the exact value of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

$$\boxed{3x^2 - \pi^2 = 12\sum_{n=1}^{\infty}\left[\frac{(-1)^n \cos nx}{n^2}\right]}, \quad \boxed{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}}$$



a) IF $f(x)$ IS PIECEWISE CONTINUOUS ON $(-\pi,\pi)$ THEN

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos nx + b_n \sin nx\right]$$

WHERE $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad n=1,2,3,\ldots$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \quad n=1,2,3,\ldots$

b) • $f(x) = 3x^2 - \pi^2$ (which is EVEN)

$\therefore \boxed{b_n = 0}$ FOR ALL n, SINCE $\frac{1}{\pi}\int_{-\pi}^{\pi}(3x^2-\pi^2)\sin nx\,dx = 0$ (EVEN, ODD)

• $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} 3x^2-\pi^2\,dx = \frac{2}{\pi}\int_0^{\pi} 3x^2-\pi^2\,dx$

$= \frac{2}{\pi}\left[x^3-\pi^2 x\right]_0^{\pi} = \frac{2}{\pi}\left[(\pi^3-\pi^3)-(0-0)\right] = 0$

$\therefore \boxed{a_0 = 0}$

• FINALLY

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(3x^2-\pi^2)\cos nx$ .... BY PARTS ...

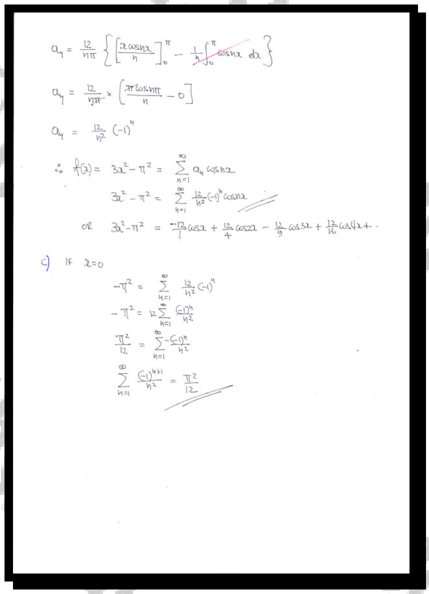
$3x^2-\pi^2$	$6x$
$\frac{1}{n}\sin nx$	$\cos nx$

$a_n = \frac{1}{\pi}\left\{\left[\frac{1}{n}(3x^2-\pi^2)\sin nx\right]_{-\pi}^{\pi} - \frac{6}{n}\int_{-\pi}^{\pi} x\sin nx\,dx\right.$

$a_n = -\frac{6}{n\pi}\int_{-\pi}^{\pi} x\sin nx\,dx$ ... EVEN INTEGRAND ...

$a_n = \frac{12}{n\pi}\int_0^{\pi} -x\sin nx$ ... BY PARTS AGAIN ...

x	1
$\frac{1}{n}\cos nx$	$-\sin nx$



$a_n = \frac{12}{n\pi}\left\{\left[\frac{x\cos nx}{n}\right]_0^{\pi} - \frac{1}{n}\int_0^{\pi}\cos nx\,dx\right\}$

$a_n = \frac{12}{n\pi}\times\left[\frac{\pi\cos n\pi}{n} - 0\right]$

$a_n = \frac{12}{n^2}(-1)^n$

$\therefore f(x) = 3x^2-\pi^2 = \sum_{n=1}^{\infty} a_n\cos nx$

$3x^2-\pi^2 = \sum_{n=1}^{\infty}\frac{12}{n^2}(-1)^n\cos nx$

OR $3x^2-\pi^2 = -\frac{12}{1}\cos x + \frac{12}{4}\cos 2x - \frac{12}{9}\cos 3x + \frac{12}{16}\cos 4x + \ldots$

c) IF $x=0$

$-\pi^2 = \sum_{n=1}^{\infty}\frac{12}{n^2}(-1)^n$

$-\pi^2 = 12\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$

$\frac{\pi^2}{12} = \sum_{n=1}^{\infty}\frac{-(-1)^n}{n^2}$

$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$

Question 15

$$f(x) = |x|, \; x \in \mathbb{R}, \; -\pi \le x \le \pi .$$

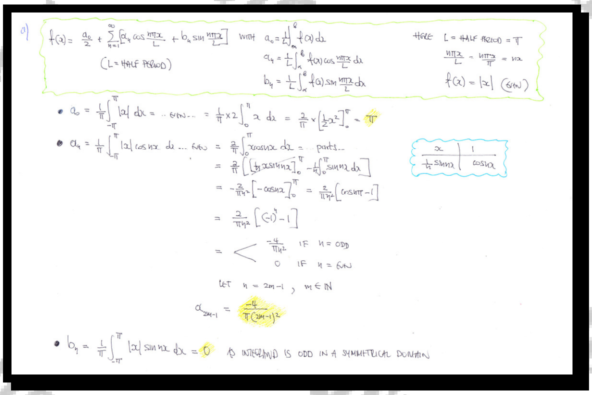
$$f(x) = f(x + 2\pi).$$

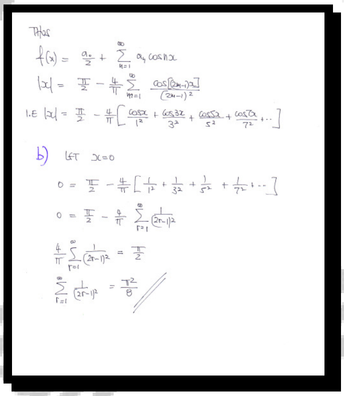
a) Determine the Fourier series expansion of $f(x)$.

b) Hence determine the exact value of

$$\sum_{r=1}^{\infty} \frac{1}{(2r-1)^2} .$$

$$\boxed{f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos\left[(2n-1)x\right]}{(2n-1)^2}}, \quad \boxed{\sum_{r=1}^{\infty} \frac{1}{(2r-1)^2} = \frac{\pi^2}{8}}$$





Question 16

$$f(x) = x, \; x \in \mathbb{R}, \; -1 \le x \le 1.$$

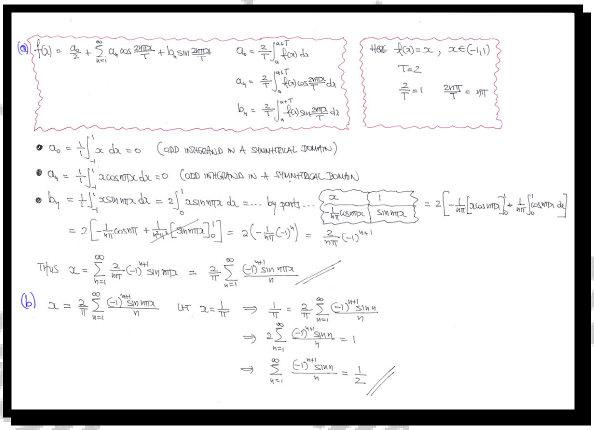
$$f(x) = f(x+2).$$

a) Determine the Fourier series expansion of $f(x)$.

b) Hence determine the exact value of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin n}{n}.$$

$$\boxed{f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin n\pi x}{n}}, \quad \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin n}{n} = \frac{1}{2}}$$


(a) $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{2n\pi x}{T} + b_n \sin \frac{2n\pi x}{T}$ $\quad a_0 = \frac{2}{T}\int_a^{a+T} f(x)\,dx$, $\quad a_n = \frac{2}{T}\int_a^{a+T} f(x) \cos\frac{2n\pi x}{T}\,dx$, $\quad b_n = \frac{2}{T}\int_a^{a+T} f(x)\sin\frac{2n\pi x}{T}\,dx$

HERE $f(x) = x$, $x \in (-1,1)$, $T = 2$, $\frac{2}{T} = 1$, $\frac{2n\pi}{T} = n\pi$

- $a_0 = \frac{1}{1}\int_{-1}^{1} x\,dx = 0$ (ODD INTEGRAND IN A SYMMETRICAL DOMAIN)
- $a_n = \frac{1}{1}\int_{-1}^{1} x\cos n\pi x\,dx = 0$ (ODD INTEGRAND IN A SYMMETRICAL DOMAIN)
- $b_n = \frac{1}{1}\int_{-1}^{1} x\sin n\pi x\,dx = 2\int_0^1 x \sin n\pi x\,dx = \ldots$ by parts $\ldots$

x	1
$-\frac{1}{n\pi}\cos n\pi x$	$\sin n\pi x$

$= 2\left[-\frac{1}{n\pi}\left[x\cos n\pi x\right]_0^1 + \frac{1}{n\pi}\int_0^1 \cos n\pi x\,dx\right]$

$= 2\left[-\frac{1}{n\pi}\cos n\pi + \frac{1}{n^2\pi^2}\left[\sin n\pi x\right]_0^1\right] = 2\left(-\frac{1}{n\pi}(-1)^n\right) = \frac{2}{n\pi}(-1)^{n+1}$

Thus $x = \sum_{n=1}^{\infty} \frac{2}{n\pi}(-1)^{n+1}\sin n\pi x = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin n\pi x}{n}$

(b) $x = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin n\pi x}{n}$ LET $x = \frac{1}{\pi}$ $\Rightarrow \frac{1}{\pi} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin n}{n}$

$\Rightarrow 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin n}{n} = 1$

$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin n}{n} = \frac{1}{2}$

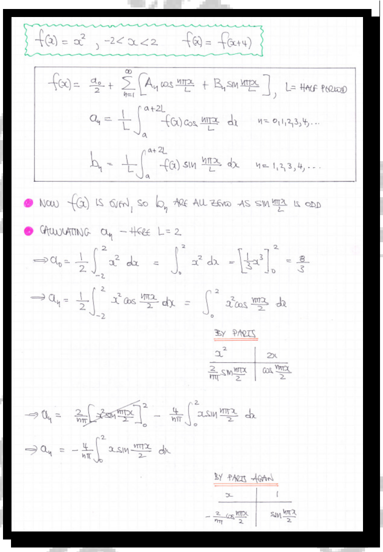
Question 17

$$f(x) = x^2, \ x \in \mathbb{R}, \ -2 \le x \le 2.$$

$$f(x) = f(x+4).$$

Determine the Fourier series expansion of $f(x)$.

$$\boxed{f(x) = \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos\left(\frac{1}{2}n\pi x\right)}$$



$f(x) = x^2$, $-2 < x < 2$ $\quad f(x) = f(x+4)$

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}\right]$, L = HALF PERIOD

$a_n = \frac{1}{L}\int_a^{a+2L} f(x)\cos\frac{n\pi x}{L}\,dx \quad n = 0,1,2,3,4,\dots$

$b_n = \frac{1}{L}\int_a^{a+2L} f(x)\sin\frac{n\pi x}{L}\,dx \quad n = 1,2,3,4,\dots$

- NOW $f(x)$ IS EVEN, SO b_n ARE ALL ZERO AS $\sin\frac{n\pi x}{2}$ IS ODD
- CALCULATING a_n – HERE $L = 2$

$\Rightarrow a_0 = \frac{1}{2}\int_{-2}^{2} x^2\,dx = \int_0^2 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^2 = \frac{8}{3}$

$\Rightarrow a_n = \frac{1}{2}\int_{-2}^{2} x^2\cos\frac{n\pi x}{2}\,dx = \int_0^2 x^2\cos\frac{n\pi x}{2}\,dx$

BY PARTS

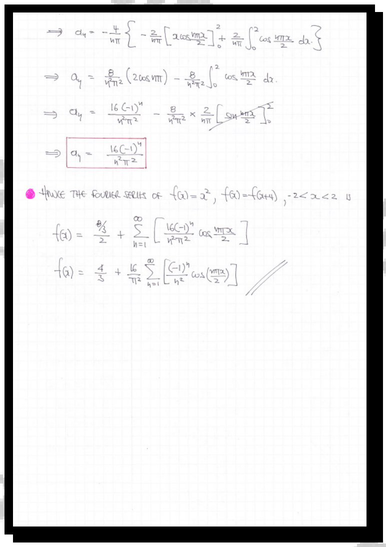
x^2	$2x$
$\frac{2}{n\pi}\sin\frac{n\pi x}{2}$	$\cos\frac{n\pi x}{2}$

$\Rightarrow a_n = \frac{2}{n\pi}\left[x^2\sin\frac{n\pi x}{2}\right]_0^2 - \frac{4}{n\pi}\int_0^2 x\sin\frac{n\pi x}{2}\,dx$

$\Rightarrow a_n = -\frac{4}{n\pi}\int_0^2 x\sin\frac{n\pi x}{2}\,dx$

BY PARTS AGAIN

x	1
$-\frac{2}{n\pi}\cos\frac{n\pi x}{2}$	$\sin\frac{n\pi x}{2}$



$\Rightarrow a_n = -\frac{4}{n\pi}\left\{-\frac{2}{n\pi}\left[x\cos\frac{n\pi x}{2}\right]_0^2 + \frac{2}{n\pi}\int_0^2 \cos\frac{n\pi x}{2}\,dx\right\}$

$\Rightarrow a_n = \frac{8}{n^2\pi^2}(2\cos n\pi) - \frac{8}{n^2\pi^2}\int_0^2 \cos\frac{n\pi x}{2}\,dx$

$\Rightarrow a_n = \frac{16(-1)^n}{n^2\pi^2} - \frac{8}{n^2\pi^2}\times\frac{2}{n\pi}\left[\sin\frac{n\pi x}{2}\right]_0^2$

$\Rightarrow \boxed{a_n = \frac{16(-1)^n}{n^2\pi^2}}$

- HENCE THE FOURIER SERIES OF $f(x) = x^2$, $f(x) = f(x+4)$, $-2 < x < 2$ IS

$f(x) = \frac{8/3}{2} + \sum_{n=1}^{\infty}\left[\frac{16(-1)^n}{n^2\pi^2}\cos\frac{n\pi x}{2}\right]$

$f(x) = \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{2}\right)\right]$

Question 18

A function $f(x)$ is defined in the interval $(-\pi, \pi)$.

a) State the general formula for the Fourier series of $f(x)$ in $(-\pi, \pi)$, giving general expressions for the coefficients of the series.

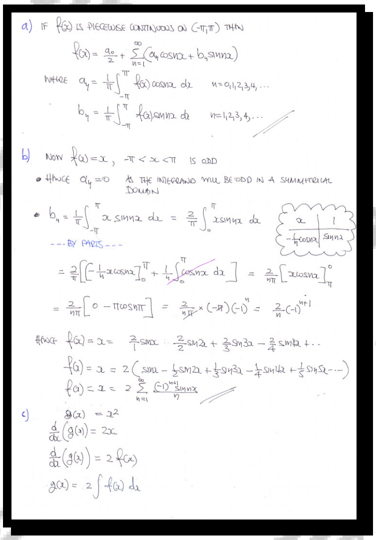
b) Find the Fourier series of

$$f(x) = x, \ -\pi \le x \le \pi .$$

c) Hence determine the exact value of

$$g(x) = x^2, \ -\pi \le x \le \pi .$$

$$\boxed{f(x) = 2\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}\sin nx}{n}\right]}, \quad \boxed{g(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\left[\frac{(-1)^n \cos nx}{n^2}\right]}$$



a) IF $f(x)$ IS PIECEWISE CONTINUOUS ON $(-\pi,\pi)$ THEN

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

WHERE $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx \quad n = 0,1,2,3,4,\dots$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx \quad n = 1,2,3,4,\dots$

b) NOW $f(x) = x$, $-\pi < x < \pi$ IS ODD

- HENCE $a_n = 0$ AS THE INTEGRAND WILL BE ODD IN A SYMMETRICAL DOMAIN
- $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx = \frac{2}{\pi}\int_{0}^{\pi} x \sin nx \, dx$

--- BY PARTS ---

x	1
$-\frac{1}{n}\cos nx$	$\sin nx$

$= \frac{2}{\pi}\left[\left[-\frac{1}{n}x\cos nx\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx \, dx\right] = \frac{2}{n\pi}\left[x\cos nx\right]_{\pi}^{0}$

$= \frac{2}{n\pi}\left[0 - \pi\cos n\pi\right] = \frac{2}{n\pi}\times(-\pi)(-1)^n = \frac{2}{n}(-1)^{n+1}$

HENCE $f(x) = x = \frac{2}{1}\sin x - \frac{2}{2}\sin 2x + \frac{2}{3}\sin 3x - \frac{2}{4}\sin 4x + \dots$

$f(x) = x = 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \frac{1}{5}\sin 5x - \dots\right)$

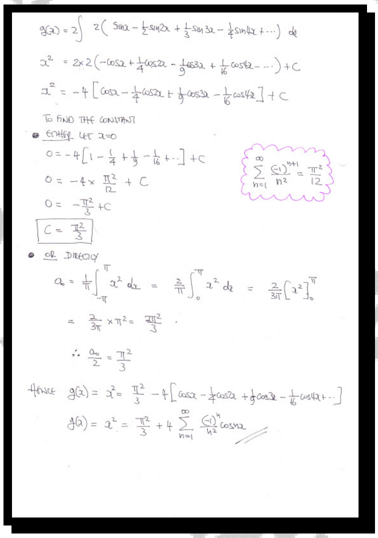
$f(x) = x = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin nx}{n}$

c) $g(x) = x^2$

$\frac{d}{dx}(g(x)) = 2x$

$\frac{d}{dx}(g(x)) = 2f(x)$

$g(x) = 2\int f(x) \, dx$



$g(x) = 2\int 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \dots\right) dx$

$x^2 = 2\times 2\left(-\cos x + \frac{1}{4}\cos 2x - \frac{1}{9}\cos 3x + \frac{1}{16}\cos 4x - \dots\right) + C$

$x^2 = -4\left[\cos x - \frac{1}{4}\cos 2x + \frac{1}{9}\cos 3x - \frac{1}{16}\cos 4x\right] + C$

TO FIND THE CONSTANT

- EITHER LET $x = 0$

$0 = -4\left[1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \dots\right] + C$

$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$

$0 = -4\times\frac{\pi^2}{12} + C$

$0 = -\frac{\pi^2}{3} + C$

$\boxed{C = \frac{\pi^2}{3}}$

- OR DIRECTLY

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \, dx = \frac{2}{\pi}\int_0^{\pi} x^2 \, dx = \frac{2}{3\pi}\left[x^3\right]_0^{\pi}$

$= \frac{2}{3\pi}\times\pi^2 = \frac{2\pi^2}{3}$

$\therefore \frac{a_0}{2} = \frac{\pi^2}{3}$

HENCE $g(x) = x^2 = \frac{\pi^2}{3} - 4\left[\cos x - \frac{1}{4}\cos 2x + \frac{1}{9}\cos 3x - \frac{1}{16}\cos 4x + \dots\right]$

$g(x) = x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$

Question 19

$$f(x) = x^2, \; x \in \mathbb{R}, \; 0 \le x \le 1.$$

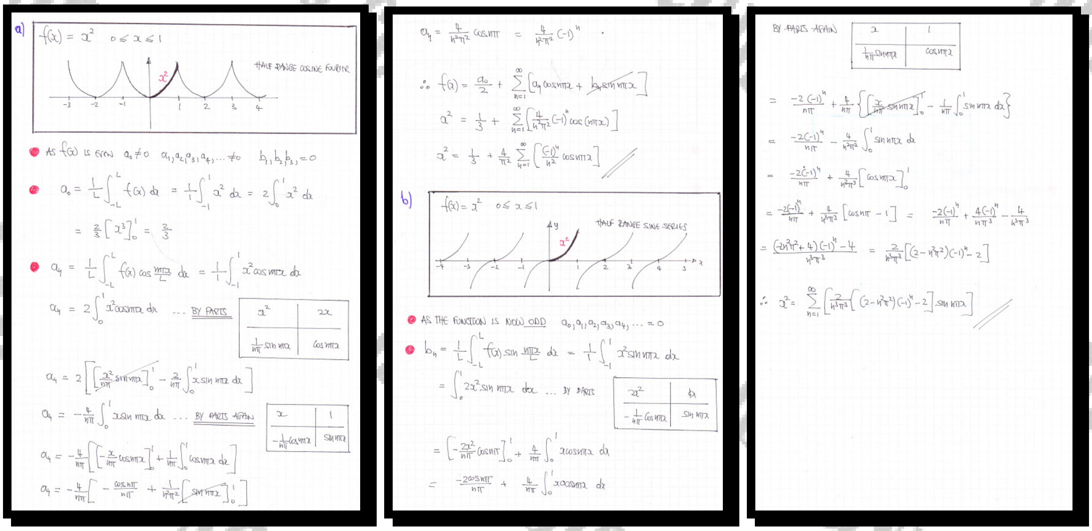
Determine the Fourier series of $f(x)$ as

a) … as half range cosine expansion.

b) … as half range sine expansion.

$$\boxed{f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^2}\cos(n\pi x)\right]},$$

$$\boxed{f(x) = \frac{2}{\pi^3}\sum_{n=1}^{\infty}\left[\frac{2}{n^3}\left[\left(2 - n^2\pi^2\right)(-1)^n - 2\right]\sin(n\pi x)\right]}$$



Question 20

$$f(x) = \begin{cases} \pi - x & 0 \le x \le \pi \\ 0 & \pi \le x \le 2\pi \end{cases}, \quad x \in \mathbb{R}.$$

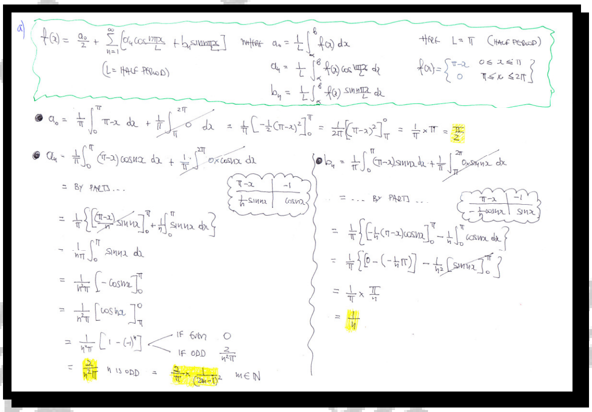
$$f(x) = f(x + 2\pi).$$

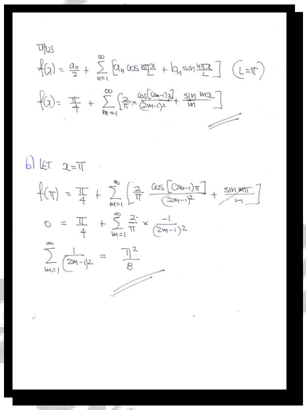
a) Determine the Fourier series expansion of $f(x)$.

b) Hence determine the exact value of

$$\sum_{r=1}^{\infty} \frac{1}{(2r-1)^2}.$$

$$\boxed{f(x) = \frac{\pi}{4} + \sum_{m=1}^{\infty}\left[\frac{2\cos\left[(2m-1)x\right]}{\pi(2m-1)^2} + \frac{\sin mx}{m}\right]}, \quad \boxed{\sum_{r=1}^{\infty} \frac{1}{(2r-1)^2} = \frac{\pi^2}{8}}$$





Question 21

$$f(x) = x^2, \; x \in \mathbb{R}, \; -\pi \le x \le \pi .$$

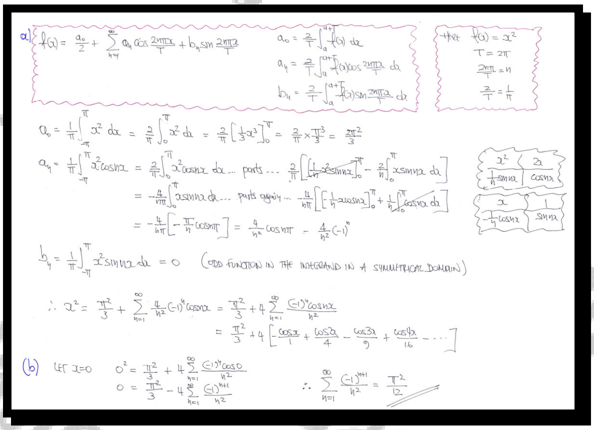
$$f(x) = f(x + 2\pi) .$$

a) Determine the Fourier series expansion of $f(x)$.

b) Hence determine the exact value of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} .$$

$$\boxed{f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}}, \quad \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}}$$



(a) $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{T} + b_n \sin\frac{2n\pi x}{T}$

$a_0 = \frac{2}{T}\int_a^{a+T} f(x)\,dx$

$a_n = \frac{2}{T}\int_a^{a+T} f(x)\cos\frac{2n\pi x}{T}\,dx$

$b_n = \frac{2}{T}\int_a^{a+T} f(x)\sin\frac{2n\pi x}{T}\,dx$

HERE $f(x) = x^2$, $T = 2\pi$, $\frac{2n\pi}{T} = n$, $\frac{2}{T} = \frac{1}{\pi}$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{1}{3}x^3\right]_0^{\pi} = \frac{2}{\pi}\times\frac{\pi^3}{3} = \frac{2\pi^2}{3}$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx \ldots$ parts $\ldots \frac{2}{\pi}\left[\left[\frac{1}{n}x^2\sin nx\right]_0^{\pi} - \frac{2}{n}\int_0^{\pi} x\sin nx\,dx\right]$

$= -\frac{4}{n\pi}\int_0^{\pi} x\sin nx\,dx \ldots$ parts again $\ldots -\frac{4}{n\pi}\left[\left[-\frac{1}{n}x\cos nx\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,dx\right]$

$= -\frac{4}{n\pi}\left[-\frac{\pi}{n}\cos n\pi\right] = \frac{4}{n^2}\cos n\pi = \frac{4}{n^2}(-1)^n$

x^2	$2x$
$\frac{1}{n}\sin nx$	$\cos nx$

x	1
$-\frac{1}{n}\cos nx$	$\sin nx$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin nx\,dx = 0$ (ODD FUNCTION IN THE INTEGRAND IN A SYMMETRICAL DOMAIN)

$\therefore x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{n^2}$

$= \frac{\pi^2}{3} + 4\left[-\frac{\cos x}{1} + \frac{\cos 2x}{4} - \frac{\cos 3x}{9} + \frac{\cos 4x}{16} - \ldots\right]$

(b) LET $x=0$

$0^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n\cos 0}{n^2}$

$0 = \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$

$\therefore \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$

Question 22

$$f(x) = x^2,\ x \in \mathbb{R},\ 0 \le x \le 2\pi .$$

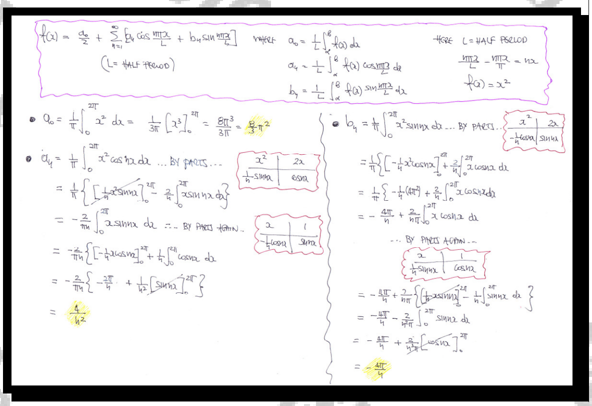
$$f(x) = f(x + 2\pi).$$

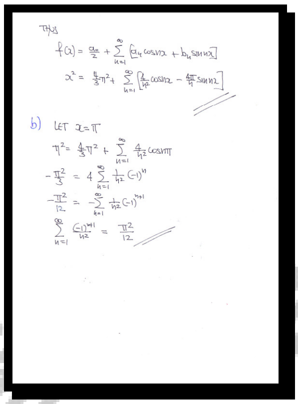
a) Determine the Fourier series expansion of $f(x)$.

b) Hence determine the exact value of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

$$f(x) = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\left[\frac{\cos nx}{n^2} - \frac{\pi \sin nx}{n}\right], \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$





Question 23

It is given that for $x \in \mathbb{R}$, $-\pi \le x \le \pi$,

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos\left[(2n-1)x\right]}{(2n-1)^2}, \qquad |x| = |x+2\pi|.$$

a) Use the above Fourier series expansion to deduce the Fourier series expansion of $\text{sgn}(x)$.

b) Verify the answer of part **(a)** by obtaining directly the Fourier series expansion of $\text{sgn}(x)$.

c) Hence determine the exact value of

$$\sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{2r-1}.$$

$\boxed{}$, $\boxed{\text{sgn}(x) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin\left[(2n-1)x\right]}{(2n-1)^2}}$, $\boxed{\sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{2r-1} = \frac{\pi}{4}}$

a) $|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{\cos[(2n-1)x]}{(2n-1)^2}\right]$

$\text{sgn}(x) = \frac{d}{dx}\left[|x|\right] = \frac{d}{dx}\left[\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\left(\frac{\cos[(2n-1)x]}{(2n-1)^2}\right)\right]$

$= -\frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{-(2n-1)\sin[(2n-1)x]}{(2n-1)^2}\right]$

$= \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{\sin[(2n-1)x]}{(2n-1)^2}\right]$ //

b) NOW FIND THE FOURIER EXPANSION DIRECTLY

$f(x) = \text{sign}\,x, \ -\pi < x < \pi, \ f(x+2\pi) = f(x)$

- $a_0 = \frac{1}{L}\int_a^b f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \text{sign}(x) = 0$ ODD FUNCTION IN A SYMMETRICAL DOMAIN
- $a_n = \frac{1}{L}\int_a^b f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\text{sign}(x)}_{\text{ODD}}\underbrace{\cos(nx)}_{\text{EVEN}}\,dx = 0$ ODD INTEGRAND IN A SYMMETRICAL DOMAIN
- $b_n = \frac{1}{L}\int_a^b f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\text{sign}(x)}_{\text{ODD}}\underbrace{\sin(nx)}_{\text{ODD}}\,dx$

$= \frac{1}{\pi}\times 2\int_0^{\pi} 1\times\sin(nx)\,dx = \frac{2}{\pi}\left[-\frac{1}{n}\cos nx\right]_0^{\pi}$

$= \frac{2}{\pi n}\left[\cos nx\right]_{\pi}^{0} = \frac{2}{\pi n}\left[1-\cos n\pi\right] = \frac{2}{\pi n}\left[1-(-1)^n\right]$

$= \begin{cases} 0 & \text{IF } n \text{ IS EVEN} \\ \frac{4}{\pi n} & \text{IF } n \text{ IS ODD} \end{cases}$

HENCE WE REWRITE THE COEFFICIENT AS $b_m = \frac{4}{\pi(2m-1)}, \ m = 1,2,3,4,\dots$

$\left[\text{OR } b_n = \frac{4}{\pi(2n-1)}, \ n = 1,2,3,\dots\right]$

HENCE WE CAN SUBSTITUTE INTO THE FOURIER FORMULA

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right]$

$\text{sign}(x) = \sum_{n=1}^{\infty}\left[\frac{4}{\pi(2n-1)}\sin[(2n-1)x]\right]$

$\text{sign}(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{\sin[(2n-1)x]}{(2n-1)}\right]$ // AS BEFORE

c) SUBSTITUTING $x = \frac{\pi}{2}$ INTO THE ABOVE FORMULA GIVES

$\text{sign}\left(\frac{\pi}{2}\right) = \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{(2n-1)}\sin\left[\frac{\pi}{2}(2n-1)\right]\right]$

$1 = \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{(2n-1)}(-1)^{n+1}\right]$

$\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{2n-1}\right] = \frac{\pi}{4}$

OR $\sum_{r=1}^{\infty}\left[\frac{(-1)^{r+1}}{2r-1}\right] = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$ //

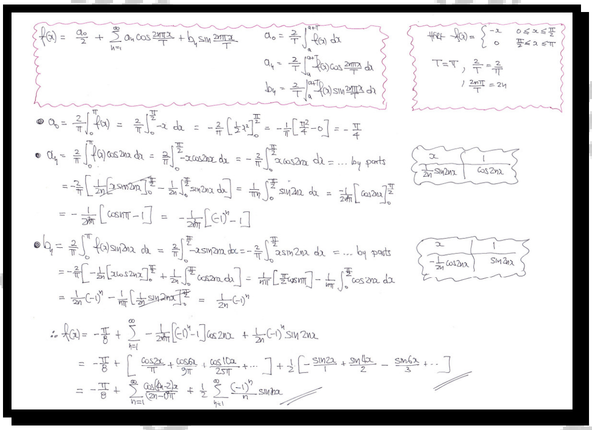
Question 24

$$f(x) = \begin{cases} -x & 0 \le x \le \dfrac{\pi}{2} \\ 0 & \dfrac{\pi}{2} < x < \pi \end{cases}$$

$$f(x) = f(x+\pi).$$

Determine the Fourier series expansion of $f(x)$.

$$\boxed{f(x) = -\frac{\pi}{8} + \sum_{n=1}^{\infty} \frac{\cos[(4n-2)x]}{(2n-1)^2 \pi} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^n \sin 2nx}{n}}$$



$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{T} + b_n \sin\frac{2n\pi x}{T}$

$a_0 = \frac{2}{T}\int_a^{a+T} f(x)\,dx$

$a_n = \frac{2}{T}\int_a^{a+T} f(x)\cos\frac{2n\pi x}{T}\,dx$

$b_n = \frac{2}{T}\int_a^{a+T} f(x)\sin\frac{2n\pi x}{T}\,dx$

Here $f(x) = \begin{cases} -x & 0 \le x \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \le x \le \pi \end{cases}$

$T = \pi$, $\frac{2}{T} = \frac{2}{\pi}$, $\frac{2n\pi}{T} = 2n$

- $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} -x\,dx = -\frac{2}{\pi}\left[\frac{1}{2}x^2\right]_0^{\frac{\pi}{2}} = -\frac{1}{\pi}\left[\frac{\pi^2}{4} - 0\right] = -\frac{\pi}{4}$
- $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos 2nx\,dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} -x\cos 2nx\,dx = -\frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\cos 2nx\,dx = \dots$ by parts

x	1
$\frac{1}{2n}\sin 2nx$	$\cos 2nx$

$= -\frac{2}{\pi}\left[\frac{1}{2n}\left[x\sin 2nx\right]_0^{\frac{\pi}{2}} - \frac{1}{2n}\int_0^{\frac{\pi}{2}} \sin 2nx\,dx\right] = \frac{1}{\pi n}\int_0^{\frac{\pi}{2}} \sin 2nx\,dx = \frac{-1}{2n^2\pi}\left[\cos 2nx\right]_0^{\frac{\pi}{2}}$

$= -\frac{1}{2n^2\pi}\left[\cos n\pi - 1\right] = -\frac{1}{2n^2\pi}\left[(-1)^n - 1\right]$

- $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin 2nx\,dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} -x\sin 2nx\,dx = -\frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\sin 2nx\,dx = \dots$ by parts

x	1
$-\frac{1}{2n}\cos 2nx$	$\sin 2nx$

$= -\frac{2}{\pi}\left[-\frac{1}{2n}\left[x\cos 2nx\right]_0^{\frac{\pi}{2}} + \frac{1}{2n}\int_0^{\frac{\pi}{2}} \cos 2nx\,dx\right] = \frac{1}{n\pi}\left[\frac{\pi}{2}\cos n\pi\right] - \frac{1}{n\pi}\int_0^{\frac{\pi}{2}} \cos 2nx\,dx$

$= \frac{1}{2n}(-1)^n - \frac{1}{n\pi}\left[\frac{1}{2n}\sin 2nx\right]_0^{\frac{\pi}{2}} = \frac{1}{2n}(-1)^n$

$\therefore f(x) = -\frac{\pi}{8} + \sum_{n=1}^{\infty} -\frac{1}{2n^2\pi}\left[(-1)^n - 1\right]\cos 2nx + \frac{1}{2n}(-1)^n \sin 2nx$

$= -\frac{\pi}{8} + \left[\frac{\cos 2x}{\pi} + \frac{\cos 6x}{9\pi} + \frac{\cos 10x}{25\pi} + \dots\right] + \frac{1}{2}\left[-\frac{\sin 2x}{1} + \frac{\sin 4x}{2} - \frac{\sin 6x}{3} + \dots\right]$

$= -\frac{\pi}{8} + \sum_{n=1}^{\infty} \frac{\cos(4n-2)x}{(2n-1)^2\pi} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\sin 2nx$

Question 25

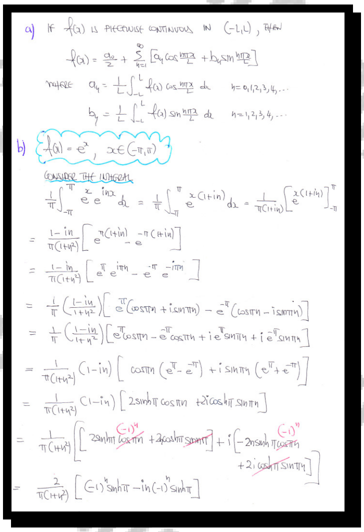
A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

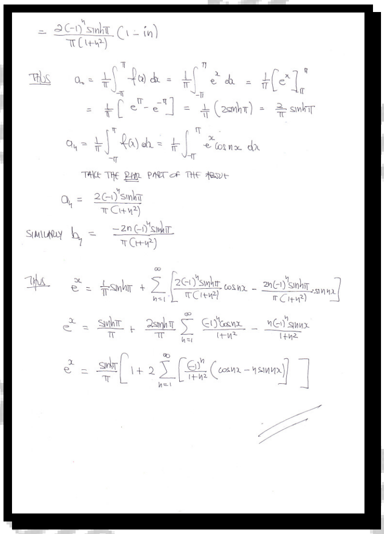
a) State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.

b) Determine the Fourier series of

$$f(x) = e^x, \quad -\pi \le x \le \pi.$$

$$\boxed{e^x = \frac{\sinh \pi}{\pi} + \frac{2\sinh \pi}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \left[\cos(nx) - n\sin(nx) \right]}{1+n^2} \right]}$$





Question 26

A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

a) State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.

b) Show that

$$\int_{-\pi}^{\pi} \mathrm{e}^{ax}\,\mathrm{e}^{inx}\,dx = \frac{2(a-ni)(-1)^n}{a^2+n^2}\sinh(a\pi)$$

c) Determine the Fourier series of

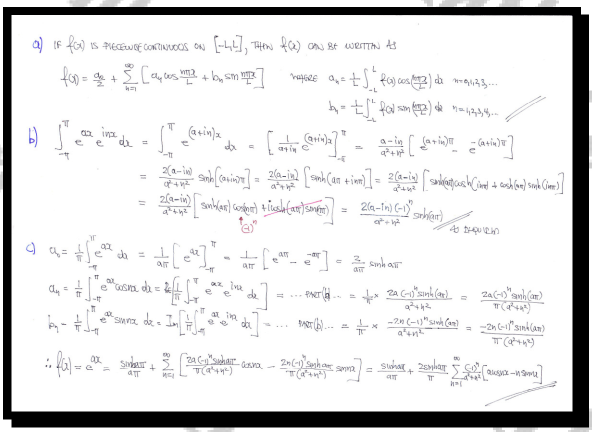
$$f(x) = \mathrm{e}^{ax},\ a > 0,\ -\pi \le x \le \pi.$$

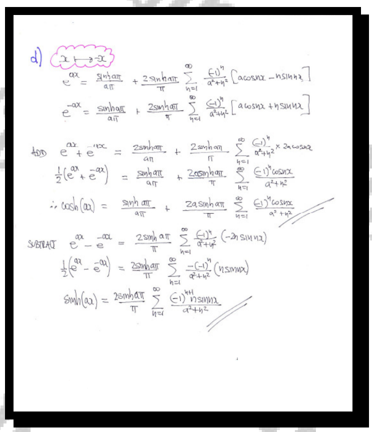
d) Hence find the Fourier series of $\cosh(ax)$ and $\sinh(ax)$, for $-\pi \le x \le \pi$.

$$\boxed{\mathrm{e}^{ax} = \frac{\sinh(a\pi)}{a\pi} + \frac{2\sinh(a\pi)}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n\left[a\cos(nx) - n\sin(nx)\right]}{a^2+n^2}\right]},$$

$$\boxed{\cosh(ax) = \frac{\sinh(a\pi)}{a\pi} + \frac{2a\sinh(a\pi)}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n\cos(nx)}{a^2+n^2}\right]},$$

$$\boxed{\sinh(ax) = \frac{2\sinh(a\pi)}{\pi}\sum_{n=1}^{\infty}\left[\frac{n(-1)^{n+1}\sin(nx)}{a^2+n^2}\right]}$$





Question 27

A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

a) State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.

b) Determine the Fourier series of

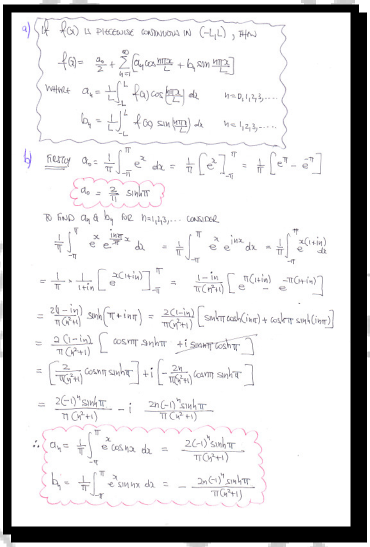
$$f(x) = e^{x}, \ -\pi \le x \le \pi .$$

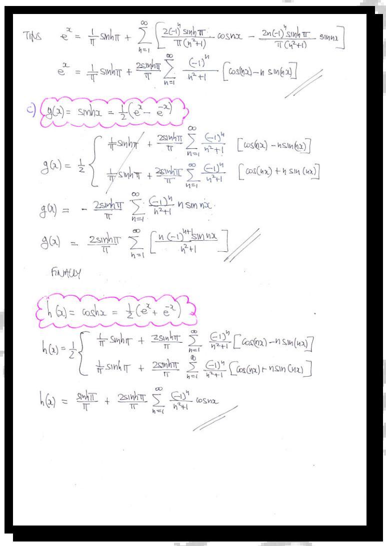
c) Hence find the Fourier series of $\sinh x$ and $\cosh x$, for $-\pi \le x \le \pi$.

$$e^{x} = \frac{\sinh \pi}{\pi} + \frac{2\sinh \pi}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^{n}\left[\cos(nx) - n\sin(nx)\right]}{1+n^{2}} \right],$$

$$\sinh x = \frac{2\sinh \pi}{\pi} \sum_{n=1}^{\infty} \left[\frac{n(-1)^{n+1}\sin(nx)}{1+n^{2}} \right],$$

$$\cosh x = \frac{\sinh \pi}{\pi} + \frac{2\sinh \pi}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^{n}\cos(nx)}{1+n^{2}} \right],$$





Question 28

A function f is defined by

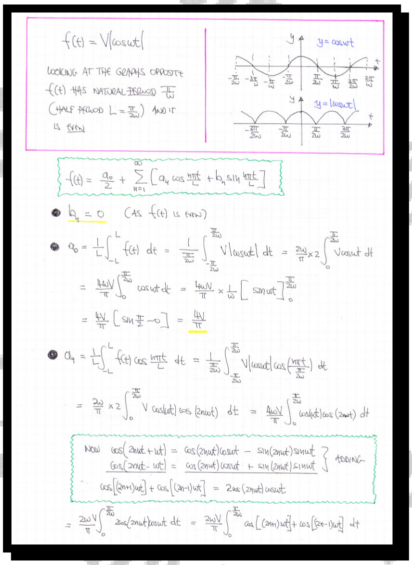
$$f(t) = V\left|\cos \omega t\right|, \; t \in \mathbb{R},$$

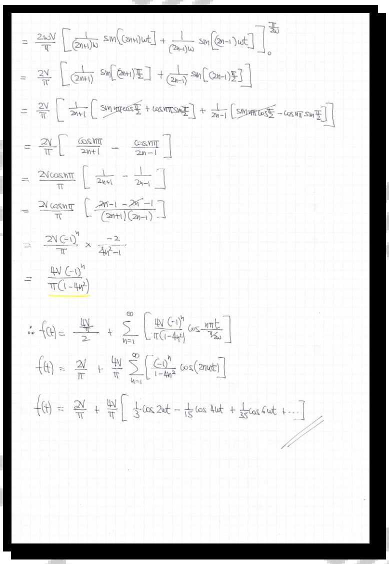
where V and ω are positive constants.

Show that the Fourier series of f is given by

$$f(t) = \frac{2V}{\pi} + \frac{4V}{\pi}\left[\frac{1}{3}\cos(2\omega t) - \frac{1}{15}\cos(4\omega t) + \frac{1}{35}\cos(6\omega t) + \ldots\right]$$

proof





PARSEVAL'S IDENTITY

Question 1

A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

a) State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.

b) Find the Fourier series of

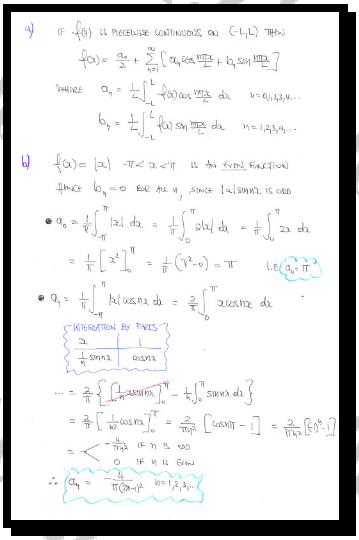
$$f(x) = |x|, \quad -\pi \le x \le \pi .$$

c) State Parseval's identity for the Fourier series of $f(x)$ from part (**a**).

d) Hence show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^4}{96} .$$

$$\boxed{|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{\cos[(2n-1)x]}{(2n-1)^2}\right]}$$


a) IF $f(x)$ IS PIECEWISE CONTINUOUS ON $(-L,L)$ THEN

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right]$$

WHERE $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \quad n = 0,1,2,3,4 \ldots$

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \quad n = 1,2,3,4,\ldots$

b) $f(x) = |x| \quad -\pi < x < \pi$ IS AN EVEN FUNCTION

HENCE $b_n = 0$ FOR ALL n, SINCE $|x|\sin nx$ IS ODD

• $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\,dx = \frac{1}{\pi}\int_0^{\pi} 2|x|\,dx = \frac{1}{\pi}\int_0^{\pi} 2x\,dx$

$= \frac{1}{\pi}\left[x^2\right]_0^{\pi} = \frac{1}{\pi}(\pi^2 - 0) = \pi \quad$ I.E. $a_0 = \pi$

• $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$

INTEGRATION BY PARTS

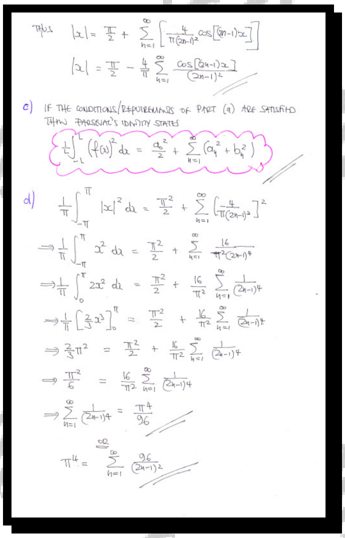
x	1
$\frac{1}{n}\sin nx$	$\cos nx$

$\ldots = \frac{2}{\pi}\left\{\left[\frac{1}{n}x\sin nx\right]_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin nx\,dx\right\}$

$= \frac{2}{\pi}\left[\frac{1}{n^2}\cos nx\right]_0^{\pi} = \frac{2}{\pi n^2}\left[\cos n\pi - 1\right] = \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$

$= -\frac{4}{\pi n^2}$ IF n IS ODD; 0 IF n IS EVEN

$\therefore\ a_n = -\frac{4}{\pi(2n-1)^2} \quad n = 1,2,3,\ldots$


THUS $|x| = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left[-\frac{4}{\pi(2n-1)^2}\cos[(2n-1)x]\right]$

$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos[(2n-1)x]}{(2n-1)^2}$

c) IF THE CONDITIONS/REQUIREMENTS OF PART (a) ARE SATISFIED THEN PARSEVAL'S IDENTITY STATES

$\frac{1}{L}\int_{-L}^{L}(f(x))^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$

d) $\frac{1}{\pi}\int_{-\pi}^{\pi}|x|^2\,dx = \frac{\pi^2}{2} + \sum_{n=1}^{\infty}\left[-\frac{4}{\pi(2n-1)^2}\right]^2$

$\Rightarrow \frac{1}{\pi}\int_{-\pi}^{\pi}x^2\,dx = \frac{\pi^2}{2} + \sum_{n=1}^{\infty}\frac{16}{\pi^2(2n-1)^4}$

$\Rightarrow \frac{1}{\pi}\int_0^{\pi}2x^2\,dx = \frac{\pi^2}{2} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}$

$\Rightarrow \frac{1}{\pi}\left[\frac{2}{3}x^3\right]_0^{\pi} = \frac{\pi^2}{2} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}$

$\Rightarrow \frac{2}{3}\pi^2 = \frac{\pi^2}{2} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}$

$\Rightarrow \frac{\pi^2}{6} = \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}$

$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$

OR

$\pi^4 = \sum_{n=1}^{\infty}\frac{96}{(2n-1)^2}$

Question 2

A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

a) State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.

b) Find the Fourier series of

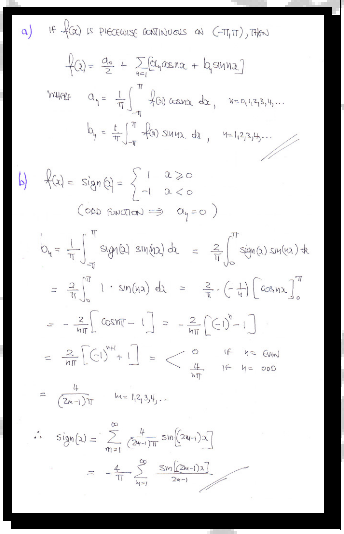
$$f(x) = \text{sign}(x), \quad -\pi \le x \le \pi .$$

c) Prove Parseval's identity for the Fourier series of $f(x)$ in $(-\pi, \pi)$.

d) Hence show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8} .$$

$$\boxed{\text{sign}(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \left[\frac{\sin\left[(2n-1)x\right]}{(2n-1)} \right]}$$



a) IF $f(x)$ IS PIECEWISE CONTINUOUS ON $(-\pi, \pi)$, THEN

$$f(x) = \frac{a_0}{2} + \sum_{n=1} [a_n \cos nx + b_n \sin nx]$$

WHERE $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx, \quad n = 0, 1, 2, 3, 4, \ldots$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx, \quad n = 1, 2, 3, 4, \ldots$

b) $f(x) = \text{sign}(x) = \begin{cases} 1 & x \ge 0 \\ -1 & x < 0 \end{cases}$

(ODD FUNCTION $\Rightarrow a_n = 0$)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \text{sign}(x)\sin(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} \text{sign}(x)\sin(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} 1 \cdot \sin(nx)\,dx = \frac{2}{\pi}\cdot\left(-\frac{1}{n}\right)\left[\cos nx\right]_0^{\pi}$$

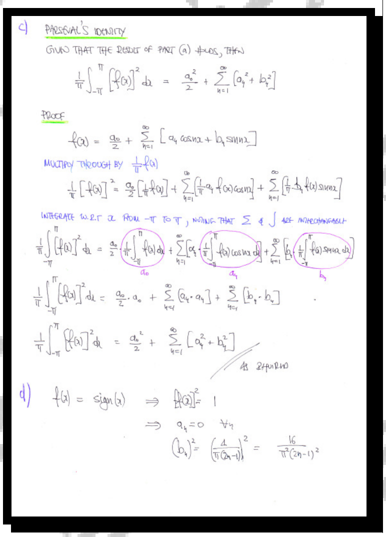
$$= -\frac{2}{n\pi}\left[\cos n\pi - 1\right] = -\frac{2}{n\pi}\left[(-1)^n - 1\right]$$

$$= \frac{2}{n\pi}\left[(-1)^{n+1} + 1\right] = \begin{cases} 0 & \text{IF } n = \text{EVEN} \\ \frac{4}{n\pi} & \text{IF } n = \text{ODD} \end{cases}$$

$$= \frac{4}{(2m-1)\pi} \quad m = 1, 2, 3, 4, \ldots$$

$$\therefore \text{sign}(x) = \sum_{m=1}^{\infty} \frac{4}{(2m-1)\pi}\sin\left[(2m-1)x\right]$$

$$= \frac{4}{\pi}\sum_{m=1}^{\infty} \frac{\sin\left[(2m-1)x\right]}{2m-1}$$



c) PARSEVAL'S IDENTITY

GIVEN THAT THE RESULT OF PART (a) HOLDS, THEN

$$\frac{1}{\pi}\int_{-\pi}^{\pi} [f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left[a_n^2 + b_n^2\right]$$

PROOF

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos nx + b_n \sin nx\right]$$

MULTIPLY THROUGH BY $\frac{1}{\pi}f(x)$

$$\frac{1}{\pi}[f(x)]^2 = \frac{a_0}{2}\left[\frac{1}{\pi}f(x)\right] + \sum_{n=1}^{\infty}\left[\frac{1}{\pi}a_n f(x)\cos nx\right] + \sum_{n=1}^{\infty}\left[\frac{1}{\pi}b_n f(x)\sin nx\right]$$

INTEGRATE W.R.T x FROM $-\pi$ TO π, NOTING THAT Σ & $\int$ ARE INTERCHANGEABLE

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx = \frac{a_0}{2}\cdot\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx\right) + \sum_{n=1}^{\infty}\left[a_n\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx\right)\right] + \sum_{n=1}^{\infty}\left[b_n\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx\right)\right]$$

$a_0 \qquad a_n \qquad b_n$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx = \frac{a_0}{2}\cdot a_0 + \sum_{n=1}^{\infty}\left[a_n \cdot a_n\right] + \sum_{n=1}^{\infty}\left[b_n \cdot b_n\right]$$

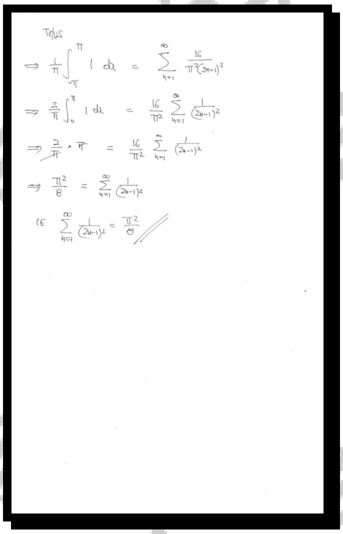
$$\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left[a_n^2 + b_n^2\right]$$

AS REQUIRED

d) $f(x) = \text{sign}(x) \Rightarrow [f(x)]^2 = 1$

$\Rightarrow a_n = 0 \quad \forall n$

$$(b_n)^2 = \left(\frac{4}{\pi(2n-1)}\right)^2 = \frac{16}{\pi^2(2n-1)^2}$$



THUS

$$\Rightarrow \frac{1}{\pi}\int_{-\pi}^{\pi} 1\,dx = \sum_{n=1}^{\infty}\frac{16}{\pi^2(2n-1)^2}$$

$$\Rightarrow \frac{2}{\pi}\int_0^{\pi} 1\,dx = \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

$$\Rightarrow \frac{2}{\pi}\cdot\pi = \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

$$\Rightarrow \frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

$$\text{IE} \quad \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

Question 3

A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

a) State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.

b) Find the Fourier series of

$$f(x) = \tfrac{1}{2} + \tfrac{1}{2}\operatorname{sign}(x), \quad -\pi \le x \le \pi .$$

c) Prove the validity of Parseval's identity for the Fourier series of $f(x)$ in the interval $(-L, L)$.

d) Hence show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8} .$$

$$\boxed{}, \quad \boxed{\frac{1}{2} + \frac{1}{2}\operatorname{sign}(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{\sin\left[(2n-1)x\right]}{(2n-1)}\right]}$$

a) STATING THE "FOURIER SERIES THEOREM"

IF $f(x)$ IS PIECEWISE CONTINUOUS ON $(-L, L)$, $L>0$, THEN

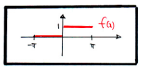
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right]$$

WHERE

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad n = 0, 1, 2, 3, \ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \qquad n = 1, 2, 3, 4, \ldots$$

b) $f(x) = \frac{1}{2} + \frac{1}{2}\operatorname{sign}x$



NOTE $\operatorname{sign}x = \begin{cases} 1 & x \ge 0 \\ -1 & x < 0 \end{cases}$

- $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{0}^{\pi} 1\,dx = \frac{1}{\pi}\times\pi = 1$
- $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{0}^{\pi} 1\times\cos nx\,dx = \frac{1}{n\pi}\left[\sin nx\right]_0^{\pi} = \frac{1}{n\pi}\left[\sin n\pi - 0\right] = 0$
- $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{0}^{\pi} 1\times\sin nx\,dx = -\frac{1}{\pi n}\left[\cos nx\right]_0^{\pi} = -\frac{1}{n\pi}\left[\cos n\pi - 1\right] = \frac{1-\cos n\pi}{n\pi} = \frac{1-(-1)^n}{n\pi}$

$$= \begin{cases} \frac{2}{n\pi} & \text{IF } n \text{ IS ODD} \\ 0 & \text{IF } n \text{ IS EVEN} \end{cases} \qquad \therefore b_n = \frac{2}{(2n-1)\pi}$$

$$\therefore f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\left[\frac{2}{(2n-1)\pi}\sin\left[(2n-1)x\right]\right]$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left[(2n-1)x\right]}{2n-1}$$

c) START FROM THE FOURIER THEOREM STATEMENT

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right]$$

MULTIPLY THROUGH BY $\frac{1}{L}f(x)$ & INTEGRATE W.R.T x, BETWEEN $-L$ & L

$$\Rightarrow \frac{1}{L}\int_{-L}^{L}\left[f(x)\right]^2 dx = \frac{a_0}{2L}\int_{-L}^{L} f(x)\,dx + \frac{1}{L}\int_{-L}^{L} f(x)\sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right]dx$$

INTERCHANGE INTEGRATION AND SUMMATION

$$\Rightarrow \frac{1}{L}\int_{-L}^{L}\left[f(x)\right]^2 dx = \frac{a_0}{2}\left[\frac{1}{L}\int_{-L}^{L} f(x)\,dx\right] + \sum_{n=1}^{\infty}\left[a_n\frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx + b_n\frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx\right]$$

$$\Rightarrow \frac{1}{L}\int_{-L}^{L}\left[f(x)\right]^2 dx = \frac{a_0}{2}\times a_0 + \sum_{n=1}^{\infty}\left[a_n\times a_n + b_n\,b_n\right]$$

$$\Rightarrow \frac{1}{L}\int_{-L}^{L}\left[f(x)\right]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left[a_n^2 + b_n^2\right]$$

WHICH IS PARSEVAL'S IDENTITY

d) USING PARSEVAL'S IDENTITY WITH $f(x) = \frac{1}{2} + \frac{1}{2}\operatorname{sign}(x)$ IN THE INTERVAL $(-\pi, \pi)$

$$\Rightarrow \frac{1}{\pi}\int_{-\pi}^{\pi}\left[f(x)\right]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

$$\Rightarrow \frac{1}{\pi}\int_{0}^{\pi} 1^2\,dx = \frac{1^2}{2} + \sum_{n=1}^{\infty}\left[0^2 + \left[\frac{2}{(2n-1)\pi}\right]^2\right]$$

$$\Rightarrow \frac{1}{\pi}\int_{0}^{\pi} 1\,dx = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{4}{\pi^2(2n-1)^2}$$

$$\Rightarrow \frac{1}{\pi}\times\pi = \frac{1}{2} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

$$\Rightarrow 1 = \frac{1}{2} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

$$\Rightarrow \frac{1}{2} = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

AS REQUIRED

Question 4

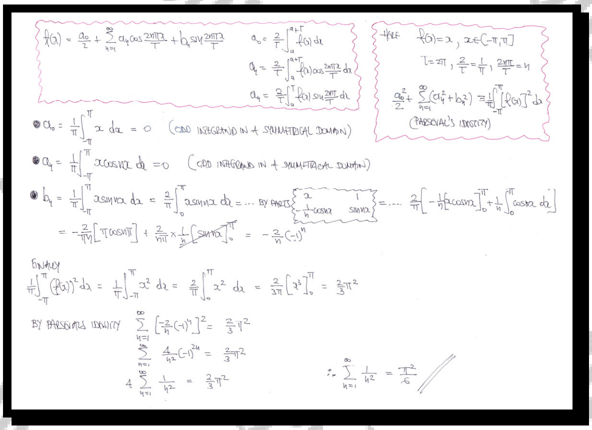
$$f(x) = x, \ x \in \mathbb{R}, \ -\pi \le x \le \pi .$$

$$f(x) = f(x + 2\pi).$$

Use Parseval's identity for the Fourier coefficients of $f(x)$ to determine the exact value of

$$\sum_{n=1}^{\infty} \frac{1}{n^2} .$$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$$


$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{T} + b_n \sin\frac{2n\pi x}{T}$, $a_0 = \frac{2}{T}\int_a^{a+T} f(x)\,dx$, $a_n = \frac{2}{T}\int_a^{a+T} f(x)\cos\frac{2n\pi x}{T}\,dx$, $b_n = \frac{2}{T}\int_a^{T} f(x)\sin\frac{2n\pi x}{T}\,dx$

HERE $f(x) = x$, $x \in (-\pi, \pi]$

$T = 2\pi$, $\frac{2}{T} = \frac{1}{\pi}$, $\frac{2n\pi}{T} = n$

$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx$ (PARSEVAL'S IDENTITY)

- $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = 0$ (ODD INTEGRAND IN A SYMMETRICAL DOMAIN)
- $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx = 0$ (ODD INTEGRAND IN A SYMMETRICAL DOMAIN)
- $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx = \dots$ BY PARTS $\left\{ \begin{matrix} x & 1 \\ -\frac{1}{n}\cos nx & \sin nx \end{matrix} \right\} = \dots \frac{2}{\pi}\left[-\frac{1}{n}\left[x\cos nx\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,dx \right]$

$= -\frac{2}{\pi n}\left[\pi\cos n\pi\right] + \frac{2}{n\pi}\times\frac{1}{n}\left[\sin nx\right]_0^{\pi} = -\frac{2}{n}(-1)^n$

FINALLY

$\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{3\pi}\left[x^3\right]_0^{\pi} = \frac{2}{3}\pi^2$

BY PARSEVAL'S IDENTITY $\sum_{n=1}^{\infty}\left[\frac{-2}{n}(-1)^n\right]^2 = \frac{2}{3}\pi^2$

$\sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^{2n} = \frac{2}{3}\pi^2$

$4\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{2}{3}\pi^2$

$\therefore \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$

Question 5

$$f(x) = x^2, \ x \in \mathbb{R}, \ -\pi \le x \le \pi .$$

$$f(x) = f(x + 2\pi) .$$

Use Parseval's identity for the Fourier coefficients of $f(x)$ to determine the exact value of

$$\sum_{n=1}^{\infty} \frac{1}{n^4} .$$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{T} + b_n \sin\frac{2n\pi x}{T}$$

$$a_0 = \frac{2}{T}\int_a^{a+T} f(x)\,dx \qquad a_n = \frac{2}{T}\int_a^{a+T} f(x)\cos\frac{2n\pi x}{T}\,dx \qquad b_n = \frac{2}{T}\int_a^{a+T} f(x)\sin\frac{2n\pi x}{T}\,dx$$

$f(x) = x^2, \ x \in [-\pi, \pi]$

$T = 2\pi, \ \frac{2}{T} = \frac{1}{\pi}, \ \frac{2n\pi}{T} = n$

PARSEVAL'S IDENTITY

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \equiv \frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\,dx$$

- $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{3\pi}\left[x^3\right]_0^{\pi} = \frac{2}{3}\pi^2$

- $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx = \frac{2}{\pi}\left[\frac{1}{n}\left[x^2\sin nx\right]_0^{\pi} - \frac{2}{n}\int_0^{\pi} x\sin nx\,dx\right]$ (BY PARTS)

$= \frac{-4}{n\pi}\int_0^{\pi} x\sin nx\,dx = -\frac{4}{n\pi}\left[-\frac{1}{n}\left[x\cos nx\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,dx\right]$ (BY PARTS AGAIN)

$= \frac{4}{n^2\pi}\left[\pi\cos n\pi\right] = \frac{4}{n^2}(-1)^n$

x^2	$2x$
$\frac{1}{n}\sin nx$	$\cos nx$

x	1
$-\frac{1}{n}\cos nx$	$\sin nx$

- $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin nx\,dx = 0$ (ODD INTEGRAND IN A SYMMETRICAL DOMAIN)

FINALLY $\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^4\,dx = \frac{2}{\pi}\int_0^{\pi} x^4\,dx = \frac{2}{5\pi}\left[x^5\right]_0^{\pi} = \frac{2}{5\pi}\left[\pi^5 - 0\right] = \frac{2}{5}\pi^4$

BY PARSEVAL'S IDENTITY:

$$\frac{1}{2}\left(\frac{2}{3}\pi^2\right)^2 + \sum_{n=1}^{\infty}\left[\frac{4}{n^2}(-1)^n\right]^2 = \frac{2}{5}\pi^4$$

$$\frac{2}{9}\pi^4 + \sum_{n=1}^{\infty}\frac{16}{n^4}(-1)^{2n} = \frac{2}{5}\pi^4$$

$$16\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{8}{45}\pi^4$$

$$\therefore \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

Question 6

A function $f(x)$ is defined in an interval $(-L,L)$, $L>0$.

a) State the general formula for the Fourier series of $f(x)$ in $(-L,L)$, giving general expressions for the coefficients of the series.

b) Prove the validity of Parseval's identity for the Fourier series of $f(x)$ in the interval $(-L,L)$.

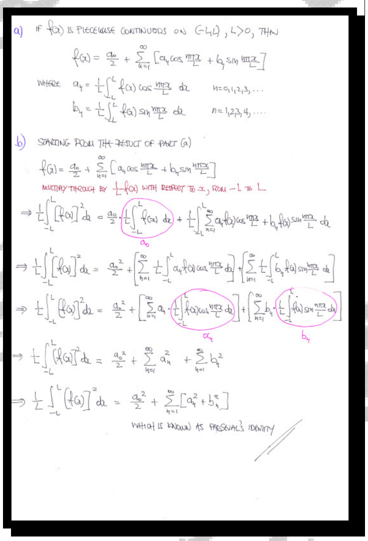
c) Find the Fourier series of

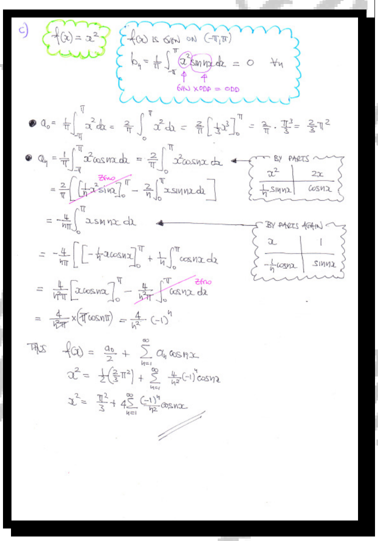
$$f(x)=x^2,\quad -\pi \le x \le \pi .$$

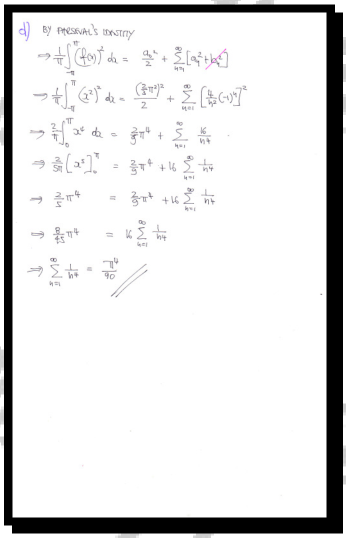
d) Hence show that

$$\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}.$$

$$\boxed{x^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^2}\cos nx\right]}$$







Question 7

A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

a) State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.

b) State and prove Parseval's identity for the Fourier series of $f(x)$ in $(-L, L)$.

c) By considering the Fourier series of

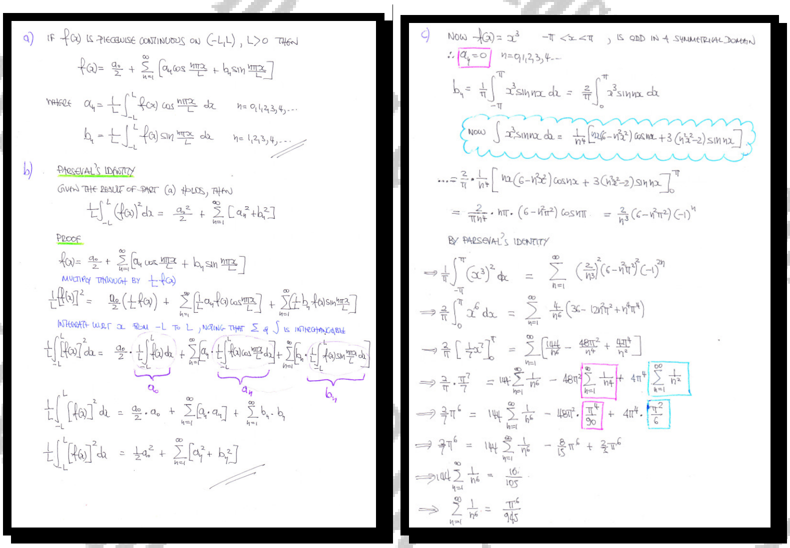
$$f(x) = x^3, \; -\pi \le x \le \pi,$$

show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$.

You may use without proof the following results.

- $\displaystyle\int x^3 \sin nx \, dx = \frac{1}{n^4}\left[nx\left(6 - n^2x^2\right)\cos nx + 3\left(n^2x^2 - 2\right)\sin nx\right] + C$
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$

proof



FOURIER SERIES
Complex Expansions

Question 1

A periodic function $f(t)$ is defined in the interval $(-L, L)$, $L > 0$, $f(t+2L) = f(t)$.

It is further given that $f(t)$ is continuous or piecewise continuous in $(-L, L)$ and has Fourier series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)\right],$$

where $a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)\, dt$, $n = 0, 1, 2, 3, \ldots$

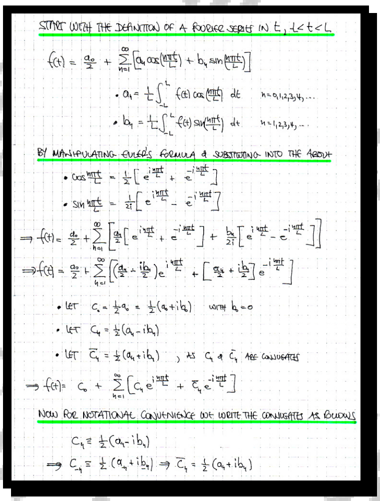
and $b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)\, dt$, $n = 1, 2, 3, 4, \ldots$

Show that the complex Fourier series expansion of $f(t)$ is

$$f(t) = \sum_{n=-\infty}^{\infty}\left[c_n \mathrm{e}^{\frac{\mathrm{i}n\pi t}{L}}\right],$$

where $c_n = \frac{1}{2L}\int_{-L}^{L} f(t)\, \mathrm{e}^{-\frac{\mathrm{i}n\pi t}{L}}\, dt$, $n \in \mathbb{Z}$

[ ], [proof]



START WITH THE DEFINITION OF A FOURIER SERIES IN t, $-L<t<L$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)\right]$$

- $a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right) dt$ $\quad n=0,1,2,3,4,\ldots$
- $b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right) dt$ $\quad n=1,2,3,4,\ldots$

BY MANIPULATING EULER'S FORMULA & SUBSTITUTING INTO THE ABOVE

- $\cos\frac{n\pi t}{L} = \frac{1}{2}\left[e^{i\frac{n\pi t}{L}} + e^{-i\frac{n\pi t}{L}}\right]$
- $\sin\frac{n\pi t}{L} = \frac{1}{2i}\left[e^{i\frac{n\pi t}{L}} - e^{-i\frac{n\pi t}{L}}\right]$

$$\Rightarrow f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\frac{a_n}{2}\left[e^{i\frac{n\pi t}{L}} + e^{-i\frac{n\pi t}{L}}\right] + \frac{b_n}{2i}\left[e^{i\frac{n\pi t}{L}} - e^{-i\frac{n\pi t}{L}}\right]\right]$$

$$\Rightarrow f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\left(\frac{a_n}{2} - \frac{ib_n}{2}\right)e^{i\frac{n\pi t}{L}} + \left[\frac{a_n}{2} + \frac{ib_n}{2}\right]e^{-i\frac{n\pi t}{L}}\right]$$

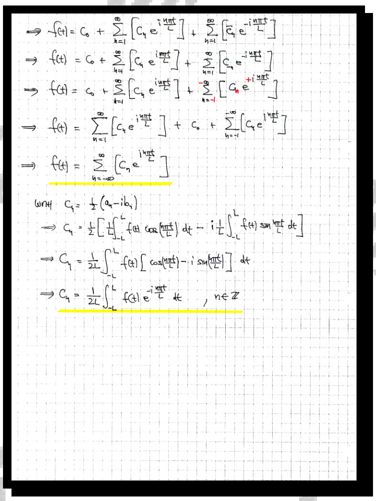
- LET $c_0 = \frac{1}{2}a_0 = \frac{1}{2}(a_0 + ib_0)$ WITH $b_0 = 0$
- LET $c_n = \frac{1}{2}(a_n - ib_n)$
- LET $\bar{c}_n = \frac{1}{2}(a_n + ib_n)$, AS c_n & $\bar{c}_n$ ARE CONJUGATES

$$\Rightarrow f(t) = c_0 + \sum_{n=1}^{\infty}\left[c_n e^{i\frac{n\pi t}{L}} + \bar{c}_n e^{-i\frac{n\pi t}{L}}\right]$$

NOW FOR NOTATIONAL CONVENIENCE WE WRITE THE CONJUGATES AS FOLLOWS

$c_n \equiv \frac{1}{2}(a_n - ib_n)$

$\Rightarrow c_{-n} \equiv \frac{1}{2}(a_{-n} + ib_{-n}) \Rightarrow \bar{c}_n = \frac{1}{2}(a_n + ib_n)$



$$\Rightarrow f(t) = c_0 + \sum_{n=1}^{\infty}\left[c_n e^{i\frac{n\pi t}{L}}\right] + \sum_{n=1}^{\infty}\left[\bar{c}_n e^{-i\frac{n\pi t}{L}}\right]$$

$$\Rightarrow f(t) = c_0 + \sum_{n=1}^{\infty}\left[c_n e^{i\frac{n\pi t}{L}}\right] + \sum_{n=1}^{\infty}\left[c_{-n} e^{-i\frac{n\pi t}{L}}\right]$$

$$\Rightarrow f(t) = c_0 + \sum_{n=1}^{\infty}\left[c_n e^{i\frac{n\pi t}{L}}\right] + \sum_{n=-1}^{-\infty}\left[c_n e^{+i\frac{n\pi t}{L}}\right]$$

$$\Rightarrow f(t) = \sum_{n=1}^{\infty}\left[c_n e^{i\frac{n\pi t}{L}}\right] + c_0 + \sum_{n=-1}^{-\infty}\left[c_n e^{i\frac{n\pi t}{L}}\right]$$

$$\Rightarrow f(t) = \sum_{n=-\infty}^{\infty}\left[c_n e^{i\frac{n\pi t}{L}}\right]$$

WITH $c_n = \frac{1}{2}(a_n - ib_n)$

$$\Rightarrow c_n = \frac{1}{2}\left[\frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right) dt - i\frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\, dt\right]$$

$$\Rightarrow c_n = \frac{1}{2L}\int_{-L}^{L} f(t)\left[\cos\left(\frac{n\pi t}{L}\right) - i\sin\left(\frac{n\pi t}{L}\right)\right] dt$$

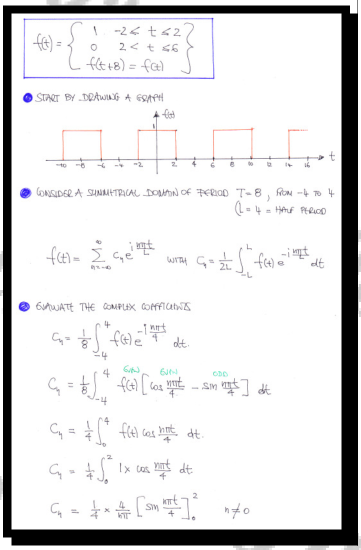
$$\Rightarrow c_n = \frac{1}{2L}\int_{-L}^{L} f(t)\, e^{-i\frac{n\pi t}{L}}\, dt \quad , n \in \mathbb{Z}$$

Question 2

$$f(t)=\begin{cases}1 & -2\le t\le 2\\ 0 & 2<t<6\end{cases}, \qquad f(t+8)=f(t).$$

Determine the complex Fourier series expansion of $f(t)$.

$$\boxed{f(t)=\tfrac{1}{2}+\tfrac{1}{2}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left[e^{\frac{1}{4}n\pi t \mathrm{i}}\ \mathrm{sinc}\left(\tfrac{1}{2}n\pi\right)\right]}$$



$$f(t)=\begin{cases}1 & -2\le t\le 2\\ 0 & 2<t\le 6\\ f(t+8)=f(t)\end{cases}$$

- START BY DRAWING A GRAPH

- CONSIDER A SYMMETRICAL DOMAIN OF PERIOD $T=8$, FROM -4 TO 4 ($L=4$ = HALF PERIOD)

$$f(t)=\sum_{n=-\infty}^{\infty} C_n e^{i\frac{n\pi t}{L}} \quad \text{WITH} \quad C_n=\frac{1}{2L}\int_{-L}^{L} f(t)e^{-i\frac{n\pi t}{L}}\,dt$$

- EVALUATE THE COMPLEX COEFFICIENTS

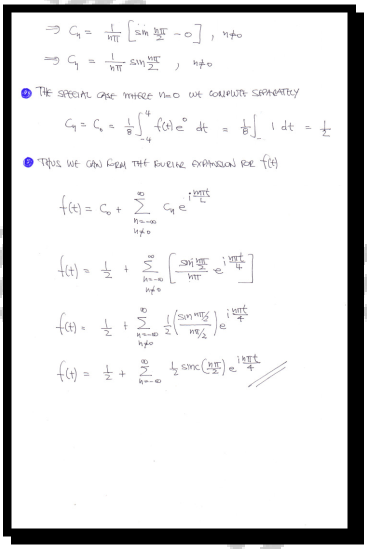
$$C_n=\frac{1}{8}\int_{-4}^{4} f(t)e^{-i\frac{n\pi t}{4}}\,dt.$$

$$C_n=\frac{1}{8}\int_{-4}^{4} f(t)\left[\cos\frac{n\pi t}{4}-\sin\frac{n\pi t}{4}\right]dt \quad (\text{EVEN, EVEN, ODD})$$

$$C_n=\frac{1}{4}\int_{0}^{4} f(t)\cos\frac{n\pi t}{4}\,dt.$$

$$C_n=\frac{1}{4}\int_{0}^{2} 1\times\cos\frac{n\pi t}{4}\,dt$$

$$C_n=\frac{1}{4}\times\frac{4}{n\pi}\left[\sin\frac{n\pi t}{4}\right]_0^2 \quad n\neq 0$$



$$\Rightarrow C_n=\frac{1}{n\pi}\left[\sin\frac{n\pi}{2}-0\right],\ n\neq 0$$

$$\Rightarrow C_n=\frac{1}{n\pi}\sin\frac{n\pi}{2},\ n\neq 0$$

- THE SPECIAL CASE WHERE $n=0$ WE COMPUTE SEPARATELY

$$C_n=C_0=\frac{1}{8}\int_{-4}^{4} f(t)e^{0}\,dt=\frac{1}{8}\int 1\,dt=\frac{1}{2}$$

- THUS WE CAN FORM THE FOURIER EXPANSION FOR $f(t)$

$$f(t)=C_0+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} C_n e^{i\frac{n\pi t}{L}}$$

$$f(t)=\frac{1}{2}+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left[\frac{\sin\frac{n\pi}{2}}{n\pi}e^{i\frac{n\pi t}{4}}\right]$$

$$f(t)=\frac{1}{2}+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{1}{2}\left(\frac{\sin n\pi/2}{n\pi/2}\right)e^{i\frac{n\pi t}{4}}$$

$$f(t)=\frac{1}{2}+\sum_{n=-\infty}^{\infty}\frac{1}{2}\mathrm{sinc}\left(\frac{n\pi}{2}\right)e^{i\frac{n\pi t}{4}}$$

Question 3

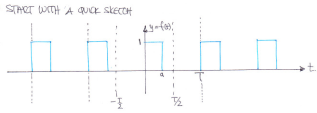
$$f(t) = \begin{cases} 1 & 0 \le t \le a \\ 0 & a < t < T \end{cases}, \qquad a < \tfrac{1}{2}T, \qquad f(t+T) = f(t).$$

Determine the complex Fourier series expansion of $f(t)$.

$$\boxed{f(t) = \frac{a}{T} + \frac{a}{T}\sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left[\exp\left(\frac{n\pi(2t-a)}{T} \right) \operatorname{sinc}\left(\frac{n\pi a}{T} \right) \right]}$$

$$f(t) = \begin{cases} 1 & 0 \le t \le a \\ 0 & a < t < T, \ a < \frac{T}{2} \\ f(t+T) = f(t) & \end{cases}$$

- START WITH A QUICK SKETCH



- THIS FUNCTION IS NEITHER ODD, NOR EVEN — PERIOD IS T — HALF PERIOD $\frac{T}{2} = L$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi t}{L}} \quad \text{with} \quad c_n = \frac{1}{2L}\int_{-L}^{L} f(t)\, e^{-i\frac{n\pi t}{L}}\, dt$$

- EVALUATE THE COMPLEX COEFFICIENTS

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\, e^{-i\frac{2n\pi t}{T}}\, dt.$$

$$c_n = \frac{1}{T}\int_{0}^{a} 1 \times e^{-i\frac{2n\pi t}{T}}\, dt$$

$$c_n = \frac{1}{T} \times \left[-\frac{T}{2n\pi i} e^{-i\frac{2n\pi t}{T}} \right]_0^a, \quad n \neq 0.$$

$$c_n = \frac{1}{2n\pi i}\left[e^{-i\frac{2n\pi t}{T}} \right]_a^0, \quad n \neq 0$$

$$c_n = \frac{1}{2n\pi i}\left[1 - e^{-i\frac{2n\pi}{T}} \right], \quad n \neq 0$$

$$\Rightarrow c_n = \frac{1}{2n\pi i} e^{-\frac{in\pi a}{T}} \left[e^{\frac{in\pi a}{T}} - e^{-i\frac{n\pi a}{T}} \right], \quad n \neq 0$$

$$\Rightarrow c_n = \frac{e^{-i\frac{n\pi a}{T}}}{n\pi} \times \frac{1}{2i}\left[e^{\frac{in\pi a}{T}} - e^{-i\frac{n\pi a}{T}} \right], \quad n \neq 0$$

$$\Rightarrow c_n = e^{-i\frac{n\pi a}{T}} \sin\left(\frac{n\pi a}{T}\right) \times \frac{1}{n\pi}, \quad n \neq 0$$

$$\Rightarrow c_n = e^{-i\frac{n\pi a}{T}} \sin\left(\frac{n\pi a}{T}\right) \times \frac{T}{n\pi a} \times \frac{a}{T}, \quad n \neq 0$$

$$\Rightarrow c_n = \frac{a}{T} e^{-i\frac{n\pi a}{T}} \operatorname{sinc}\left(\frac{n\pi a}{T}\right), \quad n \neq 0$$

- THE SPECIAL CASE WHERE $n=0$, WE CONSIDER SEPARATELY

$$c_0 = \frac{1}{T}\int_{-T/2}^{T/2} 1 \times e^0\, dt = \frac{1}{T}\int_0^a 1\, dt = \frac{a}{T}$$

- FINALLY WE HAVE A COMPLEX FOURIER SERIES

$$f(t) = c_0 + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} c_n e^{i\frac{n\pi t}{L}}$$

$$f(t) = \frac{a}{T} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left[\frac{a}{T} e^{-\frac{in\pi a}{T}} \operatorname{sinc}\frac{n\pi a}{T} \right] e^{\frac{i2n\pi t}{T}}$$

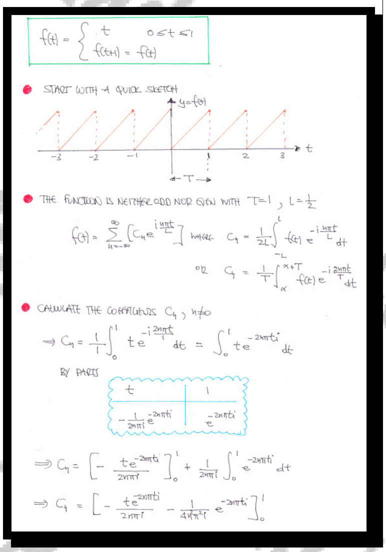
$$f(t) = \frac{a}{T} + \frac{a}{T} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left[e^{i\frac{n\pi}{T}(2t-a)} \operatorname{sinc}\frac{n\pi a}{T} \right]$$

Question 4

$$f(t)=t\,,\quad 0\le t<1\,,\qquad f(t+1)=f(t)\,.$$

Determine the complex Fourier series expansion of $f(t)$.

$$\boxed{f(t)=\frac{1}{2}+\frac{\mathrm{i}}{2\pi}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left[\frac{\mathrm{e}^{2n\pi t\mathrm{i}}}{n}\right]}$$



$$f(t)=\begin{cases} t & 0\le t\le 1\\ f(t+1)=f(t) \end{cases}$$

- START WITH A QUICK SKETCH

- THE FUNCTION IS NEITHER ODD NOR EVEN WITH $T=1$, $L=\frac{1}{2}$

$$f(t)=\sum_{n=-\infty}^{\infty}\left[C_n e^{i\frac{n\pi t}{L}}\right] \text{ WHERE } C_n=\frac{1}{2L}\int_{-L}^{L} f(t)\,e^{-i\frac{n\pi t}{L}}\,dt$$

$$\text{OR } C_n=\frac{1}{T}\int_{\alpha}^{\alpha+T} f(t)\,e^{-i\frac{2n\pi t}{T}}\,dt$$

- CALCULATE THE COEFFICIENTS C_n, $n\neq 0$

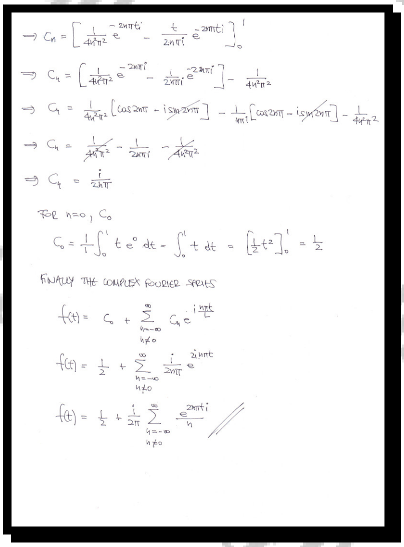
$$\Rightarrow C_n=\frac{1}{1}\int_0^1 t\,e^{-i\frac{2n\pi t}{1}}\,dt=\int_0^1 t\,e^{-2n\pi t i}\,dt$$

BY PARTS

t	1
$-\frac{1}{2n\pi i}e^{-2n\pi t i}$	$e^{-2n\pi t i}$

$$\Rightarrow C_n=\left[-\frac{t e^{-2n\pi t i}}{2n\pi i}\right]_0^1+\frac{1}{2n\pi i}\int_0^1 e^{-2n\pi t i}\,dt$$

$$\Rightarrow C_n=\left[-\frac{t e^{-2n\pi t i}}{2n\pi i}-\frac{1}{4n^2\pi^2 i}e^{-2n\pi t i}\right]_0^1$$



$$\Rightarrow C_n=\left[\frac{1}{4n^2\pi^2}e^{-2n\pi t i}-\frac{t}{2n\pi i}e^{-2n\pi t i}\right]_0^1$$

$$\Rightarrow C_n=\left[\frac{1}{4n^2\pi^2}e^{-2n\pi i}-\frac{1}{2n\pi i}e^{-2n\pi i}\right]-\frac{1}{4n^2\pi^2}$$

$$\Rightarrow C_n=\frac{1}{4n^2\pi^2}\left[\cos 2n\pi-i\sin 2n\pi\right]-\frac{1}{n\pi i}\left[\cos 2n\pi-i\sin 2n\pi\right]-\frac{1}{4n^2\pi^2}$$

$$\Rightarrow C_n=\frac{1}{4n^2\pi^2}-\frac{1}{2n\pi i}-\frac{1}{4n^2\pi^2}$$

$$\Rightarrow C_n=\frac{i}{2n\pi}$$

FOR $n=0$, C_0

$$C_0=\frac{1}{1}\int_0^1 t\,e^0\,dt=\int_0^1 t\,dt=\left[\frac{1}{2}t^2\right]_0^1=\frac{1}{2}$$

FINALLY THE COMPLEX FOURIER SERIES

$$f(t)=C_0+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} C_n e^{i\frac{n\pi t}{L}}$$

$$f(t)=\frac{1}{2}+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{i}{2n\pi}e^{2in\pi t}$$

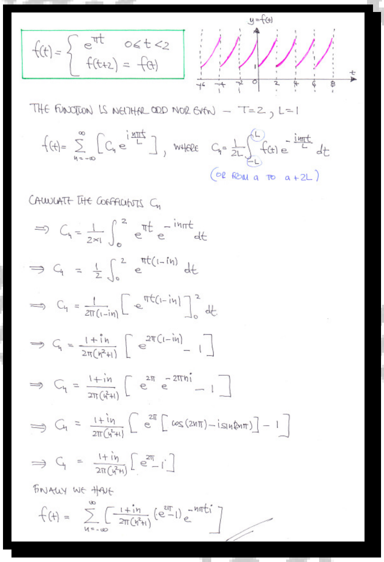
$$f(t)=\frac{1}{2}+\frac{i}{2\pi}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2n\pi t i}}{n}$$

Question 5

$$f(t) = e^{\pi t} \quad , \quad 0 \le t < 2 \, , \quad f(t+2) = f(t) \, .$$

Determine the complex Fourier series expansion of $f(t)$.

$$\boxed{f(t) = \frac{\left(e^{2\pi} - 1\right)}{2\pi} \sum_{n=-\infty}^{\infty} \left[\frac{(1 + in) e^{2n\pi t i}}{\left(n^2 + 1\right)} \right]}$$



$f(t) = \begin{cases} e^{\pi t} & 0 \le t < 2 \\ f(t+2) = f(t) \end{cases}$

THE FUNCTION IS NEITHER ODD NOR EVEN — $T = 2, L = 1$

$f(t) = \sum_{n=-\infty}^{\infty} \left[C_n e^{i\frac{n\pi t}{L}} \right]$, WHERE $C_n = \frac{1}{2L}\int_{-L}^{L} f(t) e^{-\frac{in\pi t}{L}} \, dt$ (OR FROM a TO a+2L)

CALCULATE THE COEFFICIENTS C_n

$\Rightarrow C_n = \frac{1}{2\times 1}\int_0^2 e^{\pi t} e^{-in\pi t} \, dt$

$\Rightarrow C_n = \frac{1}{2}\int_0^2 e^{\pi t(1-in)} \, dt$

$\Rightarrow C_n = \frac{1}{2\pi(1-in)}\left[e^{\pi t(1-in)} \right]_0^2 dt$

$\Rightarrow C_n = \frac{1+in}{2\pi(n^2+1)}\left[e^{2\pi(1-in)} - 1 \right]$

$\Rightarrow C_n = \frac{1+in}{2\pi(n^2+1)}\left[e^{2\pi} e^{-2\pi n i} - 1 \right]$

$\Rightarrow C_n = \frac{1+in}{2\pi(n^2+1)}\left[e^{2\pi}\left[\cos(2n\pi) - i\sin(2n\pi)\right] - 1 \right]$

$\Rightarrow C_n = \frac{1+in}{2\pi(n^2+1)}\left[e^{2\pi} - 1 \right]$

FINALLY WE HAVE

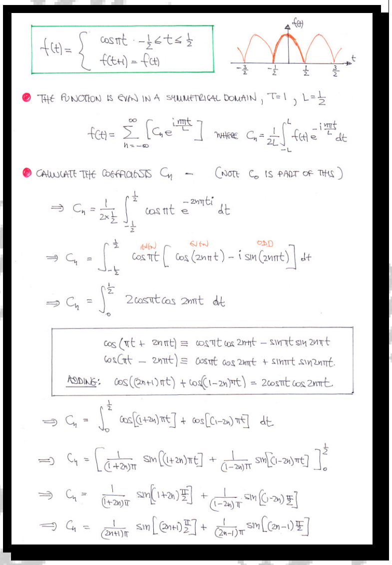
$f(t) = \sum_{n=-\infty}^{\infty}\left[\frac{1+in}{2\pi(n^2+1)}(e^{2\pi}-1) e^{-n\pi t i} \right]$

Question 6

$$f(t) = \cos(\pi t) \ , \ -\tfrac{1}{2} \le t < \tfrac{1}{2}, \quad f(t+1) = f(t).$$

Determine the complex Fourier series expansion of $f(t)$.

$$f(t) = \frac{2}{\pi} \sum_{n=-\infty}^{\infty} \left[\frac{(-1)^{n+1} e^{2n\pi t i}}{4n^2 - 1} \right]$$



$f(t) = \begin{cases} \cos \pi t & -\frac{1}{2} \le t \le \frac{1}{2} \\ f(t+1) = f(t) \end{cases}$

- THE FUNCTION IS EVEN IN A SYMMETRICAL DOMAIN, $T=1$, $L=\frac{1}{2}$

$f(t) = \sum_{n=-\infty}^{\infty} \left[C_n e^{i\frac{n\pi t}{L}} \right]$ WHERE $C_n = \frac{1}{2L} \int_{-L}^{L} f(t) e^{-i\frac{n\pi t}{L}} \, dt$

- CALCULATE THE COEFFICIENTS C_n – (NOTE C_0 IS PART OF THIS)

$\Rightarrow C_n = \frac{1}{2 \times \frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos \pi t \, e^{-2n\pi t i} \, dt$

$\Rightarrow C_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos \pi t \left[\cos(2n\pi t) - i \sin(2n\pi t) \right] dt$ (EVEN, EVEN, ODD)

$\Rightarrow C_n = \int_0^{\frac{1}{2}} 2 \cos \pi t \cos 2n\pi t \, dt$

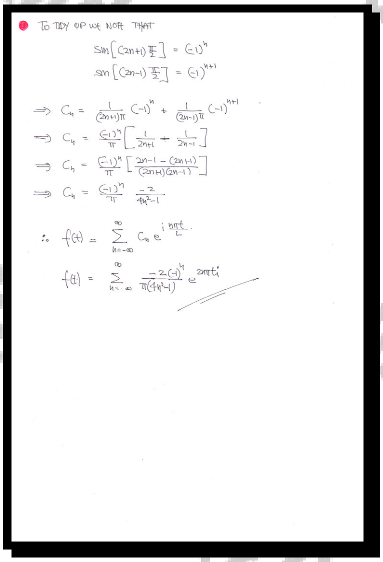
$\cos(\pi t + 2n\pi t) \equiv \cos \pi t \cos 2n\pi t - \sin \pi t \sin 2n\pi t$
$\cos(\pi t - 2n\pi t) \equiv \cos \pi t \cos 2n\pi t + \sin \pi t \sin 2n\pi t$
ADDING: $\cos((2n+1)\pi t) + \cos((1-2n)\pi t) = 2 \cos \pi t \cos 2n\pi t$

$\Rightarrow C_n = \int_0^{\frac{1}{2}} \cos[(1+2n)\pi t] + \cos[(1-2n)\pi t] \, dt$

$\Rightarrow C_n = \left[\frac{1}{(1+2n)\pi} \sin[(1+2n)\pi t] + \frac{1}{(1-2n)\pi} \sin[(1-2n)\pi t] \right]_0^{\frac{1}{2}}$

$\Rightarrow C_n = \frac{1}{(1+2n)\pi} \sin\left[(1+2n)\frac{\pi}{2}\right] + \frac{1}{(1-2n)\pi} \sin\left[(1-2n)\frac{\pi}{2}\right]$

$\Rightarrow C_n = \frac{1}{(2n+1)\pi} \sin\left[(2n+1)\frac{\pi}{2}\right] + \frac{1}{(2n-1)\pi} \sin\left[(2n-1)\frac{\pi}{2}\right]$



- TO TIDY UP WE NOTE THAT

$\sin\left[(2n+1)\frac{\pi}{2}\right] = (-1)^n$
$\sin\left[(2n-1)\frac{\pi}{2}\right] = (-1)^{n+1}$

$\Rightarrow C_n = \frac{1}{(2n+1)\pi}(-1)^n + \frac{1}{(2n-1)\pi}(-1)^{n+1}$

$\Rightarrow C_n = \frac{(-1)^n}{\pi} \left[\frac{1}{2n+1} - \frac{1}{2n-1} \right]$

$\Rightarrow C_n = \frac{(-1)^n}{\pi} \left[\frac{2n-1-(2n+1)}{(2n+1)(2n-1)} \right]$

$\Rightarrow C_n = \frac{(-1)^n}{\pi} \frac{-2}{4n^2-1}$

$\therefore f(t) = \sum_{n=-\infty}^{\infty} C_n e^{i\frac{n\pi t}{L}}$

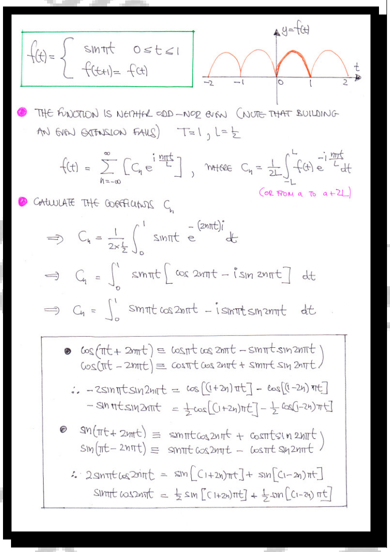
$f(t) = \sum_{n=-\infty}^{\infty} \frac{-2(-1)^n}{\pi(4n^2-1)} e^{2n\pi t i}$

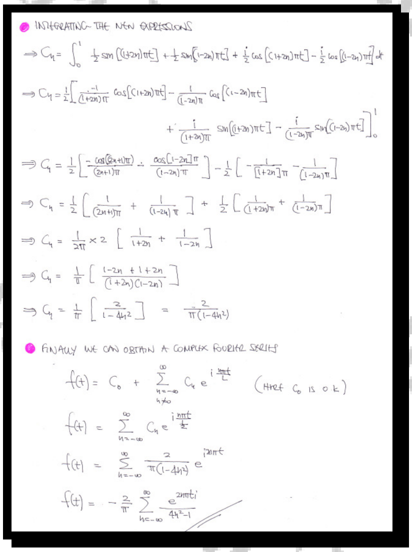
Question 7

$$f(t) = \sin(\pi t) \ , \ 0 \le t < 1 , \quad f(t+1) = f(t) .$$

Determine the complex Fourier series expansion of $f(t)$.

$$f(t) = -\frac{2}{\pi} \sum_{n=-\infty}^{\infty} \left[\frac{e^{2n\pi t i}}{4n^2 - 1} \right]$$





Question 8

The function f is defined as

$$f(t) = V\cos\left(\frac{\pi t}{T}\right) \ , \ -\tfrac{1}{2}T \le t < \tfrac{1}{2}T \ , \quad f(t) = f(t+T),$$

where V and T are positive constants.

Determine the complex Fourier series expansion of $f(t)$.

$$\boxed{f(t) = \frac{2V}{\pi}\sum_{n=-\infty}^{\infty}\left[\frac{e^{2n\pi t i}}{\left(1-4n^2\right)}\right]}$$