LAPLACE TRANSFORMS APPLICATIONS

SUMMARY OF THE LAPLACE TRANFORM

The Laplace Transform of a function $f(t)$, $t \geq 0$ is defined as

$$\mathcal{L}\left[f(t)\right] \equiv \overline{f}(s) \equiv \int_0^\infty e^{-st} f(t)\ dt,$$

where $s \in \mathbb{C}$, with $\text{Re}(s)$ sufficiently large for the integral to converge.

The Laplace Transform is a linear operation

$$\mathcal{L}\left[a\,f(t) + b\,g(t)\right] \equiv a\mathcal{L}\left[f(t)\right] + b\mathcal{L}\left[g(t)\right].$$

Laplace Transforms of Common Functions

- $\mathcal{L}\left(t^n\right) = \dfrac{n}{s^{n+1}}$

$$\mathcal{L}(1) = \frac{1}{s},\quad \mathcal{L}(a) = \frac{a}{s},\quad \mathcal{L}(t) = \frac{1}{s^2},\quad \mathcal{L}\left(t^2\right) = \frac{2}{s^3},\quad \mathcal{L}\left(t^3\right) = \frac{3}{s^4}, \ldots.$$

- $\mathcal{L}\left(e^{at}\right) = \dfrac{1}{s-a}$, $\mathcal{L}\left(e^{-at}\right) = \dfrac{1}{s+a}$
- $\mathcal{L}(\cos at) = \dfrac{s}{s^2+a^2}$, $\mathcal{L}(\sin at) = \dfrac{a}{s^2+a^2}$
- $\mathcal{L}(\cosh at) = \dfrac{s}{s^2-a^2}$, $\mathcal{L}(\sinh at) = \dfrac{a}{s^2-a^2}$

Laplace Transforms of Derivatives

- $\mathcal{L}\left[x(t)\right] = \overline{x}(t)$
- $\mathcal{L}\left[\dot{x}(t)\right] = s\overline{x}(t) - x(0)$
- $\mathcal{L}\left[\ddot{x}(t)\right] = s^2\overline{x}(t) - sx(0) - \dot{x}(0)$
- $\mathcal{L}\left[\dddot{x}(t)\right] = s^3\overline{x}(t) - s^2x(0) - s\dot{x}(0) - \ddot{x}(0)$

Laplace Transforms Theorems

- 1^{st} Shift Theorem

$$\mathcal{L}\left[\mathrm{e}^{-at} f(t)\right] = \bar{f}(s+a) \quad \text{or} \quad \mathcal{L}\left[\mathrm{e}^{at} F(t)\right] = \bar{f}(s-a)$$

- 2^{nd} Shift Theorem

$$\mathcal{L}\left[f(t-a)\right] = \mathrm{e}^{-as}\,\bar{f}(s),\ t>a \quad \text{or} \quad \mathcal{L}\left[f(t+a)\right] = \mathrm{e}^{as}\,\bar{f}(s),\ t>-a.$$

$$\mathcal{L}\left[\mathrm{H}(t-a)f(t-a)\right] = \mathrm{e}^{-as}\,\bar{f}(s) \quad \text{or} \quad \mathcal{L}\left[\mathrm{H}(t+a)f(t+a)\right] = \mathrm{e}^{as}\,\bar{f}(s)$$

- Multiplication by t^n

$$\mathcal{L}\left[t^n f(t)\right] = \left(-\frac{d}{ds}\right)^n \left[\bar{f}(s)\right] \quad \text{or} \quad \mathcal{L}\left[t\,f(t)\right] = -\frac{d}{ds}\left[\bar{f}(s)\right]$$

- Division by t

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(\sigma)\ d\sigma$$

provided that $\lim_{t\to 0}\left(\frac{f(t)}{t}\right)$ exists and the integral converges.

- Initial/Final value theorem

$$\lim_{t\to 0}\left[f(t)\right] = \lim_{s\to\infty}\left[s\,\bar{f}(s)\right] \quad \text{and} \quad \lim_{t\to\infty}\left[f(t)\right] = \lim_{s\to 0}\left[s\,\bar{f}(s)\right]$$

The Impulse Function / The Dirac Function

1. $\delta(t-c)=\begin{cases}\infty & t=c\\ 0 & t\neq c\end{cases}$, $\delta(t)=\begin{cases}\infty & t=0\\ 0 & t\neq 0\end{cases}$

2. $\int_a^b \delta(t-c)\ dt=\begin{cases}1 & a\le c\le b\\ 0 & \text{otherwise}\end{cases}$

3. $\int_a^b f(t)\delta(t-c)\ dt=\begin{cases}f(a) & a\le c\le b\\ 0 & \text{otherwise}\end{cases}$

4. $\mathcal{L}\left[\delta(t-c)\right]=\mathrm{e}^{-cs}$

5. $\mathcal{L}\left[f(t)\delta(t-c)\right]=f(c)\mathrm{e}^{-cs}$

6. $\frac{d}{dt}\left[\mathrm{H}(t-c)\right]=\delta(t-c)$

SOLVING SIMPLE O.D.E.s

Question 1

Use Laplace transforms to solve the differential equation

$$\frac{dx}{dt} - 2x = 4 \,, \; t \geq 0 \,,$$

subject to the initial condition $x = 1$ at $t = 0$.

$\boxed{}$, $\boxed{x = 3e^{2t} - 2}$

TAKING THE LAPLACE TRANSFORM OF THE O.D.E, W.R.T t

$\Rightarrow \frac{dx}{dt} - 2x = 4 \qquad [t=0, x=1]$

$\Rightarrow \mathcal{L}\left[\frac{dx}{dt}\right] - \mathcal{L}[2x] = \mathcal{L}[4]$

$\Rightarrow s\bar{x} - x_0 - 2\bar{x} = \frac{4}{s}$

$\Rightarrow s\bar{x} - 1 - 2\bar{x} = \frac{4}{s}$

$\Rightarrow (s-2)\bar{x} = \frac{4}{s} + 1$

$\Rightarrow (s-2)\bar{x} = \frac{4+s}{s}$

$\Rightarrow \bar{x} = \frac{s+4}{s(s-2)}$

INVERT BY PARTIAL FRACTIONS (COVER UP)

$\Rightarrow \bar{x} = \frac{3}{s-2} - \frac{2}{s}$

$\Rightarrow x = \mathcal{L}^{-1}\left[\frac{3}{s-2} - \frac{2}{s}\right]$

THESE ARE SIMPLE STANDARD RESULTS

$\Rightarrow x(t) = 3e^{2t} - 2$

Question 2

Use Laplace transforms to solve the differential equation

$$\frac{dy}{dx} + 2y = 10e^{3x} \,, \; x \geq 0 \,,$$

subject to the boundary condition $y = 6$ at $x = 0$.

$\boxed{y = 2e^{3x} + 4e^{-2x}}$

$\frac{dy}{dx} + 2y = 10e^{3x}$, SUBJECT TO $x=0, y=6$

$\Rightarrow y' + 2y = 10e^{3x}$

$\Rightarrow s\bar{y} - y_0 + 2\bar{y} = \frac{10}{s-3}$ $\quad y_0 = 6$

$\Rightarrow s\bar{y} - 6 + 2\bar{y} = \frac{10}{s-3}$

$\Rightarrow (s+2)\bar{y} = \frac{10}{s-3} + 6$

$\Rightarrow (s+2)\bar{y} = \frac{6s-8}{s-3}$

$\Rightarrow \bar{y} = \frac{6s-8}{(s-3)(s+2)}$

$\Rightarrow \bar{y} = \frac{2}{s-3} + \frac{4}{s+2}$ (BY COVER UP)

$\Rightarrow y = \mathcal{L}^{-1}\left[\frac{2}{s-3} + \frac{4}{s+2}\right]$

$\Rightarrow y = 2e^{3x} + 4e^{-2x}$

Question 3

Use Laplace transforms to solve the differential equation

$$\frac{dy}{dx} - 4y = 2e^{2x} + e^{4x}, \quad x \geq 0,$$

subject to the boundary condition $y = 0$ at $x = 0$.

$$\boxed{y = xe^{4x} + e^{4x} - 2e^{2x}}$$

$\frac{dy}{dx} - 4y = 2e^{2x} + e^{4x}$, SUBJECT TO $x=0, y=0$

$\Rightarrow y' - 4y = 2e^{2x} + e^{4x}$

$\Rightarrow s\bar{y} - y_0 - 4\bar{y} = \frac{2}{s-2} + \frac{1}{s-4}$ $\quad y_0 = 0$

$\Rightarrow s\bar{y} - 0 - 4\bar{y} = \frac{2}{s-2} + \frac{1}{s-4}$

$\Rightarrow (s-4)\bar{y} = \frac{2}{s-2} + \frac{1}{s-4}$

$\Rightarrow \bar{y} = \frac{2}{(s-2)(s-4)} + \frac{1}{(s-4)^2}$ (BY COVER UP)

$\Rightarrow \bar{y} = \frac{1}{s-4} - \frac{1}{s-2} + \frac{1}{(s-4)^2}$

$\Rightarrow y = \mathcal{L}^{-1}\left[\frac{1}{s-4} - \frac{1}{s-2} + \frac{1}{(s-4)^2}\right]$

$\Rightarrow y = e^{4x} - e^{2x} + xe^{4x}$

Question 4

Use Laplace transforms to solve the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2e^{3x}, \quad x \geq 0,$$

subject to the boundary conditions $y = 5$, $\frac{dy}{dx} = 7$ at $x = 0$.

$$\boxed{y = 2e^{3x} + 4e^{x}}$$

$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2e^{3x}$, SUBJECT TO $x=0, y=5, \frac{dy}{dx}=7$

$\Rightarrow y'' - 3y' + 2y = 2e^{3x}$

$\Rightarrow s^2\bar{y} - sy_0 - y'_0 - 3(s\bar{y} - y_0) + 2\bar{y} = \frac{2}{s-3}$ $\quad y_0 = 5, \ y'_0 = 7$

$\Rightarrow s^2\bar{y} - 5s - 7 - 3s\bar{y} + 15 + 2\bar{y} = \frac{2}{s-3}$

$\Rightarrow \bar{y}(s^2 - 3s + 2) = \frac{2}{s-3} + 5s - 8$

$\Rightarrow \bar{y}(s-2)(s-1) = \frac{2}{s-3} + 5s - 8$

$\Rightarrow \bar{y} = \frac{2}{(s-1)(s-2)(s-3)} + \frac{5s-8}{(s-2)(s-1)}$

$\Rightarrow \bar{y} = \frac{\frac{2}{2\times1}}{s-3} + \frac{\frac{2}{1(-1)}}{s-2} + \frac{\frac{2}{(-1)(-2)}}{s-1} + \frac{2}{s-2} + \frac{3}{s-1}$ (BY COVER UP)

$\Rightarrow \bar{y} = \frac{2}{s-3} - \cancel{\frac{2}{s-2}} + \frac{1}{s-1} + \cancel{\frac{2}{s-2}} + \frac{3}{s-1}$

$\Rightarrow \bar{y} = \frac{2}{s-3} + \frac{4}{s-1}$

$\Rightarrow y = \mathcal{L}^{-1}\left[\frac{2}{s-3} + \frac{4}{s-1}\right]$

$\Rightarrow y = 2e^{3x} + 4e^{x}$

Question 5

Use Laplace transforms to solve the differential equation

$$\frac{d^2z}{dt^2} - 2\frac{dz}{dt} + 10z = 10e^{2t},$$

subject to the initial conditions $z = 0$, $\frac{dz}{dt} = 1$ at $t = 0$.

$$\boxed{y = e^{2t} + \cos 3t + \sin 3t}$$

$\frac{d^2z}{dt^2} - 2\frac{dz}{dt} + 10z = 10e^{2t}$ SUBJECT TO $t=0$, $z=0$ $\frac{dz}{dt}=1$

$z_0 = 0$
$z_0' = 1$

$$z'' - 2z' + 10z = 10e^{2t}$$
$$(s^2\bar{z} - sz_0 - z_0') - 2(s\bar{z} - z_0) + 10\bar{z} = 10 \times \frac{1}{s-2}$$
$$s^2\bar{z} - 1 - 2s\bar{z} + 10\bar{z} = \frac{10}{s-2}$$
$$(s^2 - 2s + 10)\bar{z} = 1 + \frac{10}{s-2}$$
$$\bar{z} = \frac{1}{s^2-2s+10} + \frac{10}{(s-2)(s^2-2s+10)}$$
$$\bar{z} = \frac{1}{(s-1)^2+9} + \frac{1}{s-2} + \frac{As+B}{s^2-2s+10} \quad \text{(BY INSPECTION)}$$
$$\bar{z} = \frac{1}{(s-1)^2+3^2} + \frac{1}{s-2} + \frac{-s}{(s-1)^2+}$$
$$z = \mathcal{L}^{-1}\left[\frac{1}{(s-1)^2+3^2} + \frac{1}{s-2} - \frac{s}{(s-1)^2+3^2}\right]$$
$$z = e^{2t} + \cos 3t + \sin 3t$$

Question 6

Use Laplace transforms to solve the differential equation

$$\frac{d^2y}{dx^2} - 4y = 24\cos 2x\,,\ \ x \geq 0\,,$$

subject to the boundary conditions $y = 3$, $\frac{dy}{dx} = 4$ at $x = 0$.

$$\boxed{}\,,\ \boxed{y = 4e^{2x} + 2e^{-2x} - 3\cos 2x}$$

$$\boxed{\frac{d^2y}{dx^2} - 4y = 24\cos 2x\ ,\ x \geq 0\ ,\ x=0,\ y=3,\ \frac{dy}{dx}=4}$$

WRITE THE O.D.E IN COMPACT FORM, & TAKE LAPLACE TRANSFORMS IN x

$$\Rightarrow y'' - 4y = 24\cos 2x$$

$$\Rightarrow s^2\bar{y} - sy_0 - y'_0 - 4\bar{y} = \mathcal{L}[24\cos 2x]$$

$$\Rightarrow s^2\bar{y} - 3s - 4 - 4\bar{y} = 24 \times \frac{s}{s^2+4}$$

$$\Rightarrow (s^2-4)\bar{y} = 3s + 4 + \frac{24s}{s^2+4}$$

$$\Rightarrow \bar{y} = \frac{3s+4}{s^2-4} + \frac{24s}{(s^2-4)(s^2+4)}$$

$$\Rightarrow \bar{y} = \frac{3s+4}{(s-2)(s+2)} + \frac{24s}{(s-2)(s+2)(s^2+4)}$$

PARTIAL FRACTIONS MAINLY BY INSPECTION (COVER UP)

$$\Rightarrow \bar{y} = \frac{\frac{10}{4}}{s-2} + \frac{\frac{-2}{-4}}{s+2} + \frac{\frac{48}{4\times 8}}{s-2} + \frac{\frac{-48}{-4\times 8}}{s+2} + \frac{As+B}{s^2+4}$$

$$\Rightarrow \bar{y} = \frac{\frac{5}{2}}{s-2} + \frac{\frac{1}{2}}{s+2} + \underbrace{\frac{\frac{3}{2}}{s-2} + \frac{\frac{3}{2}}{s+2} + \frac{As+B}{s^2+4}}$$

$$24s \equiv \tfrac{3}{2}(s+2)(s^2+4) + \tfrac{3}{2}(s-2)(s^2+4) + (s^2-4)(As+B)$$

- IF $s=0 \Rightarrow 0 = 12 - 12 - 4B$
  $\Rightarrow 4B = 0$
  $\Rightarrow B = 0$
- IF $s=1 \Rightarrow 24 = \frac{45}{2} - \frac{15}{2} - 3(A)$
  $\Rightarrow 24 = 15 - 3A$
  $\Rightarrow 3A = -9$
  $\Rightarrow A = -3$

COLLECTING ALL RESULTS

$$\bar{y} = \frac{\frac{5}{2}}{s-2} + \frac{\frac{1}{2}}{s+2} + \frac{\frac{3}{2}}{s-2} + \frac{\frac{3}{2}}{s+2} - \frac{3s}{s^2+4}$$

$$\bar{y} = \frac{4}{s-2} + \frac{2}{s+2} - 3\left(\frac{s}{s^2+4}\right)$$

INVERTING (ALL ONLY SIMPLE STANDARD RESULTS)

$$y = 4e^{2x} + 2e^{-2x} - 3\cos 2x$$

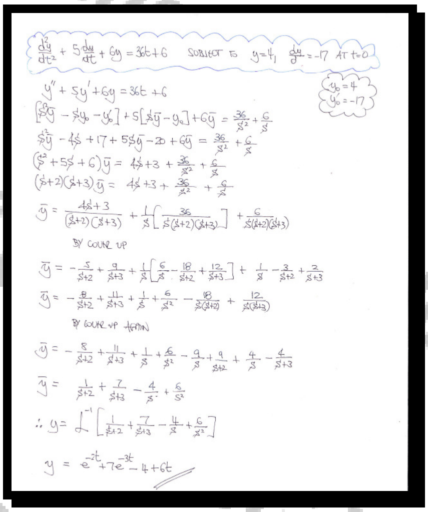
Question 7

Use Laplace transforms to solve the differential equation

$$\frac{d^2y}{dt^2}+5\frac{dy}{dt}+6y=36t+6,$$

subject to the initial conditions $y=4$, $\frac{dy}{dt}=-17$ at $t=0$.

$$\boxed{y=e^{-2t}+7e^{-3t}+6t-4}$$



Question 8

By using Laplace transforms, or otherwise, solve the following simultaneous differential equations, subject to the initial conditions $x = -1$, $y = 2$ at $t = 0$.

$$\frac{dx}{dt} = x - 2y \qquad \text{and} \qquad \frac{dy}{dt} = 5x - y.$$

$$\boxed{x = -\cos 3t - \frac{5}{3}\sin 3t, \quad y = 2\cos 3t - \frac{7}{3}\sin 3t}$$

$$\boxed{\begin{aligned} \frac{dx}{dt} &= x - 2y & x(0) &= -1 \\ \frac{dy}{dt} &= 5x - y & y(0) &= 2 \end{aligned}}$$

- TAKING LAPLACE TRANSFORMS OF EACH OF THE COUPLED O.D.E.s

$$\left.\begin{aligned} s\bar{x} - x(0) &= \bar{x} - 2\bar{y} \\ s\bar{y} - y(0) &= 5\bar{x} - \bar{y} \end{aligned}\right\} \Rightarrow \left.\begin{aligned} s\bar{x} + 1 &= \bar{x} - 2\bar{y} \\ s\bar{y} - 2 &= 5\bar{x} - \bar{y} \end{aligned}\right\} \Rightarrow$$

- TIDY THE EQUATIONS

$$\left.\begin{aligned} (s-1)\bar{x} + 2\bar{y} &= -1 \\ (s+1)\bar{y} - 5\bar{x} &= 2 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} (s-1)\bar{x} + 2\bar{y} &= -1 \\ -5\bar{x} + (s+1)\bar{y} &= 2 \end{aligned}\right\} \Rightarrow$$

- SOLVE IT USING MATRICES

$$\Rightarrow \begin{bmatrix} s-1 & 2 \\ -5 & s+1 \end{bmatrix}\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \underbrace{\underbrace{\frac{1}{(s-1)(s+1)+10}}_{\text{DETERMINANT}} \begin{bmatrix} s+1 & -2 \\ 5 & s-1 \end{bmatrix}}_{\text{INVERSE MATRIX}}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \frac{1}{s^2+9}\begin{bmatrix} -(s+1) - 4 \\ -5 + 2(s-1) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} -\dfrac{s+5}{s^2+9} \\ \dfrac{2s-7}{s^2+9} \end{bmatrix}$$

- INVERTING EACH OF THE TRANSFORMS

$$x(t) = \mathcal{L}^{-1}\left[-\frac{s+5}{s^2+9}\right] = \mathcal{L}^{-1}\left[\frac{-s}{s^2+9} - \frac{5}{s^2+9}\right]$$

$$x(t) = \mathcal{L}^{-1}\left[-\frac{s}{s^2+9} - \frac{5}{3}\left(\frac{3}{s^2+9}\right)\right]$$

$$x(t) = -\cos 3t - \frac{5}{3}\sin 3t$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{2s-7}{s^2+9}\right] = \mathcal{L}^{-1}\left[2\left(\frac{s}{s^2+9}\right) - \frac{7}{s^2+9}\right]$$

$$y(t) = \mathcal{L}^{-1}\left[2\left(\frac{s}{s^2+9}\right) - \frac{7}{3}\left(\frac{3}{s^2+9}\right)\right]$$

$$y(t) = 2\cos 3t - \frac{7}{3}\sin 3t$$

Question 9

$$\frac{dx}{dt} + y = e^{-t} \qquad \text{and} \qquad \frac{dy}{dt} - x = e^{t}.$$

Use Laplace transformations to solve the above simultaneous differential equations, subject to the initial conditions $x = 0$, $y = 0$ at $t = 0$.

$\boxed{x = -\cosh t + \sin t + \cos t, \quad y = \cosh t + \sin t - \cos t}$

$$\boxed{\frac{dy}{dt} - x = e^{t}, \quad \frac{dx}{dt} + y = e^{-t}, \quad \text{SUBJECT TO } t=0,\ x=0,\ y=0}$$

WRITE IN COMPACT NOTATION & TAKE LAPLACE TRANSFORMS IN t

$$\left.\begin{array}{l} \dot{y} - x = e^{t} \\ \dot{x} + y = e^{-t} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} s\bar{y} - y_0 - \bar{x} = \frac{1}{s-1} \\ s\bar{x} - x_0 + \bar{y} = \frac{1}{s+1} \end{array}\right\} \qquad y_0 = x_0 = 0$$

$$\Rightarrow \left\{\begin{array}{l} s\bar{y} - \bar{x} = \frac{1}{s-1} \\ s\bar{x} + \bar{y} = \frac{1}{s+1} \end{array}\right\} \leftarrow (\times s)$$

$$\Rightarrow \left\{\begin{array}{l} s^2\bar{y} - s\bar{x} = \frac{s}{s-1} \\ s\bar{x} + \bar{y} = \frac{1}{s+1} \end{array}\right\} \text{ ADDING}$$

$$\Rightarrow (s^2+1)\bar{y} = \frac{s}{s-1} + \frac{1}{s+1}$$

$$\Rightarrow (s^2+1)\bar{y} = \frac{s^2+s+s-1}{(s-1)(s+1)}$$

$$\Rightarrow \bar{y} = \frac{s^2+2s-1}{(s^2+1)(s-1)(s+1)}$$

SPLIT BY PARTIAL FRACTIONS IN ORDER TO INVERT

$$\frac{s^2+2s-1}{(s^2+1)(s+1)(s-1)} \equiv \frac{A}{s+1} + \frac{B}{s-1} + \frac{Cs+D}{s^2+1}$$

$$s^2+2s-1 \equiv A(s-1)(s^2+1) + B(s+1)(s^2+1) + (s^2-1)(Cs+D)$$

- If $s=1$, $2 = 4B \Rightarrow B = \frac{1}{2}$
- If $s=-1$, $-2 = -4A \Rightarrow A = \frac{1}{2}$
- IF $s=0$, $-1 = -A+B-D$
  $D = 1-A+B = 1-\frac{1}{2}+\frac{1}{2}$
  $D = 1$
- IF $s=2$ $\quad 4+4-1 = 5A + 15B + 3(2C+D)$
  $7 = \frac{5}{2} + \frac{15}{2} + 3(2C+1)$
  $7 = 10 + 3(2C+1)$
  $-3 = 3(2C+1)$
  $-1 = 2C+1$
  $-2 = 2C$
  $C = -1$

INVERTING THE TRANSFORM, USING STANDARD RESULTS

$$\Rightarrow \bar{y} = \frac{\frac{1}{2}}{s-1} + \frac{\frac{1}{2}}{s+1} - \frac{s-1}{s^2+1}$$

$$\Rightarrow \bar{y} = \frac{1}{2}\left(\frac{1}{s-1}\right) + \frac{1}{2}\left(\frac{1}{s-1}\right) - \left(\frac{s}{s^2+1}\right) + \left(\frac{1}{s^2+1}\right)$$

$$\Rightarrow y = \frac{1}{2}e^{t} + \frac{1}{2}e^{-t} - \cos t + \sin t$$

$$\Rightarrow y = \cosh t - \cos t + \sin t$$

TO FIND THE OTHER SOLUTION, USE THE FIRST O.D.E

$$\Rightarrow x = \frac{dy}{dt} - e^{t}$$

$$\Rightarrow x = \sinh t + \sin t + \cos t - e^{t}$$

$$\Rightarrow x = \frac{1}{2}e^{t} - \frac{1}{2}e^{-t} - e^{t} + \sin t + \cos t$$

$$\Rightarrow x = -\frac{1}{2}e^{t} - \frac{1}{2}e^{-t} + \sin t + \cos t$$

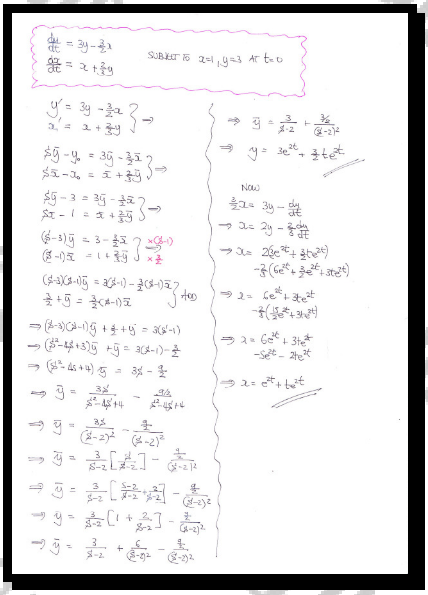
$$\Rightarrow x = -\cosh t + \cos t + \sin t$$

Question 10

$$\frac{dx}{dt} = x + \frac{2}{3}y \qquad \text{and} \qquad \frac{dy}{dt} = 3y - \frac{3}{2}x .$$

Use Laplace transformations to solve the above simultaneous differential equations, subject to the initial conditions $x=1$, $y=3$ at $t=0$.

$$\boxed{x = e^{2t} + t\,e^{2t}, \quad y = 3e^{2t} + \frac{3}{2}t\,e^{2t}}$$



$\frac{dy}{dt} = 3y - \frac{3}{2}x$

$\frac{dx}{dt} = x + \frac{2}{3}y$

SUBJECT TO $x=1$, $y=3$ AT $t=0$

$y' = 3y - \frac{3}{2}x$, $x' = x + \frac{2}{3}y$ $\Rightarrow$

$s\bar{y} - y_0 = 3\bar{y} - \frac{3}{2}\bar{x}$, $s\bar{x} - x_0 = \bar{x} + \frac{2}{3}\bar{y}$ $\Rightarrow$

$s\bar{y} - 3 = 3\bar{y} - \frac{3}{2}\bar{x}$, $s\bar{x} - 1 = \bar{x} + \frac{2}{3}\bar{y}$ $\Rightarrow$

$(s-3)\bar{y} = 3 - \frac{3}{2}\bar{x}$, $(s-1)\bar{x} = 1 + \frac{2}{3}\bar{y}$ $\xrightarrow[\times\frac{3}{2}]{\times(s-1)}$

$(s-3)(s-1)\bar{y} = 3(s-1) - \frac{3}{2}(s-1)\bar{x}$, $\frac{3}{2} + \bar{y} = \frac{3}{2}(s-1)\bar{x}$ ADD

$\Rightarrow (s-3)(s-1)\bar{y} + \frac{3}{2} + \bar{y} = 3(s-1)$

$\Rightarrow (s^2-4s+3)\bar{y} + \bar{y} = 3(s-1) - \frac{3}{2}$

$\Rightarrow (s^2-4s+4)\bar{y} = 3s - \frac{9}{2}$

$\Rightarrow \bar{y} = \frac{3s}{s^2-4s+4} - \frac{9/2}{s^2-4s+4}$

$\Rightarrow \bar{y} = \frac{3s}{(s-2)^2} - \frac{\frac{9}{2}}{(s-2)^2}$

$\Rightarrow \bar{y} = \frac{3}{s-2}\left[\frac{s}{s-2}\right] - \frac{\frac{9}{2}}{(s-2)^2}$

$\Rightarrow \bar{y} = \frac{3}{s-2}\left[\frac{s-2}{s-2} + \frac{2}{s-2}\right] - \frac{\frac{9}{2}}{(s-2)^2}$

$\Rightarrow \bar{y} = \frac{3}{s-2}\left[1 + \frac{2}{s-2}\right] - \frac{\frac{9}{2}}{(s-2)^2}$

$\Rightarrow \bar{y} = \frac{3}{s-2} + \frac{6}{(s-2)^2} - \frac{\frac{9}{2}}{(s-2)^2}$

$\Rightarrow \bar{y} = \frac{3}{s-2} + \frac{3/2}{(s-2)^2}$

$\Rightarrow y = 3e^{2t} + \frac{3}{2}te^{2t}$

NOW

$\frac{3}{2}x = 3y - \frac{dy}{dt}$

$\Rightarrow x = 2y - \frac{2}{3}\frac{dy}{dt}$

$\Rightarrow x = 2(3e^{2t} + \frac{3}{2}te^{2t}) - \frac{2}{3}(6e^{2t} + \frac{3}{2}e^{2t} + 3te^{2t})$

$\Rightarrow x = 6e^{2t} + 3te^{2t} - \frac{2}{3}(\frac{15}{2}e^{2t} + 3te^{2t})$

$\Rightarrow x = 6e^{2t} + 3te^{2t} - 5e^{2t} - 2te^{2t}$

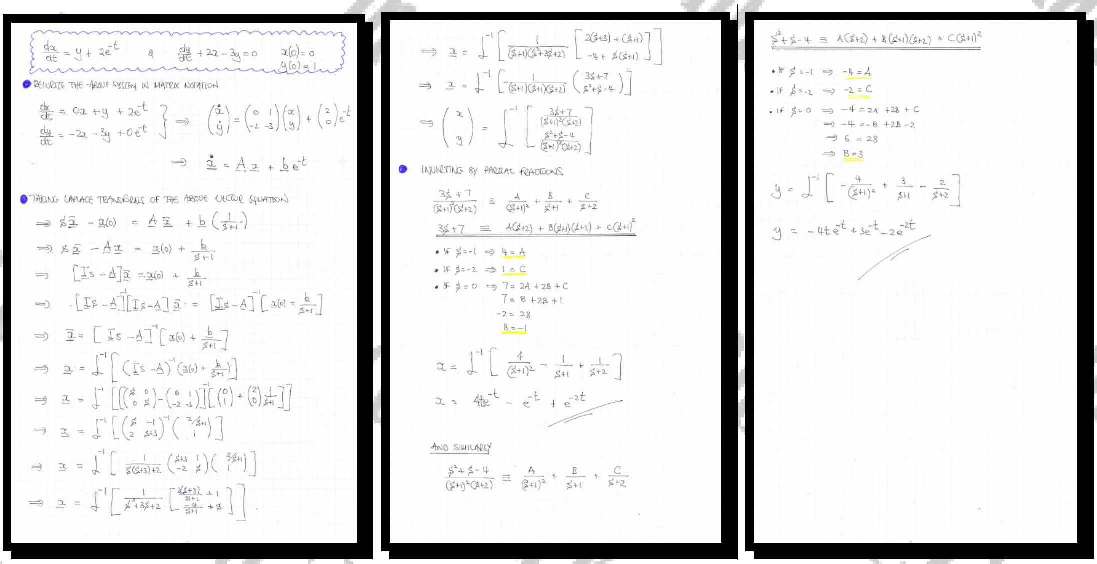
$\Rightarrow x = e^{2t} + te^{2t}$

Question 11

$$\frac{dx}{dt} = y + 2e^{-t} \qquad \text{and} \qquad \frac{dy}{dt} + 2x - 3y = 0.$$

Use Laplace transformations to solve the above simultaneous differential equations, subject to the initial conditions $x = 0$, $y = 1$ at $t = 0$.

$$\boxed{x = 4t\,e^{-t} - e^{-t} + e^{-2t}, \quad y = -4t\,e^{-t} + 3e^{-t} - 2e^{-2t}}$$



Question 12

$$\frac{d^2x}{dt^2} = 15\frac{dy}{dt} - 9y + 22e^{t} \qquad \text{and} \qquad \frac{d^2y}{dt^2} = 2x + e^{3t}.$$

The functions $x = f(t)$ and $y = g(t)$ satisfy the above simultaneous differential equations, subject to the initial conditions

$$x = 2, \quad y = -3, \quad \frac{dx}{dt} = 10, \quad \frac{dy}{dt} = -1 \quad \text{at } t = 0.$$

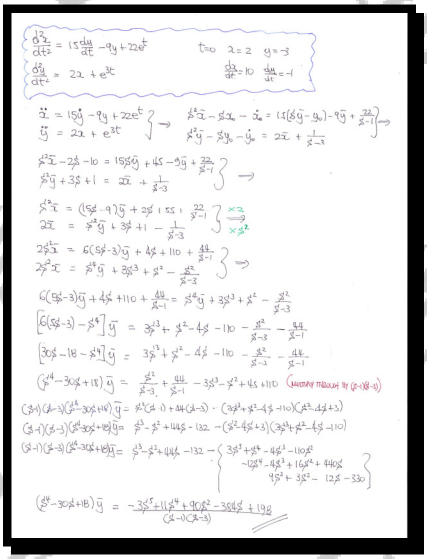
a) By using Laplace transforms, show that

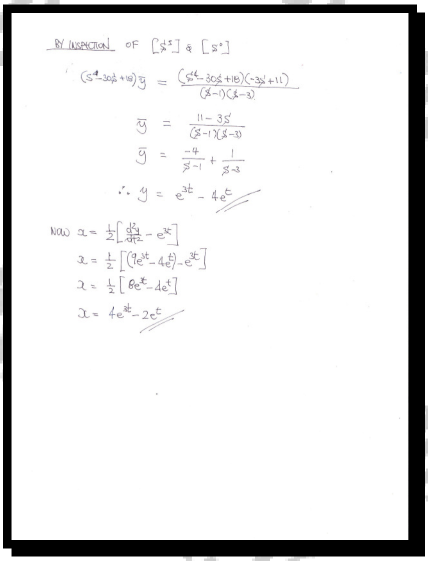
$$\left(s^4 - 30s + 18\right)\bar{y} = \frac{-3s^5 + 11s^4 + 90s^2 - 384s + 198}{(s-1)(s-3)},$$

where $\bar{y} = \mathcal{L}\left[g(t)\right]$.

b) Given further that $s^4 - 30s + 18$ is a factor of $-3s^5 + 11s^4 + 90s^2 - 384s + 198$, find expressions for x and y.

$$\boxed{x = 4e^{3t} - 2e^{t}, \quad y = e^{3t} - 4e^{t}}$$





Question 13

Use Laplace transforms to solve the differential equation

$$\frac{d^2x}{dt^2} + x = f(t),$$

given further that $x = 1$, $\dfrac{dx}{dt} = 1$ at $t = 0$, and

$$f(t) = \begin{cases} 0 & t < 0 \\ t & 0 \le t \le \pi \\ \pi & t > \pi \end{cases}$$

$$\boxed{x = t + \cos t - (t-\pi)\mathrm{H}(t-\pi) + \sin(t-\pi)\mathrm{H}(t-\pi)}$$

$$\boxed{\frac{d^2x}{dt^2} + x = f(t) \quad \text{WHERE} \quad f(t) = \begin{cases} t & 0 \le t \le \pi \\ \pi & t > \pi \\ 0 & t < 0 \end{cases} \qquad \text{SUBJECT TO } x=1,\ \frac{dx}{dt}=1 \text{ AT } t=0}$$

$$\ddot{x} + x = f(t)$$

TAKING LAPLACE TRANSFORMS

$$\Rightarrow \mathcal{L}[\ddot{x}] + \mathcal{L}[x] = \mathcal{L}[f(t)]$$

$$\Rightarrow s^2\bar{x} - sx_0 - \dot{x}_0 + \bar{x} = \int_0^\infty f(t)e^{-st}\,dt$$

$$\Rightarrow s^2\bar{x} - s - 1 + \bar{x} = \int_0^\pi te^{-st}\,dt + \int_\pi^\infty \pi e^{-st}\,dt$$

$$\Rightarrow (1+s^2)\bar{x} - (1+s) = \int_0^\pi -\frac{d}{ds}\left[e^{-st}\right]dt + \frac{\pi}{s}\left[e^{-st}\right]_\infty^\pi$$

$$\Rightarrow (1+s^2)\bar{x} = (s+1) - \frac{d}{ds}\int_0^\pi e^{-st}\,dt + \frac{\pi}{s}\left[e^{-s\pi} - 0\right]$$

$$\Rightarrow (1+s^2)\bar{x} = s+1 - \frac{d}{ds}\left[-\frac{1}{s}\left[e^{-st}\right]_0^\pi\right] + \frac{\pi}{s}\left[e^{-s\pi}\right]$$

$$\Rightarrow (1+s^2)\bar{x} = s+1 - \frac{d}{ds}\left[\frac{1}{s} - \frac{1}{s}e^{-s\pi}\right] + \frac{\pi e^{-s\pi}}{s}$$

$$\Rightarrow (1+s^2)\bar{x} = s+1 - \left[-\frac{1}{s^2} + \frac{1}{s^2}e^{-s\pi} + \frac{\pi}{s}e^{-s\pi}\right] + \frac{\pi}{s}e^{-s\pi}$$

$$\Rightarrow (1+s^2)\bar{x} = s+1 + \frac{1}{s^2} - \frac{1}{s^2}e^{-s\pi} \cancel{- \frac{\pi}{s}e^{-s\pi}} \cancel{+ \frac{\pi}{s}e^{-s\pi}}$$

$$\Rightarrow \bar{x} = \frac{s+1 + \frac{1}{s^2}\left(1 - e^{-s\pi}\right)}{s^2+1}$$

$$\Rightarrow \bar{x} = \frac{s+1}{s^2+1} + \frac{1}{s^2(s^2+1)}\left(1 - e^{-s\pi}\right)$$

PARTIAL FRACTIONS BY COVER UP SINCE IT CAN BE TREATED AS $\dfrac{1}{A(A+1)}$

$$\Rightarrow \bar{x} = \frac{s}{s^2+1} + \frac{1}{s^2+1} + \frac{1}{s^2}\left(1 - e^{-s\pi}\right) - \frac{1}{s^2+1}\left(1 - e^{-s\pi}\right)$$

$$\Rightarrow \bar{x} = \frac{s}{s^2+1} + \cancel{\frac{1}{s^2+1}} + \frac{1}{s^2} - \frac{e^{-s\pi}}{s^2} - \cancel{\frac{1}{s^2+1}} + \frac{e^{-s\pi}}{s^2+1}$$

$$\Rightarrow \bar{x} = \underbrace{\frac{s}{s^2+1}}_{\text{STANDARD } \cos t} + \underbrace{\frac{1}{s^2}}_{\text{STANDARD } t} - \underbrace{\frac{e^{-s\pi}}{s^2}}_{t \text{ "HEAVISIDED"}} + \frac{e^{-s\pi}}{s^2+1}$$

$\sin t$ "HEAVISIDED"

INVERTING

$$x(t) = \cos t + t - (t-\pi)\mathrm{H}(t-\pi) + \sin(t-\pi)\mathrm{H}(t-\pi)$$

$$\boxed{\text{SINCE } \mathcal{L}\left[f(t-c)\mathrm{H}(t-c)\right] = e^{-cs}\bar{f}(s)}$$

WHICH CAN ALSO BE WRITTEN AS

$$x(t) = \begin{cases} \cos t + t & 0 \le t \le \pi \\ \pi + \cos t - \sin t & t \ge \pi \end{cases}$$

$\sin(t-\pi) = -\sin t$

Question 14

Use Laplace transforms to solve the differential equation

$$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+5x=\delta(t-2),$$

given further that $x=0$, $\frac{dx}{dt}=1$ at $t=0$.

$$\boxed{x=\mathrm{e}^{-t}\left[\sin 2t-\mathrm{e}^{4}\sin(2t-4)\mathrm{H}(t-2)\right]}$$

$$\ddot{x}+2\dot{x}+5x=\delta(t-2) \qquad t=0,\ x_0=0,\ \dot{x}_0=1$$

TAKING LAPLACE TRANSFORMS

$$\Rightarrow \left[s^2\bar{x}-sx_0-\dot{x}_0\right]+2\left[s\bar{x}-x_0\right]+5\bar{x}=\mathcal{L}\left[\delta(t-2)\right]$$

$$\Rightarrow s^2\bar{x}-1+2s\bar{x}+5\bar{x}=e^{-2s}$$

$$\Rightarrow \bar{x}(s^2+2s+5)=1-e^{-2s}$$

$$\Rightarrow \bar{x}=\frac{1-e^{-2s}}{s^2+2s+5}$$

$$\Rightarrow \bar{x}=\frac{1-e^{-2s}}{(s+1)^2+4}$$

$$\Rightarrow \bar{x}=\frac{1}{(s+1)^2+4}-\frac{e^{-2s}}{(s+1)^2+4}$$

INVERTING

$$\Rightarrow x=e^{-t}\sin 2t-e^{-(t-2)}\sin 2(t-2)\,H(t-2)$$

$$\Rightarrow x=e^{-t}\sin 2t-e^{-t}e^{4}\sin(2t-4)\,H(t-2)$$

Question 15

Use Laplace transforms to solve the differential equation

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = 2\delta(t-6),$$

given further that $x = 0$, $\frac{dx}{dt} = 2$ at $t = 0$.

$$\boxed{x = \mathrm{e}^{-3t}\left[\mathrm{e}^{2t} - 1\right] + \mathrm{e}^{-3t}\mathrm{e}^{6}\left[\mathrm{e}^{12} - \mathrm{e}^{2t}\right]\mathrm{H}(t-6)}$$

$\ddot{x} + 4\dot{x} + 3x = 2\delta(t-6)$ SUBJECT TO $t=0$, $x_0=0$, $\dot{x}_0=2$

TAKING LAPLACE TRANSFORMS

$$\Rightarrow \left[s^2\bar{x} - sx_0 - \dot{x}_0\right] + 4\left[s\bar{x} - x_0\right] + 3\bar{x} = \mathcal{L}\left[2\delta(t-6)\right]$$

$$\Rightarrow s^2\bar{x} - 2 + 4s\bar{x} + 3\bar{x} = 2e^{-6s}$$

$$\Rightarrow \bar{x}(s^2 + 4s + 3) = 2 - 2e^{-6s}$$

$$\Rightarrow \bar{x} = \frac{2(1 - e^{-6s})}{s^2 + 4s + 3}$$

$$\Rightarrow \bar{x} = 2(1 - e^{-6s}) \times \frac{1}{(s+1)(s+3)}$$ ← PARTIAL FRACTIONS

$$\Rightarrow \bar{x} = 2(1 - e^{-6s}) \times \left[\frac{\frac{1}{2}}{s+1} - \frac{\frac{1}{2}}{s+3}\right]$$

$$\Rightarrow \bar{x} = \frac{1 - e^{-6s}}{s+1} - \frac{1 - e^{-6s}}{s+3}$$

$$\Rightarrow \bar{x} = \frac{1}{s+1} - \frac{e^{-6s}}{s+1} - \frac{1}{s+3} + \frac{e^{-6s}}{s+3}$$

INVERTING...

$$x(t) = e^{-t} - e^{-(t-6)}H(t-6) - e^{-3t} + e^{-3(t-6)}H(t-6)$$

$$x(t) = e^{-t} - e^{-3t} + e^{-3t}e^{18}H(t-6) - e^{-t}e^{6}H(t-6)$$

$$x(t) = e^{-3t}\left[e^{2t} - 1\right] + e^{-3t}e^{6}H(t-6)\left[e^{12} - e^{2t}\right]$$

Question 16

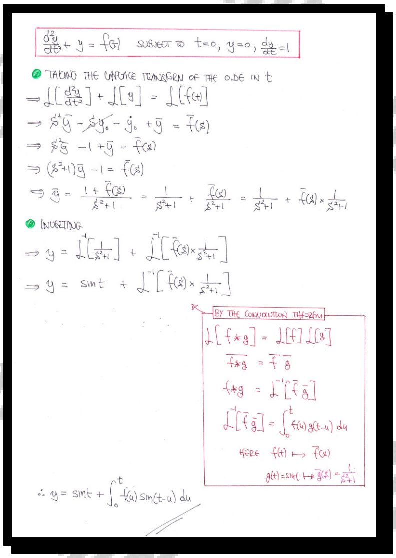
Use Laplace transforms to solve the differential equation

$$\frac{d^2y}{dt^2} + y = f(t),$$

given further that $y = 0$, $\frac{dy}{dt} = 1$ at $t = 0$, and $f(t)$ is a known function which has a Laplace transform.

You may leave the final answer containing a convolution type integral.

$$\boxed{y = \sin t + \int_0^t f(u)\sin(t-u)\ du}$$



$$\boxed{\frac{d^2y}{dt^2} + y = f(t) \quad \text{SUBJECT TO} \quad t=0,\ y=0,\ \frac{dy}{dt}=1}$$

TAKING THE LAPLACE TRANSFORM OF THE O.D.E IN t

$$\Rightarrow \mathcal{L}\left[\frac{d^2y}{dt^2}\right] + \mathcal{L}[y] = \mathcal{L}[f(t)]$$

$$\Rightarrow s^2\bar{y} - \cancel{s y_0} - \dot{y}_0 + \bar{y} = \bar{f}(s)$$

$$\Rightarrow s^2\bar{y} - 1 + \bar{y} = \bar{f}(s)$$

$$\Rightarrow (s^2+1)\bar{y} - 1 = \bar{f}(s)$$

$$\Rightarrow \bar{y} = \frac{1+\bar{f}(s)}{s^2+1} = \frac{1}{s^2+1} + \frac{\bar{f}(s)}{s^2+1} = \frac{1}{s^2+1} + \bar{f}(s) \times \frac{1}{s^2+1}$$

INVERTING

$$\Rightarrow y = \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] + \mathcal{L}^{-1}\left[\bar{f}(s) \times \frac{1}{s^2+1}\right]$$

$$\Rightarrow y = \sin t + \mathcal{L}^{-1}\left[\bar{f}(s) \times \frac{1}{s^2+1}\right]$$

BY THE CONVOLUTION THEOREM

$$\mathcal{L}[f * g] = \mathcal{L}[f]\,\mathcal{L}[g]$$

$$\overline{f*g} = \bar{f}\,\bar{g}$$

$$f*g = \mathcal{L}^{-1}[\bar{f}\,\bar{g}]$$

$$\mathcal{L}^{-1}[\bar{f}\,\bar{g}] = \int_0^t f(u)\,g(t-u)\,du$$

HERE $f(t) \mapsto \bar{f}(s)$

$g(t) = \sin t \mapsto \bar{g}(s) = \frac{1}{s^2+1}$

$$\therefore y = \sin t + \int_0^t f(u)\sin(t-u)\,du$$

Question 17

$$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+2x=f(t).$$

a) Use Laplace transforms to solve the above differential equation, given further that $x=0$, $\frac{dx}{dt}=0$ at $t=0$, and $f(t)$ is a known function which has a Laplace transform.

You may leave the answer containing a convolution type integral.

b) If $f(t)=\mathrm{e}^{2t}$ find $x=x(t)$ explicitly.

$$\boxed{x=\int_0^t f(t-u)\,\mathrm{e}^{-u}\sin u\ du}\ ,\ \boxed{x=-\frac{1}{10}\mathrm{e}^{-t}[3\sin t+\cos t]+\frac{1}{10}\mathrm{e}^{2t}}$$

a) $\boxed{\frac{d^2x}{dt^2}+2\frac{dx}{dt}+2x=f(t) \text{ SUBJECT TO } t=0,\ x=0,\ \dot{x}=0}$

TAKING THE LAPLACE TRANSFORM OF THE EQUATION

$\Rightarrow \mathcal{L}\left[\frac{d^2x}{dt^2}\right]+2\mathcal{L}\left[\frac{dx}{dt}\right]+2\mathcal{L}[x]=\mathcal{L}[f(t)]$

$\Rightarrow s^2\bar{x}-\cancel{sx_0}-\cancel{\dot{x}_0}+2[s\bar{x}-\cancel{x_0}]+2\bar{x}=\bar{f}(s)$

$\Rightarrow (s^2+2s+2)\bar{x}=\bar{f}(s)$

$\Rightarrow \bar{x}=\frac{\bar{f}(s)}{s^2+2s+2}$

$\Rightarrow \bar{x}=\bar{f}(s)\times\frac{1}{s^2+2s+2}$

$\Rightarrow \bar{x}=\underbrace{\bar{f}(s)}_{\bar{f}(s)}\times\underbrace{\frac{1}{(s+1)^2+1}}_{\bar{g}(s)}$

$\bar{g}(s)\longrightarrow g(t)=\mathrm{e}^{-t}\sin t$ (BY INSPECTION)

BY THE CONVOLUTION THEOREM

$\Rightarrow \mathcal{L}[f*g]=\mathcal{L}[f]\,\mathcal{L}[g]$

$\Rightarrow \overline{f*g}=\bar{f}\,\bar{g}$

$\Rightarrow \mathcal{L}^{-1}[\overline{f*g}]=\mathcal{L}^{-1}[\bar{f}\,\bar{g}]$

$\Rightarrow f*g=\mathcal{L}^{-1}[\bar{f}\,\bar{g}]$

$\Rightarrow f*g \quad \mathcal{L}^{-1}(\bar{x})$

$\Rightarrow \int_0^t f(u)g(t-u)\,du=x$

$\Rightarrow x=\int_0^t f(t-u)g(u)\,du$ ("COMMUTATIVE")

$\Rightarrow x=\int_0^t f(t-u)\,\mathrm{e}^{-u}\sin u\,du$

b) Now $f(t)=\mathrm{e}^{2t}$ so $f(t-u)=\mathrm{e}^{2(t-u)}$

$\bar{x}=\int_0^t \mathrm{e}^{2t-2u}\mathrm{e}^{-u}\sin u\,du=\mathrm{e}^{2t}\int_0^t \mathrm{e}^{-3u}\sin u\,du$

$=\mathrm{e}^{2t}\,\mathrm{Im}\left[\int_0^t \mathrm{e}^{-3u}\mathrm{e}^{iu}\,du\right]=\mathrm{e}^{2t}\,\mathrm{Im}\left[\int_0^t \mathrm{e}^{u(-3+i)}\,du\right]$

$=\mathrm{e}^{2t}\,\mathrm{Im}\left[\frac{1}{-3+i}\mathrm{e}^{u(-3+i)}\right]_0^t=\mathrm{e}^{2t}\,\mathrm{Im}\left[\frac{-3-i}{10}\mathrm{e}^{-3u}\mathrm{e}^{iu}\right]_0^t$

$=\mathrm{e}^{2t}\,\mathrm{Im}\left[\frac{1}{10}(-3-i)\mathrm{e}^{-3u}(\cos u+i\sin u)\right]_0^t$

SIMPLIFYING

$=\mathrm{e}^{2t}\,\mathrm{Im}\left[\frac{1}{10}\mathrm{e}^{-3t}(-3-i)(\cos t+i\sin t)+\frac{1}{10}(-3-i)\right]$

$=\frac{1}{10}\mathrm{e}^{2t}\left[\mathrm{e}^{-3t}(-3\sin t-\cos t)+1\right]$

$=\frac{1}{10}\left[\mathrm{e}^{2t}+\mathrm{e}^{-t}(-3\sin t-\cos t)\right]$

$=-\frac{1}{10}\mathrm{e}^{-t}(3\sin t+\cos t)+\frac{1}{10}\mathrm{e}^{2t}$

Question 18

Use Laplace transforms to solve the differential equation

$$\frac{d^2x}{dt^2} + 16x = f(t),$$

given further that $x = 0$, $\dfrac{dx}{dt} = 1$ at $t = 0$, and

$$f(t) = \begin{cases} \cos 4t & 0 \le t \le \pi \\ 0 & t > \pi \end{cases}$$

[*You may find the Laplace transform of* $t\sin 4t$ *useful in this question.*]

$$\boxed{x(t) = \frac{1}{4}\sin 4t + \frac{1}{8}t\sin 4t - \frac{1}{8}(t-\pi)\,\mathrm{H}(t-\pi)\,\sin 4t}$$

$$\boxed{\frac{d^2x}{dt^2} + 16x = f(t), \quad x(0)=0, \quad \frac{dx}{dt}(0)=1, \quad f(t) = \begin{cases} \cos 4t & 0 \le t \le \pi \\ 0 & t \geqslant \pi \end{cases}}$$

- TAKING LAPLACE TRANSFORMS IN t

$$\mathcal{L}\left[\frac{d^2x}{dt^2}\right] + 16\mathcal{L}[x] = \mathcal{L}[f(t)]$$

$$s^2\bar{x} - s\cancel{x(0)} - \frac{dx}{dt}(0) + 16\bar{x} = \bar{f}(s)$$

$$s^2\bar{x} - 1 + 16\bar{x} = \bar{f}(s)$$

$$(s^2+16)\bar{x} = 1 + \bar{f}(s)$$

$$\bar{x} = \frac{1+\bar{f}(s)}{s^2+16} = \frac{1}{s^2+16} + \frac{1}{s^2+16}\bar{f}(s)$$

$$\boxed{\begin{aligned} \text{NOW } f(t) &= \cos 4t - \mathrm{H}(t-\pi)\cos 4t \\ f(t) &= \cos 4t - \mathrm{H}(t-\pi)\cos[4(t-\pi)] \\ \mathcal{L}[f(t)] = \bar{f}(s) &= \frac{s}{s^2+16} - \frac{se^{-\pi s}}{s^2+16} = \frac{s - se^{-\pi s}}{s^2+16} \end{aligned}}$$

- THUS WE OBTAIN

$$\bar{x} = \frac{1}{s^2+16} + \frac{1}{s^2+16}\left[\frac{s}{s^2+16} - \frac{se^{-\pi s}}{s^2+16}\right]$$

$$\bar{x} = \frac{1}{s^2+16} + \frac{s}{(s^2+16)^2} - \frac{se^{-\pi s}}{(s^2+16)^2}$$

- NEXT CONSIDER THE LAPLACE TRANSFORM OF $t\sin 4t$

$$\mathcal{L}[t\sin 4t] = -\frac{d}{ds}\left[\frac{4}{s^2+16}\right]$$

$$= -4\frac{d}{ds}\left[(s^2+16)^{-1}\right]$$

$$= \frac{4\times 2s}{(s^2+16)^2}$$

- INVERTING WE OBTAIN

$$\bar{x}(s) = \frac{1}{4}\left[\frac{4}{s^2+16}\right] + \frac{1}{8}\left[\frac{8s}{(s^2+16)^2}\right] - \frac{1}{8}\left[\frac{8se^{-\pi s}}{(s^2+16)^2}\right]$$

$$x(t) = \frac{1}{4}\sin 4t + \frac{1}{8}t\sin 4t - \frac{1}{8}\mathrm{H}(t-\pi)(t-\pi)\sin[4(t-\pi)]$$

$$x(t) = \frac{1}{4}\sin 4t + \frac{1}{8}t\sin 4t - \frac{1}{8}\mathrm{H}(t-\pi)(t-\pi)\sin(4t-4\pi)$$

$$x(t) = \frac{1}{4}\sin 4t + \frac{1}{8}t\sin 4t - \frac{1}{8}(t-\pi)\mathrm{H}(t-\pi)\sin 4t$$

Question 19

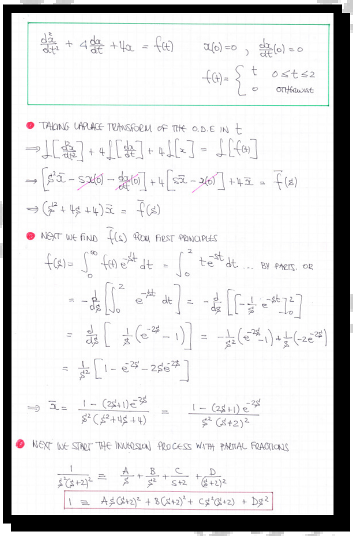
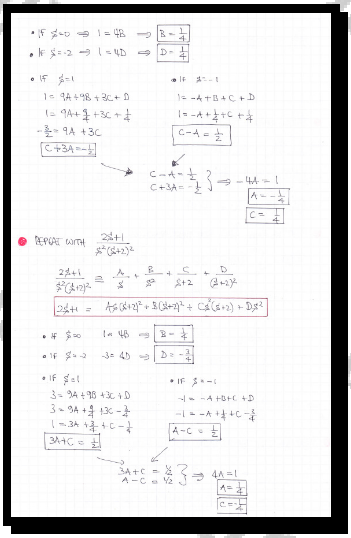
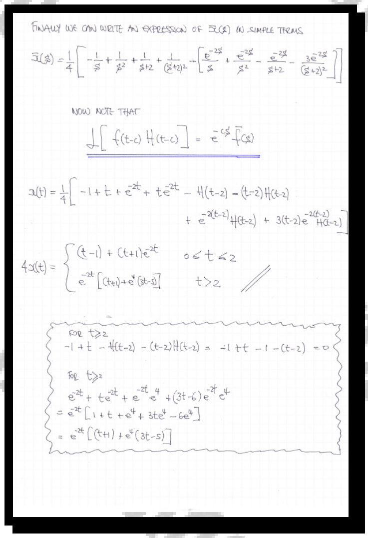
Use Laplace transforms to solve the differential equation

$$\frac{d^2x}{dt^2}+4\frac{dx}{dt}+4x=f(t),$$

given further that $x=0$, $\dfrac{dx}{dt}=0$ at $t=0$, and

$$f(t)=\begin{cases}0 & t<0\\ t & 0\le t\le 2\\ 0 & t>2\end{cases}$$

$$\boxed{x(t)=\begin{cases}0 & t<0\\ t-1+(t+1)\mathrm{e}^{-2t} & 0\le t\le 2\\ \mathrm{e}^{-2t}\left[t+1+\mathrm{e}^{4}(3t-5)\right] & t>2\end{cases}}$$

EVALUATION OF INTEGRALS

Question 1

$$\int_0^{\infty} t\,\mathrm{e}^{-2t}\cos t \;\, dt \,.$$

Given that the above integral is finite, use Laplace transform techniques to find its exact value.

$$\boxed{\frac{3}{25}}$$

$$\boxed{\int_0^{\infty} t e^{-2t}\cos t \, dt = \frac{3}{25}}$$

- CONSIDER THE LAPLACE TRANSFORM OF $t\cos t$

$$\Rightarrow \mathcal{L}\left[t\cos t\right] = -\frac{d}{ds}\left[\mathcal{L}\left[\cos t\right]\right]$$

$$\Rightarrow \int_0^{\infty} (t\cos t)e^{-st}\, dt = -\frac{d}{ds}\left[\frac{s}{s^2+1}\right]$$

$$\Rightarrow \int_0^{\infty} t e^{-st}\cos t \, dt = -\frac{(s^2+1)\times 1 - s(2s)}{(s^2+1)^2}$$

$$\Rightarrow \int_0^{\infty} t e^{-st}\cos t \, dt = -\frac{s^2+1-2s^2}{(s^2+1)^2}$$

$$\Rightarrow \int_0^{\infty} t e^{-st}\cos t \, dt = -\frac{1-s^2}{(1+s^2)^2}$$

$$\Rightarrow \int_0^{\infty} t e^{-st}\cos t \, dt = \frac{s^2-1}{(s^2+1)^2}$$

- LET $s=2$

$$\Rightarrow \int_0^{\infty} t e^{-2t}\cos t \, dt = \frac{2^2-1}{(2^2+1)^2}$$

$$\Rightarrow \int_0^{\infty} t e^{-2t}\cos t \, dt = \frac{3}{25}$$

Question 2

$$\int_0^\infty x\mathrm{e}^{-3x}\sin 2x \;\, dx\,.$$

Given that the above integral is finite, use Laplace transform techniques to find its exact value.

$$\boxed{\frac{12}{169}}$$

$$\boxed{\int_0^\infty x e^{-3x}\sin 2x \, dx = ?}$$

- CONSIDER THE LAPLACE TRANSFORM OF $t\sin 2t$

$$\Rightarrow \mathcal{L}\left[t\sin 2t\right] = -\frac{d}{ds}\mathcal{L}\left[\sin 2t\right]$$

← MULTIPLICATION RULE BY t

$$\Rightarrow \int_0^\infty (t\sin 2t)e^{-st}\,dt = -\frac{d}{ds}\left[\frac{2}{s^2+4}\right]$$

$$\Rightarrow \int_0^\infty te^{-st}\sin 2t\,dt = -2\left(\frac{-2s}{(s^2+4)^2}\right)$$

$$\Rightarrow \int_0^\infty te^{-st}\sin 2t\,dt = \frac{4s}{(s^2+4)^2}$$

- REWRITE THE INTEGRAL IN x

$$\Rightarrow \int_0^\infty xe^{-sx}\sin 2x\,dx = \frac{4s}{(s^2+4)^2}$$

- LET $s=3$

$$\Rightarrow \int_0^\infty xe^{-3x}\sin 2x\,dx = \frac{4\times 3}{13^2}$$

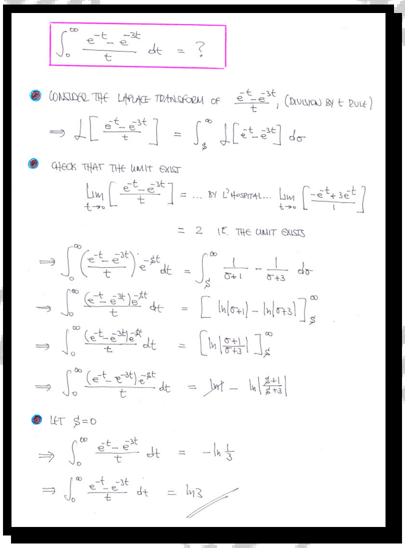
$$\Rightarrow \int_0^\infty xe^{-3x}\sin 2x\,dx = \frac{12}{169}$$

Question 3

$$\int_0^\infty \frac{e^{-t} - e^{-3t}}{t} \, dt \, .$$

Given that the above integral is finite, use Laplace transform techniques to find its exact value.

$\boxed{\ln 3}$



$$\boxed{\int_0^\infty \frac{e^{-t} - e^{-3t}}{t} \, dt = ?}$$

- CONSIDER THE LAPLACE TRANSFORM OF $\frac{e^{-t} - e^{-3t}}{t}$, (DIVISION BY t RULE)

$$\Rightarrow \mathcal{L}\left[\frac{e^{-t} - e^{-3t}}{t}\right] = \int_s^\infty \mathcal{L}\left[e^{-t} - e^{-3t}\right] d\sigma$$

- CHECK THAT THE LIMIT EXIST

$$\lim_{t \to 0}\left[\frac{e^{-t} - e^{-3t}}{t}\right] = \dots \text{ BY L'HOSPITAL} \dots \lim_{t \to 0}\left[\frac{-e^{-t} + 3e^{-t}}{1}\right]$$

$$= 2 \quad \text{I.E. THE LIMIT EXISTS}$$

$$\Rightarrow \int_0^\infty \left(\frac{e^{-t} - e^{-3t}}{t}\right) e^{-st} \, dt = \int_s^\infty \frac{1}{\sigma + 1} - \frac{1}{\sigma + 3} \, d\sigma$$

$$\Rightarrow \int_0^\infty \frac{(e^{-t} - e^{-3t})e^{-st}}{t} \, dt = \Big[\ln|\sigma + 1| - \ln|\sigma + 3|\Big]_s^\infty$$

$$\Rightarrow \int_0^\infty \frac{(e^{-t} - e^{-3t})e^{-st}}{t} \, dt = \left[\ln\left|\frac{\sigma + 1}{\sigma + 3}\right|\right]_s^\infty$$

$$\Rightarrow \int_0^\infty \frac{(e^{-t} - e^{-3t})e^{-st}}{t} \, dt = \cancel{\ln 1} - \ln\left|\frac{s + 1}{s + 3}\right|$$

- LET $s = 0$

$$\Rightarrow \int_0^\infty \frac{e^{-t} - e^{-3t}}{t} \, dt = -\ln\frac{1}{3}$$

$$\Rightarrow \int_0^\infty \frac{e^{-t} - e^{-3t}}{t} \, dt = \ln 3$$

Question 4

$$\int_0^{\infty} \frac{\sin x}{x} \, dx .$$

Given that the above integral is finite, use Laplace transform techniques to find its exact value.

$$\boxed{\frac{\pi}{2}}$$

$$\boxed{\int_0^{\infty} \frac{\sin x}{x} \, dx = \frac{\pi}{2}}$$

- CONSIDER THE LAPLACE TRANSFORM OF $\frac{\sin t}{t}$

$$\Rightarrow \mathcal{L}\left[\frac{\sin t}{t}\right] = \int_s^{\infty} \mathcal{L}[\sin t] \, d\sigma$$ (DIVISION BY t RULE)

NOTE $\lim_{t \to 0}\left(\frac{\sin t}{t}\right)$ EXIST FOR THIS RULE TO APPLY

$$\Rightarrow \int_0^{\infty} \left(\frac{\sin t}{t}\right) e^{-st} \, dt = \int_s^{\infty} \frac{1}{\sigma^2+1} \, d\sigma$$

$$\Rightarrow \int_0^{\infty} \frac{e^{-st}\sin t}{t} \, dt = \Big[\arctan \sigma\Big]_{\sigma=s}^{\sigma=\infty}$$

$$\Rightarrow \int_0^{\infty} \frac{e^{-st}\sin t}{t} \, dt = \arctan(\infty) - \arctan s$$

$$\Rightarrow \int_0^{\infty} \frac{e^{-st}\sin t}{t} \, dt = \frac{\pi}{2} - \arctan s$$

- WRITE THE INTEGRAL IN x

$$\Rightarrow \int_0^{\infty} \frac{e^{-sx}\sin x}{x} \, dx = \frac{\pi}{2} - \arctan s$$

- LET $s = 0$

$$\Rightarrow \int_0^{\infty} \frac{\sin x}{x} \, dx = \frac{\pi}{2} - \arctan 0$$

$$\Rightarrow \int_0^{\infty} \frac{\sin x}{x} \, dx = \frac{\pi}{2}$$

Question 5

$$\int_0^{\infty} \frac{e^{-\frac{1}{3}x}\sin x}{x}\,dx\,.$$

Given that the above integral is finite, use Laplace transform techniques to find its exact value.

$\boxed{\arctan 3}$

$$\boxed{\int_0^{\infty} \frac{e^{-\frac{1}{3}x}\sin x}{x}\,dx}$$

- CONSIDER THE LAPLACE TRANSFORM OF $\frac{\sin t}{t}$

$$\Rightarrow \mathcal{L}\left[\frac{\sin t}{t}\right] = \int_s^{\infty} \mathcal{L}[\sin t]\,ds$$

AS $\lim_{t \to 0}\left[\frac{\sin t}{t}\right] = 1$ I.E IT EXISTS

$$\Rightarrow \int_0^{\infty}\left(\frac{\sin t}{t}\right)e^{-st}\,dt = \int_s^{\infty} \frac{1}{1+s^2}\,ds$$

$$\Rightarrow \int_0^{\infty} \frac{e^{-st}\sin t}{t}\,dt = \Big[\arctan s\Big]_s^{\infty}$$

$$\Rightarrow \int_0^{\infty} \frac{e^{-st}\sin t}{t}\,dt = \frac{\pi}{2} - \arctan s$$

- REWRITE THE INTEGRAL WITH t AS x

$$\Rightarrow \int_0^{\infty} \frac{e^{-sx}\sin x}{x}\,dx = \frac{\pi}{2} - \arctan s$$

- LET $s = \frac{1}{3}$

$$\Rightarrow \int_0^{\infty} \frac{e^{-\frac{1}{3}x}\sin x}{x}\,dx = \frac{\pi}{2} - \arctan\frac{1}{3} = \text{arccot}\,\frac{1}{3}$$

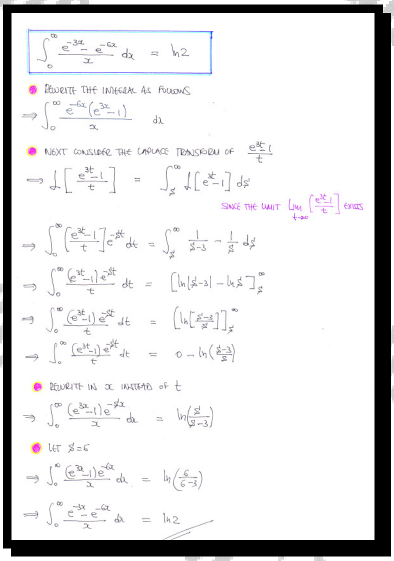
$$\Rightarrow \int_0^{\infty} \frac{e^{-\frac{1}{3}x}\sin x}{x}\,dx = \arctan 3$$

Question 6

$$\int_0^{\infty} \frac{e^{-3x} - e^{-6x}}{x} \, dx .$$

Given that the above integral is finite, use Laplace transform techniques to find its exact value.

$\boxed{\ln 2}$



$$\boxed{\int_0^{\infty} \frac{e^{-3x} - e^{-6x}}{x} \, dx = \ln 2}$$

- REWRITE THE INTEGRAL AS FOLLOWS

$$\Rightarrow \int_0^{\infty} \frac{e^{-6x}\left(e^{3x}-1\right)}{x} \, dx$$

- NEXT CONSIDER THE LAPLACE TRANSFORM OF $\frac{e^{3t}-1}{t}$

$$\Rightarrow \mathcal{L}\left[\frac{e^{3t}-1}{t}\right] = \int_s^{\infty} \mathcal{L}\left[e^{3t}-1\right] ds$$

SINCE THE LIMIT $\lim_{t \to 0}\left[\frac{e^{3t}-1}{t}\right]$ EXISTS

$$\Rightarrow \int_0^{\infty}\left[\frac{e^{3t}-1}{t}\right]e^{-st} \, dt = \int_s^{\infty} \frac{1}{s-3} - \frac{1}{s} \, ds$$

$$\Rightarrow \int_0^{\infty} \frac{\left(e^{3t}-1\right)e^{-st}}{t} \, dt = \Big[\ln|s-3| - \ln s\Big]_s^{\infty}$$

$$\Rightarrow \int_0^{\infty} \frac{\left(e^{3t}-1\right)e^{-st}}{t} \, dt = \left[\ln\left[\frac{s-3}{s}\right]\right]_s^{\infty}$$

$$\Rightarrow \int_0^{\infty} \frac{\left(e^{3t}-1\right)e^{-st}}{t} \, dt = 0 - \ln\left(\frac{s-3}{s}\right)$$

- REWRITE IN x INSTEAD OF t

$$\Rightarrow \int_0^{\infty} \frac{\left(e^{3x}-1\right)e^{-sx}}{x} \, dx = \ln\left(\frac{s}{s-3}\right)$$

- LET $s=6$

$$\Rightarrow \int_0^{\infty} \frac{\left(e^{3x}-1\right)e^{-6x}}{x} \, dx = \ln\left(\frac{6}{6-3}\right)$$

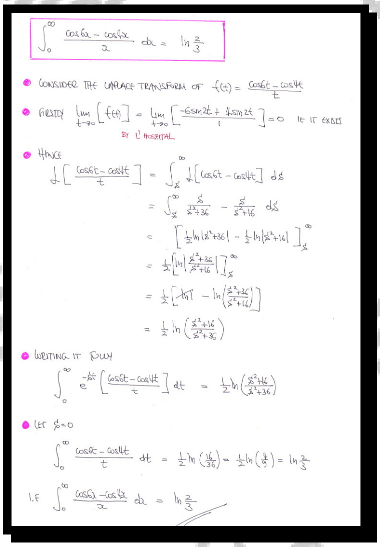
$$\Rightarrow \int_0^{\infty} \frac{e^{-3x}-e^{-6x}}{x} \, dx = \ln 2$$

Question 7

$$\int_{0}^{\infty} \frac{\cos 6x - \cos 4x}{x} \, dx \, .$$

Given that the above integral is finite, use Laplace transform techniques to find its exact value.

$$\boxed{\ln \frac{2}{3}}$$



$$\boxed{\int_0^\infty \frac{\cos 6x - \cos 4x}{x} \, dx = \ln \frac{2}{3}}$$

- CONSIDER THE LAPLACE TRANSFORM OF $f(t) = \frac{\cos 6t - \cos 4t}{t}$
- FIRSTLY $\lim_{t \to 0} [f(t)] = \lim_{t \to 0} \left[\frac{-6\sin 2t + 4\sin 2t}{1} \right] = 0$ IE IT EXISTS

  BY L'HOSPITAL
- HENCE

$$\mathcal{L}\left[\frac{\cos 6t - \cos 4t}{t} \right] = \int_s^\infty \mathcal{L}[\cos 6t - \cos 4t] \, ds$$
$$= \int_s^\infty \frac{s}{s^2+36} - \frac{s}{s^2+16} \, ds$$
$$= \left[\frac{1}{2}\ln|s^2+36| - \frac{1}{2}\ln|s^2+16| \right]_s^\infty$$
$$= \frac{1}{2}\left[\ln\left| \frac{s^2+36}{s^2+16} \right| \right]_s^\infty$$
$$= \frac{1}{2}\left[\cancel{\ln 1} - \ln\left(\frac{s^2+36}{s^2+16} \right) \right]$$
$$= \frac{1}{2}\ln\left(\frac{s^2+16}{s^2+36} \right)$$

- WRITING IT FULLY

$$\int_0^\infty e^{-st}\left[\frac{\cos 6t - \cos 4t}{t} \right] dt = \frac{1}{2}\ln\left(\frac{s^2+16}{s^2+36} \right)$$

- LET $s = 0$

$$\int_0^\infty \frac{\cos 6t - \cos 4t}{t} \, dt = \frac{1}{2}\ln\left(\frac{16}{36} \right) = \frac{1}{2}\ln\left(\frac{4}{9} \right) = \ln \frac{2}{3}$$

I.E $\int_0^\infty \frac{\cos 6x - \cos 4x}{x} \, dx = \ln \frac{2}{3}$

Question 8

$$\int_0^{\infty} x^3 \, e^{-ax} \sin x \; dx \, , \; a > 0 \, .$$

Given that the value of the above integral is zero, use Laplace transform techniques to find the value of a.

$\boxed{a = 1}$

$$\boxed{\int_0^{\infty} x^3 e^{-ax} \sin x \; dx = 0 \quad , \; a > 0}$$

CONSIDER THE LAPLACE TRANSFORM OF $t^3 \sin t$

$\Rightarrow \mathcal{L}\left[t^3 \sin t\right] = \left(-\frac{d}{ds}\right)^3 \mathcal{L}\left[\sin t\right]$ (RULE FOR MULTIPLYING BY t)

$\Rightarrow \int_0^{\infty} (t^3 \sin t) e^{-st} \, dt = -\frac{d^3}{ds^3}\left[\frac{1}{s^2+1}\right]$

$\Rightarrow \int_0^{\infty} t^3 e^{-st} \sin t \, dt = -\frac{d^2}{ds^2}\left[\frac{-2s}{(s^2+1)^2}\right]$

$\Rightarrow \int_0^{\infty} t^3 e^{-st} \sin t \, dt = 2\frac{d^2}{ds^2}\left[\frac{s}{(s^2+1)^2}\right]$

$\Rightarrow \int_0^{\infty} t^3 e^{-st} \sin t \, dt = 2\frac{d}{ds}\left[\frac{1 \times (s^2+1)^2 - s \times 4s(s^2+1)}{(s^2+1)^4}\right]$

$\Rightarrow \int_0^{\infty} t^3 e^{-st} \sin t \, dt = 2\frac{d}{ds}\left[\frac{(s^2+1) - 4s^2}{(s^2+1)^3}\right] = 2\frac{d}{ds}\left[\frac{1 - 3s^2}{(s^2+1)^3}\right]$

$\Rightarrow \int_0^{\infty} t^3 e^{-st} \sin t \, dt = 2\left[\frac{(s^2+1)^3(-6s) - (1-3s^2) \times 6s(s^2+1)^2}{(s^2+1)^6}\right]$

$\Rightarrow \int_0^{\infty} t^3 e^{-st} \sin t \, dt = 2\left[\frac{-6s(1+s^2) - 6s(1-3s^2)}{(s^2+1)^4}\right]$

$\Rightarrow \int_0^{\infty} t^3 e^{-st} \sin t \, dt = 2\left[\frac{-6s\left[2 - 2s^2\right]}{(s^2+1)^4}\right]$

$\Rightarrow \int_0^{\infty} t^3 e^{-st} \sin t \, dt = -\frac{24s(1-s^2)}{(s^2+1)^4}$

CHANGING t for x & s for a

$\Rightarrow \int_0^{\infty} x^3 e^{-ax} \sin x \, dx = \frac{24a(a^2-1)}{(a^2+1)^4}$

$\therefore$ IF $a > 0$ $\quad a = 1$

Question 9

Use Laplace transforms techniques to show that

$$\int_0^{\infty} u\,\mathrm{e}^{-u^2}\,\mathrm{erf}(u)\;du \;=\; \tfrac{1}{4}\sqrt{2}\,.$$

You may assume that

$$\mathcal{L}\left[\mathrm{erf}\,\sqrt{t}\,\right] = \frac{1}{s\sqrt{s+1}}\,.$$

$\boxed{\text{proof}}$

$\int_0^{\infty} u e^{-u^2} \operatorname{erf}(u)\, du$ ... BY SUBSTITUTION

$u^2 = t$
$u = t^{\frac{1}{2}}$
$du = \frac{1}{2}t^{-\frac{1}{2}}\,dt$
LIMITS UNCHANGED

$= \ldots \int_0^{\infty} t^{\frac{1}{2}} e^{-t} \operatorname{erf}(\sqrt{t}) \left(\frac{1}{2}t^{-\frac{1}{2}}\,dt\right)$

$= \int_0^{\infty} \frac{1}{2} e^{-t} \operatorname{erf}(\sqrt{t})\, dt$

$= \frac{1}{2}\int_0^{\infty} e^{-t} \operatorname{erf}(\sqrt{t})\, dt$

$= \frac{1}{2}\int_0^{\infty} e^{-st} \operatorname{erf}(\sqrt{t})\, dt$, WITH $s=1$

$= \frac{1}{2}\mathcal{L}\left[\operatorname{erf}\sqrt{t}\right]$, WITH $s=1$

$= \frac{1}{2}\,\frac{1}{s\sqrt{s+1}}$, WITH $s=1$

$= \frac{1}{2} \times \frac{1}{\sqrt{2}}$

$= \frac{\sqrt{2}}{4}$ // AS REQUIRED

Question 10

$$\int_0^{\infty} \int_0^{t} \frac{e^{-t} \sin u}{u} \, du \, dt .$$

Given that the above integral is finite, use Laplace transform techniques to find its exact value.

$$\boxed{\frac{\pi}{4}}$$

$$\boxed{\int_0^{\infty} \int_0^{t} \frac{e^{-t} \sin u}{u} \, du \, dt = \frac{\pi}{4}}$$

- START BY RE-WRITING THE DOUBLE INTEGRAL

$$\int_{t=0}^{\infty} e^{-t} \left[\int_{u=0}^{u=t} \frac{\sin u}{u} \, du \right] dt$$

- CONSIDER THE LAPLACE TRANSFORM OF $f(t) = \int_0^t \frac{\sin u}{u} \, du$.

I.E $\quad \mathcal{L}[f(t)] = \int_0^{\infty} e^{-st} \int_0^t \frac{\sin u}{u} \, du \, dt$

WHERE $s = 1$

- TO FIND THE LAPLACE TRANSFORM WE PROCEED AS FOLLOWS

$$\Rightarrow f(t) = \int_0^t \frac{\sin u}{u} \, du$$

$$\Rightarrow \frac{d}{dt}[f(t)] = \frac{d}{dt} \int_0^t \frac{\sin u}{u} \, du$$

$$\Rightarrow f'(t) = \frac{\sin t}{t}$$

$$\Rightarrow t f'(t) = \sin t$$

- TAKING THE LAPLACE TRANSFORM OF THE ABOVE EXPRESSION

$$\Rightarrow \mathcal{L}[t f'(t)] = \mathcal{L}[\sin t]$$

$$\boxed{\mathcal{L}[t g(t)] = -\frac{d}{ds}\left[\mathcal{L}[g(t)]\right]}$$

$$\Rightarrow -\frac{d}{ds}\left[\mathcal{L}[f'(t)]\right] = \frac{1}{s^2+1}$$

$$\boxed{\mathcal{L}[g'(t)] = s\mathcal{L}[g(t)] - g(0)}$$

$$\Rightarrow -\frac{d}{ds}\left[s\mathcal{L}[f(t)] - \cancel{f(0)} \right] = \frac{1}{s^2+1}$$

$$\Rightarrow s\mathcal{L}[f(t)] = \int -\frac{1}{s^2+1} \, ds$$

$$\Rightarrow \boxed{s\bar{f}(s) = -\arctan s + C, \quad \text{WHERE } \bar{f}(s) = \mathcal{L}[f(t)]}$$

- TO FIND THE CONSTANT WE USE THE INITIAL/FINAL THEOREM.

$$\boxed{\lim_{s \to \infty}\left[s\bar{f}(s)\right] = \lim_{t \to 0}\left[f(t)\right]}$$

$$\lim_{s \to \infty}\left[s\bar{f}(s)\right] = \lim_{s \to \infty}\left[-\arctan s + C\right] = -\frac{\pi}{2} + C$$

$$\lim_{t \to 0}\left[f(t)\right] = \lim_{t \to 0}\left[\int_0^t \frac{\sin u}{u} \, du\right] = 0$$

$$\therefore -\frac{\pi}{2} + C = 0$$

$$C = \frac{\pi}{2}$$

- FINALLY WE OBTAIN

$$\Rightarrow s\bar{f}(s) = \frac{\pi}{2} - \arctan s$$

$$\Rightarrow \bar{f}(s) = \frac{1}{s}\left[\frac{\pi}{2} - \arctan s\right]$$

$$\Rightarrow \mathcal{L}[f(t)] = \frac{1}{s}\arctan\left(\frac{1}{s}\right)$$

$$\Rightarrow \int_0^{\infty} e^{-st}\left[\int_0^t \frac{\sin u}{u} \, du\right] dt = \frac{1}{s}\arctan\frac{1}{s}$$

- LET $s = 1$

$$\Rightarrow \int_0^{\infty} e^{-t} \int_0^t \frac{\sin u}{u} \, du \, dt = \frac{\pi}{4}$$

Question 11

$$\int_0^{\infty} \frac{e^{-x}\sin^2 x}{x}\,dx\,.$$

Given that the above integral is finite, use Laplace transform techniques to find its exact value.

$$\boxed{\frac{1}{4}\ln 5}$$

- FIRSTLY THE LAPLACE TRANSFORM OF

$$\boxed{\mathcal{L}\left[\frac{\sin^2 t}{t}\right] = \mathcal{L}\left[\frac{\frac{1}{2}-\frac{1}{2}\cos 2t}{t}\right]}$$

- NEED TO CHECK THE LIMIT AS $t \to 0$

$$\lim_{t\to 0}\left[\frac{\frac{1}{2}-\frac{1}{2}\cos 2t}{t}\right] = \frac{0}{0} \text{ BY L'HOSPITAL } = \lim_{t\to 0}\left[\frac{\frac{1}{4}\sin 2t}{1}\right]$$

$$= 0 \quad \text{(I.E THE LIMIT EXISTS)}$$

- HENCE WE OBTAIN

$$\mathcal{L}\left[\frac{\sin^2 t}{t}\right] = \int_s^{\infty} \mathcal{L}\left[\tfrac{1}{2}-\tfrac{1}{2}\cos 2t\right]\,ds$$

$$= \frac{1}{2}\int_s^{\infty} \frac{1}{s} - \frac{s}{s^2+4}\,ds$$

$$= \frac{1}{2}\left[\ln s - \frac{1}{2}\ln(s^2+4)\right]_s^{\infty}$$

$$= \frac{1}{4}\left[\ln s^2 - \ln(s^2+4)\right]_s^{\infty}$$

$$= \frac{1}{4}\left[\ln\frac{s^2}{s^2+4}\right]_s^{\infty}$$

$$= \frac{1}{4}\left[\cancel{\ln 1} - \ln\left(\frac{s^2}{s^2+4}\right)\right]$$

$$= -\frac{1}{4}\ln\left(\frac{s^2}{s^2+4}\right)$$

$$= \frac{1}{4}\ln\frac{s^2+4}{s^2}$$

- NOW WE CAN ATTEMPT THE INTEGRAL

$$\Longrightarrow \int_0^{\infty} \frac{e^{-t}\sin^2 t}{t}\,dt = \mathcal{L}\left[\frac{\sin^2 t}{t}\right] \text{ WITH } s=1$$

$$\Longrightarrow \int_0^{\infty} \frac{e^{-st}\sin^2 t}{t}\,dt = \frac{1}{4}\ln\left(\frac{s^2+4}{s^2}\right)$$

- LET $s=1$

$$\Longrightarrow \int_0^{\infty} \frac{e^{-t}\sin^2 t}{t}\,dt = \frac{1}{4}\ln 5$$

I.E

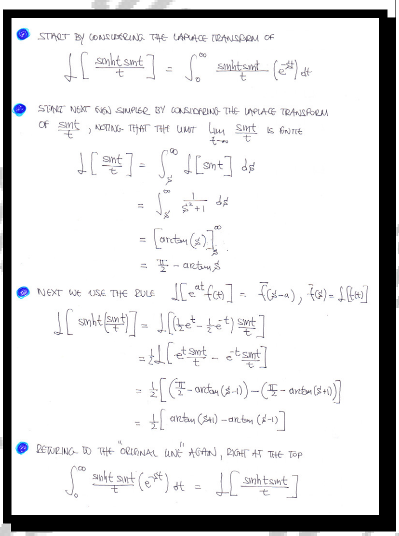
$$\Longrightarrow \int_0^{\infty} \frac{e^{-x}\sin^2 x}{x}\,dx = \frac{1}{4}\ln 5$$

Question 12

$$\int_0^\infty \frac{e^{-\sqrt{2}x} \sinh x \sin x}{x} \, dx .$$

Given that the above integral is finite, use Laplace transform techniques to find its exact value.

$$\boxed{\frac{\pi}{8}}$$



START BY CONSIDERING THE LAPLACE TRANSFORM OF

$$\mathcal{L}\left[\frac{\sinh t \sin t}{t}\right] = \int_0^\infty \frac{\sinh t \sin t}{t}\left(e^{-st}\right) dt$$

START NEXT EVEN SIMPLER BY CONSIDERING THE LAPLACE TRANSFORM OF $\frac{\sin t}{t}$, NOTING THAT THE LIMIT $\lim_{t \to \infty} \frac{\sin t}{t}$ IS FINITE

$$\mathcal{L}\left[\frac{\sin t}{t}\right] = \int_s^\infty \mathcal{L}[\sin t] \, ds$$

$$= \int_s^\infty \frac{1}{s^2+1} \, ds$$

$$= \left[\arctan(s)\right]_s^\infty$$

$$= \frac{\pi}{2} - \arctan s$$

NEXT WE USE THE RULE $\mathcal{L}\left[e^{at} f(t)\right] = \bar{f}(s-a)$, $\bar{f}(s) = \mathcal{L}[f(t)]$

$$\mathcal{L}\left[\sinh t\left(\frac{\sin t}{t}\right)\right] = \mathcal{L}\left[\left(\tfrac{1}{2}e^{t} - \tfrac{1}{2}e^{-t}\right)\frac{\sin t}{t}\right]$$

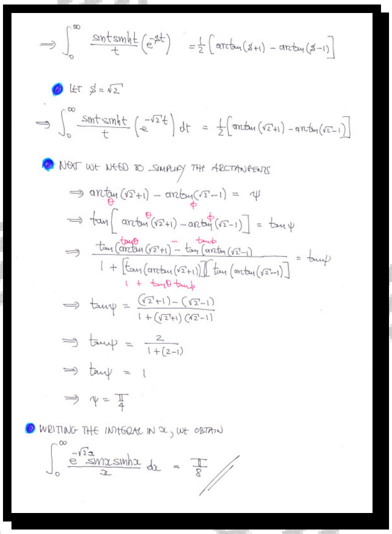
$$= \tfrac{1}{2}\mathcal{L}\left[e^{t}\frac{\sin t}{t} - e^{-t}\frac{\sin t}{t}\right]$$

$$= \tfrac{1}{2}\left[\left(\tfrac{\pi}{2} - \arctan(s-1)\right) - \left(\tfrac{\pi}{2} - \arctan(s+1)\right)\right]$$

$$= \tfrac{1}{2}\left[\arctan(s+1) - \arctan(s-1)\right]$$

RETURNING TO THE "ORIGINAL LINE" AGAIN, RIGHT AT THE TOP

$$\int_0^\infty \frac{\sinh t \sin t}{t}\left(e^{-st}\right) dt = \mathcal{L}\left[\frac{\sinh t \sin t}{t}\right]$$



$$\Rightarrow \int_0^\infty \frac{\sin t \sinh t}{t}\left(e^{-st}\right) = \tfrac{1}{2}\left[\arctan(s+1) - \arctan(s-1)\right]$$

LET $s = \sqrt{2}$

$$\Rightarrow \int_0^\infty \frac{\sin t \sinh t}{t}\left(e^{-\sqrt{2}t}\right) dt = \tfrac{1}{2}\left[\arctan(\sqrt{2}+1) - \arctan(\sqrt{2}-1)\right]$$

NEXT WE NEED TO SIMPLIFY THE ARCTANGENTS

$$\Rightarrow \underbrace{\arctan(\sqrt{2}+1)}_{\theta} - \underbrace{\arctan(\sqrt{2}-1)}_{\phi} = \psi$$

$$\Rightarrow \tan\left[\overset{\theta}{\arctan(\sqrt{2}+1)} - \overset{\phi}{\arctan(\sqrt{2}-1)}\right] = \tan\psi$$

$$\Rightarrow \frac{\overset{\tan\theta}{\tan(\arctan(\sqrt{2}+1))} \overset{-}{-} \overset{\tan\phi}{\tan(\arctan(\sqrt{2}-1))}}{\underbrace{1 + \left[\tan(\arctan(\sqrt{2}+1))\right]\left[\tan(\arctan(\sqrt{2}-1))\right]}_{1 + \tan\theta\tan\phi}} = \tan\psi$$

$$\Rightarrow \tan\psi = \frac{(\sqrt{2}+1) - (\sqrt{2}-1)}{1 + (\sqrt{2}+1)(\sqrt{2}-1)}$$

$$\Rightarrow \tan\psi = \frac{2}{1+(2-1)}$$

$$\Rightarrow \tan\psi = 1$$

$$\Rightarrow \psi = \frac{\pi}{4}$$

WRITING THE INTEGRAL IN x, WE OBTAIN

$$\int_0^\infty \frac{e^{-\sqrt{2}x}\sin x \sinh x}{x} \, dx = \frac{\pi}{8}$$

Question 13

$$f(t) \equiv \int_0^{\infty} e^{-tx^2} \, dx .$$

By considering the Laplace transform of $f(t)$, show that

$$\int_0^{\infty} e^{-x^2} \, dx = \frac{1}{2}\sqrt{\pi} .$$

proof

$$\boxed{f(t) = \int_0^{\infty} e^{-tx^2} \, dx}$$

TAKING THE LAPLACE TRANSFORM OF $f(t)$

$$\Rightarrow \mathcal{L}[f(t)] = \int_0^{\infty} f(t) e^{-st} \, dt$$

$$\Rightarrow \mathcal{L}[f(t)] = \int_0^{\infty} \left[\int_0^{\infty} e^{-tx^2} \, dx \right] e^{-st} \, dt$$

$$\Rightarrow \mathcal{L}[f(t)] = \int_0^{\infty} \left[\int_0^{\infty} e^{-tx^2} e^{-st} \, dt \right] dx$$

$$\Rightarrow \mathcal{L}[f(t)] = \int_0^{\infty} \mathcal{L}\left[e^{-tx^2} \right] dx$$

$$\Rightarrow \mathcal{L}[f(t)] = \int_0^{\infty} \frac{1}{s + x^2} \, dx$$

$$\Rightarrow \mathcal{L}[f(t)] = \frac{1}{\sqrt{s}} \left[\arctan \frac{x}{\sqrt{s}} \right]_0^{\infty}$$

$$\Rightarrow \mathcal{L}[f(t)] = \frac{\pi}{2\sqrt{s}}$$

$$\Rightarrow f(t) = \mathcal{L}^{-1}\left[\frac{\pi}{2\sqrt{s}} \right]$$

$$\Rightarrow f(t) = \frac{\pi}{2} \mathcal{L}^{-1}\left[\frac{1}{s^{\frac{1}{2}}} \right]$$

$$\Rightarrow f(t) = \frac{\pi}{2} \mathcal{L}^{-1}\left[\frac{\Gamma(\frac{1}{2})}{s^{\frac{1}{2}}} \times \frac{1}{\Gamma(\frac{1}{2})} \right]$$

$$\Rightarrow f(t) = \frac{\pi}{2} \times \frac{1}{\Gamma(\frac{1}{2})} \times \mathcal{L}^{-1}\left[\frac{\Gamma(\frac{1}{2})}{s^{\frac{1}{2}}} \right]$$

NOW $\boxed{\mathcal{L}[t^n] = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}}$

$$\Rightarrow f(t) = \frac{\pi}{2} \times \frac{1}{\sqrt{\pi}} \times t^{\frac{1}{2}}$$

$$\Rightarrow \int_0^{\infty} e^{-x^2 t} \, dx = \frac{1}{2}\sqrt{\pi}\, t^{\frac{1}{2}}$$

LETTING $t=1$

$$\Rightarrow \int_0^{\infty} e^{-x^2 t} \, dx = \frac{1}{2}\sqrt{\pi}$$

AS REQUIRED

Question 14

$$F(t) \equiv \int_0^{\infty} \cos\left(tx^2\right)\, dx \, .$$

By considering the Laplace transform of $F(t)$, show that

$$\int_0^{\infty} \cos\left(x^2\right)\, dx = \sqrt{\frac{\pi}{8}} \, .$$

proof

$$\boxed{F(t) = \int_0^{\infty} \cos(tx^2)\, dx}$$

TAKING THE LAPLACE TRANSFORM OF $F(t)$

$$\Rightarrow \mathcal{L}[F(t)] = \int_0^{\infty} F(t)\, e^{-st}\, dt.$$

$$\Rightarrow \mathcal{L}[F(t)] = \int_0^{\infty}\left[\int_0^{\infty} \cos(tx^2)\, dx\right] e^{-st}\, dt$$

$$\Rightarrow \mathcal{L}[F(t)] = \int_{x=0}^{\infty}\left[\int_{t=0}^{\infty} e^{-st}\cos(tx^2)\, dt\right] dx$$

$$\Rightarrow \mathcal{L}[F(t)] = \int_{x=0}^{\infty} \mathcal{L}[\cos tx^2]\, dx$$

$$\Rightarrow \mathcal{L}[F(t)] = \int_{x=0}^{\infty} \frac{s}{s^2 + x^4}\, dx$$

USING A SUBSTITUTION

$$\boxed{\begin{aligned} &x^2 = s\tan\theta \\ &x = \sqrt{s}\,(\tan\theta)^{\frac{1}{2}} \\ &dx = \tfrac{1}{2}\sqrt{s}\,(\tan\theta)^{-\frac{1}{2}}\sec^2\theta\, d\theta \\ &x = 0,\ \theta = 0 \\ &x = \infty,\ \theta = \tfrac{\pi}{2} \end{aligned}}$$

$$\Rightarrow \mathcal{L}[F(t)] = \int_0^{\frac{\pi}{2}} \frac{s}{s^2 + s^2\tan^2\theta}\left(\tfrac{1}{2}\sqrt{s}\,(\tan\theta)^{-\frac{1}{2}}\sec^2\theta\, d\theta\right)$$

$$\Rightarrow \mathcal{L}[F(t)] = \int_0^{\frac{\pi}{2}} \frac{s^{3/2}\cancel{\sec^2\theta}\,(\tan\theta)^{-\frac{1}{2}}}{2s^2\cancel{(1+\tan^2\theta)}}\, d\theta.$$

$$\Rightarrow \mathcal{L}[F(t)] = \int_0^{\frac{\pi}{2}} \frac{1}{2\sqrt{s}}\left(\frac{\cos\theta}{\sin\theta}\right)^{\frac{1}{2}} d\theta$$

$$\Rightarrow \mathcal{L}[F(t)] = \frac{1}{4\sqrt{s}}\int_0^{\frac{\pi}{2}} 2(\cos\theta)^{\frac{1}{2}}(\sin\theta)^{-\frac{1}{2}}\, d\theta$$

$$\Rightarrow \mathcal{L}[F(t)] = \frac{1}{4\sqrt{s}}\int_0^{\frac{\pi}{2}} 2(\cos\theta)^{2\times\frac{3}{4}-1}(\sin\theta)^{2\times\frac{1}{4}-1}\, d\theta$$

$$\Rightarrow \mathcal{L}[F(t)] = \frac{1}{4\sqrt{s}}\,B\left(\tfrac{3}{4},\tfrac{1}{4}\right) = \frac{1}{4\sqrt{s}}\,\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)} \quad \leftarrow \frac{\pi}{\sin\frac{\pi}{4}} \quad \leftarrow 0! = 1$$

$$\Rightarrow \mathcal{L}[F(t)] = \frac{1}{4\sqrt{s}} \times \frac{\pi}{\frac{1}{\sqrt{2}}}$$

$$\Rightarrow \mathcal{L}[F(t)] = \frac{\pi\sqrt{2}}{4\sqrt{s}}$$

$$\Rightarrow F(t) = \mathcal{L}^{-1}\left[\frac{\pi\sqrt{2}}{4\sqrt{s}}\right]$$

$$\Rightarrow \int_0^{\infty}\cos(tx^2)\, dx = \frac{\pi\sqrt{2}}{4}\,\mathcal{L}^{-1}\left[\frac{1}{s^{\frac{1}{2}}}\right]$$

NOW $\boxed{\mathcal{L}[t^n] = \dfrac{n!}{s^{n+1}} = \dfrac{\Gamma(n+1)}{s^{n+1}}}$

$$\Rightarrow \int_0^{\infty}\cos(tx^2)\, dx = \frac{\pi\sqrt{2}}{4}\,\mathcal{L}^{-1}\left[\frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}}\right] \times \frac{1}{\Gamma\left(\frac{1}{2}\right)}$$

$$\Rightarrow \int_0^{\infty}\cos(tx^2)\, dx = \frac{\pi\sqrt{2}}{4} \times \frac{1}{\Gamma\left(\frac{1}{2}\right)} \times t^{-\frac{1}{2}}$$

$$\Rightarrow \int_0^{\infty}\cos(tx^2)\, dx = \frac{\pi\sqrt{2}}{4} \times \frac{1}{\sqrt{\pi}} \times \frac{1}{\sqrt{t}}$$

$$\Rightarrow \int_0^{\infty}\cos(tx^2)\, dx = \sqrt{\frac{2\pi}{t}} \times \frac{1}{4}.$$

LET $t=1$ YIELDS

$$\Rightarrow \int_0^{\infty}\cos(x^2)\, dx = \frac{1}{4}\sqrt{2\pi} = \sqrt{\frac{\pi}{8}}$$

Question 15

$$I(t) \equiv \int_0^{\infty} \sin\left(tx^2\right)\ dx\ .$$

By considering the Laplace transform of $I(t)$, show that

$$\int_0^{\infty} \sin\left(x^2\right)\ dx = \sqrt{\frac{\pi}{8}}\ .$$

$\boxed{\text{proof}}$

- DEFINE THE INTEGRAL $I(t)$ AS FOLLOWS

$$I(t) = \int_0^{\infty} \sin(tx^2)\ dx$$

- TAKING THE LAPLACE TRANSFORM OF $I(t)$ WITH RESPECT TO t

$$\Rightarrow \mathcal{L}[I(t)] = \int_0^{\infty} I(t)\, e^{-st}\, dt$$

$$\Rightarrow \mathcal{L}[I(t)] = \int_0^{\infty}\left[\int_0^{\infty} \sin(tx^2)\ dx\right] e^{-st}\, dt$$

$$\Rightarrow \mathcal{L}[I(t)] = \int_{x=0}^{\infty}\left[\int_{t=0}^{\infty} e^{-st}\sin(tx^2)\ dt\right] dx$$

$$\Rightarrow \mathcal{L}[I(t)] = \int_{x=0}^{\infty} \mathcal{L}[\sin(tx^2)]\ dx$$

$$\Rightarrow \mathcal{L}[I(t)] = \int_{x=0}^{\infty} \frac{x^2}{s^2+x^4}\ dx$$

- NOW USING A SUBSTITUTION

$$\boxed{\begin{aligned} x^2 &= s\tan\theta \\ x &= \sqrt{s}(\tan\theta)^{\frac{1}{2}} \\ dx &= \tfrac{1}{2}\sqrt{s}(\tan\theta)^{-\frac{1}{2}}\sec^2\theta\ d\theta \\ x=0 &,\ \theta=0 \\ x=\infty &,\ \theta=\tfrac{\pi}{2} \end{aligned}}$$

$$\Rightarrow \mathcal{L}[I(t)] = \int_0^{\frac{\pi}{2}} \frac{s\tan\theta}{s^2+s^2\tan^2\theta}\left(\tfrac{1}{2}\sqrt{s}(\tan\theta)^{-\frac{1}{2}}\sec^2\theta\ d\theta\right)$$

$$\Rightarrow \mathcal{L}[I(t)] = \int_0^{\frac{\pi}{2}} \frac{s^{3/2}(\tan\theta)^{\frac{1}{2}}\cancel{\sec^2\theta}}{2s^2\cancel{(1+\tan^2\theta)}}\ d\theta$$

$$\Rightarrow \mathcal{L}[I(t)] = \int_0^{\frac{\pi}{2}} \frac{1}{2\sqrt{s}}(\tan\theta)^{\frac{1}{2}}\ d\theta$$

$$\Rightarrow \mathcal{L}[I(t)] = \frac{1}{4\sqrt{s}}\int_0^{\frac{\pi}{2}} 2(\sin\theta)^{\frac{1}{2}}(\cos\theta)^{-\frac{1}{2}}\ d\theta$$

$$\Rightarrow \mathcal{L}[I(t)] = \frac{1}{4\sqrt{s}}\int_0^{\frac{\pi}{2}} 2(\cos\theta)^{2\times\frac{1}{4}-1}(\sin\theta)^{2\times\frac{3}{4}-1}\ d\theta$$

$$\Rightarrow \mathcal{L}[I(t)] = \frac{1}{4\sqrt{s}}B\left(\tfrac{1}{4},\tfrac{3}{4}\right) = \frac{1}{4\sqrt{s}}\left[\frac{\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})}{\Gamma(1)}\right]$$

NOW $\left\{\Gamma(x)\Gamma(1-x) \equiv \dfrac{\pi}{\sin\pi x}\right\}$

$$\Rightarrow \mathcal{L}[I(t)] = \frac{1}{4\sqrt{s}} \times \frac{\frac{\pi}{\sin\frac{\pi}{4}}}{1} = \frac{1}{4\sqrt{s}} \times \frac{\pi}{\frac{1}{\sqrt{2}}} = \frac{\pi\sqrt{2}}{4\sqrt{s}}$$

- INVERTING THE TRANSFORM

$$\Rightarrow I(t) = \mathcal{L}^{-1}\left[\frac{\pi}{4}\sqrt{2}\times\frac{1}{\sqrt{s}}\right]$$

$$\Rightarrow I(t) = \frac{\pi\sqrt{2}}{4}\mathcal{L}^{-1}\left[\frac{1}{s^{\frac{1}{2}}}\right]$$

- NOW $\boxed{\mathcal{L}[t^n] = \dfrac{\Gamma(n+1)}{s^{n+1}} = \dfrac{n!}{s^{n+1}}}$

$$\Rightarrow I(t) = \frac{\pi\sqrt{2}}{4}\mathcal{L}^{-1}\left[\frac{1}{\Gamma(\frac{1}{2})}\ \frac{\Gamma(\frac{1}{2})}{s^{\frac{1}{2}}}\right]$$

$$\Rightarrow I(t) = \frac{\pi\sqrt{2}}{4}\times\frac{1}{\Gamma(\frac{1}{2})}\mathcal{L}^{-1}\left[\frac{\Gamma(\frac{1}{2})}{s^{\frac{1}{2}}}\right]$$

$$\Rightarrow I(t) = \frac{\pi\sqrt{2}}{4}\times\frac{1}{\sqrt{\pi}}\times t^{-\frac{1}{2}}$$

$$\Rightarrow \int_0^{\infty} \sin(tx^2)\ dx = \frac{1}{4}\sqrt{\frac{2\pi}{t}}$$

- FINALLY LET $t=1$

$$\Rightarrow \int_0^{\infty} \sin x^2\ dx = \frac{1}{4}\sqrt{2\pi} = \sqrt{\frac{\pi}{8}}$$

Question 16

$$f(t) \equiv \int_0^{\infty} x\cos\left(tx^3\right)\,dx .$$

By considering the Laplace transform of $f(t)$, show that

$$\int_0^{\infty} x\cos\left(x^3\right)\,dx = \frac{\pi\sqrt{3}}{9\Gamma\left(\frac{1}{3}\right)} .$$

proof

$$\boxed{\int_0^{\infty} x\cos x^3\,dx = \frac{\pi\sqrt{3}}{9\Gamma(\frac{1}{3})}}$$

- LET $f(t) = \int_0^{\infty} x\cos(tx^3)\,dx$
- TAKE THE LAPLACE TRANSFORM W.R.T t

$$\Rightarrow \mathcal{L}[f(t)] = \int_0^{\infty} f(t)e^{-st}\,dt$$

$$\Rightarrow \mathcal{L}[f(t)] = \int_0^{\infty}\left[\int_0^{\infty} x\cos(tx^3)\,dx\right]e^{-st}\,dt$$

- REVERSING THE ORDER OF INTEGRATION

$$\Rightarrow \mathcal{L}[f(t)] = \int_0^{\infty} x\left[\int_0^{\infty} e^{-st}\cos(tx^3)\,dt\right]dx$$

$$\Rightarrow \mathcal{L}[f(t)] = \int_0^{\infty} x\,\mathcal{L}[\cos(tx^3)]\,dx$$

$$\Rightarrow \mathcal{L}[f(t)] = \int_0^{\infty} x\left[\frac{s}{s^2+x^6}\right]dx$$

$$\Rightarrow \mathcal{L}[f(t)] = \int_0^{\infty} \frac{xs}{s^2+x^6}\,dx$$

- NOW A SUBSTITUTION FOR THE INTEGRAL

$$\boxed{\begin{array}{l} x^3 = s\tan\theta \\ x = s^{\frac{1}{3}}(\tan\theta)^{\frac{1}{3}} \\ dx = \frac{1}{3}s^{\frac{1}{3}}(\tan\theta)^{-\frac{2}{3}}\sec^2\theta \\ x=0 \mapsto \theta=0 \\ x=\infty \mapsto \theta=\frac{\pi}{2} \end{array}}$$

$$\Rightarrow \mathcal{L}[f(t)] = \int_0^{\frac{\pi}{2}} \frac{s^{\frac{4}{3}}(\tan\theta)^{\frac{4}{3}}}{s^2+s^2\tan^2\theta}\left(\frac{1}{3}s^{\frac{1}{3}}(\tan\theta)^{-\frac{2}{3}}\sec^2\theta\,d\theta\right)$$

$$\Rightarrow \mathcal{L}[f(t)] = \int_0^{\infty} \frac{s^{\frac{5}{3}}(\tan\theta)^{-\frac{1}{3}}\cancel{\sec^2\theta}}{3s^2\cancel{(1+\tan^2\theta)}}\,d\theta$$

$$\Rightarrow \mathcal{L}[f(t)] = \frac{1}{3}s^{-\frac{1}{3}}\int_0^{\infty}\frac{1}{(\tan\theta)^{\frac{1}{3}}}\,d\theta$$

$$\Rightarrow \mathcal{L}[f(t)] = \frac{1}{3}s^{-\frac{1}{3}}\int_0^{\infty}(\cos\theta)^{\frac{1}{3}}(\sin\theta)^{-\frac{1}{3}}\,d\theta$$

$$\Rightarrow \mathcal{L}[f(t)] = \frac{1}{6}s^{-\frac{1}{3}}\int_0^{\infty}2(\cos\theta)^{(2\times\frac{2}{3})-1}(\sin\theta)^{(2\times\frac{1}{3})-1}\,d\theta .$$

$$\Rightarrow \mathcal{L}[f(t)] = \frac{1}{6}s^{-\frac{1}{3}}B\left(\frac{2}{3},\frac{1}{3}\right)$$

$$\boxed{B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}}$$

$$\Rightarrow \mathcal{L}[f(t)] = \frac{1}{6}s^{-\frac{1}{3}}\frac{\Gamma(\frac{2}{3})\Gamma(\frac{1}{3})}{\Gamma(1)} \leftarrow 0! = 1$$

$$\boxed{\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin\pi x}}$$

$$\Rightarrow \mathcal{L}[f(t)] = \frac{1}{6}s^{-\frac{1}{3}}\times\frac{\pi}{\sin\frac{\pi}{3}}$$

$$\Rightarrow \mathcal{L}[f(t)] = \frac{1}{6}s^{-\frac{1}{3}}\times\frac{\pi}{\frac{\sqrt{3}}{2}}$$

$$\Rightarrow \mathcal{L}[f(t)] = \frac{\pi}{3\sqrt{3}}s^{-\frac{1}{3}}$$

- NEXT WE NEED TO INVERT $s^{-\frac{1}{3}}$

$$\mathcal{L}^{-1}\left[\frac{1}{s^{\frac{1}{3}}}\right] = \frac{1}{\Gamma(\frac{1}{3})}\mathcal{L}^{-1}\left[\frac{\Gamma(\frac{1}{3})}{s^{\frac{1}{3}}}\right] = \frac{1}{\Gamma(\frac{1}{3})}t^{-\frac{2}{3}}$$

SINCE $\boxed{\mathcal{L}[t^n] = \frac{\Gamma(n)}{s^{n+1}}}$

$$\Rightarrow f(t) = \frac{\pi}{3\sqrt{3}}\times\frac{1}{\Gamma(\frac{1}{3})}t^{-\frac{2}{3}}$$

$$\Rightarrow \int_0^{\infty} x\cos(tx^3)\,dx = \frac{\pi\sqrt{3}}{9}\times\frac{1}{\Gamma(\frac{1}{3})}\times t^{-\frac{2}{3}}$$

- FINALLY LET $t=1$

$$\Rightarrow \int_0^{\infty} x\cos x^3\,dx = \frac{\pi\sqrt{3}}{9\Gamma(\frac{1}{3})}$$

AS REQUIRED

Question 17

$$I = \int_{0}^{\frac{1}{2}\pi} \frac{\exp\left(-\frac{1}{\sqrt{3}}\tan x\right) - \exp\left(-\sqrt{3}\tan x\right)}{\sin 2x} \, dx .$$

Use Laplace transforms to show that

$$I = \frac{1}{2}\ln 3 .$$

$\boxed{\text{proof}}$

$$\boxed{\int_0^{\frac{\pi}{2}} \frac{e^{-\frac{\tan x}{\sqrt{3}}} - e^{-\sqrt{3}\tan x}}{\sin 2x}\, dx = \frac{1}{2}\ln 3}$$

- START BY THE OBVIOUS SUBSTITUTION

$$\boxed{\begin{array}{l} t = \tan x \\ dt = \sec^2 x \, dx \\ dx = \cos^2 x \, dt \\ x = 0 \longmapsto t = 0 \\ x = \frac{\pi}{2} \longmapsto t = \infty \end{array}}$$

- THUS WE NOW HAVE

$$\int_0^{\infty} \frac{e^{-\frac{t}{\sqrt{3}}} - e^{-\sqrt{3}t}}{\sin 2x} \times \left(\cos^2 x \, dt\right)$$

$$= \int_0^{\infty} \left(e^{-\frac{t}{\sqrt{3}}} - e^{-\sqrt{3}t}\right) \times \left(\frac{\cos^2 x}{2\sin x \cos x}\right) dt$$

$$= \int_0^{\infty} \left(e^{-\frac{t}{\sqrt{3}}} - e^{-\sqrt{3}t}\right) \times \left(\frac{1}{2}\cot x\right) dt$$

$$= \int_0^{\infty} \left(e^{-\frac{t}{\sqrt{3}}} - e^{-\sqrt{3}t}\right) \times \frac{1}{2t} \, dt$$

$$= \frac{1}{2}\int_0^{\infty} \frac{e^{-\frac{t}{\sqrt{3}}} - e^{-\sqrt{3}t}}{t} \, dt$$

- CONSIDER THE LAPLACE TRANSFORM OF $\frac{e^{-\frac{1}{\sqrt{3}}t} - e^{-\sqrt{3}t}}{t}$ (USING THE DIVISION BY t RULE)

$$\Rightarrow \mathcal{L}\left[\frac{e^{-\frac{1}{\sqrt{3}}t} - e^{-\sqrt{3}t}}{t}\right] = \int_s^{\infty} \mathcal{L}\left[e^{-\frac{1}{\sqrt{3}}t} - e^{-\sqrt{3}t}\right] d\sigma$$

- CHECK THAT THE LIMIT EXISTS

$$\lim_{t \to 0}\left[\frac{e^{-\frac{1}{\sqrt{3}}t} - e^{-\sqrt{3}t}}{t}\right] = \dots \text{ BY L'HOSPITAL}$$

$$= \lim_{t \to 0}\left[\frac{-\frac{1}{\sqrt{3}}e^{-\frac{1}{\sqrt{3}}t} + \sqrt{3}e^{-\sqrt{3}t}}{1}\right]$$

$$= -\frac{1}{\sqrt{3}} + \sqrt{3} = \frac{2}{3}\sqrt{3} \quad \underline{\text{IT EXISTS}}$$

- CONTINUING THE TRANSFORM

$$\Rightarrow \mathcal{L}\left[\frac{e^{-\frac{1}{\sqrt{3}}t} - e^{-\sqrt{3}t}}{t}\right] = \int_s^{\infty} \frac{1}{\sigma + \frac{1}{\sqrt{3}}} - \frac{1}{\sigma + \sqrt{3}} \, d\sigma$$

$$\Rightarrow \int_0^{\infty} e^{-st}\left[\frac{e^{-\frac{1}{\sqrt{3}}t} - e^{-\sqrt{3}t}}{t}\right] dt = \left[\ln\left[\frac{\sigma + \frac{1}{\sqrt{3}}}{\sigma + \sqrt{3}}\right]\right]_s^{\infty}$$

$$\Rightarrow \int_0^{\infty} e^{-st}\left[\frac{e^{-\frac{1}{\sqrt{3}}t} - e^{-\sqrt{3}t}}{t}\right] dt = \cancel{\ln 1} - \ln\left[\frac{s + \frac{1}{\sqrt{3}}}{s + \sqrt{3}}\right]$$

- FINALLY LET $s = 0$

$$\Rightarrow \int_0^{\infty} \frac{e^{-\frac{1}{\sqrt{3}}t} - e^{-\sqrt{3}t}}{t} \, dt = -\ln\left(\frac{\frac{1}{\sqrt{3}}}{\sqrt{3}}\right) = -\ln\frac{1}{3} = \ln 3$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{e^{-\frac{1}{\sqrt{3}}\tan x} - e^{-\sqrt{3}\tan x}}{\sin 2x} \, dx = \frac{1}{2}\int_0^{\infty} \frac{e^{-\frac{1}{\sqrt{3}}t} - e^{-\sqrt{3}t}}{t} \, dt = \frac{1}{2}\ln 3$$

Question 18

$$\int_0^\infty \frac{\sin^2 x}{x^2}\, dx\,.$$

Given that the above integral is finite, use Laplace transform techniques to find its exact value.

$$\boxed{\frac{\pi}{2}}$$

$$\boxed{\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}}$$

- CONSIDER THE LAPLACE TRANSFORM OF $\dfrac{\sin^2 t}{t}$

$$\mathcal{L}\left[\frac{\sin^2 t}{t}\right] = \mathcal{L}\left[\frac{\frac{1}{2}-\frac{1}{2}\cos 2t}{t}\right] = \frac{1}{2}\mathcal{L}\left[\frac{1-\cos 2t}{t}\right]$$

- NEXT CHECK THE LIMIT AS $t \to 0$

$$\lim_{t\to 0}\left[\frac{1-\cos 2t}{t}\right] = \frac{0}{0} = \text{BY L'HOSPITAL} = \lim_{t\to 0}\left[\frac{2\sin 2t}{1}\right] = 0$$

IE $\underline{\text{LIMIT EXISTS}}$

- HENCE WE OBTAIN

$$\mathcal{L}\left[\frac{\sin^2 t}{t}\right] = \int_s^\infty \frac{1}{2}\mathcal{L}\left[1-\cos 2t\right]\,ds = \frac{1}{2}\int_s^\infty \frac{1}{s} - \frac{s}{s^2+4}\,ds$$

$$= \frac{1}{2}\left[\ln s - \frac{1}{2}\ln(s^2+4)\right]_s^\infty$$

$$= \frac{1}{4}\left[\ln s^2 - \ln(s^2+4)\right]_s^\infty = \left[\frac{1}{4}\ln\left(\frac{s^2}{s^2+4}\right)\right]_s^\infty$$

$$= \frac{1}{4}\left[\cancel{\ln 1} - \ln\frac{s^2}{s^2+4}\right] = \frac{1}{4}\ln\left(\frac{s^2+4}{s^2}\right)$$

- NEXT CONSIDER THE LAPLACE TRANSFORM OF $\dfrac{\frac{\sin^2 t}{t}}{t} = \dfrac{\sin^2 t}{t^2}$

CHECK THE LIMIT AS $t \to 0$

$$\lim_{t\to 0}\left[\frac{\sin^2 t}{t^2}\right] = \lim_{t\to 0}\left[\frac{\sin t}{t}\times\frac{\sin t}{t}\right] = \text{PROPERTIES OF LIMITS}$$

$$= 1\times 1 = 1 \quad \text{IE } \underline{\text{LIMIT EXIST}}$$

- THUS THE LAPLACE TRANSFORM NOW GIVES

$$\mathcal{L}\left[\frac{\sin^2 t}{t^2}\right] = \int_s^\infty \mathcal{L}\left[\frac{\sin^2 t}{t}\right]ds = \frac{1}{4}\int_s^\infty \ln\left(\frac{s^2+4}{s^2}\right)ds$$

- TO DO THIS INTEGRAL (IGNORE THE LIMITS & THE $\frac{1}{4}$)

$$\int \ln\left(\frac{s^2+4}{s^2}\right)ds = \int \ln(s^2+4)\,ds - \int \ln s^2\,ds$$

BY PARTS

$\ln(s^2+4)$	$\dfrac{2s}{s^2+4}$
s	1

BY PARTS (OR STANDARD RESULT)

$$\int \ln x\,dx = x\ln x - x + C$$

$$\dots = s\ln(s^2+4) - \int\frac{2s^2}{s^2+4}\,ds - \int 2\ln s\,ds$$

$$= s\ln(s^2+4) - \int\frac{2(s^2+4)-8}{s^2+4}\,ds - 2\left[s\ln s - s\right]$$

$$= s\ln(s^2+4) - \int 2 - \frac{8}{s^2+4}\,ds - 2s\ln s + 2s$$

$$= s\ln(s^2+4) - \cancel{2s} + 8\times\frac{1}{2}\arctan\frac{s}{2} - 2s\ln s + \cancel{2s} + C$$

$$= s\ln(s^2+4) - s\ln s^2 + 4\arctan\frac{s}{2} + C$$

$$= s\ln\left(\frac{s^2+4}{s^2}\right) + 4\arctan\frac{s}{2} + C$$

- THUS WE NOW HAVE

$$\mathcal{L}\left[\frac{\sin^2 t}{t^2}\right] = \frac{1}{4}\left[s\ln\left(\frac{s^2+4}{s^2}\right) + 4\arctan\frac{s}{2}\right]_s^\infty$$

$$\int_0^\infty \frac{\sin^2 t}{t^2}e^{-st}\,dt = \frac{1}{4}\left[\left(0 + 4\times\frac{\pi}{2}\right) - \left(s\ln\frac{s^2+4}{s^2} + 4\arctan\frac{s}{2}\right)\right]$$

$$\int_0^\infty \frac{\sin^2 t}{t^2}e^{-st}\,dt = \frac{\pi}{2} - \frac{1}{4}s\ln\left(\frac{s^2+4}{s^2}\right) + \arctan\frac{s}{2}$$

- LET $s = 0$ & WRITE IT IN x

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$$

NOTE

$$\lim_{s\to\infty}\left[s\ln\frac{s^2+4}{s^2}\right] = 0$$

$$\lim_{s\to 0}\left[s\ln\frac{s^2+4}{s^2}\right] = 0$$

Question 19

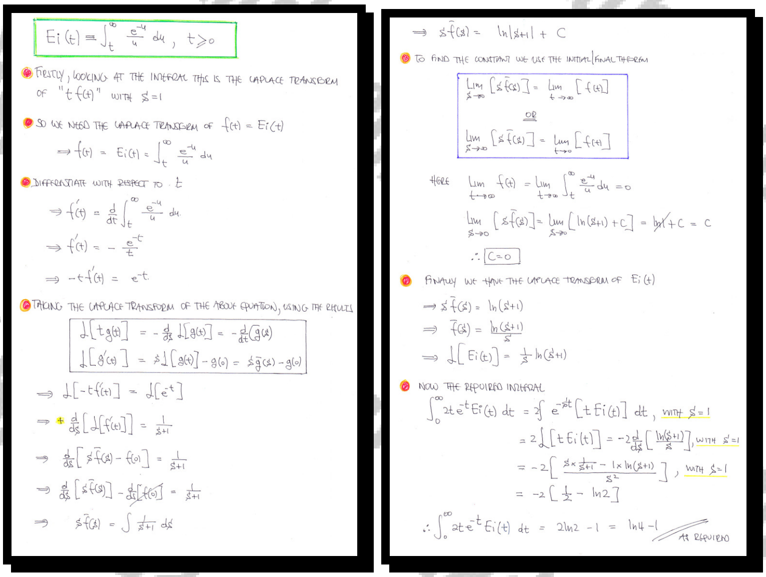
The Exponential integral function $\text{Ei}(t)$ is defined as

$$\text{Ei}(t) \equiv \int_t^{\infty} \frac{e^{-u}}{u}\, du \,,\; t \geq 0 \,.$$

By considering the Laplace transform of $\text{Ei}(t)$, show that

$$\int_0^{\infty} 2t e^{-t}\, \text{Ei}(t)\, dt = \ln 4 - 1 \,.$$

proof



$$\boxed{\text{Ei}(t) \equiv \int_t^{\infty} \frac{e^{-u}}{u}\, du \,,\; t \geq 0}$$

FIRSTLY, LOOKING AT THE INTEGRAL THIS IS THE LAPLACE TRANSFORM OF "$t f(t)$" WITH $s=1$

SO WE NEED THE LAPLACE TRANSFORM OF $f(t) = \text{Ei}(t)$

$$\Rightarrow f(t) = \text{Ei}(t) = \int_t^{\infty} \frac{e^{-u}}{u}\, du$$

DIFFERENTIATE WITH RESPECT TO t

$$\Rightarrow f'(t) = \frac{d}{dt}\int_t^{\infty} \frac{e^{-u}}{u}\, du$$

$$\Rightarrow f'(t) = -\frac{e^{-t}}{t}$$

$$\Rightarrow -t f'(t) = e^{-t}$$

TAKING THE LAPLACE TRANSFORM OF THE ABOVE EQUATION, USING THE RESULTS

$$\boxed{\begin{aligned} \mathcal{L}[t g(t)] &= -\frac{d}{ds}\mathcal{L}[g(t)] = -\frac{d}{dt}(\bar{g}(s)) \\ \mathcal{L}[g'(t)] &= s\mathcal{L}[g(t)] - g(0) = s\bar{g}(s) - g(0) \end{aligned}}$$

$$\Rightarrow \mathcal{L}[-t f'(t)] = \mathcal{L}[e^{-t}]$$

$$\Rightarrow + \frac{d}{ds}\left[\mathcal{L}[f'(t)]\right] = \frac{1}{s+1}$$

$$\Rightarrow \frac{d}{ds}\left[s\bar{f}(s) - f(0)\right] = \frac{1}{s+1}$$

$$\Rightarrow \frac{d}{ds}\left[s\bar{f}(s)\right] - \frac{d}{ds}\cancel{[f(0)]} = \frac{1}{s+1}$$

$$\Rightarrow s\bar{f}(s) = \int \frac{1}{s+1}\, ds$$

$$\Rightarrow s\bar{f}(s) = \ln|s+1| + C$$

TO FIND THE CONSTANT WE USE THE INITIAL/FINAL THEOREM

$$\boxed{\begin{gathered} \lim_{s\to 0}[s\bar{f}(s)] = \lim_{t\to\infty}[f(t)] \\ \underline{\text{OR}} \\ \lim_{s\to\infty}[s\bar{f}(s)] = \lim_{t\to 0}[f(t)] \end{gathered}}$$

HERE $\displaystyle\lim_{t\to\infty} f(t) = \lim_{t\to\infty}\int_t^{\infty}\frac{e^{-u}}{u}\, du = 0$

$$\lim_{s\to 0}[s\bar{f}(s)] = \lim_{s\to 0}[\ln(s+1) + C] = \cancel{\ln 1} + C = C$$

$$\therefore \boxed{C = 0}$$

FINALLY WE HAVE THE LAPLACE TRANSFORM OF $\text{Ei}(t)$

$$\Rightarrow s\bar{f}(s) = \ln(s+1)$$

$$\Rightarrow \bar{f}(s) = \frac{\ln(s+1)}{s}$$

$$\Rightarrow \mathcal{L}[\text{Ei}(t)] = \frac{1}{s}\ln(s+1)$$

NOW THE REQUIRED INTEGRAL

$$\begin{aligned} \int_0^{\infty} 2t e^{-t}\, \text{Ei}(t)\, dt &= 2\int e^{-st}[t\, \text{Ei}(t)]\, dt \,, \quad \text{WITH } s=1 \\ &= 2\mathcal{L}[t\, \text{Ei}(t)] = -2\frac{d}{ds}\left[\frac{\ln(s+1)}{s}\right], \quad \text{WITH } s=1 \\ &= -2\left[\frac{s\times\frac{1}{s+1} - 1\times\ln(s+1)}{s^2}\right], \quad \text{WITH } s=1 \\ &= -2\left[\frac{1}{2} - \ln 2\right] \end{aligned}$$

$$\therefore \int_0^{\infty} 2t e^{-t}\, \text{Ei}(t)\, dt = 2\ln 2 - 1 = \ln 4 - 1$$

AS REQUIRED

Question 20

$$\int_0^\infty \frac{x(1+x)\sin(\ln x)}{\ln x}\,dx.$$

Given that the above integral is finite, use Laplace transform techniques to find its exact value.

$$\boxed{\frac{\pi}{4}}$$

- $\displaystyle\int_0^1 \frac{x(1+x)\sin(\ln x)}{\ln x}\,dx$ ... START BY A SUBSTITUTION

$$\boxed{\begin{array}{l} -t=\ln x \\ e^{-t}=x \\ dx=-e^{-t}\,dt \\ \cdots\cdots\cdots \\ x=0 \mapsto t=\infty \\ x=1 \mapsto t=0 \end{array}}$$

$$= \int_\infty^0 \frac{e^{-t}(1+e^{-t})\sin(-t)}{-t}\,(-e^{-t}\,dt)$$

$$= \int_0^\infty \frac{e^{-2t}(e^{-t}+1)\sin t}{t}\,dt$$

$$= \int_0^\infty e^{-3t}\frac{\sin t}{t}\,dt + \int_0^\infty e^{-2t}\frac{\sin t}{t}\,dt$$

- NOW CONSIDER THE LAPLACE TRANSFORM OF $\frac{\sin t}{t}$ AND NOTE THAT THE $\lim_{t\to 0}\left(\frac{\sin t}{t}\right)$ EXISTS

- THEN $\displaystyle\mathcal{L}\left[\frac{\sin t}{t}\right] = \int_s^\infty \mathcal{L}[\sin t]\,ds = \int_s^\infty \frac{1}{s^2+1}\,ds$

$$= \Big[\arctan s\Big]_s^\infty = \frac{\pi}{2} - \arctan s$$

$$= \operatorname{arccot} s = \arctan\frac{1}{s}$$

- RETURNING TO THE ABOVE INTEGRALS WE HAVE

$$\dots = \underbrace{\int_0^\infty e^{-st}\left(\frac{\sin t}{t}\right)dt}_{\text{WITH } s=3} + \underbrace{\int_0^\infty e^{-st}\left(\frac{\sin t}{t}\right)}_{\text{WITH } s=2}$$

$$= \arctan\frac{1}{3} + \arctan\frac{1}{2} = \arctan\left[\frac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{3}\times\frac{1}{2}}\right]$$

$$= \arctan\left(\frac{\frac{5}{6}}{1-\frac{1}{6}}\right) = \arctan 1 = \frac{\pi}{4}$$

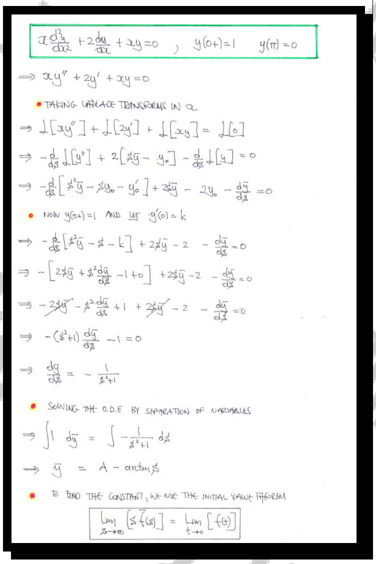
SOLVING O.D.E.s, P.D.E.s & INTEGRAL EQUATIONS

Question 1

Use Laplace transforms to solve the following differential equation

$$x\frac{d^2y}{dx^2}+2\frac{dy}{dx}+xy=0, \quad y(0+)=1, \quad y(\pi)=0.$$

$$\boxed{y(x)=\frac{\sin x}{x}}$$



$$\boxed{x\frac{d^2y}{dx^2}+2\frac{dy}{dx}+xy=0 \,, \quad y(0+)=1 \quad y(\pi)=0}$$

$\Rightarrow xy''+2y'+xy=0$

- TAKING LAPLACE TRANSFORMS IN x

$\Rightarrow \mathcal{L}[xy'']+\mathcal{L}[2y']+\mathcal{L}[xy]=\mathcal{L}[0]$

$\Rightarrow -\frac{d}{ds}\mathcal{L}[y'']+2[s\bar{y}-y_0]-\frac{d}{ds}\mathcal{L}[y]=0$

$\Rightarrow -\frac{d}{ds}[s^2\bar{y}-sy_0-y'_0]+2s\bar{y}-2y_0-\frac{d\bar{y}}{ds}=0$

- NOW $y(0+)=1$ AND LET $y'(0)=k$

$\Rightarrow -\frac{d}{ds}[s^2\bar{y}-s-k]+2s\bar{y}-2-\frac{d\bar{y}}{ds}=0$

$\Rightarrow -\left[2s\bar{y}+s^2\frac{d\bar{y}}{ds}-1+0\right]+2s\bar{y}-2-\frac{d\bar{y}}{ds}=0$

$\Rightarrow \cancel{-2s\bar{y}}-s^2\frac{d\bar{y}}{ds}+1+\cancel{2s\bar{y}}-2-\frac{d\bar{y}}{ds}=0$

$\Rightarrow -(s^2+1)\frac{d\bar{y}}{ds}-1=0$

$\Rightarrow \frac{d\bar{y}}{ds}=-\frac{1}{s^2+1}$

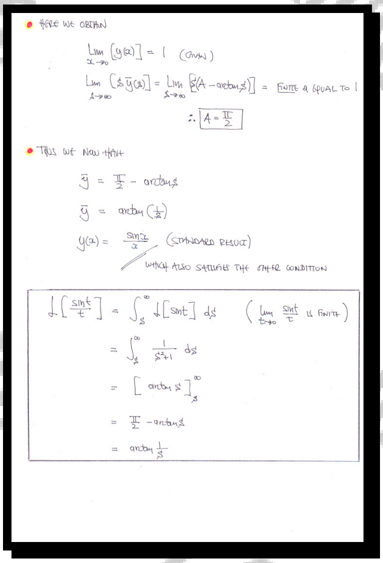
- SOLVING THE O.D.E BY SEPARATION OF VARIABLES

$\Rightarrow \int 1\,d\bar{y}=\int -\frac{1}{s^2+1}\,ds$

$\Rightarrow \bar{y}=A-\arctan s$

- TO FIND THE CONSTANT, WE USE THE INITIAL VALUE THEOREM

$$\boxed{\lim_{s\to\infty}[s\bar{f}(s)]=\lim_{t\to 0}[f(t)]}$$



- HERE WE OBTAIN

$\lim_{x\to 0}[y(x)]=1$ (GIVEN)

$\lim_{s\to\infty}[s\bar{y}(s)]=\lim_{s\to\infty}[s(A-\arctan s)]$ = FINITE & EQUAL TO 1

$\therefore \boxed{A=\frac{\pi}{2}}$

- THUS WE NOW HAVE

$\bar{y}=\frac{\pi}{2}-\arctan s$

$\bar{y}=\arctan\left(\frac{1}{s}\right)$

$y(x)=\frac{\sin x}{x}$ (STANDARD RESULT)

WHICH ALSO SATISFIES THE OTHER CONDITION

$\mathcal{L}\left[\frac{\sin t}{t}\right]=\int_s^\infty \mathcal{L}[\sin t]\,ds \quad \left(\lim_{t\to 0}\frac{\sin t}{t} \text{ is finite}\right)$

$=\int_s^\infty \frac{1}{s^2+1}\,ds$

$=\left[\arctan s\right]_s^\infty$

$=\frac{\pi}{2}-\arctan s$

$=\arctan\frac{1}{s}$

Question 2

Use Laplace transforms to solve the following differential equation

$$t\frac{d^2y}{dt^2}+\frac{dy}{dt}+4ty=0\,,\quad y(0)=1\,,\quad \frac{dy}{dt}(0)=0\,.$$

$$\boxed{y(t)=J_0(2t)}$$

$$\boxed{t\frac{d^2y}{dt^2}+\frac{dy}{dt}+4ty=0 \qquad y(0)=1\,,\ \frac{dy}{dt}(0)=0}$$

$\Rightarrow ty''+y'+4ty=0$

- TAKING LAPLACE TRANSFORMS

$\Rightarrow \mathcal{L}[ty'']+\mathcal{L}[y']+4\mathcal{L}[ty]=\mathcal{L}[0]$

$\Rightarrow -\frac{d}{ds}\left[s^2\bar{y}-sy_0-\dot{y}_0\right]+\left[s\bar{y}-y_0\right]-4\frac{d}{ds}\left[\bar{y}\right]=0$

$\Rightarrow -\frac{d}{ds}\left[s^2\bar{y}-s-0\right]+s\bar{y}-1-4\frac{d\bar{y}}{ds}=0$

$\Rightarrow -\left[2s\bar{y}+s^2\frac{d\bar{y}}{ds}-1\right]+s\bar{y}-1-4\frac{d\bar{y}}{ds}=0$

$\Rightarrow -2s\bar{y}-s^2\frac{d\bar{y}}{ds}+1+s\bar{y}-1-4\frac{d\bar{y}}{ds}=0$

$\Rightarrow -s\bar{y}=(s^2+4)\frac{d\bar{y}}{ds}$

- SEPARATE THE VARIABLES TO SOLVE THE O.D.E.

$\Rightarrow \frac{1}{\bar{y}}\,d\bar{y}=-\frac{s}{s^2+4}$

$\Rightarrow \ln\bar{y}=-\frac{1}{2}\ln(s^2+4)+C$

$\Rightarrow \ln\bar{y}=\ln(s^2+4)^{-\frac{1}{2}}+\ln A$

$\Rightarrow \bar{y}=\frac{A}{\sqrt{s^2+4}}$

$\left.\begin{array}{l}\Rightarrow \bar{y}=\dfrac{A}{\sqrt{4\left(\frac{s^2}{4}+1\right)}}\\ \Rightarrow \bar{y}=\dfrac{A}{2\sqrt{\left(\frac{s}{2}\right)^2+1}}\end{array}\right\}$ THIS LOOKS LIKE "BESSEL" WITH SCALING

- NOW BY INSPECTION

$\mathcal{L}[J_0(t)]=\frac{1}{\sqrt{s^2+1}}$

& $\mathcal{L}[f(at)]=\frac{1}{a}\bar{f}(s/a)$

$\therefore \mathcal{L}[J_0(2t)]=\frac{1}{2}\times\frac{1}{\sqrt{\left(\frac{s}{2}\right)^2+1}}=\frac{1}{2}\times\frac{1}{\sqrt{\frac{s^2}{4}+1}}$

$=\frac{1}{2}\times\frac{1}{\frac{\sqrt{s^2+4}}{2}}=\frac{1}{\sqrt{s^2+4}}$

- RETURNING TO THE SOLUTION

$\bar{y}=\frac{A}{\sqrt{s^2+4}}$

$y(t)=AJ_0(2t)$

- WE ARE NOW GIVEN

$y(0)=1$

$1=AJ_0(0)$

$1=A\times1$

$A=1$

$\therefore\ y(t)=J_0(2t)$ //

Question 3

Use Laplace transforms to solve the following differential equation

$$\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 1, \quad y(0) = 1, \quad \frac{dy}{dx}(0) = 2 .$$

$$\boxed{y(x) = 2x + 1}$$

$$\boxed{\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 1 \qquad y(0)=1\ ,\ y'(0)=2}$$

$\Rightarrow y'' - xy' + y = 1$

- TAKING LAPLACE TRANSFORMS IN x

$\Rightarrow \mathcal{L}[y''] - \mathcal{L}[xy'] + \mathcal{L}[y] = \mathcal{L}[1]$

$\Rightarrow s^2\bar{y} - sy_0 - y'_0 + \frac{d}{ds}\left[\mathcal{L}[y']\right] + \bar{y} = \frac{1}{s}$

$\Rightarrow s^2\bar{y} - sy_0 - y'_0 + \frac{d}{ds}\left[s\bar{y} - y_0\right] + \bar{y} = \frac{1}{s}$

- APPLY CONDITIONS & CARRY OUT THE DIFFERENTIATION

$\Rightarrow s^2\bar{y} - s - 2 + \frac{d}{ds}\left[s\bar{y} - 1\right] + \bar{y} = \frac{1}{s}$

$\Rightarrow s^2\bar{y} - s - 2 + \bar{y} + s\frac{d\bar{y}}{ds} + \bar{y} = \frac{1}{s}$

$\Rightarrow s\frac{d\bar{y}}{ds} + (s^2+2)\bar{y} = s + 2 + \frac{1}{s}$

$\Rightarrow \frac{d\bar{y}}{ds} + \left(s + \frac{2}{s}\right)\bar{y} = 1 + \frac{2}{s} + \frac{1}{s^2}$

- THIS IS A STANDARD FIRST ORDER LINEAR O.D.E

<u>INTEGRATING FACTOR</u> $e^{\int s + \frac{2}{s}\,ds} = e^{\frac{1}{2}s^2 + 2\ln s} = s^2 e^{\frac{1}{2}s^2}$

$\Rightarrow \frac{d}{ds}\left[s^2 e^{\frac{1}{2}s^2}\bar{y}\right] = \left(1 + \frac{2}{s} + \frac{1}{s^2}\right) \times s^2 e^{\frac{1}{2}s^2}$

$\Rightarrow \frac{d}{ds}\left[s^2 e^{\frac{1}{2}s^2}\bar{y}\right] = (s^2 + 2s + 1)e^{\frac{1}{2}s^2}$

$\Rightarrow s^2 e^{\frac{1}{2}s^2}\bar{y} = \int s^2 e^{\frac{1}{2}s^2} + 2s e^{\frac{1}{2}s^2} + e^{\frac{1}{2}s^2}\,ds$

- CARRYING OUT THE INTEGRATION ON THE RHS BY PARTS/RECOGNITION

$\int s^2 e^{\frac{1}{2}s^2} + 2s e^{\frac{1}{2}s^2} + e^{\frac{1}{2}s^2}\,ds$

$= \int s\left(s e^{\frac{1}{2}s^2}\right)ds + \int 2s e^{\frac{1}{2}s^2}\,ds + \int e^{\frac{1}{2}s^2}\,ds$

BY PARTS

s	1
$e^{\frac{1}{2}s^2}$	$s e^{\frac{1}{2}s^2}$

$= s e^{\frac{1}{2}s^2} - \cancel{\int e^{\frac{1}{2}s^2}\,ds} + 2e^{\frac{1}{2}s^2} + \cancel{\int e^{\frac{1}{2}s^2}\,ds}$

- RETURNING TO THE MAIN SOLUTION

$\Rightarrow \bar{y}\, s^2 e^{\frac{1}{2}s^2} = (s+2)e^{\frac{1}{2}s^2} + A$

$\Rightarrow \bar{y} = \frac{s+2}{s^2} + \frac{A}{s^2}e^{-\frac{1}{2}s^2}$

- EXPAND AS A SERIES

$\Rightarrow \bar{y} = \frac{1}{s} + \frac{2}{s^2} + \frac{A}{s^2}\left[1 - \frac{1}{2}s^2 + \frac{1}{8}s^4 - \frac{1}{48}s^6 + \dots\right]$

$\Rightarrow \bar{y} = \frac{1}{s} + \frac{2}{s^2} + A\left[\frac{1}{s^2} - \frac{1}{2} + \frac{1}{8}s^2 - \frac{1}{48}s^4 + \dots\right]$

$\Rightarrow \bar{y} = \frac{1}{s} + \frac{A+2}{s^2} + A\left[-\frac{1}{2} + \frac{1}{8}s^2 - \frac{1}{48}s^4 + \dots\right]$

$$\boxed{\text{Now } \mathcal{L}^{-1}\left[s^n\right] = 0 \qquad k = 0,1,2,3,4,\dots}$$

$\Rightarrow \bar{y} = \frac{1}{s} + \frac{A+2}{s^2}$

- INVERTING THE TRANSFORM

$y(x) = 1 + (A+2)x$

- APPLY THE INITIAL CONDITION $y(0) = 1$ MERELY VERIFIES

<u>DIFFERENTIATE</u>

$y' = A + 2$

$y'_0 = 2$

$2 = A + 2$

$\boxed{A = 0}$

$\therefore y = 1 + 2x$

Question 4

The function $u = u(t, y)$ satisfies the partial differential equation

$$\frac{\partial u}{\partial t} + y\frac{\partial u}{\partial y} = y, \quad t \geq 0, \quad y > 0,$$

subject to the following conditions

i. $u(0, y) = 1 + y^2$, $y > 0$.

ii. $u(t, 0) = 1$, $t \geq 0$.

Use Laplace transforms in t to show that

$$u(t, y) = 1 + y - y\mathrm{e}^{-t} + y^2\mathrm{e}^{-2t}.$$

$\boxed{}$, $\boxed{\text{proof}}$

START BY TAKING THE LAPLACE TRANSFORM OF THE P.D.E, W.R.T t

$$\Rightarrow \frac{\partial u}{\partial t} + y\frac{\partial u}{\partial y} = y$$

$$\Rightarrow \mathcal{L}\left[\frac{\partial u}{\partial t}\right] + \mathcal{L}\left[y\frac{\partial u}{\partial y}\right] = \mathcal{L}[y]$$

$$\Rightarrow \left[s\bar{u}(s,y) - u(0,y)\right] + y\frac{\partial}{\partial y}\left(\bar{u}(s,y)\right) = y\mathcal{L}[1]$$

$$\Rightarrow s\bar{u} - (1+y^2) + y\frac{\partial \bar{u}}{\partial y} = \frac{y}{s}$$

$$\Rightarrow y\frac{\partial \bar{u}}{\partial y} + s\bar{u} = 1 + y^2 + \frac{y}{s}$$

$$\Rightarrow \frac{\partial \bar{u}}{\partial y} + \frac{s}{y}\bar{u} = \frac{1}{y} + y + \frac{1}{s}$$

TREAT THE ABOVE AS AN O.D.E FOR $\bar{u} = f(y)$, AS s IS A CONSTANT, AND LOOK FOR AN INTEGRATING FACTOR

$$e^{\int \frac{s}{y}\,dy} = e^{s\ln y} = e^{\ln y^s} = y^s$$

THUS WE NOW HAVE

$$\Rightarrow \frac{\partial}{\partial y}\left[\bar{u}y^s\right] = y^s\left(\frac{1}{y} + y + \frac{1}{s}\right)$$

$$\Rightarrow \frac{\partial}{\partial y}\left[\bar{u}y^s\right] = y^{s-1} + y^{s+1} + \frac{y^s}{s}$$

$$\Rightarrow \bar{u}y^s = \int y^{s-1} + y^{s+1} + \frac{y^s}{s}\,\partial y$$

$$\Rightarrow \bar{u}y^s = \frac{1}{s}y^s + \frac{1}{s+2}y^{s+2} + \frac{1}{s(s+1)}y^{s+1} + A(s)$$

$$\Rightarrow \boxed{\bar{u}(s,y) = \frac{1}{s} + \frac{y^2}{s+2} + \frac{y}{s(s+1)} + A(s)y^{-s}}$$

NEXT WE APPLY THE BOUNDARY CONDITION $u(t,0) = 1$

$$\Rightarrow u(t,0) = 1$$

$$\Rightarrow \bar{u}(s,0) = \frac{1}{s}$$

$$\Rightarrow \frac{1}{s} = \frac{1}{s} + \cancel{\frac{1}{s+2}\times 0^2} + \cancel{\frac{1}{s(1+s)}\times 0} + A(s)\times 0^{-s}$$

$$\times\frac{1}{0^s} \to \infty$$

$$\therefore A(s) = 0$$

$$\therefore \boxed{\bar{u}(s,y) = \frac{1}{s} + \frac{1}{s+2}y^2 + \frac{1}{s(s+1)}y}$$

INVERTING BY PARTIAL FRACTIONS & INSPECTION

$$\bar{u}(s,y) = \frac{1}{s} + \frac{1}{s+2}y^2 + \left(\frac{1}{s} - \frac{1}{s+1}\right)y$$

$$\bar{u}(s,y) = \frac{1}{s}(1+y) + \frac{1}{s+2}y^2 - \frac{1}{s+1}y$$

$$u(t,y) = 1 + y + e^{-2t}y^2 - e^{-t}y$$

// AS REQUIRED

Question 5

The function $z = z(x,t)$ satisfies the partial differential equation

$$\frac{\partial u}{\partial x} = 2\frac{\partial z}{\partial t} + z, \quad x \geq 0, \quad t \geq 0,$$

subject to the following conditions

i. $z(x,0) = 6\mathrm{e}^{-3x}$, $x > 0$.

ii. $z(x,t)$, is bounded for all $x \geq 0$ and $t \geq 0$.

Find the solution of partial differential equation by using Laplace transforms.

$\boxed{}$, $\boxed{z(x,t) = 6\mathrm{e}^{-(3x+2t)}}$

$$\frac{\partial z}{\partial x} = 2\frac{\partial z}{\partial t} + z \quad \underline{\text{SUBJECT TO}} \quad z(x,0) = 6e^{-3x}, \ x \geq 0$$

$$z = z(x,t), \ x \geq 0, \ t \geq 0 \qquad z(x,t) \text{ IS BOUNDED } x \geq 0, \ t \geq 0$$

- TAKING LAPLACE TRANSFORMS OF THE P.D.E W.R.T t

$$\Rightarrow \mathcal{L}\left[\frac{\partial z}{\partial x}\right] = \mathcal{L}\left[2\frac{\partial z}{\partial t}\right] + \mathcal{L}[z]$$

$$\Rightarrow \frac{\partial}{\partial x}\bar{z} = 2\left[s\bar{z} - z(x,0)\right] + \bar{z}$$

$$\Rightarrow \frac{\partial \bar{z}}{\partial x} = 2s\bar{z} - 12e^{-3x} + \bar{z}$$

$$\Rightarrow \frac{\partial \bar{z}}{\partial x} - (2s+1)\bar{z} = -12e^{-3x}$$

- THIS IS A FIRST ORDER O.D.E FOR $\bar{z} = \bar{z}(x,s)$, WHERE s IS TREATED AS A CONSTANT — LOOK FOR AN INTEGRATING FACTOR

$$e^{\int -(2s+1)\,dx} = e^{-(2s+1)x}$$

- HENCE WE OBTAIN

$$\Rightarrow \frac{\partial}{\partial x}\left[\bar{z}e^{-(2s+1)x}\right] = -12e^{-3x}e^{-(2s+1)x}$$

$$\Rightarrow \frac{\partial}{\partial x}\left[\bar{z}e^{-(2s+1)x}\right] = -12e^{-(2s+4)x}$$

$$\Rightarrow \bar{z}e^{-2(s+1)x} = \int -12e^{-(2s+4)x}\,dx$$

$$\Rightarrow \bar{z}e^{-2(s+1)x} = \frac{12}{2s+4}e^{-(2s+4)x} + A(s)$$

$$\Rightarrow \bar{z} = \frac{6}{s+2}e^{-(2s+4)x} \times e^{(2s+1)x} + A(s)e^{(2s+1)x}$$

$$\Rightarrow \bar{z}(x,s) = \frac{6}{s+2}e^{-3x} + A(s)e^{(2s+1)x}$$

- NOW $A(s) = 0$ SINCE $z(x,t)$ IS BOUNDED AS $x \to \infty$, SO MUST $\bar{z}(x,s)$ AS $x \to \infty$

$$\Rightarrow \bar{z}(x,s) = \frac{6}{s+2}e^{-3x}$$

- INVERTING BACK INTO t, NOTING x IS A CONSTANT WITH RESPECT TO THE TRANSFORM

$$\Rightarrow z(x,t) = 6e^{-2t}e^{-3x}$$

$$\Rightarrow z(x,t) = 6e^{-(2t+3x)}$$

Question 6

$$\theta(x) = 8\sin(2\pi x), \quad 0 \le x \le 1$$

The above equation represents the temperature distribution θ °C, maintained along the 1 m length of a thin rod.

At time $t = 0$, the temperature θ is suddenly dropped to $\theta = 0$ °C at both the ends of the rod at $x = 0$, and at $x = 1$, and the source which was previously maintaining the temperature distribution is removed.

The new temperature distribution along the rod $\theta(x,t)$, satisfies the heat equation

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial \theta}{\partial t}, \quad 0 \le x \le 1, \quad t \ge 0.$$

Use Laplace transforms to determine an expression for $\theta(x,t)$.

$$\boxed{\theta(x,t) = 8e^{-4\pi^2 t}\sin(2\pi x)}$$

START BY TAKING LAPLACE TRANSFORM OF THE P.D.E W.R.T. t

$$\Rightarrow \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial \theta}{\partial t}$$

$$\Rightarrow \mathcal{L}\left[\frac{\partial^2 \theta}{\partial x^2}\right] = \mathcal{L}\left[\frac{\partial \theta}{\partial t}\right]$$

$$\Rightarrow \frac{\partial^2}{\partial x^2}\left[\mathcal{L}[\theta]\right] = s\mathcal{L}[\theta] - \theta(x,0)$$

$$\Rightarrow \frac{\partial^2 \bar{\theta}}{\partial x^2} = s\bar{\theta} - 8\sin(2\pi x) \qquad \theta(x,0) = 8\sin(2\pi x)$$

THIS IS A STANDARD 2ND ORDER O.D.E FOR $\bar{\theta}(x,s)$, WHERE s IS TREATED AS A POSITIVE CONSTANT

$$\Rightarrow \frac{\partial^2 \bar{\theta}}{\partial x^2} - s\bar{\theta} = -8\sin(2\pi x)$$

$$\Rightarrow \bar{\theta}(x,s) = A(s)e^{\sqrt{s}x} + B(s)e^{-\sqrt{s}x} + \text{PARTICULAR INTEGRAL}$$

TO FIND THE PARTICULAR INTEGRAL TRY $\bar{\theta}(x,s) = P(s)\sin(2\pi x)$, AS NO COSINE TERM IS NEEDED DUE TO THE ABSENCE OF THE FIRST DERIVATIVE

$$\Rightarrow \frac{\partial^2 \bar{\theta}}{\partial x^2} = -4\pi^2 P(s)\sin(2\pi x)$$

SUBSTITUTE INTO THE O.D.E.

$$\Rightarrow -4\pi^2 P(s)\sin(2\pi x) - sP(s)\sin(2\pi x) \equiv -8\sin(2\pi x)$$

$$\Rightarrow (4\pi^2 - s)P(s) = -8$$

$$\Rightarrow P(s) = \frac{8}{4\pi^2 + s}$$

∴ THE GENERAL SOLUTION OF THE O.D.E IS

$$\bar{\theta}(x,s) = A(s)e^{\sqrt{s}x} + B(s)e^{-\sqrt{s}x} + \frac{8}{4\pi^2 + s}\sin(2\pi x)$$

NEXT WE NEED TO TAKE THE LAPLACE TRANSFORM OF THE BOUNDARY CONDITIONS WHICH INVOLVE t

- $\theta(0,t) = 0$
  $\mathcal{L}[\theta(0,t)] = \mathcal{L}[0]$
  $\bar{\theta}(0,s) = 0$
- $\theta(1,t) = 0$
  $\mathcal{L}[\theta(1,t)] = \mathcal{L}[0]$
  $\bar{\theta}(1,s) = 0$

APPLYING THE TO THE SOLUTION

$$\bar{\theta}(0,s) = 0 \Rightarrow 0 = A(s) + B(s) + 0$$

$$\Rightarrow A(s) = -B(s)$$

$$\bar{\theta}(1,s) = 0 \Rightarrow 0 = A(s)e^{\sqrt{s}} + B(s)e^{-\sqrt{s}} + 0$$

$$\Rightarrow 0 = -B(s)e^{\sqrt{s}} + B(s)e^{-\sqrt{s}}$$

$$\Rightarrow 0 = B(s)\left[e^{-\sqrt{s}} - e^{\sqrt{s}}\right]$$

$$\Rightarrow 0 = -B(s)\left[e^{\sqrt{s}} - e^{-\sqrt{s}}\right]$$

$$\Rightarrow 0 = -2B(s)\sinh\sqrt{s}$$

$$\Rightarrow B(s) = 0 \qquad (\sinh\sqrt{s} \ne 0, \text{ AS } s \ne 0)$$

$$\Rightarrow A(s) = 0$$

$$\Rightarrow \boxed{\bar{\theta}(x,s) = \frac{8}{4\pi^2 + s}\sin(2\pi x)}$$

INVERTING THE TRANSFORM, NOTING THAT x IS A CONSTANT

$$\boxed{\theta(x,t) = 8e^{-4\pi^2 t}\sin(2\pi x)}$$

Question 7

The temperature $\theta(x,t)$ in a semi-infinite thin rod satisfies the heat equation

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial \theta}{\partial t}, \quad x \geq 0, \quad t \geq 0.$$

The initial temperature of the rod is $0\ ^\circ\text{C}$, and for $t > 0$ the endpoint at $x = 0$ is maintained at $T\ ^\circ\text{C}$.

Assuming the rod is insulated along its length, use Laplace transforms to find an expression for $\theta(x,t)$.

You may assume that

- $\mathcal{L}^{-1}\left[\dfrac{e^{-\sqrt{s}}}{s}\right] = \text{erfc}\left(\dfrac{1}{2\sqrt{t}}\right)$
- $\mathcal{L}^{-1}\left[\bar{f}(ks)\right] = \dfrac{1}{k} f\left(\dfrac{t}{k}\right)$, where k is a constant.

$$\boxed{\theta(x,t) = \frac{2T}{\sqrt{\pi}} \int_{\frac{x}{2\alpha\sqrt{t}}}^{\infty} e^{-u^2}\, du \quad = \quad T\,\text{erfc}\left(\frac{x}{2\alpha\sqrt{t}}\right)}$$

$$\boxed{\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial \theta}{\partial t}, \text{ FOR } \theta = \theta(x,t), \; t \geq 0, \; x \geq 0 \quad \text{SUBJECT TO } \theta(x,0) = 0, \; \theta(0,t) = T}$$

- TAKING THE LAPLACE TRANSFORM OF THE P.D.E, W.R.T t

$$\Rightarrow \mathcal{L}\left[\frac{\partial^2 \theta}{\partial x^2}\right] = \frac{1}{\alpha^2}\mathcal{L}\left[\frac{\partial \theta}{\partial t}\right]$$

$$\Rightarrow \frac{\partial^2}{\partial x^2}\left[\mathcal{L}[\theta]\right] = \frac{1}{\alpha^2}\left[s\mathcal{L}[\theta] - \cancel{\theta(x,0)}\right]$$

$$\Rightarrow \frac{\partial^2 \bar{\theta}}{\partial x^2} = \frac{s}{\alpha^2}\bar{\theta}$$

- THIS HAS CHANGED THE PDE INTO AN O.D.E FOR $\bar{\theta} = \bar{\theta}(x,s)$, WHERE s IS TREATED AS A CONSTANT

$$\Rightarrow \boxed{\bar{\theta}(x,s) = A(s)e^{\frac{\sqrt{s}}{\alpha}x} + B(s)e^{-\frac{\sqrt{s}}{\alpha}x}}$$

- AS SOLUTION CANNOT BE UNBOUNDED AS $x \to \infty$, $A(s) = 0$, SINCE $\bar{\theta}(x,s)$ CANNOT ALSO BE UNBOUNDED AS $x \to \infty$

$$\Rightarrow \boxed{\bar{\theta}(x,s) = B(s)e^{-\frac{\sqrt{s}}{\alpha}x}}$$

- APPLY THE LAPLACE TRANSFORM IN THE BOUNDARY CONDITION

$$\theta(0,t) = T$$
$$\mathcal{L}[\theta(0,t)] = \mathcal{L}[T]$$
$$\bar{\theta}(0,s) = \frac{T}{s}$$

HENCE IF $x = 0$

$$\bar{\theta}(0,s) = B(s)e^0$$
$$\frac{T}{s} = B(s)$$

$$\Rightarrow \boxed{\bar{\theta}(x,s) = \frac{T}{s}e^{-\frac{\sqrt{s}}{\alpha}x}}$$

- WE HAVE NOW REACHED INVERSION STAGE — FOR THIS WE USE THE FOLLOWING STANDARD RESULTS

$$\mathcal{L}^{-1}\left[\frac{e^{-\sqrt{s}}}{s}\right] = \text{erfc}\left(\frac{1}{2\sqrt{t}}\right)$$

$$\& \quad \mathcal{L}^{-1}\left[\bar{f}(ks)\right] = \frac{1}{k}f\left(\frac{t}{k}\right).$$

- IN OUR CASE

$$\Rightarrow \bar{\theta}(x,s) = T \times \frac{e^{-\frac{x}{\alpha}\sqrt{s}}}{s}$$

$$\Rightarrow \bar{\theta}(x,s) = T \times \frac{e^{-\sqrt{\frac{x^2 s}{\alpha^2}}}}{s} = T \times \frac{x^2}{\alpha^2} \times \boxed{\frac{e^{-\sqrt{\frac{x^2 s}{\alpha^2}}}}{\frac{x^2}{\alpha^2}s}}$$

$\bar{f}(ks)$, $k = \frac{x^2}{\alpha^2}$

$$\Rightarrow \theta(x,t) = T \times \frac{x^2}{\alpha^2} \times \underbrace{\frac{\alpha^2}{x^2}}_{\frac{1}{k}} \times \text{erfc}\left(\frac{1}{2\sqrt{\underbrace{\frac{t\alpha^2}{x^2}}_{\frac{t}{k}}}}\right)$$

$$\Rightarrow \theta(x,t) = T \times \text{erfc}\left(\frac{1}{2\frac{\alpha}{x}\sqrt{t}}\right)$$

$$\Rightarrow \theta(x,t) = T\,\text{erfc}\left(\frac{x}{2\alpha\sqrt{t}}\right)$$

$$\text{OR} \quad \theta(x,t) = T \times \frac{2}{\sqrt{\pi}}\int_{\frac{x}{2\alpha\sqrt{t}}}^{\infty} e^{-u^2}\, du$$

Question 8

The function $x = f(t)$ satisfies the differential equation

$$\frac{d^2x}{dt^2} + x = t\,\mathrm{H}(t-a),\ t \ge 0,$$

where $\mathrm{H}(t)$ is the Heaviside function and a is a positive constant.

Use Laplace transforms followed by inversion using complex variable to show that

$$x = t\,\mathrm{H}(t-a) - \mathrm{H}(t-a)\sin(t-a) + a\,\mathrm{H}(t-a)\cos(t-a).$$

$\boxed{\text{proof}}$

$$\boxed{\frac{d^2x}{dt^2} + x = tH(t-a),\quad x(0)=0,\ \frac{dx}{dt}(0)=0}$$

- TAKE THE LAPLACE TRANSFORM OF THE O.D.E IN t

$$\mathcal{L}\left[\frac{d^2x}{dt^2}\right] + \mathcal{L}[x] = \mathcal{L}[tH(t-a)]$$

$$s^2\bar{x} - \cancel{sx(0)} - \cancel{\dot{x}(0)} + \bar{x} = \mathcal{L}[tH(t-a)]$$

- USING STANDARD RESULTS ON THE HEAVISIDE ON THE RHS

$$\mathcal{L}[H(t)] = \frac{1}{s}$$

$$\mathcal{L}[H(t-a)] = e^{-as} \times \frac{1}{s}$$

$$\mathcal{L}[tH(t-a)] = -\frac{d}{ds}\left[\frac{e^{-as}}{s}\right] = -\left[\frac{-sae^{-as} - e^{-as}}{s^2}\right] = \frac{e^{-as}}{s^2}(as+1)$$

- RETURNING TO THE "MAIN" PROBLEM

$$(s^2+1)\bar{x} = \frac{e^{-as}(as+1)}{s^2}$$

$$\bar{x} = \frac{e^{-as}(as+1)}{s^2(s^2+1)}$$

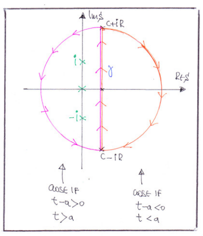
- INVERT BY THE BROMWICH INVERSION FORMULA

$$x(t) = \frac{1}{2\pi i}\int_\gamma \bar{f}(s)e^{at}\,ds$$

$$\boxed{\bar{f}(s) = \frac{e^{-as}(as+1)}{s^2(s^2+1)},\quad a>0}$$

- $$x(t) = \frac{1}{2\pi i}\int_\gamma \bar{f}(s)e^{st}\,dt = \frac{1}{2\pi i}\int_\gamma \frac{e^{-as}(as+1)e^{st}}{s^2(s^2+1)}\,ds = \frac{1}{2\pi i}\int_\gamma \frac{e^{s(t-a)}(1+as)}{s^2(s^2+1)}\,ds$$

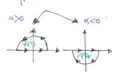
WHERE γ IS A STRAIGHT LINE TO THE "RIGHT" OF ALL THE SINGULARITIES OF THE INTEGRAND AS SHOWN IN THE FIGURE OPPOSITE



- WE REQUIRE THE CONTRIBUTION OF THE STRAIGHT LINE γ, AS $R \to \infty$

THE CONTRIBUTIONS FOR THE ARCS WOULD NOT BE CONTRIBUTING AS $R \to \infty$, DEPENDING ON THE EXPONENT'S COEFFICIENT BEING POSITIVE OR NEGATIVE, IN A SIMILAR FASHION TO THAT OF JORDAN'S LEMMA

$$\int_\Gamma \frac{e^{i\alpha z}}{f(z)}\,dz$$



- THE INTEGRAND HAS A DOUBLE POLE AT 0, AND SIMPLE POLES AT $\pm i$

RESIDUE AT i

$$\lim_{s\to i}\left[\cancel{(s-i)}\frac{e^{s(t-a)}(1+as)}{\cancel{(s-i)}(s+i)s^2}\right] = \frac{e^{i(t-a)}(1+ai)}{-2i}$$

RESIDUE AT $-i$

$$\lim_{s\to -i}\left[\cancel{(s+i)}\frac{e^{s(t-a)}(1+as)}{(s-i)\cancel{(s+i)}s^2}\right] = \frac{e^{-i(t-a)}(1-ai)}{2i}$$

RESIDUE AT 0 (DOUBLE POLE)

$$\lim_{s\to 0}\frac{d}{ds}\left[\cancel{s^2}\frac{e^{s(t-a)}(1+as)}{\cancel{s^2}(1+s^2)}\right] = \dots \text{QUOTIENT RULE} \dots$$

$$= \lim_{s\to 0}\left[\frac{(1+s^2)\left[e^{s(t-a)}\times(t-a)(1+as) + ae^{s(t-a)}\right] - e^{s(t-a)}(1+as)(2s)}{(1+s^2)^4}\right]$$

$$= \frac{1[1\times(t-a)\times 1 + a] - 1\times 1\times 0}{1} = t-a+a = t$$

- WE RETURN TO THE INVERSION

IF $t<a$ $\quad f(t)=0$

(THE SUM OF RESIDUES IS ZERO, SO THE INTEGRAL OVER THE CLOSED LOOP ON THE RIGHT YIELDS ZERO. BUT THE ARC DOES NOT CONTRIBUTE AS $R \to \infty$, SO THE STRAIGHT LINE SEGMENT (γ) FROM $-\infty$ TO ∞ WHICH GIVES THE INVERSION MUST ALSO BE ZERO)

IF $t>a$

THE ARC AGAIN DOESN'T CONTRIBUTE AS $R \to \infty$, SO THIS TIME THE CONTRIBUTION OF γ (STRAIGHT LINE FROM $-\infty$ TO ∞) WHICH GIVES $x(t)$ MUST EQUAL

$2\pi i \times \sum$ RESIDUES

$$\therefore f(t) = \underset{\text{IN FRONT OF THE FORMULA}}{\underbrace{2\pi i \times \frac{1}{2\pi i}}} \times \left[t + \frac{(1-ai)e^{-i(t-a)}}{2i} - \frac{(1+ai)e^{i(t-a)}}{2i}\right]$$

$$f(t) = t + \frac{1}{2i}e^{-i(t-a)} - \frac{a}{2}e^{-i(t-a)} - \frac{1}{2i}e^{i(t-a)} - \frac{a}{2}e^{i(t-a)}$$

$$f(t) = t - \frac{1}{2i}\left[e^{i(t-a)} - e^{-i(t-a)}\right] - \frac{a}{2}\left[e^{i(t-a)} + e^{-i(t-a)}\right]$$

$$f(t) = t - \sin(t-a) - a\cos(t-a)$$

$$f(t) = \begin{cases} t - \sin(t-a) - a\cos(t-a) & t>a \\ 0 & t<a \end{cases}$$

$$f(t) = tH(t-a) - H(t-a)\sin(t-a) - aH(t-a)\cos(t-a)$$

Question 9

The function $x_n = f(t,n)$ satisfies the differential equation

$$t\frac{d^2x}{dt^2}+(1-t)\frac{dx}{dt}+nx=0\,,\ t\geq 0\,,\ n\in\mathbb{N}\,.$$

Use Laplace transforms in t, followed by inversion using a unit circle contour, to show that

$$x_n=\frac{\mathrm{e}^t}{n!}\frac{d^n}{dt^n}\left(t^n\mathrm{e}^{-t}\right).$$

You may assume that

$$\oint_C \frac{\mathrm{e}^{st}}{s^{n+1}}\,ds=\frac{t^n}{n!}(2\pi\mathrm{i})\,,\quad \text{where}\quad C: s=\mathrm{e}^{\mathrm{i}\theta}\,,-\pi<\infty\leq\pi$$

proof

$$\boxed{t\frac{d^2x}{dt^2}+(1-t)\frac{dx}{dt}+nx=0}$$

$$\Rightarrow t\frac{d^2x}{dt^2}-t\frac{dx}{dt}+\frac{dx}{dt}+nx=0$$

TAKING THE LAPLACE TRANSFORM OF THE O.D.E.

$$\Rightarrow \underbrace{x_0-\frac{d}{ds}\left(s^2\bar{x}\right)}_{t\frac{d^2x}{dt^2}}-\left\{\underbrace{-\frac{d}{ds}\left[s\bar{x}-x_0\right]}_{t\frac{dx}{dt}}\right\}+\underbrace{s\bar{x}-x_0}_{\frac{dx}{dt}}+n\bar{x}=0$$

$$\Rightarrow \cancel{x_0}-\frac{d}{ds}\left(s^2\bar{x}\right)+\frac{d}{ds}\left[s\bar{x}-x_0\right]+s\bar{x}-\cancel{x_0}+n\bar{x}=0$$

$$\Rightarrow -2s\bar{x}-s^2\frac{d\bar{x}}{ds}+\bar{x}+s\frac{d\bar{x}}{ds}+s\bar{x}+n\bar{x}=0$$

$$\Rightarrow (s-s^2)\frac{d\bar{x}}{ds}+(n+1-s)\bar{x}=0$$

$$\Rightarrow s(1-s)\frac{d\bar{x}}{ds}=(s-n-1)\bar{x}$$

$$\Rightarrow \frac{1}{\bar{x}}\,d\bar{x}=\frac{s-n-1}{s(1-s)}\,ds$$

$$\Rightarrow \frac{1}{\bar{x}}\,d\bar{x}=(s-n-1)\left[\frac{1}{s}+\frac{1}{1-s}\right]ds$$

$$\Rightarrow \frac{1}{\bar{x}}\,d\bar{x}=\left[1-\frac{n}{s}-\frac{1}{s}+\frac{s}{1-s}-\frac{n}{1-s}-\frac{1}{1-s}\right]ds$$

$$\Rightarrow \frac{1}{\bar{x}}\,d\bar{x}=\left[\cancel{1}-\frac{n}{s}-\frac{n}{1-s}-\frac{1}{s}+\cancel{\frac{s-1}{1-s}}\right]ds$$

$$\Rightarrow \frac{1}{\bar{x}}\,d\bar{x}=\left[\frac{n}{s-1}-\frac{n}{s}-\frac{1}{s}\right]ds$$

INTEGRATING BOTH SIDES

$$\Rightarrow \ln\bar{x}=n\ln|s-1|-n\ln s-\ln s+\ln A$$

$$\Rightarrow \ln\bar{x}=\ln|s-1|^n-\ln s^n-\ln s+\ln A$$

$$\Rightarrow \ln\bar{x}=\ln\left[\frac{A(s-1)^n}{s^{n+1}}\right]$$

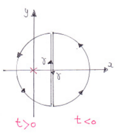
$$\Rightarrow \bar{x}=\frac{A(s-1)^n}{s^{n+1}}$$

$$\Rightarrow x(t)=\mathcal{L}^{-1}\left[\frac{A(s-1)^n}{s^{n+1}}\right]$$

c) INVERTING BY COMPLEX VARIABLE

$$\Rightarrow x(t)=\frac{1}{2\pi i}\int_{\gamma}\frac{(s-1)^n}{s^{n+1}}e^{st}\,ds$$

POLE AT THE ORIGIN OF ORDER $n+1$



IF $t<0$ WE CLOSE THE BROMWICH CONTOUR TO THE RIGHT, WHICH PRODUCES ZERO. IF $t>0$, WE CLOSE THE CONTOUR TO THE LEFT

$$x(t)=\frac{1}{2\pi i}\times 2\pi i\sum\left(\text{RESIDUES OF THE INTEGRAND INSIDE}\right)$$

IN THIS EXAMPLE IT IS NOT PHYSICALLY POSSIBLE TO FIND THIS RESIDUES. USUALLY WE OBTAIN THE ANSWER BY LETTING THE ARC "TEND TO INFINITY" WITH NO CONTRIBUTION AND GET THE INVERSION FROM THE STRAIGHT LINE SEGMENT.

HOWEVER WE CAN ALSO GET THE ANSWER (INVERSION) BY DEFORMING THE CONTOUR INTO ANY SHAPE SO LONG AS IT CONTAINS THE POLE

DEFORMING INTO A UNIT CIRCLE, CENTRE AT THE ORIGIN

$$x(t)=\frac{1}{2\pi i}\oint_C\frac{(s-1)^n e^{st}}{s^{n+1}}\,ds\,,\quad C \text{ IS A UNIT CIRCLE AT } 0$$

LET $s=u+1$	$C: s=e^{i\theta}$
$ds=du$	$C': u=-1+e^{i\theta}$

$$\Rightarrow x(t)=\frac{1}{2\pi i}\oint_{C'}\frac{u^n e^{(u+1)t}}{(u+1)^{n+1}}\,du$$

$$\Rightarrow x(t)=\frac{1}{2\pi i}\oint_{C'}\frac{u^n e^{ut}e^t}{(u+1)^{n+1}}\,du$$

$$\Rightarrow x(t)=\frac{e^t}{2\pi i}\oint_{C'}\frac{u^n e^{ut}}{(u+1)^{n+1}}\,du$$

$$\Rightarrow x(t)=\frac{e^t}{2\pi i}\oint_{C'}\frac{\frac{d^n}{dt^n}\left(e^{ut}\right)}{(u+1)^{n+1}}\,du$$

$$\Rightarrow x(t)=\frac{e^t}{2\pi i}\frac{d^n}{dt^n}\oint_{C'}\frac{e^{ut}}{(u+1)^{n+1}}\,du$$

NOW WE ARE GIVEN THE RESULT

$$\oint_C\frac{e^{st}}{s^{n+1}}\,ds=2\pi i\frac{t^n}{n!}$$

USING THE SUBSTITUTION $s=u+1$ AS EARLIER

$$\int_{C'}\frac{e^{(u+1)t}}{(u+1)^{n+1}}\,du=2\pi i\frac{t^n}{n!}\quad\left(\begin{array}{l}\text{AS THE R.H.S IS}\\ \text{INDEPENDENT OF } u/s\end{array}\right)$$

RETURNING TO THE INVERSION

$$x(t)=\frac{e^t}{2\pi i}\frac{d^n}{dt^n}\oint_{C'}\frac{e^{ut}}{(u+1)^{n+1}}\,du$$

$$x(t)=\frac{e^t}{2\pi i}\frac{d^n}{dt^n}\left[e^{-t}\oint_{C'}\frac{e^{ut}e^t}{(u+1)^{n+1}}\,du\right]$$

$$x(t)=\frac{e^t}{2\pi i}\frac{d^n}{dt^n}\left[e^{-t}\oint_{C'}\frac{e^{(u+1)t}}{(u+1)^{n+1}}\,du\right]$$

$$x(t)=\frac{e^t}{\cancel{2\pi i}}\frac{d^n}{dt^n}\left[e^{-t}\times\cancel{2\pi i}\frac{t^n}{n!}\right]$$

$$x(t)=\frac{e^t}{n!}\frac{d^n}{dt^n}\left(e^{-t}t^n\right)$$

AS REQUIRED

Question 10

The function $y = y(t)$, $t \geq 0$ satisfies the following equation.

$$\frac{d^2y}{dt^2} - y + 2\int_0^t \sin(t-u)\, y(u)\, du = \cos t\,.$$

Use Laplace transforms to show that

$$y(t) = \sin\left(\frac{t}{\sqrt{2}}\right)\sinh\left(\frac{t}{\sqrt{2}}\right).$$

$\boxed{\text{proof}}$

$$\frac{d^2y}{dt^2} - y + 2\int_0^t \sin(t-u)\, y(u)\, du = \cos t \qquad y(0)=0, \quad \frac{dy}{dt}(0)=0$$

- TAKE THE LAPLACE TRANSFORM OF THE EQUATION

$\Rightarrow \left[s^2\bar{y} - \cancel{sy(0)} - \cancel{y'(0)}\right] - \bar{y} + 2\mathcal{L}\Big[\underbrace{\int_0^t \sin(t-u)\,y(u)\,du}_{\text{CONVOLUTION}}\Big] = \mathcal{L}[\cos t]$

$\Rightarrow s^2\bar{y} - \bar{y} + 2\mathcal{L}[\sin t]\,\mathcal{L}[y(t)] = \mathcal{L}[\cos t]$

$\Rightarrow s^2\bar{y} - \bar{y} + 2\left(\frac{1}{s^2+1}\right)\bar{y} = \frac{s}{s^2+1}$

$\Rightarrow \bar{y}\left[s^2 - 1 + \frac{2}{s^2+1}\right] = \frac{s}{s^2+1}$

$\Rightarrow \bar{y}\left[\frac{(s^2-1)(s^2+1)+2}{s^2+1}\right] = \frac{s}{s^2+1}$

$\Rightarrow \bar{y}\left[\frac{s^4+1}{s^2+1}\right] = \frac{s}{s^2+1}$

$\Rightarrow \bar{y} = \frac{s}{s^4+1}$

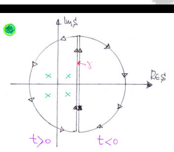
- WE NOW NEED TO INVERT, USING THE STANDARD FORMULA

$$\boxed{y = \frac{1}{2\pi i}\int_\gamma \left(\frac{s}{s^4+1}\right)e^{st}\,ds}$$

- $f(s) = \frac{s}{s^4+1}$ HAS SIMPLE POLES AT $s^4 = -1 = e^{i(\pi + 2k\pi)} = e^{i\pi(2k+1)}$

$s = e^{i\frac{\pi}{4}(2k+1)}$

$s = e^{i\frac{\pi}{4}},\ e^{i\frac{3\pi}{4}},\ e^{-i\frac{\pi}{4}},\ e^{-\frac{3\pi}{4}}$



- FOR $t<0$ WE CLOSE THE CONTOUR TO THE "RIGHT" OF γ, PICKING NO CONTRIBUTION AS THERE ARE NO POLES THERE
- FOR $t>0$ WE CLOSE THE CONTOUR TO THE "LEFT" OF γ, PICKING CONTRIBUTIONS
- NOTE THE CHOICE OF DIRECTION OF THE SEMICIRCLE IS SUCH SO THAT THE ARC DOES NOT CONTRIBUTE AS ITS RADIUS GETS INFINITE (LIKE IN JORDAN'S LEMMA)

- THUS WE GET FOR $t>0$

$y(t) = \frac{1}{2\pi i}\int_\gamma \left(\frac{s}{1+s^4}\right)e^{st}\,ds = \frac{1}{\cancel{2\pi i}}\left[\cancel{2\pi i}\sum \text{RESIDUES OF } \frac{se^{st}}{1+s^4}\right]$

- RESIDUES AT $e^{\pm i\frac{\pi}{4}},\ e^{\pm\frac{3\pi}{4}i}$ – FIND IT BY A GENERAL METHOD

LET THE POLE BE AT $s = s_0$

$\lim_{s\to s_0}\left[\frac{(s-s_0)s\,e^{st}}{1+s^4}\right]$ WHICH OF COURSE GIVES $\frac{0}{0}$, AS $(s-s_0)$ IS A FACTOR OF $1+s^4$.

BY L'HOSPITAL RULE

$\ldots = \lim_{s\to s_0} \frac{\frac{d}{ds}\left[(s-s_0)s\,e^{st}\right]}{\frac{d}{ds}\left[1+s^4\right]} = \lim_{s\to s_0} \frac{se^{st} + (s-s_0)e^{st} + (s-s_0)tse^{st}}{4s^3}$

$= \frac{s_0 e^{s_0 t}}{4s_0^3} = \boxed{\frac{e^{s_0 t}}{4s_0^2}}$

- HENCE WE OBTAIN THE RESIDUE AT EACH POLE

- $s_0 = e^{i\frac{\pi}{4}}$, $\frac{e^{te^{i\frac{\pi}{4}}}}{4e^{i\frac{\pi}{2}}} = \frac{e^{t\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)}}{4i}$
- $s_0 = e^{-i\frac{\pi}{4}}$, $\frac{e^{te^{-i\frac{\pi}{4}}}}{4e^{-i\frac{\pi}{2}}} = \frac{e^{t\left(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)}}{-4i}$
- $s_0 = e^{i\frac{3\pi}{4}}$, $\frac{e^{te^{i\frac{3\pi}{4}}}}{4e^{\frac{3\pi}{2}i}} = \frac{e^{t\left(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)}}{-4i}$
- $s_0 = e^{-i\frac{3\pi}{4}}$, $\frac{e^{te^{-i\frac{3\pi}{4}}}}{4e^{-\frac{3\pi}{2}i}} = \frac{e^{t\left(-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)}}{4i}$

- $y(t) = \sum(\text{RESIDUES AT THE 4 POLES})$

$= \frac{1}{4i}\left[e^{t\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)} - e^{t\left(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)} - e^{t\left(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)} + e^{t\left(-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)}\right]$

$= \frac{1}{4i}\left[\left[e^{at}e^{iat} - e^{at}e^{-iat}\right] + \left[e^{-at}e^{-iat} - e^{-at}e^{ati}\right]\right]$

WHERE $a = \frac{\sqrt{2}}{2}$

$= \frac{1}{4i}\left[e^{at}\left(e^{iat} - e^{-iat}\right) - e^{-at}\left(e^{ati} - e^{-ati}\right)\right]$

$= \frac{1}{4i}\left[\left(e^{iat} - e^{-iat}\right)\left(e^{at} - e^{-at}\right)\right]$

$= \frac{1}{4i}\left[2\sinh(iat)\right]\left[2\sinh(at)\right]$

$= \frac{1}{i}\sinh(iat)\sinh(at)$

$= \frac{i}{i}\sin(at)\sinh(at)$

$= \sin(at)\sinh(at)$

$= \sin\left(\frac{t}{\sqrt{2}}\right)\sinh\left(\frac{t}{\sqrt{2}}\right)$ //

Question 11

The one dimensional heat equation for the temperature, $T(x,t)$, satisfies

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\sigma}\frac{\partial T}{\partial t}, \ t \geq 0,$$

where t is the time, x is a spatial dimension and σ is a positive constant.

The temperature $T(x,t)$ is subject to the following conditions.

i. $\lim_{x \to \infty} \left[T(x,t) \right] = 0$

ii. $T(0,t) = 1$

iii. $T(x,0) = 0$

a) Use Laplace transforms to show that

$$\mathcal{L}\left[T(x,t)\right] = \bar{T}(x,s) = \frac{1}{s}\exp\left[-\sqrt{\frac{s}{\sigma}}\, x\right].$$

b) Use contour integration to show further that

$$T(x,t) = 1 - \text{erf}\left[\frac{x}{4\sigma t}\right].$$

You may assume without proof that

- $\int_0^{\infty} e^{-ax^2} \cos kx \ dx = \sqrt{\frac{\pi}{4a}} \exp\left[-\frac{k^2}{4a}\right]$
- $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2} \ d\xi$

$\boxed{\text{proof}}$

[solution overleaf]

a) $$\boxed{\frac{\partial^2 T}{\partial x^2} = \frac{1}{\sigma}\frac{\partial T}{\partial t} \qquad \text{SUBJECT} \quad T(x,0)=0, \quad T(0,t)=1, \quad T(x,t) \to 0 \text{ AS } x \to \infty}$$

$$\Rightarrow \sigma \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

- TAKING LAPLACE TRANSFORMS IN t (x UNAFFECTED)

$$\Rightarrow \mathcal{L}\left[\sigma \frac{\partial^2 T}{\partial x^2}\right] = \mathcal{L}\left[\frac{\partial T}{\partial t}\right]$$

$$\Rightarrow \sigma \frac{\partial^2 \bar{T}}{\partial x^2} = s\bar{T} - \cancel{T(x,0)}$$

$$\Rightarrow \frac{\partial^2 \bar{T}}{\partial x^2} = \frac{s}{\sigma}\bar{T}$$

IE A SECOND ORDER O.D.E IN $\bar{T}$, WITH EXPONENTIAL SOLUTIONS

$$\Rightarrow \boxed{\bar{T}(x,s) = A(s)e^{-\sqrt{\frac{s}{\sigma}}x} + B(s)e^{\sqrt{\frac{s}{\sigma}}x}}$$

- APPLY $T \to 0$ AS $x \to \infty$

$\Rightarrow \bar{T} \to 0$ AS $x \to \infty$ $\quad \therefore \boxed{B(s)=0}$

$$\Rightarrow \boxed{\bar{T}(x,s) = A(s)e^{-\sqrt{\frac{s}{\sigma}}x}}$$

- APPLY $T(0,t)=1 \Rightarrow \bar{T}(0,s) = \frac{1}{s}$

$$\Rightarrow \boxed{\frac{1}{s} = A(s)}$$

$$\Rightarrow \boxed{\bar{T}(x,s) = \frac{1}{s}e^{-\sqrt{\frac{s}{\sigma}}x}}$$

b) $$\boxed{\bar{f}(x,s) = \frac{1}{s}e^{-\sqrt{\frac{s}{\sigma}}x}}$$

- THIS IS NOT RECOGNISABLE SO WE NEED CONTOUR INTEGRATION

THERE IS ANOTHER PROBLEM
$\bar{f}(s)$ HAS A NON INTEGRABLE SINGULARITY AT $s=0$, WHICH WILL BECOME A PROBLEM EVENTUALLY IF WE WENT AHEAD AS PER USUAL

$$\frac{\partial \bar{T}}{\partial x} = \frac{\partial}{\partial x}\left[\bar{f}(x,s)\right] = -\frac{1}{\sqrt{\sigma s}}e^{-\sqrt{\frac{s}{\sigma}}x} = -\frac{e^{-\sqrt{\frac{s}{\sigma}}x}}{\sqrt{s\sigma}}$$

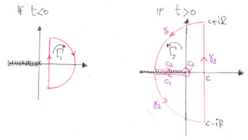
- SO WE WILL INVERT $\frac{\partial \bar{T}}{\partial x}$ TO GET THE TEMPERATURE GRADIENT I.E $\frac{\partial T}{\partial x}$ (THE TRANSFORM IS NOT AFFECTED AS THE LAPLACE TRANSFORM IS WITH RESPECT TO t

$$\Rightarrow \frac{\partial \bar{T}}{\partial x} = -\frac{e^{-\sqrt{\frac{s}{\sigma}}x}}{\sqrt{s\sigma}}$$

$$\Rightarrow \frac{\partial T}{\partial x} = \frac{1}{2\pi i}\int_{\gamma} -\frac{e^{-\sqrt{\frac{s}{\sigma}}x}}{\sqrt{s\sigma}}e^{st}\,ds = \frac{-1}{2\pi\sqrt{\sigma}\,i}\int_{\gamma}\frac{e^{-\sqrt{\frac{s}{\sigma}}x+st}}{\sqrt{s}}\,ds$$

- THE INTEGRAND CONTAINS A NON INTEGER POWER, SO WE HAVE A BRANCH POINT AT $s=0$ AND HENCE A BRANCH CUT.

IF $t<0$ $\qquad$ IF $t>0$



- INVERTING FOR $t<0$ WE GET

$$\frac{\partial T}{\partial x} = \frac{1}{2\pi i}\times 0 = 0 \quad \text{SINCE THE ARC DOES NOT CONTRIBUTE, THEN THE STRAIGHT LINE YIELDS ZERO}$$

$\Longrightarrow T(x,t) = \text{CONSTANT}$ (WE WILL GET THIS FROM THE SECOND SECTION)

- INVERTING FOR $t>0$ LOOKING ONLY AT $\int \frac{e^{st}e^{-\frac{x}{\sqrt{\sigma}}\sqrt{s}}}{\sqrt{s}}\,ds$

NO POLES INSIDE Γ_2 SO BY CAUCHY'S THEOREM THE TOTAL CONTRIBUTIONS IS 0

IT CAN BE SHOWN THAT THE CONTRIBUTION FROM γ_1 & γ_2 VANISHES AS $R \to \infty$

FOR C_3 AS $\varepsilon \to 0$
$s = \varepsilon e^{i\theta}$
$ds = \varepsilon i e^{i\theta}\,d\theta$

$$\left|\int_{C_3}\frac{e^{st}e^{-\frac{x}{\sqrt{\sigma}}\sqrt{s}}}{\sqrt{s}}\,ds\right| \leqslant \int_{\pi}^{-\pi}\left|\frac{e^{t\varepsilon e^{i\theta}}\,e^{-\frac{x}{\sqrt{\sigma}}\varepsilon^{\frac{1}{2}}e^{i\frac{\theta}{2}}}}{\varepsilon^{\frac{1}{2}}e^{i\frac{\theta}{2}}}\varepsilon i e^{i\theta}\,d\theta\right|$$

$$= \int_{\pi}^{-\pi}\left|\frac{e^{t\varepsilon\cos\theta}\times e^{it\varepsilon\sin\theta}\times e^{-\frac{x}{\sqrt{\sigma}}\varepsilon^{\frac{1}{2}}\cos\frac{\theta}{2}}\times e^{-i\frac{x}{\sqrt{\sigma}}\varepsilon^{\frac{1}{2}}\sin\frac{\theta}{2}}\times \varepsilon i e^{i\theta}}{\varepsilon^{\frac{1}{2}}e^{i\frac{\theta}{2}}}\right|d\theta$$

$$= \int_{\pi}^{-\pi}\frac{\left|e^{t\varepsilon\cos\theta}\right|\left|e^{it\varepsilon\sin\theta}\right|\left|e^{-\frac{x}{\sqrt{\sigma}}\varepsilon^{\frac{1}{2}}\cos\frac{\theta}{2}}\right|\left|e^{-i\frac{x}{\sqrt{\sigma}}\varepsilon^{\frac{1}{2}}\sin\frac{\theta}{2}}\right|\left|\varepsilon i e^{i\theta}\right|}{\left|\varepsilon^{\frac{1}{2}}\right|\left|e^{i\frac{\theta}{2}}\right|}\,d\theta$$

$$= \int_{\pi}^{-\pi}\frac{\left|e^{t\varepsilon\cos\theta}\right|\times 1\times\left|e^{-\frac{x}{\sqrt{\sigma}}\varepsilon^{\frac{1}{2}}\cos\frac{\theta}{2}}\right|\times 1\times\varepsilon}{\varepsilon^{\frac{1}{2}}\times 1}\,d\theta$$

$$\leqslant \int_{\pi}^{-\pi}\frac{e^{t\varepsilon}\times 1\times e^{-\frac{x}{\sqrt{\sigma}}\varepsilon^{\frac{1}{2}}}\times 1\times\varepsilon}{\varepsilon^{\frac{1}{2}}}\,d\theta$$

$$= e^{t\varepsilon}\times e^{-\frac{x}{\sqrt{\sigma}}\varepsilon^{\frac{1}{2}}}\times\varepsilon^{\frac{1}{2}}\int_{\pi}^{-\pi}1\,d\theta = e^{t\varepsilon}\times e^{-\frac{x}{\sqrt{\sigma}}\varepsilon^{\frac{1}{2}}}\times\varepsilon^{\frac{1}{2}}\times 2\pi$$

$$= O(\varepsilon^{\frac{1}{2}}) \to 0 \text{ AS } \varepsilon \to 0$$

- THUS THE REQUIRED ANSWER (STRAIGHT LINE THROUGH c) MUST EQUAL

$$-\left[\int_{C_1} + \int_{C_2}\right]$$

$$\text{SO } \frac{\partial T}{\partial x} = -\frac{1}{2\pi i\sqrt{\sigma}}\int_{\gamma_3}\bar{f}(s)e^{st}\,dt = -\frac{1}{2\pi i\sqrt{\sigma}}\int_{\gamma_3}\frac{e^{st}e^{-\frac{x}{\sqrt{\sigma}}\sqrt{s}}}{\sqrt{\sigma}}\,ds$$

$$\Rightarrow \frac{\partial T}{\partial x} = \frac{1}{2\pi i\sqrt{\sigma}}\int_{C_2+C_1}\bar{f}(s)e^{st}\,dt$$

PARAMETRIZE EACH SECTION

C_2: $s = ue^{i\pi}$, $ds = e^{i\pi}\,du$, $ds = -du$, u FROM ∞ to 0

C_1: $s = e^{-i\pi}$, $ds = e^{i\pi}\,du$, $ds = -du$, u = FROM 0 to ∞

$$\Rightarrow \frac{\partial T}{\partial x} = \frac{1}{2\pi i\sqrt{\sigma}}\left[\int_{\infty}^{0}\frac{e^{tue^{i\pi}}\cdot e^{-\frac{x}{\sqrt{\sigma}}u^{\frac{1}{2}}e^{i\frac{\pi}{2}}}}{u^{\frac{1}{2}}e^{i\frac{\pi}{2}}}(-du) + \int_{0}^{\infty}\frac{e^{tue^{-i\pi}}\cdot e^{-\frac{x}{\sqrt{\sigma}}u^{\frac{1}{2}}e^{-i\frac{\pi}{2}}}}{u^{\frac{1}{2}}e^{-i\frac{\pi}{2}}}(-du)\right]$$

$$\Rightarrow \frac{\partial T}{\partial x} = \frac{1}{2\pi i\sqrt{\sigma}}\left[\int_{0}^{\infty}\frac{e^{-ut}e^{-i\frac{x}{\sqrt{\sigma}}u^{\frac{1}{2}}}}{iu^{\frac{1}{2}}}\,du - \int_{0}^{\infty}\frac{e^{-ut}e^{\frac{x}{\sqrt{\sigma}}u^{\frac{1}{2}}}}{-iu^{\frac{1}{2}}}\,du\right]$$

$$e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$$
$$e^{-i\frac{\pi}{2}} = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2} = -i$$

$$\Rightarrow \frac{\partial T}{\partial x} = \frac{1}{2\pi i\sqrt{\sigma}}\int_{0}^{\infty}\frac{e^{-ut}}{iu^{\frac{1}{2}}}\left(e^{\frac{x}{\sqrt{\sigma}}u^{\frac{1}{2}}} + e^{-\frac{x}{\sqrt{\sigma}}u^{\frac{1}{2}}}\right)du$$

$$\Rightarrow \frac{\partial T}{\partial x} = -\frac{1}{2\pi\sqrt{\sigma}}\int_{0}^{\infty}\frac{e^{-ut}}{u^{\frac{1}{2}}}\times 2\cos\left[\sqrt{\frac{u}{\sigma}}x\right]du$$

$$\Longrightarrow \frac{\partial T}{\partial x} = -\frac{1}{\pi\sqrt{\sigma}}\int_{0}^{\infty}\frac{e^{-ut}}{\sqrt{u}}\cos\left[\sqrt{\frac{u}{\sigma}}x\right]du$$

USE THE SUBSTITUTION $\sqrt{u} = v$
$u = v^2$
$du = 2v\,dv$
LIMITS UNCHANGED

$$\Rightarrow \frac{\partial T}{\partial x} = -\frac{1}{\pi\sqrt{\sigma}}\int_{0}^{\infty}\frac{e^{-v^2t}}{\cancel{v}}\cos\left(\frac{xv}{\sqrt{\sigma}}\right)2\cancel{v}\,dv$$

$$\Rightarrow \frac{\partial T}{\partial x} = -\frac{2}{\pi\sqrt{\sigma}}\int_{0}^{\infty}e^{-tv^2}\cos\left(\frac{x}{\sqrt{\sigma}}v\right)dv.$$

NOW $$\boxed{\int_{0}^{\infty}e^{-ax^2}\cos kx\,dx = \sqrt{\frac{\pi}{4a}}\exp\left(-\frac{k^2}{4a}\right)}$$

$$\Rightarrow \frac{\partial T}{\partial x} = -\frac{2}{\pi\sqrt{\sigma}}\times\sqrt{\frac{\pi}{4t}}\times\exp\left(-\frac{\frac{x^2}{\sigma}}{4t}\right)$$

$$\Rightarrow \frac{\partial T}{\partial x} = -\frac{2}{\pi\sqrt{\sigma}}\sqrt{\frac{\pi}{4t}}\times\exp\left(-\frac{x^2}{4\sigma t}\right)$$

$$\Rightarrow \frac{\partial T}{\partial x} = -\frac{1}{\sqrt{\sigma\pi t}}e^{-\frac{x^2}{4\sigma t}}$$

- INTEGRATE WITH RESPECT TO x & USE FACT $T(x,0)=1$

$$\Rightarrow T = C - \frac{1}{\sqrt{\sigma\pi t}}\int_{0}^{x}e^{-\frac{q^2}{4\sigma t}}\,dq$$

WHEN $x=0$, $T=1 \Rightarrow 1 = C + 0$
$\Rightarrow \boxed{C=1}$

$$\Rightarrow T = 1 - \frac{1}{\sqrt{\sigma\pi t}}\int_{0}^{x}e^{-\frac{q^2}{4\sigma t}}\,dq$$

LET $\xi^2 = \frac{q^2}{4\sigma t}$
$\xi = \frac{q}{2\sqrt{\sigma t}}$
$d\xi = \frac{1}{2\sqrt{\sigma t}}\,dq$
$q=0 \mapsto \xi=0$
$q=x \mapsto \xi = \frac{x}{\sqrt{4\sigma t}}$

- SO WE OBTAIN

$$T = 1 - \frac{1}{\sqrt{\sigma\pi t}}\int_{0}^{\frac{x}{\sqrt{4\sigma t}}}e^{-\xi^2}\left(2\sqrt{\sigma t}\,d\xi\right)$$

$$T = 1 - \frac{2}{\sqrt{\pi}}\int_{0}^{\frac{x}{\sqrt{4\sigma t}}}e^{\xi^2}\,d\xi$$

$$T = 1 - \operatorname{erf}\left(\frac{x}{4\sigma t}\right)$$

$$\left(T = \operatorname{erfc}\left(\frac{x}{4\sigma t}\right)\right)$$