IMPULSE FUNCTION

The Impulse Function / The Dirac Function

1. $\delta(t-c) = \begin{cases} \infty & t = c \\ 0 & t \neq c \end{cases}$, $\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$

2. $\delta(t) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \left[\frac{\varepsilon}{\varepsilon^2 + t^2} \right]$

3. $\int_a^b \delta(t-c) \; dt = \begin{cases} 1 & a \le c \le b \\ 0 & \text{otherwise} \end{cases}$

4. $\int_a^b f(t)\delta(t-c) \; dt = \begin{cases} f(c) & a \le c \le b \\ 0 & \text{otherwise} \end{cases}$

5. $\mathcal{L}\left[\delta(t-c)\right] = \mathrm{e}^{-cs}$

6. $\mathcal{L}\left[f(t)\delta(t-c)\right] = f(c)\mathrm{e}^{-cs}$

7. $\mathcal{F}\left[\delta(x)\right] = \frac{1}{\sqrt{2\pi}}$

8. $\mathcal{F}^{-1}\left[\delta(k)\right] = \frac{1}{\sqrt{2\pi}}$

9. $\frac{d}{dt}\left[\mathrm{H}(t-c)\right] = \delta(t-c)$

Question 1

Evaluate the following integral

$$\int_0^5 \left(t^2+1\right)\delta(t-1)\ dt .$$

$\boxed{2}$

$$\int_0^5 \underbrace{(t^2+1)}_{f(t)}\delta(t-1) = \dots$$ IMPULSE OCCURS AT $t=1$,

• USING

$$\int_a^b f(t)\delta(t-c)\,dt = \begin{cases} f(c) & a \le c \le b \\ 0 & \text{OTHERWISE} \end{cases}$$

$$= 1^2+1 = 2$$

Question 2

Evaluate the following integral

$$\int_0^{\pi} \sin\left(\frac{1}{3}t\right)\delta\left(t-\frac{\pi}{2}\right)\ dt .$$

$\boxed{\frac{1}{2}}$

$$\int_0^{\pi} \underbrace{\sin\left(\tfrac{t}{3}\right)}_{f(t)}\delta\left(t-\tfrac{\pi}{2}\right) = \dots$$ IMPULSE OCCURS AT $t=\frac{\pi}{2}$

• USING

$$\int_a^b f(t)\delta(t-c)\,dt = \begin{cases} f(c) & a \le c \le b \\ 0 & \text{OTHERWISE} \end{cases}$$

$$= \sin\frac{\pi/2}{3}$$

$$= \sin\frac{\pi}{6}$$

$$= \frac{1}{2}$$

Question 3

Find the Laplace transform of $\delta(t-c)$, where c is a positive constant, and hence state the Laplace transform of $\delta(t)$.

$$\boxed{\mathcal{L}\left[\delta(t-c)\right]=\mathrm{e}^{-cs}}, \quad \boxed{\mathcal{L}\left[\delta(t)\right]=1}$$

$$\mathcal{L}[\delta(t-c)] = \int_0^\infty e^{-st}\delta(t-c)\,dt$$

• USING

$$\int_a^b f(t)\,\delta(t-c)\,dt = \begin{cases} f(c) & a \le c \le b \\ 0 & \text{otherwise} \end{cases}$$

$$= e^{-cs}$$

HENCE $\mathcal{L}[\delta(t)] = e^{-0s} = 1$

Question 4

Given that $F(t)$ is a piecewise continuous function defined for $t \ge 0$, find the Laplace transform of $F(t)\,\delta(t-c)$, where c is a positive constant.

$$\boxed{\mathcal{L}\left[F(t)\,\delta(t-c)\right]=F(c)\mathrm{e}^{-cs}}$$

$$\mathcal{L}[F(t)\,\delta(t-c)] = \int_0^\infty e^{-st}F(t)\,\delta(t-c)\,dt$$

$$= \int_0^\infty G(t)\,\delta(t-c)\,dt \qquad \text{WHERE } G(t) = e^{-st}F(t)$$

$$= G(c)$$

$$= F(c)\,e^{-cs}$$

Question 5

Find the Laplace transform of $\cos 3t \ \delta\left(t - \frac{\pi}{3}\right)$.

$$\boxed{\mathcal{L}\left[\cos 3t \ \delta\left(t - \frac{\pi}{3}\right)\right] = -e^{-\frac{1}{3}\pi s}}$$

$$\mathcal{L}\left[\cos 3t \ \delta\left(t - \frac{\pi}{3}\right)\right] = \int_0^\infty e^{-st} \cos 3t \ \delta(t - \pi/3) \, dt$$
$$= e^{-\frac{\pi}{3}s} \cos(\pi)$$
$$= -e^{-\frac{\pi}{3}s}$$

Question 6

Find the Laplace transform of $t^3 \ \delta(t-3)$.

$$\boxed{\mathcal{L}\left[t^3 \ \delta(t-3)\right] = 27e^{-3s}}$$

$$\mathcal{L}\left[t^3 \delta(t-3)\right] = \int_0^\infty e^{-st} t^3 \ \delta(t-3) \, dt$$
$$= e^{-3s} \times 3^3 = 27e^{-3s}$$

Question 7

Use Laplace transforms to solve the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = \delta(t-2),$$

given further that $x = 0$, $\frac{dx}{dt} = 1$ at $t = 0$.

$\boxed{}$, $\boxed{x = \frac{1}{2}\mathrm{e}^{-t}\sin 2t + \frac{1}{2}\mathrm{e}^{2-t}\sin(2t-4)\mathrm{H}(t-2)}$

$\ddot{x} + 2\dot{x} + 5x = \delta(t-2)$ SUBJECT TO $t=0$, $x_0 = 0$, $\dot{x} = 1$

TAKING LAPLACE TRANSFORMS IN t

$$\Rightarrow \mathcal{L}[\ddot{x} + 2\dot{x} + 5x] = \mathcal{L}[\delta(t-2)]$$

$$\Rightarrow (s^2\bar{x} - sx_0 - \dot{x}_0) + 2(s\bar{x} - x_0) + 5\bar{x} = e^{-2s}$$

$$\Rightarrow s^2\bar{x} - \cancel{sx_0} - \dot{x}_0 + 2s\bar{x} - \cancel{2x_0} + 5\bar{x} = e^{-2s} \qquad (x_0 = 0, \dot{x}_0 = 1)$$

$$\Rightarrow \bar{x}[s^2 + 2s + 5] - 1 = e^{-2s}$$

$$\Rightarrow \bar{x} = \frac{1 + e^{-2s}}{s^2 + 2s + 5}$$

$$\Rightarrow \bar{x} = \frac{1 + e^{-2s}}{(s+1)^2 + 4}$$

$$\Rightarrow \bar{x} = \frac{1}{(s+1)^2 + 2^2} + \frac{e^{-2s}}{(s+1)^2 + 2^2}$$

INVERTING THE ABOVE

$$\Rightarrow \bar{x} = \frac{1}{2}\left[\frac{2}{(s+1)^2+2^2} + \frac{2e^{-2s}}{(s+1)^2+2^2}\right] = \frac{1}{2}\left[\frac{2}{(s+1)^2+2^2} + e^{-2s} \times \frac{2}{(s+1)^2+2^2}\right]$$

$$\Rightarrow x = \frac{1}{2}e^{-t}\sin 2t + e^{-(t-2)}H(t-2)\sin[2(t-2)]$$

$$\Rightarrow x = \frac{1}{2}e^{-t}\sin 2t + \frac{1}{2}e^{-t+2}H(t-2)\sin(2t-4)$$

$$\mathcal{L}[H(t-c)F(t-c)] = e^{-cs}f(s) \qquad \text{WITH } f(s) = \mathcal{L}[F(t)]$$

$$e^{-2s} \times \frac{2}{(s+1)^2+4} \qquad \frac{2}{(s+1)^2+2^2} = \mathcal{L}[e^{-t}\sin 2t]$$

$$\mathcal{L}[H(t-2)F(t-2)] = e^{-2s} \times \frac{2}{(s+1)^2+2^2}$$

$$H(t-2)e^{-(t-2)}\sin[2(t-2)] = e^{-2s} \times \frac{2}{(s+1)^2+2^2}$$

Question 8

Use Laplace transforms to solve the differential equation

$$\frac{d^2x}{dt^2}+4\frac{dx}{dt}+3x=2\delta(t-6),$$

given further that $x=0$, $\dfrac{dx}{dt}=2$ at $t=0$.

$\boxed{}$, $\boxed{x=\mathrm{e}^{-t}-\mathrm{e}^{-3t}+\mathrm{e}^{6-t}\,\mathrm{H}(t-6)+\mathrm{e}^{18-3t}\,\mathrm{H}(t-6)}$

$\ddot{x}+4\dot{x}+3x=2\delta(t-6)$ SUBJECT TO $t=0, x_0=0, \dot{x}=2$

$\Rightarrow \mathcal{L}[\ddot{x}+4\dot{x}+3x]=\mathcal{L}[2\delta(t-6)]$

$\Rightarrow (s^2\bar{x}-\cancel{sx_0}-\dot{x}_0)+4(s\bar{x}-\cancel{x_0})+3\bar{x}=2e^{-6s}$

$\Rightarrow s^2\bar{x}-4s\bar{x}+3\bar{x}-2=2e^{-6s}$

$\Rightarrow \bar{x}[s^2-4s+3]=2+2e^{-6s}$

$\Rightarrow \bar{x}=\dfrac{2(1+e^{-6s})}{s^2+4s+3}=2(1+e^{-6s})\times\dfrac{1}{(s+1)(s+3)}$

PROCEED BY PARTIAL FRACTIONS

$\Rightarrow \bar{x}=2(1+e^{-6s})\times\left[\dfrac{\frac{1}{2}}{s+1}-\dfrac{\frac{1}{2}}{s+3}\right]$

$\Rightarrow \bar{x}=\dfrac{1+e^{-6s}}{s+1}-\dfrac{1-e^{-6s}}{s+3}$

$\Rightarrow \bar{x}=\dfrac{1}{s+1}-\dfrac{1}{s+3}+\dfrac{e^{-6s}}{s+1}+\dfrac{e^{-6s}}{s+3}$

INVERTING ABOVE

$x(t)=e^{-t}-e^{-3t}+e^{-(t-6)}H(t-6)+e^{-3(t-6)}H(t-6)$

$x(t)=e^{-t}-e^{-3t}+e^{6-t}H(t-6)+e^{18-3t}H(t-6)$

OR EQUIVALENT

Question 9

The function f is defined as

$$f(x) = \frac{1}{\pi}\left[\frac{\varepsilon}{\varepsilon^2 + x^2}\right],$$

where ε is a positive parameter.

a) Show that $\lim\limits_{\varepsilon \to 0}\left[f(x)\right] = \delta(x)$.

The function g is defined as

$$g(x) = \frac{\lambda}{\sqrt{\pi}}\mathrm{e}^{-\lambda^2 x^2},$$

where λ is a positive parameter.

b) Show that $\lim\limits_{\lambda \to \infty}\left[g(x)\right] = \delta(x)$.

proof

a) $\boxed{f(x) = \frac{1}{\pi}\left(\frac{\varepsilon}{\varepsilon^2+x^2}\right)}$

• IF $x=0$

$$\lim_{\varepsilon \to 0}\left[\frac{1}{\pi}\frac{\varepsilon}{\varepsilon^2+\cancel{x^2}}\right] = \lim_{\varepsilon \to 0}\left[\frac{1}{\pi}\times\frac{1}{\varepsilon}\right] = \infty$$

• IF $x \neq 0$

$$\lim_{\varepsilon \to 0}\left[\frac{1}{\pi}\frac{\varepsilon}{\cancel{\varepsilon^2}+x^2}\right] = \lim_{\varepsilon \to 0}\left[\frac{\varepsilon}{\pi x^2}\right] = 0$$

• $$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty}\frac{\varepsilon}{\pi}\frac{1}{\varepsilon^2+x^2}\,dx = \frac{\varepsilon}{\pi}\int_{-\infty}^{\infty}\frac{1}{\varepsilon^2+x^2}\,dx$$

$$= \frac{\cancel{\varepsilon}}{\pi}\times\frac{1}{\cancel{\varepsilon}}\left[\arctan\left(\frac{x}{\varepsilon}\right)\right]_{-\infty}^{\infty} = \frac{1}{\pi}\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right]$$

$$= 1$$

$\therefore$ $f(x)$ IS AN "INFINITE HEIGHT" SPIKE AT $x=0$, WITH AREA 1

$$\therefore\ \delta(x) = \lim_{\varepsilon \to 0}\left[\frac{1}{\pi}\frac{\varepsilon}{\varepsilon^2+x^2}\right]$$

b) $\boxed{g(x) = \frac{\lambda}{\sqrt{\pi}}e^{-\lambda^2 x^2}}$

• IF $x=0$

$$\lim_{\lambda \to \infty}\left[\frac{\lambda}{\sqrt{\pi}}\right] = \infty$$

• IF $x \neq 0$

$$\lim_{\lambda \to \infty}\left[\frac{\lambda}{\sqrt{\pi}}e^{-\lambda^2 x^2}\right] = 0 \quad \text{SINCE } e^{-\lambda^2x^2} \to 0 \text{ FASTER THAN } \frac{\lambda}{\sqrt{\pi}} \to \infty$$

• $$\int_{-\infty}^{\infty} g(x)\,dx = \int_{-\infty}^{\infty}\frac{\lambda}{\sqrt{\pi}}e^{-\lambda^2x^2}\,dx = \frac{\lambda}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-(\lambda x)^2}\,dx$$

BY SUBSTITUTION

$u = \lambda x$

$\frac{du}{dx} = \lambda$

$dx = \frac{du}{\lambda}$

LIMITS UNCHANGED

$$= \frac{\lambda}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-u^2}\,\frac{du}{\lambda}$$

$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-u^2}\,du$$

$$= \frac{1}{\sqrt{\pi}}\times\sqrt{\pi}$$

$$= 1$$

$\therefore$ $g(x)$ IS A SPIKE OF INFINITE HEIGHT IF $x=0$, WITH AREA 1

$$\therefore\ \delta(x) = \lim_{\lambda \to \infty}\left[\frac{\lambda}{\sqrt{\pi}}e^{-\lambda^2x^2}\right]$$

Question 10

The impulse function $\delta(x)$ is defined by

$$\delta(x)=\begin{cases}\infty & x=0\\ 0 & x\neq 0\end{cases}$$

a) Determine

i. … $\mathcal{F}\left[\delta(x)\right]$.

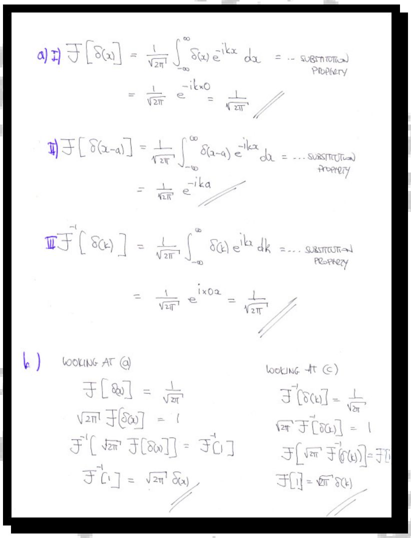
ii. … $\mathcal{F}\left[\delta(x-a)\right]$, where a is a positive constant.

iii. … $\mathcal{F}^{-1}\left[\delta(k)\right]$.

b) Use the above results to deduce $\mathcal{F}[1]$ and $\mathcal{F}^{-1}[1]$.

$$\boxed{\mathcal{F}\left[\delta(x)\right]=\frac{1}{\sqrt{2\pi}}},\ \boxed{\mathcal{F}\left[\delta(x-a)\right]=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\mathrm{i}ka}},\ \boxed{\mathcal{F}^{-1}\left[\delta(k)\right]=\frac{1}{\sqrt{2\pi}}},$$

$$\boxed{\mathcal{F}[1]=\sqrt{2\pi}\,\delta(k)},\ \boxed{\mathcal{F}^{-1}[1]=\sqrt{2\pi}\,\delta(x)}$$



a) I) $\mathcal{F}[\delta(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x)e^{-ikx}\,dx = \ldots$ SUBSTITUTION PROPERTY

$= \frac{1}{\sqrt{2\pi}}e^{-ik\times 0} = \frac{1}{\sqrt{2\pi}}$

II) $\mathcal{F}[\delta(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x-a)e^{-ikx}\,dx = \ldots$ SUBSTITUTION PROPERTY

$= \frac{1}{\sqrt{2\pi}}e^{-ika}$

III) $\mathcal{F}^{-1}[\delta(k)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(k)e^{ikx}\,dk = \ldots$ SUBSTITUTION PROPERTY

$= \frac{1}{\sqrt{2\pi}}e^{i\times 0x} = \frac{1}{\sqrt{2\pi}}$

b) LOOKING AT (a)

$\mathcal{F}[\delta(x)] = \frac{1}{\sqrt{2\pi}}$

$\sqrt{2\pi}\,\mathcal{F}[\delta(x)] = 1$

$\mathcal{F}^{-1}[\sqrt{2\pi}\,\mathcal{F}[\delta(x)]] = \mathcal{F}^{-1}[1]$

$\mathcal{F}^{-1}[1] = \sqrt{2\pi}\,\delta(x)$

LOOKING AT (c)

$\mathcal{F}^{-1}[\delta(k)] = \frac{1}{\sqrt{2\pi}}$

$\sqrt{2\pi}\,\mathcal{F}^{-1}[\delta(k)] = 1$

$\mathcal{F}[\sqrt{2\pi}\,\mathcal{F}^{-1}(\delta(k))] = \mathcal{F}[1]$

$\mathcal{F}[1] = \sqrt{2\pi}\,\delta(k)$

Question 11

The impulse function $\delta(x)$ is defined by

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$$

a) Determine the inverse Fourier transform of the impulse function $\mathcal{F}^{-1}[\delta(k)]$, and use it to deduce the Fourier transform of $f(x) = 1$.

b) Find directly the Fourier transform of $f(x) = 1$, by introducing the converging factor $e^{-\varepsilon|x|}$ and letting $\varepsilon \to 0$.

$$\boxed{\mathcal{F}[1] = \sqrt{2\pi}\,\delta(k)}$$

a) CONSIDER THE INVERSE FOURIER TRANSFORM OF $\delta(k)$

$$\mathcal{F}^{-1}[\delta(k)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \delta(k)\, e^{ikx}\, dk \quad = \text{SUBSTITUTION PROPERTY}$$

$$= \frac{1}{\sqrt{2\pi}} e^{i0x} = \frac{1}{\sqrt{2\pi}}$$

NOW

$$\mathcal{F}^{-1}[\delta(k)] = \frac{1}{\sqrt{2\pi}}$$

$$\sqrt{2\pi}\,\mathcal{F}^{-1}[\delta(k)] = 1$$

$$\mathcal{F}\left[\sqrt{2\pi}\,\mathcal{F}^{-1}[\delta(k)]\right] = \mathcal{F}[1]$$

$$\sqrt{2\pi}\,\delta(k) = \mathcal{F}[1]$$

$$\mathcal{F}[1] = \sqrt{2\pi}\,\delta(k)$$

b)

$$\mathcal{F}[1] = \lim_{\varepsilon \to 0}\left[\mathcal{F}\left[1 \times e^{-\varepsilon|x|}\right]\right] =$$

$$= \lim_{\varepsilon \to 0}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\varepsilon|x|} e^{-ikx}\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\lim_{\varepsilon \to 0}\left[\int_0^{\infty} 2e^{-\varepsilon x}\cos kx\, dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\lim_{\varepsilon \to 0}\left[\operatorname{Re}\int_0^{\infty} e^{-\varepsilon x} e^{ikx}\, dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\lim_{\varepsilon \to 0}\left[\operatorname{Re}\left[\int_0^{\infty} e^{x(-\varepsilon+ik)}\, dx\right]\right]$$

$$= \sqrt{\frac{2}{\pi}}\lim_{\varepsilon \to 0}\left[\operatorname{Re}\left[\frac{1}{-\varepsilon+ik} e^{x(-\varepsilon+ik)}\right]_0^{\infty}\right]$$

$$= \sqrt{\frac{2}{\pi}}\lim_{\varepsilon \to 0}\left[\operatorname{Re}\left[\frac{-\varepsilon-ik}{\varepsilon^2+k^2} e^{-\varepsilon x}(\cos kx + i\sin kx)\right]_0^{\infty}\right]$$

$$= \sqrt{\frac{2}{\pi}}\lim_{\varepsilon \to 0}\left[\operatorname{Re}\left[\frac{-\varepsilon-ik}{\varepsilon^2+k^2}(0-1)\right]\right]$$

$$= \sqrt{\frac{2}{\pi}}\lim_{\varepsilon \to 0}\left[\operatorname{Re}\frac{\varepsilon+ik}{\varepsilon^2+k^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\lim_{\varepsilon \to 0}\left[\frac{\varepsilon}{\varepsilon^2+k^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \times \pi \times \frac{1}{\pi}\lim_{\varepsilon \to 0}\left[\frac{\varepsilon}{\varepsilon^2+k^2}\right]$$

$$= \sqrt{2\pi}\,\delta(k)$$

NOTE $\delta(x) = \frac{1}{\pi}\lim_{\varepsilon \to 0}\left[\frac{\varepsilon}{x^2+\varepsilon^2}\right]$