FOURIER TRANSFORM

Fourier Transform Summary

Definitions

- $\mathcal{F}\left[f(x)\right] = \hat{f}(k) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-\mathrm{i}kx}\ dx$

- $\mathcal{F}^{-1}\left[\hat{f}(k)\right] = f(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(k)\,\mathrm{e}^{\mathrm{i}kx}\ dk$

Useful Results

- $\mathcal{F}\left[f'(x)\right] = \mathrm{i}k\,\hat{f}(k)$

- $\mathcal{F}\left[x f(x)\right] = \mathrm{i}\dfrac{d}{dk}\left[\hat{f}(k)\right]$

Shift Results

- $\mathcal{F}\left[f(x+c)\right] = \mathrm{e}^{\mathrm{i}kc}\,\hat{f}(k)$

- $\mathcal{F}^{-1}\left[\hat{f}(k+c)\right] = \mathrm{e}^{-\mathrm{i}cx}\,f(x)$

Convolution Theorem

$$\mathcal{F}\{[f*g](x)\} = \sqrt{2\pi}\,\mathcal{F}\left[f(x)\right]\mathcal{F}\left[g(x)\right]$$

where $[f*g](x) = \displaystyle\int_{-\infty}^{\infty} f(x-y)\,g(y)\ dy$

Parseval's Theorem

$$\int_{-\infty}^{\infty} h(y)\,g(y)\ dy = \int_{-\infty}^{\infty} \overline{\hat{h}}(k)\,\hat{g}(k)\ dk \quad \text{or} \quad \int_{-\infty}^{\infty} \left|h(y)\right|^2\ dy = \int_{-\infty}^{\infty} \left|\hat{h}(k)\right|^2\ dk$$

FINDING FOURIER TRANSFORMS and INVERSES

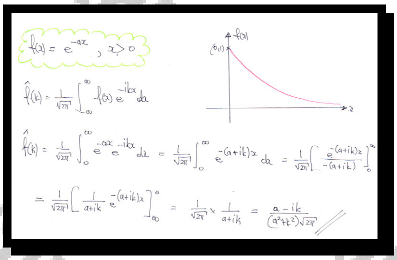
Question 1

$$f(x) = \mathrm{e}^{-ax}, \; x > 0,$$

where a is a positive constant.

Find the Fourier transform of $f(x)$.

$$\boxed{\hat{f}(k) = \frac{a - \mathrm{i}k}{\left(a^2 + k^2\right)\sqrt{2\pi}}}$$



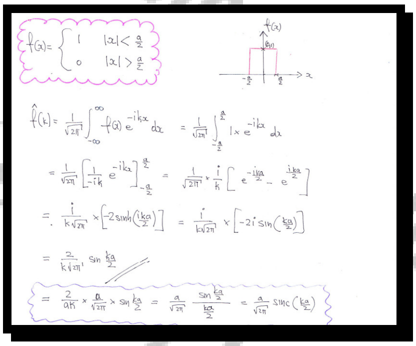
Question 2

$$f(x) = \begin{cases} 1 & |x| < \frac{1}{2}a \\ 0 & |x| > \frac{1}{2}a \end{cases},$$

where a is a positive constant.

Find the Fourier transform of $f(x)$.

$$\boxed{\hat{f}(k) = \frac{2}{k\sqrt{2\pi}}\sin\left(\frac{1}{2}ka\right) = \frac{a}{\sqrt{2\pi}}\operatorname{sinc}\left(\frac{1}{2}ka\right)}$$

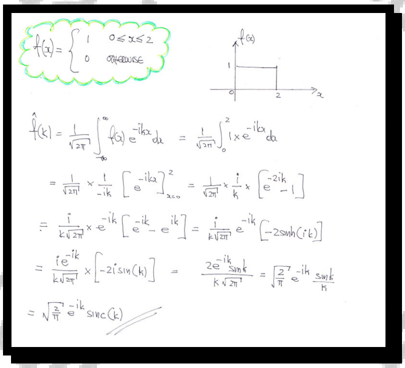


Question 3

$$f(x) = \begin{cases} 1 & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}.$$

Find the Fourier transform of $f(x)$.

$$\boxed{\hat{f}(k) = \sqrt{\frac{2}{\pi}}\, \mathrm{e}^{-\mathrm{i}k} \operatorname{sinc} k}$$



$f(x) = \begin{cases} 1 & 0 \le x \le 2 \\ 0 & \text{OTHERWISE} \end{cases}$

$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx = \frac{1}{\sqrt{2\pi}} \int_0^2 1 \times e^{-ikx}\, dx$

$= \frac{1}{\sqrt{2\pi}} \times \frac{1}{-ik} \left[e^{-ikx} \right]_{x=0}^{2} = \frac{1}{\sqrt{2\pi}} \times \frac{i}{k} \times \left[e^{-2ik} - 1 \right]$

$= \frac{i}{k\sqrt{2\pi}} \times e^{-ik} \left[e^{-ik} - e^{ik} \right] = \frac{i}{k\sqrt{2\pi}} e^{-ik} \left[-2\sinh(ik) \right]$

$= \frac{ie^{-ik}}{k\sqrt{2\pi}} \times \left[-2i\sin(k) \right] = \frac{2e^{-ik}\sin k}{k\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}} e^{-ik} \frac{\sin k}{k}$

$= \sqrt{\frac{2}{\pi}} e^{-ik} \operatorname{sinc}(k)$

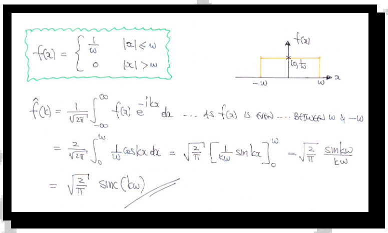
Question 4

$$f(x) = \begin{cases} \dfrac{1}{\omega} & |x| \le \omega \\ 0 & |x| > \omega \end{cases}$$

where ω is a positive constant.

Find the Fourier transform of $f(x)$.

$$\boxed{\hat{f}(k) = \sqrt{\frac{2}{\pi}} \operatorname{sinc} \omega}$$



$f(x) = \begin{cases} \frac{1}{\omega} & |x| \le \omega \\ 0 & |x| > \omega \end{cases}$

$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx$ ... AS $f(x)$ IS EVEN .... BETWEEN ω & $-\omega$

$= \frac{2}{\sqrt{2\pi}} \int_0^{\omega} \frac{1}{\omega} \cos kx\, dx = \sqrt{\frac{2}{\pi}} \left[\frac{1}{k\omega} \sin kx \right]_0^{\omega} = \sqrt{\frac{2}{\pi}} \frac{\sin k\omega}{k\omega}$

$= \sqrt{\frac{2}{\pi}} \operatorname{sinc}(k\omega)$

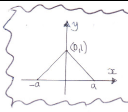
Question 5

The function $f(x)$ is defined in terms of the positive constant a, by

$$f(x)=\begin{cases} 1-\dfrac{|x|}{a} & |x|\le a \\ 0 & |x|>a \end{cases}$$

Find the Fourier transform of $f(x)$.

$$\boxed{\mathcal{F}\left[f(x)\right]=\hat{f}(k)=\sqrt{\frac{2}{\pi}}\,\frac{1}{ak^2}\left[1-\cos(ak)\right] = \frac{a}{\sqrt{2\pi}}\operatorname{sinc}^2\left(\tfrac{1}{2}ka\right)}$$



$$f(x)=\begin{cases} 1-\frac{|x|}{a} & |x|\le a \\ 0 & |x|>a \end{cases}$$

$$\hat{f}(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{-ikx}\left(1-\frac{|x|}{a}\right)dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{0}^{a} 2\left(1-\frac{x}{a}\right)\cos kx$$ (AS THE ODD INTEGRAND VANISHES & THE EVEN WE MAY DOUBLE UP)

BY PARTS

$2\left(1-\frac{x}{a}\right)$	$-\frac{2}{a}$
$\frac{1}{k}\sin kx$	$\cos kx$

$$= \frac{1}{\sqrt{2\pi}}\left\{\cancel{\left[\frac{2}{k}\left(1-\frac{x}{a}\right)\sin kx\right]_0^a} + \frac{2}{ak}\int_0^a \sin kx\right\}$$

$$= \frac{2}{ak\sqrt{2\pi}}\left[-\frac{1}{k}\cos kx\right]_0^a$$

$$= \frac{2}{ak^2\sqrt{2\pi}}\left[\cos kx\right]_a^0$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{1}{ak^2}\left[1-\cos ka\right]$$

OR CONTINUE THE MANIPULATION

$$= \sqrt{\frac{2}{\pi}}\,\frac{1}{ak^2}\left[1-\left(1-2\sin^2\frac{ka}{2}\right)\right]$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{2}{ak^2}\sin^2\left(\frac{ka}{2}\right)$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{4a}{2a^2k^2}\sin^2\left(\frac{ka}{2}\right)$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\,a \times \frac{4}{a^2k^2}\sin^2\left(\frac{ka}{2}\right) = \frac{a}{\sqrt{2\pi}}\operatorname{sinc}^2\left(\frac{ka}{2}\right)$$

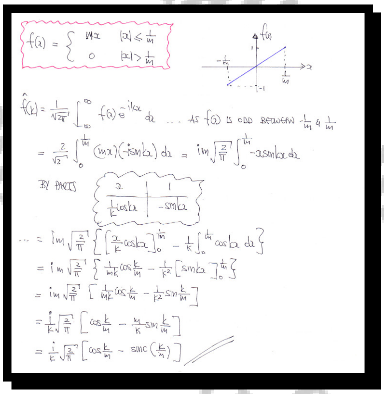
Question 6

$$f(x) = \begin{cases} mx & |x| \le \dfrac{1}{m} \\ 0 & |x| > \dfrac{1}{m} \end{cases}.$$

where m is a positive constant.

Find the Fourier transform of $f(x)$.

$$\boxed{\hat{f}(k) = \frac{\mathrm{i}}{k}\sqrt{\frac{2}{\pi}}\left[\cos\left(\frac{k}{m}\right) - \operatorname{sinc}\left(\frac{k}{m}\right)\right]}$$



Question 7

$$f(x) = x\mathrm{e}^{-2x}, \;\; x > 0 .$$

Find, by direct integration, the Fourier transform of $f(x)$.

$$\boxed{\hat{f}(k) = \frac{1}{(2+\mathrm{i}k)^2\sqrt{2\pi}}}$$

BY THE DEFINITION OF THE FOURIER TRANSFORM

$$\boxed{f(x) = xe^{-2x}, \;\; x>0}$$

$$\Rightarrow \hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

$$\Rightarrow \hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} xe^{-2x} e^{-ikx}\, dx$$

$$\Rightarrow \hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} x e^{(-2-ik)x}\, dx$$

PROCEED BY INTEGRATION BY PARTS (IGNORING THE $\frac{1}{\sqrt{2\pi}}$)

$$\Rightarrow \hat{f}(k) = \cancel{\left[\frac{xe^{(-2-ik)x}}{-2-ik}\right]_0^{\infty}} - \int_0^{\infty} \frac{e^{(-2-ik)x}}{-2-ik}\, dx$$

x	1
$\frac{e^{(-2-ik)x}}{-2-ik}$	$e^{(-2-ik)x}$

$$\Rightarrow \hat{f}(k) = \frac{-1}{\sqrt{2\pi}}\int_0^{\infty} \frac{e^{(-2-ik)x}}{-2-ik}\, dx$$

$$\Rightarrow \hat{f}(k) = \frac{-1}{\sqrt{2\pi}(-2-ik)^2}\left[e^{(-2-ik)x}\right]_0^{\infty}$$

$$\Rightarrow \hat{f}(k) = \frac{-1}{\sqrt{2\pi}(2+ik)^2}\left[0-1\right]$$

$$\Rightarrow \hat{f}(k) = \frac{1}{\sqrt{2\pi}(2+ik)^2}$$

Question 8

The triangle function $\Lambda_n(x)$ is defined as

$$\Lambda_n(x) = \begin{cases} \dfrac{1}{n^2}(n+x) & -n < x < 0 \\ \dfrac{1}{n^2}(n-x) & 0 < x < n \\ 0 & \text{otherwise} \end{cases}$$

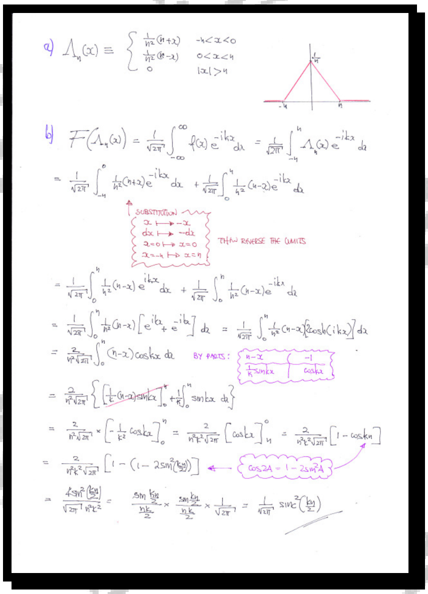
where n is a positive constant.

a) Sketch the graph of $\Lambda_n(x)$.

b) Show that the Fourier transform of $\Lambda_n(x)$ is

$$\frac{1}{\sqrt{2\pi}} \operatorname{sinc}^2\left(\tfrac{1}{2}kn\right).$$

proof



a) $\Lambda_n(x) \equiv \begin{cases} \frac{1}{n^2}(n+x) & -n<x<0 \\ \frac{1}{n^2}(n-x) & 0<x<n \\ 0 & |x|>n \end{cases}$

b) $\mathcal{F}(\Lambda_n(x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-n}^{n} \Lambda_n(x)e^{-ikx}\,dx$

$= \frac{1}{\sqrt{2\pi}}\int_{-n}^{0} \frac{1}{n^2}(n+x)e^{-ikx}\,dx + \frac{1}{\sqrt{2\pi}}\int_{0}^{n} \frac{1}{n^2}(n-x)e^{-ikx}\,dx$

SUBSTITUTION
$x \mapsto -x$
$dx \mapsto -dx$
$x=0 \mapsto x=0$
$x=-n \mapsto x=n$

THEN REVERSE THE LIMITS

$= \frac{1}{\sqrt{2\pi}}\int_{0}^{n} \frac{1}{n^2}(n-x)e^{ikx}\,dx + \frac{1}{\sqrt{2\pi}}\int_{0}^{n} \frac{1}{n^2}(n-x)e^{-ikx}\,dx$

$= \frac{1}{\sqrt{2\pi}}\int_{0}^{n} \frac{1}{n^2}(n-x)\left[e^{ikx}+e^{-ikx}\right]dx = \frac{1}{\sqrt{2\pi}}\int_{0}^{n} \frac{1}{n^2}(n-x)\left[2\cosh(ikx)\right]dx$

$= \frac{2}{n^2\sqrt{2\pi}}\int_{0}^{n} (n-x)\cos kx\,dx$

BY PARTS:

$n-x$	-1
$\frac{1}{k}\sin kx$	$\cos kx$

$= \frac{2}{n^2\sqrt{2\pi}}\left\{\cancel{\left[\frac{1}{k}(n-x)\sin kx\right]_0^n} + \frac{1}{k}\int_0^n \sin kx\,dx\right\}$

$= \frac{2}{n^2\sqrt{2\pi}} \times \left[-\frac{1}{k^2}\cos kx\right]_0^n = \frac{2}{n^2k^2\sqrt{2\pi}}\left[\cos kx\right]_n^0 = \frac{2}{n^2k^2\sqrt{2\pi}}\left[1-\cos kn\right]$

$= \frac{2}{n^2k^2\sqrt{2\pi}}\left[1-\left(1-2\sin^2\left(\frac{kn}{2}\right)\right)\right]$ $\longleftarrow$ $\cos 2A = 1 - 2\sin^2 A$

$= \frac{4\sin^2\left(\frac{kn}{2}\right)}{\sqrt{2\pi}\,n^2k^2} = \frac{\sin\frac{kn}{2}}{\frac{nk}{2}} \times \frac{\sin\frac{kn}{2}}{\frac{nk}{2}} \times \frac{1}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}}\operatorname{sinc}^2\left(\frac{kn}{2}\right)$

Question 9

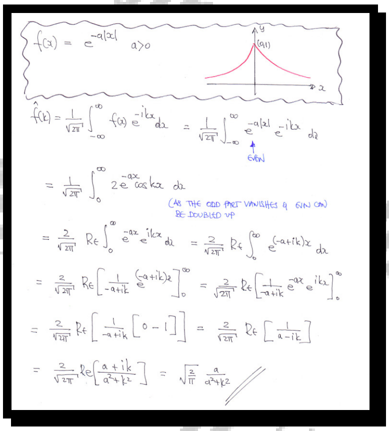
The function f is defined by

$$f(x) = \mathrm{e}^{-a|x|},$$

where a is a positive constant.

Find the Fourier transform of $f(x)$.

$$\boxed{\mathcal{F}\left[\mathrm{e}^{-a|x|}\right] = \hat{f}(k) = \sqrt{\frac{2}{\pi}}\,\frac{a}{a^2+k^2}}$$



Question 10

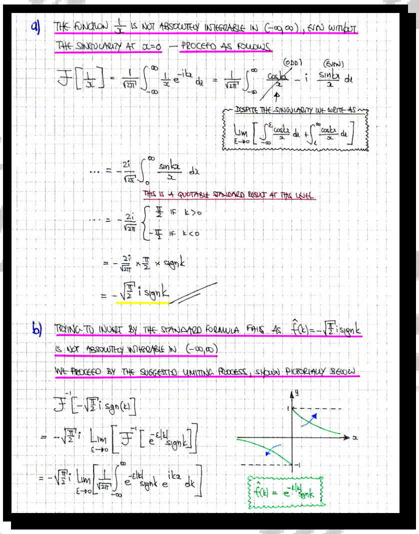
The function f is defined by

$$f(x) = \frac{1}{x}, \quad x \neq 0.$$

a) Determine the Fourier transform of $f(x)$, assuming without proof any standard results about $\displaystyle\int_0^\infty \frac{\sin ax}{x}\, dx$.

b) By introducing the converging factor $\mathrm{e}^{-\varepsilon|x|}$ and letting $\varepsilon \to 0$, invert the answer of part **(a)** to obtain f.

MM2-F , $\boxed{\mathcal{F}\left[\frac{1}{x}\right] = \hat{f}(k) = -\mathrm{i}\sqrt{\frac{\pi}{2}}\,\mathrm{sign}(k)}$



a) THE FUNCTION $\frac{1}{x}$ IS NOT ABSOLUTELY INTEGRABLE IN $(-\infty,\infty)$, EVEN WITHOUT THE SINGULARITY AT $x=0$ — PROCEED AS FOLLOWS

$$\mathcal{F}\left[\frac{1}{x}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{x}e^{-ikx}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\underbrace{\cancel{\frac{\cos kx}{x}}}_{(\text{ODD})} - i\underbrace{\frac{\sin kx}{x}}_{(\text{EVEN})}\,dx$$

DESPITE THE SINGULARITY WE WRITE AS

$$\lim_{\varepsilon\to 0}\left[\int_{-\infty}^{\varepsilon}\frac{\cos kx}{x}\,dx + \int_{\varepsilon}^{\infty}\frac{\cos kx}{x}\,dx\right]$$

$$\ldots = -\frac{2i}{\sqrt{2\pi}}\int_0^\infty \frac{\sin kx}{x}\,dx$$

THIS IS A QUOTABLE STANDARD RESULT AT THIS LEVEL

$$\ldots = -\frac{2i}{\sqrt{2\pi}}\begin{cases}\frac{\pi}{2} & \text{IF } k>0\\ -\frac{\pi}{2} & \text{IF } k<0\end{cases}$$

$$= -\frac{2i}{\sqrt{2\pi}}\times\frac{\pi}{2}\times \mathrm{sign}\,k$$

$$= -\sqrt{\frac{\pi}{2}}\,i\,\mathrm{sign}\,k$$

b) TRYING TO INVERT BY THE STANDARD FORMULA FAILS AS $\hat{f}(k) = -\sqrt{\frac{\pi}{2}}\,i\,\mathrm{sign}\,k$ IS NOT ABSOLUTELY INTEGRABLE IN $(-\infty,\infty)$

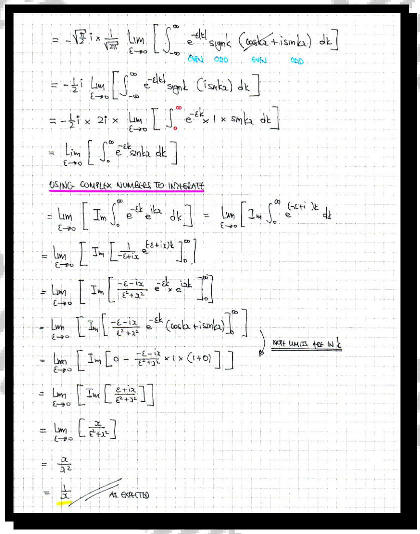
WE PROCEED BY THE SUGGESTED LIMITING PROCESS, SHOWN PICTORIALLY BELOW

$$\mathcal{F}^{-1}\left[-\sqrt{\frac{\pi}{2}}\,i\,\mathrm{sgn}(k)\right]$$

$$= -\sqrt{\frac{\pi}{2}}\,i\lim_{\varepsilon\to 0}\left[\mathcal{F}^{-1}\left[e^{-\varepsilon|k|}\mathrm{sign}\,k\right]\right]$$

$$= -\sqrt{\frac{\pi}{2}}\,i\lim_{\varepsilon\to 0}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\varepsilon|k|}\mathrm{sign}\,k\; e^{ikx}\,dk\right]$$

$\boxed{\hat{f}(k) = e^{-\varepsilon|k|}\mathrm{sign}\,k}$



$$= -\sqrt{\frac{\pi}{2}}\,i\times\frac{1}{\sqrt{2\pi}}\lim_{\varepsilon\to 0}\left[\int_{-\infty}^{\infty}\underbrace{e^{-\varepsilon|k|}}_{\text{EVEN}}\underbrace{\mathrm{sign}\,k}_{\text{ODD}}\,(\underbrace{\cancel{\cos kx}}_{\text{EVEN}} + i\underbrace{\sin kx}_{\text{ODD}})\,dk\right]$$

$$= -\frac{1}{2}i\lim_{\varepsilon\to 0}\left[\int_{-\infty}^{\infty}e^{-\varepsilon|k|}\mathrm{sign}\,k\,(i\sin kx)\,dk\right]$$

$$= -\frac{1}{2}i\times 2i\times\lim_{\varepsilon\to 0}\left[\int_0^\infty e^{-\varepsilon k}\times 1\times\sin kx\,dk\right]$$

$$= \lim_{\varepsilon\to 0}\left[\int_0^\infty e^{-\varepsilon k}\sin kx\,dk\right]$$

USING COMPLEX NUMBERS TO INTEGRATE

$$= \lim_{\varepsilon\to 0}\left[\mathrm{Im}\int_0^\infty e^{-\varepsilon k}e^{ikx}\,dk\right] = \lim_{\varepsilon\to 0}\left[\mathrm{Im}\int_0^\infty e^{(-\varepsilon+ix)k}\,dk\right]$$

$$= \lim_{\varepsilon\to 0}\left[\mathrm{Im}\left[\frac{1}{-\varepsilon+ix}e^{(-\varepsilon+ix)k}\right]_0^\infty\right]$$

$$= \lim_{\varepsilon\to 0}\left[\mathrm{Im}\left[\frac{-\varepsilon-ix}{\varepsilon^2+x^2}e^{-\varepsilon k}\times e^{ixk}\right]_0^\infty\right]$$

$$= \lim_{\varepsilon\to 0}\left[\mathrm{Im}\left[\frac{-\varepsilon-ix}{\varepsilon^2+x^2}e^{-\varepsilon k}(\cos kx + i\sin kx)\right]_0^\infty\right]$$

NOTE LIMITS ARE IN k

$$= \lim_{\varepsilon\to 0}\left[\mathrm{Im}\left[0 - \frac{-\varepsilon-ix}{\varepsilon^2+x^2}\times 1\times(1+0)\right]\right]$$

$$= \lim_{\varepsilon\to 0}\left[\mathrm{Im}\left[\frac{\varepsilon+ix}{\varepsilon^2+x^2}\right]\right]$$

$$= \lim_{\varepsilon\to 0}\left[\frac{x}{\varepsilon^2+x^2}\right]$$

$$= \frac{x}{x^2}$$

$$= \frac{1}{x}$$ AS EXPECTED

Question 11

The impulse function $\delta(x)$ is defined by

$$\delta(x)=\begin{cases}\infty & x=0\\ 0 & x\neq 0\end{cases}$$

a) Determine

i. … $\mathcal{F}\left[\delta(x)\right]$.

ii. … $\mathcal{F}\left[\delta(x-a)\right]$, where a is a positive constant.

iii. … $\mathcal{F}^{-1}\left[\delta(k)\right]$.

b) Use the above results to deduce $\mathcal{F}[1]$ and $\mathcal{F}^{-1}[1]$.

$\boxed{\mathcal{F}\left[\delta(x)\right]=\frac{1}{\sqrt{2\pi}}}$, $\boxed{\mathcal{F}\left[\delta(x-a)\right]=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\mathrm{i}ka}}$, $\boxed{\mathcal{F}^{-1}\left[\delta(k)\right]=\frac{1}{\sqrt{2\pi}}}$,

$\boxed{\mathcal{F}[1]=\sqrt{2\pi}\,\delta(k)}$, $\boxed{\mathcal{F}^{-1}[1]=\sqrt{2\pi}\,\delta(x)}$

a) I) $\mathcal{F}[\delta(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x)e^{-ikx}\,dx = \ldots$ SUBSTITUTION PROPERTY

$= \frac{1}{\sqrt{2\pi}}e^{-ik\times 0} = \frac{1}{\sqrt{2\pi}}$

II) $\mathcal{F}[\delta(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x-a)e^{-ikx}\,dx = \ldots$ SUBSTITUTION PROPERTY

$= \frac{1}{\sqrt{2\pi}}e^{-ika}$

III) $\mathcal{F}^{-1}[\delta(k)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(k)e^{ikx}\,dk = \ldots$ SUBSTITUTION PROPERTY

$= \frac{1}{\sqrt{2\pi}}e^{ix0x} = \frac{1}{\sqrt{2\pi}}$

b) LOOKING AT (a)

$\mathcal{F}[\delta(x)] = \frac{1}{\sqrt{2\pi}}$

$\sqrt{2\pi}\,\mathcal{F}[\delta(x)] = 1$

$\mathcal{F}^{-1}[\sqrt{2\pi}\,\mathcal{F}[\delta(x)]] = \mathcal{F}^{-1}[1]$

$\mathcal{F}^{-1}[1] = \sqrt{2\pi}\,\delta(x)$

LOOKING AT (c)

$\mathcal{F}^{-1}[\delta(k)] = \frac{1}{\sqrt{2\pi}}$

$\sqrt{2\pi}\,\mathcal{F}^{-1}[\delta(k)] = 1$

$\mathcal{F}[\sqrt{2\pi}\,\mathcal{F}^{-1}[\delta(k)]] = \mathcal{F}[1]$

$\mathcal{F}[1] = \sqrt{2\pi}\,\delta(k)$

Question 12

The signum function $\text{sign}(x)$ is defined by

$$\text{sign}(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

By introducing the converging factor $e^{-\varepsilon|x|}$ and letting $\varepsilon \to 0$, determine the Fourier transform of $\text{sign}(x)$.

$$\boxed{\mathcal{F}\left[\text{sign}(x)\right] = -\frac{i}{k}\sqrt{\frac{1}{\pi}}}$$

$$\mathcal{F}[\text{sign}(x)] = \lim_{\varepsilon \to 0}\left[e^{-\varepsilon|x|}\,\text{sign}\,x\right]$$

$$= \lim_{\varepsilon \to 0}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \underbrace{e^{-\varepsilon|x|}}_{\text{EVEN}}\,\underbrace{\text{sign}\,x\; e^{-ikx}}_{\text{ODD}}\,dx\right]$$

$$= \lim_{\varepsilon \to 0}\left[\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} 2e^{-\varepsilon x} \times 1 \times (-i\sin kx)\,dx\right]$$

$$= \frac{-2i}{\sqrt{2\pi}}\lim_{\varepsilon \to 0}\left[\int_0^{\infty} e^{-\varepsilon x}\sin kx\,dx\right]$$

$$= \frac{-2i}{\sqrt{2\pi}}\lim_{\varepsilon \to 0}\left[\text{Im}\left[\int_0^{\infty} e^{-\varepsilon x}e^{ikx}\,dx\right]\right]$$

$$= \frac{-2i}{\sqrt{2\pi}}\lim_{\varepsilon \to 0}\left[\text{Im}\left[\int_0^{\infty} e^{x(-\varepsilon+ik)}\,dx\right]\right]$$

$$= \frac{-2i}{\sqrt{2\pi}}\lim_{\varepsilon \to 0}\left[\text{Im}\left[\frac{1}{-\varepsilon+ik}e^{x(-\varepsilon+ik)}\right]_0^{\infty}\right]$$

$$= \frac{-2i}{\sqrt{2\pi}}\lim_{\varepsilon \to 0}\left[\text{Im}\left[\frac{-\varepsilon-ik}{\varepsilon^2+k^2}e^{-\varepsilon x}(\cos kx + i\sin kx)\right]_0^{\infty}\right]$$

$$= \frac{-2i}{\sqrt{2\pi}}\lim_{\varepsilon \to 0}\left[\text{Im}\left[\frac{-\varepsilon-ik}{\varepsilon^2+k^2}(0-1)\right]\right]$$

$$= \frac{-2i}{\sqrt{2\pi}}\lim_{\varepsilon \to 0}\left[\text{Im}\left[\frac{\varepsilon+ik}{\varepsilon^2+k^2}\right]\right]$$

$$= \frac{-2i}{\sqrt{2\pi}}\lim_{\varepsilon \to 0}\left(\frac{k}{\varepsilon^2+k^2}\right)$$

$$= \frac{-2i}{\sqrt{2\pi}}\,\frac{k}{k^2}$$

$$= -\frac{i}{k}\sqrt{\frac{2}{\pi}}$$

Question 13

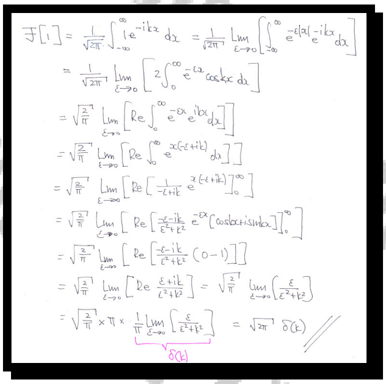
The Unit function $\mathrm{U}(x)$ is defined by

$$\mathrm{U}(x) = 1.$$

By introducing the converging factor $\mathrm{e}^{-\varepsilon|x|}$ and letting $\varepsilon \to 0$, determine the Fourier transform of $\mathrm{U}(x)$.

You may assume that $\delta(t) = \dfrac{1}{\pi} \lim\limits_{\varepsilon \to 0}\left[\dfrac{\varepsilon}{\varepsilon^2 + t^2}\right]$.

$$\boxed{\mathcal{F}\left[\mathrm{U}(x)\right] = \sqrt{2\pi}\,\delta(k)}$$



Question 14

The Unit function $\mathrm{U}(x)$ is defined by

$$\mathrm{U}(x)=1.$$

By introducing the converging factor $\mathrm{e}^{-\varepsilon|k|}$ and letting $\varepsilon \to 0$, find $\mathcal{F}^{-1}\left[\mathrm{U}(k)\right]$.

You may assume that $\delta(t)=\dfrac{1}{\pi}\lim\limits_{\varepsilon\to 0}\left[\dfrac{\varepsilon}{\varepsilon^2+t^2}\right]$.

$$\boxed{\mathcal{F}^{-1}\left[\mathrm{U}(k)\right]=\sqrt{2\pi}\,\delta(x)}$$

$$\begin{aligned}
\mathcal{F}^{-1}[1] &= \frac{1}{\sqrt{2\pi}}\int_0^{\infty} 1\cdot e^{ikx}\,dk \\
&= \lim_{\varepsilon\to 0}\left[\frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{-\varepsilon|k|}e^{ikx}\,dk\right] \\
&= \frac{1}{\sqrt{2\pi}}\lim_{\varepsilon\to 0}\left[\int_{-\infty}^{\infty} e^{-\varepsilon|k|}(\cos kx + i\sin kx)\,dk\right] \\
&= \frac{1}{\sqrt{2\pi}}\lim_{\varepsilon\to 0}\left[2\int_0^{\infty} e^{-\varepsilon k}\cos kx\,dk\right] \\
&= \sqrt{\frac{2}{\pi}}\lim_{\varepsilon\to 0}\left[\mathrm{Re}\left[\int_0^{\infty} e^{-\varepsilon k}e^{ikx}\,dk\right]\right] \\
&= \sqrt{\frac{2}{\pi}}\lim_{\varepsilon\to 0}\left[\mathrm{Re}\left[\int_0^{\infty} e^{k(-\varepsilon+ix)}\,dk\right]\right] \\
&= \sqrt{\frac{2}{\pi}}\lim_{\varepsilon\to 0}\left[\mathrm{Re}\left[\frac{1}{-\varepsilon+ix}e^{k(-\varepsilon+ix)}\right]_0^{\infty}\right] \\
&= \sqrt{\frac{2}{\pi}}\lim_{\varepsilon\to 0}\left[\mathrm{Re}\left[\frac{1}{-\varepsilon+ix}e^{-\varepsilon k}[\cos kx + i\sin kx]\right]_0^{\infty}\right] \\
&= \sqrt{\frac{2}{\pi}}\lim_{\varepsilon\to 0}\left[\mathrm{Re}\ \frac{-\varepsilon-ix}{\varepsilon^2+x^2}[0-1]\right] \\
&= \sqrt{\frac{2}{\pi}}\lim_{\varepsilon\to 0}\left[\frac{\varepsilon}{\varepsilon^2+x^2}\right] \\
&= \sqrt{\frac{2}{\pi}}\times\pi\times\left[\frac{1}{\pi}\lim_{\varepsilon\to 0}\left[\frac{\varepsilon}{\varepsilon^2+x^2}\right]\right] \\
&= \sqrt{2\pi}\,\delta(x)
\end{aligned}$$

Question 15

The function $g(x)$ has Fourier transform given by

$$\hat{g}(k) = -\mathrm{i}\,\mathrm{sign}(k)\,.$$

By introducing the converging factor $e^{-\varepsilon|k|}$ and letting $\varepsilon \to 0$, find $\mathcal{F}^{-1}\left[\hat{g}(k)\right]$.

$$\boxed{\mathcal{F}^{-1}\left[\hat{g}(k)\right] = \sqrt{\frac{2}{\pi}}\,\frac{1}{x}}$$

AS $\hat{g}(k) = -\mathrm{i}\,\mathrm{sign}\,k$ IS NOT ABSOLUTELY INTEGRABLE, WE INTRODUCE A CONVERGENCE FACTOR $e^{-\varepsilon|k|}$, AND LET $\varepsilon \to 0$ AT THE END

$$\Rightarrow g(x) = \lim_{\varepsilon \to 0}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\big(\underbrace{-\mathrm{i}\,\mathrm{sign}\,k}_{(\text{ODD})}\ \underbrace{e^{-\varepsilon|k|}}_{(\text{EVEN})}\big)e^{\mathrm{i}kx}\,dk\right]$$

ONLY THE ODD PART SURVIVES

$$\Rightarrow g(x) = \lim_{\varepsilon \to 0}\left[-\frac{2\mathrm{i}}{\sqrt{2\pi}}\int_{0}^{\infty}\mathrm{sign}\,k\ e^{-\varepsilon|k|}(\mathrm{i}\sin kx)\,dk\right]$$

$$\Rightarrow g(x) = \lim_{\varepsilon \to 0}\left[-\frac{2\mathrm{i}^2}{\sqrt{2\pi}}\int_{0}^{\infty}1 \times e^{-\varepsilon k}\sin kx\,dk\right]$$

$$\Rightarrow g(x) = \lim_{\varepsilon \to 0}\left[\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}e^{-\varepsilon k}\sin kx\,dk\right]$$

CARRY THE INTEGRATION BY COMPLEX NUMBERS (OR TWICE BY PARTS)

$$\Rightarrow g(x) = \lim_{\varepsilon \to 0}\left[\sqrt{\frac{2}{\pi}}\,\mathrm{Im}\left[\int_{0}^{\infty}e^{-\varepsilon k}e^{\mathrm{i}kx}\,dk\right]\right]$$

$$\Rightarrow g(x) = \sqrt{\frac{2}{\pi}}\lim_{\varepsilon \to 0}\left[\mathrm{Im}\left[\int_{0}^{\infty}e^{(-\varepsilon+\mathrm{i}x)k}\,dk\right]\right]$$

$$\Rightarrow g(x) = \sqrt{\frac{2}{\pi}}\lim_{\varepsilon \to 0}\left[\mathrm{Im}\left[\frac{1}{-\varepsilon+\mathrm{i}x}e^{(-\varepsilon+\mathrm{i}x)k}\right]_{k=0}^{k=\infty}\right]$$

$$\Rightarrow g(x) = \sqrt{\frac{2}{\pi}}\lim_{\varepsilon \to 0}\left[\mathrm{Im}\left[\frac{-\varepsilon-\mathrm{i}x}{\varepsilon^2+x^2}e^{-\varepsilon k}(\cos kx + \mathrm{i}\sin kx)\right]_{k=0}^{\infty}\right]$$

$$\Rightarrow g(x) = \sqrt{\frac{2}{\pi}}\lim_{\varepsilon \to 0}\left[\left[\frac{-e^{-\varepsilon k}}{\varepsilon^2+x^2}(\varepsilon\sin kx + x\cos kx)\right]_{k=0}^{k=\infty}\right]$$

$$\Rightarrow g(x) = \sqrt{\frac{2}{\pi}}\lim_{\varepsilon \to 0}\left[\left[\frac{e^{-\varepsilon k}}{\varepsilon^2+x^2}(\varepsilon\sin kx + x\cos kx)\right]_{k=\infty}^{k=0}\right]$$

$$\Rightarrow g(x) = \sqrt{\frac{2}{\pi}}\lim_{\varepsilon \to 0}\left[\frac{1}{\varepsilon^2+x^2}(0+x) - 0\right]$$

$$\Rightarrow g(x) = \sqrt{\frac{2}{\pi}}\lim_{\varepsilon \to 0}\left[\frac{x}{\varepsilon^2+x^2}\right]$$

$$\Rightarrow g(x) = \sqrt{\frac{2}{\pi}} \times \frac{x}{x^2}$$

$$\Rightarrow g(x) = \sqrt{\frac{2}{\pi}}\,\frac{1}{x}$$

Question 16

The Heaviside function $\mathrm{H}(x)$ is defined by

$$\mathrm{H}(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

By introducing the converging factor $\mathrm{e}^{-\varepsilon x}$ and letting $\varepsilon \to 0$, determine the Fourier transform of $\mathrm{H}(x)$.

You may assume that $\delta(t) = \dfrac{1}{\pi} \displaystyle\lim_{\varepsilon \to 0}\left[\dfrac{\varepsilon}{\varepsilon^2 + t^2}\right]$.

$$\boxed{\mathcal{F}\left[\mathrm{H}(x)\right] = \frac{1}{\sqrt{2\pi}}\left[\pi\delta(k) - \frac{\mathrm{i}}{k}\right]}$$

$$\mathcal{F}(H(x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} H(x)^{-ikx}\,dx = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{-ikx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\lim_{\varepsilon \to 0}\left[\int_0^{\infty} e^{-\varepsilon x} e^{-ikx}\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\lim_{\varepsilon \to 0}\left[\int_0^{\infty} e^{x(-\varepsilon - ik)}\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\lim_{\varepsilon \to 0}\left[\frac{1}{-\varepsilon - ik} e^{x(-\varepsilon - ik)}\right]_0^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}}\lim_{\varepsilon \to 0}\left[\frac{-\varepsilon + ik}{\varepsilon^2 + k^2} e^{-\varepsilon x} e^{-ikx}\right]_0^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}}\lim_{\varepsilon \to 0}\left[\frac{-\varepsilon + ik}{\varepsilon^2 + k^2} e^{-\varepsilon x}(\cos kx - i\sin kx)\right]_0^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}}\lim_{\varepsilon \to 0}\left[\frac{-\varepsilon + ik}{\varepsilon^2 + k^2}(0 - 1)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\lim_{\varepsilon \to 0}\left[\frac{\varepsilon - ik}{\varepsilon^2 + k^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\lim_{\varepsilon \to 0}\left[\frac{\varepsilon}{\varepsilon^2 + k^2}\right] - ik\lim_{\varepsilon \to 0}\left[\frac{1}{\varepsilon^2 + k^2}\right]\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\pi \times \underbrace{\frac{1}{\pi}\lim_{\varepsilon \to 0}\left[\frac{\varepsilon}{\varepsilon^2 + k^2}\right]}_{\delta(k)} - ik \times \frac{1}{k^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\pi\delta(k) - \frac{i}{k}\right]$$

Question 17

The impulse function $\delta(x)$ is defined by

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$$

a) Determine the inverse Fourier transform of the impulse function $\mathcal{F}^{-1}\left[\delta(k)\right]$, and use it to deduce the Fourier transform of $f(x) = 1$.

b) Find directly the Fourier transform of $f(x) = 1$, by introducing the converging factor $e^{-\varepsilon|x|}$ and letting $\varepsilon \to 0$.

$$\boxed{\mathcal{F}[1] = \sqrt{2\pi}\,\delta(k)}$$

a) CONSIDER THE INVERSE FOURIER TRANSFORM OF $\delta(k)$

$$\mathcal{F}^{-1}[\delta(k)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \delta(k)\,e^{ikx}\,dk \quad = \text{SUBSTITUTION PROPERTY}$$

$$= \frac{1}{\sqrt{2\pi}}e^{i0x} = \frac{1}{\sqrt{2\pi}}$$

NOW

$$\mathcal{F}^{-1}[\delta(k)] = \frac{1}{\sqrt{2\pi}}$$

$$\sqrt{2\pi}\,\mathcal{F}^{-1}[\delta(k)] = 1$$

$$\mathcal{F}\left[\sqrt{2\pi}\,\mathcal{F}^{-1}[\delta(k)]\right] = \mathcal{F}[1]$$

$$\sqrt{2\pi}\,\delta(k) = \mathcal{F}[1]$$

$$\mathcal{F}[1] = \sqrt{2\pi}\,\delta(k)$$

b)

$$\mathcal{F}[1] = \lim_{\varepsilon\to 0}\left[\mathcal{F}\left[1\times e^{-\varepsilon|x|}\right]\right] =$$

$$= \lim_{\varepsilon\to 0}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\varepsilon|x|}e^{-ikx}\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\lim_{\varepsilon\to 0}\left[\int_0^{\infty} 2e^{-\varepsilon x}\cos kx\,dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\lim_{\varepsilon\to 0}\left[\mathrm{Re}\int_0^{\infty} e^{-\varepsilon x}e^{ikx}\,dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\lim_{\varepsilon\to 0}\left[\mathrm{Re}\left[\int_0^{\infty} e^{x(-\varepsilon+ik)}\,dx\right]\right]$$

$$= \sqrt{\frac{2}{\pi}}\lim_{\varepsilon\to 0}\left[\mathrm{Re}\left[\frac{1}{-\varepsilon+ik}e^{x(-\varepsilon+ik)}\right]_0^{\infty}\right]$$

$$= \sqrt{\frac{2}{\pi}}\lim_{\varepsilon\to 0}\left[\mathrm{Re}\left[\frac{-\varepsilon-ik}{\varepsilon^2+k^2}e^{-\varepsilon x}(\cos kx + i\sin kx)\right]_0^{\infty}\right]$$

$$= \sqrt{\frac{2}{\pi}}\lim_{\varepsilon\to 0}\left[\mathrm{Re}\left[\frac{-\varepsilon-ik}{\varepsilon^2+k^2}(0-1)\right]\right]$$

$$= \sqrt{\frac{2}{\pi}}\lim_{\varepsilon\to 0}\left[\mathrm{Re}\,\frac{\varepsilon+ik}{\varepsilon^2+k^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\lim_{\varepsilon\to 0}\left[\frac{\varepsilon}{\varepsilon^2+k^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\times\pi\times\frac{1}{\pi}\lim_{\varepsilon\to 0}\left[\frac{\varepsilon}{\varepsilon^2+k^2}\right]$$

$$= \sqrt{2\pi}\,\delta(k)$$

NOTE $\delta(x) = \frac{1}{\pi}\lim_{\varepsilon\to 0}\left[\frac{\varepsilon}{x^2+\varepsilon^2}\right]$

Question 18

The function f is defined by

$$f(x) = \text{sign}(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

a) By introducing the converging factor $\mathrm{e}^{-\varepsilon|x|}$ and letting $\varepsilon \to 0$, find the Fourier transform of f.

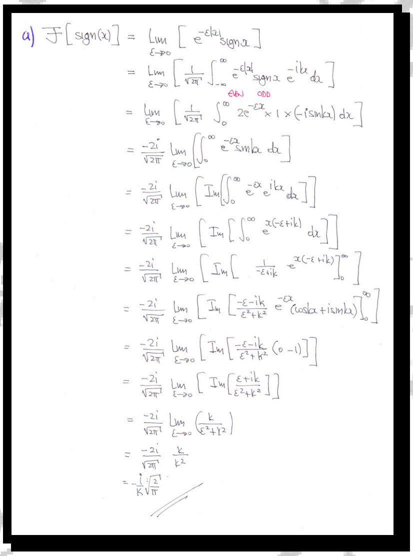
b) By introducing the converging factor $\mathrm{e}^{-\varepsilon|x|}$ and letting $\varepsilon \to 0$, find the Fourier transform of $g(x) = 1$.

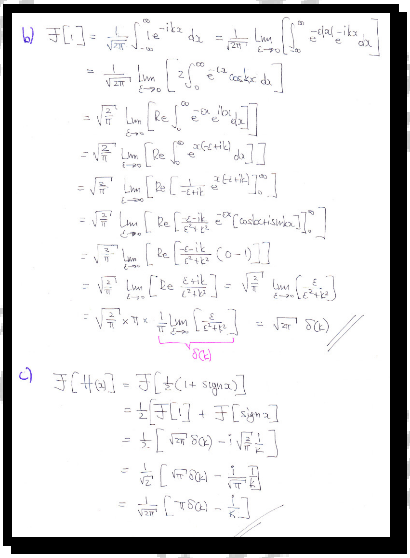
You may assume that $\delta(t) = \dfrac{1}{\pi} \lim\limits_{\varepsilon \to 0}\left[\dfrac{\varepsilon}{\varepsilon^2 + t^2}\right]$.

c) Hence determine the Fourier transform of the Heaviside function $\mathrm{H}(x)$,

$$\mathrm{H}(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$\boxed{\mathcal{F}[\text{sign}(x)] = -\dfrac{\mathrm{i}}{k}\sqrt{\dfrac{1}{\pi}}}$, $\boxed{\mathcal{F}[1] = \sqrt{2\pi}\,\delta(k)}$, $\boxed{\mathcal{F}[\mathrm{H}(x)] = \dfrac{1}{\sqrt{2\pi}}\left[\pi\delta(k) - \dfrac{\mathrm{i}}{k}\right]}$





Question 19

The Fourier transforms of the functions $f(x)$ and $g(x)$ are

$$\hat{f}(k) = \delta(k) \quad \text{and} \quad \hat{g}(k) = \frac{1}{\mathrm{i}k},$$

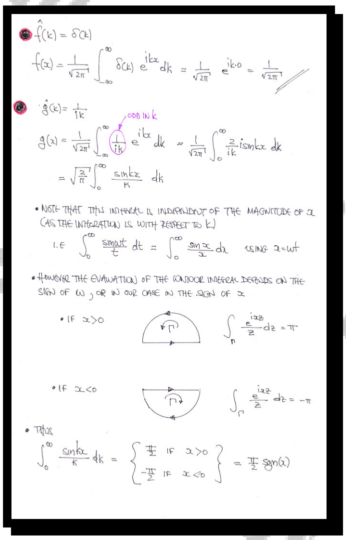
where $\delta(x)$ denotes the impulse function.

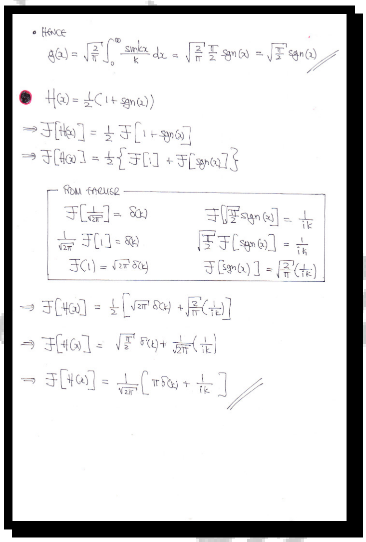
Find simplified expressions for $f(x)$ and $g(x)$, and use them to show that

$$\mathcal{F}[\mathrm{H}(x)] = \frac{1}{\sqrt{2\pi}}\left[\pi\delta(k) + \frac{1}{\mathrm{i}k}\right],$$

where $\mathrm{H}(x)$ denotes the Heaviside function.

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}}}, \quad \boxed{g(x) = \frac{1}{2}\pi\,\mathrm{sgn}(x)}$$





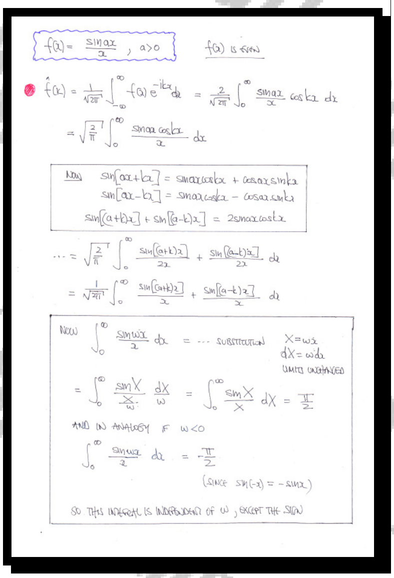
Question 20

The function f is defined by

$$f(x) = \frac{\sin ax}{x}, \ a > 0 .$$

Find the Fourier transform of $f(x)$, stating clearly any results used.

$$\boxed{\mathcal{F}\left[\frac{\sin ax}{x}\right] = \begin{cases} \sqrt{\dfrac{\pi}{2}} & |k| < a \\ \sqrt{\dfrac{\pi}{8}} & |k| = a \\ 0 & |k| > a \end{cases}}$$



$$\boxed{f(x) = \frac{\sin ax}{x}, \ a>0} \qquad \underline{f(x) \text{ is EVEN}}$$

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} \frac{\sin ax}{x} \cos kx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\sin ax \cos kx}{x} \, dx$$

NOW
$$\sin[ax + kx] = \sin ax \cos kx + \cos ax \sin kx$$
$$\sin[ax - kx] = \sin ax \cos kx - \cos ax \sin kx$$
$$\sin[(a+k)x] + \sin[(a-k)x] = 2 \sin ax \cos kx$$

$$\dots = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\sin[(a+k)x]}{2x} + \frac{\sin[(a-k)x]}{2x} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{\sin[(a+k)x]}{x} + \frac{\sin[(a-k)x]}{x} \, dx$$

NOW $\int_0^{\infty} \frac{\sin \omega x}{x} \, dx = \dots$ SUBSTITUTION $X = \omega x$, $dX = \omega \, dx$, LIMITS UNCHANGED

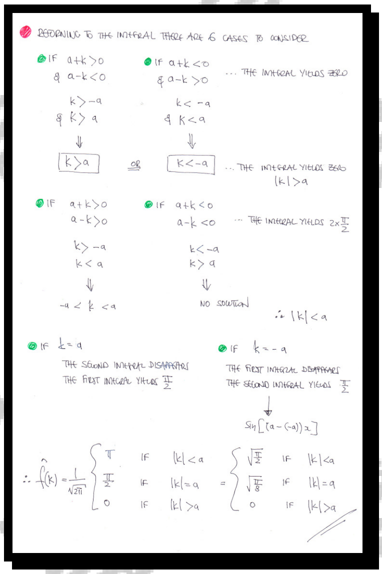
$$= \int_0^{\infty} \frac{\sin X}{\frac{X}{\omega}} \frac{dX}{\omega} = \int_0^{\infty} \frac{\sin X}{X} \, dX = \frac{\pi}{2}$$

AND IN ANALOGY IF $\omega < 0$

$$\int_0^{\infty} \frac{\sin \omega x}{x} \, dx = -\frac{\pi}{2}$$

(SINCE $\sin(-x) = -\sin x$)

SO THIS INTEGRAL IS INDEPENDENT OF ω, EXCEPT THE SIGN



RETURNING TO THE INTEGRAL THERE ARE 6 CASES TO CONSIDER

IF $a+k>0$ & $a-k<0$: $k>-a$ & $k>a$ $\Downarrow$ $\boxed{k>a}$ OR IF $a+k<0$ & $a-k>0$: $k<-a$ & $k<a$ $\Downarrow$ $\boxed{k<-a}$ … THE INTEGRAL YIELDS ZERO

… THE INTEGRAL YIELDS ZERO $|k|>a$

IF $a+k>0$, $a-k>0$: $k>-a$, $k<a$ $\Downarrow$ $-a<k<a$

IF $a+k<0$, $a-k<0$: $k<-a$, $k>a$ $\Downarrow$ NO SOLUTION

… THE INTEGRAL YIELDS $2 \times \frac{\pi}{2}$

$\therefore |k|<a$

IF $k=a$: THE SECOND INTEGRAL DISAPPEARS, THE FIRST INTEGRAL YIELDS $\frac{\pi}{2}$

IF $k=-a$: THE FIRST INTEGRAL DISAPPEARS, THE SECOND INTEGRAL YIELDS $\frac{\pi}{2}$ $\downarrow$ $\sin[(a-(-a))x]$

$$\therefore \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \begin{cases} \pi & \text{IF } |k|<a \\ \frac{\pi}{2} & \text{IF } |k|=a \\ 0 & \text{IF } |k|>a \end{cases} = \begin{cases} \sqrt{\frac{\pi}{2}} & \text{IF } |k|<a \\ \sqrt{\frac{\pi}{8}} & \text{IF } |k|=a \\ 0 & \text{IF } |k|>a \end{cases}$$

Question 21

Given that l is a non zero constant, show that

$$\mathcal{F}\left[\frac{\exp\left(-\frac{x^2}{l^2}\right)}{l\sqrt{\pi}}\right] = \hat{f}(k) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{k^2l^2}{4}\right).$$

$\boxed{\text{proof}}$

$$\mathcal{F}\left[\frac{e^{-\frac{x^2}{l^2}}}{l\sqrt{\pi}}\right] = \frac{1}{l\sqrt{\pi}}\mathcal{F}\left[e^{-\frac{x^2}{l^2}}\right] = \frac{1}{l\sqrt{\pi}} \times \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{l^2}} e^{-ikx}\,dx$$

$$= \frac{1}{\pi l\sqrt{2}}\int_{-\infty}^{\infty} \underbrace{e^{-\frac{x^2}{l^2}}}_{\text{EVEN}} e^{-ikx}\,dx = \frac{2}{\pi l\sqrt{2}}\int_{0}^{\infty} e^{-\frac{x^2}{l^2}}\cos kx\,dx$$

NOW CONSIDER THE INTEGRAL TO BE FOUND (IGNORE THE CONSTANTS)

LET $I = \int_0^{\infty} e^{-\frac{x^2}{l^2}}\cos kx\,dx$

$$\Rightarrow \frac{\partial I}{\partial k} = \frac{\partial}{\partial k}\int_0^{\infty} e^{-\frac{x^2}{l^2}}\cos kx\,dx = \int_0^{\infty} e^{-\frac{x^2}{l^2}}\frac{\partial}{\partial k}(\cos kx)\,dx$$

$$\Rightarrow \frac{\partial I}{\partial k} = \int_0^{\infty} -xe^{-\frac{x^2}{l^2}}\sin kx\,dx$$

NOW BY PARTS

$\sin kx$	$k\cos kx$
$\frac{l^2}{2}e^{-\frac{x^2}{l^2}}$	$-xe^{-\frac{x^2}{l^2}}$

$$\Rightarrow \frac{\partial I}{\partial k} = \cancel{\left[\frac{l^2}{2}e^{-\frac{x^2}{l^2}}\sin kx\right]_0^{\infty}} - \frac{1}{2}kl^2\int_0^{\infty} e^{-\frac{x^2}{l^2}}\cos kx\,dx$$

$$\Rightarrow \frac{\partial I}{\partial k} = -\frac{1}{2}kl^2 I$$

THIS IS A STANDARD ODE WHICH CAN BE SOLVED BY SEPARATION

$$\Rightarrow \frac{1}{I}\partial I = -\frac{1}{2}kl^2\,\partial k$$

$$\Rightarrow \ln I = -\frac{1}{4}k^2l^2 + C$$

$$\Rightarrow I = Ae^{-\frac{1}{4}k^2l^2}$$

$$\Rightarrow \int_0^{\infty} e^{-\frac{x^2}{l^2}}\cos kx\,dx = Ae^{-\frac{1}{4}k^2l^2}$$

WE NEED TO FIND A — SET $k=0$

$$\int_0^{\infty} e^{-\frac{x^2}{l^2}}\,dx = A$$

$$A = \int_0^{\infty} e^{-u^2}\,l\,du$$

$$A = l\int_0^{\infty} e^{-u^2}\,du$$

$$A = l \times \frac{\sqrt{\pi}}{2}$$

BY SUBSTITUTION

$u = \frac{x}{l}$

$du = \frac{1}{l}\,dx$

$dx = l\,du$

LIMITS UNCHANGED

$$\Rightarrow I = \frac{1}{2}l\sqrt{\pi}\,e^{-\frac{1}{4}k^2l^2}$$

$$\Rightarrow \mathcal{F}\left[\frac{1}{l\sqrt{\pi}}e^{-\frac{x^2}{l^2}}\right] = \frac{\cancel{2}}{\pi\cancel{l}\sqrt{2}} \times \frac{1}{\cancel{2}}\cancel{l}\sqrt{\pi}\,e^{-\frac{1}{4}k^2l^2} = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{4}k^2l^2}$$

Question 22

The Gaussian function $f(x)$ is defined by

$$f(x) = A\mathrm{e}^{-\alpha x^2},$$

where A and α are positive constants.

Find the Fourier transform of $f(x)$.

$$\boxed{\mathcal{F}\left[A\mathrm{e}^{-\alpha x^2}\right] = \hat{f}(k) = \frac{A}{\sqrt{2\alpha}}\exp\left(-\frac{k^2}{4\alpha}\right)}$$

- START BY THE DEFINITION OF THE FOURIER TRANSFORM

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\left(Ae^{-\alpha x^2}\right)dx$$

$$= \frac{A}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} e^{-\alpha x^2}\,dx = \frac{2A}{\sqrt{2\pi}}\int_0^{\infty} e^{-\alpha x^2}\cos kx\,dx$$

(EVEN INTEGRAND)

- THIS INTEGRAL CAN BE DONE BY DIFFERENTIATION UNDER THE INTEGRAL SIGN, WHICH WILL THEN ALLOW INTEGRATION BY PARTS

$$\Rightarrow I = \int_0^{\infty} e^{-\alpha x^2}\cos kx\,dx$$

$$\Rightarrow \frac{\partial I}{\partial k} = \frac{\partial}{\partial k}\int_0^{\infty} e^{-\alpha x^2}\cos kx\,dx = \int_0^{\infty} e^{-\alpha x^2}\frac{\partial}{\partial k}(\cos kx)\,dx$$

$$\Rightarrow \frac{\partial I}{\partial k} = \int_0^{\infty}(-x\sin kx)\,e^{-\alpha x^2}\,dx = \int_0^{\infty} -xe^{-\alpha x^2}\sin kx\,dx$$

- FOLLOWING BY INTEGRATION BY PARTS

$\sin kx$	$k\cos kx$
$\frac{1}{2\alpha}e^{-\alpha x^2}$	$-xe^{-\alpha x^2}$

$$\Rightarrow \frac{\partial I}{\partial k} = \left[\cancel{\frac{1}{2\alpha}e^{-\alpha x^2}\sin kx}\right]_0^{\infty} - \frac{k}{2\alpha}\int_0^{\infty} e^{-\alpha x^2}\cos kx\,dx$$

$$\Rightarrow \frac{\partial I}{\partial k} = -\frac{k}{2\alpha}I$$

- SOLVING THE O.D.E FOR I

$$\Rightarrow \frac{dI}{dk} = -\frac{k}{2\alpha}I$$

$$\Rightarrow \int \frac{dI}{I} = \int -\frac{k}{2\alpha}\,dk$$

$$\Rightarrow \ln I = -\frac{k^2}{4\alpha} + C$$

$$\Rightarrow I = Be^{-\frac{k^2}{4\alpha}} \quad (B \text{ ARBITRARY})$$

- HENCE SO FAR WE HAVE

$$I = \int_0^{\infty} e^{-\alpha x^2}\cos kx\,dx = Be^{-\frac{k^2}{4\alpha}}$$

LET $k=0$

$$\int_0^{\infty} e^{-\alpha x^2}\,dx = B$$

USING A SUBSTITUTION

$y^2 = \alpha x^2$	$B = \int_0^{\infty} e^{-y^2}\frac{dy}{\sqrt{\alpha}}$
$y = \sqrt{\alpha}\,x$	$B = \frac{1}{\sqrt{\alpha}}\int_0^{\infty} e^{-y^2}\,dy$
$dy = \sqrt{\alpha}\,dx$	$B = \frac{1}{\sqrt{\alpha}}\left(\frac{\sqrt{\pi}}{2}\right)$
LIMITS UNCHANGED	

- FINALLY WE HAVE

$$\hat{f}(k) = \frac{2A}{\sqrt{2\pi}}I = \frac{\cancel{2}A}{\sqrt{2\pi}}\left(\frac{1}{\sqrt{\alpha}}\frac{\sqrt{\pi}}{\cancel{2}}\right)e^{-\frac{k^2}{4\alpha}} = \frac{A}{\sqrt{2\alpha}}e^{-\frac{k^2}{4\alpha}}$$

ALTERNATIVE APPROACH BY CONTOUR INTEGRATION

- STARTING BY THE DEFINITION OF THE FOURIER TRANSFORM

$$\hat{f}(k) = \frac{A}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\alpha x^2}e^{-ikx}\,dx = \frac{A}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\alpha\left[x^2+\frac{ik}{\alpha}x\right]}\,dx$$

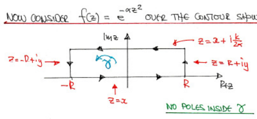
- COMPLETING THE SQUARE IN THE EXPONENT, IN x

$$x^2 + \frac{ik}{\alpha}x = \left(x+\frac{ik}{2\alpha}\right)^2 - \left(\frac{ik}{2\alpha}\right)^2 = \left(x+\frac{ik}{2\alpha}\right)^2 + \frac{k^2}{4\alpha^2}$$

- RETURNING TO THE TRANSFORM

$$\hat{f}(k) = \frac{A}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\alpha\left(x+\frac{ik}{2\alpha}\right)^2}\times e^{-\frac{k^2}{4\alpha}}\,dx = \frac{Ae^{-\frac{k^2}{4\alpha}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\alpha\left(x+\frac{k}{2\alpha}i\right)^2}\,dx$$

- NOW CONSIDER $f(z) = e^{-\alpha z^2}$ OVER THE CONTOUR SHOWN BELOW



NO POLES INSIDE γ

$$\Rightarrow \int_{\gamma} e^{-\alpha z^2}\,dz = 0$$

$$\Rightarrow \int_{-R}^{R} e^{-\alpha x^2}\,dx + \int_{y=0}^{y=\frac{k}{2\alpha}} e^{-\alpha(R+iy)^2}(i\,dy) + \int_{R}^{-R} e^{-\alpha\left(x+\frac{ik}{2\alpha}\right)^2}\,dx + \int_{y=\frac{k}{2\alpha}}^{y=0} e^{-\alpha(-R+iy)^2}(i\,dy) = 0$$

$z = x$, $dz = dx$; $z = R + iy$, $dz = i\,dy$; $z = x + i\frac{k}{2\alpha}$, $dz = dx$; $z = -R + iy$, $dz = i\,dy$

- NOW AS $R \to \infty$, THE 2ND & 4TH INTEGRALS VANISH BECAUSE OF THE TERM $-\alpha R^2$ IN THE EXPONENT

$$\Rightarrow \int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx + \int_{\infty}^{-\infty} e^{-\alpha\left(x+i\frac{k}{2\alpha}\right)^2}\,dx = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-\alpha\left(x+i\frac{k}{2\alpha}\right)^2}\,dx = \int_{-\infty}^{\infty} e^{-\alpha x^2}$$

- RETURNING TO THE TRANSFORM

$$\hat{f}(k) = \frac{Ae^{-\frac{k^2}{4\alpha}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\alpha\left(x+i\frac{k}{2\alpha}\right)^2}\,dx = \frac{Ae^{-\frac{k^2}{4\alpha}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx$$

$$\hat{f}(k) = \frac{Ae^{-\frac{k^2}{4\alpha}}}{\sqrt{2\pi}}\times\frac{1}{\sqrt{\alpha}}\int_{-\infty}^{\infty} e^{-y^2}\,dy$$

LET $y = \sqrt{\alpha}\,x$
$dy = \sqrt{\alpha}\,dx$
$dx = \frac{1}{\sqrt{\alpha}}\,dy$
LIMITS UNCHANGED

$$\hat{f}(k) = \frac{Ae^{-\frac{k^2}{4\alpha}}}{\sqrt{2\pi}}\times\frac{1}{\sqrt{\alpha}}\times\sqrt{\pi}$$

$$\hat{f}(k) = \frac{A}{\sqrt{2\alpha}}e^{-\frac{k^2}{4\alpha}}$$

AS BEFORE

Question 23

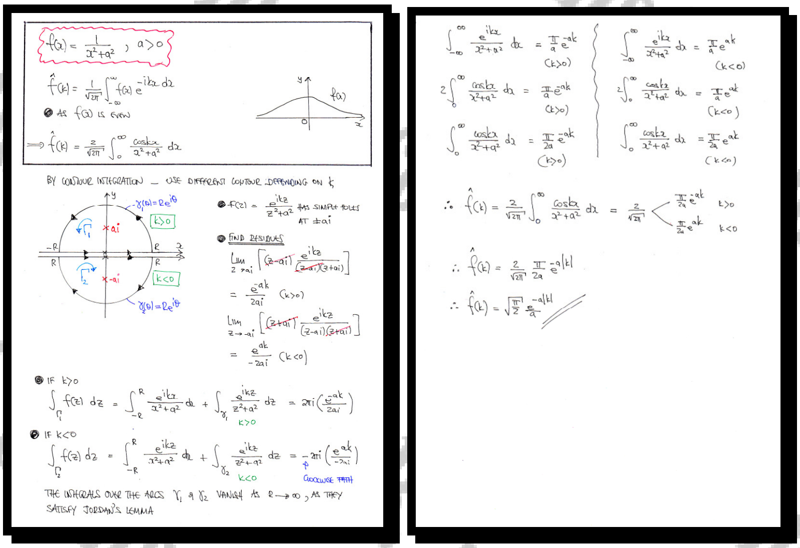
The function f is defined by

$$f(x) = \frac{1}{x^2 + a^2},$$

where a is a positive constant.

Use contour integration to find the Fourier transform of $f(x)$.

$$\boxed{\mathcal{F}\left[\frac{1}{x^2+a^2}\right] = \hat{f}(k) = \sqrt{\frac{\pi}{2}}\,\frac{e^{-a|k|}}{a}}$$



Question 24

The function f is defined by

$$f(x) = x\mathrm{e}^{-x^2}, \quad x \in \mathbb{R}.$$

Find the Fourier transform of $f(x)$, stating clearly any results used.

$$\boxed{\mathcal{F}\left[x\mathrm{e}^{-x^2}\right] = \frac{1}{4}k\sqrt{2}\,\mathrm{e}^{-\frac{1}{4}k^2}}$$

$$\mathcal{F}\left[x e^{-x^2}\right] = i\frac{d}{dk}\left[\mathcal{F}\left(e^{-x^2}\right)\right]$$

$$\boxed{\mathcal{F}(xf(x)) = i\frac{d}{dk}\left[\hat{f}(k)\right]}$$

- SO WE NEED THE FOURIER TRANSFORM OF e^{-x^2}

$$\mathcal{F}\left[e^{-x^2}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2} e^{-ikx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2}(\cos kx + i\sin kx)\, dx$$

$$= \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-x^2}\cos kx\, dx$$

- NOW LET $I = \int_0^\infty e^{-x^2}\cos kx\, dx$

$$\Rightarrow \frac{\partial I}{\partial k} = \frac{\partial}{\partial k}\int_0^\infty e^{-x^2}\cos kx\, dx$$

$$\Rightarrow \frac{\partial I}{\partial k} = \int_0^\infty e^{-x^2}\frac{\partial}{\partial k}\left[\cos kx\right] dx$$

$$\Rightarrow \frac{\partial I}{\partial k} = \int_0^\infty e^{-x^2}\left[-x\sin kx\right] dx$$

$$\Rightarrow \frac{\partial I}{\partial k} = \int_0^\infty \left(-xe^{-x^2}\right)\sin kx\, dx$$

- BY PARTS NEXT

$\sin kx$	$k\cos kx$
$\frac{1}{2}e^{-x^2}$	$-xe^{-x^2}$

$$\Rightarrow \frac{\partial I}{\partial k} = \cancel{\left[\frac{1}{2}e^{-x^2}\sin kx\right]_0^\infty} - \int_0^\infty \frac{k}{2}e^{-x^2}\cos kx\, dx$$

$$\Rightarrow \frac{\partial I}{\partial k} = -\frac{k}{2}\int_0^\infty e^{-x^2}\cos kx\, dx$$

$$\Rightarrow \frac{\partial I}{\partial k} = -\frac{k}{2}I$$

- SEPARATE VARIABLES & INTEGRATE THE O.D.E

$$\Rightarrow \frac{1}{I}\partial I = -\frac{k}{2}\partial k$$

$$\Rightarrow \ln I = -\frac{1}{4}k^2 + C$$

$$\Rightarrow I = Ae^{-\frac{1}{4}k^2} \quad \text{(A IS ARBITRARY)}$$

$$\Rightarrow \boxed{\int_0^\infty e^{-x^2}\cos kx\, dx = Ae^{-\frac{1}{4}k^2}}$$

- TO FIND THE CONSTANT A, LET $k=0$

$$\int_0^\infty e^{-x^2}\, dx = A$$

$$A = \frac{1}{2}\sqrt{\pi} \quad \text{(STANDARD RESULT)}$$

$$\Rightarrow \mathcal{F}\left[e^{-x^2}\right] = \frac{2}{\sqrt{2\pi}}\int_0^\infty e^{-x^2}\cos kx$$

$$= \frac{2}{\sqrt{2\pi}}\left(\frac{1}{2}\sqrt{\pi}\right)e^{-\frac{1}{4}k^2}$$

$$= \frac{1}{\sqrt{2}}e^{-\frac{1}{4}k^2}$$

- HENCE

$$\mathcal{F}\left[xe^{-x^2}\right] = i\frac{d}{dk}\left[\frac{1}{\sqrt{2}}e^{-\frac{1}{4}k^2}\right]$$

$$= \frac{i}{\sqrt{2}}\left(-\frac{1}{2}ke^{-\frac{1}{4}k^2}\right)$$

$$= i\frac{\sqrt{2}}{4}ke^{-\frac{1}{4}k^2}$$

Question 25

The function f is defined by

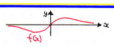
$$f(x) = \frac{x}{x^2 + a^2},$$

where a is a positive constant.

Use contour integration to find the Fourier transform of $f(x)$.

$$\mathcal{F}\left[\frac{x}{x^2+a^2}\right] = \hat{f}(k) = -\mathrm{i}\sqrt{\frac{\pi}{2}}\,\frac{\mathrm{e}^{-a|k|}\operatorname{sign} k}{a}$$

$f(x) = \dfrac{x}{x^2+a^2}$, $a > 0$

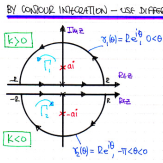


$\hat{f}(k) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$

AS $f(x)$ IS ODD & $e^{-ikx} = \cos kx - i \sin kx$

$\hat{f}(k) = \dfrac{2i}{\sqrt{2\pi}} \displaystyle\int_0^{\infty} \dfrac{-x \sin kx}{x^2+a^2}\, dx = -\sqrt{\dfrac{2}{\pi}}\, i \displaystyle\int_0^{\infty} \dfrac{x \sin kx}{x^2+a^2}\, dx$

BY CONTOUR INTEGRATION – USE DIFFERENT CONTOUR DEPENDING ON THE SIGN OF k



$f(z) = \dfrac{z e^{ikz}}{z^2+a^2}$ HAS SIMPLE POLES AT $\pm ai$, ONE IN EACH OF THE CONTOURS SHOWN OPPOSITE

RESIDUE AT $z = ai$

$\lim_{z \to ai} \left[(z-ai) f(z) \right] = \lim_{z \to ai} \left[\cancel{(z-ai)} \dfrac{z e^{ikz}}{\cancel{(z-ai)}(z+ai)} \right] = \dfrac{ai\, e^{ik(ai)}}{ai+ai}$

$= \dfrac{ai\, e^{-ak}}{2ai} = \dfrac{1}{2} e^{-ak}$

RESIDUE AT $z = -ai$

$\lim_{z \to -ai} \left[(z+ai) f(z) \right] = \lim_{z \to -ai} \left[\cancel{(z+ai)} \dfrac{z e^{ikz}}{(z-ai)\cancel{(z+ai)}} \right] = \dfrac{-ai\, e^{ik(-ai)}}{-ai-ai}$

$= \dfrac{-ai\, e^{ak}}{-2ai} = \dfrac{1}{2} e^{ak}$

- IF $k>0$, USING THE "TOP" CONTOUR

$\displaystyle\int_{\Gamma_1} f(z)\, dz = \int_{-R}^{R} \frac{x e^{ikx}}{x^2+a^2}\, dx + \int_{\gamma_1} \frac{z e^{ikz}}{z^2+a^2}\, dz = 2\pi i \left(\tfrac{1}{2} e^{-ak}\right)$

- IF $k<0$, USING THE "BOTTOM" CONTOUR

$\displaystyle\int_{\Gamma_2} f(z)\, dz = \int_{-R}^{R} \frac{x e^{ikx}}{x^2+a^2}\, dx + \int_{\gamma_2} \frac{z e^{ikz}}{z^2+a^2}\, dz = -2\pi i \left(\tfrac{1}{2} e^{ak}\right)$

NOTE THIS PATH IS CLOCKWISE

THE INTEGRALS OVER THE ARCS γ_1 & γ_2 VANISH AS $R \to \infty$, AS BOTH SATISFY THE CONDITIONS OF JORDAN'S LEMMA, FOR THE CORRECT SIGN OF k IN EACH CONTOUR

DEALING WITH EACH CASE SEPARATELY AS $R \to \infty$

- IF $k>0$

$\displaystyle\int_{-\infty}^{\infty} \frac{x e^{ikx}}{x^2+a^2}\, dx = i\pi e^{-ak}$

$\displaystyle\int_{-\infty}^{\infty} \frac{x(\cos kx + i \sin kx)}{x^2+a^2}\, dx = i\pi e^{-ak}$

ONLY EVEN PART SURVIVES

$\displaystyle\int^{\infty} \frac{2xi \sin kx}{x^2+a^2}\, dx = i\pi e^{-ak}$

$\displaystyle\int_0^{\infty} \frac{x \sin kx}{x^2+a^2}\, dx = \frac{\pi}{2} e^{-ak}$

- IF $k<0$

$\displaystyle\int_{-\infty}^{\infty} \frac{x e^{ikx}}{x^2+a^2}\, dx = -i\pi e^{ak}$

⋮

$\displaystyle\int_0^{\infty} \frac{x \sin kx}{x^2+a^2} = -\frac{\pi}{2} e^{ak}$

COLLECTING RESULTS FOR ALL k

$\hat{f}(k) = -\sqrt{\dfrac{2}{\pi}}\, i \displaystyle\int_0^{\infty} \dfrac{x \sin kx}{x^2+a^2}\, dx$

$\hat{f}(k) = -\dfrac{\sqrt{2}}{\sqrt{\pi}}\, i \times \dfrac{\pi}{2} e^{-ak}$ $(k>0)$ OR $-\dfrac{\sqrt{2}}{\sqrt{\pi}}\, i \left(-\dfrac{\pi}{2} e^{ak}\right)$ $(k<0)$

$\hat{f}(k) = -\sqrt{\dfrac{\pi}{2}}\, i e^{-ak}$ $(k>0)$ OR $\sqrt{\dfrac{\pi}{2}}\, i\, e^{ak}$ $(k<0)$

$\therefore\ \hat{f}(k) = -\sqrt{\dfrac{\pi}{2}}\, i\, e^{-a|k|} \operatorname{sign} k$

Question 26

Find the inverse Fourier transform of

$$\hat{g}(k) = e^{-k^2\sigma^2 t},$$

where σ and t are positive constants.

$$\boxed{\mathcal{F}^{-1}\left[e^{-k^2\sigma^2 t}\right] = \frac{1}{\sqrt{2t}\,\sigma}\exp\left(-\frac{x^2}{4t\sigma^2}\right)}$$

proof

$$\boxed{\hat{g}(k) = e^{-k^2\sigma^2 t}}$$

$$\Rightarrow g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{g}(k)e^{ikx}\,dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-k^2\sigma^2 t}e^{ikx}\,dk$$

- AS $e^{-k^2\sigma^2 t}$ IS EVEN IN k WE MAY SIMPLIFY

$$\Rightarrow g(x) = \frac{2}{\sqrt{2\pi}}\int_0^{\infty}e^{-k^2\sigma^2 t}\cos kx\,dk$$

- LET $I = \int_0^{\infty}e^{-k^2\sigma^2 t}\cos kx\,dk$

$$\Rightarrow \frac{\partial I}{\partial x} = \int_0^{\infty}e^{-k^2\sigma^2 t}\left[-k\sin kx\right]dk$$

$$\Rightarrow \frac{\partial I}{\partial x} = \int_0^{\infty}-ke^{-k^2\sigma^2 t}\sin kx\,dk$$

- BY PARTS (W.R.T k)

$\sin kx$	$x\cos kx$
$\frac{1}{2\sigma^2 t}e^{-k^2\sigma^2 t}$	$-ke^{-k^2\sigma^2 t}$

$$\Rightarrow \frac{\partial I}{\partial x} = \cancel{\left[\frac{1}{2\sigma^2 t}e^{-k^2\sigma^2 t}\sin kx\right]_0^{\infty}} - \frac{x}{2\sigma^2 t}\int_0^{\infty}e^{-k^2\sigma^2 t}\cos kx\,dk$$

$$\Rightarrow \frac{\partial I}{\partial x} = -\frac{x}{2\sigma^2 t}I$$

- SOLVING THE O.D.E BY SEPARATION OF VARIABLES

$$\Rightarrow \frac{1}{I}\partial I = -\frac{x}{2\sigma^2 t}\partial x$$

$$\Rightarrow \ln I = -\frac{x^2}{4\sigma^2 t} + C$$

$$\Rightarrow I = Ae^{-\frac{x^2}{4\sigma^2 t}} \qquad (A = e^C)$$

$$\Rightarrow \int_0^{\infty}e^{-k^2\sigma^2 t}\cos kx\,dk = Ae^{-\frac{x^2}{4\sigma^2 t}}$$

- EVALUATE AT $x=0$

$$\Rightarrow \int_0^{\infty}e^{-k^2\sigma^2 t}\,dk = A.$$

- USE A SUBSTITUTION $u^2 = k^2\sigma^2 t$ OR $u = k\sigma\sqrt{t}$

$du = dk\,\sigma\sqrt{t}$

$dk = \frac{du}{\sigma\sqrt{t}}$

LIMITS UNCHANGED

$$\Rightarrow A = \frac{1}{\sigma\sqrt{t}}\int_0^{\infty}e^{-u^2}\,du$$

$$\Rightarrow \boxed{A = \frac{1}{\sigma\sqrt{t}}\frac{\sqrt{\pi}}{2}}$$

$$\Rightarrow I = \frac{1}{\sigma\sqrt{t}}\frac{\sqrt{\pi}}{2}e^{-\frac{x^2}{4\sigma^2 t}}$$

$$\Rightarrow g(x) = \frac{2}{\sqrt{2\pi}}I = \frac{2}{\sqrt{2\pi}}\frac{1}{\sigma\sqrt{t}}\frac{\sqrt{\pi}}{2}e^{-\frac{x^2}{4\sigma^2 t}}$$

$$\Rightarrow g(x) = \frac{1}{\sqrt{2t}\,\sigma}e^{-\frac{x^2}{4\sigma^2 t}}$$

Question 27

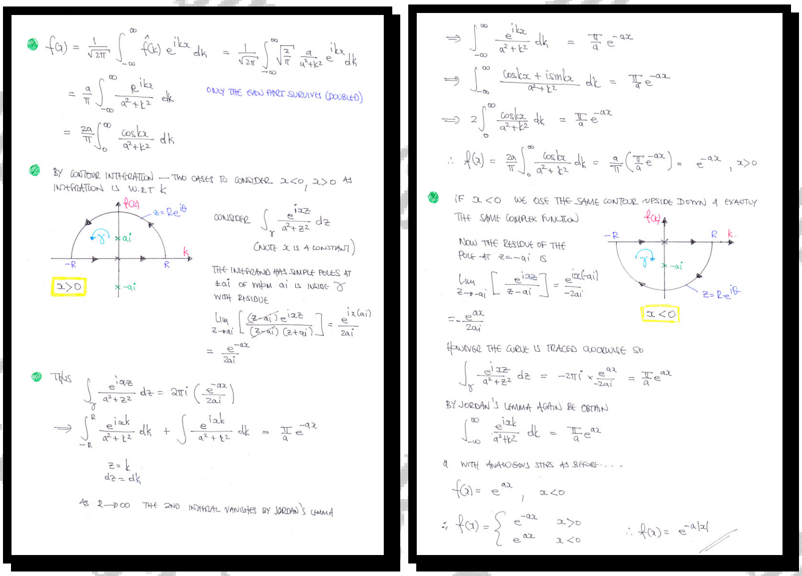
The Fourier transform $\hat{f}(k)$, of function $f(x)$ is

$$\hat{f}(k) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + k^2},$$

where a is a positive constant.

Use contour integration to find an expression for $f(x)$.

$$\boxed{f(x) = e^{a|x|}}$$



$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k)\, e^{ikx}\, dk = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{a}{a^2+k^2} e^{ikx}\, dk$

$= \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{e^{ikx}}{a^2+k^2}\, dk$ ONLY THE EVEN PART SURVIVES (DOUBLED)

$= \frac{2a}{\pi} \int_{0}^{\infty} \frac{\cos kx}{a^2+k^2}\, dk$

BY CONTOUR INTEGRATION — TWO CASES TO CONSIDER $x<0$, $x>0$ AS INTEGRATION IS W.R.T k

CONSIDER $\int_{\gamma} \frac{e^{ixz}}{a^2+z^2}\, dz$

(NOTE x IS A CONSTANT)

THE INTEGRAND HAS SIMPLE POLES AT $\pm ai$ OF WHICH ai IS INSIDE γ WITH RESIDUE

$\lim_{z \to ai} \left[\frac{\cancel{(z-ai)}\, e^{ixz}}{\cancel{(z-ai)}(z+ai)} \right] = \frac{e^{ix(ai)}}{2ai}$

$= \frac{e^{-ax}}{2ai}$

THUS

$\int_{\gamma} \frac{e^{ixz}}{a^2+z^2}\, dz = 2\pi i \left(\frac{e^{-ax}}{2ai} \right)$

$\Rightarrow \int_{-R}^{R} \frac{e^{ixk}}{a^2+k^2}\, dk + \int \frac{e^{ixk}}{a^2+k^2}\, dk = \frac{\pi}{a} e^{-ax}$

$z = k$
$dz = dk$

AS $R \to \infty$ THE 2ND INTEGRAL VANISHES BY JORDAN'S LEMMA

$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{ikx}}{a^2+k^2}\, dk = \frac{\pi}{a} e^{-ax}$

$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos kx + i\sin kx}{a^2+k^2}\, dk = \frac{\pi}{a} e^{-ax}$

$\Rightarrow 2\int_{0}^{\infty} \frac{\cos kx}{a^2+k^2}\, dk = \frac{\pi}{a} e^{-ax}$

$\therefore f(x) = \frac{2a}{\pi} \int_{0}^{\infty} \frac{\cos kx}{a^2+k^2}\, dk = \frac{a}{\pi}\left(\frac{\pi}{a} e^{-ax} \right) = e^{-ax}, \quad x>0$

IF $x<0$ WE USE THE SAME CONTOUR UPSIDE DOWN & EXACTLY THE SAME COMPLEX FUNCTION

NOW THE RESIDUE OF THE POLE AT $z=-ai$ IS

$\lim_{z \to -ai} \left[\frac{e^{ixz}}{z-ai} \right] = \frac{e^{ix(-ai)}}{-2ai}$

$= -\frac{e^{ax}}{2ai}$

HOWEVER THE CURVE IS TRACED CLOCKWISE SO

$\int_{\gamma} \frac{e^{ixz}}{a^2+z^2}\, dz = -2\pi i \times \frac{e^{ax}}{-2ai} = \frac{\pi}{a} e^{ax}$

BY JORDAN'S LEMMA AGAIN WE OBTAIN

$\int_{-\infty}^{\infty} \frac{e^{ixk}}{a^2+k^2}\, dk = \frac{\pi}{a} e^{ax}$

& WITH ANALOGOUS STEPS AS BEFORE....

$f(x) = e^{ax}, \quad x<0$

$\therefore f(x) = \begin{cases} e^{-ax} & x>0 \\ e^{ax} & x<0 \end{cases}$ $\therefore f(x) = e^{-a|x|}$

Question 28

The function f is defined by

$$f(x) = \frac{1}{\left(x^2 + a^2\right)^2},$$

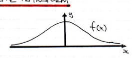
where a is a positive constant.

Use contour integration to find the Fourier transform of $f(x)$.

$$\boxed{}, \quad \boxed{\mathcal{F}\left[\frac{1}{\left(x^2+a^2\right)^2}\right] = \hat{f}(k) = \sqrt{\frac{\pi}{8}}\,\frac{(1+a|k|)\mathrm{e}^{-a|k|}}{a^3}}$$

BY THE INTEGRAL DEFINITION OF FOURIER TRANSFORM

$$f(x) = \frac{1}{(a^2+x^2)^2}, \quad a>0$$



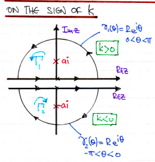
$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx$$

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} \frac{\cos kx - i\sin kx}{(a^2+x^2)^2}\,dx$$

ONLY EVEN PART SURVIVES

$$\hat{f}(k) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} \frac{\cos kx}{(a^2+x^2)^2}\,dx$$

PROCEED BY CONTOUR INTEGRATION – USE DIFFERENT CONTOUR DEPENDING ON THE SIGN OF k



$f(z) = \frac{e^{ikz}}{(z^2+a^2)^2}$ HAS DOUBLE POLES AT $\pm ai$, ONE IN EACH OF THE CONTOURS SHOWN OPPOSITE

RESIDUE AT $z = ai$

$$\lim_{z\to ai}\left[\frac{d}{dz}\left[f(z)(z-ai)^2\right]\right] = \lim_{z\to ai}\left[\frac{d}{dz}\left[\frac{e^{ikz}\cancel{(z-ai)^2}}{\cancel{(z-ai)^2}(z+ai)^2}\right]\right]$$

$$= \lim_{z\to ai}\left[\frac{d}{dz}\left(\frac{e^{ikz}}{(z+ai)^2}\right)\right]$$

$$= \lim_{z\to ai}\left[\frac{ike^{ikz}(z+ai)^2 - 2(z+ai)e^{ikz}}{(z+ai)^4}\right]$$

$$= \lim_{z\to ai}\left[\frac{ike^{ikz}(z+ai) - 2e^{ikz}}{(z+ai)^3}\right] = \lim_{z\to ai}\left[\frac{e^{ikz}(ikz - ak - 2)}{(z+ai)^3}\right]$$

$$= \frac{e^{-ak}(-2ak-2)}{(2ai)^3} = \frac{-2e^{-ak}(ak+1)}{-8a^3i} = \frac{(ak+1)e^{-ak}}{4a^3i}$$

IN ALMOST ANALOGOUS FASHION THE RESIDUE AT $-ai$ FOR USE IN THE BOTTOM CONTOUR

$$\dots = \lim_{z\to -ai}\left[\frac{ike^{ikz}(z-ai) - 2e^{ikz}}{(z-ai)^3}\right]$$

$$= \lim_{z\to -ai}\left[\frac{e^{ikz}(ikz + ak - 2)}{(z-ai)^3}\right] = \frac{e^{ak}(2ak-2)}{(-2ai)^3}$$

$$= \frac{2(ak-1)e^{ak}}{8a^3i} = \frac{(ak-1)e^{ak}}{4a^3i}$$

- IF $k>0$ USING THE "TOP" CONTOUR

$$\int_{\Gamma_1} f(z)\,dz = \int_{-R}^{R}\frac{e^{ikx}}{(x^2+a^2)^2}\,dx + \int_{\gamma_1}\frac{e^{ikz}}{(z^2+a^2)^2}\,dz = 2\pi i\left(\frac{(ak+1)e^{-ak}}{4a^3i}\right)$$

- IF $k<0$ USING THE "BOTTOM" CONTOUR

$$\int_{\Gamma_2} f(z)\,dz = \int_{-R}^{R}\frac{e^{ikx}}{(x^2+a^2)^2}\,dx + \int_{\gamma_2}\frac{e^{ikz}}{(z^2+a^2)^2}\,dz = -2\pi i\left(\frac{(ak-1)e^{ak}}{4a^3i}\right)$$

CLOCKWISE PATH

THE INTEGRALS OVER THE ARCS γ_1 & γ_2 VANISH AS $R\to\infty$, AS BOTH SATISFY THE CONDITIONS OF JORDAN'S LEMMA, FOR THE CORRECT SIGN OF k IN EACH CONTOUR

DEALING WITH EACH CASE SEPARATELY AS $R\to\infty$

- IF $k>0$

$$\int_{-\infty}^{\infty}\frac{e^{ikx}}{(x^2+a^2)^2}\,dx = \frac{\pi}{2a^3}(ak+1)e^{-ak}$$

$$\int_{0}^{\infty}\frac{2\cos kx}{(x^2+a^2)^2}\,dx = \frac{\pi e^{-ak}(ak+1)}{2a^3}$$

$$\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\frac{2\cos kx}{(x^2+a^2)^2}\,dx = \sqrt{\frac{\pi}{2}}\,\frac{e^{-ak}(ak+1)}{a^3}$$

$$\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\frac{\cos kx}{(x^2+a^2)^2}\,dx = \sqrt{\frac{\pi}{8}}\,\frac{e^{-ak}(ak+1)}{a^3}$$

- IF $k<0$

$$\int_{-\infty}^{\infty}\frac{e^{ikx}}{(x^2+a^2)^2}\,dx = \frac{\pi}{2a^3}(-ak+1)e^{ak}$$

$\vdots$

$$\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\frac{\cos kx}{(x^2+a^2)^2}\,dx = \sqrt{\frac{\pi}{8}}\,\frac{e^{ak}(-ak+1)}{a^3}$$

COLLECTING THE RESULTS FOR ALL k

$$\hat{f}(k) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\frac{\cos kx}{(x^2+a^2)^2}\,dx = \begin{cases}\sqrt{\frac{\pi}{8}}\,\frac{e^{-ak}(ak+1)}{a^3} & k>0\\ \sqrt{\frac{\pi}{8}}\,\frac{e^{ak}(-ak+1)}{a^3} & k<0\end{cases}$$

$$\hat{f}(k) = \sqrt{\frac{\pi}{8}}\,\frac{(1+a|k|)e^{-a|k|}}{a^3}$$

VARIOUS PROBLEMS on FOURIER TRANSFORMS

Question 1

Find the Fourier transform of an arbitrary function $f(x)$ if

i. $f(x)$ is even.

ii. $f(x)$ is odd.

Give the answers as a simplified integral form.

$$\boxed{\hat{f}(k)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos kx \ dx}, \quad \boxed{\hat{f}(k)=-\mathrm{i}\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin kx \ dx}$$

$$\hat{f}(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\left[\cos kx - i\sin kx\right]dx$$

- IF $f(x)$ IS EVEN

$$\dots=\frac{1}{\sqrt{2\pi}}\int_0^{\infty} 2f(x)\cos kx\,dx=\frac{2}{\sqrt{2\pi}}\int_0^{\infty} f(x)\cos kx\,dx=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos kx\,dx$$

- IF $f(x)$ IS ODD

$$\dots=\frac{1}{\sqrt{2\pi}}\int_0^{\infty} -2if(x)\sin kx\,dx=\frac{-2i}{\sqrt{2\pi}}\int_0^{\infty} f(x)\sin kx\,dx=-i\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin kx\,dx$$

Question 2

Use the definition of the Fourier transform, of an absolutely integrable function $f(x)$, to show that

$$\mathcal{F}\left[f'(x)\right]=\mathrm{i}k\mathcal{F}\left[f(x)\right].$$

$\boxed{\text{proof}}$

$$\mathcal{F}\left[f'(x)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x)e^{-ikx}\,dx$$

BY PARTS

e^{-ikx}	$-ike^{-ikx}$
$f(x)$	$f'(x)$

$$\dots=\frac{1}{\sqrt{2\pi}}\left\{\cancel{\left[f(x)e^{-ikx}\right]_{-\infty}^{\infty}}+ik\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx\right\}$$

THIS IS ZERO SINCE $\left|e^{-ikx}\right|=1$ & SINCE $\int_{-\infty}^{\infty}|f(x)|\,dx \leq M$ IT IMPLIES THAT $|f(x)| \to 0$ AS $|x| \to \infty$

NOTE PROPERTY OF $f(x)$ TO HAVE A FOURIER TRANSFORM IS TO BE ABSOLUTELY INTEGRABLE FOR $(-\infty,\infty)$, IE $\int_{-\infty}^{\infty}|f(x)|\,dx \leq M$

$$=ik\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx\right]=ik\hat{f}(k)$$

$$\therefore \mathcal{F}\left[f'(x)\right]=ik\mathcal{F}\left[f(x)\right]$$

Question 3

The Fourier transform of an absolutely integrable function $f(x)$, is denoted by $\hat{f}(k)$.

Show that

$$\mathcal{F}\left[x f(x)\right] = \mathrm{i}\frac{d}{dk}\left[\hat{f}(k)\right].$$

proof

$$\frac{d}{dk}\left[\hat{f}(k)\right] = \frac{d}{dk}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx = \int_{-\infty}^{\infty} f(x)\frac{\partial}{\partial k}\left(e^{-ikx}\right)dx$$

$$= \int_{-\infty}^{\infty} f(x)(-ix)e^{-ikx}\,dx$$

$$= \mathcal{F}\left(-ixf(x)\right) = -i\,\mathcal{F}\left(xf(x)\right)$$

$$\mathcal{F}\left[xf(x)\right] = i\frac{d}{dk}\left[\hat{f}(k)\right]$$

Question 4

Given that c is a constant show that

$$\mathcal{F}\left[f(x+c)\right] = \mathrm{e}^{\mathrm{i}kc}\,\mathcal{F}\left[f(x)\right].$$

proof

$$\mathcal{F}\left[f(x+c)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x+c)\,e^{-ikx}\,dx$$

SUBSTITUTION
$u = x+c$
$du = dx$
LIMITS UNCHANGED

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\,e^{-ik(u-c)}\,du$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{-iku}e^{ikc}\,du$$

$$= \frac{1}{\sqrt{2\pi}}e^{ikc}\int_{-\infty}^{\infty} f(u)\,e^{-iku}\,du$$

u IS A DUMMY VARIABLE

$$= e^{ikc}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-ikx}\,dx\right]$$

$$= e^{ikc}\,\mathcal{F}\left[f(x)\right]$$

Question 5

Given that c is a constant show that

$$\mathcal{F}^{-1}\left[\hat{f}(k+c)\right] = e^{icx} f(x),$$

where $\hat{f}(k) \equiv \mathcal{F}\left[f(x)\right]$

proof

$$\mathcal{F}^{-1}\left[\hat{f}(k+c)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(k+c)\, e^{ikx}\, dk$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(u)\, e^{ix(u-c)}\, du$$

SUBSTITUTION
$u = k+c$
$du = dk$
LIMITS UNCHANGED

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(u)\, e^{ixu}\, e^{-icx}\, du$$

$$= \frac{e^{-icx}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(u)\, e^{ixu}\, du$$

$$= e^{-icx}\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(k)\, e^{ikx}\, dk$$

u IS A DUMMY VARIABLE

$$= e^{-icx}\,\mathcal{F}^{-1}\left[f(\hat{k})\right]$$

$$= e^{-icx} f(x)$$

Question 6

Given that c is a constant prove the validity of the two shift theorems

a) $\mathcal{F}\left[f(x+c)\right] = \mathrm{e}^{\mathrm{i}kc}\,\mathcal{F}\left[f(x)\right]$.

b) $\mathcal{F}^{-1}\left[\hat{f}(k+c)\right] = \mathrm{e}^{\mathrm{i}cx}\,f(x)$.

Note that $\hat{f}(k) \equiv \mathcal{F}\left[f(x)\right]$.

proof

a)
$$\mathcal{F}[f(x+c)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x+c)\,e^{-ikx}\,dx$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\,e^{-ik(u-c)}\,du$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\,e^{-iku}\,e^{ikc}\,du$$
$$= e^{ikc}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\,e^{-iku}\,du\right]$$
$$= e^{ikc}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-ikx}\,dx\right]$$
$$= e^{ikc}\,\mathcal{F}[f(x)]$$

BY SUBSTITUTION
$u = x+c$
$x = u-c$
$dx = du$
LIMITS UNCHANGED

u IS A DUMMY VARIABLE

b)
$$\boxed{\mathcal{F}[f(x)] = \hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-ikx}\,dx}$$
$$\Rightarrow \hat{f}(k+c) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-i(k+c)x}\,dx$$
$$\Rightarrow \hat{f}(k+c) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-ikx}\,e^{-icx}\,dx$$
$$\Rightarrow \hat{f}(k+c) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \left[f(x)\,e^{-icx}\right] e^{-ikx}\,dx$$
$$\Rightarrow \hat{f}(k+c) = \mathcal{F}\left[e^{-icx} f(x)\right]$$

Thus

$$\mathcal{F}^{-1}\left(\hat{f}(k+c)\right) = e^{-icx} f(x)$$

Question 7

The convolution $[f*g](x)$, of two functions $f(x)$ and $g(x)$ is defined as

$$[f*g](x)=\int_{-\infty}^{\infty} f(x-y)\,g(y)\;dy\,.$$

Show that

$$\mathcal{F}\{[f*g](x)\} \;=\; \sqrt{2\pi}\,\mathcal{F}\big[f(x)\big]\mathcal{F}\big[g(x)\big] \;=\; \sqrt{2\pi}\,\hat{f}(k)\,\hat{g}(k)\,.$$

proof

$$\boxed{(f*g)(x)=\int_{-\infty}^{\infty} f(x-y)\,g(y)\,dy}$$

- TAKING THE FOURIER TRANSFORM OF THE CONVOLUTION

$$\mathcal{F}\big[(f*g)(x)\big]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\left[\int_{-\infty}^{\infty} f(x-y)\,g(y)\,dy\right]dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-y)\,g(y)\,e^{-ikx}\,dy\,dx$$

- REVERSE THE ORDER OF INTEGRATION (LIMITS UNCHANGED)

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(y)\int_{-\infty}^{\infty} f(x-y)\,e^{-ikx}\,dx\,dy$$

$$=\int_{-\infty}^{\infty} g(y)\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-y)\,e^{-ikx}\,dx\right]dy$$

$$=\int_{-\infty}^{\infty} g(y)\,\mathcal{F}\big[f(x-y)\big]\,dy$$

- NOW USING THE RESULT

$$\boxed{\mathcal{F}\big[f(x+c)\big]=e^{+ikc}\,\mathcal{F}\big[f(x)\big]=e^{+ikc}\,\hat{f}(k)}$$

$$=\int_{-\infty}^{\infty} g(y)\left[e^{-iky}\,\hat{f}(k)\right]dy$$

$$=\int_{-\infty}^{\infty}\hat{f}(k)\left[e^{-iky}\,g(y)\right]dy$$

$$=\sqrt{2\pi}\,\hat{f}(k)\times\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iky}\,g(y)\,dy$$

$$=\hat{f}(k)\sqrt{2\pi}\times\hat{g}(k)$$

$$=\sqrt{2\pi}\,\hat{f}(k)\,\hat{g}(k)$$

- HENCE WE OBTAIN

$$\mathcal{F}\big[(f*g)(x)\big]=\sqrt{2\pi}\,\hat{f}(k)\,\hat{g}(k)$$

OR

$$\mathcal{F}\big[(f*g)(x)\big]=\sqrt{2\pi}\,\mathcal{F}\big[f(x)\big]\,\mathcal{F}\big[g(x)\big]$$

ALTERNATIVE FROM PREVIOUSLY OBTAINED RESULT

$$\dots=\frac{1}{\sqrt{4\pi}}\int_{-\infty}^{\infty} g(y)\left[\int_{-\infty}^{\infty} f(x-y)\,e^{-ikx}\,dx\right]dy$$

- LET $u=x-y$ IN THE INNER INTEGRAL, $du=dx$, LIMITS UNCHANGED

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(y)\left[\int_{-\infty}^{\infty} f(u)\,e^{-ik(u+y)}\,du\right]dy$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(y)\int_{-\infty}^{\infty} f(u)\,e^{-iku}\,e^{-iky}\,du\,dy$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(y)\,e^{-iky}\int_{-\infty}^{\infty} f(u)\,e^{-iku}\,du\,dy$$

$$=\sqrt{2\pi}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(y)\,e^{-iky}\,dy\right]\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\,e^{-iku}\,du\right]$$

$$=\sqrt{2\pi}\,\hat{g}(k)\,\hat{f}(k)$$

Question 8

It is given that c is a constant and $\hat{f}(k) \equiv \mathcal{F}\left[f(x)\right]$.

a) Prove the validity of the inversion shift theorem

$$\mathcal{F}^{-1}\left[\hat{f}(k+c)\right] = \mathrm{e}^{\mathrm{i}cx} f(x).$$

b) Hence determine an expression for

$$\mathcal{F}^{-1}\left[\mathrm{e}^{-(k-a)^2}\right],$$

where a is a positive constant.

$$\boxed{\mathcal{F}^{-1}\left[\mathrm{e}^{-(k-a)^2}\right] = \frac{1}{\sqrt{2}}\mathrm{e}^{-\frac{1}{4}x^2}\left[\cos ax + \mathrm{i}\sin ax\right]}$$

a) $\mathcal{F}^{-1}\left[\hat{f}(k+c)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(k+c)\, e^{ikx}\, dk$

SUBSTITUTION
$u = k+c$
$k = u-c$
$dk = du$
LIMITS UNCHANGED

$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(u)\, e^{ix(u-c)}\, du$

$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(u)\, e^{ixu} e^{-icx}\, du$

$= e^{-icx} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(u)\, e^{ixu}\, du$

u IS A DUMMY VARIABLE

$= e^{-icx} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(k)\, e^{ikx}\, dk$

$= e^{-icx} \mathcal{F}^{-1}\left[\hat{f}(k)\right]$

$= e^{-icx} f(x)$

AS REQUIRED

ALTERNATIVE

BY DEFINITION $\mathcal{F}\left[f(x)\right] = \hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$

THUS $\hat{f}(k+c) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-i(k+c)x}\, dx$

$\hat{f}(k+c) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-ikx} e^{-icx}\, dx$

$\hat{f}(k+c) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \left[e^{-icx} f(x)\right] e^{-ikx}\, dx$

$\hat{f}(k+c) = \mathcal{F}\left[e^{-icx} f(x)\right]$

$\mathcal{F}^{-1}\left[\hat{f}(k+c)\right] = e^{-icx} f(x)$

AS BEFORE

b) $\boxed{\mathcal{F}^{-1}\left[\hat{f}(k+c)\right] = e^{iax} f(x) \quad \text{WHERE } f(x) \text{ IS THE INVERSE OF } \hat{f}(k)}$

- SO WE NEED THE INVERSE OF $\hat{f}(k) = e^{-k^2}$

$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-k^2} e^{ikx}\, dk = \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-k^2} \cos kx\, dk$

ODD PART VANISHES
EVEN PART DOUBLES

- BY INTEGRATION UNDER THE INTEGRAL SIGN

$\Rightarrow I = \int_{0}^{\infty} e^{-k^2} \cos kx\, dk$

$\Rightarrow \frac{\partial I}{\partial x} = \frac{\partial}{\partial x}\int_{0}^{\infty} e^{-k^2} \cos kx\, dk = \int_{0}^{\infty} e^{-k^2} \frac{\partial}{\partial x}(\cos kx)\, dk$

$\Rightarrow \frac{\partial I}{\partial x} = \int_{0}^{\infty} -k e^{-k^2} \sin kx\, dk$

- BY PARTS NOW

$\sin kx$	$x\cos kx$
$\frac{1}{2}e^{-k^2}$	$-ke^{-k^2}$

$\Rightarrow \frac{\partial I}{\partial x} = \cancel{\left[\frac{1}{2}e^{-k^2} \sin kx\right]_{k=0}^{k=\infty}} - \frac{1}{2}x\int_{0}^{\infty} e^{-k^2} \cos kx\, dk$

$\Rightarrow \frac{\partial I}{\partial x} = -\frac{1}{2}xI$

- SEPARATE VARIABLES

$\Rightarrow \frac{1}{I}\, dI = -\frac{1}{2}x\, dx$

$\Rightarrow \ln I = -\frac{1}{4}x^2 + C$

$\Rightarrow I = Ae^{-\frac{1}{4}x^2}$ ("A" ARBITRARY)

$\Rightarrow \int_{0}^{\infty} e^{-k^2} \cos kx\, dk = Ae^{-\frac{1}{4}x^2}$

USE A SUITABLE VALUE TO EVALUATE THE CONSTANT, SAY $x=0$

$\int_{0}^{\infty} e^{-k^2}\, dk = A$

$\boxed{A = \frac{1}{2}\sqrt{\pi}}$

$\Rightarrow \int_{0}^{\infty} e^{-k^2} \cos kx\, dk = \frac{\sqrt{\pi}}{2} e^{-\frac{1}{4}x^2}$

$\Rightarrow \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-k^2} \cos kx\, dk = \frac{2}{\sqrt{2\pi}} \frac{\sqrt{\pi}}{2} e^{-\frac{1}{4}x^2}$

$\Rightarrow f(x) = \frac{1}{\sqrt{2}} e^{-\frac{1}{4}x^2}$

$\therefore \mathcal{F}^{-1}\left[e^{-(k-a)^2}\right] = e^{iax} \frac{1}{\sqrt{2}} e^{-\frac{1}{4}x^2}$

$= \frac{1}{\sqrt{2}} e^{-\frac{1}{4}x^2}\left[\cos ax + i\sin ax\right]$

Question 9

The convolution theorem for two functions $f(x)$ and $g(x)$ asserts that

$$\mathcal{F}\{[f*g](x)\} = \sqrt{2\pi}\,\mathcal{F}\big[f(x)\big]\mathcal{F}\big[g(x)\big],$$

where

$$[f*g](x) = \int_{-\infty}^{\infty} f(x-y)\,g(y)\ dy\,.$$

a) Starting from the convolution theorem prove Parseval's Theorem

$$\int_{-\infty}^{\infty} \left|h(y)\right|^2\ dy = \int_{-\infty}^{\infty} \left|\hat{h}(k)\right|^2\ dk\,.$$

b) Use Parseval's Theorem to evaluate

$$\int_0^{\infty} \frac{1}{x^2+a^2}\ dx\,.$$

You may assume that if $f(x) = e^{-a|x|}$, then $\hat{f}(k) = \sqrt{\frac{2}{\pi}}\,\frac{a}{a^2+k^2}$.

$$\boxed{\frac{\pi}{4a^3}}$$

a)

- STARTING FROM THE FOURIER CONVOLUTION

$$\boxed{\mathcal{F}[(f*g)(x)] = \sqrt{2\pi}\,\mathcal{F}[f(x)]\,\mathcal{F}[g(x)] = \sqrt{2\pi}\,\hat{f}(k)\,\hat{g}(k)}$$

- INVERTING BOTH SIDES OF THE CONVOLUTION EQUATION

$$(f*g)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\sqrt{2\pi}\,\hat{f}(k)\,\hat{g}(k)\right]e^{ikx}\,dk$$

$$\int_{-\infty}^{\infty} f(x-y)\,g(y)\,dy = \int_{-\infty}^{\infty} e^{ikx}\,\hat{f}(k)\,\hat{g}(k)\,dk$$

- IN THIS EXPRESSION x IS A PARAMETER AS THE LHS IS W.R.T. y AND THE R.H.S IS W.R.T k — SO EVALUATE AT $x=0$

$$\int_{-\infty}^{\infty} f(-y)\,g(y)\,dy = \int_{-\infty}^{\infty} \hat{f}(k)\,\hat{g}(k)\,dk$$

- NOW LET $h(y) = f(-y) \Longrightarrow \hat{h}(-k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)\,e^{-iky}\,dy$
  $\hat{h}(-k) = \hat{f}(k)$ BY DEFINITION

$$\int_{-\infty}^{\infty} h(y)\,g(y)\,dy = \int_{-\infty}^{\infty} \hat{h}(-k)\,\hat{g}(k)\,dk$$

- NOW IF h IS REAL $\hat{h}(-k) = \overline{\hat{h}(k)}$ (CONJUGATE)

$$\int_{-\infty}^{\infty} h(y)\,g(y)\,dy = \int_{-\infty}^{\infty} \overline{\hat{h}(k)}\,\hat{g}(k)\,dk$$

- FINALLY TAKING $g(y) = h(y) \Rightarrow \hat{g}(k) = \hat{h}(k)$

$$\int_{-\infty}^{\infty} [h(y)]^2\,dy = \int_{-\infty}^{\infty} \overline{\hat{h}(k)}\,\hat{h}(k)\,dk$$

$$\int_{-\infty}^{\infty} |h(y)|^2\,dy = \int_{-\infty}^{\infty} |\hat{h}(k)|^2\,dk \quad \text{AS REQUIRED}$$

(OR REPLACE "MODS" WITH BRACKETS)

b)

$$\boxed{\text{NOW IF } f(x) = e^{-a|x|} \text{ THEN } \hat{f}(k) = \sqrt{\tfrac{2}{\pi}}\,\tfrac{a}{a^2+k^2}}$$

USING PARSEVAL'S THEOREM

$$\Rightarrow \int_{-\infty}^{\infty} [f(x)]^2\,dx = \int_{-\infty}^{\infty} [\hat{f}(k)]^2\,dk$$

$$\Rightarrow \int_{-\infty}^{\infty} \left[e^{-a|x|}\right]^2\,dx = \int_{-\infty}^{\infty} \left[\sqrt{\frac{2}{\pi}}\,\frac{a}{a^2+k^2}\right]^2\,dk$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-2a|x|}\,dx = \int_{-\infty}^{\infty} \frac{2a^2}{\pi}\times\frac{a}{a^2+k^2}\,dk$$

$$\Rightarrow \int_{-\infty}^{\infty} \left(\frac{2a^2}{\pi}\right)\frac{1}{a^2+k^2}\,dk = \int_0^{\infty} 2e^{-2ax}\,dx$$

$$\Rightarrow \frac{4a^2}{\pi}\int_0^{\infty} \frac{1}{a^2+k^2}\,dk = \left[-\frac{1}{a}e^{-2ax}\right]_0^{\infty}$$

$$\Rightarrow \int_0^{\infty} \frac{1}{a^2+k^2}\,dk = \frac{\pi}{4a^2}\times\frac{1}{a}\left[e^{-2ax}\right]_{\infty}^{0}$$

$$\Rightarrow \int_0^{\infty} \frac{1}{a^2+k^2}\,dk = \frac{\pi}{4a^3}\left[1-0\right]$$

$$\Rightarrow \int_0^{\infty} \frac{1}{a^2+k^2}\,dk = \frac{\pi}{4a^3}$$

OR $\displaystyle\int_0^{\infty} \frac{1}{a^2+x^2}\,dx = \frac{\pi}{4a^3}$

Question 10

The convolution $[f*g](x)$, of two functions $f(x)$ and $g(x)$ is defined as

$$[f*g](x)=\int_{-\infty}^{\infty} f(x-y)\,g(y)\;dy\,.$$

a) Show that

$$\mathcal{F}\{[f*g](x)\} \;=\; \sqrt{2\pi}\,\mathcal{F}\big[f(x)\big]\mathcal{F}\big[g(x)\big] \;=\; \sqrt{2\pi}\,\hat{f}(k)\,\hat{g}(k)\,.$$

b) Hence prove Parseval's Theorem

$$\int_{-\infty}^{\infty} h(y)\,g(y)\;dy \;=\; \int_{-\infty}^{\infty} \overline{\hat{h}}(k)\,\hat{g}(k)\;dk\,.$$

c) Use Parseval's Theorem to evaluate

$$\int_{0}^{\infty} \frac{1}{\left(x^2+a^2\right)\left(x^2+b^2\right)}\;dx\,.$$

You may assume that if $f(x)=\mathrm{e}^{-a|x|}$, then $\hat{f}(k)=\sqrt{\frac{2}{\pi}}\,\frac{a}{a^2+k^2}$.

$$\boxed{\frac{\pi}{2ab(a+b)}}$$

a)

CONVOLUTION OF f & g: $(f*g)(x) \equiv \int_{-\infty}^{\infty} f(x-y)\,g(y)\,dy$

$\mathcal{F}\big[(f*g)(x)\big] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\left[\int_{-\infty}^{\infty} f(x-y)\,g(y)\,dy\right]dx$

$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(y)\int_{-\infty}^{\infty} f(x-y)\,e^{-ikx}\,dx\,dy$

$= \int_{-\infty}^{\infty} g(y)\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-y)\,e^{-ikx}\,dx\right]dy$

$= \int_{-\infty}^{\infty} g(y)\,\mathcal{F}\big[f(x-y)\big]\,dy$

BUT $\mathcal{F}(x+c) = e^{ikc}\hat{f}(k) = e^{ikc}\,\mathcal{F}\big[f(x)\big]$

$= \int_{-\infty}^{\infty} g(y)\left[e^{-iky}\hat{f}(k)\right]dy$

$= \hat{f}(k)\int_{-\infty}^{\infty} g(y)\,e^{-iky}\,dy$

$= \hat{f}(k)\sqrt{2\pi} \times \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(y)\,e^{-iky}\,dy\right]$

$= \sqrt{2\pi}\,\hat{f}(k)\,\hat{g}(k)$

$\therefore\ \mathcal{F}\big[(f*g)(x)\big] = \sqrt{2\pi}\,\hat{f}(k)\,g(k)$

OR $\mathcal{F}\big[(f*g)(x)\big] = \sqrt{2\pi}\,\mathcal{F}\big[f(x)\big]\,\mathcal{F}\big[g(x)\big]$

b)

CONTINUING BY THE CONVOLUTION THEOREM

$\boxed{\mathcal{F}\big[(f*g)(x)\big] = \sqrt{2\pi}\,\hat{f}(k)\,\hat{g}(k)}$

INVERTING BOTH SIDES

$(f*g)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\sqrt{2\pi}\,\hat{f}(k)\,\hat{g}(k)\right]e^{ikx}\,dk$

$\int_{-\infty}^{\infty} f(x-y)\,g(y)\,dy = \int_{-\infty}^{\infty} e^{ikx}\hat{f}(k)\,\hat{g}(k)\,dk$

AS x IS A PARAMETER IN THE ABOVE EQUATION WE MAY EVALUATE IT, SAY AT $x=0$

$\int_{-\infty}^{\infty} f(-y)\,g(y)\,dy = \int_{-\infty}^{\infty} \hat{f}(k)\,\hat{g}(k)\,dk$

NEXT LET $h(y)=f(-y)$ NOW $\hat{h}(-k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)\,e^{-iky}\,dy$

$\hat{h}(-k) = \hat{f}(k)$ (BY DEFINITION)

$\int_{-\infty}^{\infty} h(y)\,g(y)\,dy = \int_{-\infty}^{\infty} \hat{h}(-k)\,\hat{g}(k)\,dk$

NOW IF h IS REAL $\hat{h}(-k) = \overline{\hat{h}}(k)$ (CONJUGATE)

$\int_{-\infty}^{\infty} h(y)\,g(y)\,dy = \int_{-\infty}^{\infty} \overline{\hat{h}}(k)\,\hat{g}(k)\,dk$

AS REQUIRED

c)

NOW IF $f(x)=e^{-a|x|}$ THEN $\hat{f}(k)=\sqrt{\frac{2}{\pi}}\,\frac{a}{a^2+k^2}$

USING PARSEVAL'S THEOREM

$\Rightarrow \int_{-\infty}^{\infty} h(y)\,g(y)\,dy = \int_{-\infty}^{\infty} \overline{\hat{h}}(k)\,\hat{g}(k)\,dk$

$\Rightarrow \int_{-\infty}^{\infty} e^{-a|y|}e^{-b|y|}\,dy = \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}}\,\frac{a}{a^2+k^2}\,\sqrt{\frac{2}{\pi}}\,\frac{b}{b^2+k^2}\,dk$

(SAME AS IT IS REAL)

$\Rightarrow \int_{-\infty}^{\infty} e^{-(a+b)|y|}\,dy = \int_{-\infty}^{\infty} \frac{2ab}{\pi}\,\frac{dk}{(a^2+k^2)(b^2+k^2)}$

BOTH SIDES ARE EVEN — (DIVIDE BOTH SIDES BY 2)

$\Rightarrow \int_{0}^{\infty} e^{-(a+b)y}\,dy = \frac{2ab}{\pi}\int_{0}^{\infty}\frac{dk}{(a^2+k^2)(b^2+k^2)}$

$\Rightarrow \frac{2ab}{\pi}\int_{0}^{\infty}\frac{dk}{(a^2+k^2)(b^2+k^2)} = \left[\frac{1}{a+b}\,e^{-(a+b)y}\right]_{\infty}^{0}$

$\Rightarrow \int_{0}^{\infty}\frac{dk}{(a^2+k^2)(b^2+k^2)} = \left[\frac{1}{a+b} - 0\right]\times\frac{\pi}{2ab}$

$\Rightarrow \int_{0}^{\infty}\frac{dk}{(a^2+k^2)(b^2+k^2)} = \frac{\pi}{2ab(a+b)}$

OR IN x

$\Rightarrow \int_{0}^{\infty}\frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}$

APPLICATIONS of FOURIER TRANSFORMS

Question 1

The function $\varphi = \varphi(x,y)$ satisfies Laplace's equation in Cartesian coordinates

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 .$$

Use Fourier transforms to convert the above partial differential equation into an ordinary differential equation for $\hat{\varphi}(k,y)$, where $\hat{\varphi}(k,y)$ is the Fourier transform of $\varphi(x,y)$ with respect to x.

$$\boxed{\frac{d^2\hat{\varphi}}{dx^2} - k^2\hat{\varphi} = 0}$$

$$\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0}$$

- TAKING THE FOURIER TRANSFORM OF THIS P.D.E, I.E MULTIPLY BY $\frac{1}{\sqrt{2\pi}} e^{-ikx}$ & INTEGRATE FROM $-\infty$ TO ∞, WITH RESPECT TO x

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 \phi}{\partial x^2} e^{-ikx} \, dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 \phi}{\partial y^2} e^{-ikx} \, dx = \cancel{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 0 e^{-ikx} \, dx}$$

$$\boxed{\begin{aligned} \text{NOW } \mathcal{F}[f'(x)] &= ik\hat{f}(k) \\ \mathcal{F}[f''(x)] &= (ik)^2 \hat{f}(k) \end{aligned}}$$

$$\cancel{\frac{1}{\sqrt{2\pi}}} (ik)^2 \int_{-\infty}^{\infty} \phi(x,y) e^{-ikx} \, dx + \cancel{\frac{1}{\sqrt{2\pi}}} \frac{\partial}{\partial y^2} \int_{-\infty}^{\infty} \phi(x,y) e^{-ikx} \, dx = 0$$

$$-k^2 \hat{\phi}(k,y) + \frac{\partial}{\partial y^2}\left(\hat{\phi}(k,y) \right) = 0$$

- k IS A CONSTANT, SO AN O.D.E IN y

$$\frac{d^2\hat{\phi}}{dy^2} - k^2\hat{\phi} = 0 \quad \text{FOR } \hat{\phi}(k,y)$$

Question 2

The function $\varphi = \varphi(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0,$$

in the part of the x-y plane for which $y \geq 0$.

It is further given that

- $\varphi(x, y) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$
- $\varphi(x, 0) = \begin{cases} \frac{1}{2} & |x| < 1 \\ 0 & |x| > 1 \end{cases}$

Use Fourier transforms to show that

$$\varphi(x, y) = \frac{1}{\pi} \int_0^{\infty} \frac{1}{k} \mathrm{e}^{-ky} \sin k \cos kx \; dk,$$

and hence deduce the value of $\varphi(\pm 1, 0)$.

$\boxed{}$, $\boxed{\varphi(\pm 1, 0) = \frac{1}{4}}$

$$\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{SUBJECT TO} \quad \begin{array}{l} y \geq 0 \\ \phi(x,y) \to 0, \text{ AS } \sqrt{x^2+y^2} \to \infty \\ \phi(x,0) = \begin{cases} \frac{1}{2} & |x|<1 \\ 0 & |x|>1 \end{cases} \end{array}}$$

START BY TAKING THE FOURIER TRANSFORM OF THE P.D.E IN x

$$\Rightarrow \mathcal{F}\left[\frac{\partial^2 \phi}{\partial x^2}\right] + \mathcal{F}\left[\frac{\partial^2 \phi}{\partial y^2}\right] = \mathcal{F}[0]$$

$$\Rightarrow (ik)^2 \hat{\phi}(k,y) + \frac{\partial^2}{\partial y^2} \hat{\phi}(k,y) = 0$$

$$\Rightarrow \frac{\partial^2 \hat{\phi}}{\partial y^2} - k^2 \hat{\phi} = 0$$

$$\Rightarrow \hat{\phi}(k,y) = A(k) e^{-|k|y} + B(k) e^{|k|y}$$

AS $\sqrt{x^2+y^2} \to \infty$, $\phi(x,y) \to 0 \Rightarrow$ AS $\sqrt{k^2+y^2} \to \infty$, $\hat{\phi}(k,y) \to 0$

$$\Rightarrow B(k) = 0$$

$$\Rightarrow \hat{\phi}(k,y) = A(k) e^{-|k|y}$$

NEXT WE TAKING THE FOURIER TRANSFORM OF $\phi(x,0) = g(x)$

$$\Rightarrow \hat{\phi}(k,0) = \hat{g}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-ikx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \frac{1}{2} e^{-ikx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \frac{1}{2}\cos kx - \cancel{\frac{1}{2} i \sin kx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{1} \cos kx\, dx = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{k} \sin kx\right]_0^1$$

$$= \frac{1}{\sqrt{2\pi}} \frac{\sin k}{k}$$

$$\Rightarrow \hat{\phi}(k,0) = A(k) = \frac{1}{\sqrt{2\pi}} \frac{\sin k}{k}$$

INVERTING $\hat{\phi}(k,y)$ DIRECTLY FROM THE DEFINITION

$$\Rightarrow \hat{\phi}(k,y) = \frac{1}{\sqrt{2\pi}} \frac{\sin k}{k} e^{-|k|y}$$

$$\Rightarrow \phi(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Big[\frac{1}{\sqrt{2\pi}} \underbrace{\frac{\sin k}{k} e^{-|k|y}}_{\text{EVEN}}\Big] e^{ikx}\, dk$$

ONLY EVEN PART SURVIVES $\times 2$

$$\Rightarrow \phi(x,y) = \frac{1}{2\pi} \int_0^{\infty} 2 \times \left(\frac{\sin k}{k} e^{-ky}\right) \cos kx\, dk$$

$$\Rightarrow \phi(x,y) = \frac{1}{\pi} \int_0^{\infty} \frac{1}{k} e^{-ky} \sin k \cos kx\, dk$$

AS REQUIRED

FINALLY FIND $\phi(\pm 1, 0)$

$$\Rightarrow \phi(\pm 1,0) = \frac{1}{\pi} \int_0^{\infty} \frac{1}{k} \times 1 \times \sin k \times \underbrace{\cos(\pm k)}_{\text{(EVEN)}}\, dk$$

$$\Rightarrow \phi(\pm 1,0) = \frac{1}{2\pi} \int_0^{\infty} \frac{2 \sin k \cos k}{k}\, dk$$

$$\Rightarrow \phi(\pm 1,0) = \frac{1}{2\pi} \int_0^{\infty} \frac{\sin 2k}{k}\, dk$$

PROCEED BY SUBSTITUTION

$$\boxed{\begin{array}{l} t = 2k \\ k = \frac{1}{2}t \\ dk = \frac{1}{2}dt \\ \text{LIMITS UNCHANGED} \end{array}}$$

$$\Rightarrow \phi(\pm 1,0) = \frac{1}{2\pi} \int_0^{\infty} \frac{\sin t}{\frac{1}{2}t} \frac{1}{2} dt$$

$$\Rightarrow \phi(\pm 1,0) = \frac{1}{2\pi} \int_0^{\infty} \frac{\sin t}{t}\, dt$$

$$\Rightarrow \phi(\pm 1,0) = \frac{1}{2\pi} \times \frac{\pi}{2}$$

$$\Rightarrow \phi(\pm 1,0) = \frac{1}{4}$$

Question 3

The Airy function $\text{Ai}(x)$ satisfies the differential equation

$$\frac{d^2 y}{dx^2} - xy = 0 .$$

Use Fourier transforms to show that

$$\text{Ai}(x) = \frac{1}{\pi}\int_0^{\infty} \cos\left(\tfrac{1}{3}t^3 + xt\right)\, dt ,$$

for suitable boundary conditions.

You may assume that $\mathcal{F}\left[x\, f(x)\right] = \text{i}\dfrac{d}{dk}\left\{\mathcal{F}\left[f(x)\right]\right\}$.

proof

$$\boxed{\frac{d^2y}{dx^2} - xy = 0}$$

TAKING FOURIER TRANSFORM IN x

$$\boxed{\begin{array}{l}\mathcal{F}[f'(x)] = ik\,\mathcal{F}[f(x)] \\ \mathcal{F}[xf(x)] = i\frac{d}{dk}\mathcal{F}[f(x)]\end{array}}$$

$\Rightarrow \mathcal{F}\left[\frac{d^2y}{dx^2}\right] - \mathcal{F}\left[xy\right] = 0$

$\Rightarrow (ik)^2\mathcal{F}[y] - i\frac{d}{dk}\mathcal{F}(y) = 0$

$\Rightarrow -k^2\hat{f}(k) - i\frac{d}{dk}\hat{f}(k) = 0.$

$\Rightarrow \frac{d\hat{f}(k)}{dk} = -\frac{k^2}{i}\hat{f}(k)$

SEPARATE VARIABLES

$\Rightarrow \frac{1}{\hat{f}}\,d\hat{f} = -\frac{k^2}{i}\,dk$

$\Rightarrow \ln\hat{f} = -\frac{k^3}{3i} + C$

$\Rightarrow \ln\hat{f} = \frac{1}{3}ik^3 + C$

$\Rightarrow \hat{f} = Ae^{\frac{1}{3}ik^3} \quad (A = e^C)$

NEED TO INVERT NOW

$\Rightarrow \hat{f}(k) = Ae^{\frac{1}{3}ik^3}$

$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} Ae^{\frac{1}{3}ik^3}e^{ikx}\,dk$

$\Rightarrow f(x) = \frac{A}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\left(\frac{1}{3}k^3 + kx\right)}\,dk$

$\Rightarrow f(x) = \frac{A}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \underset{\uparrow\ \text{EVEN}}{\cos\left[\tfrac{1}{3}k^3+kx\right]} + i\underset{\uparrow\ \text{ODD}}{\sin\left[\tfrac{1}{3}k^3+kx\right]}\,dk$

$\Rightarrow f(x) = \frac{2A}{\sqrt{2\pi}}\int_0^{\infty}\cos\left(\tfrac{1}{3}k^3 + kx\right)\,dk$

HERE CHOOSE CONDITIONS SUCH THAT $A = \frac{1}{\sqrt{2\pi}}$

$\Rightarrow f(x) = \frac{2}{\sqrt{2\pi}} \times \frac{1}{\sqrt{2\pi}}\int_0^{\infty}\cos\left(\tfrac{1}{3}k^3 + kx\right)\,dk$

$\Rightarrow f(x) = \frac{1}{\pi}\int_0^{\infty}\cos\left(\tfrac{1}{3}k^3 + kx\right)\,dk$

I.E THE AIRY FUNCTION

$\text{Ai}(x) = \frac{1}{\pi}\int_0^{\infty}\cos\left(\tfrac{1}{3}t^3 + xt\right)\,dt$ //

Question 4

The function $\psi = \psi(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0,$$

in the part of the x-y plane for which $y \geq 0$.

It is further given that

- $\psi(x,0) = \delta(x)$
- $\psi(x, y) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$

Use Fourier transforms to convert the above partial differential equation into an ordinary differential equation and hence show that

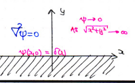
$$\psi(x, y) = \frac{1}{\pi}\left(\frac{y}{x^2 + y^2}\right).$$

, proof

SOLVING LAPLACE'S EQUATION

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

- $\psi(x,0) = \delta(x)$
- $\psi(x,y) \to 0$ AS $\sqrt{x^2+y^2} \to \infty$



TAKING FOURIER TRANSFORM OF THE P.D.E IN x

$$\Rightarrow \mathcal{F}\left[\frac{\partial^2 \psi}{\partial x^2}\right] + \mathcal{F}\left[\frac{\partial^2 \psi}{\partial y^2}\right] = \mathcal{F}[0]$$

$$\Rightarrow (ik)^2 \hat{\psi}(k,y) + \frac{\partial^2}{\partial y^2}\left[\hat{\psi}(k,y)\right] = 0$$

$$\Rightarrow \frac{\partial^2 \hat{\psi}}{\partial y^2} - k^2 \hat{\psi} = 0 \quad , \quad \hat{\psi} = \hat{\psi}(k,y)$$

SOLVING THE O.D.E, AS k IS A CONSTANT

$$\Rightarrow \hat{\psi}(k,y) = A(k)e^{-|k|y} + B(k)e^{|k|y}$$

AS $\psi(x,y)$ VANISHES AS "LARGE" DISTANCES, SO WOULD $\hat{\psi}(k,y)$, SO THIS IMPLIES THAT $B(k)=0$

$$\Rightarrow \hat{\psi}(k,y) = A(k)e^{-|k|y}$$

NEXT WE TAKE THE FOURIER TRANSFORM OF THE CONDITION $\psi(x,0) = \delta(x)$

$$\underline{\psi(x,0) = \delta(x)} \Rightarrow \hat{\psi}(k,0) = \mathcal{F}(\delta(x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \delta(x) e^{-ikx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \times e^{-ik\times 0} = \frac{1}{\sqrt{2\pi}}$$

HENCE

$$\hat{\psi}(k,0) = \frac{1}{\sqrt{2\pi}}$$

$$\frac{1}{\sqrt{2\pi}} = A(k)e^{0}$$

$$\boxed{A(k) = \frac{1}{\sqrt{2\pi}}}$$

$$\Rightarrow \hat{\psi}(k,y) = \frac{1}{\sqrt{2\pi}} e^{-|k|y}$$

INVERTING THE TRANSFORM ABOVE

$$\Rightarrow \psi(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} e^{-|k|y}\right) e^{ikx}\, dk$$

$$\Rightarrow \psi(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|k|y} e^{ikx}\, dk$$

$$\Rightarrow \psi(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|k|y} (\cos kx + i\sin kx)\, dk$$

INTEGRATING THE EVEN PART TWICE FROM $k=0$ TO $k=\infty$

$$\Rightarrow \psi(x,y) = \frac{1}{\pi}\int_{0}^{\infty} e^{-ky}\cos kx\, dk$$

$$\Rightarrow \psi(x,y) = \frac{1}{\pi}\operatorname{Re}\left[\int_{0}^{\infty} e^{-ky} e^{ikx}\, dk\right]$$

$$\Rightarrow \psi(x,y) = \frac{1}{\pi}\operatorname{Re}\left[\int_{0}^{\infty} e^{k(-y+ix)}\, dk\right]$$

$$\Rightarrow \psi(x,y) = \frac{1}{\pi}\operatorname{Re}\left[\frac{1}{-y+ix}\left[e^{k(-y+ix)}\right]_{0}^{\infty}\right]$$

$$\Rightarrow \psi(x,y) = \frac{1}{\pi}\operatorname{Re}\left[\frac{-y-ix}{y^2+x^2}\left[e^{-ky}(\cos kx + i\sin kx)\right]_{0}^{\infty}\right]$$

$$\Rightarrow \psi(x,y) = \frac{1}{\pi}\operatorname{Re}\left[\frac{-y-ix}{y^2+x^2}\left[0-1\right]\right]$$

$$\Rightarrow \psi(x,y) = \frac{1}{\pi}\operatorname{Re}\left[\frac{y+ix}{y^2+x^2}\right]$$

$$\Rightarrow \psi(x,y) = \frac{1}{\pi}\left(\frac{y}{x^2+y^2}\right)$$

Question 5

The function $u = u(x,t)$ satisfies the partial differential equation

$$\frac{\partial u}{\partial t} + \frac{1}{3}\frac{\partial^3 u}{\partial x^3} = 0.$$

It is further given that

- $u(x,0) = \delta(x)$
- $u(x,t) \to 0$ as $|x| \to \infty$

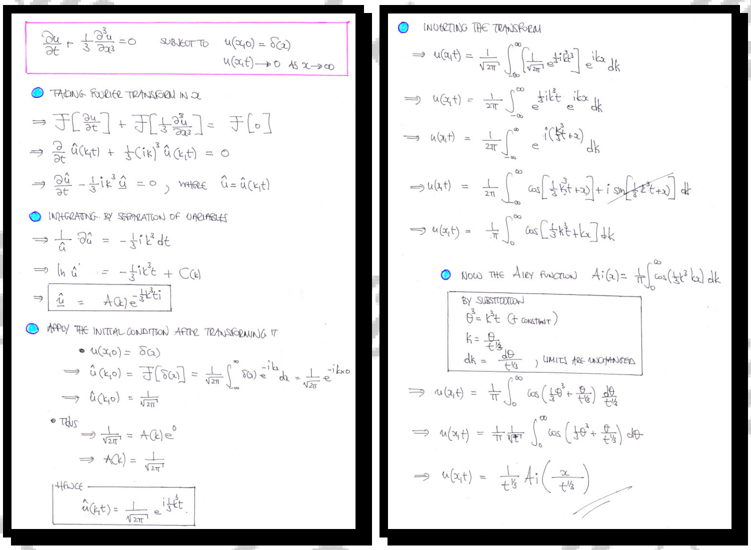
Use Fourier transforms to convert the above partial differential equation into an ordinary differential equation and hence show that

$$u(x,t) = \frac{1}{t^{\frac{1}{3}}}\,\mathrm{Ai}\left(\frac{x}{t^{\frac{1}{3}}}\right),$$

where the $\mathrm{Ai}(x)$ is the Airy function, defined as

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^{\infty} \cos\left[\frac{1}{3}k^3 + kx\right]\,dk.$$

proof



Question 6

The function $\varphi = \varphi(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0,$$

in the part of the x-y plane for which $x \geq 0$ and $y \geq 0$.

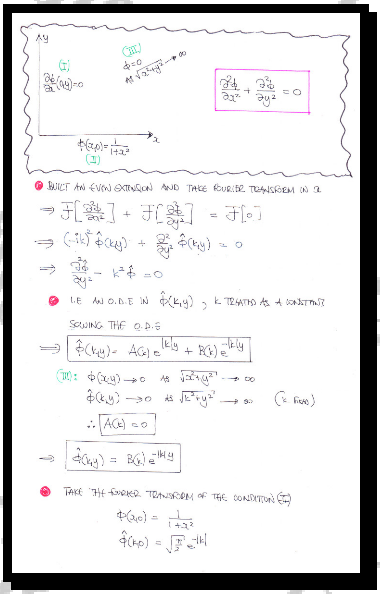
It is further given that

- $\varphi(x, 0) = \dfrac{1}{1+x^2}$
- $\varphi(x, y) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$
- $\dfrac{\partial}{\partial x}\big[\varphi(x, 0)\big] = 0$

Use Fourier transforms to convert the above partial differential equation into an ordinary differential equation and hence show that

$$\varphi(x, y) = \frac{y+1}{x^2 + (y+1)^2}.$$

proof



- BUILT AN EVEN EXTENSION AND TAKE FOURIER TRANSFORM IN x

$$\Rightarrow \mathcal{F}\left[\frac{\partial^2 \phi}{\partial x^2}\right] + \mathcal{F}\left[\frac{\partial^2 \phi}{\partial y^2}\right] = \mathcal{F}[0]$$

$$\Rightarrow (-ik)^2 \hat{\phi}(k,y) + \frac{\partial^2}{\partial y^2}\hat{\phi}(k,y) = 0$$

$$\Rightarrow \frac{\partial^2 \hat{\phi}}{\partial y^2} - k^2 \hat{\phi} = 0$$

- I.E AN O.D.E IN $\hat{\phi}(k,y)$, k TREATED AS A CONSTANT

SOLVING THE O.D.E

$$\Rightarrow \boxed{\hat{\phi}(k,y) = A(k)e^{|k|y} + B(k)e^{-|k|y}}$$

(III): $\phi(x,y) \to 0$ AS $\sqrt{x^2+y^2} \to \infty$

$\hat{\phi}(k,y) \to 0$ AS $\sqrt{k^2+y^2} \to \infty$ (k FIXED)

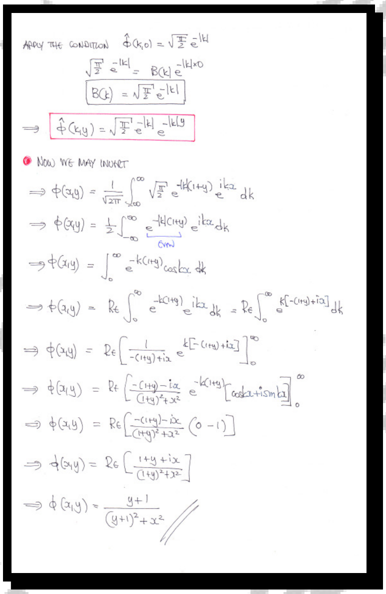
$\therefore \boxed{A(k) = 0}$

$$\Rightarrow \boxed{\hat{\phi}(k,y) = B(k)e^{-|k|y}}$$

- TAKE THE FOURIER TRANSFORM OF THE CONDITION (II)

$$\phi(x,0) = \frac{1}{1+x^2}$$

$$\hat{\phi}(k,0) = \sqrt{\frac{\pi}{2}}\,e^{-|k|}$$



APPLY THE CONDITION $\hat{\phi}(k,0) = \sqrt{\frac{\pi}{2}}\,e^{-|k|}$

$$\sqrt{\frac{\pi}{2}}\,e^{-|k|} = B(k)e^{-|k| \times 0}$$

$$\boxed{B(k) = \sqrt{\frac{\pi}{2}}\,e^{-|k|}}$$

$$\Rightarrow \boxed{\hat{\phi}(k,y) = \sqrt{\frac{\pi}{2}}\,e^{-|k|}e^{-|k|y}}$$

- NOW WE MAY INVERT

$$\Rightarrow \phi(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \sqrt{\frac{\pi}{2}}\,e^{-|k|(1+y)}e^{ikx}\,dk$$

$$\Rightarrow \phi(x,y) = \frac{1}{2}\int_{-\infty}^{\infty} \underbrace{e^{-|k|(1+y)}}_{\text{EVEN}} e^{ikx}\,dk$$

$$\Rightarrow \phi(x,y) = \int_0^{\infty} e^{-k(1+y)}\cos kx\,dk$$

$$\Rightarrow \phi(x,y) = \mathrm{Re}\int_0^{\infty} e^{-k(1+y)}e^{ikx}\,dk = \mathrm{Re}\int_0^{\infty} e^{k[-(1+y)+ix]}\,dk$$

$$\Rightarrow \phi(x,y) = \mathrm{Re}\left[\frac{1}{-(1+y)+ix}e^{k[-(1+y)+ix]}\right]_0^{\infty}$$

$$\Rightarrow \phi(x,y) = \mathrm{Re}\left[\frac{-(1+y)-ix}{(1+y)^2+x^2}e^{-k(1+y)}\left[\cos kx + i\sin kx\right]\right]_0^{\infty}$$

$$\Rightarrow \phi(x,y) = \mathrm{Re}\left[\frac{-(1+y)-ix}{(1+y)^2+x^2}(0-1)\right]$$

$$\Rightarrow \phi(x,y) = \mathrm{Re}\left[\frac{1+y+ix}{(1+y)^2+x^2}\right]$$

$$\Rightarrow \phi(x,y) = \frac{y+1}{(y+1)^2+x^2}$$

Question 7

The function $\Phi = \Phi(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0,$$

in the part of the x-y plane for which $y \geq 0$.

It is further given that

- $\Phi(x, 0) = \delta(x)$
- $\Phi(x, y) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$

Use Fourier transforms to find the solution of the above partial differential equation and hence show that

$$\delta(x) = \lim_{\alpha \to 0} \left[\frac{1}{\pi\alpha} \left(1 + \frac{y^2}{\alpha^2} \right)^{-1} \right].$$

proof

$$\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad -\infty < x < \infty, \; y \geq 0 \qquad \text{SUBJECT TO } \phi(x,0) = \delta(x), \; \phi \to 0 \text{ AS } \sqrt{x^2+y^2} \to \infty}$$

- TAKING THE FOURIER TRANSFORM OF THE EQUATION IN x

$$\Rightarrow \mathcal{F}\left[\frac{\partial^2 \phi}{\partial x^2}\right] + \mathcal{F}\left[\frac{\partial^2 \phi}{\partial y^2}\right] = \mathcal{F}[0]$$

$$\Rightarrow (ik)^2 \hat{\phi}(k,y) + \frac{\partial^2}{\partial y^2}\hat{\phi}(k,y) = 0$$

$$\Rightarrow -k^2 \hat{\phi}(k,y) + \frac{\partial^2}{\partial y^2}\hat{\phi}(k,y) = 0$$

$$\Rightarrow \frac{\partial^2 \hat{\phi}}{\partial y^2} - k^2 \hat{\phi} = 0 \quad \text{IE AN O.D.E IN } \hat{\phi}(k,y)$$

$$\Rightarrow \boxed{\hat{\phi}(k,y) = A(k)e^{-|k|y} + B(k)e^{|k|y}}$$

- AS $\phi(x,y) \to 0$ AS $\sqrt{x^2+y^2} \to \infty$, $\hat{\phi}(k,y) \to 0$ AS $\sqrt{k^2+y^2} \to \infty$ $\Big\}$ $B(k) = 0$.

$$\Rightarrow \boxed{\hat{\phi}(k,y) = A(k)e^{-|k|y}}$$

- TO APPLY THE NEXT CONDITION WE NEED TO TAKE ITS FOURIER TRANSFORM FIRST

$$\phi(x,0) = \delta(x)$$

$$\hat{\phi}(k,0) = \mathcal{F}[\delta(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \delta(x) e^{-ikx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-ik \times 0} = \frac{1}{\sqrt{2\pi}}$$

$$\hat{\phi}(k,0) = A(k)e^0$$

$$\frac{1}{\sqrt{2\pi}} = A(k)$$

- THUS WE OBTAIN $\boxed{\hat{\phi}(k,y) = \frac{1}{\sqrt{2\pi}} e^{-|k|y}}$

- INVERTING THE TRANSFORM DIRECTLY

$$\Rightarrow \phi(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(k,y) e^{ikx}\,dk$$

$$\Rightarrow \phi(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-|k|y} e^{ikx}\,dk$$

$$\Rightarrow \phi(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{e^{-|k|y}}_{\text{EVEN}} e^{ikx}\,dk$$

$$\Rightarrow \phi(x,y) = \frac{1}{\pi}\int_{0}^{\infty} e^{-ky} \cos kx\,dk$$

$$\Rightarrow \phi(x,y) = \frac{1}{\pi}\int_{0}^{\infty} \mathrm{Re}\left[e \; e^{ikx}\right] dk$$

$$\Rightarrow \phi(x,y) = \frac{1}{\pi}\mathrm{Re}\int_{0}^{\infty} e^{k(y+ix)}\,dk$$

$$\Rightarrow \phi(x,y) = \frac{1}{\pi}\mathrm{Re}\left[\frac{1}{-y+ix} e^{k(y+ix)}\right]_0^{\infty}$$

$$\Rightarrow \phi(x,y) = \frac{1}{\pi}\mathrm{Re}\left[\frac{-y-ix}{y^2+x^2} e^{-ky} e^{ikx}\right]_0^{\infty}$$

$$\Rightarrow \phi(x,y) = \frac{1}{\pi}\mathrm{Re}\left[\frac{-y-ix}{y^2+x^2}(0-1)\right]$$

$$\Rightarrow \phi(x,y) = \frac{y}{\pi(y^2+x^2)}$$

- FINALLY

$$\phi(x,0) = \delta(x) \Rightarrow \delta(x) = \lim_{y \to 0}\left(\frac{1}{\pi}\frac{y}{x^2+y^2}\right) = \lim_{y \to 0}\left[\frac{1}{\pi}\frac{y/y^2}{\frac{x^2}{y^2}+\frac{y^2}{y^2}}\right]$$

$$= \lim_{y \to 0}\left[\frac{1}{\pi}\frac{\frac{1}{y}}{\frac{x^2}{y^2}+1}\right] = \lim_{y \to 0}\left[\frac{1}{\pi y}\frac{1}{1+\frac{x^2}{y^2}}\right]$$

$$\therefore \delta(x) = \lim_{\alpha \to 0}\left[\frac{1}{\pi\alpha}\frac{1}{1+\frac{x^2}{\alpha^2}}\right]$$

Question 8

The function $y = y(x)$ satisfies the differential equation

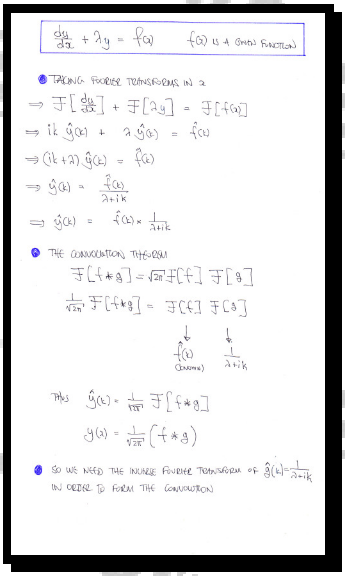
$$\frac{dy}{dx} + \lambda y = f(x),$$

where $f(x)$ is a given function and λ is a real constant.

Use Fourier transforms to show that

$$y(x) = \int_0^{\infty} e^{\lambda t} f(x-t) \, dt .$$

$\boxed{\text{proof}}$



$\boxed{\frac{dy}{dx} + \lambda y = f(x) \qquad f(x) \text{ IS A GIVEN FUNCTION}}$

TAKING FOURIER TRANSFORMS IN x

$\Rightarrow \mathcal{F}\left[\frac{dy}{dx}\right] + \mathcal{F}[\lambda y] = \mathcal{F}[f(x)]$

$\Rightarrow ik\,\hat{y}(k) + \lambda \hat{y}(k) = \hat{f}(k)$

$\Rightarrow (ik+\lambda)\hat{y}(k) = \hat{f}(k)$

$\Rightarrow \hat{y}(k) = \frac{\hat{f}(k)}{\lambda + ik}$

$\Rightarrow \hat{y}(k) = \hat{f}(k) \times \frac{1}{\lambda + ik}$

THE CONVOLUTION THEOREM

$\mathcal{F}[f*g] = \sqrt{2\pi}\,\mathcal{F}[f]\,\mathcal{F}[g]$

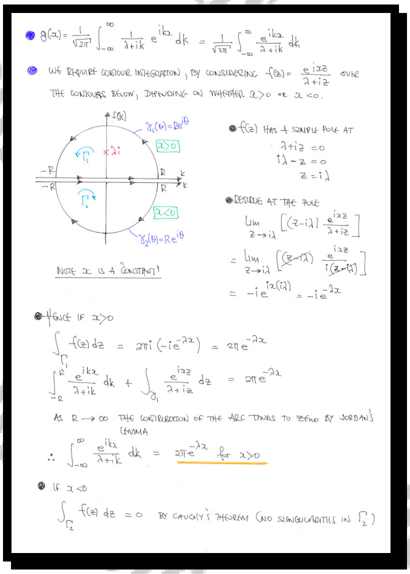
$\frac{1}{\sqrt{2\pi}}\mathcal{F}[f*g] = \mathcal{F}[f]\,\mathcal{F}[g]$

$\mathcal{F}[f] \rightarrow \hat{f}(k)$ (KNOWN), $\quad \mathcal{F}[g] \rightarrow \frac{1}{\lambda+ik}$

THUS $\hat{y}(k) = \frac{1}{\sqrt{2\pi}}\mathcal{F}[f*g]$

$y(x) = \frac{1}{\sqrt{2\pi}}(f*g)$

SO WE NEED THE INVERSE FOURIER TRANSFORM OF $\hat{g}(k) = \frac{1}{\lambda+ik}$ IN ORDER TO FORM THE CONVOLUTION



$g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{\lambda+ik} e^{ikx}\,dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{e^{ikx}}{\lambda+ik}\,dk$

WE REQUIRE CONTOUR INTEGRATION, BY CONSIDERING $f(z) = \frac{e^{ixz}}{\lambda+iz}$ OVER THE CONTOURS BELOW, DEPENDING ON WHETHER $x>0$ OR $x<0$.

NOTE x IS A "CONSTANT"

$f(z)$ HAS A SIMPLE POLE AT

$\lambda + iz = 0$

$i\lambda - z = 0$

$z = i\lambda$

RESIDUE AT THE POLE

$\lim_{z\to i\lambda}\left[(z-i\lambda)\frac{e^{ixz}}{\lambda+iz}\right]$

$= \lim_{z\to i\lambda}\left[\cancel{(z-i\lambda)}\frac{e^{ixz}}{i\cancel{(z-i\lambda)}}\right]$

$= -ie^{ix(i\lambda)} = -ie^{-\lambda x}$

HENCE IF $x>0$

$\int_{\Gamma_1} f(z)\,dz = 2\pi i(-ie^{-\lambda x}) = 2\pi e^{-\lambda x}$

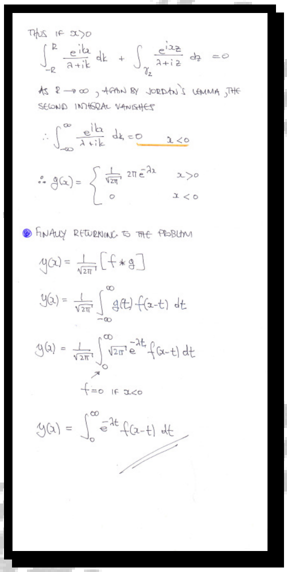
$\int_{-R}^{R}\frac{e^{ikx}}{\lambda+ik}\,dk + \int_{\gamma_1}\frac{e^{ixz}}{\lambda+iz}\,dz = 2\pi e^{-\lambda x}$

AS $R\to\infty$ THE CONTRIBUTION OF THE ARC TENDS TO ZERO BY JORDAN'S LEMMA

$\therefore \int_{-\infty}^{\infty}\frac{e^{ikx}}{\lambda+ik}\,dk = 2\pi e^{-\lambda x}$ for $x>0$

IF $x<0$

$\int_{\Gamma_2} f(z)\,dz = 0$ BY CAUCHY'S THEOREM (NO SINGULARITIES IN Γ_2)



THUS IF $x>0$

$\int_{-R}^{R}\frac{e^{ikx}}{\lambda+ik}\,dk + \int_{\gamma_2}\frac{e^{ixz}}{\lambda+iz}\,dz = 0$

AS $R\to\infty$, AGAIN BY JORDAN'S LEMMA, THE SECOND INTEGRAL VANISHES

$\therefore \int_{-\infty}^{\infty}\frac{e^{ikx}}{\lambda+ik}\,dk = 0 \qquad x<0$

$\therefore g(x) = \begin{cases}\frac{1}{\sqrt{2\pi}}2\pi e^{-\lambda x} & x>0 \\ 0 & x<0\end{cases}$

FINALLY RETURNING TO THE PROBLEM

$y(x) = \frac{1}{\sqrt{2\pi}}[f*g]$

$y(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(t)f(x-t)\,dt$

$y(x) = \frac{1}{\sqrt{2\pi}}\int_0^{\infty}\sqrt{2\pi}\,e^{-\lambda t}f(x-t)\,dt$

$f=0$ IF $x<0$

$y(x) = \int_0^{\infty} e^{-\lambda t}f(x-t)\,dt$

Question 9

The function $\varphi = \varphi(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0,$$

in the semi-infinite region of the x-y plane for which $y \geq 0$.

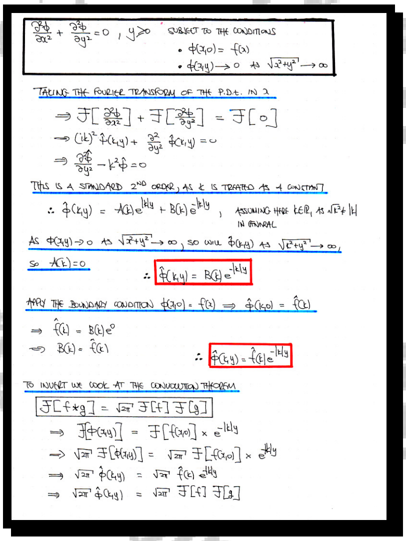
It is further given that

- $\varphi(x, 0) = f(x)$
- $\varphi(x, y) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$

Use Fourier transforms to convert the above partial differential equation into an ordinary differential equation and hence show that

$$\varphi(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(x-u)}{u^2 + y^2} \, du .$$

, proof



$$\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \ y \geq 0}$$ SUBJECT TO THE CONDITIONS

- $\phi(x,0) = f(x)$
- $\phi(x,y) \to 0$ AS $\sqrt{x^2+y^2} \to \infty$

TAKING THE FOURIER TRANSFORM OF THE P.D.E. IN x

$$\Rightarrow \mathcal{F}\left[\frac{\partial^2 \phi}{\partial x^2}\right] + \mathcal{F}\left[\frac{\partial^2 \phi}{\partial y^2}\right] = \mathcal{F}[0]$$
$$\Rightarrow (ik)^2 \hat{\phi}(k,y) + \frac{\partial^2}{\partial y^2}\hat{\phi}(k,y) = 0$$
$$\Rightarrow \frac{\partial^2 \hat{\phi}}{\partial y^2} - k^2 \hat{\phi} = 0$$

THIS IS A STANDARD 2ND ORDER, AS k IS TREATED AS A CONSTANT

$$\therefore \hat{\phi}(k,y) = A(k)e^{|k|y} + B(k)e^{-|k|y}$$, ASSUMING HERE $k \in \mathbb{R}$, AS $\sqrt{k^2} \neq |k|$ IN GENERAL

AS $\phi(x,y) \to 0$ AS $\sqrt{x^2+y^2} \to \infty$, SO WILL $\hat{\phi}(k,y)$ AS $\sqrt{k^2+y^2} \to \infty$,

SO $A(k) = 0$

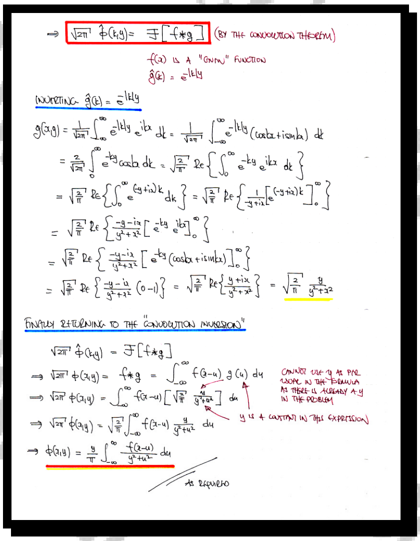
$$\therefore \boxed{\hat{\phi}(k,y) = B(k)e^{-|k|y}}$$

APPLY THE BOUNDARY CONDITION $\phi(x,0) = f(x) \Rightarrow \hat{\phi}(k,0) = \hat{f}(k)$

$$\Rightarrow \hat{f}(k) = B(k)e^0$$
$$\Rightarrow B(k) = \hat{f}(k)$$
$$\therefore \boxed{\hat{\phi}(k,y) = \hat{f}(k)e^{-|k|y}}$$

TO INVERT WE LOOK AT THE CONVOLUTION THEOREM

$$\boxed{\mathcal{F}[f*g] = \sqrt{2\pi}\,\mathcal{F}[f]\,\mathcal{F}[g]}$$
$$\Rightarrow \mathcal{F}[\phi(x,y)] = \mathcal{F}[f(x,0)] \times e^{-|k|y}$$
$$\Rightarrow \sqrt{2\pi}\,\mathcal{F}[\phi(x,y)] = \sqrt{2\pi}\,\mathcal{F}[f(x,0)] \times e^{-|k|y}$$
$$\Rightarrow \sqrt{2\pi}\,\hat{\phi}(k,y) = \sqrt{2\pi}\,\hat{f}(k)\,e^{-|k|y}$$
$$\Rightarrow \sqrt{2\pi}\,\hat{\phi}(k,y) = \sqrt{2\pi}\,\mathcal{F}[f]\,\mathcal{F}[g]$$



$$\Rightarrow \boxed{\sqrt{2\pi}\,\hat{\phi}(k,y) = \mathcal{F}[f*g]}$$ (BY THE CONVOLUTION THEOREM)

$f(x)$ IS A "GIVEN" FUNCTION

$\hat{g}(k) = e^{-|k|y}$

INVERTING $\hat{g}(k) = e^{-|k|y}$

$$g(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|k|y} e^{ikx}\,dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|k|y}(\cos kx + i\sin kx)\,dk$$
$$= \frac{2}{\sqrt{2\pi}}\int_0^{\infty} e^{-ky}\cos kx\,dk = \sqrt{\frac{2}{\pi}}\,\mathrm{Re}\left\{\int_0^{\infty} e^{-ky} e^{ikx}\,dk\right\}$$
$$= \sqrt{\frac{2}{\pi}}\,\mathrm{Re}\left\{\int_0^{\infty} e^{(-y+ix)k}\,dk\right\} = \sqrt{\frac{2}{\pi}}\,\mathrm{Re}\left\{\frac{1}{-y+ix}\left[e^{(-y+ix)k}\right]_0^{\infty}\right\}$$
$$= \sqrt{\frac{2}{\pi}}\,\mathrm{Re}\left\{\frac{-y-ix}{y^2+x^2}\left[e^{-ky}e^{ikx}\right]_0^{\infty}\right\}$$
$$= \sqrt{\frac{2}{\pi}}\,\mathrm{Re}\left\{\frac{-y-ix}{y^2+x^2}\left[e^{-ky}(\cos kx + i\sin kx)\right]_0^{\infty}\right\}$$
$$= \sqrt{\frac{2}{\pi}}\,\mathrm{Re}\left\{\frac{-y-ix}{y^2+x^2}(0-1)\right\} = \sqrt{\frac{2}{\pi}}\,\mathrm{Re}\left\{\frac{y+ix}{y^2+x^2}\right\} = \sqrt{\frac{2}{\pi}}\,\frac{y}{y^2+x^2}$$

FINALLY RETURNING TO THE "CONVOLUTION INVERSION"

$$\sqrt{2\pi}\,\hat{\phi}(k,y) = \mathcal{F}[f*g]$$
$$\Rightarrow \sqrt{2\pi}\,\phi(x,y) = f*g = \int_{-\infty}^{\infty} f(x-u)\,g(u)\,du$$

CANNOT USE y AS PER USUAL IN THE FORMULA AS THERE IS ALREADY A y IN THE PROBLEM

$$\Rightarrow \sqrt{2\pi}\,\phi(x,y) = \int_{-\infty}^{\infty} f(x-u)\left[\sqrt{\frac{2}{\pi}}\,\frac{y}{y^2+u^2}\right]du$$

y IS A CONSTANT IN THIS EXPRESSION

$$\Rightarrow \sqrt{2\pi}\,\phi(x,y) = \sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} f(x-u)\,\frac{y}{y^2+u^2}\,du$$
$$\Rightarrow \phi(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(x-u)}{y^2+u^2}\,du$$

AS REQUIRED

Question 10

The function $\varphi = \varphi(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0,$$

in the semi-infinite region of the x-y plane for which $y \geq 0$.

It is further given that for a given function $f = f(x)$

- $\frac{\partial}{\partial y}\left[\varphi(x,0)\right] = \frac{\partial}{\partial x}\left[f(x)\right]$
- $\varphi(x, y) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$

Use Fourier transforms to convert the above partial differential equation into an ordinary differential equation and hence show that

$$\varphi(x,0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(u)}{x-u}\, du .$$

$\boxed{\text{proof}}$

[**solution overleaf**]

$$\boxed{\begin{array}{ll} \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{SUBJECT TO} & \phi(x,y) \longrightarrow 0 \text{ AS } \sqrt{x^2+y^2} \to \infty \\ \begin{bmatrix} -\infty < x < \infty \\ y > 0 \end{bmatrix} & \dfrac{\partial \phi}{\partial y}(x,0) = \dfrac{\partial f}{\partial x} \\ & \text{WHERE } f(x) \text{ IS A KNOWN FUNCTION} \end{array}}$$

- TAKING FOURIER TRANSFORM IN x

$$\Rightarrow \mathcal{F}\left[\frac{\partial^2 \phi}{\partial x^2}\right] + \mathcal{F}\left[\frac{\partial^2 \phi}{\partial y^2}\right] = \mathcal{F}[0]$$

$$\Rightarrow (ik)^2 \hat{\phi}(k,y) + \frac{\partial^2}{\partial y^2} \hat{\phi}(k,y) = 0$$

$$\Rightarrow \frac{\partial^2 \hat{\phi}}{\partial y^2} - k^2 \hat{\phi} = 0 \quad \text{IE AN O.D.E IN } \hat{\phi} = \hat{\phi}(k,y),\ k \text{ CONSTANT}$$

- STANDARD SOLUTION GIVES EXPONENTIALS

$$\Rightarrow \boxed{\hat{\phi}(k,y) = A(k)e^{-|k|y} + B(k)e^{|k|y}}$$

- AS $\phi(x,y)$ IS FINITE AT INFINITY SO WILL ITS TRANSFORM $\hat{\phi}(k,y)$, SO $B(k) = 0$

$$\Rightarrow \boxed{\hat{\phi}(k,y) = A(k)e^{-|k|y}}$$

- NEXT TAKE THE SECOND BOUNDARY CONDITION, AND TAKE ITS TRANSFORM

  - $\dfrac{\partial}{\partial y}[\phi(x,0)] = \dfrac{\partial f}{\partial x}$ $\quad \boxed{f = f(x), \text{ A KNOWN FUNCTION}}$

$$\Rightarrow \mathcal{F}\left[\frac{\partial}{\partial y}[\phi(x,0)]\right] = \mathcal{F}\left[\frac{\partial f}{\partial x}\right]$$

$$\Rightarrow \frac{\partial}{\partial y}\left[\hat{\phi}(k,0)\right] = ik\hat{f}(k)$$

  - NOW DIFFERENTIATE $\hat{\phi}(k,y)$ WITH RESPECT TO y TO APPLY IT

$$\Rightarrow \frac{\partial}{\partial y}\left[\hat{\phi}(k,y)\right] = -A(k)|k|e^{-|k|y}$$

$$\Rightarrow -A(k)|k|e^{-|k| \times 0} = ik\hat{f}(k) \quad (\text{AT } y=0)$$

$$\Rightarrow A(k) = -i\frac{k}{|k|}\hat{f}(k)$$

$$\Rightarrow A(k) = -i \operatorname{sgn} k\, \hat{f}(k) \qquad \left\{ \operatorname{sgn} x \equiv \frac{x}{|x|} \quad x \neq 0 \right\}$$

HENCE WE OBTAIN

$$\boxed{\hat{\phi}(k,y) = -i \operatorname{sgn} k\, \hat{f}(k)\, e^{-|k|y}}$$

- INVERTING THE SPECIAL CASE $\phi(x,0)$

$$\hat{\phi}(k,0) = -i \operatorname{sgn} k\, \hat{f}(k)\, e^{-|k|0}$$

$$\hat{\phi}(k,0) = \underbrace{(-i \operatorname{sgn} k)\, \hat{f}(k)}_{\text{PRODUCT OF TWO FOURIER TRANSFORMS}}$$

CONVOLUTION THEOREM

$$\mathcal{F}[f * g] = \sqrt{2\pi}\, \mathcal{F}[f][g]$$

$$\underbrace{\frac{1}{\sqrt{2\pi}} \mathcal{F}[f*g]}_{\hat{\phi}(k,0)} = \underbrace{\mathcal{F}[f]}_{\hat{f}(k)}\, \underbrace{\mathcal{F}[g]}_{\hat{g}(k) = -i \operatorname{sgn} k}$$

$$\text{SO } \phi(x,0) = \frac{1}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} f(u)\, g(x-u)\, du}_{f * g}$$

NEED $g(x)$ SO WE NEED TO INVERT $\hat{g}(k) = -i \operatorname{sign} k$

$\hat{g}(k) = -i \operatorname{sign} k$ IS NOT ABSOLUTELY INTEGRABLE, SO USE A CONVERGENCE FACTOR $e^{\varepsilon|k|}$ AND LET $\varepsilon \to 0$

$$g(x) = \lim_{\varepsilon \to 0} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Big[\underbrace{-i \operatorname{sign} k}_{\text{ODD}}\ \underbrace{e^{-\varepsilon|k|}}_{\text{EVEN}} \Big] e^{ikx}\, dk \right]$$

$$= \lim_{\varepsilon \to 0} \left[\frac{-2i}{\sqrt{2\pi}} \int_0^{\infty} 1 \times e^{-\varepsilon k} (i \sin kx)\, dk \right]$$

$$= \lim_{\varepsilon \to 0} \left[\sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-\varepsilon k} \sin kx\, dk \right]$$

$$= \lim_{\varepsilon \to 0} \left[\sqrt{\frac{2}{\pi}}\, \mathrm{Im} \int_0^{\infty} e^{-\varepsilon k} e^{ikx}\, dk \right]$$

$$= \sqrt{\frac{2}{\pi}} \lim_{\varepsilon \to 0} \left[\mathrm{Im} \int_0^{\infty} e^{k(-\varepsilon + ix)}\, dk \right]$$

$$= \sqrt{\frac{2}{\pi}} \lim_{\varepsilon \to 0} \left[\mathrm{Im} \left[\frac{1}{-\varepsilon + ix} e^{k(-\varepsilon + ix)} \right]_{k=0}^{k=\infty} \right]$$

$$= \sqrt{\frac{2}{\pi}} \lim_{\varepsilon \to 0} \left[\mathrm{Im} \left[\frac{-\varepsilon - ix}{\varepsilon^2 + x^2} e^{-\varepsilon k} (\cos kx + i \sin kx) \right]_0^{\infty} \right]$$

$$= \sqrt{\frac{2}{\pi}} \lim_{\varepsilon \to 0} \left[\mathrm{Im} \left[\frac{-\varepsilon - ix}{\varepsilon^2 + x^2} (0 - 1) \right] \right]$$

$$= \sqrt{\frac{2}{\pi}} \lim_{\varepsilon \to 0} \left[\mathrm{Im} \left[\frac{\varepsilon + ix}{\varepsilon^2 + x^2} \right] \right]$$

$$= \sqrt{\frac{2}{\pi}} \lim_{\varepsilon \to 0} \left[\frac{x}{\varepsilon^2 + x^2} \right]$$

$$= \sqrt{\frac{2}{\pi}}\, \frac{x}{x^2}$$

$$= \sqrt{\frac{2}{\pi}}\, \frac{1}{x}.$$

- FINALLY

$$\phi(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) \times \overbrace{\left[\sqrt{\frac{2}{\pi}}\, \frac{1}{x-u} \right]}^{g(x-u)} du. \qquad \left\{ f(x) = \sqrt{\frac{2}{\pi}}\, \frac{1}{x} \right\}$$

$$\phi(x,0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(u)}{x-u}\, du$$

AS REQUIRED

Question 11

The function $\varphi = \varphi(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0, \quad -\infty < x < \infty, \; y \geq 0.$$

It is further given that

- $\varphi(x, y) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$

- $\varphi(x, 0) = \mathrm{H}(x)$, the Heaviside function.

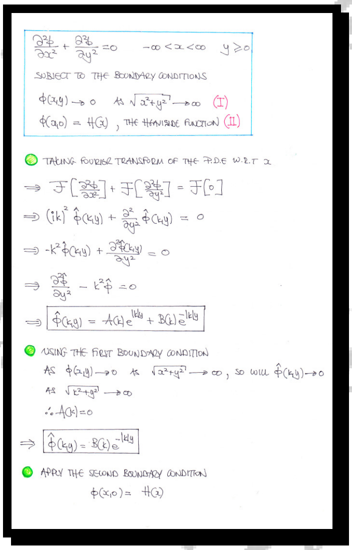
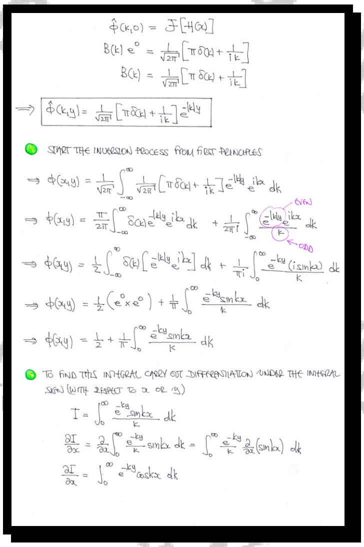
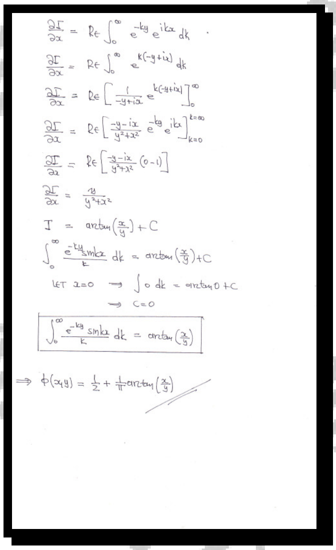
Use Fourier transforms to show that

$$\varphi(x, y) = \frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{x}{y}\right).$$

You may assume that

$$\mathcal{F}\left[\mathrm{H}(x)\right] = \frac{1}{\sqrt{2\pi}}\left[\pi\,\delta(k) + \frac{1}{\mathrm{i}\,k}\right].$$

proof

Question 12

The function $u = u(x,y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad -\infty < x < \infty, \quad 0 < y < 1.$$

It is further given that

- $u(x,0) = 0$
- $u(x,1) = f(x)$

  where $f(-x) = f(x)$ and $f(x) \to 0$ as $x \to \infty$

a) Use Fourier transforms to show that

$$u(x,y) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{\hat{f}(k) \cos kx \sinh ky}{\sinh k} \, dk, \quad \hat{f}(k) = \mathcal{F}\left[f(x)\right].$$

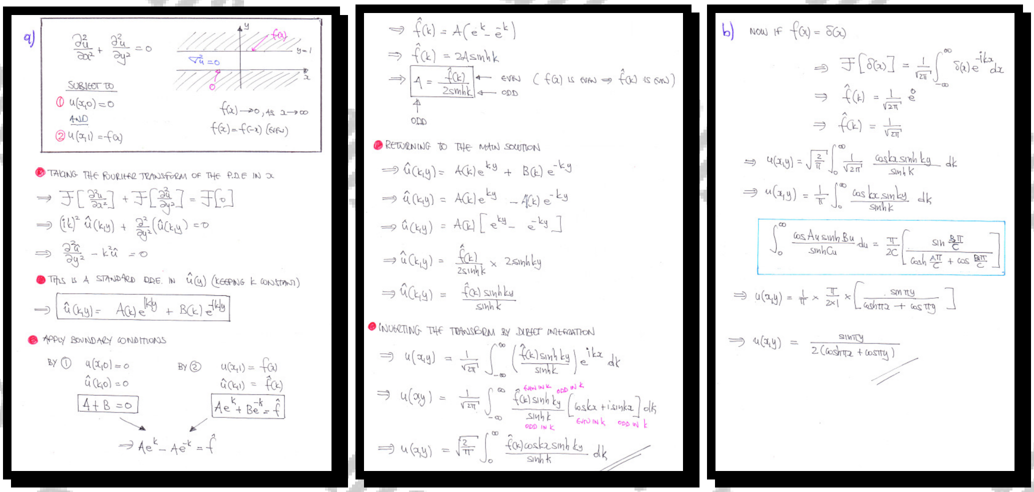
b) Given that $f(x) = \delta(x)$ show further that

$$u(x,y) = \frac{\sin \pi y}{2\left[\cosh \pi x + \cos \pi y\right]}.$$

You may assume without proof

$$\int_0^{\infty} \frac{\cos Au \sinh Bu}{\sinh Cu} \, du = \frac{\pi}{2C}\left[\frac{\sin(B\pi/C)}{\cosh(A\pi/C) + \cos(B\pi/C)}\right], \quad 0 \le B < C.$$

proof



Question 13

The function $\psi = \psi(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0,$$

in the part of the x-y plane for which $y \geq 0$.

It is further given that

- $\psi(x,0) = f(x)$
- $\psi(x, y) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$

c) Use Fourier transforms to convert the above partial differential equation into an ordinary differential equation and hence show that

$$\psi(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(u)}{(x-u)^2 + y^2} \, du.$$

d) Evaluate the above integral for …

i. … $f(x) = 1$.

ii. … $f(x) = \operatorname{sgn} x$

iii. … $f(x) = \mathrm{H}(x)$

commenting further whether these answers are consistent.

$\boxed{\psi(x, y) = 1}$, $\boxed{\psi(x, y) = \frac{2}{\pi}\arctan\left(\frac{x}{y}\right)}$, $\boxed{\psi(x, y) = \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{x}{y}\right)}$

[solution overleaf]

a) $\dfrac{\partial^2\psi}{\partial x^2} + \dfrac{\partial^2\psi}{\partial y^2} = 0$ SUBJECT TO $\psi(x,0) = f(x)$

$\psi(x,y) \to 0$ AS $\sqrt{x^2+y^2} \to \infty$

- TAKING FOURIER TRANSFORM IN x

$$\Rightarrow \mathcal{F}\left[\frac{\partial^2\psi}{\partial x^2}\right] + \mathcal{F}\left[\frac{\partial^2\psi}{\partial y^2}\right] = \mathcal{F}[0]$$

$$\Rightarrow (ik)^2\,\hat{\psi}(k,y) + \frac{\partial^2}{\partial y^2}\hat{\psi}(k,y) = 0$$

$$\Rightarrow \frac{\partial^2\hat{\psi}}{\partial y^2} - k^2\hat{\psi} = 0 \qquad \hat{\psi} = \hat{\psi}(k,y)$$

- SOLVING THE O.D.E

$$\Rightarrow \hat{\psi}(k,y) = A(k)e^{-|k|y} + B(k)e^{|k|y}$$

- AS ψ VANISHES AT INFINITY SO WOULD $\hat{\psi}$, SO $B(k)=0$

$$\Rightarrow \boxed{\hat{\psi}(k,y) = A(k)e^{-|k|y}}$$

- NEXT TAKE THE FOURIER TRANSFORM OF THE CONDITION ON THE x AXIS

$$\psi(x,0) = f(x)$$
$$\Rightarrow \hat{\psi}(k,0) = \hat{f}(k)$$
$$\Rightarrow \boxed{A(k) = \hat{f}(k)}$$
$$\therefore \boxed{\hat{\psi}(k,y) = \hat{f}(k)e^{-|k|y}}$$

- START INVERTING DIRECTLY OR BY THE CONVOLUTION THEOREM

$$\Rightarrow \psi(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{\psi}(k,y)\,e^{ikx}\,dk$$

$$\Rightarrow \psi(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(k)\,e^{-|k|y}\,e^{ikx}\,dk$$

$$\Rightarrow \psi(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(u)\,e^{-iku}\,du\right]e^{-|k|y}\,e^{ikx}\,dk$$

$$\Rightarrow \psi(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(u)\,e^{-iku}\,e^{-|k|y}\,e^{ikx}\,du\,dk$$

$$\Rightarrow \psi(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}f(u)\left[\int_{-\infty}^{\infty}e^{ik(x-u)}\,e^{-|k|y}\,dk\right]du$$

- ONLY THE EVEN PART SURVIVES IN THE INNER INTEGRAL IN k

$$\Rightarrow \psi(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty}f(u)\left[\int_{0}^{\infty}e^{-ky}\cos[k(x-u)]\,dk\right]du$$

$$\Rightarrow \psi(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty}f(u)\,\mathrm{Re}\int_{0}^{\infty}e^{-ky}\,e^{ik(x-u)}\,dk\,du$$

$$\Rightarrow \psi(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty}f(u)\,\mathrm{Re}\int_{0}^{\infty}e^{k[-y+i(x-u)]}\,dk\,du$$

$$\Rightarrow \psi(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty}f(u)\,\mathrm{Re}\left[\frac{1}{-y+i(x-u)}e^{k(-y+i(x-u))}\right]_{k=0}^{\infty}du$$

$$\Rightarrow \psi(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty}f(u)\,\mathrm{Re}\left[\frac{-y-i(x-u)}{y^2+(x-u)^2}e^{-ky}\left(\cos k(x-u) + i\sin k(x-u)\right)\right]_{k=0}^{k=\infty}du$$

$$\Rightarrow \psi(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty}f(u)\,\mathrm{Re}\left[\frac{-y-i(x-u)}{y^2+(x-u)^2}(0-1)\right]du$$

$$\Rightarrow \psi(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty}f(u)\,\mathrm{Re}\left[\frac{y+i(x-u)}{y^2+(x-u)^2}\right]du$$

$$\Rightarrow \psi(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty}f(u)\frac{y}{y^2+(x-u)^2}\,du$$

$$\Rightarrow \psi(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(u)}{(x-u)^2+y^2}\,du \quad \text{// AS REQUIRED}$$

ALTERNATIVE BY THE CONVOLUTION THEOREM

$$\hat{\psi}(k,y) = \hat{f}(k)\,e^{-|k|y}$$

CONVOLUTION THEOREM

$$\mathcal{F}[f*g] = \sqrt{2\pi}\,\mathcal{F}(f)\,\mathcal{F}(g)$$

$$\frac{\mathcal{F}[f*g]}{\sqrt{2\pi}} = \mathcal{F}(f)\,\mathcal{F}(g)$$

$$\hat{\psi} \qquad \hat{f} \qquad \hat{g} = e^{-|k|y}$$

SO $\hat{g}(k,y) = e^{-|k|y}$

- $g(x,y) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty}e^{-|k|y}\,e^{ikx}\,dk = \dfrac{2}{\sqrt{2\pi}}\displaystyle\int_{0}^{\infty}e^{-ky}\cos kx\,dk$

$= \ldots$ COMPLEX NUMBERS...

$= \dfrac{2}{\sqrt{2\pi}}\,\dfrac{y}{y^2+x^2}$

- $f(x)$ IS GIVEN

$$\therefore \psi(x,y) = \frac{1}{\sqrt{2\pi}}(f*g)(x) \quad (y \text{ IS A CONSTANT HERE})$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(u)\,g(x-u)\,du$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(u)\left[\frac{2}{\sqrt{2\pi}}\,\frac{y}{y^2+(x-u)^2}\right]du$$

$$= \frac{2y}{2\pi}\int_{-\infty}^{\infty}f(u)\left[\frac{1}{y^2+(x-u)^2}\right]du$$

$$= \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(u)}{y^2+(x-u)^2}\,du \quad \text{// AS BEFORE.}$$

b) $\boxed{f(x) = 1 \Rightarrow f(u) = 1}$

$$\psi(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{1}{(x-u)^2+y^2}\,du = \ldots \text{ SUBSTITUTION}$$

LET $t = x-u$
$dt = -du$
LIMITS SWAP
$u = x-t$

$$= \frac{y}{\pi}\int_{\infty}^{-\infty}\frac{1}{t^2+y^2}(-dt) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{1}{t^2+y^2}\,dt$$

$$= \frac{2y}{\pi}\int_{0}^{\infty}\frac{1}{t^2+y^2}\,dt = \frac{2y}{\pi}\left[\frac{1}{y}\arctan\frac{t}{y}\right]_0^{\infty}$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2} - 0\right] = 1 \quad //$$

$\boxed{f(x) = \mathrm{sgn}(x) \Rightarrow f(u) = \mathrm{sgn}(u)}$

$$\psi(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{sgn}(u)}{(x-u)^2+y^2}\,du = \frac{y}{\pi}\int_{-\infty}^{0}\frac{-1}{(x-u)^2+y^2}\,du + \frac{y}{\pi}\int_{0}^{\infty}\frac{1}{(x-u)^2+y^2}\,du$$

$= \ldots$ SAME SUBSTITUTIONS AS ABOVE.

$$= \frac{y}{\pi}\int_{\infty}^{x}\frac{-1}{t^2+y^2}(-dt) + \frac{y}{\pi}\int_{x}^{-\infty}\frac{1}{t^2+y^2}(-dt)$$

$$= \frac{y}{\pi}\int_{\infty}^{x}\frac{1}{t^2+y^2}\,dt + \frac{y}{\pi}\int_{-\infty}^{x}\frac{1}{t^2+y^2}\,dt$$

$$= \frac{y}{\pi}\left\{\frac{1}{y}\left[\arctan\frac{t}{y}\right]_{\infty}^{x} + \frac{1}{y}\left[\arctan\frac{t}{y}\right]_{-\infty}^{x}\right\}$$

$$= \frac{y}{\pi}\times\frac{1}{y}\left[\arctan\left(\frac{x}{y}\right) - \frac{\pi}{2} + \arctan\left(\frac{x}{y}\right) - \left(-\frac{\pi}{2}\right)\right]$$

$$= \frac{1}{\pi}\times 2\arctan\left(\frac{x}{y}\right)$$

$$= \frac{2}{\pi}\arctan\left(\frac{x}{y}\right) \quad //$$

$\boxed{f(x) = H(x) \quad \text{i.e.} \quad f(u) = H(u)}$

$$\psi(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{H(u)}{(x-u)^2+y^2}\,du = \frac{y}{\pi}\int_{0}^{\infty}\frac{1}{(x-u)^2+y^2}\,du$$

SAME SUBSTITUTION AS BEFORE

$$= \frac{y}{\pi}\int_{x}^{-\infty}\frac{1}{t^2+y^2}(-dt) = \frac{y}{\pi}\int_{-\infty}^{x}\frac{1}{t^2+y^2}\,dt$$

$$= \frac{y}{\pi}\times\frac{1}{y}\left[\arctan\left(\frac{t}{y}\right)\right]_{t=-\infty}^{x} = \frac{1}{\pi}\left[\arctan\frac{x}{y} - \left(-\frac{\pi}{2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} + \arctan\left(\frac{x}{y}\right)\right] = \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{x}{y}\right)$$

AND RESULTS ARE CONSISTENT WITH

$$H(x) = \frac{1}{2}(1 + \mathrm{sgn}\,x)$$

SINCE $\dfrac{1}{2}\left[1 + \dfrac{2}{\pi}\arctan\left(\dfrac{x}{y}\right)\right]$

$$= \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{x}{y}\right) \quad //$$

Question 14

The function $\theta = \theta(x,t)$ satisfies the heat equation in one spatial dimension,

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\sigma^2}\frac{\partial \theta}{\partial t}, \quad -\infty < x < \infty \,, \; t \geq 0 \,,$$

where σ is a positive constant.

Given further that $\theta(x,0) = f(x)$, use Fourier transforms to convert the above partial differential equation into an ordinary differential equation and hence show that

$$\theta(x,t) = \frac{1}{2\sigma\sqrt{\pi t}} \int_{-\infty}^{\infty} f(x-u)\,\exp\left(\frac{u^2}{4t\sigma^2}\right)\, du \,.$$

proof

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\sigma^2}\frac{\partial \theta}{\partial t} \qquad -\infty < x < \infty, \; t > 0$$

SUBJECT TO THE INITIAL CONDITION $\theta(x,0) = f(x)$

- TAKE THE FOURIER TRANSFORM OF THE P.D.E IN x

$$\Rightarrow \mathcal{F}\left[\frac{\partial^2 \theta}{\partial x^2}\right] = \frac{1}{\sigma^2}\mathcal{F}\left[\frac{\partial \theta}{\partial t}\right]$$

$$\Rightarrow (ik)^2\, \hat{\theta}(k,t) = \frac{1}{\sigma^2}\frac{\partial}{\partial t}\hat{\theta}(k,t)$$

$$\Rightarrow -k^2\hat{\theta} = \frac{1}{\sigma^2}\frac{\partial \hat{\theta}}{\partial t}$$

$$\Rightarrow \frac{\partial \hat{\theta}}{\partial t} = -k^2\sigma^2\hat{\theta}$$

(SIMPLE EXPONENTIAL O.D.E AS k IS TREATED AS A CONSTANT)

$$\Rightarrow \hat{\theta}(k,t) = A(k)e^{-k^2\sigma^2 t}$$

- APPLY THE INITIAL CONDITION, $\theta(x,0) = f(x)$

$$\theta(x,0) = f(x) \Rightarrow \hat{\theta}(k,0) = \hat{f}(k)$$

SO

$$\hat{\theta}(k,0) = A(k)e^{0}$$

$$\boxed{\hat{f}(k) = A(k)}$$

$$\hat{\theta}(k,t) = \hat{f}(k)e^{-k^2\sigma^2 t}$$

- TO INVERT WE LOOK AT THE CONVOLUTION THEOREM

$$\mathcal{F}[f * g] = \sqrt{2\pi}\,\mathcal{F}(f)\,\mathcal{F}(g)$$

COMPARING WITH OUR SOLUTION

$$\mathcal{F}[\theta(x,t)] = \hat{f}(k)\, e^{-k^2\sigma^2 t}$$

$$\sqrt{2\pi}\,\mathcal{F}[\theta(x,t)] = \sqrt{2\pi}\,\hat{f}(k)\, e^{-k^2\sigma^2 t}$$

$$\therefore \boxed{\theta(x,t) \times \sqrt{2\pi} = f * g}$$

- $f(x)$ WE HAVE (GIVEN)
- $g(x)$ WE MUST INVERT $\hat{g}(k) = e^{-k^2\sigma^2 t}$

$$\boxed{\hat{g}(k) = e^{-k^2\sigma^2 t}}$$

$$\Rightarrow g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{g}(k)\,e^{ikx}\,dk = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty} e^{-k^2\sigma^2 t}e^{ikx}\,dk$$

- AS $e^{-k^2\sigma^2 t}$ IS EVEN IN k WE MAY SIMPLIFY

$$\Rightarrow g(x) = \frac{2}{\sqrt{2\pi}}\int_0^{\infty} e^{-k^2\sigma^2 t}\cos kx\,dk$$

- LET $I = \int_0^{\infty} e^{-k^2\sigma^2 t}\cos kx\,dk$

$$\Rightarrow \frac{\partial I}{\partial x} = \int_0^{\infty} e^{-k^2\sigma^2 t}\left[-k\sin kx\right]dk$$

$$\Rightarrow \frac{\partial I}{\partial x} = \int_0^{\infty} -ke^{-k^2\sigma^2 t}\sin kx\,dk$$

- BY PARTS (W.R.T k)

$\sin kx$	$x\cos kx$
$\frac{1}{2\sigma^2 t}e^{-k^2\sigma^2 t}$	$-ke^{-k^2\sigma^2 t}$

$$\Rightarrow \frac{\partial I}{\partial x} = \cancel{\left[\frac{1}{2\sigma^2 t}e^{-k^2\sigma^2 t}\sin kx\right]_0^{\infty}} - \frac{x}{2\sigma^2 t}\int_0^{\infty} e^{-k^2\sigma^2 t}\cos kx\,dk$$

$$\Rightarrow \frac{\partial I}{\partial x} = -\frac{x}{2\sigma^2 t}I$$

- SOLVING THE O.D.E BY SEPARATION OF VARIABLES

$$\Rightarrow \frac{1}{I}\partial I = -\frac{x}{2\sigma^2 t}\partial x$$

$$\Rightarrow \ln I = -\frac{x^2}{4\sigma^2 t} + C$$

$$\Rightarrow I = Ae^{-\frac{x^2}{4\sigma^2 t}} \qquad (A = e^{C})$$

$$\Rightarrow \int_0^{\infty} e^{-k^2\sigma^2 t}\cos kx\,dk = Ae^{-\frac{x^2}{4\sigma^2 t}}$$

- EVALUATE AT $x=0$

$$\Rightarrow \int_0^{\infty} e^{-k^2\sigma^2 t}\,dk = A.$$

- USE A SUBSTITUTION $u^2 = k^2\sigma^2 t$ OR $u = k\sigma\sqrt{t}$

$du = dk\,\sigma\sqrt{t}$

$dk = \frac{du}{\sigma\sqrt{t}}$

LIMITS UNCHANGED

$$\Rightarrow A = \frac{1}{\sigma\sqrt{t}}\int_0^{\infty} e^{-u^2}\,du$$

$$\Rightarrow \boxed{A = \frac{1}{\sigma\sqrt{t}}\frac{\sqrt{\pi}}{2}}$$

$$\Rightarrow I = \frac{1}{\sigma\sqrt{t}}\frac{\sqrt{\pi}}{2}e^{-\frac{x^2}{4\sigma^2 t}}$$

$$\Rightarrow g(x) = \frac{2}{\sqrt{2\pi}}I = \frac{2}{\sqrt{2\pi}}\frac{1}{\sigma\sqrt{t}}\frac{\sqrt{\pi}}{2}e^{-\frac{x^2}{4\sigma^2 t}}$$

$$\Rightarrow g(x) = \frac{1}{\sqrt{2t}\,\sigma}e^{-\frac{x^2}{4\sigma^2 t}}$$

- RETURNING TO THE INVERSION

$$\boxed{\sqrt{2\pi}\,\theta(x,t) = f * g}$$

- $f(x)$ IS GIVEN
- $g(x) = \frac{1}{\sqrt{2t}\,\sigma}e^{-\frac{x^2}{4\sigma^2 t}}$ (TREAT t AS A CONSTANT)

$$\Rightarrow \theta(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-y)\,g(y)\,dy$$

$$\Rightarrow \theta(x,t) = \frac{1}{\sqrt{2\pi}}\int f(x-y) \times \frac{1}{\sqrt{2t}\,\sigma}e^{-\frac{y^2}{4\sigma^2 t}}\,dy$$

$$\Rightarrow \theta(x,t) = \frac{1}{2\sigma\sqrt{\pi t}}\int_{-\infty}^{\infty} f(x-y)\,e^{-\frac{y^2}{4\sigma^2 t}}\,dy$$

(IF $f(x)$ IS GIVEN EXPLICITLY, THE INTEGRAL MAY BE EVALUATED)

Question 15

The function $u = u(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

in the part of the x-y plane for which $x \geq 0$ and $y \geq 0$.

It is further given that

- $u(0, y) = 0$
- $u(x, y) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$
- $u(x,0) = f(x)$, $f(0) = 0$, $f(x) \to 0$ as $x \to \infty$

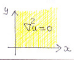
Use Fourier transforms to show that

$$u(x, y) = \frac{y}{\pi}\int_0^{\infty} f(w)\left[\frac{1}{y^2 + (x-w)^2} - \frac{1}{y^2 + (x+w)^2}\right] dw.$$

$\boxed{\text{proof}}$

[solution overleaf]

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$



SUBJECT TO

1. $u(x,y) \to 0$ AS $\sqrt{x^2+y^2} \to \infty$
2. $u(0,y) = 0$
3. $u(x,0) = f(x)$, $f(0) = 0$, $f(x) \to 0$ AS $x \to \infty$

- ALTHOUGH THE REGION IS NOT SYMMETRICAL IN x (OR IN y), EXTEND $u(x,y)$ & $f(x)$ IN THE NEGATIVE x DIRECTION, SO BOTH ARE ODD
- TAKE FOURIER TRANSFORM OF THE P.D.E IN x

$$\Rightarrow \mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] + \mathcal{F}\left[\frac{\partial^2 u}{\partial y^2}\right] = \mathcal{F}[0]$$

$$\Rightarrow (ik)^2 \hat{u}(k,y) + \frac{\partial^2}{\partial y^2}\hat{u}(k,y) = 0$$

$$\Rightarrow \frac{\partial^2 \hat{u}}{\partial y^2} - k^2 \hat{u} = 0$$

$$\Rightarrow \boxed{\hat{u}(k,y) = A(k)e^{|k|y} + B(k)e^{-|k|y}}$$

- APPLY BOUNDARY CONDITION ①

IF $u(x,y) \to 0$ AS $\sqrt{x^2+y^2} \to \infty$, THEN $\hat{u}(k,y) \to 0$ AS $\sqrt{k^2+y^2} \to \infty$

$\therefore A(k) = 0$

$$\Rightarrow \boxed{\hat{u}(k,y) = B(k)e^{-|k|y}}$$

- APPLY BOUNDARY CONDITION ③

$$\Rightarrow u(x,0) = f(x)$$

$$\Rightarrow \hat{u}(k,0) = \hat{f}(k)$$

$$\Rightarrow \hat{f}(k) = B(k)e^{0}$$

$$\Rightarrow B(k) = \hat{f}(k)$$

$$\therefore \boxed{u(k,y) = \hat{f}(k)e^{-|k|y}}$$

- AS WE DO NOT KNOW $f(x)$ EXPLICITLY, WE START THE INVERSION FROM FIRST PRINCIPLES

$$\Rightarrow u(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{u}(k,y)e^{ikx}\,dk$$

- AS $u(x,y)$ IS ODD (WE BUILT ODD EXTENSION), $\hat{u}(k,y)$ WILL ALSO BE ODD

$$\Rightarrow u(x,y) = \frac{2}{\sqrt{2\pi}}\int_0^{\infty} \hat{u}(k,y)\,i\sin kx\,dk$$

$$\Rightarrow u(x,y) = \sqrt{\frac{2}{\pi}}\,i\int_0^{\infty} \hat{f}(k)\,e^{-ky}\sin kx\,dk$$

$$\Rightarrow u(x,y) = \sqrt{\frac{2}{\pi}}\,i\int_0^{\infty} e^{-ky}\sin kx\underbrace{\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(w)e^{-ikw}\,dw\right]}_{\hat{f}(k)}dk$$

- AS f IS "BUILT" ODD, ONLY THE ODD (SINE PART) SURVIVES

$$\Rightarrow u(x,y) = \sqrt{\frac{2}{\pi}}\,i\int_0^{\infty} e^{-ky}\sin kx\left[-\sqrt{\frac{2}{\pi}}\,i\int_0^{\infty} f(w)\sin kw\,dw\right]dk$$

$$\Rightarrow u(x,y) = -\frac{2}{\pi}(i)^2\int_0^{\infty} e^{-ky}\sin kx\left[\int_0^{\infty} f(w)\sin kw\,dw\right]dk$$

$$\Rightarrow u(x,y) = \frac{2}{\pi}\int_0^{\infty}\int_0^{\infty} e^{-ky}f(w)\sin kx\sin kw\,dw\,dk$$

- REVERSING THE ORDER OF INTEGRATION NOTING THAT THE LIMITS ARE UNCHANGED ("BOX" REGION FROM 0 TO ∞)

$$\Rightarrow u(x,y) = \frac{2}{\pi}\int_{w=0}^{\infty} f(w)\left[\int_{k=0}^{\infty} e^{-ky}\sin kx\sin kw\,dk\right]dw$$

NEED TO DERIVE AN IDENTITY

$$\cos(kx+kw) \equiv \cos kx\cos kw - \sin kx\sin kw$$

$$\cos(kx-kw) \equiv \cos kx\cos kw + \sin kx\sin kw$$

$$\cos(kx-kw) - \cos(kx+kw) \equiv 2\sin kx\sin kw$$

$$\Rightarrow u(x,y) = \frac{1}{\pi}\int_{w=0}^{\infty} f(w)\left[\int_{k=0}^{\infty} e^{-ky}\cos[k(x-w)] - e^{-ky}\cos[k(x+w)]\,dk\right]dw$$

- LOOKING AT EACH OF THE "INNER" INTEGRALS

$$\int_{k=0}^{\infty} e^{-ky}\cos[k(x-w)]\,dk = \mathrm{Re}\int_{k=0}^{\infty} e^{-ky}e^{ik(x-w)}\,dk$$

$$= \mathrm{Re}\int_{k=0}^{\infty} e^{k[-y+i(x-w)]}\,dk = \mathrm{Re}\left[\frac{1}{-y+i(x-w)}e^{k(-y+i(x-w))}\right]_{k=0}^{\infty}$$

$$= \mathrm{Re}\left[\frac{-y-i(x-w)}{y^2+(x-w)^2}e^{-ky}e^{ik(x-w)}\right]_{k=0}^{\infty} = \mathrm{Re}\left[\frac{-y-i(x-w)}{y^2+(x-w)^2}(0-1)\right]$$

$$= \frac{y}{y^2+(x-w)^2}$$

SIMILARLY THE OTHER INTEGRAL EXCEPT THAT WE HAVE $(x+w)$ INSTEAD OF $(x-w)$

- $$u(x,y) = \frac{1}{\pi}\int_0^{\infty} f(w)\left[\frac{y}{y^2+(x-w)^2} - \frac{y}{y^2+(x+w)^2}\right]dw$$

$$u(x,y) = \frac{y}{\pi}\int_0^{\infty} f(w)\left[\frac{1}{y^2+(x-w)^2} - \frac{1}{y^2+(x+w)^2}\right]dw$$

Question 16

The function $T = T(x,t)$ satisfies the heat equation in one spatial dimension,

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\sigma}\frac{\partial \theta}{\partial t}, \quad x \geq 0, \ t \geq 0,$$

where σ is a positive constant.

It is further given that

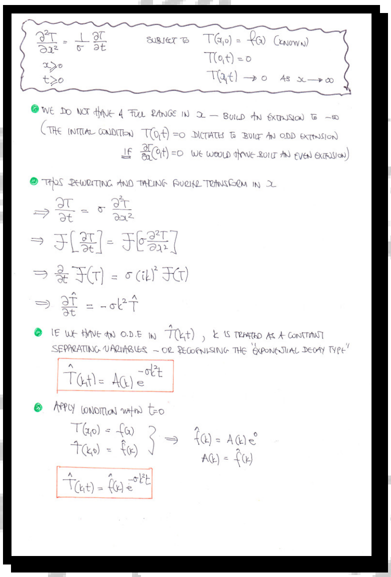
- $T(x,0) = f(x)$
- $T(0,t) = 0$
- $T(x,t) \to 0$ as $x \to \infty$

Use Fourier transforms to convert the above partial differential equation into an ordinary differential equation and hence show that

$$T(x,t) = \frac{1}{\sqrt{4\pi\sigma t}}\int_{-\infty}^{\infty} f(u)\exp\left[\frac{(x-u)^2}{4t\sigma}\right]du .$$

You may assume that $\mathcal{F}\left[e^{ax^2}\right] = \frac{1}{\sqrt{2a}}e^{\frac{k^2}{4a}}$.

proof



$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\sigma}\frac{\partial T}{\partial t} \qquad \text{SUBJECT TO} \quad T(x,0) = f(x) \text{ (KNOWN)}, \quad T(0,t) = 0, \quad T(x,t) \to 0 \text{ AS } x \to \infty$$

$x \geq 0$, $t \geq 0$

- WE DO NOT HAVE A FULL RANGE IN x — BUILD AN EXTENSION TO $-\infty$ (THE INITIAL CONDITION $T(0,t) = 0$ DICTATES TO BUILT AN ODD EXTENSION; IF $\frac{\partial T}{\partial x}(0,t) = 0$ WE WOULD HAVE BUILT AN EVEN EXTENSION)
- THUS REWRITING AND TAKING FOURIER TRANSFORM IN x

$$\Rightarrow \frac{\partial T}{\partial t} = \sigma\frac{\partial^2 T}{\partial x^2}$$

$$\Rightarrow \mathcal{F}\left[\frac{\partial T}{\partial t}\right] = \mathcal{F}\left[\sigma\frac{\partial^2 T}{\partial x^2}\right]$$

$$\Rightarrow \frac{\partial}{\partial t}\mathcal{F}(T) = \sigma(ik)^2\mathcal{F}(T)$$

$$\Rightarrow \frac{\partial \hat{T}}{\partial t} = -\sigma k^2\hat{T}$$

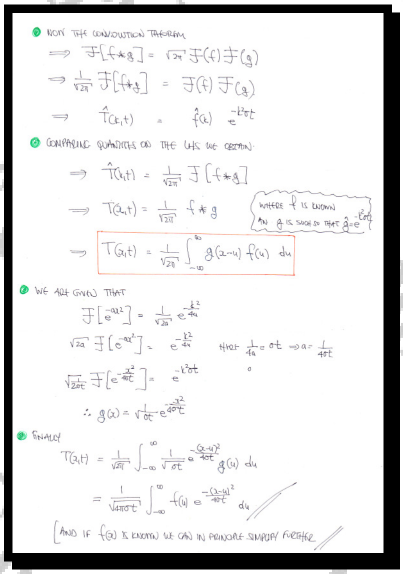
- IF WE HAVE AN O.D.E IN $\hat{T}(k,t)$, k IS TREATED AS A CONSTANT. SEPARATING VARIABLES – OR RECOGNISING THE "EXPONENTIAL DECAY TYPE"

$$\boxed{\hat{T}(k,t) = A(k)e^{-\sigma k^2 t}}$$

- APPLY CONDITION WHEN $t=0$

$$\left.\begin{array}{l} T(x,0) = f(x) \\ \hat{T}(k,0) = \hat{f}(k) \end{array}\right\} \Rightarrow \begin{array}{l} \hat{f}(k) = A(k)e^{0} \\ A(k) = \hat{f}(k) \end{array}$$

$$\boxed{\hat{T}(k,t) = \hat{f}(k)e^{-\sigma k^2 t}}$$



- NOW THE CONVOLUTION THEOREM

$$\Rightarrow \mathcal{F}[f*g] = \sqrt{2\pi}\,\mathcal{F}(f)\mathcal{F}(g)$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}}\mathcal{F}[f*g] = \mathcal{F}(f)\mathcal{F}(g)$$

$$\Rightarrow \hat{T}(k,t) = \hat{f}(k)\ e^{-k^2\sigma t}$$

- COMPARING QUANTITIES ON THE LHS WE OBTAIN

$$\Rightarrow \hat{T}(k,t) = \frac{1}{\sqrt{2\pi}}\mathcal{F}[f*g]$$

$$\Rightarrow T(x,t) = \frac{1}{\sqrt{2\pi}} f*g$$

WHERE f IS KNOWN AND g IS SUCH SO THAT $\hat{g} = e^{-k^2\sigma t}$

$$\Rightarrow \boxed{T(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x-u)f(u)\,du}$$

- WE ARE GIVEN THAT

$$\mathcal{F}\left[e^{-ax^2}\right] = \frac{1}{\sqrt{2a}}e^{-\frac{k^2}{4a}}$$

$$\sqrt{2a}\,\mathcal{F}\left[e^{-ax^2}\right] = e^{-\frac{k^2}{4a}} \qquad \text{THEN } \frac{1}{4a} = \sigma t \Rightarrow a = \frac{1}{4\sigma t}$$

$$\sqrt{\frac{1}{2\sigma t}}\,\mathcal{F}\left[e^{-\frac{x^2}{4\sigma t}}\right] = e^{-k^2\sigma t}$$

$$\therefore g(x) = \sqrt{\frac{1}{\sigma t}}\,e^{-\frac{x^2}{4\sigma t}}$$

- FINALLY

$$T(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{\sigma t}}e^{-\frac{(x-u)^2}{4\sigma t}}g(u)\,du$$

$$= \frac{1}{\sqrt{4\pi\sigma t}}\int_{-\infty}^{\infty} f(u)\,e^{\frac{-(x-u)^2}{4\sigma t}}\,du$$

[AND IF $f(x)$ IS KNOWN WE CAN IN PRINCIPLE SIMPLIFY FURTHER

Question 17

The function $f = f(x)$ satisfies the integral equation

$$\int_{-\infty}^{\infty} \frac{f(t)}{(x-t)^2+1}\, dt = \frac{1}{x^2+4},$$

where $f(x) \to 0$ as $x \to \infty$

Use Fourier transforms to find the solution of the above integral equation.

You may assume that $\mathcal{F}\left[\dfrac{1}{x^2+a^2}\right] = \dfrac{1}{a}\sqrt{\dfrac{\pi}{2}}\, \mathrm{e}^{-a|k|}$.

$$\boxed{f(x) = \frac{1}{2\pi\left(1+x^2\right)}}$$

$$\boxed{\int_{-\infty}^{\infty} \frac{f(t)}{(x-t)^2+1}\, dt = \frac{1}{x^2+4}}$$

- THE CONVOLUTION THEOREM STATES

$$\mathcal{F}\big[(f*g)(x)\big] = \sqrt{2\pi}\, \mathcal{F}\big[f(x)\big]\, \mathcal{F}\big[g(x)\big]$$

WHERE $(f*g)(x) = \displaystyle\int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy$

- HENCE

$$\Rightarrow \underbrace{\int_{-\infty}^{\infty} f(t) \times \frac{1}{(x-t)^2+1}\, dt}_{(f*g)(x)} = \underbrace{\frac{1}{x^2+4}}_{h(x)} \qquad \boxed{\text{WHERE } g(x) = \frac{1}{x^2+1}}$$

- TAKING FOURIER TRANSFORM IN x FOR THE INTEGRAL EQUATION

$$\Rightarrow \sqrt{2\pi}\, \hat{f}(k)\, \hat{g}(k) = \hat{h}(k)$$

$$\Rightarrow \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \frac{\hat{h}(k)}{\hat{g}(k)}$$

- USING THE RESULT $\boxed{\mathcal{F}\left[\dfrac{1}{x^2+a^2}\right] = \dfrac{1}{a}\sqrt{\dfrac{\pi}{2}}\, e^{-a|k|}}$

$$\hat{h}(k) = \sqrt{\frac{\pi}{2}}\, e^{-2|k|}$$

$$\hat{g}(k) = \sqrt{\frac{\pi}{2}}\, e^{-|k|}$$

$$\Rightarrow \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \frac{\cancel{\sqrt{\frac{\pi}{2}}}\frac{1}{2} e^{-2|k|}}{\cancel{\sqrt{\frac{\pi}{2}}}\, e^{-|k|}}$$

$$\Rightarrow \hat{f}(k) = \frac{1}{2\sqrt{2\pi}}\, e^{-|k|}$$

- FINALLY INVERTING

$$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k)\, e^{ikx}\, dk$$

$$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{2\sqrt{2\pi}}\, e^{-|k|}\, e^{ikx}\, dk$$

- TIDYING & NOTING THAT THE INTEGRAND IS EVEN

$$\Rightarrow f(x) = \frac{1}{2\pi} \int_{0}^{\infty} e^{-k} \cos kx\, dk$$

$$\Rightarrow f(x) = \frac{1}{2\pi} \mathrm{Re} \int_{0}^{\infty} e^{k(-1+ix)}\, dk$$

$$\Rightarrow f(x) = \frac{1}{2\pi} \mathrm{Re}\left[\frac{1}{-1+ix}\, e^{k(-1+ix)}\right]_{0}^{\infty}$$

$$\Rightarrow f(x) = \frac{1}{2\pi} \mathrm{Re}\left[\frac{-1-ix}{1+x^2}\, e^{-k}\, e^{ikx}\right]_{k=0}^{\infty}$$

$$\Rightarrow f(x) = \frac{1}{2\pi} \mathrm{Re}\left[0 - \frac{-1-ix}{1+x^2}\right]$$

$$\Rightarrow f(x) = \frac{1}{2\pi} \mathrm{Re}\left[\frac{1+ix}{1+x^2}\right]$$

$$\Rightarrow f(x) = \frac{1}{2\pi}\left(\frac{1}{1+x^2}\right)$$

Question 18

The function $f = f(x)$ satisfies the integral equation

$$\int_{-\infty}^{\infty} f(x-u)\,f(u)\;du \;=\; \frac{1}{1+x^2}\,,$$

where $f(x) \to 0$ as $x \to \infty$

Use Fourier transforms to find the solution of the above integral equation.

You may assume that

$$\int_0^{\infty} \frac{\cos kx}{x^2+1}\;dx = \frac{1}{2}\pi\,\mathrm{e}^{|k|}.$$

$$\boxed{f(x) = \frac{2}{\left(1+4x^2\right)\sqrt{\pi}}}$$

$\boxed{\begin{array}{l}\text{CONVOLUTION OF } f \text{ \& } g\\ (f*g)(x) = \int_{-\infty}^{\infty} f(x-y)\,g(y)\,dy\end{array}}$

$\boxed{\begin{array}{l}\text{FOURIER TRANSFORM OF THE CONVOLUTION}\\ \mathcal{F}[f*g] = \sqrt{2\pi}\,\hat{f}(k)\,\hat{g}(k)\end{array}}$

- STARTING WITH THE INTEGRAL EQUATION

$\Rightarrow \int_{-\infty}^{\infty} f(x-u)\,f(u)\,du = \frac{1}{1+x^2}$

$\Rightarrow (f*f)(x) = \frac{1}{1+x^2}$

- TAKING THE FOURIER TRANSFORM OF THE EQUATION

$\Rightarrow \mathcal{F}[(f*f)(x)] = \mathcal{F}\left[\frac{1}{1+x^2}\right]$

$\Rightarrow \sqrt{2\pi}\,\hat{f}(k)\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \underbrace{\frac{1}{1+x^2}}_{\text{EVEN FUNCTION}}\,e^{-ikx}\,dx$

$\Rightarrow \sqrt{2\pi}\left[\hat{f}(k)\right]^2 = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} 2\times\frac{1}{1+x^2}\cos kx\,dx$

$\Rightarrow \left[\hat{f}(k)\right]^2 = \frac{1}{\pi}\int_0^{\infty}\frac{\cos kx}{1+x^2}\,dx$

$\Rightarrow \left[\hat{f}(k)\right]^2 = \frac{1}{\pi}\left(\frac{\pi}{2}e^{-|k|}\right)$

$\Rightarrow \left[\hat{f}(k)\right]^2 = \frac{1}{2}e^{-|k|}$

$\Rightarrow \hat{f}(k) = \frac{1}{\sqrt{2}}e^{-\frac{|k|}{2}}$

- START INVERTING THE TRANSFORM

$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\underbrace{\left[\frac{1}{\sqrt{2}}e^{-\frac{|k|}{2}}\right]}_{\text{EVEN}}e^{ikx}\,dk$

$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi}}\int_0^{\infty}\frac{2}{\sqrt{2}}e^{-\frac{k}{2}}\cos kx\,dk$

$\Rightarrow f(x) = \frac{1}{\sqrt{\pi}}\int_0^{\infty} e^{-\frac{k}{2}}\cos kx\,dk$

- BY USING COMPLEX NUMBERS TO INTEGRATE

$\Rightarrow f(x) = \frac{1}{\sqrt{\pi}}\,\mathrm{Re}\int_0^{\infty} e^{-\frac{k}{2}}e^{ikx}\,dk = \frac{1}{\sqrt{\pi}}\,\mathrm{Re}\int_0^{\infty} e^{k\left(-\frac{1}{2}+ix\right)}\,dk$

$\Rightarrow f(x) = \frac{1}{\sqrt{\pi}}\,\mathrm{Re}\left[\frac{1}{-\frac{1}{2}+ix}e^{k\left(-\frac{1}{2}+ix\right)}\right]_{k=0}^{k=\infty}$

$\Rightarrow f(x) = \frac{1}{\sqrt{\pi}}\,\mathrm{Re}\left[\frac{-\frac{1}{2}-ix}{\frac{1}{4}+x^2}e^{-\frac{1}{2}k}e^{ikx}\right]_{k=0}^{\infty}$

$\Rightarrow f(x) = \frac{1}{\sqrt{\pi}}\,\mathrm{Re}\left[\frac{-\frac{1}{2}-ix}{\frac{1}{4}+x^2}(0-1)\right]$

$\Rightarrow f(x) = \frac{1}{\sqrt{\pi}}\,\mathrm{Re}\left[\frac{\frac{1}{2}+ix}{\frac{1}{4}+x^2}\right]$

$\Rightarrow f(x) = \frac{1}{\sqrt{\pi}}\,\frac{\frac{1}{2}}{\frac{1}{4}+x^2}$

$\Rightarrow f(x) = \frac{2}{\sqrt{\pi}\,(1+4x^2)}$

Question 19

The function $f = f(x)$ satisfies the integral equation

$$e^{-\frac{1}{2}x^2} = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-u|} f(u)\ du\,,$$

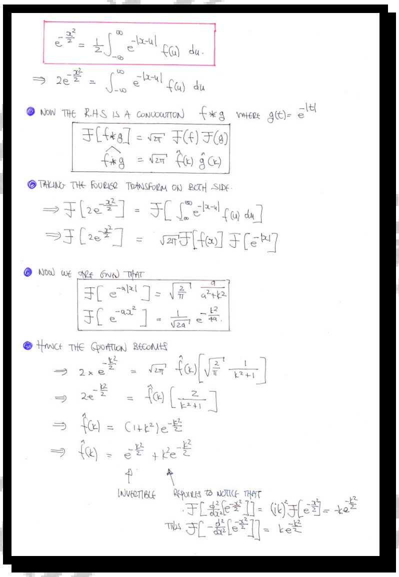
where $f(x) \to 0$ as $x \to \infty$

Use Fourier transforms to find the solution of the above integral equation.

You may assume that

- $\mathcal{F}\left[e^{ax^2}\right] = \frac{1}{\sqrt{2a}} e^{\frac{k^2}{4a}}$.
- $\mathcal{F}\left[e^{a|x|}\right] = \sqrt{\frac{2}{\pi}}\,\frac{a}{a^2+k^2}$.

$$\boxed{f(x) = \left(2 - x^2\right)e^{-\frac{1}{2}x^2}}$$



$$\boxed{e^{-\frac{x^2}{2}} = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-u|} f(u)\ du.}$$

$$\Rightarrow 2e^{-\frac{x^2}{2}} = \int_{-\infty}^{\infty} e^{-|x-u|} f(u)\ du$$

- NOW THE R.H.S IS A CONVOLUTION $f * g$ WHERE $g(t) = e^{-|t|}$

$$\boxed{\begin{aligned} \mathcal{F}[f*g] &= \sqrt{2\pi}\,\mathcal{F}(f)\,\mathcal{F}(g) \\ \widehat{f*g} &= \sqrt{2\pi}\,\hat{f}(k)\,\hat{g}(k) \end{aligned}}$$

- TAKING THE FOURIER TRANSFORM ON BOTH SIDE.

$$\Rightarrow \mathcal{F}\left[2e^{-\frac{x^2}{2}}\right] = \mathcal{F}\left[\int_{-\infty}^{\infty} e^{-|x-u|} f(u)\ du\right]$$

$$\Rightarrow \mathcal{F}\left[2e^{-\frac{x^2}{2}}\right] = \sqrt{2\pi}\,\mathcal{F}[f(x)]\,\mathcal{F}\left[e^{-|x|}\right]$$

- NOW WE ARE GIVEN THAT

$$\boxed{\begin{aligned} \mathcal{F}\left[e^{-a|x|}\right] &= \sqrt{\frac{2}{\pi}}\,\frac{a}{a^2+k^2} \\ \mathcal{F}\left[e^{-ax^2}\right] &= \frac{1}{\sqrt{2a}}\,e^{-\frac{k^2}{4a}}. \end{aligned}}$$

- HENCE THE EQUATION BECOMES

$$\Rightarrow 2 \times e^{-\frac{k^2}{2}} = \sqrt{2\pi}\,\hat{f}(k)\left[\sqrt{\frac{2}{\pi}}\,\frac{1}{k^2+1}\right]$$

$$\Rightarrow 2e^{-\frac{k^2}{2}} = \hat{f}(k)\left[\frac{2}{k^2+1}\right]$$

$$\Rightarrow \hat{f}(k) = (1+k^2)e^{-\frac{k^2}{2}}$$

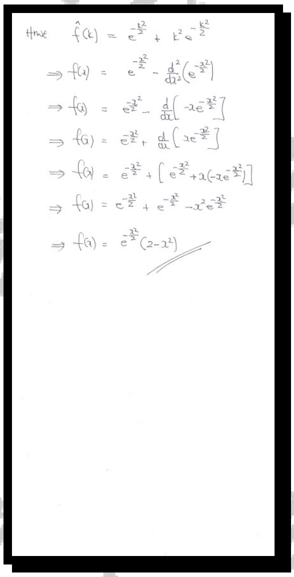
$$\Rightarrow \hat{f}(k) = e^{-\frac{k^2}{2}} + k^2 e^{-\frac{k^2}{2}}$$

INVERTIBLE

REQUIRES TO NOTICE THAT

$$\mathcal{F}\left[\frac{d^2}{dx^2}\left[e^{-\frac{x^2}{2}}\right]\right] = (ik)^2\,\mathcal{F}\left[e^{-\frac{x^2}{2}}\right] = -ke^{-\frac{k^2}{2}}$$

THUS $\mathcal{F}\left[-\frac{d^2}{dx^2}\left[e^{-\frac{x^2}{2}}\right]\right] = ke^{-\frac{k^2}{2}}$



HENCE $\hat{f}(k) = e^{-\frac{k^2}{2}} + k^2 e^{-\frac{k^2}{2}}$

$$\Rightarrow f(x) = e^{-\frac{x^2}{2}} - \frac{d^2}{dx^2}\left(e^{-\frac{x^2}{2}}\right)$$

$$\Rightarrow f(x) = e^{-\frac{x^2}{2}} - \frac{d}{dx}\left[-xe^{-\frac{x^2}{2}}\right]$$

$$\Rightarrow f(x) = e^{-\frac{x^2}{2}} + \frac{d}{dx}\left[xe^{-\frac{x^2}{2}}\right]$$

$$\Rightarrow f(x) = e^{-\frac{x^2}{2}} + \left[e^{-\frac{x^2}{2}} + x\left(-xe^{-\frac{x^2}{2}}\right)\right]$$

$$\Rightarrow f(x) = e^{-\frac{x^2}{2}} + e^{-\frac{x^2}{2}} - x^2 e^{-\frac{x^2}{2}}$$

$$\Rightarrow f(x) = e^{-\frac{x^2}{2}}(2-x^2)$$