INDEX SUMMATION NOTATION

Question 1

Use index summation notation to prove the validity of the following vector identity

$$\nabla \cdot (\varphi \mathbf{A}) \equiv \nabla \varphi \cdot \mathbf{A} + \varphi (\nabla \cdot \mathbf{A}),$$

where $\varphi = \varphi(x, y, z)$ is a smooth scalar function and $\mathbf{A} = \mathbf{A}(x, y, z)$ is a smooth vector function.

$\boxed{\text{proof}}$

$$\nabla \cdot (\phi A) = \frac{\partial}{\partial x_i}(\phi A_i)$$

BY THE PRODUCT RULE

$$= \frac{\partial \phi}{\partial x_i} A_i + \phi \frac{\partial A_i}{\partial x_i} = \nabla \phi \cdot A + \phi \nabla \cdot A$$

$$= A \cdot \nabla \phi + \phi \nabla \cdot A$$

Question 2

Use index summation notation to prove the validity of the following vector identity

$$\nabla \wedge (\varphi \mathbf{A}) \equiv \nabla \varphi \wedge \mathbf{A} + \varphi (\nabla \wedge \mathbf{A}),$$

where $\varphi = \varphi(x, y, z)$ is a smooth scalar function and $\mathbf{A} = \mathbf{A}(x, y, z)$ is a smooth vector function.

$\boxed{}$, $\boxed{\text{proof}}$

CONSIDER THE k^{TH} COMPONENT OF $\nabla \wedge (\phi A)$

$$[\nabla \wedge (\phi A)]_k = \varepsilon_{ijk} \frac{\partial}{\partial x_i}(\phi A_j) \qquad \boxed{(A \wedge B)_k = \varepsilon_{ijk} A_i B_j}$$

APPLYING THE PRODUCT RULE WITH SCALARS

$$\ldots = \varepsilon_{ijk}\left[\frac{\partial \phi}{\partial x_i} A_j + \phi \frac{\partial A_j}{\partial x_i}\right]$$

$$= \varepsilon_{ijk} \frac{\partial \phi}{\partial x_i} A_j + \phi\, \varepsilon_{ijk} \frac{\partial A_j}{\partial x_i}$$

$$= \varepsilon_{ijk} \frac{\partial \phi}{\partial x_i} A_j + \phi\, \varepsilon_{ijk} \frac{\partial}{\partial x_i} A_j$$

BACK INTO "CROSS PRODUCTS"

$$= [\nabla \phi \wedge A]_k + \phi [\nabla \wedge A]_k$$

$$= \nabla \phi \wedge A + \phi \nabla \wedge A$$

AS REQUIRED

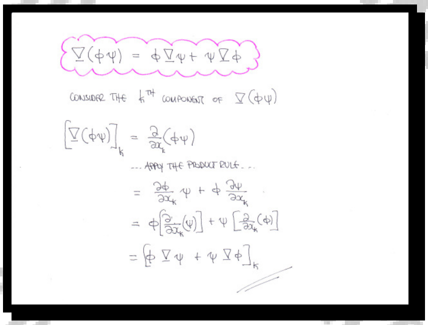
Question 3

Use index summation notation to prove the validity of the following vector identity

$$\nabla(\varphi\psi) \equiv \varphi\nabla\psi + \psi\nabla\varphi,$$

where $\varphi = \varphi(x, y, z)$ and $\psi = \psi(x, y, z)$ are smooth scalar functions.

proof



$$\underline{\nabla}(\phi\psi) = \phi\underline{\nabla}\psi + \psi\underline{\nabla}\phi$$

CONSIDER THE k^{TH} COMPONENT OF $\underline{\nabla}(\phi\psi)$

$$\left[\underline{\nabla}(\phi\psi)\right]_k = \frac{\partial}{\partial x_k}(\phi\psi)$$

... APPLY THE PRODUCT RULE ...

$$= \frac{\partial\phi}{\partial x_k}\psi + \phi\frac{\partial\psi}{\partial x_k}$$

$$= \phi\left[\frac{\partial}{\partial x_k}(\psi)\right] + \psi\left[\frac{\partial}{\partial x_k}(\phi)\right]$$

$$= \left[\phi\underline{\nabla}\psi + \psi\underline{\nabla}\phi\right]_k$$

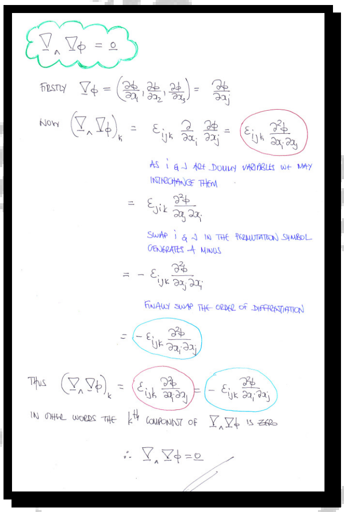
Question 4

Use index summation notation to prove the validity of the following vector identity

$$\nabla \wedge \nabla\varphi \equiv \mathbf{0},$$

where $\varphi = \varphi(x, y, z)$ is a smooth scalar function.

proof



$$\underline{\nabla} \wedge \underline{\nabla}\phi = \underline{0}$$

FIRSTLY $\underline{\nabla}\phi = \left(\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}, \frac{\partial\phi}{\partial x_3}\right) = \frac{\partial\phi}{\partial x_j}$

NOW $\left(\underline{\nabla} \wedge \underline{\nabla}\phi\right)_k = \varepsilon_{ijk}\frac{\partial}{\partial x_i}\frac{\partial\phi}{\partial x_j} = \varepsilon_{ijk}\frac{\partial^2\phi}{\partial x_i\partial x_j}$

AS i & j ARE DUMMY VARIABLES WE MAY INTERCHANGE THEM

$$= \varepsilon_{jik}\frac{\partial^2\phi}{\partial x_j\partial x_i}$$

SWAP i & j IN THE PERMUTATION SYMBOL GENERATES A MINUS

$$= -\varepsilon_{ijk}\frac{\partial^2\phi}{\partial x_j\partial x_i}$$

FINALLY SWAP THE ORDER OF DIFFERENTIATION

$$= -\varepsilon_{ijk}\frac{\partial^2\phi}{\partial x_i\partial x_j}$$

THUS $\left(\underline{\nabla} \wedge \underline{\nabla}\phi\right)_k = \varepsilon_{ijk}\frac{\partial^2\phi}{\partial x_i\partial x_j} = -\varepsilon_{ijk}\frac{\partial^2\phi}{\partial x_i\partial x_j}$

IN OTHER WORDS THE k^{th} COMPONENT OF $\underline{\nabla} \wedge \underline{\nabla}\phi$ IS ZERO

$$\therefore \underline{\nabla} \wedge \underline{\nabla}\phi = \underline{0}$$

Question 5

Use index summation notation to prove the validity of the following vector identity

$$\nabla \cdot [\nabla \wedge \mathbf{F}] \equiv 0,$$

where $\mathbf{F} = \mathbf{F}(x, y, z)$ is a smooth vector function.

$\boxed{}$, $\boxed{\text{proof}}$

- FIRSTLY WE HAVE IN SUMMATION NOTATION
  - THE k^{th} COMPONENT OF $\underline{A} \wedge \underline{B}$ IS $(\underline{A} \wedge \underline{B})_k = \varepsilon_{ijk} A_i B_j$
  - THE DIVERGENCE OF A VECTOR u_k IS $\underline{\nabla} \cdot \underline{u} = \frac{\partial}{\partial x_k} u_k$
- USING THESE RESULTS WE NOW HAVE

$$\underline{\nabla} \cdot (\underline{\nabla} \wedge \underline{F}) = \frac{\partial}{\partial x_k}\left[\varepsilon_{ijk} \frac{\partial}{\partial x_i} F_j\right] = \varepsilon_{ijk} \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_i} F_j$$

$$= \varepsilon_{ijk} \frac{\partial^2 F_j}{\partial x_k \partial x_i}$$

AS i & k ARE DUMMY VARIABLES WE MAY INTERCHANGE THEM

$$= \varepsilon_{kji} \frac{\partial^2 F_j}{\partial x_i \partial x_k}$$

SWAP i & k IN THE PERMUTATION SYMBOL GENERATES A MINUS

$$= -\varepsilon_{ijk} \frac{\partial^2 F_j}{\partial x_i \partial x_k}$$

REVERSE THE DIFFERENTIATION ORDER IN THE PARTIAL DERIVATIVE

$$= -\varepsilon_{ijk} \frac{\partial^2 F_j}{\partial x_k \partial x_i}$$

$$= -\underline{\nabla} \cdot (\underline{\nabla} \wedge \underline{F})$$

- CONCLUDING THE ARGUMENT

$\underline{\nabla} \cdot (\underline{\nabla} \wedge \underline{F}) = -\underline{\nabla} \cdot (\underline{\nabla} \wedge \underline{F})$ FOR ANY SMOOTH VECTOR FIELD $\underline{B}$

$$\therefore \underline{\nabla} \cdot (\underline{\nabla} \wedge \underline{F}) = \underline{0}$$

Question 6

Use index summation notation to prove the validity of the following vector identity.

$$(\mathbf{A} \wedge \mathbf{B}) \cdot (\mathbf{C} \wedge \mathbf{D}) \equiv (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}) .$$

[illegible], proof

USING THE STANDARD INDEX NOTATION DEFINITION OF A CROSS PRODUCT, FOR ITS k^{TH} COMPONENT

$$(\underline{p} \wedge \underline{q})_k = \varepsilon_{ijk} \, p_i \, q_j$$

THUS WE HAVE

$$
\begin{aligned}
(\underline{A} \wedge \underline{B}) \cdot (\underline{C} \wedge \underline{D}) &= (\varepsilon_{ijk} A_i B_j)(\varepsilon_{lnk} C_l D_n) \\
&= \varepsilon_{ijk} \varepsilon_{lnk} A_i B_j C_l D_n \\
&= \begin{vmatrix} \delta_{il} & \delta_{in} \\ \delta_{jl} & \delta_{jn} \end{vmatrix} A_i B_j C_l D_n \\
&= [\delta_{il}\delta_{jn} - \delta_{jl}\delta_{in}] A_i B_j C_l D_n \\
&= \delta_{il}\delta_{jn} A_i B_j C_l D_n - \delta_{jl}\delta_{in} A_i B_j C_l D_n \\
&= A_l B_j C_l D_j - A_i B_l C_l D_i \\
&= A_l C_l B_j D_j - A_i \; {}_i B_l C_l \\
&= (\underline{A} \cdot \underline{C})(\underline{B} \cdot \underline{D}) - (\underline{A} \cdot \underline{D})(\underline{B} \cdot \underline{C})
\end{aligned}
$$

"STANDARD" IDENTITY

AS REQUIRED

Question 7

Use index summation notation to prove the validity of the following vector identity

$$\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) \equiv (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} .$$

$\boxed{\text{proof}}$

$$\boxed{\underline{a} \wedge (\underline{b} \wedge \underline{c}) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}}$$

CONSIDER THE kth COMPONENT OF $b \wedge c$

$$(\underline{b} \wedge \underline{c})_k = \varepsilon_{ijk} b_i c_j$$

NEXT THE CROSS PRODUCT OF THE CROSS PRODUCT

$$\varepsilon_{mkn}\, a_m \underbrace{\left[\varepsilon_{ijk} b_i c_j\right]}_{\text{FREE VARIABLE } k \text{ IN THIS TERM}} = \left[a \wedge (\underline{b} \wedge \underline{c})\right]_n$$

NEW DUMMY

NEW FREE VARIABLE

$$= \varepsilon_{mkn} \varepsilon_{ijk} a_m b_i c_j$$

$$= -\varepsilon_{mnk} \varepsilon_{ijk} a_m b_i c_j$$

$$\varepsilon_{ijk} \varepsilon_{lmk} \equiv \begin{vmatrix} \delta_{il} & \delta_{im} \\ \delta_{jl} & \delta_{jm} \end{vmatrix}$$

USE IDENTITY OPPOSITE

$$= -\begin{vmatrix} \delta_{mi} & \delta_{mj} \\ \delta_{ni} & \delta_{nj} \end{vmatrix} a_m b_i c_j$$

$$= -\left[\delta_{mi}\delta_{nj} - \delta_{ni}\delta_{mj}\right] a_m b_i c_j$$

$$= \left[\delta_{ni}\delta_{mj} - \delta_{mi}\delta_{nj}\right] a_m b_i c_j$$

$$= \delta_{ni}\delta_{mj} a_m b_i c_j - \delta_{mi}\delta_{nj} a_m b_i c_j$$

SUBSTITUTION PROPERTY

$$\delta_{ij} T_{ik} \equiv T_{jk}$$

$$= a_j b_n c_j - a_i b_i c_n$$

$$= (\underline{a} \cdot \underline{c}) b_n - (\underline{a} \cdot \underline{b}) c_n$$

$$\therefore \left[\underline{a} \wedge (\underline{b} \wedge \underline{c})\right]_n = (\underline{a} \cdot \underline{c}) b_n - (\underline{a} \cdot \underline{b}) c_n$$

Question 8

Use index summation notation to prove the validity of the following vector identity

$$\nabla \cdot [\mathbf{A} \wedge \mathbf{B}] \equiv \mathbf{B} \cdot (\nabla \wedge \mathbf{A}) - \mathbf{A} \cdot (\nabla \wedge \mathbf{B}),$$

where $\mathbf{A} = \mathbf{A}(x, y, z)$ and $\mathbf{B} = \mathbf{B}(x, y, z)$ are smooth vector functions.

$\boxed{\text{proof}}$

- FIRSTLY THE kth COMPONENT OF $\underline{A} \wedge \underline{B}$ IS

$$(\underline{A} \wedge \underline{B})_k = \varepsilon_{ijk} A_i B_j$$

- HENCE THE DIVERGENCE OF A VECTOR u_k BECOMES

$$\underline{\nabla} \cdot \underline{u} = \frac{\partial}{\partial x_k} u_k$$

- LET $u_k = (\underline{A} \wedge \underline{B})_k$

THUS

$$\underline{\nabla} \cdot (\underline{A} \wedge \underline{B}) = \frac{\partial}{\partial x_k}\left[\varepsilon_{ijk} A_i B_j\right] = \varepsilon_{ijk} \frac{\partial}{\partial x_k}\left[A_i B_j\right]$$

BY THE PRODUCT RULE WE OBTAIN

$$= \varepsilon_{ijk} \frac{\partial A_i}{\partial x_k} B_j + \varepsilon_{ijk} A_i \frac{\partial B_j}{\partial x_k}$$

$$= B_j \varepsilon_{ijk} \frac{\partial}{\partial x_k} A_i + A_i \varepsilon_{ijk} \frac{\partial}{\partial x_k} B_j$$

SWAP k & j SWAP i & k

$$= B_k \varepsilon_{ikj} \frac{\partial}{\partial x_j} A_i + A_k \varepsilon_{kji} \frac{\partial}{\partial x_i} B_j$$

SWAP i & j CHANGE THE PERMUTATION

$$= B_k \varepsilon_{jki} \frac{\partial}{\partial x_i} A_j + A_k \left(-\varepsilon_{ijk} \frac{\partial}{\partial x_i} B_j\right)$$

SAME

$$= B_k \varepsilon_{ijk} \frac{\partial}{\partial x_i} A_j - A_k \varepsilon_{ijk} \frac{\partial}{\partial x_i} B_j$$

$$= B_k (\underline{\nabla} \wedge \underline{A})_k - A_k (\underline{\nabla} \wedge \underline{B})_k$$

$$= \underline{B} \cdot (\underline{\nabla} \wedge \underline{A}) - \underline{A} \cdot (\underline{\nabla} \wedge \underline{B})$$

Question 9

Use index summation notation to prove the validity of the following vector identity

$$\nabla \cdot [\nabla f \wedge \nabla g] \equiv 0,$$

where $f = f(x, y, z)$ and $g = g(x, y, z)$ are smooth scalar functions.

Any additional results used must be clearly stated.

$\boxed{}$, $\boxed{\text{proof}}$

- START THE PROOF BY WRITING THE DOT & CROSS PRODUCT IN INDEX NOTATION

$$\nabla \cdot [\nabla f \wedge \nabla g] = \frac{\partial}{\partial x_k}\left[\varepsilon_{ijk} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}\right]$$

- APPLY THE PRODUCT RULE

$$\dots = \varepsilon_{ijk} \frac{\partial^2 f}{\partial x_k \partial x_i} \frac{\partial g}{\partial x_j} + \varepsilon_{ijk} \frac{\partial f}{\partial x_i} \frac{\partial^2 g}{\partial x_k \partial x_j}$$

- REWRITE AS FOLLOWS

$$\dots = \frac{\partial g}{\partial x_j}\left[\varepsilon_{ijk} \frac{\partial}{\partial x_k}\left(\frac{\partial f}{\partial x_i}\right)\right] + \frac{\partial f}{\partial x_i}\left[\varepsilon_{ijk} \frac{\partial}{\partial x_k}\left(\frac{\partial g}{\partial x_j}\right)\right]$$

- NOW $\varepsilon_{ijk} = \varepsilon_{kij}$ AND $\varepsilon_{ijk} = -\varepsilon_{kji}$

$$\dots = \frac{\partial g}{\partial x_j}\left[\varepsilon_{kij} \frac{\partial}{\partial x_k}\left(\frac{\partial f}{\partial x_i}\right)\right] + \frac{\partial f}{\partial x_i}\left[-\varepsilon_{kji} \frac{\partial}{\partial x_k} \frac{\partial g}{\partial x_j}\right]$$

$$= \nabla g \cdot [\nabla \wedge \nabla f] + \nabla f \cdot [-\nabla \wedge \nabla g]$$

$$= 0$$

SINCE $\nabla \wedge \nabla u = \mathbf{0}$

Question 10

Use index summation notation to prove that

$$\mathbf{A} \wedge (\mathbf{B} \wedge \mathbf{C}) \equiv (\mathbf{C} \cdot \mathbf{A})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C},$$

and hence deduce that

$$\mathbf{A} \wedge (\mathbf{B} \wedge \mathbf{C}) \equiv (\mathbf{C} \wedge \mathbf{A}) \wedge \mathbf{B} + (\mathbf{A} \wedge \mathbf{B}) \wedge \mathbf{C}$$

$\boxed{}$, $\boxed{\text{proof}}$

CONSIDER THE nth COMPONENT OF $\underline{A} \wedge (\underline{B} \wedge \underline{C})$

$$\left[\varepsilon_{\ell k n} A_\ell (\varepsilon_{ijk} B_i C_j)\right] = \varepsilon_{\ell k n}\varepsilon_{ijk} A_\ell B_i C_j$$

SWAP k & n THE FIRST PERMUTATION SYMBOL IN ORDER TO USE IDENTITY

$$\boxed{\varepsilon_{ab\alpha}\varepsilon_{cd\alpha} \equiv \begin{vmatrix} \delta_{ac} & \delta_{ad} \\ \delta_{bc} & \delta_{bd} \end{vmatrix}}$$

$$\ldots = -\varepsilon_{\ell n k}\varepsilon_{ijk} A_\ell B_i C_j$$

$$= -\begin{vmatrix} \delta_{\ell i} & \delta_{\ell j} \\ \delta_{ni} & \delta_{nj} \end{vmatrix} A_\ell B_i C_j$$

$$= \left[\delta_{ni}\delta_{\ell j} - \delta_{\ell i}\delta_{nj}\right] A_\ell B_i C_j$$

$$= \delta_{ni}\delta_{\ell j} A_\ell B_i C_j - \delta_{\ell i}\delta_{nj} A_\ell B_i C_j$$

NOW USE δ SUBSTITUTION PROPERTY IN EACH TERM

$$= A_j B_n C_j - A_i B_i C_n$$

$$= A_j C_j B_n - A_i B_i C_n$$

$$= (\underline{A} \cdot \underline{C}) B_n - (\underline{A} \cdot \underline{B}) C_n$$

$$= \left[(\underline{A} \cdot \underline{C})\underline{B} - (A \cdot B)\underline{C}\right]_n$$

$$\therefore \left[\underline{A} \wedge (\underline{B} \wedge \underline{C})\right]_n = \left[(\underline{A} \cdot \underline{C})\underline{B} - (A \cdot B)C\right]_n$$

$$\therefore \underline{A} \wedge (\underline{B} \wedge \underline{C}) = (\underline{C} \cdot \underline{A})\underline{B} - (\underline{A} \cdot \underline{B})\underline{C}$$

// AS REQUIRED

FINALLY TO DEDUCE THE 2ND IDENTITY CONSIDER THE CYCLIC EXPRESSION OF 3 TRIPLE VECTOR PRODUCTS

$$\underline{A} \wedge (\underline{B} \wedge \underline{C}) + \underline{B} \wedge (\underline{C} \wedge \underline{A}) + \underline{C} \wedge (\underline{A} \wedge \underline{B})$$

$$= \cancel{(\underline{C} \cdot \underline{A})\underline{B}} - \cancel{(\underline{A} \cdot \underline{B})\underline{C}} + \cancel{(\underline{A} \cdot \underline{B})\underline{C}} - \cancel{(\underline{B} \cdot \underline{C})\underline{A}} + \cancel{(\underline{C} \cdot \underline{B})\underline{A}} - \cancel{(\underline{C} \cdot \underline{A})\underline{B}}$$

$$= \underline{0}$$

THUS $\underline{A} \wedge (\underline{B} \wedge \underline{C}) + \underline{B} \wedge (\underline{C} \wedge \underline{A}) + \underline{C} \wedge (\underline{A} \wedge \underline{B}) = \underline{0}$

$$\underline{A} \wedge (\underline{B} \wedge \underline{C}) = -\underline{B} \wedge (\underline{C} \wedge \underline{A}) - \underline{C} \wedge (\underline{A} \wedge \underline{B})$$

$$= (\underline{C} \wedge \underline{A}) \wedge \underline{B} + (\underline{A} \wedge \underline{B}) \wedge \underline{C}$$

// AS REQUIRED

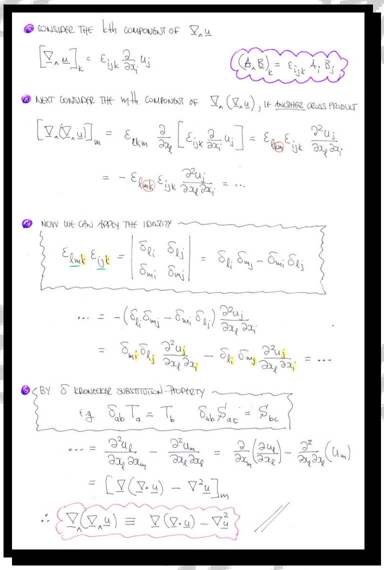
Question 11

Use index summation notation to prove the validity of the following vector identity

$$\nabla \wedge [\nabla \wedge \mathbf{u}] \equiv \nabla(\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u},$$

where $\mathbf{u} = \mathbf{u}(x, y, z)$ is a smooth vector function.

proof



CONSIDER THE kth COMPONENT OF $\nabla \wedge \underline{u}$

$$[\nabla \wedge \underline{u}]_k = \varepsilon_{ijk} \frac{\partial}{\partial x_i} u_j \qquad (\underline{A} \wedge \underline{B})_k = \varepsilon_{ijk} A_i B_j$$

NEXT CONSIDER THE mth COMPONENT OF $\nabla \wedge (\nabla \wedge \underline{u})$, IE ANOTHER CROSS PRODUCT

$$[\nabla \wedge (\nabla \wedge \underline{u})]_m = \varepsilon_{lkm} \frac{\partial}{\partial x_l}\left[\varepsilon_{ijk} \frac{\partial}{\partial x_i} u_j\right] = \varepsilon_{lkm} \varepsilon_{ijk} \frac{\partial^2 u_j}{\partial x_l \partial x_i}$$

$$= -\varepsilon_{lmk} \varepsilon_{ijk} \frac{\partial^2 u_j}{\partial x_l \partial x_i} = \dots$$

NOW WE CAN APPLY THE IDENTITY

$$\varepsilon_{lmk}\varepsilon_{ijk} = \begin{vmatrix} \delta_{li} & \delta_{lj} \\ \delta_{mi} & \delta_{mj} \end{vmatrix} = \delta_{li}\delta_{mj} - \delta_{mi}\delta_{lj}$$

$$\dots = -(\delta_{li}\delta_{mj} - \delta_{mi}\delta_{lj}) \frac{\partial^2 u_j}{\partial x_l \partial x_i}$$

$$= \delta_{mi}\delta_{lj} \frac{\partial^2 u_j}{\partial x_l \partial x_i} - \delta_{li}\delta_{mj} \frac{\partial^2 u_j}{\partial x_l \partial x_i} = \dots$$

BY δ KRONECKER SUBSTITUTION PROPERTY

$$\text{e.g. } \delta_{ab} T_a = T_b \qquad \delta_{ab} S_{ac} = S_{bc}$$

$$\dots = \frac{\partial^2 u_l}{\partial x_l \partial x_m} - \frac{\partial^2 u_m}{\partial x_l \partial x_l} = \frac{\partial}{\partial x_m}\left(\frac{\partial u_l}{\partial x_l}\right) - \frac{\partial^2}{\partial x_l \partial x_l}(u_m)$$

$$= [\nabla(\nabla \cdot \underline{u}) - \nabla^2 \underline{u}]_m$$

$$\therefore \boxed{\nabla \wedge (\nabla \wedge \underline{u}) \equiv \nabla(\nabla \cdot \underline{u}) - \nabla^2 \underline{u}}$$

Question 12

Use index summation notation to prove the validity of the following vector identity

$$\nabla \wedge [\mathbf{A} \wedge \mathbf{B}] \equiv \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B},$$

where $\mathbf{A} = \mathbf{A}(x, y, z)$ and $\mathbf{B} = \mathbf{B}(x, y, z)$ are smooth vector functions.

$\boxed{\text{proof}}$

- $(\underline{A} \wedge \underline{B})_k = \varepsilon_{ijk} A_i B_j$
- $\left[\underline{\nabla} \wedge (\underline{A} \wedge \underline{B})\right]_m = \varepsilon_{lkm} \frac{\partial}{\partial x_l}\left[\varepsilon_{ijk} A_i B_j\right] = \varepsilon_{lkm}\varepsilon_{ijk} \frac{\partial}{\partial x_l}\left[A_i B_j\right]$

(SWAP) PRODUCT RULE

$$= -\varepsilon_{lmk}\varepsilon_{ijk}\left[\frac{\partial A_i}{\partial x_l} \cdot B_j + A_i \frac{\partial B_j}{\partial x_l}\right]$$

$$\boxed{\varepsilon_{lmk}\varepsilon_{ijk} = \begin{vmatrix} \delta_{li} & \delta_{lj} \\ \delta_{mi} & \delta_{mj} \end{vmatrix} = \delta_{li}\delta_{mj} - \delta_{mi}\delta_{lj}}$$

$$= (\delta_{mi}\delta_{lj} - \delta_{li}\delta_{mj})\left(\frac{\partial A_i}{\partial x_l} B_j + A_i \frac{\partial B_j}{\partial x_l}\right)$$

$$= \delta_{mi}\delta_{lj}\frac{\partial A_i}{\partial x_l} B_j + \delta_{mi}\delta_{lj} A_i \frac{\partial B_j}{\partial x_l} - \delta_{li}\delta_{mj}\frac{\partial A_i}{\partial x_l} B_j - \delta_{li}\delta_{mj} A_i \frac{\partial B_j}{\partial x_l}$$

USING THE KRONECKER SUBSTITUTION PROPERTY

$$\boxed{\text{e.g. } \delta_{ab} T_b = T_a \qquad \delta_{ac} W_{cde} = W_{ade}}$$

$$= \frac{\partial A_m}{\partial x_l} B_l + A_m \frac{\partial B_l}{\partial x_l} - \frac{\partial A_l}{\partial x_l} B_m - A_l \frac{\partial B_m}{\partial x_l}$$

$$= B_l \frac{\partial}{\partial x_l} A_m + A_m \frac{\partial B_l}{\partial x_l} - B_m \frac{\partial A_l}{\partial x_l} - A_l \frac{\partial}{\partial x_l} B_m$$

$$= \underline{(\underline{B} \cdot \nabla) A_m} + \underline{A_m (\nabla \cdot \underline{B})} - \underline{B_m (\nabla \cdot \underline{A})} - \underline{(\underline{A} \cdot \nabla) B_m}$$

THUS

$$\underline{\nabla} \wedge (\underline{A} \wedge \underline{B}) = \underline{A}(\underline{\nabla} \cdot \underline{B}) - \underline{B}(\underline{\nabla} \cdot \underline{A}) + (\underline{B} \cdot \underline{\nabla})\underline{A} - (\underline{A} \cdot \underline{\nabla})\underline{B}$$

Question 13

Use index summation notation to prove the validity of the following vector identity

$$\nabla\left[\mathbf{A}\cdot\mathbf{B}\right] \equiv \left(\mathbf{B}\cdot\nabla\right)\mathbf{A} + \left(\mathbf{A}\cdot\nabla\right)\mathbf{B} + \mathbf{B}\wedge\left(\nabla\wedge\mathbf{A}\right) + \mathbf{A}\wedge\left(\nabla\wedge\mathbf{B}\right),$$

where $\mathbf{A}=\mathbf{A}(x,y,z)$ and $\mathbf{B}=\mathbf{B}(x,y,z)$ are smooth vector functions.

proof

START FROM THE R.H.S – AFTER REGROUPING THE TERMS CONSIDER THE n^{th} COMPONENT OF THE R.H.S

$$\left[(\underline{B}\cdot\underline{\nabla})\underline{A} + (\underline{A}\cdot\underline{\nabla})\underline{B} + \underline{B}\wedge(\underline{\nabla}\wedge\underline{A}) + \underline{A}\wedge(\underline{\nabla}\wedge\underline{B})\right]_n$$

$$= \left[\underline{B}\wedge(\underline{\nabla}\wedge\underline{A}) + \underline{A}\wedge(\underline{\nabla}\wedge\underline{B}) + (\underline{B}\cdot\underline{\nabla})\underline{A} + (\underline{A}\cdot\underline{\nabla})\underline{B}\right]_n$$

$$= \varepsilon_{lkn}B_l\left[\varepsilon_{ijk}\frac{\partial}{\partial x_i}A_j\right] + \varepsilon_{lkn}A_l\left[\varepsilon_{ijk}\frac{\partial}{\partial x_i}B_j\right] + \left[B_i\frac{\partial}{\partial x_i}\right]A_n + \left[A_i\frac{\partial}{\partial x_i}\right]B_n$$

$$= \varepsilon_{lkn}\varepsilon_{ijk}B_l\frac{\partial A_j}{\partial x_i} + \varepsilon_{lkn}\varepsilon_{ijk}A_l\frac{\partial B_j}{\partial x_i} + B_i\frac{\partial A_n}{\partial x_i} + A_i\frac{\partial B_n}{\partial x_i}$$

$$= -\varepsilon_{lnk}\varepsilon_{ijk}B_l\frac{\partial A_j}{\partial x_i} - \varepsilon_{lnk}\varepsilon_{ijk}A_l\frac{\partial B_j}{\partial x_i} + B_i\frac{\partial A_n}{\partial x_i} + A_i\frac{\partial B_n}{\partial x_i}$$

NOW USING THE IDENTITY

$$\varepsilon_{abc}\varepsilon_{dec} = \begin{vmatrix}\delta_{ad} & \delta_{ae}\\ \delta_{bd} & \delta_{be}\end{vmatrix} = \delta_{ad}\delta_{be} - \delta_{bd}\delta_{ae}$$

$$-\varepsilon_{lnk}\varepsilon_{ijk} = -\begin{vmatrix}\delta_{li} & \delta_{lj}\\ \delta_{ni} & \delta_{nj}\end{vmatrix} = \delta_{ni}\delta_{lj} - \delta_{li}\delta_{nj}$$

RETURNING TO THE "MAIN LINE"

$$= \left(\delta_{ni}\delta_{lj} - \delta_{li}\delta_{nj}\right)B_l\frac{\partial A_j}{\partial x_i} + \left(\delta_{ni}\delta_{lj} - \delta_{li}\delta_{nj}\right)A_l\frac{\partial B_j}{\partial x_i} + B_i\frac{\partial A_n}{\partial x_i} + A_i\frac{\partial B_n}{\partial x_i}$$

EXPAND & USE "DELTA" SUBSTITUTION PROPERTY

$$= \delta_{ni}\delta_{lj}B_l\frac{\partial A_j}{\partial x_i} - \delta_{li}\delta_{nj}B_l\frac{\partial A_j}{\partial x_i} + \delta_{ni}\delta_{lj}A_l\frac{\partial B_j}{\partial x_i} - \delta_{li}\delta_{nj}A_l\frac{\partial B_j}{\partial x_i} + B_i\frac{\partial A_n}{\partial x_i} + A_i\frac{\partial B_n}{\partial x_i}$$

$$= B_j\frac{\partial A_j}{\partial x_n} - \cancel{B_l\frac{\partial A_n}{\partial x_l}} + A_j\frac{\partial B_j}{\partial x_n} - \cancel{A_l\frac{\partial B_n}{\partial x_l}} + \cancel{B_i\frac{\partial A_n}{\partial x_i}} + \cancel{A_i\frac{\partial B_n}{\partial x_i}}$$

$$= B_j\frac{\partial A_j}{\partial x_n} + A_j\frac{\partial B_j}{\partial x_n}$$

$$= \frac{\partial}{\partial x_n}\left[A_jB_j\right]$$

$$= \left[\underline{\nabla}(\underline{A}\cdot\underline{B})\right]_n$$

$$\therefore\ \underline{\nabla}(\underline{A}\cdot\underline{B}) \equiv (\underline{B}\cdot\underline{\nabla})\underline{A} + (\underline{A}\cdot\underline{\nabla})\underline{B} + \underline{B}\wedge(\underline{\nabla}\wedge\underline{A}) + \underline{A}\wedge(\underline{\nabla}\wedge\underline{B})$$

Question 14

Use index summation notation to prove the validity of the following vector identity

$$\nabla \cdot \left[\mathbf{c} \wedge (\mathbf{r} \wedge \mathbf{c}) \right] \equiv 2c^2,$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{c}$ is a constant three dimensional vector and $c \equiv |\mathbf{c}|$.

$\boxed{}$, $\boxed{\text{proof}}$

STARTING FROM $(A \wedge B)_k = \varepsilon_{ijk} A_i B_j$

$$\Rightarrow (\underline{r} \wedge \underline{c})_k = \varepsilon_{ijk} r_i c_j = \varepsilon_{ijk} x_i c_j$$

SINCE $\underline{r} = \underline{r}_i = x_i$

$$\Rightarrow [\underline{c} \wedge (\underline{r} \wedge \underline{c})]_m = \varepsilon_{lkm} c_l [\varepsilon_{ijk} x_i c_j]$$

$$\Rightarrow [\underline{c} \wedge (\underline{r} \wedge \underline{c})]_m = \varepsilon_{lkm} \varepsilon_{ijk} c_l c_j x_i$$

$$\Rightarrow \underline{\nabla} \cdot [\underline{c} \wedge (\underline{r} \wedge \underline{c})] = \frac{\partial}{\partial x_m} [\varepsilon_{lkm} \varepsilon_{ijk} c_l c_j x_i]$$

$$= \varepsilon_{lkm} \varepsilon_{ijk} c_l c_j \frac{\partial x_i}{\partial x_m}$$

NOW $\frac{\partial x_i}{\partial x_m} \equiv \delta_{im}$

$$= \varepsilon_{lkm} \varepsilon_{ijk} c_l c_j \delta_{im}$$

USING THE δ SUBSTITUTION PROPERTY

$$= \varepsilon_{lki} \varepsilon_{ijk} c_l c_j$$

NEXT USING THE PERMUTATION TO δ IDENTITY

$$\varepsilon_{abc} \varepsilon_{ade} = \delta_{bd} \delta_{ce} - \delta_{be} \delta_{cd}$$

$$\Rightarrow \varepsilon_{lki} \varepsilon_{ijk} = (-\varepsilon_{kli})(-\varepsilon_{kji})$$

$$\Rightarrow \varepsilon_{lki} \varepsilon_{ijk} = \varepsilon_{kli} \varepsilon_{kji}$$

$$\Rightarrow \varepsilon_{lki} \varepsilon_{ijk} = \delta_{lj} \delta_{ii} - \delta_{li} \delta_{ij}$$

NOW FOR 3 DIMENSIONS $\delta_{ii} = 3$

$$\Rightarrow \varepsilon_{lki} \varepsilon_{ijk} = 3\delta_{lj} - \delta_{li} \delta_{ij}$$

RETURNING TO THE MAIN PROOF

$$\underline{\nabla} \cdot [\underline{c} \wedge (\underline{r} \wedge \underline{c})] = \varepsilon_{lki} \varepsilon_{ijk} c_l c_j$$

$$= (3\delta_{lj} - \delta_{li} \delta_{ij}) c_l c_j$$

$$= 3\delta_{lj} c_l c_j - \delta_{li} \delta_{ij} c_l c_j$$

BY THE δ SUBSTITUTION PROPERTY

$$= 3\delta_{lj} c_l c_j - \delta_{li} \delta_{ij} c_l c_j$$

$$= 3 c_j c_j - \delta_{lj} c_l c_j$$

$$= 3 c_j c_j - c_j c_j$$

$$= 2 c_j c_j$$

$$= 2 \underline{c} \cdot \underline{c}$$

$$= 2 |\underline{c}|^2$$

$$= 2c^2$$

Question 15

Use index summation notation to prove the validity of the following vector identity

$$\nabla \cdot \left[\mathbf{r} \wedge (\mathbf{r} \wedge \mathbf{c}) \right] \equiv 2\mathbf{r} \cdot \mathbf{c} \, ,$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{c}$ is a constant three dimensional vector.

proof

STARTING FROM $(\underline{A} \wedge \underline{B})_k = \varepsilon_{ijk} A_i B_j$

$\Rightarrow (\underline{r} \wedge \underline{c})_k = \varepsilon_{ijk} x_i c_j$ $\quad$ {SINCE $\underline{r} = r_i = x_i$}

$\Rightarrow [\underline{r} \wedge (\underline{r} \wedge \underline{c})]_m = \varepsilon_{lkm} x_l [\varepsilon_{ijk} x_i c_j]$

$\Rightarrow [\underline{r} \wedge (\underline{r} \wedge \underline{c})]_m = \varepsilon_{lkm} \varepsilon_{ijk} x_l x_i c_j$

$\Rightarrow \nabla \cdot [\underline{r} \wedge (\underline{r} \wedge \underline{c})] = \frac{\partial}{\partial x_m} [\varepsilon_{lkm} \varepsilon_{ijk} x_l x_i c_j]$

$= \varepsilon_{lkm} \varepsilon_{ijk} c_j \frac{\partial}{\partial x_m} [x_l x_i]$

$= c_j \varepsilon_{lkm} \varepsilon_{ijk} \frac{\partial}{\partial x_m} [x_l x_i]$

NOW IN ORDER TO USE THE "PERMUTATION – KRONECKER" IDENTITY WHICH STATES THAT

$\varepsilon_{ijk} \varepsilon_{imn} \equiv \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}$

$\Rightarrow \varepsilon_{lkm} \varepsilon_{ijk} = (-\varepsilon_{klm})(-\varepsilon_{kji})$

$\Rightarrow \varepsilon_{lkm} \varepsilon_{ijk} = \varepsilon_{klm} \varepsilon_{kji}$

$\Rightarrow \varepsilon_{lkm} \varepsilon_{ijk} = \delta_{lj} \delta_{mi} - \delta_{li} \delta_{mj}$

RETURNING TO THE "MAIN LINE" OF THE PROOF & USING "δ SUB PROPERTY"

$\Rightarrow \nabla \cdot [\underline{r} \wedge (\underline{r} \wedge \underline{c})] = c_j [\delta_{lj} \delta_{mi} - \delta_{li} \delta_{mj}] \frac{\partial}{\partial x_m} (x_l x_i)$

$= c_j \left[\delta_{lj} \delta_{mi} \frac{\partial}{\partial x_m} (x_l x_i) - \delta_{li} \delta_{mj} \frac{\partial}{\partial x_m} (x_l x_i) \right]$

$= c_j \left[\frac{\partial}{\partial x_i} (x_j x_i) - \frac{\partial}{\partial x_j} (x_i x_i) \right]$

$\nabla \cdot [\underline{r} \wedge (\underline{r} \wedge \underline{c})] = c_j \left[x_j \frac{\partial x_i}{\partial x_i} + x_i \frac{\partial x_j}{\partial x_i} - x_i \frac{\partial x_i}{\partial x_j} - x_i \frac{\partial x_i}{\partial x_j} \right]$

NOW $\frac{\partial x_i}{\partial x_i} = \delta_{ii}$ & $\frac{\partial x_j}{\partial x_i} = \delta_{ij}$ WHICH FOR THREE DIMENSIONAL VECTORS $\delta_{ii} = 3$

$\nabla \cdot [\underline{r} \wedge (\underline{r} \wedge \underline{c})] = c_j [x_j \times 3 + x_i \delta_{ij} - x_i \delta_{ij} - x_i \delta_{ij}]$

$= c_j [3x_j - x_i \delta_{ij}]$ $\quad$ "δ SUB PROPERTY"

$= c_j [3x_j - x_j]$

$= c_j \times 2x_j$

$= 2c_j x_j$

$= 2\underline{c} \cdot \underline{r}$

AS REQUIRED

Question 16

The vector field $\mathbf{F}$ exists around the open surface S, with closed boundary C.

a) State Stokes' Integral Theorem for open surfaces, fully defining all the quantities involved.

Let $\mathbf{c}$ be a constant vector and $\varphi = \varphi(x, y, z)$ a smooth scalar function.

b) By considering $\nabla \wedge (\mathbf{c}\varphi)$, use index summation notation in Stokes' Theorem to prove the validity of the following result

$$\int_S \hat{\mathbf{n}} \wedge \nabla \varphi \, dS = \oint_C \varphi \, d\mathbf{r} \, ,$$

where $\hat{\mathbf{n}}$ is a unit normal vector field to S.

proof

a) $$\boxed{\int_S \nabla \wedge \mathbf{F} \cdot \hat{\mathbf{n}} \, dS = \oint_C \mathbf{F} \cdot d\mathbf{r}}$$

WHERE $\mathbf{F}$ IS A SMOOTH VECTOR FIELD
S IS AN OPEN TWO SIDED SURFACE WITH BOUNDARY C
$\hat{\mathbf{n}}$ IS A UNIT VECTOR FIELD TO S, SO THAT THE DIRECTION OF $\hat{\mathbf{n}}$ AND THE DIRECTION OF C FORM A RIGHT HAND SET

b) LET $\mathbf{F} = \phi \mathbf{c}$ WHERE $\phi = \phi(x,y,z)$ & $\mathbf{c}$ IS A CONSTANT VECTOR

$$\Rightarrow \int_S \nabla \wedge (\phi \mathbf{c}) \cdot \hat{\mathbf{n}} \, dS = \oint_C \phi \mathbf{c} \cdot d\mathbf{r}$$

$$\Rightarrow \int_S \varepsilon_{ijk} \frac{\partial}{\partial x_i}(\phi c_j) n_k \, dS = \oint_C \phi c_j \, dx_j$$

$$\Rightarrow \int_S \varepsilon_{ijk} \left[c_j \frac{\partial \phi}{\partial x_i} \right] n_k \, dS = \oint_C \phi c_j \, dx_j$$

$$\Rightarrow \int_S c_j \varepsilon_{ijk} \frac{\partial \phi}{\partial x_i} n_k \, dS = \oint_C \phi c_j \, dx_j$$

$$\Rightarrow c_j \int_S \varepsilon_{ijk} n_k \frac{\partial \phi}{\partial x_i} \, dS = c_j \oint_C \phi \, dx_j$$

$$\Rightarrow c_j \int_S -\varepsilon_{kji} n_k \frac{\partial \phi}{\partial x_i} \, dS = c_j \oint_C \phi \, dx_j$$

$$\Rightarrow \cancel{c_j} \int_S +\varepsilon_{kij} n_k \frac{\partial \phi}{\partial x_i} \, dS = \cancel{c_j} \oint_C \phi \, dx_j$$

$$\Rightarrow \int_S \varepsilon_{kij} n_k \frac{\partial}{\partial x_i} \phi \, dS = \oint_C \phi \, dx_j$$

$$\Rightarrow \int_S \hat{\mathbf{n}} \wedge \nabla \phi \, dS = \oint_C \phi \, d\mathbf{r}$$

AS REQUIRED

Question 17

The vector field $\mathbf{F}$ exists around the open surface S, with closed boundary C.

Let $\mathbf{c}$ be a constant vector and $\mathbf{A} = \mathbf{A}(x, y, z)$ a smooth vector function.

By considering $\mathbf{F} = \mathbf{c} \wedge \mathbf{A}$, use index summation notation in Stokes' Theorem to prove the validity of the following result

$$\int_S (\mathbf{dS} \wedge \nabla) \wedge \mathbf{A} \; = \oint_C d\mathbf{r} \wedge \mathbf{A},$$

where $\mathbf{dS} = \hat{\mathbf{n}}\, dS$, $\hat{\mathbf{n}}$ is a unit normal vector field to S, forming a right hand set with the direction of C.

proof

$$\boxed{\int_S \nabla \wedge \mathbf{F} \cdot \hat{\mathbf{n}} \, dS = \oint_C \mathbf{F} \cdot d\mathbf{r}}$$

LET $\mathbf{F} = \mathbf{c} \wedge \mathbf{A}$ WHERE $\mathbf{A}$ IS A SMOOTH VECTOR FIELD & $\mathbf{c}$ = CONSTANT

$$\Rightarrow \int_S \nabla \wedge (\mathbf{c} \wedge \mathbf{A}) \cdot \hat{\mathbf{n}} \, dS = \oint_C \mathbf{c} \wedge \mathbf{A} \cdot d\mathbf{r}$$

$$\Rightarrow \int_S \varepsilon_{lkm} \frac{\partial}{\partial x_l}\left(\varepsilon_{ijk} c_i A_j\right) n_m \, dS = \oint_C e_{ijk} c_i A_j \, dx_k$$

$$\Rightarrow c_i \int_S n_m \varepsilon_{lkm} \varepsilon_{ijk} \frac{\partial A_j}{\partial x_l} \, dS = c_i \oint_C \varepsilon_{ijk} A_j \, dx_k$$

(SWAP i & k IN PERMUTATION)

$$\Rightarrow c_i \int_S \left(\varepsilon_{lkm} n_m \frac{\partial}{\partial x_l}\right) \varepsilon_{ijk} A_j \, dS = c_i \oint_C -\varepsilon_{kji} A_j \, dx_k$$

(SWAP m & l IN PERMUTATION)

$$\Rightarrow c_i \int_S \left(-\varepsilon_{mkl} n_m \frac{\partial}{\partial x_l}\right) \varepsilon_{ijk} A_j \, dS = -c_i \oint_C \varepsilon_{kji} \, dx_k \, A_j$$

SWAP k & l AGAIN IN PERMUTATION

$$\Rightarrow c_i \int_S \left(+\varepsilon_{mlk} n_m \frac{\partial}{\partial x_l}\right) \varepsilon_{ijk} A_j \, dS = -c_i \oint_C (d\mathbf{r} \wedge \mathbf{A})_i$$

$$\Rightarrow c_i \int_S (\hat{\mathbf{n}} \wedge \nabla)_k \varepsilon_{ijk} A_j \, dS = -c_i \oint_C (d\mathbf{r} \wedge \mathbf{A})_i$$

$$\Rightarrow c_i \int_S \varepsilon_{ijk} (\hat{\mathbf{n}} \wedge \nabla)_k A_j \, dS = -c_i \oint_C (d\mathbf{r} \wedge \mathbf{A})_i$$

(SWAP i & k IN PERMUTATION)

$$\Rightarrow c_i \int_S -\varepsilon_{kji} (\hat{\mathbf{n}} \wedge \nabla)_k A_j \, dS = -c_i \oint_C (d\mathbf{r} \wedge \mathbf{A})_i$$

$$\Rightarrow -\cancel{c_i} \int_S [(\hat{\mathbf{n}} \wedge \nabla) \wedge \mathbf{A}]_i \, dS = -\cancel{c_i} \oint_C (d\mathbf{r} \wedge \mathbf{A})_i$$

AS REQUIRED