INTRINSIC COORDINATES

Question 1 (**)

A curve C has Cartesian equation

$$y = \arctan 2x\,,\ x \in \mathbb{R}\,.$$

Find the magnitude of the radius of curvature at the point on C where $x = \frac{1}{2}$.

$\boxed{\sqrt{2}}$

$$y = \arctan 2x$$
$$\frac{dy}{dx} = \frac{2}{1+4x^2} = 2(1+4x^2)^{-1}$$
$$\frac{d^2y}{dx^2} = -2(1+4x^2)^{-2}(8x) = -\frac{16x}{(1+4x^2)^2}$$

$$\left.\frac{dy}{dx}\right|_{x=\frac{1}{2}} = \frac{2}{1+4\left(\frac{1}{2}\right)^2} = 1$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{2}} = -\frac{16\times\frac{1}{2}}{\left[1+4\left(\frac{1}{2}\right)^2\right]^2} = -\frac{8}{4} = -2$$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{(1+1^2)^{\frac{3}{2}}}{-2} = \frac{2\sqrt{2}}{-2} = -\sqrt{2}$$

$\therefore$ MAGNITUDE IS $\sqrt{2}$

Question 2 (**)

A curve C has Cartesian equation

$$y = \cosh x\,,\ x \in \mathbb{R}\,.$$

Find an intrinsic equation of C in the form $s = f(\psi)$, where s is measured from the point with coordinates $(0,1)$, and ψ is the angle the tangent to C makes with the positive x axis.

$\boxed{s = \tan\psi}$

$$y = \cosh x$$
$$\boxed{\frac{dy}{dx} = \sinh x} \quad \& \quad \boxed{\frac{dy}{dx} = \tan\psi} \implies \tan\psi = \sinh x$$
$$\left(\frac{dy}{dx}\right)^2 + 1 = \sinh^2 x + 1 = \cosh^2 x$$

$$\therefore s = \int_0^x \left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx = \int_0^x \cosh x \, dx = \left[\sinh x\right]_0^x$$
$$= \sinh x - \sinh 0 = \sinh x$$

$$\therefore s = \sinh x$$
$$s = \tan\psi$$

Question 3 (**)

A curve C has Cartesian equation

$$y = \cosh x, \; x \in \mathbb{R}.$$

Find a simplified expression, in terms of x, for the radius of curvature at a general point on C.

$$\boxed{\rho = \cosh^2 x}$$

$y = \cosh x$

$\frac{dy}{dx} = \sinh x$

$\frac{d^2y}{dx^2} = \cosh x$

THUS

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{(1+\sinh^2 x)^{\frac{3}{2}}}{\cosh x}$$

$$\rho = \frac{(\cosh^2 x)^{\frac{3}{2}}}{\cosh x} = \frac{\cosh^3 x}{\cosh x} = \cosh^2 x$$

Question 4 (**)

A curve C has Cartesian equation

$$y = \ln(\sec x), \; 0 \le x < \frac{\pi}{2}.$$

Find an intrinsic equation of C in the form $s = f(\psi)$, where s is measured from the point with coordinates $(0,0)$, and ψ is the angle the tangent to C makes with the positive x axis.

$$\boxed{s = \ln\left|\tan\psi + \sec\psi\right|}$$

$y = \ln(\sec x)$

$\frac{dy}{dx} = \frac{1}{\sec x}\sec x \tan x = \tan x$

$\boxed{\frac{dy}{dx} = \tan x}$ & $\boxed{\frac{dy}{dx} = \tan\psi}$ $\Longrightarrow$ $x = \psi$

$\left(\frac{dy}{dx}\right)^2 + 1 = 1 + \tan^2 x = \sec^2 x$

$\therefore s = \int_0^x \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{1}{2}} dx = \int_0^x \sec x \, dx = \Big[\ln|\sec x + \tan x|\Big]_0^x$

$= \ln|\sec x + \tan x| - \ln|\sec 0 + \cancel{\tan 0}| = \ln|\sec x + \tan x| \cancel{- \ln 1}$

$\therefore s = \ln|\sec\psi + \tan\psi|$

Question 5 (**)

A curve C has intrinsic equation

$$s = a\cos\psi \,, \ \psi \in [0,\pi],$$

where s denotes the arc length measured from some fixed point and ψ is the angle the tangent to C makes with the positive x axis.

Show that the tangent to C at the point where $s=0$ is parallel to the y axis and determine the radius of curvature at that point.

$\boxed{\rho = a}$

$s = a\cos\psi$

$\frac{ds}{d\psi} = \rho = -a\sin\psi$

• NOW $s = 0$

$0 = a\cos\psi$

$\cos\psi = 0$

$\psi = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$

$\psi = \frac{\pi}{2}$

• $\frac{dy}{dx} = \tan\psi$

$\left.\frac{dy}{dx}\right|_{\psi=\frac{\pi}{2}} = \infty$ IE PARALLEL TO y AXIS

• $\left.\frac{ds}{d\psi}\right|_{\psi=\frac{\pi}{2}} = -a$

$\therefore$ RADIUS OF CURVATURE IS a

Question 6 (**)

A curve C has intrinsic equation

$$s = 2\sin\psi, \ \psi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

where s denotes the arc length measured from some fixed point and ψ is the angle the tangent to C makes with the positive x axis.

Show clearly that

$$\rho = \sqrt{4 - s^2},$$

where ρ is the radius of curvature at a general point on C.

$\boxed{\text{proof}}$

$$s = 2\sin\psi$$
$$\Rightarrow \rho = \frac{ds}{d\psi} = 2\cos\psi$$
$$\Rightarrow \rho = 2\sqrt{1 - \sin^2\psi}$$
$$\Rightarrow \rho = 2\sqrt{1 - \left(\frac{s}{2}\right)^2}$$
$$\Rightarrow \rho = 2\sqrt{1 - \frac{s^2}{4}}$$
$$\Rightarrow \rho = 2\sqrt{\frac{4 - s^2}{4}}$$
$$\Rightarrow \rho = 2\,\frac{\sqrt{4 - s^2}}{2}$$
$$\Rightarrow \rho = \sqrt{4 - s^2}$$

Question 7 (**)

A curve C has Cartesian equation

$$y = \operatorname{arsinh} x, \ x \in \mathbb{R}.$$

Find the magnitude of the radius of curvature at the point on C where $x = \sqrt{2}$.

$\boxed{4\sqrt{2}}$

- $y = \operatorname{arsinh} x$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}} = (1+x^2)^{-\frac{1}{2}} \Rightarrow \left.\frac{dy}{dx}\right|_{x=\sqrt{2}} = \frac{1}{\sqrt{1+2}} = \frac{1}{\sqrt{3}}$$

$$\frac{d^2y}{dx^2} = -x(1+x^2)^{-\frac{3}{2}} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{x=\sqrt{2}} = \frac{-\sqrt{2}}{3\sqrt{3}}$$

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left(1 + \frac{1}{3}\right)^{\frac{3}{2}}}{-\frac{\sqrt{2}}{3\sqrt{3}}} = -\frac{\left(\frac{4}{3}\right)^{\frac{3}{2}}}{\frac{\sqrt{2}}{3\sqrt{3}}} = -\frac{\frac{8}{\cancel{3\sqrt{3}}}}{\frac{\sqrt{2}}{\cancel{3\sqrt{3}}}} = -4\sqrt{2}$$

$$\therefore |\rho| = 4\sqrt{2}$$

Question 8 (**)

A curve C has Cartesian equation

$$y = \arcsin x, \ -1 \le x \le 1.$$

Find an expression, in terms of x, for the radius of curvature on C, giving the answer as a single simplified fraction.

$$\boxed{\rho = \frac{\left(2-x^2\right)^{\frac{3}{2}}}{x}}$$

$y = \arcsin x$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}}$

$\frac{d^2y}{dx^2} = -\frac{1}{2}(-2x)(1-x^2)^{-\frac{3}{2}} = \frac{x}{(1-x^2)^{\frac{3}{2}}}$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left[1+\left(\frac{1}{\sqrt{1-x^2}}\right)^2\right]^{\frac{3}{2}}}{\frac{x}{(1-x^2)^{\frac{3}{2}}}} = \frac{\left[1+\frac{1}{1-x^2}\right]^{\frac{3}{2}}}{\frac{x}{(1-x^2)^{\frac{3}{2}}}}$$

$$= \frac{\left[\frac{1-x^2+1}{1-x^2}\right]^{\frac{3}{2}}}{\frac{x}{(1-x^2)^{\frac{3}{2}}}} = \frac{\frac{(2-x^2)^{\frac{3}{2}}}{\cancel{(1-x^2)^{\frac{3}{2}}}}}{\frac{x}{\cancel{(1-x^2)^{\frac{3}{2}}}}} = \frac{(2-x^2)^{\frac{3}{2}}}{x}$$

Question 9 (**)

A curve C has Cartesian equation

$$y = \arctan x^2, \ 0 \le y < \frac{\pi}{2}.$$

Calculate the radius of curvature at the point on C where $x = -1$.

$$\boxed{-2\sqrt{2}}$$

$y = \arctan x^2$

- $\frac{dy}{dx} = \frac{1}{1+(x^2)^2} \times 2x = \frac{2x}{1+x^4}$
- $\frac{d^2y}{dx^2} = \frac{(1+x^4)\times 2 - 2x(4x^3)}{(1+x^4)^2}$
- $\left.\frac{dy}{dx}\right|_{x=-1} = \frac{-2}{1+1} = -1$
- $\left.\frac{d^2y}{dx^2}\right|_{x=-1} = \frac{2\times 2 + 2(-4)}{4} = -1$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \quad \therefore \rho\Big|_{x=-1} = \frac{\left[1+(-1)^2\right]^{\frac{3}{2}}}{-1} = \frac{2\sqrt{2}}{-1} = -2\sqrt{2}$$

Question 10 (+)**

A curve C has parametric equation s

$$x = \cosh t - t\ ,\ \ y = \cosh t + t\ ,\ \ t \in \mathbb{R}\ .$$

Find the exact value of the radius of curvature at the point on C where $t = \ln 2$.

$\boxed{\rho = \frac{25}{16}\sqrt{2}}$

$$x = \cosh t - t \qquad \dot{x} = \sinh t - 1 \qquad \ddot{x} = \cosh t$$
$$y = \cosh t + t \qquad \dot{y} = \sinh t + 1 \qquad \ddot{y} = \cosh t$$

IF $t = \ln 2$ $\quad \sinh(\ln 2) = \frac{1}{2}\left(e^{\ln 2} - e^{-\ln 2}\right) = \frac{1}{2}\left(2 - \frac{1}{2}\right) = \frac{3}{4}$

$\cosh(\ln 2) = \frac{1}{2}\left(e^{\ln 2} + e^{-\ln 2}\right) = \frac{1}{2}\left(2 + \frac{1}{2}\right) = \frac{5}{4}$

$$\rho = \frac{\left[\dot{x}^2 + \dot{y}^2\right]^{\frac{3}{2}}}{\dot{x}\ddot{y} - \ddot{x}\dot{y}} = \frac{\left[\left(\frac{3}{4}-1\right)^2 + \left(\frac{3}{4}+1\right)^2\right]^{\frac{3}{2}}}{\left(\frac{3}{4}-1\right)\times\frac{5}{4} - \frac{5}{4}\left(\frac{3}{4}+1\right)} = \frac{\left(\frac{1}{16} + \frac{49}{16}\right)^{\frac{3}{2}}}{-\frac{5}{16} - \frac{35}{16}}$$

$$= \frac{\left(\frac{50}{16}\right)^{\frac{3}{2}}}{-\frac{40}{16}} = \frac{\frac{50\sqrt{50}}{64}}{-\frac{40}{16}} = -\frac{16 \times 50\sqrt{50}}{64 \times 40} = -\frac{4000\sqrt{2}}{2560}$$

$$= -\frac{25}{16}\sqrt{2}$$

Question 11 (+)**

A curve C has Cartesian equation

$$y = a\cosh\left(\frac{x}{a}\right),\ \text{where } a \text{ is a constant.}$$

Show that the radius of curvature on C, is given by $\frac{1}{a}y^2$

$\boxed{\text{proof}}$

$$y = a\cosh\left(\frac{x}{a}\right)$$
$$\frac{dy}{dx} = \sinh\left(\frac{x}{a}\right)$$
$$\frac{d^2y}{dx^2} = \frac{1}{a}\cosh\left(\frac{x}{a}\right)$$

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left[1 + \sinh^2\frac{x}{a}\right]^{\frac{3}{2}}}{\frac{1}{a}\cosh\left(\frac{x}{a}\right)}$$

$$\rho = \frac{\left(\cosh^2\frac{x}{a}\right)^{\frac{3}{2}}}{\frac{1}{a}\cosh\frac{x}{a}} = \frac{a\cosh^3\frac{x}{a}}{\cosh\frac{x}{a}}$$

$$\rho = a\cosh^2\frac{x}{a} = \frac{1}{a}\left(a^2\cosh^2\frac{x}{a}\right)$$

$$\rho = \frac{1}{a}y^2$$

AS REQUIRED

Question 12 (**+)

A curve C has Cartesian equation

$$y = \frac{1}{2}(x-1)^{\frac{3}{2}}, \ x \in \mathbb{R}, \ x \geq 1.$$

Find an intrinsic equation of C in the form $s = f(\psi)$, where s is measured from the point with coordinates $(1,0)$, and ψ is the angle the tangent to C makes with the positive x axis.

$$\boxed{s = \frac{2}{3}\left(\sec^3\psi - 1\right)}$$

$y = \frac{2}{3}(x-1)^{\frac{3}{2}}$

$\frac{dy}{dx} = (x-1)^{\frac{1}{2}}$

$\tan\psi = (x-1)^{\frac{1}{2}}$

$\tan^2\psi = x-1$

$x = 1 + \tan^2\psi$

$\boxed{x = \sec^2\psi}$

$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$\Rightarrow \int_{s=0}^{s} 1\,ds = \int_{x=1}^{x} \sqrt{1+(x-1)}\,dx$

$\Rightarrow [s]_0^s = \int_1^x x^{\frac{1}{2}}\,dx$

$\Rightarrow s = \left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^x$

$\Rightarrow s = \frac{2}{3}\left[x^{\frac{3}{2}} - 1\right]$

$\Rightarrow s = \frac{2}{3}\left[\sec^3\psi - 1\right]$

Question 13 (**+)

The radius of curvature at a general point on a curve C is given by

$$e^{\sin\psi}\cos\psi,$$

where ψ is the angle the tangent to C makes with the positive x axis..

It is further given that when $\psi = 0$, $s = 1$, where s is the arc length measured from some fixed point.

Find an intrinsic equation for C, in the form $s = f(\psi)$.

$$\boxed{s = e^{\sin\psi}}$$

$\rho = e^{\sin\psi}\cos\psi$ SUBJECT TO $\psi = 0$, $s = 1$

$\Rightarrow \frac{ds}{d\psi} = e^{\sin\psi}\cos\psi$

$\Rightarrow ds = e^{\sin\psi}\cos\psi$

$\Rightarrow \int_{s=1}^{s} 1\,ds = \int_{\psi=0}^{\psi} e^{\sin\psi}\cos\psi\,d\psi$

$\Rightarrow [s]_1^s = \left[e^{\sin\psi}\right]_0^\psi$

$\Rightarrow s - 1 = e^{\sin\psi} - e^0$

$\Rightarrow s - 1 = e^{\sin\psi} - 1$

$\Rightarrow s = e^{\sin\psi}$

Question 14 (***)

A curve C has Cartesian equation

$$y = \sinh^2 x, \quad x \in \mathbb{R}.$$

Express the **curvature** at a general point on C in terms of $\cosh 4x$.

$$\boxed{\kappa = \frac{4}{1+\cosh 4x}}$$

$y = \sinh^2 x$

$\frac{dy}{dx} = 2\sinh x \cosh x = \sinh 2x$

$\frac{d^2y}{dx^2} = 2\cosh 2x$

$k = \frac{1}{\rho} = \frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}$

$\Rightarrow k = \frac{2\cosh 2x}{(1+\sinh^2 2x)^{\frac{3}{2}}}$

$\Rightarrow k = \frac{2\cosh 2x}{(\cosh^2 2x)^{\frac{3}{2}}}$

$\Rightarrow k = \frac{2\cosh 2x}{\cosh^3 2x}$

$\Rightarrow k = \frac{2}{\cosh^2 2x}$

$\Rightarrow k = \frac{2}{\frac{1}{2}+\frac{1}{2}\cosh 4x}$

$\Rightarrow k = \frac{4}{1+\cosh 4x}$

AS REQUIRED

Question 15 (***)

A curve C has intrinsic equation

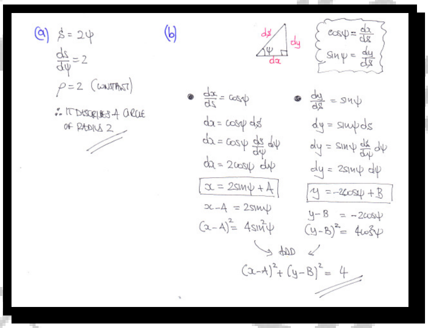
$$s = 2\psi,$$

where s is measured from some arbitrary point, and ψ is the angle the tangent to C makes with the positive x axis..

a) Describe C geometrically, with reference to $\dfrac{ds}{d\psi}$.

b) Use a calculus method to obtain a Cartesian equation for C, in terms of suitable constants.

$$\boxed{(x-a)^2 + (y-b)^2 = 4}$$



(a) $s = 2\psi$

$\dfrac{ds}{d\psi} = 2$

$\rho = 2$ (CONSTANT)

$\therefore$ IT DESCRIBES A CIRCLE OF RADIUS 2

(b)

$\cos\psi = \dfrac{dx}{ds}$

$\sin\psi = \dfrac{dy}{ds}$

- $\dfrac{dx}{ds} = \cos\psi$

$dx = \cos\psi \, ds$

$dx = \cos\psi \dfrac{ds}{d\psi} d\psi$

$dx = 2\cos\psi \, d\psi$

$\boxed{x = 2\sin\psi + A}$

$x - A = 2\sin\psi$

$(x-A)^2 = 4\sin^2\psi$

- $\dfrac{dy}{ds} = \sin\psi$

$dy = \sin\psi \, ds$

$dy = \sin\psi \dfrac{ds}{d\psi} d\psi$

$dy = 2\sin\psi \, d\psi$

$\boxed{y = -2\cos\psi + B}$

$y - B = -2\cos\psi$

$(y-B)^2 = 4\cos^2\psi$

ADD

$(x-A)^2 + (y-B)^2 = 4$

Question 16 (***)

A curve C has parametric equation s

$$x = t - \sin 2t\,, \;\; y = \cos 2t\,, \;\; 0 \le t < \frac{\pi}{2}.$$

The point P lies on C where $\cos t = \frac{1}{4}\sqrt{10}$.

Calculate the radius of curvature at P.

$$\boxed{\rho\big|_P = \frac{8}{7}}$$

$x = t - \sin 2t$
$y = \cos 2t$

$\frac{dx}{dt} = 1 - 2\cos 2t$
$\frac{dy}{dt} = -2\sin 2t$
$\frac{d^2x}{dt^2} = 4\sin 2t$
$\frac{d^2y}{dt^2} = -4\cos 2t$

$$\rho = \frac{\left[\dot{x}^2 + \dot{y}^2\right]^{\frac{3}{2}}}{\dot{x}\ddot{y} - \ddot{x}\dot{y}}$$

$$\Rightarrow \rho = \frac{\left[(1-2\cos 2t)^2 + (-2\sin 2t)^2\right]^{\frac{3}{2}}}{(1-2\cos 2t)(-4\cos 2t) - (4\sin 2t)(-2\sin 2t)}$$

$$\Rightarrow \rho = \frac{\left[1 + 4\cos^2 2t - 4\cos 2t + 4\sin^2 2t\right]^{\frac{3}{2}}}{-4\cos 2t + 8\cos^2 2t + 8\sin^2 2t}$$

$$\Rightarrow \rho = \frac{(1 + 4 - 4\cos 2t)^{\frac{3}{2}}}{8 - 4\cos 2t}$$

$$\Rightarrow \rho = \frac{(5 - 4\cos 2t)^{\frac{3}{2}}}{8 - 4\cos 2t}$$

$$\Rightarrow \rho = \frac{\left[5 - 4(2\cos^2 t - 1)\right]^{\frac{3}{2}}}{8 - 4(2\cos^2 t - 1)}$$

$$\Rightarrow \rho = \frac{(9 - 8\cos^2 t)^{\frac{3}{2}}}{12 - 8\cos^2 t}$$

BUT $\cos t = \frac{1}{4}\sqrt{10}$
$\cos^2 t = \frac{1}{16} \times 10$
$\cos^2 t = \frac{5}{8}$
$8\cos^2 t = 5$

$$\Rightarrow \rho\big|_P = \frac{(9-5)^{\frac{3}{2}}}{12 - 5} = \frac{4^{\frac{3}{2}}}{7} = \frac{8}{7}$$

Question 17 (***+)

A curve C has intrinsic equation

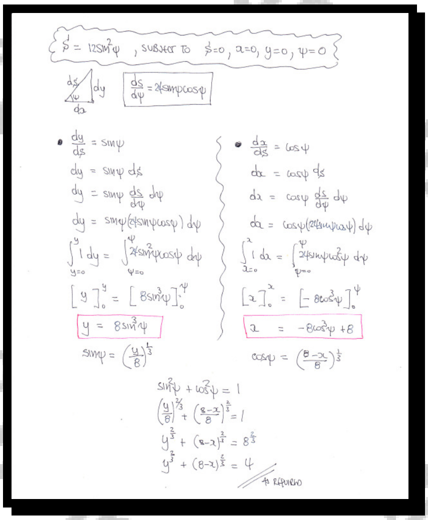
$$s = 12\sin^2\psi\,,$$

where s is measured from a Cartesian origin, and ψ is the angle the tangent to C makes with the positive x axis.

Show that a Cartesian equation of C is

$$y^{\frac{2}{3}} + (8-x)^{\frac{2}{3}} = 4\,.$$

proof



$s = 12\sin^2\psi$, SUBJECT TO $s=0$, $x=0$, $y=0$, $\psi=0$

$\boxed{\frac{ds}{d\psi} = 24\sin\psi\cos\psi}$

- $\frac{dy}{ds} = \sin\psi$
  - $dy = \sin\psi \, ds$
  - $dy = \sin\psi \frac{ds}{d\psi} \, d\psi$
  - $dy = \sin\psi(24\sin\psi\cos\psi) \, d\psi$
  - $\int_{y=0}^{y} 1 \, dy = \int_{\psi=0}^{\psi} 24\sin^2\psi\cos\psi \, d\psi$
  - $\left[y \right]_0^y = \left[8\sin^3\psi \right]_0^\psi$
  - $\boxed{y = 8\sin^3\psi}$
  - $\sin\psi = \left(\frac{y}{8}\right)^{\frac{1}{3}}$
- $\frac{dx}{ds} = \cos\psi$
  - $dx = \cos\psi \, ds$
  - $dx = \cos\psi \frac{ds}{d\psi} \, d\psi$
  - $dx = \cos\psi(24\sin\psi\cos\psi) \, d\psi$
  - $\int_{x=0}^{x} 1 \, dx = \int_{\psi=0}^{\psi} 24\sin\psi\cos^2\psi \, d\psi$
  - $\left[x \right]_0^x = \left[-8\cos^3\psi \right]_0^\psi$
  - $\boxed{x = -8\cos^3\psi + 8}$
  - $\cos\psi = \left(\frac{8-x}{8}\right)^{\frac{1}{3}}$

$\sin^2\psi + \cos^2\psi = 1$

$\left(\frac{y}{8}\right)^{2/3} + \left(\frac{8-x}{8}\right)^{\frac{2}{3}} = 1$

$y^{\frac{2}{3}} + (8-x)^{\frac{2}{3}} = 8^{\frac{2}{3}}$

$y^{\frac{2}{3}} + (8-x)^{\frac{2}{3}} = 4$

AS REQUIRED

Question 18 (*+)**

A curve C has Cartesian equation

$$y = \cosh^2 x, \ x \in \mathbb{R}.$$

Show that the radius of curvature at the point on C where $y = 4$ is $24\frac{1}{2}$.

proof

$y = \cosh^2 x$

$\frac{dy}{dx} = 2\cosh x \sinh x = \sinh 2x$

$\frac{d^2y}{dx^2} = 2\cosh 2x$

IF $y=4$ $\cosh x = 2$

$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$

$\rho = \frac{\left[1+\sinh^2 2x\right]^{\frac{3}{2}}}{2\cosh 2x}$

$\rho = \frac{\left(\cosh^2 2x\right)^{\frac{3}{2}}}{2\cosh 2x} = \frac{1}{2}\cosh^2 2x$

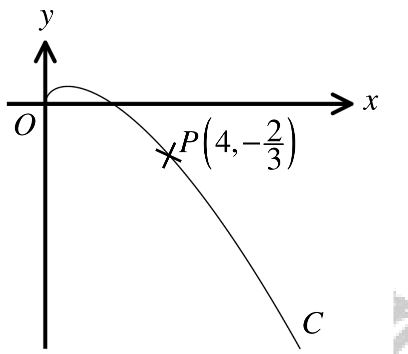
$\rho = \frac{1}{2} \times \left(2\cosh^2 x - 1\right)^2$

$\rho = \frac{1}{2}\left(4\cosh^4 x - 4\cosh^2 x + 1\right)$

$\rho = 2\cosh^4 x - 2\cosh^2 x + \frac{1}{2}$

$\therefore \rho = 2 \times 2^4 - 2 \times 2^2 + \frac{1}{2} = \frac{49}{2}$

Question 19 (***+)

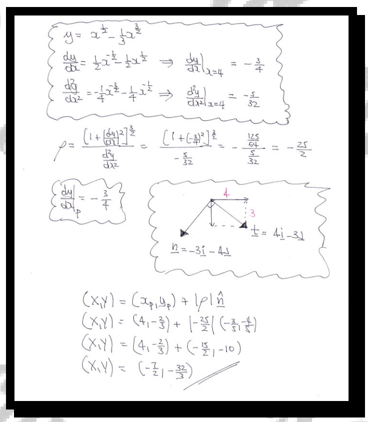


The figure above shows the curve C with Cartesian equation

$$y = x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}}, \; x \in \mathbb{R}, \; x \geq 0.$$

Show that the centre of curvature at the point $P\left(4, -\frac{2}{3}\right)$ on C, is $\left(-\frac{7}{2}, -\frac{32}{3}\right)$.

proof



Question 20 (***+)

A curve C has Cartesian equation

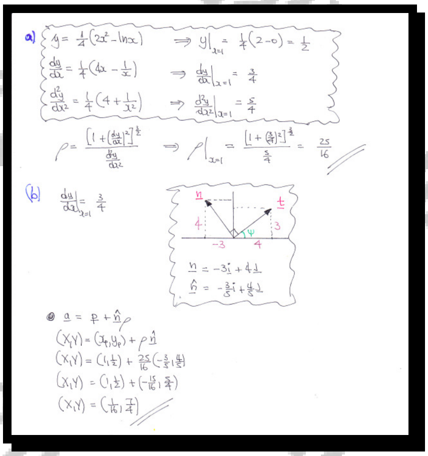
$$y = \tfrac{1}{2}\left(2x^2 - \ln x\right),\ x \in \mathbb{R},\ x > 0.$$

The point P lies on C where $x = 1$.

a) Determine the radius of curvature at P.

b) Find the exact coordinates of the centre of curvature at P.

$\boxed{\rho = \frac{25}{16}}$, $\boxed{\left(\frac{1}{16}, \frac{7}{4}\right)}$



Question 21 (***+)

A curve C has Cartesian equation

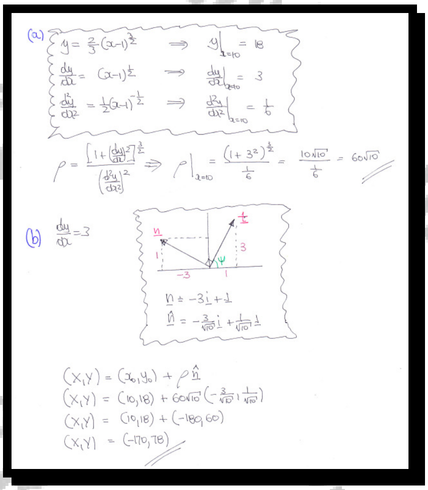
$$y = \frac{2}{3}(x-1)^{\frac{3}{2}}, \ x \in \mathbb{R}, \ x \geq 1.$$

The point P lies on C where $x = 10$.

a) Determine the radius of curvature at P.

b) Find the exact coordinates of the centre of curvature at P.

$\boxed{\rho = 60\sqrt{10}}$, $\boxed{(-170, 78)}$



(a)

$y = \frac{2}{3}(x-1)^{\frac{3}{2}} \Rightarrow y\big|_{x=10} = 18$

$\frac{dy}{dx} = (x-1)^{\frac{1}{2}} \Rightarrow \frac{dy}{dx}\big|_{x=10} = 3$

$\frac{d^2y}{dx^2} = \frac{1}{2}(x-1)^{-\frac{1}{2}} \Rightarrow \frac{d^2y}{dx^2}\big|_{x=10} = \frac{1}{6}$

$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\left(\frac{d^2y}{dx^2}\right)^2} \Rightarrow \rho\big|_{x=10} = \frac{(1+3^2)^{\frac{3}{2}}}{\frac{1}{6}} = \frac{10\sqrt{10}}{\frac{1}{6}} = 60\sqrt{10}$

(b) $\frac{dy}{dx} = 3$

$\underline{n} = -3\underline{i} + \underline{j}$

$\hat{\underline{n}} = -\frac{3}{\sqrt{10}}\underline{i} + \frac{1}{\sqrt{10}}\underline{j}$

$(X, Y) = (x_0, y_0) + \rho\hat{\underline{n}}$

$(X, Y) = (10, 18) + 60\sqrt{10}\left(-\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)$

$(X, Y) = (10, 18) + (-180, 60)$

$(X, Y) = (-170, 78)$

Question 22 (***+)

A curve C has Cartesian equation

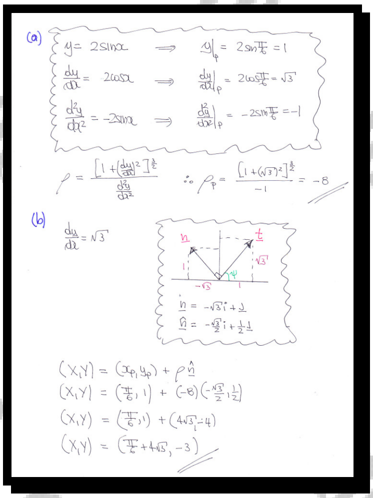
$$y = 2\sin x, \quad x \in \mathbb{R}.$$

The point P lies on C where $x = \dfrac{\pi}{6}$.

a) Determine the radius of curvature at P.

b) Find the exact coordinates of the centre of curvature at P.

$\boxed{\rho = -8}$, $\boxed{\left(\dfrac{\pi}{6} + 4\sqrt{3}, -3\right)}$



Question 23 (***+)

A cycloid C has parametric equations

$$x = 2t + 2\sin t\,,\ \ y = 2 - 2\cos t\,,\ \ 0 \le t < \frac{\pi}{2}.$$

a) Show clearly that

$$\frac{dy}{dx} = \tan\left(\tfrac{1}{2}t\right).$$

b) Find an intrinsic equation for C, in the form $s = f(\psi)$, where s is measured from a Cartesian origin, and ψ is the angle the tangent to C makes with the positive x axis.

c) Calculate the curvature at the point on C where $s = 4$.

$\boxed{s = 8\sin\psi}$, $\boxed{\kappa = \frac{1}{12}\sqrt{3}}$

(a) $x = 2t + 2\sin t$ $\qquad$ $y = 2 - 2\cos t$

$\boxed{\frac{dx}{dt} = 2 + 2\cos t}$ $\qquad$ $\boxed{\frac{dy}{dt} = 2\sin t}$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\sin t}{2 + 2\cos t} = \frac{\sin t}{1 + \cos t} = \frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{1 + (2\cos^2\frac{t}{2} - 1)} = \frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\cos^2\frac{t}{2}}$

$\therefore \frac{dy}{dx} = \tan\frac{t}{2}$

(b) $\frac{dy}{dx} = \tan\psi = \tan\frac{t}{2}$ $\quad \therefore \boxed{\psi = \frac{t}{2}}$ $\quad$ ALSO $(0,0) \Rightarrow t = 0$

$s = \int_{t=0}^{t} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^t \sqrt{(2 + 2\cos t)^2 + (2\sin t)^2}\,dt$

$= \int_0^t \sqrt{4 + 8\cos t + 4\cos^2 t + 4\sin^2 t}\,dt = \int_0^t \sqrt{8 + 8\cos t}\,dt$

$= \int_0^t \sqrt{8 + 8(2\cos^2\frac{t}{2} - 1)}\,dt = \int_0^t \sqrt{16\cos^2\frac{t}{2}}\,dt = \int_0^t 4\cos\frac{t}{2}\,dt$

$= \left[8\sin\frac{t}{2}\right]_0^t = 8\sin\frac{t}{2}$

$\therefore s = 8\sin\psi$

(c) WHEN $s = 4$ $\qquad$ $\rho = \frac{ds}{d\psi} = 8\cos\psi$

$\Rightarrow 4 = 8\sin\psi$ $\qquad$ AT $\psi = \pi/6$

$\Rightarrow \sin\psi = \frac{1}{2}$ $\qquad$ $\rho = 8\cos\frac{\pi}{6} = 4\sqrt{3}$

$\Rightarrow \psi = \frac{\pi}{6}$ $\qquad$ $\therefore \kappa = \frac{1}{12}\sqrt{3}$

Question 24 (***+)

A curve C has parametric equations

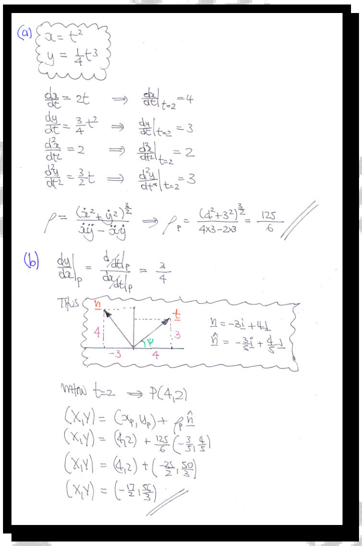
$$x = t^2, \quad y = \frac{1}{4}t^3, \quad t \in \mathbb{R}.$$

The point P lies on C where $t = 2$.

a) Determine the radius of curvature at P.

b) Find the exact coordinates of the centre of curvature at P.

$\boxed{\rho = \frac{125}{6}}$, $\boxed{\left(-\frac{17}{2}, \frac{56}{3}\right)}$



(a) $x = t^2$, $y = \frac{1}{4}t^3$

$\frac{dx}{dt} = 2t \implies \left.\frac{dx}{dt}\right|_{t=2} = 4$

$\frac{dy}{dt} = \frac{3}{4}t^2 \implies \left.\frac{dy}{dt}\right|_{t=2} = 3$

$\frac{d^2x}{dt^2} = 2 \implies \left.\frac{d^2x}{dt^2}\right|_{t=2} = 2$

$\frac{d^2y}{dt^2} = \frac{3}{2}t \implies \left.\frac{d^2y}{dt^2}\right|_{t=2} = 3$

$\rho = \frac{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}{\dot{x}\ddot{y} - \ddot{x}\dot{y}} \implies \rho_P = \frac{(4^2+3^2)^{\frac{3}{2}}}{4\times3 - 2\times3} = \frac{125}{6}$

(b) $\left.\frac{dy}{dx}\right|_P = \frac{\left.\frac{dy}{dt}\right|_P}{\left.\frac{dx}{dt}\right|_P} = \frac{3}{4}$

Thus

$\underline{n} = -3\underline{i} + 4\underline{j}$

$\hat{\underline{n}} = -\frac{3}{5}\underline{i} + \frac{4}{5}\underline{j}$

when $t=2 \implies P(4,2)$

$(X,Y) = (x_P, y_P) + \rho_P \hat{\underline{n}}$

$(X,Y) = (4,2) + \frac{125}{6}\left(-\frac{3}{5}, \frac{4}{5}\right)$

$(X,Y) = (4,2) + \left(-\frac{25}{2}, \frac{50}{3}\right)$

$(X,Y) = \left(-\frac{17}{2}, \frac{56}{3}\right)$

Question 25 (***+)

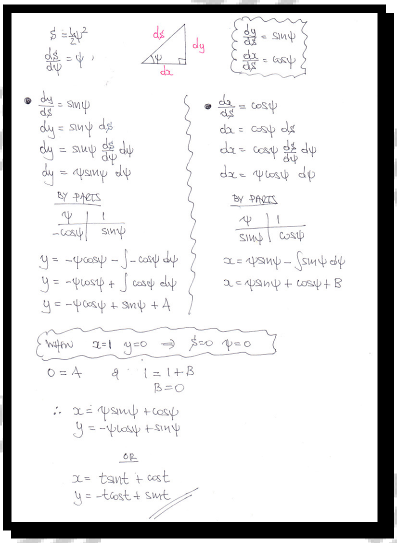
A curve C has intrinsic equation

$$s = \frac{1}{2}\psi^2 ,$$

where s is measured from the point with Cartesian coordinates $(1,0)$, and ψ is the angle the tangent to C makes with the positive x axis..

Use a calculus method to obtain two parametric equations for C, in terms of a suitable parameter.

$$\boxed{\begin{aligned} x &= t\sin t + \cos t \\ y &= -t\cos t + \sin t \end{aligned}}$$



$s = \frac{1}{2}\psi^2$

$\frac{ds}{d\psi} = \psi$,

$\boxed{\frac{dy}{ds} = \sin\psi \qquad \frac{dx}{ds} = \cos\psi}$

- $\frac{dy}{ds} = \sin\psi$

$dy = \sin\psi \, ds$

$dy = \sin\psi \frac{ds}{d\psi} \, d\psi$

$dy = \psi\sin\psi \, d\psi$

BY PARTS

ψ	1
$-\cos\psi$	$\sin\psi$

$y = -\psi\cos\psi - \int -\cos\psi \, d\psi$

$y = -\psi\cos\psi + \int \cos\psi \, d\psi$

$y = -\psi\cos\psi + \sin\psi + A$

- $\frac{dx}{ds} = \cos\psi$

$dx = \cos\psi \, ds$

$dx = \cos\psi \frac{ds}{d\psi} \, d\psi$

$dx = \psi\cos\psi \, d\psi$

BY PARTS

ψ	1
$\sin\psi$	$\cos\psi$

$x = \psi\sin\psi - \int \sin\psi \, d\psi$

$x = \psi\sin\psi + \cos\psi + B$

WHEN $x=1$ $y=0$ $\Rightarrow$ $s=0$ $\psi=0$

$0 = A$ & $1 = 1 + B$

$B = 0$

$\therefore$ $x = \psi\sin\psi + \cos\psi$

$y = -\psi\cos\psi + \sin\psi$

OR

$x = t\sin t + \cos t$

$y = -t\cos t + \sin t$

Question 26 (***+)

The radius of curvature at a general point on a curve C is given by

$$2\sin\psi,$$

where ψ is the angle the tangent to C makes with the positive x axis..

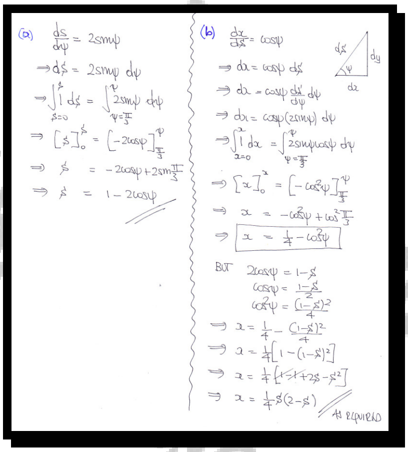
It is further given that the arc length s is measured from the point with Cartesian coordinates $(0,1)$, where the value of ψ at that point is $\frac{\pi}{3}$.

a) Find an intrinsic equation for C, in the form $s = f(\psi)$.

b) Show clearly that

$$x = \frac{1}{4}s(2-s).$$

$$\boxed{s = 1-2\cos\psi}$$



(a) $\frac{ds}{d\psi} = 2\sin\psi$

$\Rightarrow ds = 2\sin\psi \, d\psi$

$\Rightarrow \int_{s=0}^{s} 1 \, ds = \int_{\psi=\frac{\pi}{3}}^{\psi} 2\sin\psi \, d\psi$

$\Rightarrow [s]_0^s = [-2\cos\psi]_{\frac{\pi}{3}}^{\psi}$

$\Rightarrow s = -2\cos\psi + 2\sin\frac{\pi}{3}$

$\Rightarrow s = 1 - 2\cos\psi$

(b) $\frac{dx}{ds} = \cos\psi$

$\Rightarrow dx = \cos\psi \, ds$

$\Rightarrow dx = \cos\psi \frac{ds}{d\psi} d\psi$

$\Rightarrow dx = \cos\psi(2\sin\psi) \, d\psi$

$\Rightarrow \int_{x=0}^{x} 1 \, dx = \int_{\psi=\frac{\pi}{3}}^{\psi} 2\sin\psi\cos\psi \, d\psi$

$\Rightarrow [x]_0^x = [-\cos^2\psi]_{\frac{\pi}{3}}^{\psi}$

$\Rightarrow x = -\cos^2\psi + \cos^2\frac{\pi}{3}$

$\Rightarrow \boxed{x = \frac{1}{4} - \cos^2\psi}$

BUT $2\cos\psi = 1 - s$

$\cos\psi = \frac{1-s}{2}$

$\cos^2\psi = \frac{(1-s)^2}{4}$

$\Rightarrow x = \frac{1}{4} - \frac{(1-s)^2}{4}$

$\Rightarrow x = \frac{1}{4}\left[1 - (1-s)^2\right]$

$\Rightarrow x = \frac{1}{4}\left[\cancel{1} - \cancel{1} + 2s - s^2\right]$

$\Rightarrow x = \frac{1}{4}s(2-s)$

AS REQUIRED

Question 27 (***+)

The radius of curvature at a general point on a curve C is given by

$$2s+1,$$

where s is the arc length measured from the Cartesian origin.

It is further given when $s=0$, $\psi=0$, where ψ is the angle the tangent to C makes with the positive x axis..

a) Find an intrinsic equation for C, in the form $s=f(\psi)$.

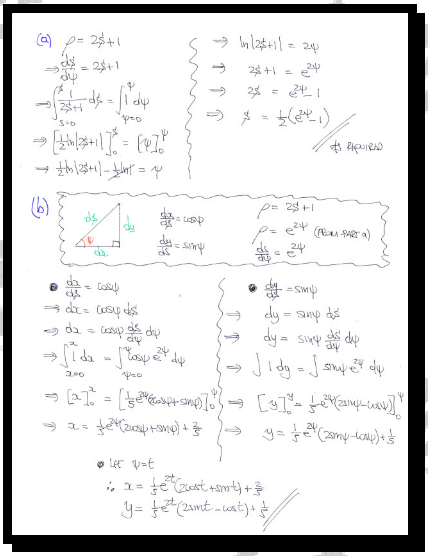
b) Determine a set of parametric equations for C.

You may assume without proof

$$\int e^{2u}\cos u\,du = \tfrac{1}{5}e^{2u}(2\cos u+\sin u)+\text{constant}$$

$$\int e^{2u}\sin u\,du = \tfrac{1}{5}e^{2u}(2\sin u-\cos u)+\text{constant}.$$

$\boxed{s=\tfrac{1}{2}\left(e^{2\psi}-1\right)}$, $\boxed{x=\tfrac{1}{5}e^{2t}(2\cos t+\sin t)+\tfrac{2}{5},\ y=\tfrac{1}{5}e^{2t}(2\sin t-\cos t)+\tfrac{2}{5}}$



(a) $\rho = 2s+1$

$\Rightarrow \frac{ds}{d\psi} = 2s+1$

$\Rightarrow \int_{s=0}^{s} \frac{1}{2s+1}\,ds = \int_{\psi=0}^{\psi} 1\,d\psi$

$\Rightarrow \left[\frac{1}{2}\ln|2s+1|\right]_0^s = [\psi]_0^\psi$

$\Rightarrow \frac{1}{2}\ln|2s+1| - \cancel{\frac{1}{2}\ln 1} = \psi$

$\Rightarrow \ln|2s+1| = 2\psi$

$\Rightarrow 2s+1 = e^{2\psi}$

$\Rightarrow 2s = e^{2\psi}-1$

$\Rightarrow s = \frac{1}{2}(e^{2\psi}-1)$ // AS REQUIRED

(b) ds, dy, dx, ψ

$\frac{dx}{ds} = \cos\psi$

$\frac{dy}{ds} = \sin\psi$

$\rho = 2s+1$

$\rho = e^{2\psi}$ (FROM PART a)

$\frac{ds}{d\psi} = e^{2\psi}$

• $\frac{dx}{ds} = \cos\psi$

$\Rightarrow dx = \cos\psi\,ds$

$\Rightarrow dx = \cos\psi \frac{ds}{d\psi}\,d\psi$

$\Rightarrow \int_{x=0}^{x} 1\,dx = \int_{\psi=0}^{\psi} \cos\psi\, e^{2\psi}\,d\psi$

$\Rightarrow [x]_0^x = \left[\frac{1}{5}e^{2\psi}(2\cos\psi+\sin\psi)\right]_0^\psi$

$\Rightarrow x = \frac{1}{5}e^{2\psi}(2\cos\psi+\sin\psi)+\frac{2}{5}$

• $\frac{dy}{ds} = \sin\psi$

$\Rightarrow dy = \sin\psi\,ds$

$\Rightarrow dy = \sin\psi\frac{ds}{d\psi}\,d\psi$

$\Rightarrow \int 1\,dy = \int \sin\psi\, e^{2\psi}\,d\psi$

$\Rightarrow [y]_0^y = \frac{1}{5}e^{2\psi}(2\sin\psi-\cos\psi)\Big]_0^\psi$

$\Rightarrow y = \frac{1}{5}e^{2\psi}(2\sin\psi-\cos\psi)+\frac{1}{5}$

• LET $\psi = t$

$\therefore x = \frac{1}{5}e^{2t}(2\cos t+\sin t)+\frac{2}{5}$

$y = \frac{1}{5}e^{2t}(2\sin t-\cos t)+\frac{1}{5}$ //

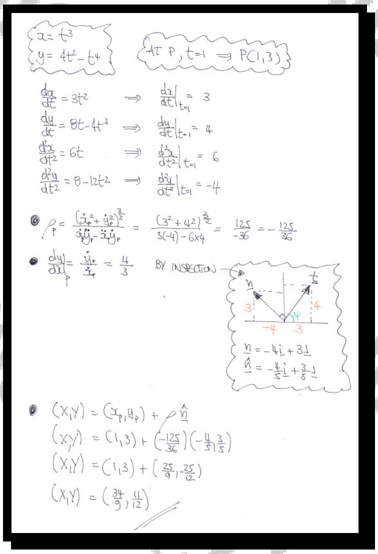
Question 28 (****)

A curve C has parametric equations

$$x = t^3, \ \ y = 4t^2 - t^4, \ \ t \in \mathbb{R}.$$

Find the exact coordinates of the centre of curvature at the point P on C where $t = 1$.

$$\boxed{\left(\frac{34}{9}, \frac{11}{12}\right)}$$



$x = t^3$
$y = 4t^2 - t^4$

AT P, $t=1 \Rightarrow P(1,3)$

$\frac{dx}{dt} = 3t^2 \Rightarrow \left.\frac{dx}{dt}\right|_{t=1} = 3$

$\frac{dy}{dt} = 8t - 4t^3 \Rightarrow \left.\frac{dy}{dt}\right|_{t=1} = 4$

$\frac{d^2x}{dt^2} = 6t \Rightarrow \left.\frac{d^2x}{dt^2}\right|_{t=1} = 6$

$\frac{d^2y}{dt^2} = 8 - 12t^2 \Rightarrow \left.\frac{d^2y}{dt^2}\right|_{t=1} = -4$

- $\rho_P = \frac{(\dot{x}_P^2 + \dot{y}_P^2)^{\frac{3}{2}}}{\dot{x}_P\ddot{y}_P - \ddot{x}_P\dot{y}_P} = \frac{(3^2+4^2)^{\frac{3}{2}}}{3(-4) - 6\times 4} = \frac{125}{-36} = -\frac{125}{36}$

- $\left.\frac{dy}{dx}\right|_P = \frac{\dot{y}_P}{\dot{x}_P} = \frac{4}{3}$ BY INSPECTION

$\underline{n} = -4\underline{i} + 3\underline{j}$

$\hat{\underline{n}} = -\frac{4}{5}\underline{i} + \frac{3}{5}\underline{j}$

- $(X,Y) = (x_P, y_P) + \rho\hat{\underline{n}}$

$(X,Y) = (1,3) + \left(-\frac{125}{36}\right)\left(-\frac{4}{5}, \frac{3}{5}\right)$

$(X,Y) = (1,3) + \left(\frac{25}{9}, -\frac{25}{12}\right)$

$(X,Y) = \left(\frac{34}{9}, \frac{11}{12}\right)$

Question 29 (****)

A cycloid C has parametric equations

$$x = \theta + \sin\theta\,,\ \ y = 1 - \cos\theta\,,\ \ 0 \le \theta \le 2\pi\,.$$

Find the exact coordinates of the centre of curvature at the point on C where $\theta = \dfrac{2\pi}{3}$.

$$\boxed{\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}, \frac{5}{2}\right)}$$

$x = \theta + \sin\theta$
$y = 1 - \cos\theta$

P HAS $\theta = \frac{2\pi}{3}$ $\Rightarrow$ $\boxed{P\left(\frac{2\pi}{3} + \frac{\sqrt{3}}{2}, \frac{3}{2}\right)}$

$$\frac{dx}{d\theta} = 1 + \cos\theta \Rightarrow \left.\frac{dx}{d\theta}\right|_P = \frac{1}{2}$$

$$\frac{dy}{d\theta} = \sin\theta \Rightarrow \left.\frac{dy}{d\theta}\right|_P = \frac{\sqrt{3}}{2}$$

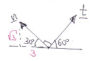
$$\frac{d^2x}{d\theta^2} = -\sin\theta \Rightarrow \left.\frac{d^2x}{d\theta^2}\right|_P = -\frac{\sqrt{3}}{2}$$

$$\frac{d^2y}{d\theta^2} = \cos\theta \Rightarrow \left.\frac{d^2y}{d\theta^2}\right|_P = -\frac{1}{2}$$

- $$\rho = \frac{\left[\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right]^{\frac{3}{2}}}{\frac{dx}{d\theta}\frac{d^2y}{d\theta^2} - \frac{d^2x}{d\theta^2}\frac{dy}{d\theta}} = \frac{\left(\frac{1}{4} + \frac{3}{4}\right)^{\frac{3}{2}}}{\frac{1}{2}\left(-\frac{1}{2}\right) - \left(-\frac{\sqrt{3}}{2}\right)\frac{\sqrt{3}}{2}} = \frac{1}{-\frac{1}{4} + \frac{3}{4}} = 2$$

- NEXT $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta}{1 + \cos\theta} \Rightarrow \left.\frac{dy}{dx}\right|_{\theta = \frac{2\pi}{3}} = \frac{\sqrt{3}/2}{\frac{1}{2}} = \sqrt{3}$$

THUS BY INSPECTION $\psi = 60°$ ($\arctan\sqrt{3}$)



$\therefore\ \underline{n} = -3\underline{i} + \sqrt{3}\underline{j}$

$|\underline{n}| = \sqrt{9 + 3} = \sqrt{12} = 2\sqrt{3}$

$\hat{\underline{n}} = -\frac{\sqrt{3}}{2}\underline{i} + \frac{1}{2}$

SO $(X, Y) = (x_P, y_P) + \rho\hat{\underline{n}}$

$(X, Y) = \left(\frac{2\pi}{3} + \frac{\sqrt{3}}{2}, \frac{3}{2}\right) + 2\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

$(X, Y) = \left(\frac{2\pi}{3} + \frac{\sqrt{3}}{2}, \frac{3}{2}\right) + \left(-\sqrt{3}, 1\right)$

$(X, Y) = \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}, \frac{5}{2}\right)$

Question 30 (****)

The gradient at every point on a curve C is given by

$$\frac{dy}{dx} = \frac{1}{2}s,$$

where s is the arc length along C measured from the point P whose Cartesian coordinates are $(0,2)$. It is further given that $\psi = 0$ at P, where ψ is the angle the tangent to C makes with the positive x axis.

a) Show clearly that

$$\frac{ds}{dx} = \frac{1}{2}\sqrt{s^2+4}.$$

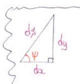
b) Express s as a function of x.

c) Deduce that

$$y = 2\cosh\left(\frac{1}{2}x\right).$$

$$\boxed{s = 2\sinh\left(\frac{1}{2}x\right)}$$

(a)



$\frac{dy}{ds} = \sin\psi$

$\frac{dx}{ds} = \cos\psi$

$\frac{dy}{dx} = \tan\psi$

BY PYTHAGORAS

$ds^2 = dx^2 + dy^2$

$\frac{ds^2}{dx^2} = \frac{dx^2}{dx^2} + \frac{dy^2}{dx^2}$

$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$

$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{1}{2}s\right)^2$

$\left(\frac{ds}{dx}\right)^2 = \frac{1}{4}(4+s^2)$

$\frac{ds}{dx} = \frac{1}{2}\sqrt{s^2+4}$ // AS REQUIRED

(b) $\frac{1}{\sqrt{s^2+4}}\,ds = \frac{1}{2}\,dx$

$\Rightarrow \int_{s=0}^{s} \frac{1}{\sqrt{s^2+4}}\,ds = \int_{x=0}^{x} \frac{1}{2}\,dx$

$\Rightarrow \left[\operatorname{arsinh}\left(\frac{s}{2}\right)\right]_0^s = \left[\frac{1}{2}x\right]_0^x$

$\Rightarrow \operatorname{arsinh}\frac{s}{2} - \operatorname{arsinh}0 = \frac{1}{2}x$

$\Rightarrow \frac{s}{2} = \sinh\frac{x}{2}$

$\Rightarrow s = 2\sinh\frac{x}{2}$ //

(c) $s = 2\sinh\frac{x}{2}$

$\Rightarrow \frac{1}{2}s = \sinh\frac{x}{2}$

$\Rightarrow \frac{dy}{dx} = \sinh\frac{x}{2}$

$\Rightarrow \int_{y=2}^{y} 1\,dy = \int_{x=0}^{x} \sinh\frac{x}{2}\,dx$

$\Rightarrow [y]_2^y = \left[2\cosh\frac{x}{2}\right]_0^x$

$\Rightarrow y - \cancel{2} = 2\cosh\frac{x}{2} - \cancel{2}$

$\Rightarrow y = 2\cosh\frac{x}{2}$ // AS REQUIRED

Question 31 (****)

An ellipse has equation

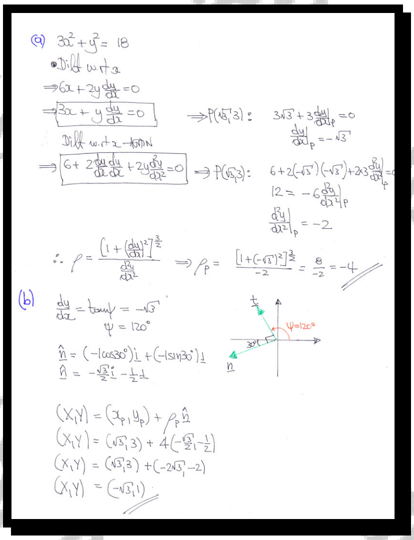
$$3x^2 + y^2 = 18.$$

The point $P\left(\sqrt{3},3\right)$ lies on C.

a) Determine the radius of curvature at P.

b) Find the exact coordinates of the centre of curvature at P.

$\boxed{\rho = -4}$, $\boxed{\left(-\sqrt{3},1\right)}$



(a) $3x^2+y^2=18$

• Diff w.r.t x

$\Rightarrow 6x+2y\frac{dy}{dx}=0$

$\Rightarrow \boxed{3x+y\frac{dy}{dx}=0}$ $\Rightarrow P(\sqrt{3},3)$: $3\sqrt{3}+3\left.\frac{dy}{dx}\right|_P=0$

$\left.\frac{dy}{dx}\right|_P=-\sqrt{3}$

Diff w.r.t x ~~AGAIN~~

$\Rightarrow \boxed{6+2\frac{dy}{dx}\frac{dy}{dx}+2y\frac{d^2y}{dx^2}=0}$ $\Rightarrow P(\sqrt{3},3)$: $6+2(-\sqrt{3})(-\sqrt{3})+2\times3\left.\frac{d^2y}{dx^2}\right|_P=0$

$12=-6\left.\frac{d^2y}{dx^2}\right|_P$

$\left.\frac{d^2y}{dx^2}\right|_P=-2$

$\therefore \rho=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \Rightarrow \rho_P=\frac{\left[1+(-\sqrt{3})^2\right]^{\frac{3}{2}}}{-2}=\frac{8}{-2}=-4$

(b) $\frac{dy}{dx}=\tan\psi=-\sqrt{3}$

$\psi=120°$

$\hat{n}=(-1\cos30°)\underline{i}+(-1\sin30°)\underline{j}$

$\hat{n}=-\frac{\sqrt{3}}{2}\underline{i}-\frac{1}{2}\underline{j}$

$(X,Y)=(x_P,y_P)+\rho_P\hat{n}$

$(X,Y)=(\sqrt{3},3)+4\left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$

$(X,Y)=(\sqrt{3},3)+(-2\sqrt{3},-2)$

$(X,Y)=(-\sqrt{3},1)$

Question 32 (****)

A curve C has intrinsic equation

$$s = \ln\left(\tan\frac{\psi}{2}\right), \ 0 < \psi < \pi ,$$

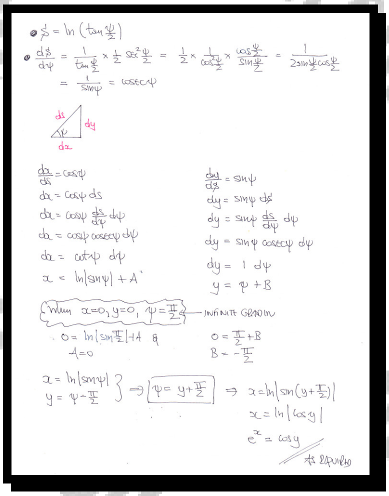
where s is measured from a fixed point, and ψ is the angle the tangent to C makes with the positive x axis.

It is given that the tangent to C at a Cartesian origin has infinite gradient.

Show that a Cartesian equation of C is

$$e^x = \cos y .$$

$\boxed{\text{proof}}$



Question 33 (****)

A curve C has Cartesian equation

$$y = \ln\left(x+1+\sqrt{x^2+2x}\right), \ x \in \mathbb{R}.$$

Determine the radius of curvature at the point on C where $x = 2$.

$\boxed{\rho = -9}$

- $y = \ln\left(x+1+\sqrt{x^2+2x}\right) = \ln\left(x+1+\sqrt{(x+1)^2-1}\right) = \operatorname{arcosh}(x+1)$
- $\frac{dy}{dx} = \frac{1}{\sqrt{(x+1)^2-1}} = \frac{1}{\sqrt{x^2+2x}} = (x^2+2x)^{-\frac{1}{2}}$
- $\frac{d^2y}{dx^2} = -\frac{1}{2}(x^2+2x)^{-\frac{3}{2}}(2x+2) = -(x+1)(x^2+2x)^{-\frac{3}{2}} = -\frac{(x+1)}{(x^2+2x)^{\frac{3}{2}}}$

- when $x=2$

$$\left.\frac{dy}{dx}\right|_{x=2} = (2^2+2\times2)^{-\frac{1}{2}} = 8^{-\frac{1}{2}} = \frac{1}{\sqrt{8}} = \frac{\sqrt{8}}{8} = \frac{\sqrt{2}}{4}$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=2} = -\frac{(2+1)}{(2^2+2\times2)^{\frac{3}{2}}} = -\frac{3}{8^{3/2}} = -\frac{3}{8\sqrt{8}} = -\frac{3\sqrt{8}}{64} = -\frac{3\sqrt{2}}{32}$$

- $\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$

At $x=2$

$$\rho = \frac{\left(1+\frac{1}{8}\right)^{\frac{3}{2}}}{-\frac{3}{32}\sqrt{2}} = \frac{\left(\frac{9}{8}\right)^{\frac{3}{2}}}{-\frac{3}{32}\sqrt{2}} = \frac{\frac{27}{8\sqrt{8}}}{-\frac{3}{32}\sqrt{2}} = -\frac{27\times32}{8\sqrt{8}\times3\sqrt{2}}$$

$$= -\frac{27\times\cancel{32}}{3\times\cancel{32}} = -9$$

Question 34 (****)

A curve C has Cartesian equation

$$\sin y = e^x, \quad x \leq 0.$$

Find an intrinsic equation for C, in the form $s = f(\psi)$, where s is measured from the point with Cartesian coordinates $\left(0, \frac{\pi}{2}\right)$, and ψ is the angle the tangent to C makes with the positive x axis.

$$\boxed{s = \ln\left|\tan\left(\frac{\psi}{2}\right)\right| \quad \text{or} \quad e^s = \tan\left(\frac{\psi}{2}\right)}$$

START BY OBTAINING A RELATIONSHIP BETWEEN ψ & EITHER x OR y

$\Rightarrow \sin y = e^x$

$\Rightarrow y = \arcsin(e^x)$

$\Rightarrow \frac{dy}{dx} = \frac{e^x}{\sqrt{1-e^{2x}}} = \frac{\sin y}{(1-\sin^2 y)^{\frac{1}{2}}} = \frac{\sin y}{\cos y} = \tan y$

BUT $\frac{dy}{dx} \equiv \tan\psi$ (ALWAYS)

$\therefore y = \psi$

USING THE ARCLENGTH FORMULA

$\Rightarrow s = \int_0^x \left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx = \int_0^x (1+\tan^2 y)^{\frac{1}{2}} dx = \int_0^x \sec y \, dx$

FROM $(0, \frac{\pi}{2})$

$= \int_{y=\frac{\pi}{2}}^{y} \sec y \frac{dx}{dy} \, dy = \int_{\frac{\pi}{2}}^{y} \sec y \left(\frac{1}{\tan y}\right) dy = \int_{\frac{\pi}{2}}^{y} \frac{1}{\cancel{\cos y}} \frac{\cancel{\cos y}}{\sin y} \, dy$

$= \int_{\frac{\pi}{2}}^{y} \operatorname{cosec} y \, dy = \left[\ln\left|\tan\frac{y}{2}\right|\right]_{\frac{\pi}{2}}^{y} = \ln\left|\tan\frac{y}{2}\right| - \cancel{\ln\left|\tan\frac{\pi}{4}\right|}$

$\Rightarrow s = \ln\left|\tan\frac{y}{2}\right|$

BUT AS $y = \psi$

$\Rightarrow s = \ln\left|\tan\frac{\psi}{2}\right|$

$\Rightarrow e^s = \tan\frac{\psi}{2}$

VARIATION

$\Rightarrow \sin y = e^x$

$\Rightarrow \ln(\sin y) = x$

$\Rightarrow \frac{dx}{dy} = \frac{1}{\sin y} \times \cos y$

$\Rightarrow \frac{dx}{dy} = \cot y$

$\Rightarrow \frac{dy}{dx} = \tan y$ (AS BEFORE) $\Rightarrow y = \psi$

& HENCE USING THE ARCLENGTH FORMULA

$\Rightarrow s = \int_{y=\frac{\pi}{2}}^{y} \left[1+\left(\frac{dx}{dy}\right)^2\right]^{\frac{1}{2}} dy$

$\Rightarrow s = \int_{\frac{\pi}{2}}^{y} (1+\cot^2 y)^{\frac{1}{2}} dy$

$\Rightarrow s = \int_{\frac{\pi}{2}}^{y} \operatorname{cosec} y \, dy$ ... WHICH MERGES WITH THE PREVIOUS

Question 35 (****)

A curve C has intrinsic equation

$$s = 8\left(\sec^3\psi - 1\right),\ 0 \le \psi < \frac{\pi}{2},$$

where s is the arc length is measured from a Cartesian origin O, and ψ is the angle the tangent to C makes with the positive x axis. It is further given that $\psi = 0$ at the origin O.

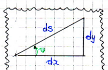
Show that a Cartesian equation of C is

$$y^2 = \frac{x^3}{27}.$$

[ ], proof

$$\boxed{s = 8\left(\sec^3\psi - 1\right) \quad 0 \le \psi < \frac{\pi}{2} \qquad x=0,\ y=0,\ s=0,\ \psi=0}$$

FORMING TWO SEPARATE DIFFERENTIAL EQUATIONS BASED ON THE TRIANGLE SHOWN BELOW



$$\frac{ds}{d\psi} = 8\left(3\sec^3\psi\tan\psi\right)$$

$$\rho = 24\sec^3\psi\tan\psi$$

$\Rightarrow \frac{dx}{ds} = \cos\psi$

$\Rightarrow dx = \cos\psi\, ds$

$\Rightarrow dx = \cos\psi \frac{ds}{d\psi}\, d\psi$

$\Rightarrow dx = \cos\psi\left(24\sec^3\psi\tan\psi\right) d\psi$

$\Rightarrow dx = 24\sec^2\psi\tan\psi\, d\psi$

$\Rightarrow \int_{x=0}^{x} 1\, dx = \int_{\psi=0}^{\psi} 24\sec^2\psi\tan\psi\, d\psi$

$\Rightarrow \left[x\right]_0^x = \left[12\sec^2\psi\right]_0^\psi$

$\Rightarrow \boxed{x = 12\sec^2\psi - 12}$ OR $\boxed{x = 12\tan^2\psi}$

SIMILARLY WE HAVE

$\Rightarrow \frac{dy}{ds} = \sin\psi$

$\Rightarrow dy = \sin\psi\, ds$

$\Rightarrow dy = \sin\psi \frac{ds}{d\psi}\, d\psi$

$\Rightarrow dy = \sin\psi\left(24\sec^3\psi\tan\psi\right) d\psi$

$\Rightarrow dy = 24\sec^2\psi\tan^2\psi\, d\psi$

$\Rightarrow \int_{y=0}^{y} 1\, dy = \int_{\psi=0}^{\psi} 24\sec^2\psi\tan^2\psi\, d\psi$

$\Rightarrow \left[y\right]_0^y = \left[8\tan^3\psi\right]_0^\psi$

$\Rightarrow \boxed{y = 8\tan^3\psi}$

TREATING ψ AS A PARAMETER ELIMINATE INTO CARTESIAN

$\Rightarrow \begin{cases} x = 12\tan^2\psi \\ y = 8\tan^3\psi \end{cases}$

$\Rightarrow \begin{cases} x^3 = 1728\tan^6\psi \\ y^2 = 64\tan^6\psi \end{cases}$

$\Rightarrow \frac{x^3}{y^2} = \frac{1728}{64}$

$\Rightarrow \frac{x^3}{y^2} = 27$

$\Rightarrow y^2 = \frac{x^3}{27}$ // AS REQUIRED

Question 36 (****)

A curve C has intrinsic equation

$$s = 4\sin\psi, \quad 0 \le \psi \le \pi,$$

where s is the arc length is measured from the Cartesian origin O, and ψ is the angle the tangent to C makes with the positive x axis.

It is given that the tangent to C at O has zero gradient.

Show that the parametric equations of C are

$$x = t + \sin t, \quad y = 1 - \cos t, \quad 0 \le t \le 2\pi.$$

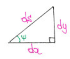
MM1A, proof

- WE ARE GIVEN THE INTRINSIC EQUATION & CONDITIONS – PROCEED WITH THE STANDARD SET UP

$s = 4\sin\psi$

$s=0, \psi=0, x=0, y=0$

$\rho = \frac{ds}{d\psi} = 4\cos\psi$



- $\frac{dy}{ds} = \sin\psi$
- $\frac{dx}{ds} = \cos\psi$

- SOLVING EACH OF THE DIFFERENTIAL EQUATIONS

$\Rightarrow \frac{dx}{ds} = \cos\psi$

$\Rightarrow dx = \cos\psi \, ds$

$\Rightarrow dx = \cos\psi \frac{ds}{d\psi} d\psi$

$\Rightarrow dx = \cos\psi (4\cos\psi) d\psi$

$\Rightarrow dx = 4\cos^2\psi \, d\psi$

$\Rightarrow \int_{x=0}^{x} 1 \, dx = \int_{\psi=0}^{\psi} 4\left(\frac{1}{2}+\frac{1}{2}\cos 2\psi\right) d\psi$

$\Rightarrow \int_0^x 1 \, dx = \int_0^\psi 2 + 2\cos 2\psi \, d\psi$

$\Rightarrow [x]_0^x = [2\psi + \sin 2\psi]_0^\psi$

$\Rightarrow x - 0 = (2\psi + \sin 2\psi) - 0$

$\Rightarrow x = 2\psi + \sin 2\psi$

$\Rightarrow \frac{dy}{ds} = \sin\psi$

$\Rightarrow dy = \sin\psi \, ds$

$\Rightarrow dy = \sin\psi \frac{ds}{d\psi} d\psi$

$\Rightarrow dy = \sin\psi (4\cos\psi) d\psi$

$\Rightarrow dy = 2\sin 2\psi \, d\psi$

$\Rightarrow \int_{y=0}^{y} 1 \, dy = \int_{\psi=0}^{\psi} 2\sin 2\psi \, d\psi$

$\Rightarrow [y]_0^y = [-\cos 2\psi]_0^\psi$

$\Rightarrow y - 0 = -\cos 2\psi + 1$

$\Rightarrow y = 1 - \cos 2\psi$

- FINALLY WE LET $t = 2\psi$, $0 \le \psi \le \pi$, $0 \le t \le 2\pi$

HENCE THE PARAMETRIC REPRESENTATION IS

$x = t + \sin t$ & $y = 1 - \cos t$, $0 \le t \le 2\pi$

Question 37 (****)

A curve has Cartesian equation

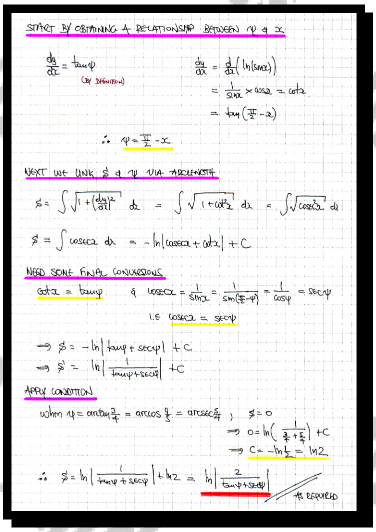
$$y = \ln(\sin x), \ 0 < x < \pi .$$

Show that an intrinsic equation of the curve is

$$s = \ln\left|\frac{2}{\tan\psi + \sec\psi}\right|,$$

where s is the arc length measured from the point where $\psi = \arctan\frac{3}{4}$, where ψ is the angle the tangent to the curve makes with the positive x axis.

proof



START BY OBTAINING A RELATIONSHIP BETWEEN ψ & x

$\frac{dy}{dx} = \tan\psi$ (BY DEFINITION)

$\frac{dy}{dx} = \frac{d}{dx}\left(\ln(\sin x)\right) = \frac{1}{\sin x} \times \cos x = \cot x = \tan\left(\frac{\pi}{2} - x\right)$

$\therefore \ \psi = \frac{\pi}{2} - x$

NEXT WE LINK s & ψ VIA ARCLENGTH

$s = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int \sqrt{1 + \cot^2 x}\, dx = \int \sqrt{\operatorname{cosec}^2 x}\, dx$

$s = \int \operatorname{cosec} x \, dx = -\ln|\operatorname{cosec} x + \cot x| + C$

NEED SOME FINAL CONVERSIONS

$\cot x = \tan\psi$ & $\operatorname{cosec} x = \frac{1}{\sin x} = \frac{1}{\sin\left(\frac{\pi}{2} - \psi\right)} = \frac{1}{\cos\psi} = \sec\psi$

I.E $\operatorname{cosec} x = \sec\psi$

$\Rightarrow s = -\ln|\tan\psi + \sec\psi| + C$

$\Rightarrow s = \ln\left|\frac{1}{\tan\psi + \sec\psi}\right| + C$

APPLY CONDITION

when $\psi = \arctan\frac{3}{4} = \arccos\frac{4}{5} = \operatorname{arcsec}\frac{5}{4}$, $s = 0$

$\Rightarrow 0 = \ln\left(\frac{1}{\frac{3}{4} + \frac{5}{4}}\right) + C$

$\Rightarrow C = -\ln\frac{1}{2} = \ln 2$

$\therefore \ s = \ln\left|\frac{1}{\tan\psi + \sec\psi}\right| + \ln 2 = \ln\left|\frac{2}{\tan\psi + \sec\psi}\right|$

AS REQUIRED

Question 38 (****)

A curve C has parametric equations

$$x = 6\tan^2 t,\ \ y = 4\tan^3 t,\ \ 0 \le t < \frac{\pi}{2}.$$

It is further given that when $t = 0$, $s = 4$, where s is the arc length measured from some fixed point.

Show clearly that

$$s = 4\sec^3 \psi,$$

where ψ is the angle the tangent to C makes with the positive x axis.

proof

$$x = 6\tan^2 t \implies \frac{dx}{dt} = 12\tan t\sec^2 t$$
$$y = 4\tan^3 t \implies \frac{dy}{dt} = 12\tan^2 t\sec^2 t$$

- $\sqrt{\dot{x}^2+\dot{y}^2} = \sqrt{144\tan^2 t\sec^4 t + 144\tan^4 t\sec^4 t} = 12\tan t\sec^2 t\sqrt{1+\tan^2 t}$
  $= 12\tan t\sec^2 t\sqrt{\sec^2 t} = 12\tan t\sec^3 t$

- $\frac{dy}{dx} = (\tan\psi) = \frac{\dot{y}}{\dot{x}} = \frac{12\tan^2 t\sec^2 t}{\cancel{12\tan t\sec^2 t}} = \tan t \implies \boxed{t = \psi}$

- $ds = \sqrt{\dot{x}^2+\dot{y}^2}\ dt$

$\implies \int_{s=4}^{s} 1\ ds = \int_{t=0}^{t} 12\tan t\sec^3 t\ dt$

$\implies [s]_4^s = [4\sec^3 t]_0^t$

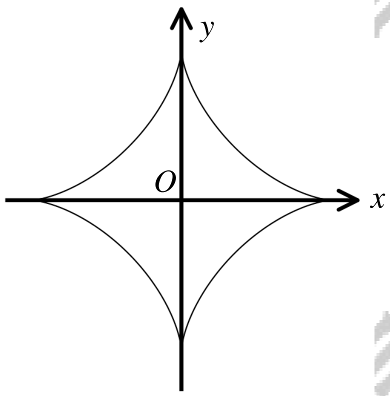
$\implies s - 4 = 4\sec^3 t - 4\sec^3 0$

$\implies s \cancel{-4} = 4\sec^3 t \cancel{-4}$

$\implies s = 4\sec^3 t$

$\implies s = 4\sec^3\psi$ // AS REQUIRED

Question 39 (**+)**



An astroid is given parametrically by

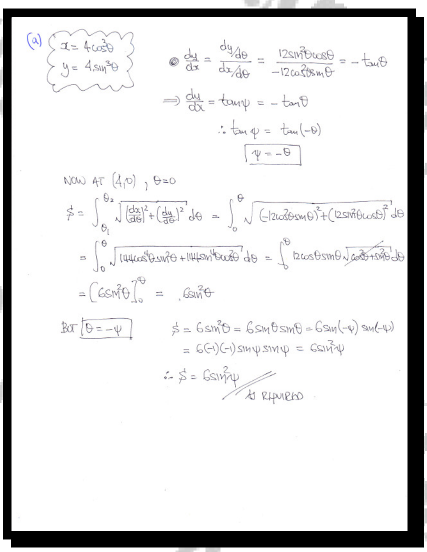
$$x = 4\cos^3\theta\,,\; x = 4\sin^3\theta\,,\; 0 \le \theta < 2\pi\,.$$

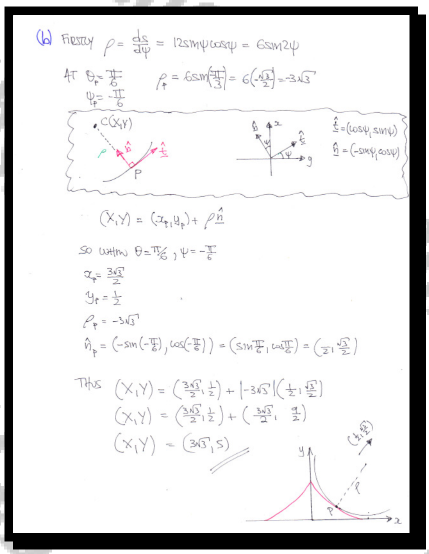
a) Show that if the arc length s is measured from the point $(4,0)$, and ψ is the angle the tangent to the astroid makes with the positive x axis, then

$$s = 6\sin^2\psi\,.$$

b) Determine the coordinates of the centre of curvature at the point P on the astroid where $\theta = \dfrac{\pi}{6}$

$$\boxed{C\left(3\sqrt{3},5\right)}$$





Question 40 (****+)

A curve C has parametric equations

$$x = 2\sinh t\,, \;\; y = \cosh^2 t\,, \;\; t \in \mathbb{R}\,.$$

It is further given that the arc length s is measured from the point where $t = 0$.

Show clearly that

$$s = \ln\left(\tan\psi + \sec\psi\right) + \tan\psi\sec\psi\,,$$

where ψ is the angle the tangent to C makes with the positive x axis.

proof

$$\begin{aligned} x &= 2\sinh t \\ y &= \cosh^2 t \end{aligned} \implies \begin{aligned} \dot{x} &= 2\cosh t \\ \dot{y} &= 2\cosh t\sinh t \end{aligned}$$

- $\sqrt{\dot{x}^2+\dot{y}^2} = \sqrt{4\cosh^2 t + 4\cosh^2 t\sinh^2 t} = 2\cosh t\sqrt{1+\sinh^2 t}$
  $= 2\cosh t\sqrt{\cosh^2 t} = 2\cosh t\cosh t = 2\cosh^2 t$

- $\frac{dy}{dx} = (\tan\psi) = \frac{\dot{y}}{\dot{x}} = \frac{2\cosh t\sinh t}{2\cosh t} = \sinh t \qquad \therefore \tan\psi = \sinh t$

- $ds = \sqrt{\dot{x}^2+\dot{y}^2}\,dt$

$\implies \int_0^s 1\,ds = \int_{t=0}^{t} 2\cosh^2 t\,dt$

$\implies [s]_0^s = \int_0^t 1+\cosh 2t\,dt$

$\implies s = \left[t + \frac{1}{2}\sinh t\right]_0^t$

$\implies s = t + \frac{1}{2}\sinh 2t$

$\implies \boxed{s = t + \sinh t\cosh t}$

$\boxed{\sinh t = \tan\psi}$

$\sinh^2 t = \tan^2\psi$

$1+\sinh^2 t = 1+\tan^2\psi$

$\cosh^2 t = \sec^2\psi$

$\boxed{\cosh t = \sec\psi}$

& $t = \text{arsinh}(\tan\psi)$

$t = \ln\left(\tan\psi + \sqrt{1+\tan^2\psi}\right)$

$\boxed{t = \ln(\tan\psi + \sec\psi)}$

$\implies s = \ln(\tan\psi + \sec\psi) + \tan\psi\sec\psi$

Question 41 (****+)

The position vector of a curve C is given by

$$\mathbf{r}(t) = \left(\frac{2}{1+t^2} - 1\right)\mathbf{i} + \left(\frac{2t}{1+t^2}\right)\mathbf{j},$$

where t is a scalar parameter with $t \in \mathbb{R}$.

Find an expression for the position vector of C, giving the answer in the form

$$\mathbf{r}(s) = f(s)\,\mathbf{i} + g(s)\,\mathbf{j},$$

where s is the arc length of a general point on C, measure from the point $(1,0)$.

$$\boxed{\mathbf{r}(s) = (\cos s)\mathbf{i} + (\sin s)\mathbf{j}}$$

$$\mathbf{r}(t) = \left[\frac{2}{1+t^2} - 1\right]\mathbf{i} + \left[\frac{2t}{1+t^2}\right]\mathbf{j}$$

- $x = \frac{2}{1+t^2} - 1 \Rightarrow \dot{x} = \frac{(1+t^2)\times 0 - 2(2t)}{(1+t^2)^2} = -\frac{4t}{(1+t^2)^2}$
- $y = \frac{2t}{1+t^2} \Rightarrow \dot{y} = \frac{(1+t^2)\times 2 - 2t(2t)}{(1+t^2)^2} = \frac{2+2t^2-4t^2}{(1+t^2)^2} = \frac{2-2t^2}{(1+t^2)^2}$

$(1,0) \Rightarrow t=0$

$$\therefore s = \int_{t=0}^{t} \sqrt{\dot{x}^2 + \dot{y}^2}\,dt = \int_0^t \sqrt{\frac{16t^2}{(1+t^2)^4} + \frac{(2-2t^2)^2}{(1+t^2)^4}}\,dt$$

$$= \int_0^t \sqrt{\frac{16t^2 + 4 - 8t^2 + 4t^4}{(1+t^2)^4}}\,dt = \int_0^t \frac{\sqrt{4t^4+8t^2+4}}{(1+t^2)^2}\,dt$$

$$= \int_0^t \frac{\sqrt{4(t^4+2t^2+1)}}{(t^2+1)^2}\,dt = \int_0^t \frac{\sqrt{4(t^2+1)^2}}{(t^2+1)^2}\,dt = \int_0^t \frac{2(t^2+1)}{(t^2+1)^2}\,dt$$

$$= \int_0^t \frac{2}{t^2+1}\,dt = \Big[2\arctan t\Big]_0^t = 2\arctan t - \arctan 0$$

Thus $s = 2\arctan t$

$\frac{s}{2} = \arctan t$

$\boxed{\tan\frac{s}{2} = t}$

- $x = \frac{2}{1+t^2} - 1 = \frac{2-1-t^2}{1+t^2} = \frac{1-t^2}{1+t^2} = \cos s$
- $y = \frac{2t}{1+t^2} = \sin s$

THESE ARE THE "LITTLE t IDENTITIES"

$\therefore \mathbf{r}(s) = \cos s\,\mathbf{i} + \sin s\,\mathbf{j}$

Question 42 (****+)

A curve C has intrinsic equation

$$s = \ln(\tan\psi + \sec\psi) + \tan\psi\sec\psi\,, \quad 0 \le \psi < \frac{\pi}{2}\,,$$

where s is the arc length is measured from the point with Cartesian coordinates $(0,1)$, and ψ is the angle the tangent to C makes with the positive x axis.

It is further given that the gradient at $(0,1)$ is zero.

Show that the Cartesian equation of C is

$$y = \frac{1}{4}x^2 + 1\,.$$

$\boxed{}$, $\boxed{\text{proof}}$

START WITH THE RADIUS OF CURVATURE

$$\Rightarrow s = \ln(\tan\psi + \sec\psi) + \tan\psi\sec\psi$$

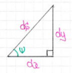
$$\Rightarrow \frac{ds}{d\psi} = \frac{\sec^2\psi + \sec\psi\tan\psi}{\tan\psi + \sec\psi} + \sec^3\psi + \sec\psi\tan^2\psi$$

$$\Rightarrow \frac{ds}{d\psi} = \frac{\sec\psi(\cancel{\sec\psi + \tan\psi})}{\cancel{\tan\psi + \sec\psi}} + \sec^3\psi + \sec\psi(\sec^2\psi - 1)$$

$$\Rightarrow \frac{ds}{d\psi} = \cancel{\sec\psi} + \sec^3\psi + \sec^3\psi - \cancel{\sec\psi}$$

$$\Rightarrow \frac{ds}{d\psi} = 2\sec^3\psi$$

NOW WE HAVE



- $\frac{dx}{ds} = \cos\psi$
- $\frac{dy}{ds} = \sin\psi$
- $\frac{ds}{d\psi} = 2\sec^3\psi$

AND CONDITIONS

AT $(0,1)$
$s = 0$
$\psi = 0$

NEXT SOLVING AN O.D.E AS FOLLOWS

$$\Rightarrow \frac{dx}{ds} = \cos\psi$$

$$\Rightarrow dx = \cos\psi\, ds$$

$$\Rightarrow dx = \cos\psi \frac{ds}{d\psi} d\psi$$

$$\Rightarrow dx = \cos\psi(2\sec^3\psi)\, d\psi$$

$$\Rightarrow dx = 2\sec^2\psi\, d\psi$$

$$\Rightarrow \int_{x=0}^{x} 1\, dx = \int_{\psi=0}^{\psi} 2\sec^2\psi\, d\psi$$

$$\Rightarrow [x]_0^x = [2\tan\psi]_0^\psi$$

$$\Rightarrow x - 0 = 2\tan\psi - 0$$

$$\Rightarrow x = 2\tan\psi$$

SIMILARLY ANOTHER O.D.E

$$\Rightarrow \frac{dy}{ds} = \sin\psi$$

$$\Rightarrow dy = \sin\psi\, ds$$

$$\Rightarrow dy = \sin\psi \frac{ds}{d\psi} d\psi$$

$$\Rightarrow dy = \sin\psi(2\sec^3\psi)\, d\psi$$

$$\Rightarrow dy = 2\frac{\sin\psi}{\cos\psi}\sec^2\psi\, d\psi$$

$$\Rightarrow \int_{y=1}^{y} 1\, dy = \int_{\psi=0}^{\psi} 2\tan\psi\sec^2\psi\, d\psi$$

$$\Rightarrow [y]_1^y = [\tan^2\psi]_0^\psi$$

$$\Rightarrow y - 1 = \tan^2\psi - 0$$

$$\Rightarrow y = 1 + \tan^2\psi$$

BUT $x = 2\tan\psi$ FROM THE "x O.D.E"

$$\Rightarrow y = 1 + \left(\frac{x}{2}\right)^2$$

$$\Rightarrow y = \frac{1}{4}x^2 + 1$$

AS REQUIRED

Question 43 (*****)

The gradient at every point on a curve C is given by

$$\frac{dy}{dx} = \frac{1}{2}s\,,$$

where s is the arc length along C measured from the point P whose Cartesian coordinates are $(0,2)$.

It is further given that $\psi = 0$ at P, where ψ is the angle the tangent to C makes with the positive x axis.

a) Show clearly that

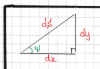
$$x = 2\ln\left|\sec\psi + \tan\psi\right|, \qquad y = 2\sec\psi\,.$$

b) Eliminate ψ to show further that

$$y = 2\cosh\left(\frac{1}{2}x\right).$$

proof

a) PREPARE ALL THE AUXILIARIES



$\frac{dy}{ds} = \sin\psi$

$\frac{dx}{ds} = \cos\psi$

$\frac{dy}{dx} = \tan\psi = \frac{1}{2}s$ (GIVEN)

$x=0$, $y=2$, $s=0$, $\psi=0$

THE RADIUS CURVATURE IS GIVEN BY

$\Rightarrow \tan\psi = \frac{1}{2}s$

$\Rightarrow \sec^2\psi\, d\psi = \frac{1}{2}ds$

$\Rightarrow \rho = \frac{ds}{d\psi} = 2\sec^2\psi$

FORMING TWO ANALOGOUS O.D.Es

$\Rightarrow \frac{dx}{ds} = \cos\psi$ | $\Rightarrow \frac{dy}{ds} = \sin\psi$

$\Rightarrow dx = \cos\psi\, ds$ | $\Rightarrow dy = \sin\psi\, ds$

$\Rightarrow dx = \cos\psi \frac{ds}{d\psi} d\psi$ | $\Rightarrow dy = \sin\psi \frac{ds}{d\psi} d\psi$

$\Rightarrow dx = \cos\psi(2\sec^2\psi)\,d\psi$ | $\Rightarrow dy = \sin\psi(2\sec^2\psi)\,d\psi$

$\Rightarrow dx = 2\sec\psi\, d\psi$ | $\Rightarrow dy = 2\tan\psi\sec\psi\, d\psi$

$\Rightarrow \int_{x=0}^{x} 1\, dx = \int_{\psi=0}^{\psi} 2\sec\psi\, d\psi$ | $\Rightarrow \int_{y=0}^{y} 1\, dy = \int_{\psi=0}^{\psi} 2\tan\psi\sec\psi\, d\psi$

$\Rightarrow [x]_0^x = [2\ln|\sec\psi + \tan\psi|]_0^\psi$ | $\Rightarrow [y]_2^y = [2\sec\psi]_0^\psi$

$\Rightarrow x = 2\ln|\sec\psi + \tan\psi| - \cancel{2\ln 1}$ | $\Rightarrow y - \cancel{2} = 2\sec\psi - \cancel{2}$

$\Rightarrow y = 2\sec\psi$

b) TO ELIMINATE THE "PARAMETER" ψ WE MANIPULATE THE x EQUATION AS FOLLOWS

$\Rightarrow x = 2\ln|\sec\psi + \tan\psi|$

$\Rightarrow -x = -2\ln|\sec\psi + \tan\psi|$

$\Rightarrow -x = 2\ln\left(\frac{1}{\sec\psi + \tan\psi}\right)$

$\Rightarrow -x = 2\ln \left[\frac{\sec\psi - \tan\psi}{(\sec\psi + \tan\psi)(\sec\psi - \tan\psi)}\right]$

$\Rightarrow -x = 2\ln\left[\frac{\sec\psi - \tan\psi}{\sec^2\psi - \tan^2\psi}\right]$

BUT $1 + \tan^2\psi \equiv \sec^2\psi$
$\sec^2\psi - \tan^2\psi \equiv 1$

$\Rightarrow -x = 2\ln(\sec\psi - \tan\psi)$

HENCE WE OBTAIN USING HYPERBOLIC FUNCTIONS

$\left.\begin{array}{l} \frac{x}{2} = \ln|\sec\psi + \tan\psi| \\ -\frac{x}{2} = \ln|\sec\psi - \tan\psi| \end{array}\right\} \Rightarrow \begin{array}{l} e^{\frac{x}{2}} = \sec\psi + \tan\psi \\ e^{-\frac{x}{2}} = \sec\psi - \tan\psi \end{array}$

$\Rightarrow e^{\frac{x}{2}} - e^{-\frac{x}{2}} = 2\sec\psi$

$\Rightarrow \sec\psi = \frac{1}{2}\left[e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right]$

$\Rightarrow \sec\psi = \cosh\frac{x}{2}$

$\Rightarrow 2\sec\psi = 2\cosh\frac{x}{2}$

$\Rightarrow y = 2\cosh\frac{x}{2}$ // AS REQUIRED

Question 44 (*****)

The radius of curvature ρ at any point on a curve with Cartesian equation $y = f(x)$ is given by

$$\frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

a) Given that the curve can be parameterized as $x = g(t)$, $y = h(t)$, for some parameter t, show that

$$\rho = \frac{\left(\dot{x}^2+\dot{y}^2\right)^{\frac{3}{2}}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}},$$

where a dot above a variable denoted differentiation with respect to t.

A curve C is given parametrically by

$$x = \cos t + t\sin t, \quad y = \sin t - t\cos t, \quad 0 \le t < 2\pi.$$

b) Find an expression for ρ on C, giving the answer in terms of t.

$\boxed{}$, $\boxed{\rho = t}$

a) LET US NOTE THAT IN PARAMETRIC

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\dot{y}}{\dot{x}} = \left(\frac{dy}{dt}\right)\left(\frac{dx}{dt}\right)^{-1}$$

SUBSTITUTING INTO THE CARTESIAN RADIUS OF CURVATURE FORMULA

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left[1+\left(\frac{\dot{y}}{\dot{x}}\right)^2\right]^{\frac{3}{2}}}{\frac{d}{dx}\left[\left(\frac{dy}{dt}\right)\left(\frac{dx}{dt}\right)^{-1}\right]} = \frac{\left[\frac{\dot{x}^2+\dot{y}^2}{\dot{x}^2}\right]^{\frac{3}{2}}}{\frac{d}{dx}\left[\left(\frac{dy}{dt}\right)\left(\frac{dx}{dt}\right)^{-1}\right]}$$

$$= \frac{\frac{(\dot{x}^2+\dot{y}^2)^{3/2}}{\dot{x}^3}}{\frac{d}{dx}\left[\left(\frac{dy}{dt}\right)\left(\frac{dx}{dt}\right)^{-1}\right]}$$

DIFFERENTIATING THE DENOMINATOR, WE OBTAIN

$$\frac{d}{dx}\left[\left(\frac{dy}{dt}\right)\left(\frac{dx}{dt}\right)^{-1}\right] = \frac{d^2y}{dt^2}\frac{dt}{dx}\left(\frac{dx}{dt}\right)^{-1} + \frac{dy}{dt}\times(-1)\left(\frac{dx}{dt}\right)^{-2}\times\frac{d^2x}{dt^2}\times\frac{dt}{dx}$$

$$= \ddot{y}\times\frac{1}{\dot{x}}\times\frac{1}{\dot{x}} - \dot{y}\times\frac{1}{\dot{x}^2}\times\ddot{x}\times\frac{1}{\dot{x}}$$

$$= \frac{\ddot{y}}{\dot{x}^2} - \frac{\dot{y}\ddot{x}}{\dot{x}^3}$$

SUBSTITUTING INTO THE DENOMINATOR OF ρ

$$\rho = \frac{\frac{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}{\dot{x}^3}}{\frac{\ddot{y}}{\dot{x}^2}-\frac{\dot{y}\ddot{x}}{\dot{x}^3}} = \dots$$ MULTIPLY "TOP & BOTTOM" OF THE FRACTION BY $\dot{x}^3$

$$= \frac{(\dot{x}^2+\dot{y}^2)^{3/2}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}}$$

AS REQUIRED

a) OBTAIN ALL THE PARAMETRIC DERIVATIVES

- $x = \cos t + t\sin t$
  $\dot{x} = -\cancel{\sin t} + \cancel{\sin t} + t\cos t$
- $\dot{x} = t\cos t$
- $\ddot{x} = \cos t - t\sin t$
- $y = \sin t - t\cos t$
  $\dot{y} = \cancel{\cos t} - \cancel{\cos t} + t\sin t$
- $\dot{y} = t\sin t$
- $\ddot{y} = \sin t + t\cos t$

SUBSTITUTE INTO THE PARAMETRIC FORM OF ρ

$$\rho = \frac{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}}$$

$$\rho = \frac{(t^2\cos^2 t + t^2\sin^2 t)^{\frac{3}{2}}}{(t\cos t)(\sin t + t\cos t)-(t\sin t)(\cos t - t\sin t)}$$

$$\rho = \frac{t^3(\cos^2 t+\sin^2 t)^{\frac{3}{2}}}{\cancel{t\cos t\sin t} + t^2\cos^2 t - \cancel{t\cos t\sin t} + t^2\sin^2 t}$$

$$\rho = \frac{t^3\times 1^{\frac{3}{2}}}{t^2(\cos^2 t+\sin^2 t)}$$

$$\rho = \frac{t^3}{t^2}$$

$$\rho = t$$

Question 45 (*****)

A curve C has Cartesian equation $y = f(x)$.

The same curve has intrinsic equation $s = g(\psi)$, where s is measured from an arbitrary point and ψ is the angle the tangent to C makes with the positive x axis.

The radius of curvature ρ at any point on C is defined as $\dfrac{ds}{d\psi}$.

a) Show clearly that $\rho = \dfrac{\left(1+\left(\dfrac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}$

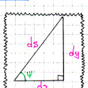
b) Given that C can be suitably parameterized as $x = h_1(t)$, $y = h_2(t)$, for some parameter t, show further that

$$\rho = \frac{\left(\dot{x}^2+\dot{y}^2\right)^{\frac{3}{2}}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}},$$

where a dot above a variable denotes differentiation with respect to t.

$\boxed{}$, $\boxed{\text{proof}}$

a) STARTING WITH THE USUAL "INTRINSIC" DIAGRAM



$\dfrac{dy}{dx} = \tan\psi$

$\dfrac{dy}{ds} = \sin\psi$

$\dfrac{dx}{ds} = \cos\psi$

IF $s = f(\psi) \Rightarrow \rho = \dfrac{ds}{d\psi}$

DIFFERENTIATE $\frac{dy}{dx}$ W.R.T s

$\Rightarrow \dfrac{dy}{dx} = \tan\psi$

$\Rightarrow \dfrac{d}{ds}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{ds}(\tan\psi)$

$\Rightarrow \dfrac{d^2y}{dx^2} \times \dfrac{dx}{ds} = \sec^2\psi \dfrac{d\psi}{ds}$

$\Rightarrow \dfrac{d^2y}{dx^2} \times \cos\psi = \sec^2\psi \dfrac{1}{\rho}$

$\Rightarrow \dfrac{1}{\rho} = \dfrac{d^2y}{dx^2} \times \cos^3\psi$

$\Rightarrow \rho = \dfrac{1}{\frac{d^2y}{dx^2} \times \cos^3\psi}$

$\Rightarrow \rho = \dfrac{\sec^3\psi}{\frac{d^2y}{dx^2}}$

$\Rightarrow \rho = \dfrac{(\sec^2\psi)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$

$\Rightarrow \rho = \dfrac{(1+\tan^2\psi)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$

$\Rightarrow \rho = \dfrac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$ AS REQUIRED

b) START BY OBTAINING EXPRESSIONS FOR $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ IN TERMS OF t

- $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{\dot{y}}{\dot{x}}$
- $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left[\dfrac{dy}{dx}\right] = \dfrac{d}{dx}\left[\dfrac{\dot{y}}{\dot{x}}\right] = \dfrac{d}{dx}\left[\dfrac{dy}{dt} \times \left(\dfrac{dx}{dt}\right)^{-1}\right]$ (EASIER AS A PRODUCT)

$= \dfrac{d}{dx}\left(\dfrac{dy}{dt}\right) \times \left(\dfrac{dx}{dt}\right)^{-1} + \dfrac{dy}{dt} \times \dfrac{d}{dx}\left[\left(\dfrac{dx}{dt}\right)^{-1}\right]$

$= \dfrac{d^2y}{dt^2} \times \dfrac{dt}{dx} \times \left(\dfrac{dx}{dt}\right)^{-1} + \dfrac{dy}{dt}\left[-\left(\dfrac{dx}{dt}\right)^{-2} \times \dfrac{d^2x}{dt^2} \times \dfrac{dt}{dx}\right]$

$= \dfrac{\frac{d^2y}{dt^2}}{\left(\frac{dx}{dt}\right)\left(\frac{dx}{dt}\right)} - \dfrac{\frac{d^2x}{dt^2} \times \frac{dy}{dt}}{\left(\frac{dx}{dt}\right)^2\left(\frac{dx}{dt}\right)}$

$= \dfrac{\ddot{y}}{\dot{x}^2} - \dfrac{\ddot{x}\dot{y}}{\dot{x}^3}$

RETURNING TO THE EXPRESSION OF PART (a)

$\rho = \dfrac{ds}{d\psi} = \dfrac{\left[1+(dy/dx)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \dfrac{\left[1+\left(\frac{\dot{y}}{\dot{x}}\right)^2\right]^{\frac{3}{2}}}{\frac{\ddot{y}}{\dot{x}^2} - \frac{\ddot{x}\dot{y}}{\dot{x}^3}}$

MULTIPLY "TOP & BOTTOM" OF THE FRACTION BY $\dot{x}^3$

$\rho = \dfrac{ds}{d\psi} = \dfrac{\dot{x}^3\left[1+\frac{\dot{y}^2}{\dot{x}^2}\right]^{\frac{3}{2}}}{\dot{x}\ddot{y}-\ddot{x}\dot{y}} = \dfrac{(\dot{x}^2)^{\frac{3}{2}}\left[1+\frac{\dot{y}^2}{\dot{x}^2}\right]^{\frac{3}{2}}}{\dot{x}\ddot{y}-\ddot{x}\dot{y}}$

$\rho = \dfrac{(\dot{x}^2+\dot{y}^2)^{3/2}}{\dot{x}\ddot{y}-\ddot{x}\dot{y}}$ AS REQUIRED