INTEGRAL EQUATIONS

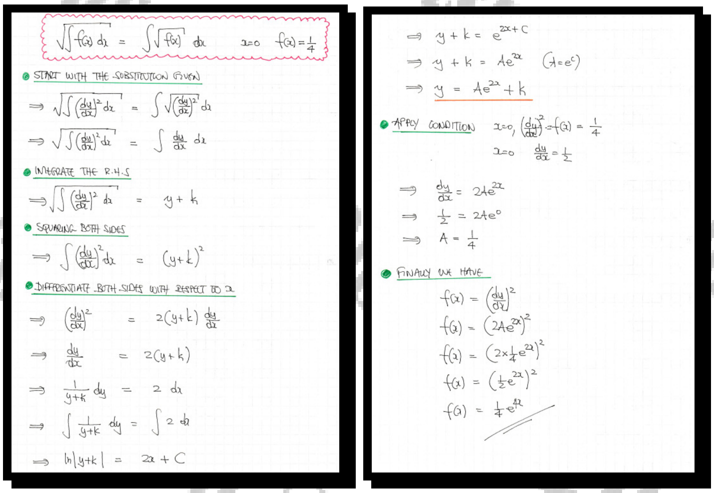
Question 1

The non zero function $f(x)$ satisfies the integral equation

$$\sqrt{\int f(x)\,dx} = \int \sqrt{f(x)}\,dx\,, \qquad f(0)=\frac{1}{4}.$$

Use the substitution $f(x)=\left(\frac{dy}{dx}\right)^2$, to find a simplified expression for $f(x)$.

$$\boxed{\quad}\,,\ \boxed{f(x)=\frac{1}{4}e^{4x}}$$



Question 2

The non zero functions $u(x)$ and $v(x)$ satisfy the integral equations

$$\int u(x)\,dx = x^2 u(x) \qquad \text{and} \qquad \int u(x)v(x)\,dx = \left[\int u(x)\,dx\right]\left[\int v(x)\,dx\right].$$

Determine, in terms of an arbitrary constant, a simplified expression for $u(x)$ and a similar expression for $\left[v(x)\right]^2$.

SP , $u(x) = \dfrac{A\mathrm{e}^{-\frac{1}{x}}}{x^2}$, $\left[v(x)\right]^2 = \dfrac{B}{(1-x)^2\left(1-x^2\right)}$

- STARTING WITH

$$\int u\,dx = ux^2$$

- DIFFERENTIATE W.R.T x

$$\Rightarrow \frac{d}{dx}\int u\,dx = \frac{d}{dx}\left(ux^2\right)$$

$$\Rightarrow u = \frac{du}{dx}x^2 + u(2x)$$

$$\Rightarrow x^2\frac{du}{dx} = u - 2ux$$

$$\Rightarrow x^2\frac{du}{dx} = u(1-2x)$$

$$\Rightarrow \frac{1}{u}\,du = \frac{1-2x}{x^2}\,dx$$

$$\Rightarrow \int \frac{1}{u}\,du = \int \frac{1}{x^2} - \frac{2}{x}\,dx$$

$$\Rightarrow \ln|u| = -\frac{1}{x} - 2\ln|x| + C$$

$$\Rightarrow u = e^{C - \frac{1}{x} + \ln x^{-2}}$$

$$\Rightarrow u = e^C \times e^{-\frac{1}{x}} \times e^{\ln\frac{1}{x^2}}$$

$$\Rightarrow u = \frac{A}{x^2}e^{-\frac{1}{x}}$$

- NEXT WE PROCEED WITH

$$\int uv\,dx = \left[\int u\,dx\right]\left[\int v\,dx\right]$$

- DIFFERENTIATING W.R.T x

$$\Rightarrow \frac{d}{dx}\int uv\,dx = \frac{d}{dx}\left[\left[\int u\,dx\right]\left[\int v\,dx\right]\right]$$

$$\Rightarrow uv = \frac{d}{dx}\left[\int u\,dx\right] \times \int v\,dx + \int u\,dx \times \frac{d}{dx}\left[\int v\,dx\right]$$

$$\Rightarrow uv = u\int v\,dx + v\int u\,dx$$

$$\Rightarrow uv = u\int v\,dx + v\left(ux^2\right)$$

$$\Rightarrow v = \int v\,dx + vx^2$$

$$\Rightarrow v - vx^2 = \int v\,dx$$

$$\Rightarrow v(1-x^2) = \int v\,dx$$

DIFFERENTIATE W.R.T x AGAIN

$$\Rightarrow \frac{dv}{dx}(1-x^2) + v(-2x) = \frac{d}{dx}\int v\,dx$$

$$\Rightarrow \frac{dv}{dx}(1-x^2) - 2vx = v$$

$$\Rightarrow \frac{dv}{dx}(1-x^2) = v + 2vx$$

$$\Rightarrow \frac{dv}{dx}(1-x^2) = v(1+2x)$$

- SEPARATING VARIABLES AND INTEGRATING

$$\Rightarrow \int \frac{1}{v}\,dv = \int \frac{2x+1}{1-x^2}\,dx$$

$$\Rightarrow \ln|v| = \int \frac{2x+1}{(1-x)(1+x)}\,dx$$

- PARTIAL FRACTIONS BY INSPECTION (COVER UP)

$$\Rightarrow \ln|v| = \int \frac{\frac{3}{2}}{1-x} - \frac{\frac{1}{2}}{1+x}\,dx$$

$$\Rightarrow 2\ln|v| = \int \frac{3}{1-x} - \frac{1}{1+x}\,dx$$

$$\Rightarrow \ln v^2 = -3\ln|1-x| - \ln|1+x| + \ln A.$$

$$\Rightarrow \ln v^2 = \ln\left|\frac{B}{(1-x)^3(1+x)}\right|$$

$$\Rightarrow v^2 = \frac{B}{(1-x)^3(1+x)}$$

OR

$$v^2 = \frac{B}{(1-x)^2(1-x^2)}$$

Question 3

The function f satisfies the following relationship.

$$f(x) = \int_{1}^{x} \left[f(t) \right]^2 \, dt \, , \quad f(2) = \tfrac{1}{2} \, .$$

Determine the value of $f\left(\tfrac{1}{2}\right)$.

$\boxed{}$, $\boxed{f\left(\tfrac{1}{2}\right) = \tfrac{2}{7}}$

DIFFERENTIATE WITH RESPECT TO x

$$f(x) = \int_{1}^{x} \left(f(t)\right)^2 \, dt$$

$$\frac{d}{dx}\left(f(x)\right) = \frac{d}{dx}\left[\int_{1}^{x} \left[f(t)\right]^2 \, dt\right]$$

BY LEIBNIZ INTEGRAL THEOREM

$$\frac{df}{dx} = \left[f(x)\right]^2 \times 1 - \left[f(1)\right]^2 \times 0$$

$$\frac{df}{dx} = f^2$$

SEPARATE VARIABLES AND INTEGRATE OVER THE LIMITS GIVEN

$$\Rightarrow \frac{1}{f^2} \, df = 1 \, dx$$

$$\Rightarrow \int_{\frac{1}{2}}^{f} \frac{1}{f^2} \, df = \int_{2}^{\frac{1}{2}} 1 \, dx$$

$$\Rightarrow \left[-\frac{1}{f}\right]_{\frac{1}{2}}^{f} = \left[x\right]_{2}^{\frac{1}{2}}$$

$$\Rightarrow -\frac{1}{f} - \left(-\frac{1}{\frac{1}{2}}\right) = \frac{1}{2} - 2$$

$$\Rightarrow -\frac{1}{f} + 2 = -\frac{3}{2}$$

$$\Rightarrow \frac{7}{2} = \frac{1}{f}$$

$$\Rightarrow f = \frac{2}{7}$$

$$\therefore f\left(\tfrac{1}{2}\right) = \frac{2}{7}$$