LAPLACE TRANSFORMS FURTHER

SUMMARY OF THE LAPLACE TRANFORM

The Laplace Transform of a function $f(t)$, $t \geq 0$ is defined as

$$\mathcal{L}\left[f(t)\right] \equiv \bar{f}(s) \equiv \int_0^\infty \mathrm{e}^{-st} f(t)\ dt,$$

where $s \in \mathbb{C}$, with $\mathrm{Re}(s)$ sufficiently large for the integral to converge.

The Laplace Transform is a linear operation

$$\mathcal{L}\left[a\,f(t) + b\,g(t)\right] \equiv a\mathcal{L}\left[f(t)\right] + b\mathcal{L}\left[g(t)\right].$$

Laplace Transforms of Common Functions

- $\mathcal{L}\left(t^n\right) = \dfrac{n}{s^{n+1}}$

$$\mathcal{L}(1) = \frac{1}{s},\quad \mathcal{L}(a) = \frac{a}{s},\quad \mathcal{L}(t) = \frac{1}{s^2},\quad \mathcal{L}\left(t^2\right) = \frac{2}{s^3},\quad \mathcal{L}\left(t^3\right) = \frac{3}{s^4},\ \ldots.$$

- $\mathcal{L}\left(\mathrm{e}^{at}\right) = \dfrac{1}{s-a}$, $\mathcal{L}\left(\mathrm{e}^{-at}\right) = \dfrac{1}{s+a}$
- $\mathcal{L}(\cos at) = \dfrac{s}{s^2+a^2}$, $\mathcal{L}(\sin at) = \dfrac{a}{s^2+a^2}$
- $\mathcal{L}(\cosh at) = \dfrac{s}{s^2-a^2}$, $\mathcal{L}(\sinh at) = \dfrac{a}{s^2-a^2}$

Laplace Transforms of Derivatives

- $\mathcal{L}\left[x(t)\right] = \bar{x}(t)$
- $\mathcal{L}\left[\dot{x}(t)\right] = s\bar{x}(t) - x(0)$
- $\mathcal{L}\left[\ddot{x}(t)\right] = s^2\bar{x}(t) - sx(0) - \dot{x}(0)$
- $\mathcal{L}\left[\dddot{x}(t)\right] = s^3\bar{x}(t) - s^2x(0) - s\dot{x}(0) - \ddot{x}(0)$

Laplace Transforms Theorems

- 1st Shift Theorem

$$\mathcal{L}\left[e^{-at} f(t)\right] = \bar{f}(s+a) \quad \text{or} \quad \mathcal{L}\left[e^{at} F(t)\right] = \bar{f}(s-a)$$

- 2nd Shift Theorem

$$\mathcal{L}\left[f(t-a)\right] = e^{-as}\,\bar{f}(s),\ t > a \quad \text{or} \quad \mathcal{L}\left[f(t+a)\right] = e^{as}\,\bar{f}(s),\ t > -a\,.$$

$$\mathcal{L}\left[\mathrm{H}(t-a)\,f(t-a)\right] = e^{-as}\,\bar{f}(s) \quad \text{or} \quad \mathcal{L}\left[\mathrm{H}(t+a)\,f(t+a)\right] = e^{as}\,\bar{f}(s)$$

- Multiplication by t^n

$$\mathcal{L}\left[t^n f(t)\right] = \left(-\frac{d}{ds}\right)^n \left[\bar{f}(s)\right] \quad \text{or} \quad \mathcal{L}\left[t\,f(t)\right] = -\frac{d}{ds}\left[\bar{f}(s)\right]$$

- Division by t

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^{\infty} \bar{f}(\sigma)\ d\sigma$$

provided that $\lim\limits_{t\to 0}\left(\dfrac{f(t)}{t}\right)$ exists and the integral converges.

- Initial/Final value theorem

$$\lim_{t\to 0}\left[f(t)\right] = \lim_{s\to\infty}\left[s\,\bar{f}(s)\right] \quad \text{and} \quad \lim_{t\to\infty}\left[f(t)\right] = \lim_{s\to 0}\left[s\,\bar{f}(s)\right]$$

The Impulse Function / The Dirac Function

1. $\delta(t-c)=\begin{cases}\infty & t=c\\ 0 & t\neq c\end{cases}$, $\delta(t)=\begin{cases}\infty & t=0\\ 0 & t\neq 0\end{cases}$

2. $\displaystyle\int_a^b \delta(t-c)\ dt=\begin{cases}1 & a\le c\le b\\ 0 & \text{otherwise}\end{cases}$

3. $\displaystyle\int_a^b f(t)\delta(t-c)\ dt=\begin{cases}f(a) & a\le c\le b\\ 0 & \text{otherwise}\end{cases}$

4. $\mathcal{L}\left[\delta(t-c)\right]=\mathrm{e}^{-cs}$

5. $\mathcal{L}\left[f(t)\delta(t-c)\right]=f(c)\mathrm{e}^{-cs}$

6. $\dfrac{d}{dt}\left[\mathrm{H}(t-c)\right]=\delta(t-c)$

VARIOUS LAPLACE TRANSFORM QUESTIONS

Question 1

The function $x = x(t)$ is suitably defined for $t \geq 0$.

a) Show from first principles that

$$\mathcal{L}\left[\frac{dx}{dt}\right] = s\,\mathcal{L}\left[x(t)\right] - x(0).$$

b) Hence show further that

$$\mathcal{L}\left[\frac{d^2x}{dt^2}\right] = s^2\,\mathcal{L}\left[x(t)\right] - s\,x(0) - \frac{dx}{dt}(0).$$

proof

$x = x(t) \quad t \geq 0$

a) $\mathcal{L}\left[\frac{dx}{dt}\right] = \int_0^{\infty} \frac{dx}{dt} e^{-st}\,dt$ ... INTEGRATING BY PARTS

e^{-st}	$-s e^{-st}$
$x(t)$	$\frac{dx}{dt}$

$$= \left[x(t)e^{-st}\right]_{t=0}^{t=\infty} - \int_0^{\infty} -s\,x(t)e^{-st}\,dt$$

$$= 0 - x(0) + s\int_0^{\infty} x(t)e^{-st}\,dt$$

$$= s\,\mathcal{L}\left[x(t)\right] - x(0)$$

b) $\mathcal{L}\left[\frac{d^2x}{dt^2}\right] = \int_0^{\infty} \frac{d^2x}{dt^2} e^{-st}\,dt$ .... BY PARTS AGAIN

e^{-st}	$-s e^{-st}$
$\frac{dx}{dt}$	$\frac{d^2x}{dt^2}$

$$= \left[\frac{dx}{dt}e^{-st}\right]_{t=0}^{\infty} - \int_0^{\infty} -s\frac{dx}{dt}e^{-st}\,dt$$

$$= 0 - \left.\frac{dx}{dt}\right|_{t=0} + s\int_0^{\infty} \frac{dx}{dt}e^{-st}\,dt$$

$$= -\left.\frac{dx}{dt}\right|_0 + s\,\mathcal{L}\left[\frac{dx}{dt}\right]$$

$$= -\left.\frac{dx}{dt}\right|_0 + s\left[s\,\mathcal{L}\left[x(t)\right] - x(0)\right]$$

$$= s^2\,\mathcal{L}\left[x(t)\right] - s\,x(0) - \frac{dx}{dt}(0)$$

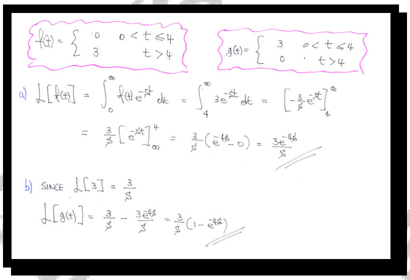
Question 2

$$f(t) \equiv \begin{cases} 0 & 0 < t \le 4 \\ 3 & t > 4 \end{cases} \quad \text{and} \quad g(t) \equiv \begin{cases} 3 & 0 < t \le 4 \\ 0 & t > 4 \end{cases}.$$

a) Find the Laplace transform of $f(t)$ from first principles.

b) Hence determine the Laplace transform of $g(t)$.

$\boxed{\mathcal{L}\left[f(t)\right] = \dfrac{3\mathrm{e}^{-4s}}{s}}$, $\boxed{\mathcal{L}\left[g(t)\right] = \dfrac{3}{s}\left(1 - \mathrm{e}^{-4s}\right)}$



$f(t) = \begin{cases} 0 & 0 < t \leqslant 4 \\ 3 & t > 4 \end{cases}$ $\qquad$ $g(t) = \begin{cases} 3 & 0 < t \leqslant 4 \\ 0 & t > 4 \end{cases}$

a) $\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}\,dt = \int_4^\infty 3e^{-st}\,dt = \left[-\frac{3}{s}e^{-st}\right]_4^\infty$

$= \frac{3}{s}\left[e^{-st}\right]_\infty^4 = \frac{3}{s}\left(e^{-4s} - 0\right) = \frac{3e^{-4s}}{s}$

b) SINCE $\mathcal{L}[3] = \frac{3}{s}$

$\mathcal{L}[g(t)] = \frac{3}{s} - \frac{3e^{-4s}}{s} = \frac{3}{s}\left(1 - e^{-4s}\right)$

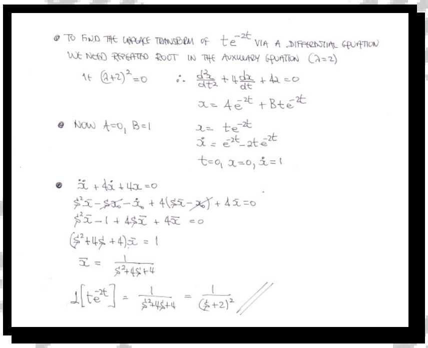
Question 3

By considering a suitable differential equation with appropriate initial conditions show clearly that

$$\mathcal{L}\left(t\,\mathrm{e}^{-2t}\right) = \frac{1}{(s-2)^2},\ t \ge 0.$$

You may not use integration in this question.

$\boxed{\text{proof}}$



• TO FIND THE LAPLACE TRANSFORM OF te^{-2t} VIA A DIFFERENTIAL EQUATION WE NEED REPEATED ROOT IN THE AUXILIARY EQUATION $(\lambda = 2)$

IE $(\lambda+2)^2 = 0$ $\quad \therefore \frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 0$

$x = Ae^{-2t} + Bte^{-2t}$

• NOW $A = 0,\ B = 1$ $\quad x = te^{-2t}$

$\dot{x} = e^{-2t} - 2te^{-2t}$

$t = 0,\ x = 0,\ \dot{x} = 1$

• $\ddot{x} + 4\dot{x} + 4x = 0$

$s^2\bar{x} - \cancel{sx_0} - \dot{x}_0 + 4(s\bar{x} - \cancel{x_0}) + 4\bar{x} = 0$

$s^2\bar{x} - 1 + 4s\bar{x} + 4\bar{x} = 0$

$(s^2 + 4s + 4)\bar{x} = 1$

$\bar{x} = \frac{1}{s^2 + 4s + 4}$

$\mathcal{L}[te^{-2t}] = \frac{1}{s^2+4s+4} = \frac{1}{(s+2)^2}$

Question 4

Use the differential equation

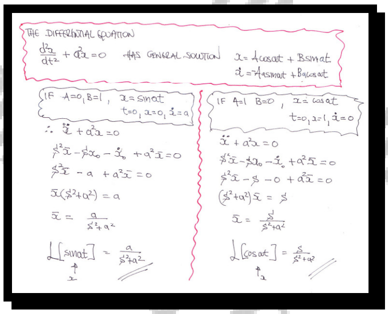
$$\frac{d^2y}{dt^2} + a^2x = 0\,,\ t \geq 0\,,$$

with appropriate initial conditions to show that

$$\mathcal{L}\left(\cos at\right) = \frac{s}{s^2+a^2} \quad \text{and} \quad \mathcal{L}\left(\sin at\right) = \frac{a}{s^2+a^2}\,.$$

You may not use integration in this question.

$\boxed{\text{proof}}$



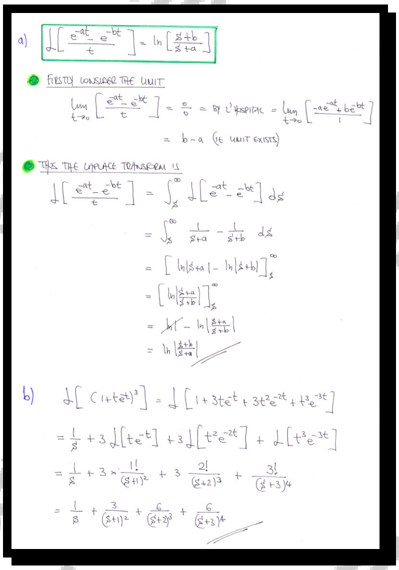
Question 5

Find each of the following Laplace transforms.

a) $\mathcal{L}\left[\dfrac{\mathrm{e}^{-at}-\mathrm{e}^{-bt}}{t}\right]$, $a>0$, $b>0$

b) $\mathcal{L}\left[\left(1+t\,\mathrm{e}^{-t}\right)^{3}\right]$

$$\boxed{\mathcal{L}\left[\frac{\mathrm{e}^{-at}-\mathrm{e}^{-bt}}{t}\right]=\ln\left[\frac{s+b}{s+a}\right]},\quad \boxed{\mathcal{L}\left[\left(1+t\,\mathrm{e}^{-t}\right)^{3}\right]=\frac{1}{s}+\frac{3}{(s+1)^{2}}+\frac{6}{(s+2)^{3}}+\frac{6}{(s+3)^{4}}}$$



a) $\boxed{\mathcal{L}\left[\dfrac{e^{-at}-e^{-bt}}{t}\right]=\ln\left[\dfrac{s+b}{s+a}\right]}$

FIRSTLY CONSIDER THE LIMIT

$$\lim_{t\to 0}\left[\frac{e^{-at}-e^{-bt}}{t}\right]=\frac{0}{0}=\text{BY L'HOSPITAL}=\lim_{t\to 0}\left[\frac{-ae^{-at}+be^{-bt}}{1}\right]$$

$= b-a$ (IE LIMIT EXISTS)

THUS THE LAPLACE TRANSFORM IS

$$\mathcal{L}\left[\frac{e^{-at}-e^{-bt}}{t}\right]=\int_s^{\infty}\mathcal{L}\left[e^{-at}-e^{-bt}\right]ds$$

$$=\int_s^{\infty}\frac{1}{s+a}-\frac{1}{s+b}\,ds$$

$$=\Big[\ln|s+a|-\ln|s+b|\Big]_s^{\infty}$$

$$=\left[\ln\left|\frac{s+a}{s+b}\right|\right]_s^{\infty}$$

$$=\cancel{\ln 1}-\ln\left|\frac{s+a}{s+b}\right|$$

$$=\ln\left|\frac{s+b}{s+a}\right|$$

b) $\mathcal{L}\left[(1+te^{-t})^3\right]=\mathcal{L}\left[1+3te^{-t}+3t^2e^{-2t}+t^3e^{-3t}\right]$

$$=\frac{1}{s}+3\mathcal{L}\left[te^{-t}\right]+3\mathcal{L}\left[t^2e^{-2t}\right]+\mathcal{L}\left[t^3e^{-3t}\right]$$

$$=\frac{1}{s}+3\times\frac{1!}{(s+1)^2}+3\frac{2!}{(s+2)^3}+\frac{3!}{(s+3)^4}$$

$$=\frac{1}{s}+\frac{3}{(s+1)^2}+\frac{6}{(s+2)^3}+\frac{6}{(s+3)^4}$$

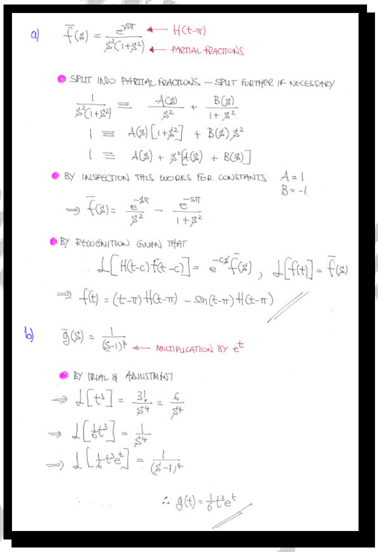
Question 6

Invert each of the following Laplace transforms.

i. $\bar{f}(s) = \dfrac{e^{-s\pi}}{s^2\left(s^2+1\right)}$

ii. $\bar{g}(s) = \dfrac{1}{(s-1)^4}$

$\boxed{f(t) = t\,\mathrm{H}(t-\pi) - \sin t\,\mathrm{H}(t-\pi)}$, $\boxed{g(t) = \frac{1}{6}t^3 e^t}$



a) $\bar{f}(s) = \dfrac{e^{-s\pi}}{s^2(1+s^2)}$ ← $H(t-\pi)$ ← PARTIAL FRACTIONS

• SPLIT INTO PARTIAL FRACTIONS – SPLIT FURTHER IF NECESSARY

$$\frac{1}{s^2(1+s^2)} \equiv \frac{A(s)}{s^2} + \frac{B(s)}{1+s^2}$$

$$1 \equiv A(s)\left[1+s^2\right] + B(s)s^2$$

$$1 \equiv A(s) + s^2\left[A(s) + B(s)\right]$$

• BY INSPECTION THIS WORKS FOR CONSTANTS $A=1$, $B=-1$

$$\Rightarrow \bar{f}(s) = \frac{e^{-s\pi}}{s^2} - \frac{e^{-s\pi}}{1+s^2}$$

• BY RECOGNITION GIVEN THAT

$$\mathcal{L}\left[H(t-c)f(t-c)\right] = e^{-cs}\bar{f}(s), \quad \mathcal{L}\left[f(t)\right] = \bar{f}(s)$$

$$\Rightarrow f(t) = (t-\pi)H(t-\pi) - \sin(t-\pi)H(t-\pi)$$

b) $\bar{g}(s) = \dfrac{1}{(s-1)^4}$ ← MULTIPLICATION BY e^t

• BY TRIAL & ADJUSTMENT

$$\Rightarrow \mathcal{L}\left[t^3\right] = \frac{3!}{s^4} = \frac{6}{s^4}$$

$$\Rightarrow \mathcal{L}\left[\tfrac{1}{6}t^3\right] = \frac{1}{s^4}$$

$$\Rightarrow \mathcal{L}\left[\tfrac{1}{6}t^3e^t\right] = \frac{1}{(s-1)^4}$$

$$\therefore g(t) = \tfrac{1}{6}t^3e^t$$

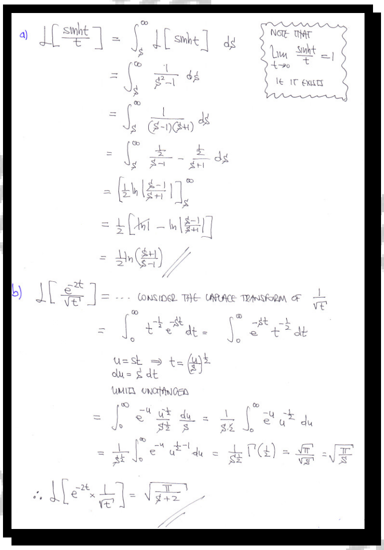
Question 7

Find each of the following Laplace transforms.

c) $\mathcal{L}\left[\dfrac{\sinh t}{t}\right]$

d) $\mathcal{L}\left[\dfrac{e^{-2t}}{\sqrt{t}}\right]$

$$\boxed{\mathcal{L}\left[\frac{\sinh t}{t}\right]=\frac{1}{2}\ln\left[\frac{s+1}{s-1}\right]},\quad \boxed{\mathcal{L}\left[\frac{e^{-2t}}{\sqrt{t}}\right]=\sqrt{\frac{\pi}{s+2}}}$$



a) $\mathcal{L}\left[\frac{\sinh t}{t}\right] = \int_s^\infty \mathcal{L}[\sinh t]\,ds$

NOTE THAT $\lim_{t\to 0}\frac{\sinh t}{t}=1$ IE IT EXISTS

$= \int_s^\infty \frac{1}{s^2-1}\,ds$

$= \int_s^\infty \frac{1}{(s-1)(s+1)}\,ds$

$= \int_s^\infty \frac{\frac{1}{2}}{s-1} - \frac{\frac{1}{2}}{s+1}\,ds$

$= \left[\frac{1}{2}\ln\left|\frac{s-1}{s+1}\right|\right]_s^\infty$

$= \frac{1}{2}\left[\cancel{\ln 1} - \ln\left|\frac{s-1}{s+1}\right|\right]$

$= \frac{1}{2}\ln\left(\frac{s+1}{s-1}\right)$

b) $\mathcal{L}\left[\frac{e^{-2t}}{\sqrt{t}}\right] = \dots$ CONSIDER THE LAPLACE TRANSFORM OF $\frac{1}{\sqrt{t}}$

$= \int_0^\infty t^{-\frac{1}{2}}e^{-st}\,dt = \int_0^\infty e^{-st}t^{-\frac{1}{2}}\,dt$

$u = st \Rightarrow t = \left(\frac{u}{s}\right)^{\frac{1}{2}}$

$du = s\,dt$

LIMITS UNCHANGED

$= \int_0^\infty e^{-u}\frac{u^{-\frac{1}{2}}}{s^{-\frac{1}{2}}}\frac{du}{s} = \frac{1}{s^{\frac{1}{2}}}\int_0^\infty e^{-u}u^{-\frac{1}{2}}\,du$

$= \frac{1}{s^{\frac{1}{2}}}\int_0^\infty e^{-u}u^{\frac{1}{2}-1}\,du = \frac{1}{s^{\frac{1}{2}}}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}$

$\therefore \mathcal{L}\left[e^{-2t}\times\frac{1}{\sqrt{t}}\right] = \sqrt{\frac{\pi}{s+2}}$

Question 8

Find the inverse following Laplace transforms of the following functions.

i. $\dfrac{2s}{s^2+4s+10}$.

ii. $\dfrac{e^{-2s}}{s^2+a^2}$.

iii. $\dfrac{1}{s^2\left(s^2+1\right)}$

$$\boxed{2e^{-2t}\left[\cos\sqrt{6}\,t-\tfrac{1}{3}\sqrt{6}\,c\sin\sqrt{6}\,t\right]},\ \boxed{\frac{1}{a}\mathrm{H}(t-a)-\sin\left[a(t-a)\right]},\ \boxed{t-\sin t}$$

I) $\mathcal{L}^{-1}\left[\dfrac{2s}{s^2+4s+10}\right]=\mathcal{L}^{-1}\left[\dfrac{2s}{(s+2)^2+6}\right]=\mathcal{L}\left[\dfrac{2(s+2)-4}{(s+2)^2+6}\right]$

$=\mathcal{L}^{-1}\left[\dfrac{2(s+2)}{(s+2)^2+6}-\dfrac{4}{(s+2)^2+6}\right]=2e^{-2t}\cos\sqrt{6}t-\dfrac{4}{\sqrt{6}}e^{-2t}\sin\sqrt{6}t$

$=2e^{-2t}\left[\cos\sqrt{6}t-\frac{\sqrt{6}}{3}\sin\sqrt{6}t\right]$

II) $\boxed{\mathcal{L}\left[H(t-a)f(t-a)\right]=e^{-as}\bar{f}(s)}$

$\mathcal{L}[\sin at]=\dfrac{a}{a^2+s^2}$

$\mathcal{L}\left[H(t-a)\sin[a(t-a)]\right]=e^{-as}\dfrac{a}{s^2+a^2}$

$\therefore \mathcal{L}^{-1}\left[\dfrac{e^{-2s}}{s^2+a^2}\right]=\dfrac{1}{a}H(t-a)\sin[a(t-a)]$

III) BY PARTIAL FRACTIONS

$\dfrac{1}{s^2(1+s^2)}\equiv\dfrac{A}{s}+\dfrac{B}{s^2}+\dfrac{Cs+D}{1+s^2}$

$1\equiv As(1+s)+B(1+s^2)+s^2(Cs+D)$

$1\equiv As+As^2+B+Bs^2+Cs^3+Ds^2$

$1\equiv Cs^3+(A+B+D)s^2+As+B$

$\therefore B=1 \quad A=0 \quad C=0 \qquad A+B+D=0$

$0+1+D=0$

$D=-1$

HENCE

$\mathcal{L}^{-1}\left[\dfrac{1}{s^2(1+s^2)}\right]=\mathcal{L}^{-1}\left[\dfrac{1}{s^2}-\dfrac{1}{1+s^2}\right]=t-\sin t$

Question 9

Find the following Laplace transform

$$\mathcal{L}\left[\frac{\sin^2 t}{t}\right].$$

$$\boxed{\mathcal{L}\left[\frac{\sin^2 t}{t}\right] = \frac{1}{4}\ln\left[\frac{s^2+4}{s^2}\right]}$$

$$\boxed{\mathcal{L}\left[\frac{\sin^2 t}{t}\right] = \mathcal{L}\left[\frac{\frac{1}{2}-\frac{1}{2}\cos 2t}{t}\right]}$$

- NEED TO CHECK THE LIMIT AS $t \to 0$

$$\lim_{t \to 0}\left[\frac{\frac{1}{2}-\frac{1}{2}\cos 2t}{t}\right] = \frac{0}{0} \text{ BY L'HOSPITAL } = \lim_{t \to 0}\left[\frac{\pm \sin 2t}{1}\right]$$

$$= 0 \quad \text{(I.E THE LIMIT EXISTS)}$$

- HENCE WE OBTAIN

$$\mathcal{L}\left[\frac{\sin^2 t}{t}\right] = \int_s^\infty \mathcal{L}\left[\frac{1}{2}-\frac{1}{2}\cos 2t\right]\,ds$$

$$= \frac{1}{2}\int_s^\infty \frac{1}{s} - \frac{s}{s^2+4}\,ds$$

$$= \frac{1}{2}\left[\ln s - \frac{1}{2}\ln(s^2+4)\right]_s^\infty$$

$$= \frac{1}{4}\left[\ln s^2 - \ln(s^2+4)\right]_s^\infty$$

$$= \frac{1}{4}\left[\ln\frac{s^2}{s^2+4}\right]_s^\infty$$

$$= \frac{1}{4}\left[\cancel{\ln 1} - \ln\left(\frac{s^2}{s^2+4}\right)\right]$$

$$= -\frac{1}{4}\ln\left(\frac{s^2}{s^2+4}\right)$$

$$= \frac{1}{4}\ln\frac{s^2+4}{s^2}$$

Question 10

It is given that

$$\mathcal{L}\left[f(t)\right] = \frac{1}{s}\exp\left(-\frac{1}{s}\right), \ t \geq 0.$$

Determine a simplified expression for

$$\mathcal{L}\left[e^{-t} f(3t)\right].$$

$$\boxed{\mathcal{L}\left[e^{-t} f(3t)\right] = \frac{1}{s+1}\exp\left(-\frac{3}{s+1}\right)}$$

$$\boxed{\mathcal{L}\left[f(t)\right] = \frac{1}{s}e^{-s^{-1}}}$$

USING $\left.\begin{array}{l}\mathcal{L}\left[e^{-kt}f(t)\right] = \bar{f}(s+k) \\ \mathcal{L}\left[f(at)\right] = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)\end{array}\right\}$ COMBINING $\frac{1}{a}\bar{f}\left(\frac{s+k}{a}\right)$

$$\therefore \mathcal{L}\left[e^{-t}f(3t)\right] = \frac{1}{3}\times\frac{1}{\frac{s+1}{3}}\times e^{-\left(\frac{s+1}{3}\right)^{-1}}$$

$$= \frac{1}{3}\times\frac{3}{s+1}\times e^{-\frac{3}{s+1}}$$

$$= \frac{\exp\left(-\frac{3}{s+1}\right)}{s+1}$$

Question 11

Find a simplified expression for

$$\mathcal{L}\left[\cosh^2 4t\right].$$

$$\boxed{\mathcal{L}\left[\cosh^2 4t\right] = \frac{s^2 - 32}{s\left(s^2 - 64\right)}}$$

$$\mathcal{L}\left[\cosh^2 4t\right] = \mathcal{L}\left[\tfrac{1}{2} + \tfrac{1}{2}\cosh 8t\right]$$

$$= \tfrac{1}{2}\mathcal{L}\left[1 + \cosh 8t\right]$$

$$= \tfrac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2 - 64}\right]$$

$$= \tfrac{1}{2}\left[\frac{s^2 - 64 + s^2}{s(s^2 - 64)}\right]$$

$$= \tfrac{1}{2}\left[\frac{2s^2 - 64}{s(s^2 - 64)}\right]$$

$$= \frac{s^2 - 32}{s(s^2 - 64)}$$

$\cos^2 A \equiv \tfrac{1}{2} + \tfrac{1}{2}\cos 2A$

$\cosh^2 A \equiv \tfrac{1}{2} + \tfrac{1}{2}\cosh 2A$

Question 12

The function $y = y(t)$ satisfies the differential equation

$$\frac{dy}{dt} + y = 1, \quad t \geq 0, \quad y(0) = 0.$$

Use the initial-final value theorem to find $\lim\limits_{t \to \infty}\left[y(t)\right]$.

$$\boxed{\lim_{t \to \infty}\left[y(t)\right] = 1}$$

$$\boxed{\frac{dy}{dt} + y = 1 \qquad y(0) = 0}$$

- TAKING THE LAPLACE TRANSFORM IN t

$$\mathcal{L}\left[\frac{dy}{dt}\right] + \mathcal{L}\left[y\right] = \mathcal{L}\left[1\right]$$

$$s\bar{y} - \cancel{y(0)} + \bar{y} = \frac{1}{s}$$

$$(s+1)\bar{y} = \frac{1}{s}$$

$$s\bar{y} = \frac{1}{s+1}$$

- BY THE INITIAL FINAL VALUE THEOREM

$$\boxed{\lim_{s \to 0}\left[s\bar{f}(s)\right] = \lim_{t \to \infty}\left[f(t)\right]}$$

- HENCE WE GET

$$\lim_{t \to \infty}\left[y(t)\right] = \lim_{s \to 0}\left[s\bar{y}(s)\right]$$

$$= \lim_{s \to 0}\left[\frac{1}{s+1}\right]$$

$$= 1$$

Question 13

The function $y = f(t)$ satisfies

$$\mathcal{L}\left[f(t)\right] = \frac{1}{\sqrt{s+2}} .$$

Determine a simplified expression for $f(t)$.

$$\boxed{f(t) = \frac{e^{-2t}}{\sqrt{\pi t}}}$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{1}{\sqrt{s+2}}\right]}$$

- THIS IS A LAPLACE OF A FUNCTION PRODUCING $\frac{1}{\sqrt{s}}$ WITH THE SHIFT FACTOR e^{-2t}
- WE SUSPECT IT MAY BE $\mathcal{L}\left[t^{n}\right] = \frac{n!}{t^{n+1}}$ (SO $n = -\frac{1}{2}$)
- TRY $\mathcal{L}\left[t^{-\frac{1}{2}}\right] = \int_0^\infty t^{-\frac{1}{2}} e^{-st}\,dt$

LET $u = st$
$t = \frac{u}{s}$
$dt = \frac{1}{s}du$
LIMITS UNCHANGED

$$= \int_0^\infty \left(\frac{u}{s}\right)^{-\frac{1}{2}} e^{-u} \frac{1}{s}\,du$$

$$= \int_0^\infty \frac{u^{-\frac{1}{2}} e^{-u}}{s^{\frac{1}{2}}}\,du$$

$$= \frac{1}{s^{\frac{1}{2}}} \int_0^\infty u^{\frac{1}{2}-1} e^{-u}\,du$$

$$= \frac{1}{s^{\frac{1}{2}}} \Gamma\left(\tfrac{1}{2}\right)$$

$$= \frac{\sqrt{\pi}}{s^{\frac{1}{2}}}$$

- ADJUSTING: $\mathcal{L}\left[\frac{1}{\sqrt{\pi}} t^{-\frac{1}{2}}\right] = \frac{1}{s^{\frac{1}{2}}} = \frac{1}{\sqrt{s}}$

$$\mathcal{L}\left[\frac{1}{\sqrt{\pi t}}\right] = \frac{1}{\sqrt{s}}$$

$$\mathcal{L}\left[\frac{e^{-2t}}{\sqrt{\pi t}}\right] = \frac{1}{\sqrt{s-2}}$$

Question 14

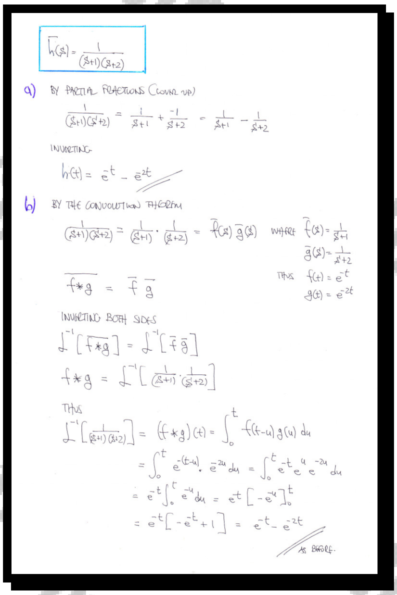
$$\bar{h}(s) = \frac{1}{(s+1)(s+2)}.$$

Invert the above Laplace transform by …

a) … partial fractions

b) … the convolution theorem

$$\boxed{h(t) = \mathrm{e}^{-t} - \mathrm{e}^{-2t}}$$



$\boxed{\bar{h}(s) = \frac{1}{(s+1)(s+2)}}$

a) BY PARTIAL FRACTIONS (COVER UP)

$$\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} + \frac{-1}{s+2} = \frac{1}{s+1} - \frac{1}{s+2}$$

INVERTING

$$h(t) = e^{-t} - e^{-2t}$$

b) BY THE CONVOLUTION THEOREM

$$\frac{1}{(s+1)(s+2)} = \frac{1}{(s+1)} \cdot \frac{1}{(s+2)} = \bar{f}(s)\,\bar{g}(s) \quad \text{WHERE} \quad \bar{f}(s) = \frac{1}{s+1}, \quad \bar{g}(s) = \frac{1}{s+2}$$

THUS $f(t) = e^{-t}$, $g(t) = e^{-2t}$

$$\overline{f*g} = \bar{f}\,\bar{g}$$

INVERTING BOTH SIDES

$$\mathcal{L}^{-1}\left[\overline{f*g}\right] = \mathcal{L}^{-1}\left[\bar{f}\,\bar{g}\right]$$

$$f*g = \mathcal{L}^{-1}\left[\frac{1}{(s+1)} \cdot \frac{1}{(s+2)}\right]$$

THUS

$$\mathcal{L}^{-1}\left[\frac{1}{(s+1)(s+2)}\right] = (f*g)(t) = \int_0^t f(t-u)\,g(u)\,du$$

$$= \int_0^t e^{-(t-u)} \cdot e^{-2u}\,du = \int_0^t e^{-t}e^{u}e^{-2u}\,du$$

$$= e^{-t}\int_0^t e^{-u}\,du = e^{-t}\left[-e^{-u}\right]_0^t$$

$$= e^{-t}\left[-e^{-t}+1\right] = e^{-t} - e^{-2t}$$

AS BEFORE.

Question 15

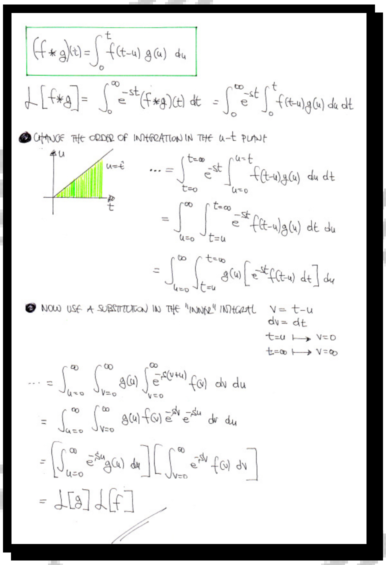
The convolution $[f*g](t)$, of two functions $f(t)$ and $g(t)$ is defined as

$$[f*g](t)=\int_0^t f(t-u)\,g(u)\;du\,.$$

Show that

$$\mathcal{L}\{[f*g](t)\} \;=\; \mathcal{L}[f(t)]\mathcal{L}[g(t)] \;=\; \bar{f}(s)\,\bar{g}(s)\,.$$

$\boxed{\text{proof}}$



$$\boxed{(f*g)(t)=\int_0^t f(t-u)\,g(u)\;du}$$

$$\mathcal{L}[f*g]=\int_0^\infty e^{-st}(f*g)(t)\,dt=\int_0^\infty e^{-st}\int_0^t f(t-u)g(u)\,du\,dt$$

- CHANGE THE ORDER OF INTEGRATION IN THE u–t PLANE

$$\ldots=\int_{t=0}^{t=\infty} e^{-st}\int_{u=0}^{u=t} f(t-u)g(u)\;du\,dt$$

$$=\int_{u=0}^{\infty}\int_{t=u}^{t=\infty} e^{-st} f(t-u)g(u)\;dt\,du$$

$$=\int_{u=0}^{\infty}\int_{t=u}^{t=\infty} g(u)\left[e^{-st}f(t-u)\;dt\right]du$$

- NOW USE A SUBSTITUTION IN THE "INNER" INTEGRAL

$v=t-u$
$dv=dt$
$t=u \mapsto v=0$
$t=\infty \mapsto v=\infty$

$$\ldots=\int_{u=0}^{\infty}\int_{v=0}^{\infty} g(u)\int_{v=0}^{\infty} e^{-s(v+u)}f(v)\;dv\,du$$

$$=\int_{u=0}^{\infty}\int_{v=0}^{\infty} g(u)f(v)e^{-sv}e^{-su}\;dv\,du$$

$$=\left[\int_{u=0}^{\infty} e^{-su}g(u)\,du\right]\left[\int_{v=0}^{\infty} e^{-sv}f(v)\,dv\right]$$

$$=\mathcal{L}[g]\,\mathcal{L}[f]$$

Question 16

Use the differential equation

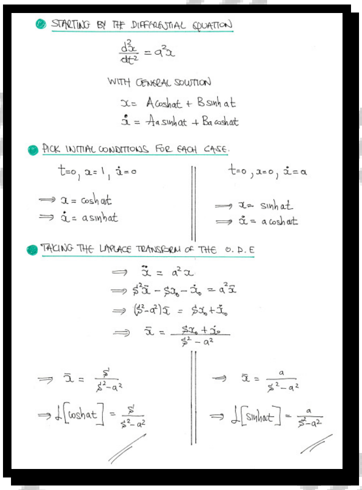
$$\frac{d^2x}{dt^2} = a^2 x, \ t \geq 0,$$

with appropriate initial conditions to show that

$$\mathcal{L}(\cosh at) = \frac{s}{s^2 - a^2} \quad \text{and} \quad \mathcal{L}(\sinh at) = \frac{a}{s^2 - a^2}.$$

You may not use integration in this question.

proof


STARTING BY THE DIFFERENTIAL EQUATION

$$\frac{d^2x}{dt^2} = a^2 x$$

WITH GENERAL SOLUTION

$$x = A\cosh at + B\sinh at$$
$$\dot{x} = Aa\sinh at + Ba\cosh at$$

PICK INITIAL CONDITIONS FOR EACH CASE.

$t=0,\ x=1,\ \dot{x}=0$: $\Rightarrow x = \cosh at$, $\Rightarrow \dot{x} = a\sinh at$

$t=0,\ x=0,\ \dot{x}=a$: $\Rightarrow x = \sinh at$, $\Rightarrow \dot{x} = a\cosh at$

TAKING THE LAPLACE TRANSFORM OF THE O.D.E

$$\Rightarrow \ddot{x} = a^2 x$$
$$\Rightarrow s^2\bar{x} - sx_0 - \dot{x}_0 = a^2\bar{x}$$
$$\Rightarrow (s^2 - a^2)\bar{x} = sx_0 + \dot{x}_0$$
$$\Rightarrow \bar{x} = \frac{sx_0 + \dot{x}_0}{s^2 - a^2}$$

$\Rightarrow \bar{x} = \dfrac{s}{s^2-a^2}$, $\Rightarrow \mathcal{L}[\cosh at] = \dfrac{s}{s^2-a^2}$

$\Rightarrow \bar{x} = \dfrac{a}{s^2-a^2}$, $\Rightarrow \mathcal{L}[\sinh at] = \dfrac{a}{s^2-a^2}$

Question 17

The function $y = f(t)$, $t \geq 0$, is twice differentiable.

a) Show from first principles that

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] = s^2\mathcal{L}\left[y(t)\right] - s\,y(0) - \frac{dy}{dt}(0)$$

A second function $g(t)$ is defined for $t \geq 0$.

b) Show further that

$$\mathcal{L}\left[\int_0^t f(t-u)\,g(u)\;du\right] = \mathcal{L}\left[f(t)\right]\mathcal{L}\left[g(t)\right].$$

$\boxed{}$, $\boxed{\text{proof}}$

a) STARTING BY THE DEFINITION OF A LAPLACE TRANSFORM

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] = \int_0^\infty \frac{d^2y}{dt^2}e^{-st}\,dt.$$

PROCEED BY INTEGRATION BY PARTS

e^{-st}	$-se^{-st}$
$\frac{dy}{dt}$	$\frac{d^2y}{dt^2}$

$$\ldots = \left[\frac{dy}{dt}e^{-st}\right]_0^\infty - \int_0^\infty -s\frac{dy}{dt}e^{-st}\,dt$$

$$= 0 - \frac{dy}{dt}\Big|_{t=0} + s\int_0^\infty \frac{dy}{dt}e^{-st}\,dt$$

INTEGRATION BY PARTS AGAIN

e^{-st}	$-se^{-st}$
y	$\frac{dy}{dt}$

$$= -\frac{dy}{dt}\Big|_{t=0} + s\left[\left[ye^{-st}\right]_0^\infty - \int -sye^{-st}\,dt\right]$$

$$= -\frac{dy}{dt}\Big|_{t=0} + s\left[0 - y\Big|_{t=0} + s\int_0^\infty ye^{-st}\,dt\right]$$

$$= -\frac{dy}{dt}\Big|_{t=0} - sy\Big|_{t=0} + s^2\int_0^\infty ye^{-st}$$

$$= -\frac{dy}{dt}\Big|_{t=0} - sy\Big|_{t=0} + s^2\mathcal{L}[y]$$

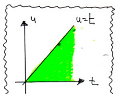
$$= s^2\mathcal{L}[y(t)] - sy(0) - \frac{dy}{dt}(0)$$

// AS REQUIRED

b) AGAIN STARTING BY THE DEFINITION

$$\mathcal{L}\left[\int_0^t f(t-u)\,g(u)\,du\right] = \int_{t=0}^\infty e^{-st}\left[\int_{u=0}^t f(t-u)g(u)\,du\right]dt$$

REVERSING THE ORDER OF INTEGRATION



FROM $u=0$ TO $u=t$
& FROM $t=0$ TO $t=\infty$

OR

FROM $t=u$ TO $t=\infty$
FROM $u=0$ TO $u=\infty$

$$\ldots = \int_{u=0}^{\infty}\int_{t=u}^{t=\infty} e^{-st}f(t-u)\,g(u)\;dt\,du$$

$$= \int_{u=0}^\infty g(u)\left[\int_{t=u}^{t=\infty} e^{-st}f(t-u)\;dt\right]du$$

USING A SUBSTITUTION IN THE "INNER" INTEGRAL

$v = t-u \iff t = v+u$

$dv = dt$ (u IS CONSTANT IN THIS INTEGRAL)

$t=u \longmapsto v=0$

$t=\infty \longmapsto v=\infty$

$$\ldots = \int_{u=0}^\infty g(u)\left[\int_{v=0}^\infty e^{-s(u+v)}f(v)\;dv\right]du$$

$$= \int_{u=0}^\infty\int_{v=0}^\infty e^{-su}g(u)\;e^{-sv}f(v)\;dv\,du$$

$$= \left[\int_{u=0}^\infty e^{-su}g(u)\;du\right]\left[\int_{v=0}^\infty e^{-sv}f(v)\,dv\right]$$

$$= \left[\int_{t=0}^\infty e^{-st}g(t)\;dt\right]\left[\int_{t=0}^\infty e^{-st}f(t)\,dt\right] = \mathcal{L}[g(t)]\,\mathcal{L}[f(t)]$$

//

Question 18

$$\mathcal{L}\left[f(t)\right] \equiv \bar{f}(s),\ t \geq 0 .$$

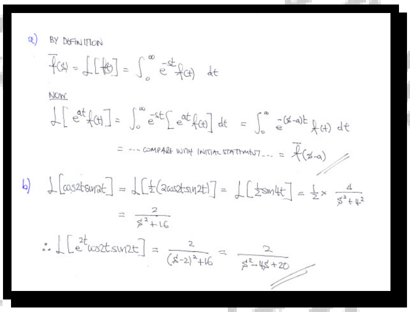
a) Show clearly that

$$\mathcal{L}\left[e^{at} f(t)\right] \equiv \bar{f}(s-a) .$$

b) Find in its simplest form

$$\mathcal{L}\left[e^{2t} \cos 2t \sin 2t\right] .$$

$$\boxed{\frac{2}{s^2 - 4s + 20}}$$



a) BY DEFINITION

$\bar{f}(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\ dt$

NOW

$\mathcal{L}[e^{at} f(t)] = \int_0^\infty e^{-st}[e^{at} f(t)]\ dt = \int_0^\infty e^{-(s-a)t} f(t)\ dt$

$= \ldots$ COMPARE WITH INITIAL STATEMENT $\ldots = \bar{f}(s-a)$

b) $\mathcal{L}[\cos 2t \sin 2t] = \mathcal{L}[\frac{1}{2}(2\cos 2t \sin 2t)] = \mathcal{L}[\frac{1}{2}\sin 4t] = \frac{1}{2} \times \frac{4}{s^2+4^2}$

$= \frac{2}{s^2+16}$

$\therefore \mathcal{L}[e^{2t}\cos 2t \sin 2t] = \frac{2}{(s-2)^2+16} = \frac{2}{s^2-4s+20}$

Question 19

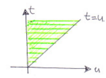
Use the definition of a Laplace transform to show that

$$\mathcal{L}\left[\int_0^t f(u)\ du\right] = \frac{1}{s}\mathcal{L}\left[f(u)\right],\ t \geq 0.$$

proof

$$\mathcal{L}\left[\int_{u=0}^{t} f(u)\,du\right] = \int_0^{\infty}\left[\int_{u=0}^{u=t} f(u)\,du\right]e^{-st}\,dt.$$

$$= \int_{t=0}^{\infty}\int_{u=0}^{u=t} e^{-st}f(u)\ du\,dt = \dots \text{ CHANGE THE INTEGRATION ORDER}$$



$$= \int_{u=0}^{\infty}\int_{t=u}^{t=\infty} e^{-st}f(u)\,dt\,du$$

$$= \int_{u=0}^{\infty}\left[-\frac{1}{s}e^{-st}f(u)\right]_{t=u}^{\infty} du$$

$$= \int_{u=0}^{\infty} 0 - \left[-\frac{1}{s}e^{-su}f(u)\right] du$$

$$= \int_{u=0}^{\infty}\frac{1}{s}e^{-su}f(u)\ du$$

$$= \frac{1}{s}\int_0^{u} e^{-su}f(u)\ du$$

$$= \frac{1}{s}\mathcal{L}\left[f(u)\right]$$

Question 20

Determine a simplified expression for

$$\mathcal{L}\left[t\,e^{2t}\cos 3t\right].$$

$$\boxed{\mathcal{L}\left[t\,e^{2t}\cos 3t\right] = \frac{s^2-4s-5}{\left(s^2-4s+13\right)^2}}$$

- STARTING WITH THE LAPLACE TRANSFORM OF $\cos 3t$

$$\mathcal{L}[\cos 3t] = \frac{s}{s^2+9}$$

- NEXT FIND THE LAPLACE TRANSFORM OF $e^{2t}\cos 3t$ USING THE RESULT

$$\mathcal{L}\left[e^{at}f(t)\right] = \bar{f}(s-a)$$

WITH $\bar{f}(s) = \mathcal{L}(f(t)]$

$$\Rightarrow \mathcal{L}\left[e^{2t}\cos 3t\right] = \frac{s-2}{(s-2)^2+9} = \frac{s-2}{s^2-4s+13}$$

- FINALLY FIND THE LAPLACE TRANSFORM OF $te^{2t}\cos 3t$ USING THE RESULT

$$\mathcal{L}\left[t\,f(t)\right] = -\frac{d}{ds}\bar{f}(s) \quad \text{WHERE} \quad \bar{f}(s) = \mathcal{L}\left[f(t)\right]$$

$$\Rightarrow \mathcal{L}\left[t\left(e^{2t}\cos 3t\right)\right] = -\frac{d}{ds}\left[\frac{s-2}{s^2-4s+13}\right]$$

$$= -\frac{(s^2-4s+13)\times 1 - (s-2)(2s-4)}{(s^2-4s+13)^2}$$

$$= -\frac{s^2-4s+13-(2s^2-8s+8)}{(s^2-4s+13)^2}$$

$$= -\frac{-s^2+4s+5}{(s^2-4s+13)^2}$$

$$= \frac{s^2-4s-5}{(s^2-4s+13)^2}$$

Question 21

Find the following inverse Laplace transform

$$\mathcal{L}^{-1}\left[\ln\left(1+\frac{1}{s^2}\right)\right].$$

$$\boxed{\mathcal{L}^{-1}\left[\ln\left(1+\frac{1}{s^2}\right)\right]=\frac{2}{t}(1-\cos t)}$$

LOOKING AT $\mathcal{L}^{-1}\left[\ln\left(1+\frac{1}{s^2}\right)\right]$ WE SUSPECT THAT THIS IS THE LAPLACE TRANSFORM OF $\frac{f(t)}{t}$ FOR SOME $f(t)$

$$\mathcal{L}\left[\frac{f(t)}{t}\right]=\int_s^{\infty}\bar{f}(x)\,dx=\bar{g}(s)$$

HERE $\bar{g}(s)$ IS $\ln\left(1+\frac{1}{s^2}\right)$

$$\frac{d}{ds}\left(\bar{g}(s)\right)=\frac{d}{ds}\left(\ln\left(1+\frac{1}{s^2}\right)\right)=\frac{d}{ds}\left[\ln\left(\frac{s^2+1}{s^2}\right)\right]$$

$$=\frac{d}{ds}\left[\ln(s^2+1)-\ln s^2\right]=\frac{d}{ds}\left[\ln(s^2+1)-2\ln s\right]$$

$$=\frac{2s}{s^2+1}-\frac{2}{s}=2\left(\frac{s}{s^2+1}\right)-2\left(\frac{1}{s}\right)$$

WE RECOGNIZE THIS AS STANDARD RESULTS

$$2\mathcal{L}[\cos t]=2\left(\frac{s}{s^2+1}\right)$$
$$2\mathcal{L}[1]=2\times\frac{1}{s}$$

HENCE WE HAVE NOTING THE REVERSAL OF THE SIGNS DUE TO THE INTEGRATION LIMITS

$$\mathcal{L}^{-1}\left[\ln\left(1+\frac{1}{s^2}\right)\right]=\frac{2(1-\cos t)}{t}$$

QUICK CHECK

$$\mathcal{L}\left[\frac{2(1-\cos t)}{t}\right]=\int_s^{\infty}\mathcal{L}\left[2(1-\cos t)\right]ds=\int_s^{\infty}\mathcal{L}\left[2-2\cos t\right]ds=\int_s^{\infty}\frac{2}{s}-\frac{2s}{s^2+1}\,ds$$

$$=\left[2\ln s-\ln(s^2+1)\right]_s^{\infty}=\left[\ln s^2-\ln(s^2+1)\right]_s^{\infty}=\left[\ln\left(\frac{s^2}{s^2+1}\right)\right]_s^{\infty}$$

$$=\cancel{\ln 1}-\ln\left(\frac{s^2}{s^2+1}\right)=-\ln\left(\frac{s^2}{s^2+1}\right)=\ln\left(\frac{s^2+1}{s^2}\right)=\ln\left(1+\frac{1}{s^2}\right)$$

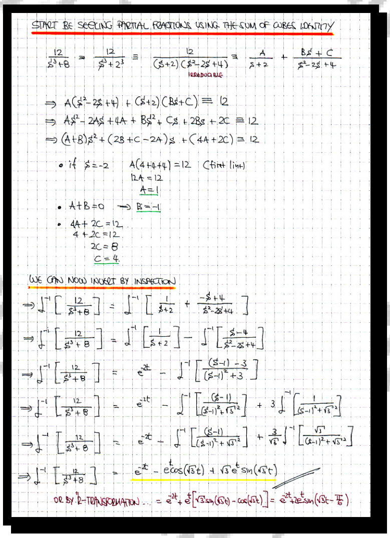
Question 22

Find the following inverse Laplace transform

$$\mathcal{L}^{-1}\left[\frac{12}{s^3+8}\right].$$

,

$$\boxed{\mathcal{L}^{-1}\left[\frac{12}{s^3+8}\right] = \mathrm{e}^{-2t} + 2\mathrm{e}^{t}\left[\sqrt{3}\sin\left(\sqrt{3}t\right) - \cos\left(\sqrt{3}t\right)\right] = \mathrm{e}^{-2t} + 2\mathrm{e}^{t}\sin\left(\sqrt{3}t - \frac{1}{6}\pi\right)}$$



START BY SEEKING PARTIAL FRACTIONS USING THE SUM OF CUBES IDENTITY

$$\frac{12}{s^3+8} = \frac{12}{s^3+2^3} \equiv \frac{12}{(s+2)\underbrace{(s^2-2s+4)}_{\text{IRREDUCIBLE}}} \equiv \frac{A}{s+2} + \frac{Bs+C}{s^2-2s+4}$$

$\Rightarrow A(s^2-2s+4) + (s+2)(Bs+C) \equiv 12$

$\Rightarrow As^2 - 2As + 4A + Bs^2 + Cs + 2Bs + 2C \equiv 12$

$\Rightarrow (A+B)s^2 + (2B+C-2A)s + (4A+2C) \equiv 12$

- if $s=-2$ $\quad A(4+4+4)=12$ (first line)
  $12A = 12$
  $A=1$
- $A+B=0 \longrightarrow B=-1$
- $4A+2C=12$
  $4+2C=12$
  $2C=8$
  $C=4$

WE CAN NOW INVERT BY INSPECTION

$\Rightarrow \mathcal{L}^{-1}\left[\frac{12}{s^3+8}\right] = \mathcal{L}^{-1}\left[\frac{1}{s+2} + \frac{-s+4}{s^2-2s+4}\right]$

$\Rightarrow \mathcal{L}^{-1}\left[\frac{12}{s^3+8}\right] = \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] - \mathcal{L}^{-1}\left[\frac{s-4}{s^2-2s+4}\right]$

$\Rightarrow \mathcal{L}^{-1}\left[\frac{12}{s^3+8}\right] = e^{-2t} - \mathcal{L}^{-1}\left[\frac{(s-1)-3}{(s-1)^2+3}\right]$

$\Rightarrow \mathcal{L}^{-1}\left[\frac{12}{s^3+8}\right] = e^{-2t} - \mathcal{L}^{-1}\left[\frac{(s-1)}{(s-1)^2+\sqrt{3}^2}\right] + 3\mathcal{L}^{-1}\left[\frac{1}{(s-1)^2+\sqrt{3}^2}\right]$

$\Rightarrow \mathcal{L}^{-1}\left[\frac{12}{s^3+8}\right] = e^{-2t} - \mathcal{L}^{-1}\left[\frac{(s-1)}{(s-1)^2+\sqrt{3}^2}\right] + \frac{3}{\sqrt{3}}\mathcal{L}^{-1}\left[\frac{\sqrt{3}}{(s-1)^2+\sqrt{3}^2}\right]$

$\Rightarrow \mathcal{L}^{-1}\left[\frac{12}{s^3+8}\right] = e^{-2t} - e^{t}\cos(\sqrt{3}t) + \sqrt{3}e^{t}\sin(\sqrt{3}t)$

OR BY "R-TRANSFORMATION" ... $= e^{-2t} + e^{t}\left[\sqrt{3}\sin(\sqrt{3}t) - \cos(\sqrt{3}t)\right] = e^{-2t} + 2e^{t}\sin\left(\sqrt{3}t - \frac{\pi}{6}\right)$

Question 23

Find and verify the following inverse Laplace transform

$$\mathcal{L}^{-1}\left[\frac{s^2}{\left(s^2+4\right)^2}\right].$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{s^2}{\left(s^2+4\right)^2}\right]=\frac{1}{2}t\cos 2t+\frac{1}{4}\sin 2t}$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{s^2}{(s^2+4)^2}\right] = \mathcal{L}^{-1}\left[\frac{s}{s^2+4}\times\frac{s}{s^2+4}\right]}$$

- BY THE CONVOLUTION THEOREM

$$\mathcal{L}[f*g] = \bar{f}(s)\,\bar{g}(s)$$
$$f*g = \mathcal{L}^{-1}\left[\bar{f}(s)\,\bar{g}(s)\right]$$

$f(t)=\cos 2t$, $g(t)=\cos 2t$

- THUS WE OBTAIN

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{s}{s^2+4}\times\frac{s}{s^2+4}\right] = \int_0^t \cos 2u\,\cos[2(t-u)]\,du$$
$$\Rightarrow \mathcal{L}^{-1}\left[\frac{s^2}{(s^2+4)^2}\right] = \int_0^t \cos 2u\,\cos(2t-2u)\,du$$

- WE NEED TO MANIPULATE THE TRIGONOMETRIC INTEGRAND

$$\cos[2u+(2t-2u)] = \cos 2u\cos(2t-2u) - \sin 2u\sin(2t-2u)$$
$$\cos[2u-(2t-2u)] = \cos 2u\cos(2t-2u) + \sin 2u\sin(2t-2u)$$

ADDING

$$\cos 2u\cos(2t-2u) = \tfrac{1}{2}\cos 2t + \tfrac{1}{2}\cos(4u-2t)$$

- RETURNING TO THE INVERSION

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{s^2}{(s^2+4)^2}\right] = \int_0^t \tfrac{1}{2}\cos 2t + \tfrac{1}{2}\cos(4u-2t)\,du$$
$$\Rightarrow \mathcal{L}^{-1}\left[\frac{s^2}{(s^2+4)^2}\right] = \left[\tfrac{1}{2}u\cos 2t + \tfrac{1}{8}\sin(4u-2t)\right]_{u=0}^{u=t}$$
$$\Rightarrow \mathcal{L}^{-1}\left[\frac{s^2}{(s^2+4)^2}\right] = \left[\tfrac{1}{2}t\cos 2t + \tfrac{1}{8}\sin 2t\right] - \left[\tfrac{1}{8}\sin(-2t)\right]$$
$$\Rightarrow \mathcal{L}^{-1}\left[\frac{s^2}{(s^2+4)^2}\right] = \tfrac{1}{2}t\cos 2t + \tfrac{1}{8}\sin 2t + \tfrac{1}{8}\sin 2t$$
$$\Rightarrow \mathcal{L}^{-1}\left[\frac{s^2}{(s^2+4)^2}\right] = \tfrac{1}{2}t\cos 2t + \tfrac{1}{4}\sin 2t$$

- $\mathcal{L}\left[\tfrac{1}{2}t\cos 2t + \tfrac{1}{4}\sin 2t\right]$

$$= \tfrac{1}{2}\left[-\frac{d}{ds}\left[\mathcal{L}[\cos 2t]\right]\right] + \tfrac{1}{4}\times\frac{2}{s^2+4}$$
$$= -\tfrac{1}{2}\frac{d}{ds}\left[\frac{s}{s^2+4}\right] + \tfrac{1}{2}\left[\frac{1}{s^2+4}\right]$$
$$= -\tfrac{1}{2}\left[\frac{(s^2+4)\times 1 - s(2s)}{(s^2+4)^2}\right] + \tfrac{1}{2}\left[\frac{1}{s^2+4}\right]$$
$$= -\tfrac{1}{2}\left[\frac{4-s^2}{(s^2+4)^2}\right] + \tfrac{1}{2}\left[\frac{1}{s^2+4}\right]$$
$$= \tfrac{1}{2}\left[\frac{s^2-4}{(s^2+4)^2}\right] + \tfrac{1}{2}\left[\frac{s^2+4}{(s^2+4)^2}\right]$$
$$= \tfrac{1}{2}\left[\frac{s^2\cancel{-4}+s^2\cancel{+4}}{(s^2+4)^2}\right]$$
$$= \tfrac{1}{2}\times\frac{2s^2}{(s^2+4)^2}$$
$$= \frac{s^2}{(s^2+4)^2}$$

WHICH VERIFIES THE INVERSION

Question 24

Use the definition of a Laplace transform to show that if $x = f(t)$ then

$$\mathcal{L}\left[t^2 \frac{d^2x}{dt^2}\right] = x_0 - \frac{d}{ds}\left[s^2 \mathcal{L}(x)\right], \quad \text{where} \quad x_0 = f(0).$$

proof

FIRSTLY THE LAPLACE TRANSFORM OF $t f(t)$

$$\mathcal{L}[t f(t)] = \int_0^\infty t f(t) e^{-st}\,dt = \int_0^\infty f(t) \times -\frac{d}{ds}\left(e^{-st}\right) dt$$

$$= -\frac{d}{ds}\int_0^\infty f(t) e^{-st}\,dt = -\frac{d}{ds}\left[\bar{f}(s)\right]$$

$$= -\frac{d}{ds}\left[\mathcal{L}[f(t)]\right]$$

NEXT THE LAPLACE TRANSFORM OF A SECOND DIFFERENTIAL

$$\mathcal{L}\left[\frac{d^2x}{dt^2}\right] = \int_0^\infty \frac{d^2x}{dt^2} e^{-st}\,dt \quad \dots \text{ BY PARTS}$$

e^{-st}	$-se^{-st}$
$\frac{dx}{dt}$	$\frac{d^2x}{dt^2}$

$$= \left[\frac{dx}{dt}e^{-st}\right]_0^\infty + s\int_0^\infty \frac{dx}{dt} e^{-st}\,dt$$

$$= -\left.\frac{dx}{dt}\right|_{t=0} + s\int_0^\infty \frac{dx}{dt} e^{-st}\,dt \quad \dots \text{ BY PARTS AGAIN}$$

e^{-st}	$-se^{-st}$
x	$\frac{dx}{dt}$

$$= -\left.\frac{dx}{dt}\right|_{t=0} + s\left\{\left[x e^{-st}\right]_0^\infty + s\int_0^\infty x e^{-st}\,dt\right\}$$

$$= -\left.\frac{dx}{dt}\right|_{t=0} + s\left\{-x\Big|_{t=0} + s\,\mathcal{L}[x(t)]\right\}$$

$$= s^2\mathcal{L}[x(t)] - s x\Big|_0 - \left.\frac{dx}{dt}\right|_0$$

COMBINING THE TWO RESULTS WE OBTAIN

$$\mathcal{L}\left[t \underbrace{\frac{d^2x}{dt^2}}_{f(t)}\right] = -\frac{d}{ds}\left[\mathcal{L}\left[\underbrace{\frac{d^2x}{dt^2}}_{f(t)}\right]\right] = -\frac{d}{ds}\left[s^2\bar{x} - s x_0 - \dot{x}_0\right]$$

$$= -\frac{d}{ds}\left[s^2\bar{x}\right] + \frac{d}{ds}\left[s x_0\right] + \cancel{\frac{d}{ds}\left[\dot{x}_0\right]}$$

$$= x_0 - \frac{d}{ds}\left[s^2\bar{x}\right]$$

Question 25

$$\mathcal{L}\left[f(t)\right] = \bar{f}(s) \equiv \int_0^{\infty} f(t)\mathrm{e}^{-st}\ dt\ ,\ t \geq 0 .$$

a) Show from the above definition that if a is a non zero constant, then

$$\mathcal{L}\left[f(at)\right] = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right).$$

b) Deduce that if k is a non zero constant, then

$$\mathcal{L}^{-1}\left[\bar{f}(ks)\right] = \frac{1}{k}f\left(\frac{t}{k}\right).$$

proof

a) STARTING FROM THE DEFINITION OF THE LAPLACE TRANSFORM

$$\boxed{\mathcal{L}[f(at)] \equiv \int_0^{\infty} f(at)e^{-st}\,dt,\quad t \geq 0}$$

BY SUBSTITUTION NEXT

$u = at$
$t = \frac{u}{a}$
$dt = \frac{1}{a}du$
LIMITS UNCHANGED

$$\Rightarrow \mathcal{L}[f(at)] = \int_0^{\infty} f(u)\,e^{-s\left(\frac{u}{a}\right)}\frac{1}{a}\,du$$

$$\Rightarrow \mathcal{L}[f(at)] = \int_0^{\infty} \frac{1}{a}f(u)\,e^{-\frac{s}{a}u}\,du$$

$$\Rightarrow \mathcal{L}[f(at)] = \frac{1}{a}\int_0^{\infty} f(u)\,e^{-\frac{s}{a}u}\,du$$

$$\Rightarrow \mathcal{L}[f(at)] = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$$

SINCE $\bar{f}(s) \equiv \int_0^{\infty} f(u)e^{-su}\,du$

b) NOW TAKE PART (a) & LET $k = \frac{1}{a}$

$$\Rightarrow \mathcal{L}\left[f\left(\frac{t}{k}\right)\right] = k\bar{f}(ks)$$

$$\Rightarrow \bar{f}(ks) = \frac{1}{k}\mathcal{L}\left[f\left(\frac{t}{k}\right)\right]$$

$$\Rightarrow \mathcal{L}^{-1}\left[\bar{f}(ks)\right] = \frac{1}{k}f\left(\frac{t}{k}\right)$$

Question 26

$$\mathcal{L}\left[f(t)\right] \equiv \bar{f}(s), \; t \geq 0 .$$

a) Show clearly that

$$\mathcal{L}\left[k^t f(t)\right] \equiv \bar{f}(s - \ln k), \; k > 0 .$$

b) Find in its simplest form

$$\mathcal{L}\left[t^3 \mathrm{e}^{-t} 2^t\right].$$

$$\boxed{\mathcal{L}\left[t^3 \mathrm{e}^{-t} 2^t\right] = \frac{6}{(s+1-\ln 2)^4}}$$

a) $$\mathcal{L}[k^t f(t)] = \int_0^\infty e^{-st}\left(k^t f(t)\right)\,dt = \int_0^\infty e^{-st} e^{t\ln k} f(t)\,dt = \int_0^\infty e^{-t(s-\ln k)} f(t)\,dt = \bar{f}(s-\ln k) \;//$$

b) $$\mathcal{L}[t^3 e^{-t} 2^t] = ?$$

FIRSTLY $$\mathcal{L}[t^3] = \frac{3!}{s^4} = \frac{6}{s^4}$$

$$\mathcal{L}[e^{-t}t^3] = \frac{6}{(s+1)^4}$$

$$\mathcal{L}[2^t(t^3 e^{-t})] = \frac{6}{[(s-\ln 2)+1]^4}$$

$$\mathcal{L}[t^3 e^{-t} 2^t] = \frac{6}{(1-\ln 2+s)^4} \;//$$

Question 27

Find the following inverse Laplace transform

$$\mathcal{L}^{-1}\left[\frac{1}{s^3\left(s^2+1\right)}\right].$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{1}{s^3\left(s^2+1\right)}\right]=\frac{1}{2}t^2-1+\cos t}$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{1}{s^3(s^2+1)}\right] = \frac{1}{2}t^2 - 1 + \cos t}$$

METHOD A

- USING THE RESULT $\mathcal{L}^{-1}\left[\frac{\bar{f}(s)}{s}\right] = \int_0^t f(u)\,du$
- HERE WE HAVE

$$\bar{f}(s) = \frac{1}{s^2+1} \implies f(t) = \sin t$$

$$\therefore \mathcal{L}^{-1}\left[\frac{1}{s}\times\frac{1}{s^2+1}\right] = \int_0^t \sin u\,du$$

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2+1)}\right] = \left[-\cos u\right]_0^t$$

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2+1)}\right] = -\cos t + 1$$

$$\therefore \mathcal{L}^{-1}\left[\frac{1}{s}\left[\frac{1}{s(s^2+1)}\right]\right] = \int_0^t (1-\cos u)\,du$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s^2+1)}\right] = \left[u - \sin u\right]_0^t$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s^2+1)}\right] = t - \sin t$$

$$\therefore \mathcal{L}^{-1}\left[\frac{1}{s}\left[\frac{1}{s^2(s^2+1)}\right]\right] = \int_0^t u - \sin u\,du$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^3(s^2+1)}\right] = \left[\frac{1}{2}u^2 + \cos u\right]_0^t$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^3(s^2+1)}\right] = \frac{1}{2}t^2 + \cos t - 1$$

METHOD B BY THE CONVOLUTION THEOREM

$$\mathcal{L}[f*g] = \bar{f}(s)\,\bar{g}(s)$$

$$\underset{\frac{1}{2}t^2}{f} * \underset{\sin t}{g} = \mathcal{L}^{-1}\left[\underset{\frac{1}{s^3}}{\bar{f}(s)}\,\underset{\frac{1}{s^2+1}}{\bar{g}(s)}\right]$$

THUS WE OBTAIN

$$\mathcal{L}^{-1}\left[\frac{1}{s^3}\times\frac{1}{s^2+1}\right] = \int_0^t \frac{1}{2}(t-u)^2\sin u\,du$$

$$= \left[-\frac{1}{2}(t-u)^2\cos u\right]_{u=0}^{u=t} + \int_0^t (u-t)\cos u\,du$$

BY PARTS

$\frac{1}{2}(t-u)^2$	$-(t-u)$
$-\cos u$	$\sin u$

$$= 0 + \frac{1}{2}t^2 + \int_0^t (u-t)\cos u\,du$$

BY PARTS AGAIN

$u-t$	1
$\sin u$	$\cos u$

$$= \frac{1}{2}t^2 + \left[(u-t)\sin u\right]_{u=0}^{u=t} - \int_{u=0}^{u=t} \sin u\,du$$

$$= \frac{1}{2}t^2 + 0 + \left[\cos u\right]_0^t$$

$$= \frac{1}{2}t^2 + \cos t - 1$$

AS BEFORE

Question 28

Find the following inverse Laplace transform

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s+1)^2}\right].$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{1}{s^2(s+1)^2}\right]=t-2+(t+1)\mathrm{e}^{-t}}$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{1}{s^2(s+1)^2}\right]=(t+2)e^{-t}+t-2}$$

METHOD A – BY THE CONVOLUTION THEOREM

$$\mathcal{L}[f*g]=\bar{f}(s)\bar{g}(s)$$

$$\mathcal{L}^{-1}[\underset{\frac{1}{(s+1)^2}}{\bar{f}(s)}\,\underset{\frac{1}{s^2}}{\bar{g}(s)}]=\underset{te^{-t}}{f}*\underset{t}{g}$$

APPLYING THE THEOREM

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\times\frac{1}{(s+1)^2}\right]=\int_0^t ue^{-u}\times(t-u)\,du$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s+1)^2}\right]=\int_0^t (ut-u^2)e^{-u}\,du$$

BY PARTS

$ut-u^2$	$t-2u$
$-e^{-u}$	e^{-u}

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s+1)^2}\right]=\left[(u^2-ut)e^{-u}\right]_0^t+\int_0^t(t-2u)e^{-u}\,du$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s+1)^2}\right]=(0-0)+\int_0^t(t-2u)e^{-u}\,du$$

BY PART AGAIN

$t-2u$	-2
$-e^{-u}$	e^{-u}

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s+1)^2}\right]=\left[(2u-t)e^{-u}\right]_{u=0}^{u=t}-\int_0^t 2e^{-u}\,du$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s+1)^2}\right]=te^{-t}+t+\left[2e^{-u}\right]_0^t$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s+1)^2}\right]=te^{-t}+t+2e^{-t}-2$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s+1)^2}\right]=(t+2)e^{-t}+t-2$$

METHOD B

USING THE RESULT $\mathcal{L}^{-1}\left[\frac{\bar{f}(s)}{s}\right]=\int_0^t f(u)\,du$

HERE $\bar{f}(s)=\frac{1}{(s+1)^2}\Rightarrow f(t)=te^{-t}$

$$\therefore\ \mathcal{L}^{-1}\left[\frac{1}{s}\left(\frac{1}{(s+1)^2}\right)\right]=\int_0^t ue^{-u}\,du$$

BY PARTS

u	1
$-e^{-u}$	e^{-u}

$$\mathcal{L}^{-1}\left[\frac{1}{s}\times\frac{1}{(s+1)^2}\right]=\left[-ue^{-u}\right]_0^t+\int_0^t e^{-u}$$

$$=-te^{-t}-0+\left[-e^{-u}\right]_0^t$$

$$=-te^{-t}-e^{-t}+1$$

$$\therefore\ \mathcal{L}^{-1}\left[\frac{1}{s}\left(\frac{1}{s(s+1)^2}\right)\right]=\int_0^t 1-e^{-u}-ue^{-u}\,du$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s+1)^2}\right]=\int_0^t 1-e^{-u}\,du-\int_0^t ue^{-u}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s+1)^2}\right]=\left[u+e^{-u}\right]_0^t-\underbrace{(-te^{-t}-e^{-t}+1)}_{\text{FROM EARLIER}}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s+1)^2}\right]=t+e^{-t}-1+te^{-t}+e^{-t}-1$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s+1)^2}\right]=te^{-t}+2e^{-t}+t-2$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s+1)^2}\right]=(t+2)e^{-t}+t-2$$

Question 29

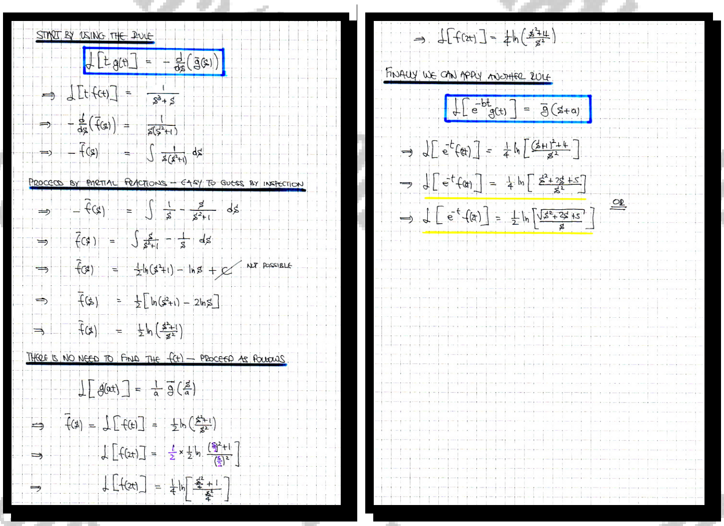
It is given that

$$\mathcal{L}\left[t\,f(t)\right] = \frac{1}{s^3+s}, \; t \geq 0 .$$

Determine a simplified expression for

$$\mathcal{L}\left[e^{-t}\,f(2t)\right].$$

$$\boxed{}, \quad \boxed{\mathcal{L}\left[e^{-t}\,f(2t)\right] = \frac{1}{2}\ln\left(\frac{\sqrt{s^2+2s+5}}{s+1}\right)}$$



START BY USING THE RULE

$$\boxed{\mathcal{L}[t\,g(t)] = -\frac{d}{ds}\left(\bar{g}(s)\right)}$$

$$\Rightarrow \mathcal{L}[t\,f(t)] = \frac{1}{s^3+s}$$

$$\Rightarrow -\frac{d}{ds}\left(\bar{f}(s)\right) = \frac{1}{s(s^2+1)}$$

$$\Rightarrow -\bar{f}(s) = \int \frac{1}{s(s^2+1)}\,ds$$

PROCEED BY PARTIAL FRACTIONS – EASY TO GUESS BY INSPECTION

$$\Rightarrow -\bar{f}(s) = \int \frac{1}{s} - \frac{s}{s^2+1}\,ds$$

$$\Rightarrow \bar{f}(s) = \int \frac{s}{s^2+1} - \frac{1}{s}\,ds$$

$$\Rightarrow \bar{f}(s) = \frac{1}{2}\ln(s^2+1) - \ln s + \cancel{C} \quad \text{NOT POSSIBLE}$$

$$\Rightarrow \bar{f}(s) = \frac{1}{2}\left[\ln(s^2+1) - 2\ln s\right]$$

$$\Rightarrow \bar{f}(s) = \frac{1}{2}\ln\left(\frac{s^2+1}{s^2}\right)$$

THERE IS NO NEED TO FIND THE $f(t)$ – PROCEED AS FOLLOWS

$$\mathcal{L}[g(at)] = \frac{1}{a}\bar{g}\left(\frac{s}{a}\right)$$

$$\Rightarrow \bar{f}(s) = \mathcal{L}[f(t)] = \frac{1}{2}\ln\left(\frac{s^2+1}{s^2}\right)$$

$$\Rightarrow \mathcal{L}[f(2t)] = \frac{1}{2}\times\frac{1}{2}\ln\left[\frac{\left(\frac{s}{2}\right)^2+1}{\left(\frac{s}{2}\right)^2}\right]$$

$$\Rightarrow \mathcal{L}[f(2t)] = \frac{1}{4}\ln\left[\frac{\frac{s^2}{4}+1}{\frac{s^2}{4}}\right]$$

$$\Rightarrow \mathcal{L}[f(2t)] = \frac{1}{4}\ln\left(\frac{s^2+4}{s^2}\right)$$

FINALLY WE CAN APPLY ANOTHER RULE

$$\boxed{\mathcal{L}\left[e^{-bt}g(t)\right] = \bar{g}(s+a)}$$

$$\Rightarrow \mathcal{L}\left[e^{-t}f(2t)\right] = \frac{1}{4}\ln\left[\frac{(s+1)^2+4}{s^2}\right]$$

$$\Rightarrow \mathcal{L}\left[e^{-t}f(2t)\right] = \frac{1}{4}\ln\left[\frac{s^2+2s+5}{s^2}\right]$$

OR

$$\Rightarrow \mathcal{L}\left[e^{-t}f(2t)\right] = \frac{1}{2}\ln\left[\frac{\sqrt{s^2+2s+5}}{s}\right]$$

Question 30

Use an appropriate method to show that

$$\mathcal{L}^{-1}\left[\frac{1}{s\sqrt{s+a}}\right] = \frac{1}{\sqrt{a}}\operatorname{erf}\left(\sqrt{at}\right),$$

where a is a positive constant.

proof

- THE LAPLACE TRANSFORM $\frac{1}{s\sqrt{s+a}}$ IS NOT RECOGNISABLE, AND WE CANNOT EVEN SPLIT IT INTO PARTIAL FRACTIONS.
- BY THE CONVOLUTION THEOREM $\mathcal{L}[f * g] = \underset{\uparrow}{\mathcal{L}[f]}\,\underset{\uparrow}{\mathcal{L}[g]}$, with $\frac{1}{\sqrt{s+a}}$ and $\frac{1}{s}$ written under them
- THUS $\bar{f}(s) = \frac{1}{(s+a)^{\frac{1}{2}}}$ IS A SHIFT OF $\frac{1}{s^{\frac{1}{2}}}$

$$\mathcal{L}\left[t^{-\frac{1}{2}}\right] = \frac{\left(-\frac{1}{2}\right)!}{s^{-\frac{1}{2}+1}} = \frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}} = \frac{\sqrt{\pi}}{s^{\frac{1}{2}}}$$

$$\therefore \mathcal{L}\left[\frac{1}{\sqrt{\pi}}t^{-\frac{1}{2}}\right] = \frac{1}{s^{\frac{1}{2}}}$$

$$\therefore \mathcal{L}\left[\frac{1}{\sqrt{\pi t}}e^{-at}\right] = \frac{1}{(s+a)^{\frac{1}{2}}}$$

$$\therefore f(t) = \frac{1}{\sqrt{\pi t}}e^{-at}$$

$$g(t) = 1$$

- SO INVERTING BY THE CONVOLUTION

$$\mathcal{L}^{-1}\left[\frac{1}{s(s+a)}\right] = \int_0^t f(u)\,g(t-u)\,du$$

$$= \int_0^t \frac{1}{\sqrt{\pi u}}e^{-au} \times 1 \, du$$

$$= \frac{1}{\sqrt{\pi}}\int_0^t \frac{1}{u^{1/2}}e^{-au}\,du$$

- BY SUBSTITUTION

LET $v^2 = au$

$v = a^{\frac{1}{2}}u^{\frac{1}{2}} \implies u^{\frac{1}{2}} = \frac{v}{a^{\frac{1}{2}}} \implies \frac{1}{u^{\frac{1}{2}}} = \frac{a^{\frac{1}{2}}}{v}$

$\frac{dv}{du} = \frac{1}{2}a^{\frac{1}{2}}u^{-\frac{1}{2}}$

$du = \frac{dv}{\frac{1}{2}a^{\frac{1}{2}}u^{-\frac{1}{2}}}$

$du = dv\left(\frac{2u^{\frac{1}{2}}}{a^{\frac{1}{2}}}\right) = \frac{2}{a^{\frac{1}{2}}}\frac{v}{a^{\frac{1}{2}}}\,dv = \frac{2v}{a}\,dv$

LIMITS $u=0 \quad v=0$

$u=t \quad v = a^{\frac{1}{2}}t^{\frac{1}{2}} = \sqrt{at}$

$$= \dots \frac{1}{\sqrt{\pi}}\int_0^{\sqrt{at}} \left(\frac{a^{\frac{1}{2}}}{\cancel{v}}\right)e^{-v^2}\left(\frac{2\cancel{v}}{a}\,dv\right)$$

$$= \frac{2}{\sqrt{\pi}}\frac{1}{\sqrt{a}}\int_0^{\sqrt{at}} e^{-v^2}\,dv$$

$$= \frac{1}{\sqrt{a}}\left[\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{at}} e^{-v^2}\,dv\right]$$

$$= \frac{1}{\sqrt{a}}\operatorname{erf}\left(\sqrt{at}\right)$$

Question 31

Use an appropriate method to show that

$$\mathcal{L}\left[\int_0^t \frac{1-e^{-u}}{u}\,du\right] = \frac{1}{s}\ln\left(s+\frac{1}{s}\right).$$

proof

$$\boxed{\mathcal{L}\left[\int_0^t \frac{1-e^{-u}}{u}\,du\right] = \frac{1}{s}\ln\left(s+\frac{1}{s}\right)}$$

- START BY DEFINING AN INTEGRAL FUNCTION

$$f(t) = \int_0^t \frac{1-e^{-u}}{u}\,du$$

- DIFFERENTIATING W.R.T t

$$\Rightarrow f'(t) = \frac{d}{dt}\int_0^t \frac{1-e^{-u}}{u}\,du$$

$$\Rightarrow f'(t) = \frac{1-e^{-t}}{t}$$

$$\Rightarrow t f'(t) = 1-e^{-t}$$

- TAKING THE LAPLACE TRANSFORM OF THE ABOVE EQUATION

$$\Rightarrow \mathcal{L}\left[t f'(t)\right] = \mathcal{L}\left[1-e^{-t}\right]$$

$$\boxed{\mathcal{L}\left[t\,g(t)\right] = -\frac{d}{ds}\left[\mathcal{L}[g(t)]\right]}$$

$$\Rightarrow -\frac{d}{ds}\left[\mathcal{L}[f'(t)]\right] = \frac{1}{s} - \frac{1}{s+1}$$

$$\boxed{\mathcal{L}[g'(t)] = s\mathcal{L}[g(t)] - g(0)}$$

$$\Rightarrow -\frac{d}{ds}\left[s\mathcal{L}[f(t)] - \cancel{f(0)}\right] = \frac{1}{s} - \frac{1}{s+1}$$

$$\Rightarrow -\frac{d}{ds}\left[s\mathcal{L}[f(t)]\right] = \frac{1}{s} - \frac{1}{s+1}$$

$$\Rightarrow s\mathcal{L}[f(t)] = \int \frac{1}{s+1} - \frac{1}{s}\,ds$$

$$\Rightarrow s\mathcal{L}[f(t)] = \ln(s+1) - \ln(s) + C$$

$$\Rightarrow s\mathcal{L}[f(t)] = \ln\left(\frac{s+1}{s}\right) + C$$

$$\therefore \boxed{s\bar{f}(s) = \ln\left(\frac{s+1}{s}\right) + C, \quad \text{WHERE } \bar{f}(s) = \mathcal{L}[f(t)] = \mathcal{L}\left[\int_0^t \frac{1-e^{-u}}{u}\,du\right]}$$

- TO EVALUATE THE CONSTANT WE USE THE INITIAL/FINAL VALUE THEOREM

$$\lim_{s\to\infty}\left[s\bar{f}(s)\right] = \lim_{t\to 0}\left[f(t)\right]$$

HERE $\displaystyle\lim_{s\to\infty}\left[s\bar{f}(s)\right] = \lim_{s\to\infty}\left[\ln\left(\frac{s+1}{s}\right) + C\right] = \cancel{\ln 1} + C = C$

$$\lim_{t\to 0}\left[f(t)\right] = \lim_{t\to 0}\left[\int_0^t \frac{1-e^{-u}}{u}\,du\right] = 0$$

$$\therefore \boxed{C=0}$$

- HENCE WE FINALLY OBTAIN

$$s\bar{f}(s) = \ln\left(\frac{s+1}{s}\right)$$

$$\bar{f}(s) = \frac{1}{s}\ln\left(1+\frac{1}{s}\right)$$

$$\mathcal{L}[f(t)] = \frac{1}{s}\ln\left(1+\frac{1}{s}\right)$$

$$\mathcal{L}\left[\int_0^t \frac{1-e^{-u}}{u}\,du\right] = \frac{1}{s}\ln\left(1+\frac{1}{s}\right)$$

// AS REQUIRED

Question 32

Use an appropriate method to show that

$$\mathcal{L}\left[\operatorname{erf}\left(\sqrt{t}\right)\right] = \frac{1}{s\sqrt{s+1}} \ .$$

proof

$$\boxed{\mathcal{L}\left[\operatorname{erf}(\sqrt{t})\right] = \frac{1}{s\sqrt{s+1}}}$$

- EXPAND THE ERROR FUNCTION AS A SERIES, BEFORE TAKING ITS TRANSFORM

$$\Rightarrow \mathcal{L}\left[\operatorname{erf}(\sqrt{t})\right] = \mathcal{L}\left[\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-u^2}\,du\right]$$

$$\dots = \frac{2}{\sqrt{\pi}}\mathcal{L}\left[\int_0^{\sqrt{t}} 1 - u^2 + \frac{u^4}{2!} - \frac{u^6}{3!} + \frac{u^8}{4!} - \dots\,du\right]$$

$$\dots = \frac{2}{\sqrt{\pi}}\mathcal{L}\left[\left[u - \frac{1}{3}u^3 + \frac{1}{5\cdot 2!}u^5 - \frac{1}{7\cdot 3!}u^7 + \frac{1}{9\cdot 4!}u^9 - \dots\right]_0^{\sqrt{t}}\right]$$

$$\dots = \frac{2}{\sqrt{\pi}}\mathcal{L}\left[t^{\frac{1}{2}} - \frac{t^{\frac{3}{2}}}{3\times 1!} + \frac{t^{\frac{5}{2}}}{5\times 2!} - \frac{t^{\frac{7}{2}}}{7\times 3!} + \frac{t^{\frac{9}{2}}}{9\times 4!} - \dots\right]$$

- NOW TAKING THE LAPLACE TRANSFORM TERM BY TERM USING THE STANDARD RESULT $\boxed{\mathcal{L}\left[t^n\right] = \frac{n!}{s^{n+1}} = \frac{\Gamma(n+1)}{s^{n+1}}}$

$$\dots = \frac{2}{\sqrt{\pi}}\left[\frac{\Gamma(\frac{3}{2})}{s^{3/2}} - \frac{\Gamma(\frac{5}{2})}{s^{\frac{5}{2}}\times 3\times 1!} + \frac{\Gamma(\frac{7}{2})}{s^{\frac{7}{2}}\times 5\times 2!} - \frac{\Gamma(\frac{9}{2})}{s^{\frac{9}{2}}\times 7\times 3!} + \frac{\Gamma(\frac{11}{2})}{s^{\frac{11}{2}}\times 9\times 4!} - \dots\right]$$

- USING THE GAMMA RECURRENCE FORMULA AND THE FACT THAT $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$$\dots = \frac{2}{\sqrt{\pi}}\left[\frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}} - \frac{\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}}{s^{\frac{5}{2}}\times 3\times 1!} + \frac{\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}}{s^{\frac{7}{2}}\times 5\times 2!} - \frac{\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}}{s^{\frac{9}{2}}\times 7\times 3!} + \dots\right]$$

$$\dots = \frac{2}{s^{\frac{3}{2}}}\left[\frac{\frac{1}{2}}{1} - \frac{\frac{1}{2}\times\frac{3}{2}}{s\times 3\times 1!} + \frac{\frac{1}{2}\times\frac{3}{2}\times\frac{5}{2}}{s^2\times 5\times 2!} - \frac{\frac{1}{2}\times\frac{3}{2}\times\frac{5}{2}\times\frac{7}{2}}{s^3\times 7\times 3!} + \frac{\frac{1}{2}\times\frac{3}{2}\times\frac{5}{2}\times\frac{7}{2}\times\frac{9}{2}}{s^4\times 9\times 4!} - \dots\right]$$

$$\dots = \frac{1}{s^{\frac{3}{2}}}\left[1 - \frac{\frac{3}{2}}{1!}\left(\frac{1}{s}\right)\left(\frac{1}{3}\right) + \frac{\frac{3}{2}\times\frac{5}{2}}{2!}\left(\frac{1}{s}\right)^2\left(\frac{1}{5}\right) - \frac{\frac{3}{2}\times\frac{5}{2}\times\frac{7}{2}}{3!}\left(\frac{1}{s}\right)^3\left(\frac{1}{7}\right) + \frac{\frac{3}{2}\times\frac{5}{2}\times\frac{7}{2}\times\frac{9}{2}}{4!}\left(\frac{1}{s^4}\right)\frac{1}{9} - \dots\right]$$

- TIDY UP THE SPARE RECIPROCAL TERM BY CANCELLING

$$\dots = \frac{1}{s^{\frac{3}{2}}}\left[1 - \frac{\frac{1}{2}}{1!}\left(\frac{1}{s}\right) + \frac{\frac{1}{2}\times\frac{3}{2}}{2!}\left(\frac{1}{s}\right)^2 - \frac{\frac{1}{2}\times\frac{3}{2}\times\frac{5}{2}}{3!}\left(\frac{1}{s}\right)^3 + \frac{\frac{1}{2}\times\frac{3}{2}\times\frac{5}{2}\times\frac{7}{2}}{4!}\left(\frac{1}{s}\right)^4 + \dots\right]$$

- REWRITE AS A BINOMIAL BY INTRODUCING MINUSES SO WE HAVE THE USUAL SEQUENCE $-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots$

$$\dots = \frac{1}{s^{\frac{3}{2}}}\left[1 + \frac{(-\frac{1}{2})}{1!}\left(\frac{1}{s}\right) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\left(\frac{1}{s}\right)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}\left(\frac{1}{s}\right)^3 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})}{4!}\left(\frac{1}{s}\right)^4 + \dots\right]$$

$$= \frac{1}{s^{\frac{3}{2}}}\left(1 + \frac{1}{s}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{s^{\frac{3}{2}}}\left(\frac{s+1}{s}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{s^{3/2}}\left(\frac{s}{s+1}\right)^{\frac{1}{2}}$$

$$= \frac{1}{s^{3/2}}\times\frac{s^{\frac{1}{2}}}{(s+1)^{\frac{1}{2}}}$$

$$= \frac{1}{s\sqrt{s+1}}$$

AS REQUIRED

Question 33

$$g(t) \equiv \int_0^t f(x)\ dx\ ,\ t \geq 0\ .$$

a) Show clearly that

$$\mathcal{L}\big(g(t)\big) = \frac{\bar{f}(s)}{s},$$

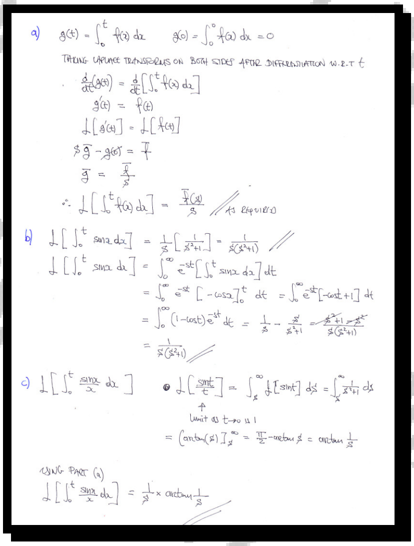
where $\bar{f}(s) = \mathcal{L}\big(f(t)\big)$.

b) Verify the validity of the result of part **(a)** by using $f(x) = \sin x$ and finding $\mathcal{L}\big(g(t)\big)$ by its integral definition.

c) Use the result of part **(a)** to determine

$$\mathcal{L}\left[\int_0^t \frac{\sin x}{x}\ dx\right].$$

$$\boxed{\frac{1}{s}\arctan\left(\frac{1}{s}\right)}$$



a) $g(t) = \int_0^t f(x)\,dx \qquad g(0) = \int_0^0 f(x)\,dx = 0$

TAKING LAPLACE TRANSFORMS ON BOTH SIDES AFTER DIFFERENTIATION W.R.T t

$\frac{d}{dt}(g(t)) = \frac{d}{dt}\left[\int_0^t f(x)\,dx\right]$

$g'(t) = f(t)$

$\mathcal{L}[g'(t)] = \mathcal{L}[f(t)]$

$s\bar{g} - \cancel{g(0)} = \bar{f}$

$\bar{g} = \frac{\bar{f}}{s}$

$\therefore \mathcal{L}\left[\int_0^t f(x)\,dx\right] = \frac{\bar{f}(s)}{s}$ // AS REQUIRED

b) $\mathcal{L}\left[\int_0^t \sin x\,dx\right] = \frac{1}{s}\left[\frac{1}{s^2+1}\right] = \frac{1}{s(s^2+1)}$ //

$\mathcal{L}\left[\int_0^t \sin x\,dx\right] = \int_0^\infty e^{-st}\left[\int_0^t \sin x\,dx\right]dt$

$= \int_0^\infty e^{-st}\left[-\cos x\right]_0^t\,dt = \int_0^\infty e^{-st}[-\cos t + 1]\,dt$

$= \int_0^\infty (1-\cos t)e^{-st}\,dt = \frac{1}{s} - \frac{s}{s^2+1} = \frac{\cancel{s^2}+1-\cancel{s^2}}{s(s^2+1)}$

$= \frac{1}{s(s^2+1)}$ //

c) $\mathcal{L}\left[\int_0^t \frac{\sin x}{x}\,dx\right]$

$\bullet\ \mathcal{L}\left[\frac{\sin t}{t}\right] = \int_s^\infty \mathcal{L}[\sin t]\,ds = \int_s^\infty \frac{1}{s^2+1}\,ds$

limit as $t \to 0$ is 1

$= \left[\arctan(s)\right]_s^\infty = \frac{\pi}{2} - \arctan s = \arctan\frac{1}{s}$

USING PART (a)

$\mathcal{L}\left[\int_0^t \frac{\sin x}{x}\,dx\right] = \frac{1}{s} \times \arctan\frac{1}{s}$ //

Question 34

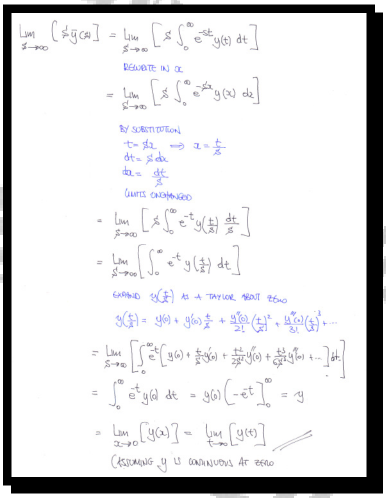
The function $y = y(t)$ is infinitely differentiable and defined for $t \geq 0$.

Show that

$$\lim_{s \to \infty} \left[s\,\overline{y}(s) \right] = \lim_{t \to 0} \left[y(t) \right],$$

where $\overline{y}(s) = \mathcal{L}\left[y(t) \right]$

$\boxed{\text{proof}}$



$$\lim_{s \to \infty} \left[s\,\overline{y}(s) \right] = \lim_{s \to \infty} \left[s \int_0^\infty e^{-st} y(t)\, dt \right]$$

REWRITE IN x

$$= \lim_{s \to \infty} \left[s \int_0^\infty e^{-sx} y(x)\, dx \right]$$

BY SUBSTITUTION

$t = sx \implies x = \frac{t}{s}$

$dt = s\, dx$

$dx = \frac{dt}{s}$

LIMITS UNCHANGED

$$= \lim_{s \to \infty} \left[s \int_0^\infty e^{-t} y\left(\frac{t}{s}\right) \frac{dt}{s} \right]$$

$$= \lim_{s \to \infty} \left[\int_0^\infty e^{-t} y\left(\frac{t}{s}\right) dt \right]$$

EXPAND $y\left(\frac{t}{s}\right)$ AS A TAYLOR ABOUT ZERO

$$y\left(\frac{t}{s}\right) = y(0) + y'(0)\frac{t}{s} + \frac{y''(0)}{2!}\left(\frac{t}{s}\right)^2 + \frac{y'''(0)}{3!}\left(\frac{t}{s}\right)^3 + \dots$$

$$= \lim_{s \to \infty} \left[\int_0^\infty e^{-t} \left[y(0) + \frac{t}{s}y'(0) + \frac{t^2}{2s^2}y''(0) + \frac{t^3}{6s^3}y'''(0) + \dots \right] dt \right]$$

$$= \int_0^\infty e^{-t} y(0)\, dt = y(0)\left[-e^{-t} \right]_0^\infty = y$$

$$= \lim_{x \to 0} \left[y(x) \right] = \lim_{t \to 0} \left[y(t) \right]$$

(ASSUMING y IS CONTINUOUS AT ZERO

Question 35

The Laplace transform of $f(t)$, $t \geq 0$, is denoted by $\bar{f}(s) = \mathcal{L}(f(t))$.

Show that the inverse Laplace transform of $\dfrac{\bar{f}(s)}{s}$ satisfies

$$\mathcal{L}^{-1}\left(\frac{\bar{f}(s)}{s}\right) = \int_0^t f(u)\ du .$$

proof

$$\boxed{\mathcal{L}^{-1}\left[\frac{\bar{f}(s)}{s}\right] = \int_0^t f(u)\ du}$$

- LET $g(t) = \int_0^t f(u)\ du$
- DIFFERENTIATE W.R.T t

$$\Rightarrow g'(t) = \frac{d}{dt}\int_0^t f(u)\ du$$

$$\Rightarrow g'(t) = f(t)$$

- TAKING THE LAPLACE TRANSFORM OF THIS EQUATION

$$\Rightarrow \mathcal{L}[g'(t)] = \mathcal{L}[f(t)]$$

$$\Rightarrow s\bar{g}(s) - \cancel{g(0)} = \bar{f}(s)$$

$$g(0) = \int_0^0 f(u)\ du = 0$$

$$\Rightarrow \bar{g}(s) = \frac{\bar{f}(s)}{s}$$

$$\Rightarrow \mathcal{L}^{-1}[\bar{g}(s)] = \mathcal{L}^{-1}\left[\frac{\bar{f}(s)}{s}\right]$$

$$\Rightarrow g(t) = \mathcal{L}^{-1}\left[\frac{\bar{f}(s)}{s}\right]$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{\bar{f}(s)}{s}\right] = \int_0^t f(u)\ du$$

Question 36

$$t\frac{d^2y}{dt^2}+\frac{dy}{dt}+ty\ ,\ t>0\ .$$

The function $y=J_0(t)$ is a solution of the above differential equation.

It is further given that $\lim\limits_{t\to 0}\left[J_0(t)\right]=1$.

By taking the Laplace transform of the above differential equation, show that

$$\mathcal{L}\left[J_0(t)\right]=\frac{1}{\sqrt{s^2+1}}\ .$$

$\boxed{\text{proof}}$

TAKE THE LAPLACE TRANSFORM OF THE O.D.E

$$\boxed{t\frac{d^2y}{dt^2}+\frac{dy}{dt}+ty=0}$$

WHOSE SOLUTION $y(t)=J_0(t)$ SUCH THAT $J_0(0)=1$

$$\Rightarrow -\frac{d}{ds}\left[s^2\bar{y}-sy_0-\dot{y}_0\right]+\left[s\bar{y}-y_0\right]-\frac{d}{ds}(\bar{y})=0$$

IT IS IRRELEVANT WHAT $\dot{y}_0$ IS, AS IT VANISHES ON DIFFERENTIATION
ALSO $y_0=1$

$$\Rightarrow -\frac{d}{ds}\left[s^2\bar{y}-s-\dot{y}_0\right]+s\bar{y}-1-\frac{d\bar{y}}{ds}=0$$

$$\Rightarrow -\left[2s\bar{y}+s^2\frac{d\bar{y}}{ds}-1+0\right]+s\bar{y}-1-\frac{d\bar{y}}{ds}=0$$

$$\Rightarrow -2s\bar{y}-s^2\frac{d\bar{y}}{ds}\cancel{+1}+s\bar{y}\cancel{-1}-\frac{d\bar{y}}{ds}=0$$

$$\Rightarrow -s\bar{y}=(s^2+1)\frac{d\bar{y}}{ds}$$

$$\Rightarrow \boxed{\frac{d\bar{y}}{ds}=-\frac{s\bar{y}}{s^2+1}}$$

SOLVE THE ODE BY SEPARATING VARIABLES

$$\Rightarrow \frac{1}{\bar{y}}\,d\bar{y}=-\frac{s}{s^2+1}\,ds$$

$$\Rightarrow \ln\bar{y}=-\tfrac{1}{2}\ln(s^2+1)+C$$

$$\Rightarrow \ln\bar{y}=\ln\left(\frac{A}{\sqrt{s^2+1}}\right)$$

$$\Rightarrow \boxed{\bar{y}=\frac{A}{\sqrt{s^2+1}}}$$

NOW WE USE THESE RESULTS TO EVALUATE THE CONSTANT A

$$\boxed{\begin{aligned}\lim_{t\to 0}f(t)&=\lim_{s\to\infty}\left(s\bar{f}(s)\right)\\ \lim_{t\to\infty}f(t)&=\lim_{s\to 0}\left(s\bar{f}(s)\right)\end{aligned}}$$

HERE WE OBTAIN

$$\lim_{s\to\infty}\left[s\bar{y}\right]=\lim_{t\to 0}\left[y(t)\right]=\lim_{t\to 0}\left[J_0(t)\right]=1$$

THUS

$$\lim_{s\to\infty}\left[\frac{As}{\sqrt{s^2+1}}\right]=1 \qquad \therefore \boxed{A=1}$$

RETURNING TO THE PROBLEM

$$\bar{y}=\frac{1}{\sqrt{s^2+1}}$$

$$\therefore \mathcal{L}\left[J_0(t)\right]=\frac{1}{\sqrt{s^2+1}}\ //$$

Question 37

By forming and taking the Laplace transform of a suitable second order differential equation, show that

$$\mathcal{L}\left[\sin\sqrt{t}\right] = \frac{\sqrt{\pi}\,\mathrm{e}^{-\frac{1}{4s}}}{2s^{\frac{3}{2}}}.$$

$\boxed{\text{proof}}$

• LET $f(t) = y = \sin\sqrt{t} = \sin(t^{\frac{1}{2}})$

$$\left.\begin{aligned} f'(t) &= \dot{y} = \tfrac{1}{2}t^{-\frac{1}{2}}\cos\sqrt{t} = \tfrac{1}{2}t^{-\frac{1}{2}}\cos(t^{\frac{1}{2}}) \\ f''(t) &= \ddot{y} = -\tfrac{1}{4}t^{-\frac{3}{2}}\cos\sqrt{t} - \tfrac{1}{4}t^{-\frac{1}{2}}\sin(t^{\frac{1}{2}}) \times t^{-\frac{1}{2}} \\ &= -\tfrac{1}{4}t^{-\frac{3}{2}}\cos\sqrt{t} - \tfrac{1}{4}t^{-1}\sin(t^{\frac{1}{2}}) \end{aligned}\right\} \text{TRY TO FORM AN O.D.E}$$

$$4t\ddot{y} = -t^{-\frac{1}{2}}\cos\sqrt{t} - \sin\sqrt{t}$$
$$y = +\sin\sqrt{t}$$
$$2\dot{y} = t^{-\frac{1}{2}}\cos\sqrt{t}$$

$$4t\ddot{y} + 2\dot{y} + y = 0$$

HAS SOLUTION $y = \sin\sqrt{t}$ SUBJECT TO $t=0$, $y=0$, $\dot{y}=\infty$

TAKING LAPLACE TRANSFORMS

$$\Rightarrow -4\frac{d}{ds}\left[s^2\bar{y} - sy_0 - \dot{y}_0\right] + 2\left[s\bar{y} - y_0\right] + \bar{y} = 0$$

$$\Rightarrow -4\frac{d}{ds}\left[s^2\bar{y} - 0 - \dot{y}_0\right] + 2s\bar{y} + \bar{y} = 0 \qquad \left(\frac{d}{ds}(\underbrace{\dot{y}_0}_{\text{CONSTANT}})\right) = 0$$

$$\Rightarrow -4\left[2s\bar{y} + s^2\frac{d\bar{y}}{ds}\right] + 2s\bar{y} + \bar{y} = 0$$

$$\Rightarrow -8s\bar{y} - 4s^2\frac{d\bar{y}}{ds} + 2s\bar{y} + \bar{y} = 0$$

$$\Rightarrow (1-6s)\bar{y} = 4s^2\frac{d\bar{y}}{ds}$$

$$\Rightarrow \frac{1-6s}{4s^2}\,ds = \frac{1}{\bar{y}}\,d\bar{y}$$

$$\Rightarrow \int \frac{1}{4}s^{-2} - \frac{3}{2s}\,ds = \int \frac{1}{\bar{y}}\,d\bar{y}$$

$$\Rightarrow -\frac{1}{4s} - \frac{3}{2}\ln s + C = \ln\bar{y}$$

$$\Rightarrow \bar{y} = e^{-\frac{1}{4s} - \frac{3}{2}\ln s + C}$$

$$\Rightarrow \bar{y} = e^{-\frac{1}{4s}} \times e^{\ln s^{-\frac{3}{2}}} \times e^{C}$$

$$\Rightarrow \bar{y} = \frac{Ae^{-\frac{1}{4s}}}{s^{3/2}}$$

NOW $\lim\limits_{t\to 0} f(t) = \lim\limits_{s\to\infty}\left[s\bar{f}(s)\right]$

SO $\lim\limits_{t\to 0}\left[\sin\sqrt{t}\right] = 0 \Rightarrow \lim\limits_{s\to\infty}\left[s\bar{f}(s)\right] = 0$

$$\Rightarrow \lim_{s\to\infty}\left[\frac{Ase^{-\frac{1}{4s}}}{s^{3/2}}\right] = 0$$

$$\Rightarrow A\lim_{s\to\infty}\left[\frac{e^{-\frac{1}{4s}}}{s^{1/2}}\right] = 0$$

METHOD FAILS! $[A \times 0 = 0]$

TRY SMALL t $(t \to 0)$

$$\sin\sqrt{t} \to \sqrt{t} = t^{\frac{1}{2}}$$

$$\mathcal{L}\left[t^{\frac{1}{2}}\right] = \frac{\Gamma(\frac{3}{2})}{s^{3/2}} = \frac{\frac{1}{2}\Gamma(\frac{1}{2})}{s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}}$$

AS $s \to \infty$ $\bar{y} \simeq \frac{A}{s^{3/2}}$ $\therefore A = \frac{\sqrt{\pi}}{2}$

$$\therefore \mathcal{L}\left[\sin\sqrt{t}\right] = \frac{\sqrt{\pi}\,e^{-\frac{1}{4s}}}{2s^{3/2}}$$

Question 38

The Sine integral function $\text{Si}(t)$ is defined as

$$\text{Si}(t) \equiv \int_0^t \frac{\sin u}{u}\,du\ ,\ t \geq 0 .$$

Show that

$$\mathcal{L}\left[\text{Si}(t)\right] = \frac{1}{s}\arctan\left(\frac{1}{s}\right).$$

proof

$$\boxed{\text{Si}(t) \equiv \int_0^t \frac{\sin u}{u}\,du,\ t \geq 0}$$

- FOR CONVENIENCE LET $f(t) = \text{Si}(t)$

NOTE THAT $\boxed{f(0) = \text{Si}(0) = 0}$

- DIFFERENTIATE THE DEFINITION W.R.T t

$$\Rightarrow f(t) = \int_0^t \frac{\sin u}{u}\,du$$

$$\Rightarrow f'(t) = \frac{d}{dt}\int_0^t \frac{\sin u}{u}\,du$$

$$\Rightarrow f'(t) = \frac{\sin t}{t}$$

$$\Rightarrow t f'(t) = \sin t$$

- NOW TAKING THE LAPLACE TRANSFORM OF THE ABOVE EQUATION AND NOTING THE RESULTS

$$\boxed{\begin{aligned}\mathcal{L}\left[t\,g(t)\right] &= -\frac{d}{ds}\left[\mathcal{L}\left[g(t)\right]\right]\\ \mathcal{L}\left[g'(t)\right] &= s\,\mathcal{L}\left[g(t)\right] - g(0)\end{aligned}}$$

$$\Rightarrow \mathcal{L}\left[t f'(t)\right] = \mathcal{L}\left[\sin t\right]$$

$$\Rightarrow -\frac{d}{ds}\left[\mathcal{L}\left[f'(t)\right]\right] = \frac{1}{s^2+1}$$

$$\Rightarrow -\frac{d}{ds}\left[s\bar{f}(s) - \cancel{f(0)}\right] = \frac{1}{s^2+1}$$

$$\Rightarrow -s\bar{f}(s) = \int \frac{1}{s^2+1}\,ds$$

$$\Rightarrow -s\bar{f}(s) = \arctan s + C$$

$$\Rightarrow \boxed{s\bar{f}(s) = c - \arctan s}$$

WHERE $\bar{f}(s) = \mathcal{L}\left[\text{Si}(t)\right]$

- TO FIND THE CONSTANT, WE USE

$$\boxed{\lim_{s \to \infty} s\bar{f}(s) = \lim_{t \to 0} f(t)}$$

← INITIAL/FINAL VALUE THEOREM

$$\lim_{s \to \infty}\left[s\bar{f}(s)\right] = \lim_{s \to \infty}\left[C - \arctan s\right] = C - \frac{\pi}{2}$$

$$\lim_{t \to 0}\left[f(t)\right] = \lim_{t \to 0}\left[\text{Si}(t)\right] = 0$$

$$\therefore C - \frac{\pi}{2} = 0$$

$$C = \frac{\pi}{2}$$

- FINALLY WE OBTAIN

$$\Rightarrow s\bar{f}(s) = \frac{\pi}{2} - \arctan s$$

$$\Rightarrow s\bar{f}(s) = \arctan\left(\frac{1}{s}\right)$$

$$\Rightarrow \bar{f}(s) = \frac{1}{s}\arctan\left(\frac{1}{s}\right)$$

$$\Rightarrow \mathcal{L}\left[\text{Si}(t)\right] = \mathcal{L}\left[\int_0^t \frac{\sin u}{u}\,du\right] = \frac{1}{s}\arctan\frac{1}{s}$$

Question 39

Find the following inverse Laplace transform by 3 different methods.

$$\mathcal{L}^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right],\ a>0\,.$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right]=\frac{t\sin at}{2a}}$$

$\mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$ CAN BE DONE BY TRICKS & INSPECTION, THE CONVOLUTION THEOREM, COMPLEX VARIABLES, DIFFERENTIATION UNDER THE INTEGRAL SIGN – STRANGELY PARTIAL FRACTION, WHICH IS THE MOST POPULAR METHOD, CANNOT BE USED

METHOD A (BY INSPECTION)

- WE OBSERVE THAT $\frac{d}{ds}\left[\frac{1}{s^2+a^2}\right]=\frac{d}{ds}\left[(s^2+a^2)^{-1}\right]$

$$= -2s(s^2+a^2)^{-2}$$
$$= -2\left[\frac{s}{(s^2+a^2)^2}\right]$$

- HENCE

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right] = \frac{1}{2}\mathcal{L}\left[-\frac{d}{ds}\left[\frac{1}{s^2+a^2}\right]\right]=\frac{1}{2}\mathcal{L}\left[-\frac{d}{ds}\left[\frac{1}{a}\frac{a}{s^2+a^2}\right]\right]$$
$$=\frac{1}{2a}\left[-\frac{d}{ds}\mathcal{L}\left[\sin at\right]\right]=\frac{1}{2a}t\sin at$$

METHOD B (BY THE CONVOLUTION THEOREM)

- $\mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]=\mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\times\frac{1}{s^2+a^2}\right]=$

$$=\frac{1}{a}\mathcal{L}^{-1}\left[\underbrace{\frac{s}{s^2+a^2}}_{\mathcal{L}[\cos at]}\times\underbrace{\frac{a}{s^2+a^2}}_{\mathcal{L}[\sin at]}\right]$$

- $\mathcal{L}[f*g]=\mathcal{L}[f]\,\mathcal{L}[g]$
  $\mathcal{L}[f*g]=\bar{f}(s)\,\bar{g}(s)$
  $\mathcal{L}^{-1}[\bar{f}(s)\bar{g}(s)]=f*g$

- HERE $\mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]=\frac{1}{a}\int_0^t f(u)\,g(t-u)\,du$

$$=\frac{1}{a}\int_0^t \cos(au)\,\sin[a(t-u)]\,du$$

- EXPAND BY THE COMPOUND ANGLE

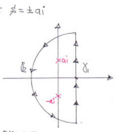
$$\mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]=\frac{1}{a}\int_0^t\cos(au)\left[\sin at\cos au-\cos at\sin au\right]du$$
$$=\frac{1}{a}\int_0^t \sin at\cos^2 au-\cos at\sin au\cos au\,du.$$
$$=\frac{\sin at}{a}\int_0^t\cos^2 au\,du-\frac{\cos at}{a}\int_0^t\sin au\cos au$$
$$=\frac{\sin at}{a}\int_0^t\frac{1}{2}+\frac{1}{2}\cos 2au\,du-\frac{\cos at}{a}\int_0^t\frac{1}{2}\sin 2au\,du$$
$$=\frac{\sin at}{2a}\int_0^t 1+\cos 2au\,du-\frac{\cos at}{2a}\int_0^t\sin 2au\,du$$
$$=\frac{\sin at}{2a}\left[u+\frac{1}{2a}\sin 2au\right]_0^t-\frac{\cos at}{2a}\left[-\frac{1}{2a}\cos 2au\right]_0^t$$
$$=\frac{\sin at}{2a}\left[t+\frac{1}{2a}\sin 2at\right]+\frac{1}{4a^2}\cos at\left[\cos 2au\right]_0^t$$
$$=\frac{t\sin at}{2a}+\frac{1}{2a}\sin 2at\sin at+\frac{1}{4a^2}\cos at\left[\cos 2at-1\right]$$
$$=\frac{t\sin at}{2a}+\frac{1}{4a^2}\left[\sin 2at\sin at+\cos 2at\cos at\right]-\frac{1}{4a^2}\cos at$$
$$=\frac{t\sin at}{2a}+\frac{1}{4a^2}\left[\cos[2at-at]\right]-\frac{1}{4a^2}\cos at$$
$$=\frac{t\sin at}{2a}+\cancel{\frac{1}{4a^2}\cos at}-\cancel{\frac{1}{4a^2}\cos at}$$
$$=\frac{1}{2a}t\sin at$$

AS BEFORE

METHOD C – BY COMPLEX INTEGRATION

- $f(t)=\frac{1}{2\pi i}\int_{\gamma_1}\bar{f}(s)e^{st}\,ds=\frac{1}{2\pi i}\int_{\gamma_1}\frac{se^{st}}{(s^2+a^2)^2}\,ds$

- THE INTEGRAND HAS TWO DOUBLE POLES AT $s=\pm ai$
  IN THIS CASE THE INVERSION IS THE SUM OF THE TWO RESIDUES



SINCE

$$f(t)=\cancel{2\pi i}\sum\left(\frac{1}{\cancel{2\pi i}}\frac{se^{st}}{(s^2+a^2)^2}\right)$$
RESIDUES INSIDE CONTOUR

SINCE γ_2 DOES NOT CONTRIBUTE AS THE RADIUS TENDS TO INFINITY & γ_1 NEEDS TO GO FROM $-\infty$ TO ∞

- RESIDUE AT ai

$$\lim_{s\to ai}\frac{d}{ds}\left[\cancel{(s-ai)^2}\frac{se^{st}}{\cancel{(s-ai)^2}(s+ai)^2}\right]=\lim_{s\to ai}\left[\frac{d}{ds}\frac{se^{st}}{(s+ai)^2}\right]$$
$$=\lim_{s\to ai}\left[\frac{(s+ai)^2\left[e^{st}+ste^{st}\right]-2(s+ai)se^{st}}{(s+ai)^4}\right]$$
$$=\lim_{s\to ai}\left[\frac{(s+ai)e^{st}(1+st)-2se^{st}}{(s+ai)^3}\right]$$
$$=\lim_{s\to ai}\left[\frac{e^{st}\left[(s+ai)(1+st)-2s\right]}{(s+ai)^3}\right]=\frac{e^{ati}\left[(2ai)(1+ati)-2ai\right]}{(2ai)^3}$$
$$=\frac{e^{ati}\left(\cancel{2ai}-2a^2t-\cancel{2ai}\right)}{-8a^3i}=\frac{te^{ati}}{4ai}$$

- RESIDUE AT $-ai$

$$\lim_{s\to -ai}\left[\frac{d}{ds}\cancel{(s+ai)^2}\frac{se^{st}}{\cancel{(s+ai)^2}(s-ai)^2}\right]=\lim_{s\to -ai}\left[\frac{d}{ds}\frac{se^{st}}{(s-ai)^2}\right]$$

THE DIFFERENTIATION WILL BE IDENTICAL AS PREVIOUSLY...

$$\lim_{s\to -ai}\left[\frac{e^{st}\left[(s-ai)(1+st)-2s\right]}{(s-ai)^3}\right]=\frac{e^{-ati}\left[-2ai(1-ati)+2ai\right]}{(-2ai)^3}$$
$$=\frac{e^{-ati}\left(\cancel{-2ai}-2a^2t+\cancel{2ai}\right)}{8a^3i}=e^{-ati}\times\frac{-2a^2t}{8a^3i}=-\frac{te^{-ati}}{4ai}$$

- COLLECTING & MANIPULATING THE RESIDUES

$$f(t)=\frac{te^{ati}}{4ai}-\frac{te^{-ati}}{4ai}=\frac{t}{4ai}\left[e^{ati}-e^{-ati}\right]$$
$$=\frac{t}{4ai}\left[2\sinh(ati)\right]=\frac{t}{4ai}\left[2i\sin(at)\right]$$
$$=\frac{2ti}{4ai}\sin(at)$$
$$=\frac{t}{2a}\sin(at)$$

METHOD D – BY DIFFERENTIATION UNDER THE INTEGRAL SIGN

- STARTING WITH $\frac{s}{s^2+a^2}=s(s^2+a^2)^{-1}$

$$\frac{d}{da}\left[\frac{s}{s^2+a^2}\right]=-\frac{2as}{(s^2+a^2)^2}$$

- TAKING INVERSE LAPLACE TRANSFORMS ON BOTH SIDES

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{d}{da}\left(\frac{s}{s^2+a^2}\right)\right]=-2a\,\mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$$
$$\Rightarrow \frac{d}{da}\left[\mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right]\right]=-2a\,\mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$$
$$\Rightarrow \frac{d}{da}\left[\cos(at)\right]=-2a\,\mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$$
$$\Rightarrow -t\sin(at)=-2a\,\mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$$
$$\Rightarrow \mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]=\frac{1}{2a}t\sin(at)$$

Question 40

The Cosine integral function $\text{Ci}(t)$ is defined as

$$\text{Ci}(t) \equiv \int_t^{\infty} \frac{\cos u}{u} \, du \ , \ t > 0 .$$

Show that

$$\mathcal{L}\left[\text{Ci}(t)\right] = \frac{\ln\left(s^2+1\right)}{2s} .$$

proof

$$\boxed{\text{Ci}(t) \equiv \int_t^{\infty} \frac{\cos u}{u} \, du \ , \ t > 0}$$

- FOR SIMPLICITY OF NOTATION, LET $f(t) = \text{Ci}(t)$
  [NOTE THAT $f(\infty) = 0$]
- DIFFERENTIATE THE COSINE INTEGRAL DEFINITION W.R.T t

$$\Rightarrow f(t) = \int_t^{\infty} \frac{\cos u}{u} \, du$$

$$\Rightarrow f'(t) = \frac{d}{dt}\int_t^{\infty} \frac{\cos u}{u} \, du$$

$$\Rightarrow f'(t) = -\frac{\cos t}{t}$$

$$\Rightarrow -t f'(t) = \cos t$$

- NEXT WE TAKE THE LAPLACE TRANSFORM, USING THE RESULTS

$$\boxed{\begin{aligned} \mathcal{L}\left[t\,g(t)\right] &= -\frac{d}{ds}\left[\mathcal{L}\left[g(t)\right]\right] \\ \mathcal{L}\left[g'(t)\right] &= s\,\mathcal{L}\left[g(t)\right] - g(0) \end{aligned}}$$

$$\Rightarrow +\frac{d}{ds}\left[\mathcal{L}\left[f'(t)\right]\right] = \mathcal{L}\left[\cos t\right]$$

$$\Rightarrow \frac{d}{ds}\left[s\bar{f}(s) - f(0)\right] = \frac{s}{s^2+1}$$

$$\Rightarrow \frac{d}{ds}\left[s\bar{f}(s)\right] - \cancel{\frac{d}{ds}f(0)} = \frac{s}{s^2+1}$$

$$\Rightarrow s\bar{f}(s) = \int \frac{s}{s^2+1} \, ds$$

$$\Rightarrow s\bar{f}(s) = \frac{1}{2}\ln\left(s^2+1\right) + \text{constant}$$

- TO FIND THE CONSTANT WE USE THE INITIAL/FINAL VALUE THEOREM

$$\boxed{\lim_{s \to 0} s\bar{f}(s) = \lim_{t \to \infty} f(t)}$$

- SO HERE: $\lim_{s \to 0}\left[s\bar{f}(s)\right] = \lim_{s \to \infty}\left[\frac{1}{2}\ln\left(s^2+1\right) + C\right] = C$

$$\& \lim_{t \to \infty} f(t) = \lim \int_t^{\infty} \frac{\cos u}{u} \, du = 0$$

$$\therefore C = 0$$

- FINALLY WE OBTAIN

$$s\bar{f}(s) = \frac{1}{2}\ln\left(s^2+1\right)$$

$$\bar{f}(s) = \frac{\ln\left(s^2+1\right)}{2s}$$

$$\mathcal{L}\left[\int_t^{\infty} \frac{\cos u}{u} \, du\right] = \mathcal{L}\left[\text{Ci}(t)\right] = \frac{\ln\left(s^2+1\right)}{2s}$$

Question 41

The Exponential integral function $\mathrm{Ei}(t)$ is defined as

$$\mathrm{Ei}(t) \equiv \int_t^{\infty} \frac{e^{-u}}{u}\, du \, , \; t \geq 0 \, .$$

Show that

$$\mathcal{L}\left[\mathrm{Ei}(t)\right] = \frac{\ln(s+1)}{s} \, .$$

proof

$$\boxed{\mathrm{Ei}(t) \equiv \int_t^{\infty} \frac{e^{-u}}{u}\, du \, , \; t \geq 0}$$

- FOR CONVENIENCE IN NOTATION LET $f(t) = \mathrm{Ei}(t)$

$$\Rightarrow f(t) = \int_t^{\infty} \frac{e^{-u}}{u}\, du$$

- DIFFERENTIATE W.R.T t

$$\Rightarrow f'(t) = \frac{d}{dt}\int_t^{\infty} \frac{e^{-u}}{u}\, du.$$

$$\Rightarrow f'(t) = -\frac{e^{-t}}{t}$$

$$\Rightarrow -t f'(t) = e^{-t}$$

- TAKING THE LAPLACE TRANSFORM, USING THESE RESULTS

$$\boxed{\begin{aligned} \mathcal{L}\left[t g(t)\right] &= -\frac{d}{ds}\left[\mathcal{L}\left[g(t)\right]\right] \\ \mathcal{L}\left[g'(t)\right] &= s\mathcal{L}\left[g(t)\right] - g(0) \end{aligned}}$$

$$\Rightarrow \mathcal{L}\left[-t f'(t)\right] = \mathcal{L}\left[e^{-t}\right]$$

$$\Rightarrow \frac{d}{ds}\left[\mathcal{L}\left[f'(t)\right]\right] = \frac{1}{s+1}$$

$$\Rightarrow \frac{d}{ds}\left[s\bar{f}(s) - f(0)\right] = \frac{1}{s+1}$$

$$\Rightarrow \frac{d}{ds}\left[s\bar{f}(s)\right] - \cancel{\frac{d}{ds}\left[f(0)\right]} = \frac{1}{s+1}$$

$$\Rightarrow s\bar{f}(s) = \int \frac{1}{s+1}\, ds$$

$$\Rightarrow s\bar{f}(s) = \ln|s+1| + C$$

- TO FIND THE CONSTANT WE USE THE INITIAL/FINAL THEOREM

$$\boxed{\begin{gathered} \lim_{s \to 0} s\bar{f}(s) = \lim_{t \to \infty} f(t) \\ \underline{\text{OR}} \\ \lim_{s \to \infty} s\bar{f}(s) = \lim_{t \to 0} f(t) \end{gathered}}$$

$$\left.\begin{aligned} \lim_{t \to \infty} f(t) &= \lim_{t \to \infty} \int_t^{\infty} \frac{e^{-u}}{u}\, du = 0 \\ \lim_{s \to 0} s\bar{f}(s) &= \lim_{s \to 0}\left[\ln|s+1| + C\right] = C \end{aligned}\right\} \Rightarrow \boxed{C = 0}$$

- FINALLY WE OBTAIN

$$\Rightarrow s\bar{f}(s) = \ln(s+1)$$

$$\Rightarrow \bar{f}(s) = \frac{\ln(s+1)}{s}$$

$$\Rightarrow \mathcal{L}\left[f(t)\right] = \mathcal{L}\left[\mathrm{Ei}(t)\right] = \mathcal{L}\left[\int_t^{\infty} \frac{e^{-u}}{u}\, du\right] = \frac{\ln(s+1)}{s}$$

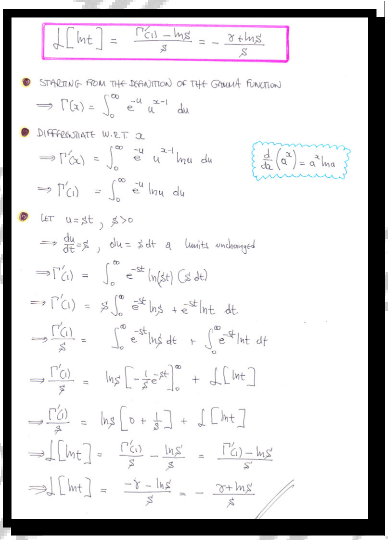
Question 42

By differentiating the integral definition of the Gamma function, $\Gamma(x)$, with respect to x, show that

$$\mathcal{L}[\ln t] = -\frac{\gamma + \ln s}{s}.$$

You may assume that $\Gamma'(1) = -\gamma$.

proof



$$\boxed{\mathcal{L}[\ln t] = \frac{\Gamma'(1) - \ln s}{s} = -\frac{\gamma + \ln s}{s}}$$

- STARTING FROM THE DEFINITION OF THE GAMMA FUNCTION

$$\Rightarrow \Gamma(x) = \int_0^\infty e^{-u} u^{x-1}\, du$$

- DIFFERENTIATE W.R.T. x

$$\Rightarrow \Gamma'(x) = \int_0^\infty e^{-u} u^{x-1} \ln u\, du$$

$\frac{d}{dx}(a^x) = a^x \ln a$

$$\Rightarrow \Gamma'(1) = \int_0^\infty e^{-u} \ln u\, du$$

- LET $u = st$, $s > 0$

$\Rightarrow \frac{du}{dt} = s$, $du = s\,dt$ & limits unchanged

$$\Rightarrow \Gamma'(1) = \int_0^\infty e^{-st} \ln(st)\,(s\,dt)$$

$$\Rightarrow \Gamma'(1) = s\int_0^\infty e^{-st}\ln s + e^{-st}\ln t\, dt$$

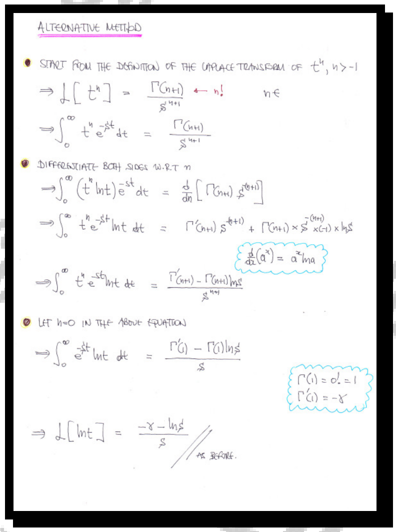
$$\Rightarrow \frac{\Gamma'(1)}{s} = \int_0^\infty e^{-st}\ln s\, dt + \int_0^\infty e^{-st}\ln t\, dt$$

$$\Rightarrow \frac{\Gamma'(1)}{s} = \ln s\left[-\frac{1}{s}e^{-st}\right]_0^\infty + \mathcal{L}[\ln t]$$

$$\Rightarrow \frac{\Gamma'(1)}{s} = \ln s\left[0 + \frac{1}{s}\right] + \mathcal{L}[\ln t]$$

$$\Rightarrow \mathcal{L}[\ln t] = \frac{\Gamma'(1)}{s} - \frac{\ln s}{s} = \frac{\Gamma'(1) - \ln s}{s}$$

$$\Rightarrow \mathcal{L}[\ln t] = \frac{-\gamma - \ln s}{s} = -\frac{\gamma + \ln s}{s}$$



ALTERNATIVE METHOD

- START FROM THE DEFINITION OF THE LAPLACE TRANSFORM OF t^n, $n > -1$

$$\Rightarrow \mathcal{L}[t^n] = \frac{\Gamma(n+1)}{s^{n+1}} \leftarrow n! \quad n \in$$

$$\Rightarrow \int_0^\infty t^n e^{-st}\, dt = \frac{\Gamma(n+1)}{s^{n+1}}$$

- DIFFERENTIATE BOTH SIDES W.R.T n

$$\Rightarrow \int_0^\infty (t^n \ln t) e^{-st}\, dt = \frac{d}{dn}\left[\Gamma(n+1)\, s^{-(n+1)}\right]$$

$$\Rightarrow \int_0^\infty t^n e^{-st} \ln t\, dt = \Gamma'(n+1)\, s^{-(n+1)} + \Gamma(n+1) \times s^{-(n+1)} \times (-1) \times \ln s$$

$\frac{d}{dx}(a^x) = a^x \ln a$

$$\Rightarrow \int_0^\infty t^n e^{-st} \ln t\, dt = \frac{\Gamma'(n+1) - \Gamma(n+1)\ln s}{s^{n+1}}$$

- LET $n = 0$ IN THE ABOVE EQUATION

$$\Rightarrow \int_0^\infty e^{-st} \ln t\, dt = \frac{\Gamma'(1) - \Gamma(1)\ln s}{s}$$

$\Gamma(1) = 0! = 1$
$\Gamma'(1) = -\gamma$

$$\Rightarrow \mathcal{L}[\ln t] = \frac{-\gamma - \ln s}{s}$$

AS BEFORE.

Question 43

$$\mathcal{L}\left[f(t)\right] = \bar{f}(s) \equiv \int_0^{\infty} f(t)\mathrm{e}^{-st}\ dt\,,\ t \geq 0\,.$$

a) Show from the above definition that if a is a non zero constant, then

$$\mathcal{L}\left[f(at)\right] = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right).$$

b) By taking the Laplace transform of Bessel's equation

$$t^2\frac{d^2x}{dt^2} + t\frac{dx}{dx} + \left(t^2 - n^2\right)x = 0\,,\ n \in \mathbb{N}\,,$$

and assuming further that $J_0(0) = 1$, show that

$$\mathcal{L}\left[J_0(t)\right] = \frac{1}{\sqrt{s^2+1}}\,.$$

c) Deduce in simplified form the Laplace transform of $J_0(at)$

$$\boxed{\mathcal{L}\left[J_0(at)\right] = \frac{1}{\sqrt{s^2+a^2}}}$$

a) $\mathcal{L}\left[f(at)\right] = \int_0^{\infty} f(at)\,e^{-st}\,dt$ (BY DEFINITION)

NOW BY A SUBSTITUTION $T = at$, $t = \frac{T}{a}$, $dt = \frac{1}{a}dT$, LIMITS UNCHANGED

$$\dots = \int_0^{\infty} f(T)\,e^{-s\left(\frac{T}{a}\right)}\,\frac{1}{a}\,dT = \frac{1}{a}\int_0^{\infty} f(T)\,e^{-\frac{s}{a}T}\,dT = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$$

b) • TAKE BESSEL'S EQUATION

$$\boxed{t^2\frac{d^2x}{dt^2} + t\frac{dx}{dt} + (t^2-n^2)x = 0}$$

• LET $n=0$

$$\Rightarrow t^2\frac{d^2x}{dt^2} + t\frac{dx}{dt} + t^2x = 0$$

$$\Rightarrow t\frac{d^2x}{dt^2} + \frac{dx}{dt} + tx = 0$$

• TAKE THE LAPLACE TRANSFORM OF THE O.D.E W.R.T t

$$\Rightarrow -\frac{d}{ds}\left[s^2\bar{x} - sx_0 - \dot{x}_0\right] + \left[s\bar{x} - x_0\right] - \frac{d}{ds}(\bar{x}) = 0$$

IRRELEVANT WHAT THIS IS AS IT WILL VANISH FROM THE DIFFERENTIATION

SO $x_0 = J_0(0) = 1$, SINCE $x(t) = J_0(t)$

$$\Rightarrow -\frac{d}{ds}\left[s^2\bar{x} - s - \dot{x}_0\right] + s\bar{x} - 1 - \frac{d\bar{x}}{ds} = 0$$

$$\Rightarrow -\left[2s\bar{x} + s^2\frac{d\bar{x}}{ds} - 1 + 0\right] + s\bar{x} - 1 - \frac{d\bar{x}}{ds} = 0$$

$$\Rightarrow -2s\bar{x} - s^2\frac{d\bar{x}}{ds} \cancel{+1} + s\bar{x} \cancel{-1} - \frac{d\bar{x}}{ds} = 0$$

$$\Rightarrow -s\bar{x} = (1+s^2)\frac{d\bar{x}}{ds}$$

$$\Rightarrow \boxed{\frac{d\bar{x}}{ds} = -\frac{s}{1+s^2}\bar{x}}$$

• SOLVE THE O.D.E BY SEPARATING VARIABLES

$$\Rightarrow \frac{1}{\bar{x}}\,d\bar{x} = -\frac{s}{s^2+1}\,ds$$

$$\Rightarrow \ln\bar{x} = -\frac{1}{2}\ln(s^2+1) + C$$

$$\Rightarrow \ln\bar{x} = \ln\left(\frac{A}{\sqrt{s^2+1}}\right)$$

$$\Rightarrow \boxed{\bar{x} = \frac{A}{\sqrt{s^2+1}}}$$

$$\boxed{\begin{aligned}\lim_{t\to 0} f(t) &= \lim_{s\to\infty}\left(s\bar{f}(s)\right)\\ \lim_{t\to\infty} f(t) &= \lim_{s\to 0}\left(s\bar{f}(s)\right)\end{aligned}}$$

HERE

$$\lim_{s\to\infty}\left[s\bar{x}\right] = \lim_{t\to 0}\left[f(t)\right] = \lim_{t\to 0}\left[J_0(t)\right] = 1$$

$$\lim_{s\to\infty}\left[\frac{As}{\sqrt{s^2+1}}\right] = 1$$

$$\boxed{A=1}$$

$$\Rightarrow \bar{x} = \frac{1}{\sqrt{s^2+1}}$$

$$\therefore\ \mathcal{L}\left[J_0(t)\right] = \frac{1}{\sqrt{s^2+1}}$$

c) • FROM PART (a)

$$\mathcal{L}\left[f(at)\right] = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$$

• FROM PART (b)

$$\mathcal{L}\left[J_0(t)\right] = \frac{1}{\sqrt{s^2+1}}$$

• COMBINING THESE RESULTS

$$\mathcal{L}\left[J_0(at)\right] = \frac{1}{a}\left[\frac{1}{\sqrt{\left(\frac{s}{a}\right)^2+1}}\right] = \frac{1}{a}\left(\frac{1}{\sqrt{\frac{s^2}{a^2}+1}}\right) = \frac{1}{a}\left(\frac{1}{\frac{\sqrt{s^2+a^2}}{a}}\right) = \frac{1}{a}\times\frac{a}{\sqrt{s^2+a^2}} = \frac{1}{\sqrt{s^2+a^2}}$$

Question 44

$$\mathcal{L}\left[f(t)\right] = \bar{f}(s) \equiv \int_0^{\infty} f(t)\mathrm{e}^{-st}\ dt\,,\ t \geq 0\,.$$

a) Show from the above definition that if k is a non zero constant, then

$$\mathcal{L}^{-1}\left[\bar{f}(ks)\right] = \frac{1}{k}f\left(\frac{t}{k}\right).$$

b) Show further that

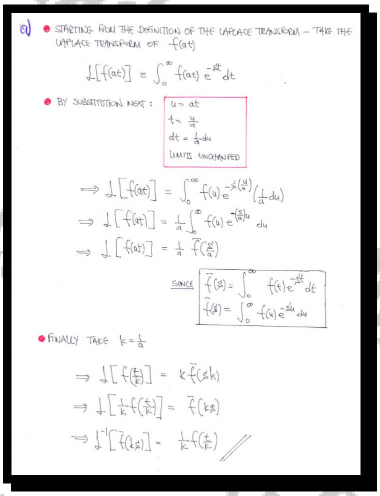
$$\mathcal{L}^{-1}\left[\frac{\bar{f}(s)}{s}\right] = \int_0^t f(u)\ du\,.$$

c) Given that $\mathcal{L}^{-1}\left[\mathrm{e}^{-\sqrt{s}}\right] = \dfrac{\mathrm{e}^{-\frac{1}{4t}}}{2t^{\frac{3}{2}}\sqrt{\pi}}$, use parts **(a)** and **(b)** to prove that

$$\mathcal{L}^{-1}\left[\frac{\mathrm{e}^{-\alpha\sqrt{s}}}{s}\right] = \mathrm{erfc}\left(\frac{\alpha}{2\sqrt{t}}\right),$$

where α is a positive constant.

proof



a) • STARTING FROM THE DEFINITION OF THE LAPLACE TRANSFORM – TAKE THE LAPLACE TRANSFORM OF $f(at)$

$$\mathcal{L}[f(at)] = \int_0^{\infty} f(at)\,e^{-st}\,dt$$

• BY SUBSTITUTION NEXT: $u = at$, $t = \frac{u}{a}$, $dt = \frac{1}{a}du$, LIMITS UNCHANGED

$$\Rightarrow \mathcal{L}[f(at)] = \int_0^{\infty} f(u)\,e^{-s\left(\frac{u}{a}\right)}\left(\tfrac{1}{a}du\right)$$

$$\Rightarrow \mathcal{L}[f(at)] = \frac{1}{a}\int_0^{\infty} f(u)\,e^{-\left(\frac{s}{a}\right)u}\,du$$

$$\Rightarrow \mathcal{L}[f(at)] = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$$

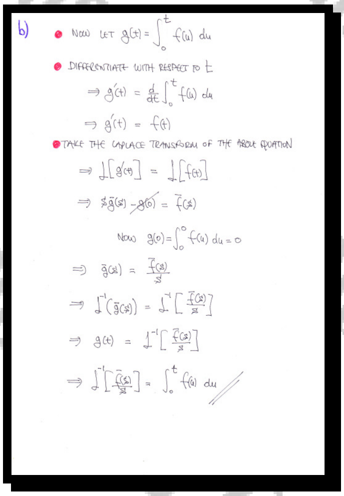
SINCE $\bar{f}(s) = \int_0^{\infty} f(t)e^{-st}dt$, $\bar{f}(s) = \int_0^{\infty} f(u)e^{-su}du$

• FINALLY TAKE $k = \frac{1}{a}$

$$\Rightarrow \mathcal{L}\left[f\left(\tfrac{t}{k}\right)\right] = k\bar{f}(sk)$$

$$\Rightarrow \mathcal{L}\left[\tfrac{1}{k}f\left(\tfrac{t}{k}\right)\right] = \bar{f}(ks)$$

$$\Rightarrow \mathcal{L}^{-1}\left[\bar{f}(ks)\right] = \frac{1}{k}f\left(\frac{t}{k}\right)$$



b) • NOW LET $g(t) = \int_0^t f(u)\,du$

• DIFFERENTIATE WITH RESPECT TO t

$$\Rightarrow g'(t) = \frac{d}{dt}\int_0^t f(u)\,du$$

$$\Rightarrow g'(t) = f(t)$$

• TAKE THE LAPLACE TRANSFORM OF THE ABOVE EQUATION

$$\Rightarrow \mathcal{L}[g'(t)] = \mathcal{L}[f(t)]$$

$$\Rightarrow s\bar{g}(s) - \cancel{g(0)} = \bar{f}(s)$$

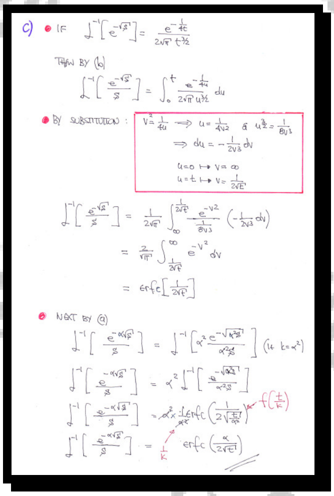
NOW $g(0) = \int_0^0 f(u)\,du = 0$

$$\Rightarrow \bar{g}(s) = \frac{\bar{f}(s)}{s}$$

$$\Rightarrow \mathcal{L}^{-1}(\bar{g}(s)) = \mathcal{L}^{-1}\left[\frac{\bar{f}(s)}{s}\right]$$

$$\Rightarrow g(t) = \mathcal{L}^{-1}\left[\frac{\bar{f}(s)}{s}\right]$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{\bar{f}(s)}{s}\right] = \int_0^t f(u)\,du$$



c) • IF $\mathcal{L}^{-1}\left[e^{-\sqrt{s}}\right] = \dfrac{e^{-\frac{1}{4t}}}{2\sqrt{\pi}\,t^{3/2}}$

THEN BY (b)

$$\mathcal{L}^{-1}\left[\frac{e^{-\sqrt{s}}}{s}\right] = \int_0^t \frac{e^{-\frac{1}{4u}}}{2\sqrt{\pi}\,u^{3/2}}\,du$$

• BY SUBSTITUTION: $v^2 = \frac{1}{4u} \Rightarrow u = \frac{1}{4v^2}$ & $u^{\frac{3}{2}} = \frac{1}{8v^3}$, $\Rightarrow du = -\frac{1}{2v^3}dv$, $u = 0 \mapsto v = \infty$, $u = t \mapsto v = \frac{1}{2\sqrt{t}}$

$$\mathcal{L}^{-1}\left[\frac{e^{-\sqrt{s}}}{s}\right] = \frac{1}{2\sqrt{\pi}}\int_{\infty}^{\frac{1}{2\sqrt{t}}} \frac{e^{-v^2}}{\frac{1}{8v^3}}\left(-\frac{1}{2v^3}dv\right)$$

$$= \frac{2}{\sqrt{\pi}}\int_{\frac{1}{2\sqrt{t}}}^{\infty} e^{-v^2}\,dv$$

$$= \mathrm{erfc}\left[\frac{1}{2\sqrt{t}}\right]$$

• NEXT BY (a)

$$\mathcal{L}^{-1}\left[\frac{e^{-\alpha\sqrt{s}}}{s}\right] = \mathcal{L}^{-1}\left[\alpha^2\frac{e^{-\sqrt{\alpha^2 s}}}{\alpha^2 s}\right] \quad (\text{i.e. } k = \alpha^2)$$

$$\mathcal{L}^{-1}\left[\frac{e^{-\alpha\sqrt{s}}}{s}\right] = \alpha^2\mathcal{L}^{-1}\left[\frac{e^{-\sqrt{\alpha^2 s}}}{\alpha^2 s}\right]$$

$$\mathcal{L}^{-1}\left[\frac{e^{-\alpha\sqrt{s}}}{s}\right] = \cancel{\alpha^2}\times\frac{1}{\cancel{\alpha^2}}\mathrm{erfc}\left(\frac{1}{2\sqrt{\frac{t}{\alpha^2}}}\right) \quad f\left(\tfrac{t}{k}\right)$$

$$\mathcal{L}^{-1}\left[\frac{e^{-\alpha\sqrt{s}}}{s}\right] = \mathrm{erfc}\left(\frac{\alpha}{2\sqrt{t}}\right) \quad \tfrac{1}{k}$$

Question 45

The Laplace transform $\bar{y}(s)$, of a function $y = y(t)$, $t \ge 0$ is given by

$$\bar{y}(s) = e^{-\sqrt{s}}, \quad s > 0.$$

a) Show that $\bar{y}(s)$ satisfies the differential equation

$$4s\,\bar{y}''(s) + 2\bar{y}'(s) - \bar{y}(s) = 0.$$

b) Hence show further that

$$4t^2 \frac{dy}{dt} + (6t - 1)\,y = 0.$$

c) Use parts **(a)** and **(b)** to prove that

$$y(t) = \mathcal{L}^{-1}\left(e^{-\sqrt{s}}\right) = \frac{e^{-\frac{1}{4t}}}{2t^{\frac{3}{2}}\sqrt{\pi}}.$$

$\boxed{\text{proof}}$

a) $\boxed{\bar{y}(s) = e^{-\sqrt{s}} = e^{-s^{\frac{1}{2}}}}$

- DIFFERENTIATE WITH RESPECT TO s

$$\bar{y}'(s) = -\tfrac{1}{2}s^{-\frac{1}{2}}e^{-s^{\frac{1}{2}}} = -\frac{1}{2s^{\frac{1}{2}}}\bar{y}(s) = \frac{-\bar{y}(s)}{2s^{\frac{1}{2}}}$$

$$\boxed{\bar{y}'(s) = \frac{-\bar{y}(s)}{2s^{\frac{1}{2}}}}$$

- DIFFERENTIATE ONCE MORE BY THE QUOTIENT RULE

$$\bar{y}''(s) = \frac{-2s^{\frac{1}{2}}\bar{y}'(s) + \bar{y}(s)s^{-\frac{1}{2}}}{4s}$$

$$4s\bar{y}''(s) = \frac{\bar{y}(s)}{s^{\frac{1}{2}}} - 2s^{\frac{1}{2}}\bar{y}'(s)$$

$$4s\bar{y}''(s) = \underbrace{-2\bar{y}'(s) + \bar{y}(s)}_{\text{FROM THE PREVIOUS "BOXED" EXPRESSION}}$$

$$4s\bar{y}''(s) + 2\bar{y}'(s) - \bar{y}(s) = 0 \quad //$$

b)
- NEXT WE CONSIDER HOW THE EXPRESSIONS IN THE ABOVE ODE CAN BE PRODUCED BY TAKING LAPLACE TRANSFORMS

$$\mathcal{L}[t^2y(t)] = \left(-\frac{d}{ds}\right)^2[\bar{y}(s)] = \bar{y}''(s)$$

$$\mathcal{L}[t\,y(t)] = -\frac{d}{ds}[\bar{y}(s)] = -\bar{y}'(s)$$

- TO PRODUCE THE TERM $s\bar{y}''(s)$ WE TRY

$$\mathcal{L}\left[t^2\frac{dy}{dt}\right] = \left(-\frac{d}{ds}\right)^2[s\bar{y}(s) - y_0] = \frac{d^2}{ds^2}[s\bar{y} - \overset{\text{CONSTANT}}{y_0}]$$

$$= \frac{d}{ds}[\bar{y}(s) + s\bar{y}'(s)] = \bar{y}'(s) + \bar{y}'(s) + s\bar{y}''(s)$$

$$= s\bar{y}''(s) + 2\bar{y}'(s)$$

- COLLECTING THE LAST 3 RESULTS

$$\bar{y}''(s) = \mathcal{L}[t^2y(t)]$$

$$\bar{y}'(s) = \mathcal{L}[-t\,y(t)]$$

$$s\bar{y}''(s) = \mathcal{L}\left[t^2\frac{dy}{dt}\right] - 2\bar{y}'(s)$$

- RETURNING TO THE O.D.E.

$$\Rightarrow 4s\bar{y}''(s) + 2\bar{y}'(s) - \bar{y}(s) = 0$$

$$\Rightarrow 4\left\{\mathcal{L}\left[t^2\frac{dy}{dt}\right] - 2\bar{y}'(s)\right\} + 2\bar{y}'(s) - \bar{y}(s) = 0$$

$$\Rightarrow 4\mathcal{L}\left[t^2\frac{dy}{dt}\right] - 6\bar{y}'(s) - \bar{y}(s) = 0$$

$$\Rightarrow 4\mathcal{L}\left[t^2\frac{dy}{dt}\right] - 6\mathcal{L}[-t\,y(t)] - \mathcal{L}[y(t)] = 0$$

- INVERTING THESE TRANSFORMS

$$\Rightarrow 4t^2\frac{dy}{dt} + 6ty - y = 0$$

$$\Rightarrow 4t^2\frac{dy}{dt} + (6t-1)y = 0 \quad //$$

c)
- SOLVING THE O.D.E

$$\Rightarrow 4t^2\frac{dy}{dt} = (1-6t)\,y$$

$$\Rightarrow \frac{1}{y}\,dy = \frac{1-6t}{4t^2}\,dt$$

$$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{1}{4t^2} - \frac{3}{2t}\,dt$$

$$\Rightarrow \ln y = -\frac{1}{4t} - \frac{3}{2}\ln t + C$$

$$\Rightarrow \ln y + \frac{3}{2}\ln t = -\frac{1}{4t} + C$$

$$\Rightarrow \ln\left(yt^{\frac{3}{2}}\right) = -\frac{1}{4t} + C$$

$$\Rightarrow \boxed{yt^{\frac{3}{2}} = Ae^{-\frac{1}{4t}}} \quad (A = e^C)$$

- TO EVALUATE THE CONSTANT WE PROCEED AS FOLLOWS

$$yt = \frac{Ae^{-\frac{1}{4t}}}{t^{1/2}}$$

$$\mathcal{L}[yt] = -\frac{d}{ds}[\mathcal{L}[y(t)]] = -\frac{d}{ds}[\bar{y}(s)] = -\frac{d}{ds}[e^{-\sqrt{s}}]$$

$$\boxed{\mathcal{L}[yt] = \frac{e^{-\sqrt{s}}}{2\sqrt{s}}}$$

AS $t \to \infty$ $\quad \dfrac{Ae^{-\frac{1}{4t}}}{t^{1/2}} \sim \dfrac{A}{t^{\frac{1}{2}}}$

$$\mathcal{L}\left[\frac{A}{t^{1/2}}\right] = \mathcal{L}\left[At^{-\frac{1}{2}}\right] = A\frac{(-\frac{1}{2})!}{s^{\frac{1}{2}}} = A\frac{\Gamma(\frac{1}{2})}{s^{\frac{1}{2}}}$$

$$\boxed{\mathcal{L}\left[\frac{A}{t^{1/2}}\right] \sim \frac{A\sqrt{\pi}}{s^{1/2}}}$$

AS $s \to 0$ $\quad \dfrac{e^{-\sqrt{s}}}{2\sqrt{s}} \sim \dfrac{1}{2s^{\frac{1}{2}}}$

$\therefore$ BY THE INITIAL/FINAL VALUE THEOREM

$$\frac{A\sqrt{\pi}}{s^{\frac{1}{2}}} = \frac{1}{2s^{1/2}}$$

$$A = \frac{1}{2\sqrt{\pi}}$$

- FINALLY WE OBTAIN

$$yt^{\frac{3}{2}} = Ae^{-\frac{1}{4t}}$$

$$y = \frac{Ae^{-\frac{1}{4t}}}{t^{3/2}}$$

$$y(t) = \mathcal{L}^{-1}\left[e^{-\sqrt{s}}\right] = \frac{e^{-\frac{1}{4t}}}{2\sqrt{\pi}\,t^{3/2}} \quad //$$

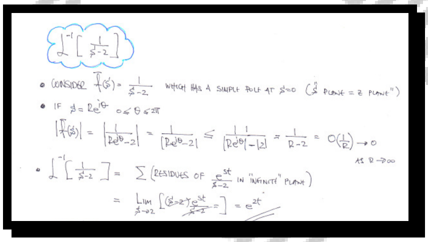
INVERSION BY COMPLEX VARIABLES

Question 1

Use the method of residues to find

$$\mathcal{L}^{-1}\left(\frac{1}{s-2}\right).$$

$$\boxed{\mathcal{L}^{-1}\left(\frac{1}{s-2}\right)=e^{2t}}$$



$\mathcal{L}^{-1}\left[\frac{1}{s-2}\right]$

- CONSIDER $\bar{f}(s)=\frac{1}{s-2}$ WHICH HAS A SIMPLE POLE AT $s=0$ ("s PLANE = z PLANE")
- IF $s=Re^{i\theta}$ $0 \le \theta \le 2\pi$

$$|\bar{f}(s)| = \left|\frac{1}{Re^{i\theta}-2}\right| = \frac{1}{|Re^{i\theta}-2|} \le \frac{1}{|Re^{i\theta}|-|2|} = \frac{1}{R-2} = O\left(\frac{1}{R}\right) \to 0 \text{ AS } R \to \infty$$

- $\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] = \sum \left(\text{RESIDUES OF } \frac{e^{st}}{s-2} \text{ IN "INFINITE" PLANE}\right)$

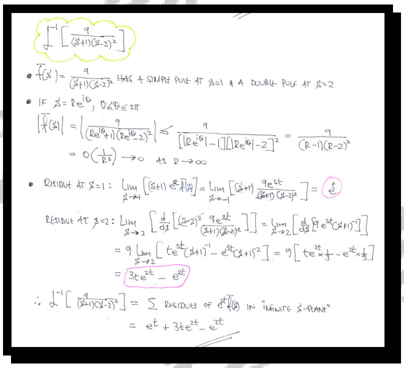
$$= \lim_{s \to 2}\left[\cancel{(s-2)}\frac{e^{st}}{\cancel{s-2}}\right] = e^{2t}$$

Question 2

Use the method of residues to find

$$\mathcal{L}^{-1}\left[\frac{9}{(s+1)(s-2)^2}\right].$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{9}{(s+1)(s-2)^2}\right]=e^{t}+(3t-1)e^{2t}}$$



$\mathcal{L}^{-1}\left[\frac{9}{(s+1)(s-2)^2}\right]$

- $\bar{f}(s)=\frac{9}{(s+1)(s-2)^2}$ HAS A SIMPLE POLE AT $s=1$ & A DOUBLE POLE AT $s=2$
- IF $s=Re^{i\theta}$, $0 \le \theta \le 2\pi$

$$|\bar{f}(s)| = \left|\frac{9}{(Re^{i\theta}+1)(Re^{i\theta}-2)^2}\right| \le \frac{9}{\left[|Re^{i\theta}|-1\right]\left[|Re^{i\theta}|-2\right]^2} = \frac{9}{(R-1)(R-2)^2}$$

$$= O\left(\frac{1}{R^3}\right) \to 0 \text{ AS } R \to \infty$$

- RESIDUE AT $s=1$: $\lim_{s \to -1}\left[(s+1)e^{st}\bar{f}(s)\right] = \lim_{s \to -1}\left[\cancel{(s+1)}\frac{9e^{st}}{\cancel{(s+1)}(s-2)^2}\right] = e^{t}$

RESIDUE AT $s=2$: $\lim_{s \to 2}\left[\frac{d}{ds}\left[\cancel{(s-2)^2}\frac{9e^{st}}{(s+1)\cancel{(s-2)^2}}\right]\right] = \lim_{s \to 2}\left[\frac{d}{ds}\left[9e^{st}(s+1)^{-1}\right]\right]$

$$= 9\lim_{s \to 2}\left[te^{st}(s+1)^{-1} - e^{st}(s+1)^{-2}\right] = 9\left[te^{2t}\times\frac{1}{3} - e^{2t}\times\frac{1}{9}\right]$$

$$= 3te^{2t} - e^{2t}$$

$\therefore \mathcal{L}^{-1}\left[\frac{9}{(s+1)(s-2)^2}\right] = \sum \text{RESIDUES OF } e^{st}\bar{f}(s) \text{ IN "INFINITE } s\text{-PLANE"}$

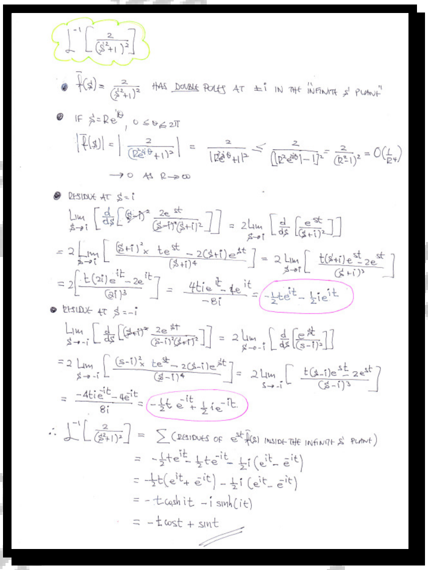
$$= e^{t} + 3te^{2t} - e^{2t}$$

Question 3

Use the method of residues to find

$$\mathcal{L}^{-1}\left[\frac{2}{\left(s^2+1\right)^2}\right].$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{2}{\left(s^2+1\right)^2}\right]=\sin t - t\cos t}$$



$\mathcal{L}^{-1}\left[\frac{2}{(s^2+1)^2}\right]$

- $\bar{f}(s)=\frac{2}{(s^2+1)^2}$ HAS DOUBLE POLES AT $\pm i$ IN THE "INFINITE s PLANE"
- IF $s=Re^{i\theta}$, $0 \leqslant \theta \leqslant 2\pi$

$|\bar{f}(s)|=\left|\frac{2}{(R^2e^{2i\theta}+1)^2}\right|=\frac{2}{|R^2e^{2i\theta}+1|^2}\leqslant\frac{2}{\left[|R^2e^{2i\theta}|-1\right]^2}=\frac{2}{(R^2-1)^2}=O\left(\frac{1}{R^4}\right)$

$\to 0$ AS $R\to\infty$

- RESIDUE AT $s=i$

$\lim_{s\to i}\left[\frac{d}{ds}\left[\cancel{(s-i)^2}\frac{2e^{st}}{\cancel{(s-i)^2}(s+i)^2}\right]\right]=2\lim_{s\to i}\left[\frac{d}{ds}\left[\frac{e^{st}}{(s+i)^2}\right]\right]$

$=2\lim_{s\to i}\left[\frac{(s+i)^2\times te^{st}-2(s+i)e^{st}}{(s+i)^4}\right]=2\lim_{s\to i}\left[\frac{t(s+i)e^{st}-2e^{st}}{(s+i)^3}\right]$

$=2\left[\frac{t(2i)e^{it}-2e^{it}}{(2i)^3}\right]=\frac{4tie^{it}-4e^{it}}{-8i}=-\frac{1}{2}te^{it}-\frac{1}{2}ie^{it}$

- RESIDUE AT $s=-i$

$\lim_{s\to -i}\left[\frac{d}{ds}\left[\cancel{(s+i)^2}\frac{2e^{st}}{(s-i)^2\cancel{(s+i)^2}}\right]\right]=2\lim_{s\to -i}\left[\frac{d}{ds}\left[\frac{e^{st}}{(s-i)^2}\right]\right]$

$=2\lim_{s\to -i}\left[\frac{(s-i)^2\times te^{st}-2(s-i)e^{st}}{(s-i)^4}\right]=2\lim_{s\to -i}\left[\frac{t(s-i)e^{st}-2e^{st}}{(s-i)^3}\right]$

$=\frac{-4tie^{-it}-4e^{-it}}{8i}=-\frac{1}{2}te^{-it}+\frac{1}{2}ie^{-it}$

$\therefore \mathcal{L}^{-1}\left[\frac{2}{(s^2+1)^2}\right]=\sum$ (RESIDUES OF $e^{st}\bar{f}(s)$ INSIDE THE INFINITE s PLANE)

$=-\frac{1}{2}te^{it}-\frac{1}{2}te^{-it}-\frac{1}{2}i\left(e^{it}-e^{-it}\right)$

$=-\frac{1}{2}t\left(e^{it}+e^{-it}\right)-\frac{1}{2}i\left(e^{it}-e^{-it}\right)$

$=-t\cosh it-i\sinh(it)$

$=-t\cos t+\sin t$

Question 4

Use complex integration to find the following inverse Laplace transform.

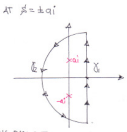
$$\mathcal{L}^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right],\ a>0\,.$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right]=\frac{t\sin at}{2a}}$$

- $f(t)=\frac{1}{2\pi i}\int_{\gamma_1}\bar{f}(s)e^{st}\,ds=\frac{1}{2\pi i}\int_{\gamma_1}\frac{se^{st}}{(s^2+a^2)^2}\,ds$

- THE INTEGRAND HAS TWO DOUBLE POLES AT $s=\pm ai$

IN THIS CASE THE INVERSION IS THE SUM OF THE TWO RESIDUES



SINCE

$$f(t)=\cancel{2\pi i}\sum_{\text{RESIDUES INSIDE CONTOUR}}\left(\frac{1}{\cancel{2\pi i}}\frac{se^{st}}{(s^2+a^2)^2}\right)$$

SINCE γ_2 DOES NOT CONTRIBUTE AS THE RADIUS TENDS TO INFINITY & γ_1 NEEDS TO GO FROM $-\infty$ TO ∞

- RESIDUE AT ai

$$\lim_{s\to ai}\frac{d}{ds}\left[\cancel{(s-ai)^2}\frac{se^{st}}{\cancel{(s-ai)^2}(s+ai)^2}\right]=\lim_{s\to ai}\left[\frac{d}{ds}\frac{se^{st}}{(s+ai)^2}\right]$$

$$=\lim_{s\to ai}\left[\frac{(s+ai)^2\left[e^{st}+ste^{st}\right]-2(s+ai)se^{st}}{(s+ai)^4}\right]$$

$$=\lim_{s\to ai}\left[\frac{(s+ai)e^{st}(1+st)-2se^{st}}{(s+ai)^3}\right]$$

$$=\lim_{s\to ai}\left[\frac{e^{st}\left[(s+ai)(1+st)-2s\right]}{(s+ai)^3}\right]=\frac{e^{ati}\left[(2ai)(1+ati)-2ai\right]}{(2ai)^3}$$

$$=\frac{e^{ati}\left(\cancel{2ai}-2a^2t-\cancel{2ai}\right)}{-8a^3i}=\frac{te^{ati}}{4ai}$$

- RESIDUE AT $-ai$

$$\lim_{s\to -ai}\left[\frac{d}{ds}\cancel{(s+ai)^2}\frac{se^{st}}{\cancel{(s+ai)^2}(s-ai)^2}\right]=\lim_{s\to -ai}\left[\frac{d}{ds}\frac{se^{st}}{(s-ai)^2}\right]$$

THE DIFFERENTIATION WILL BE IDENTICAL AS PREVIOUSLY ...

$$\lim_{s\to -ai}\left[\frac{e^{st}\left[(s+ai)(1+st)-2s\right]}{(s-ai)^3}\right]=\frac{e^{-ati}\left[-2ai(1-ati)+2ai\right]}{(-2ai)^3}$$

$$=\frac{e^{-ati}\left(\cancel{-2ai}-2a^2t+\cancel{2ai}\right)}{8a^3i}=e^{-ati}\times\frac{-2a^2t}{8a^3i}=-\frac{te^{-ati}}{4ai}$$

- COLLECTING & MANIPULATING THE RESIDUES

$$f(t)=\frac{te^{ati}}{4ai}-\frac{te^{-ati}}{4ai}=\frac{t}{4ai}\left[e^{ati}-e^{-ati}\right]$$

$$=\frac{t}{4ai}\left[2\sinh(ati)\right]=\frac{t}{4ai}\left[2i\sin(at)\right]$$

$$=\frac{2ti}{4ai}\sin(at)$$

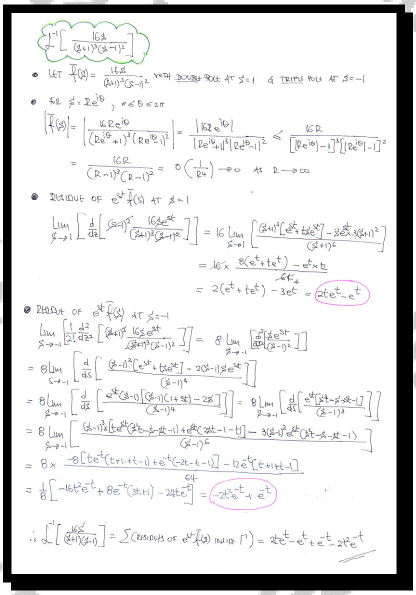
$$=\frac{t}{2a}\sin(at)$$

Question 5

Use the method of residues to find

$$\mathcal{L}^{-1}\left[\frac{16s}{(s+1)^3(s-1)^2}\right].$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{16s}{(s+1)^3(s-1)^2}\right]=(2t-1)\mathrm{e}^{t}+\left(1-2t^2\right)\mathrm{e}^{-t}}$$



$$\mathcal{L}^{-1}\left[\frac{16s}{(s+1)^3(s-1)^2}\right]$$

- LET $\bar{f}(s)=\frac{16s}{(s+1)^3(s-1)^2}$ WITH DOUBLE POLE AT $s=1$ & TRIPLE POLE AT $s=-1$
- FOR $s=Re^{i\theta}$, $0\leqslant\theta\leqslant 2\pi$

$$|\bar{f}(s)|=\left|\frac{16Re^{i\theta}}{(Re^{i\theta}+1)^3(Re^{i\theta}-1)^2}\right|=\frac{|16Re^{i\theta}|}{|Re^{i\theta}+1|^3|Re^{i\theta}-1|^2}\leqslant\frac{16R}{[|Re^{i\theta}|-1]^3[|Re^{i\theta}|-1]^2}$$

$$=\frac{16R}{(R-1)^3(R-1)^2}=O\left(\frac{1}{R^4}\right)\rightarrow 0 \text{ AS } R\rightarrow\infty$$

- RESIDUE OF $e^{st}\bar{f}(s)$ AT $s=1$

$$\lim_{s\to 1}\left[\frac{d}{ds}\left[\cancel{(s-1)^2}\frac{16se^{st}}{(s+1)^3\cancel{(s-1)^2}}\right]\right]=16\lim_{s\to 1}\left[\frac{(s+1)^3[e^{st}+tse^{st}]-se^{st}\times 3(s+1)^2}{(s+1)^6}\right]$$

$$=\cancel{16}\times\frac{8(e^t+te^t)-e^t\times 12}{\cancel{64}_4}$$

$$=2(e^t+te^t)-3e^t=2te^t-e^t$$

- RESIDUE OF $e^{st}\bar{f}(s)$ AT $s=-1$

$$\lim_{s\to -1}\left[\frac{1}{2!}\frac{d^2}{ds^2}\left[\cancel{(s+1)^3}\frac{16se^{st}}{\cancel{(s+1)^3}(s-1)^2}\right]\right]=8\lim_{s\to -1}\left[\frac{d^2}{ds^2}\left[\frac{se^{st}}{(s-1)^2}\right]\right]$$

$$=8\lim_{s\to -1}\left[\frac{d}{ds}\left[\frac{(s-1)^2[e^{st}+tse^{st}]-2(s-1)se^{st}}{(s-1)^4}\right]\right]$$

$$=8\lim_{s\to -1}\left[\frac{d}{ds}\left[\frac{e^{st}(s-1)[(s-1)(1+st)-2s]}{(s-1)^4}\right]\right]=8\lim_{s\to -1}\left[\frac{d}{ds}\left[\frac{e^{st}[s^2t-s-st-1]}{(s-1)^3}\right]\right]$$

$$=8\lim_{s\to -1}\left[\frac{(s-1)^3\times[te^{st}(s^2t-s-st-1)+e^{st}(2st-1-t)]-3(s-1)^2e^{st}(s^2t-s-st-1)}{(s-1)^6}\right]$$

$$=8\times\frac{-8[te^{-t}(t+1+t-1)+e^{-t}(-2t-t-1)]-12e^{-t}[t+1+t-1]}{64}$$

$$=\frac{1}{8}\left[-16t^2e^{-t}+8e^{-t}(3t+1)-24te^{-t}\right]=-2t^2e^{-t}+e^{-t}$$

$$\therefore\ \mathcal{L}^{-1}\left[\frac{16s}{(s+1)(s-1)}\right]=\sum(\text{RESIDUES OF } e^{st}\bar{f}(s) \text{ INSIDE } \Gamma)=2te^{t}-e^{t}+e^{-t}-2t^2e^{-t}$$

Question 6

Use complex variables to find

$$\mathcal{L}^{-1}\left[\frac{s^2-4s-5}{\left(s^2-4s+13\right)^2}\right].$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{s^2-4s-5}{\left(s^2-4s+13\right)^2}\right] = t\,\mathrm{e}^{2t}\cos 3t}$$

- $\mathcal{L}^{-1}\left[\dfrac{s^2-4s-5}{(s^2-4s+13)^2}\right]$ CAN BE FOUND SENSIBLY BY COMPLEX VARIABLES

- FACTORIZE THE DENOMINATOR

$$s^2-4s+13 = (s-2)^2+9 = (s-2)^2-(3i)^2 = (s-2-3i)(s-2+3i)$$

- AS THE TRANSFORM IS VERY STRAIGHT FORWARD, THE FUNCTION WHICH WAS TRANSFORMED IS SIMPLY

$$f(t) = \sum\left[\text{RESIDUES OF } \underbrace{\frac{(s^2-4s-5)e^{st}}{(s^2-4s+13)^2}}_{\text{DOUBLE POLES AT } 2\pm 3i}\right]$$

- RESIDUE AT DOUBLE POLE AT $2+3i$

$$\lim_{s\to(2+3i)}\left[\frac{d}{ds}\left[\cancel{(s-2-3i)^2}\frac{(s^2-4s-5)e^{st}}{\cancel{(s-2-3i)^2}(s-2+3i)^2}\right]\right]$$

$$\lim_{s\to(2+3i)}\left[\frac{(s-2+3i)^2\left[(2s-4)e^{st}+te^{st}(s^2-4s-5)\right]-e^{st}(s^2-4s-5)\times 2(s-2+3i)}{(s-2+3i)^4}\right]$$

$$\lim_{s\to(2+3i)}\left[\frac{(s-2+3i)e^{st}\left[2s-4+t(s^2-4s-5)\right]-2e^{st}(s^2-4s-5)}{(s-2+3i)^3}\right]$$

$$\boxed{\begin{aligned}&(2+3i)^2-4(2+3i)-5 = 4+\cancel{12i}-9-8-\cancel{12i}-5 = -18\\&(2+3i-2+3i)^3 = (6i)^3 = -216i\end{aligned}}$$

$$= \frac{6i\,e^{(2+3i)t}\left(4+6i-4+t(-18)\right)-2e^{(2+3i)t}(-18)}{-216i}$$

$$= \frac{6i\,e^{(2+3i)t}(6i-18t)+36e^{(2+3i)t}}{-216i} = \frac{6e^{(2+3i)t}\left[\cancel{-6}-18ti+\cancel{6}\right]}{-216i}$$

$$= \frac{-108ti\,e^{(2+3i)t}}{-216i} = \frac{1}{2}te^{2t}e^{3ti}$$

- RESIDUE OF THE DOUBLE POLE AT $s=2-3i$

$$\lim_{s\to 2-3i}\left[\frac{d}{ds}\left[(s-2+3i)^2\frac{(s^2-4s-5)e^{st}}{\cancel{(s-2+3i)^2}(s-2-3i)^2}\right]\right]$$

THE SIMPLIFIED DIFFERENTIATION IS IDENTICAL ....

$$\lim_{s\to 2-3i}\left[\frac{(s-2-3i)e^{st}\left[2s-4+t(s^2-4s-5)\right]-2e^{st}(s^2-4s-5)}{(s-2-3i)^3}\right]$$

$$\boxed{\begin{aligned}&(2-3i)^2-4(2-3i)-5 = 4-\cancel{12i}-9-8+\cancel{12i}-5 = -18\\&(2-3i-2-3i)^3 = (-6i)^3 = 216i\end{aligned}}$$

$$= \frac{6i\,e^{(2-3i)t}\left[\cancel{4}-6i\cancel{-4}+t(-18)\right]-2e^{(2-3i)t}(-18)}{216i}$$

$$= \frac{-6i\,e^{(2-3i)t}(6i-18t)+36e^{(2-3i)t}}{216i} = \frac{(-36+108ti)e^{(2-3i)t}+36e^{(2-3i)t}}{216i}$$

$$= \frac{108ti\,e^{(2-3i)t}}{216i} = \frac{1}{2}te^{2t}e^{-3ti}$$

- THUS WE HAVE $f(t)$ NOW AFTER SOME TIDYING

$$f(t) = \frac{1}{2}te^{2t}e^{3ti}+\frac{1}{2}te^{2t}e^{-3ti}$$

$$= te^{2t}\left[\frac{1}{2}e^{3ti}+\frac{1}{2}e^{-3ti}\right]$$

$$= te^{2t}\cosh(3ti)$$

$$= te^{2t}\cos 3t$$

Question 7

$$\bar{f}(s) = \frac{e^{-s\pi}}{\left(s^2+1\right)^2}.$$

Use complex variable methods to invert the above Laplace transform.

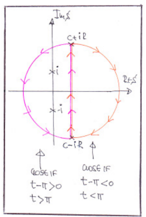
Use a detailed method, describing briefly each stage in the workings.

Give the final answer in terms of Heaviside functions.

$$\boxed{f(t) = \frac{1}{2}\mathrm{H}(t-\pi)\left[\sin(t-\pi)-(t-\pi)\cos(t-\pi)\right]}$$

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \bar{f}(s)\, e^{st}\, ds = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{-\pi s}}{(s^2+1)^2} e^{st}\, ds$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{s(t-\pi)}}{(s^2+1)^2}\, ds = \dots$$



CALCULATE THE RESIDUES AT EACH OF THE DOUBLE POLES AT $\pm i$ (IGNORE THE $2\pi i$ IN FRONT)

- RESIDUE AT i

$$\lim_{s\to i} \frac{d}{ds}\left[\cancel{(s-i)^2}\frac{e^{s(t-\pi)}}{\cancel{(s-i)^2}(s+i)^2}\right]$$

$$= \lim_{s\to i}\left[\frac{(s+i)^2(t-\pi)e^{s(t-\pi)} - e^{s(t-\pi)}\times 2(s+i)}{(s+i)^4}\right]$$

$$= \lim_{s\to i}\left[\frac{(t-\pi)(s+i)e^{s(t-\pi)} - 2e^{s(t-\pi)}}{(s+i)^3}\right]$$

$$= \frac{(t-\pi)(2i)e^{i(t-\pi)} - 2e^{i(t-\pi)}}{-8i} = -\frac{1}{4}(t-\pi)e^{i(t-\pi)} - \frac{1}{4}ie^{i(t-\pi)}$$

- RESIDUE AT $-i$

$$\lim_{s\to -i} \frac{d}{ds}\left[\cancel{(s+i)^2}\frac{e^{s(t-\pi)}}{\cancel{(s+i)^2}(s-i)^2}\right] = \lim_{s\to -i}\left[\frac{(s-i)^2(t-\pi)e^{s(t-\pi)} - e^{s(t-\pi)}\times 2(s-i)}{(s-i)^4}\right]$$

$$= \lim_{s\to -i}\left[\frac{(s-i)(t-\pi)e^{s(t-\pi)} - 2e^{s(t-\pi)}}{(s-i)^3}\right] = \frac{-2i(t-\pi)e^{-i(t-\pi)} - 2e^{-i(t-\pi)}}{8i}$$

$$= -\frac{1}{4}(t-\pi)e^{-i(t-\pi)} + \frac{1}{4}ie^{-i(t-\pi)}$$

RETURNING TO THE INVERSION

- IF $t<\pi$ $\quad f(t)=0$

(THE SUM OF RESIDUES IS ZERO SO INTEGRAL IS ZERO. BUT THE ARC DOES NOT CONTRIBUTE AS $R\to\infty$, SO THE STRAIGHT LINE THROUGH C FROM $-\infty$ TO ∞, WHICH GIVES THE INVERSION MUST ALSO BE ZERO)

- IF $t>\pi$ $\quad$ ARC AGAIN DOES NOT CONTRIBUTE BUT THIS TIME THE STRAIGHT LINE CONTRIBUTION MUST EQUAL TO $2\pi i\times$ SUM OF RESIDUES

$$\therefore f(t) = 2\pi i \sum \text{RESIDUES}$$

$$= \cancel{2\pi i}\times\frac{1}{\cancel{2\pi i}}\left[-\frac{1}{4}(t-\pi)e^{i(t-\pi)} - \frac{1}{4}(t-\pi)e^{-i(t-\pi)} - \frac{1}{4}ie^{i(t-\pi)} + \frac{1}{4}ie^{-i(t-\pi)}\right]$$

WE JUST LEFT IT OUT OF THE RESIDUE CALCULATIONS

$$= -\frac{1}{4}(t-\pi)\left[e^{i(t-\pi)} + e^{-i(t-\pi)}\right] - \frac{1}{4}i\left[e^{i(t-\pi)} - e^{-i(t-\pi)}\right]$$

$$= -\frac{1}{4}(t-\pi)\times 2\cosh[i(t-\pi)] - \frac{1}{4}i\times 2\sinh[i(t-\pi)]$$

$$= -\frac{1}{2}(t-\pi)\cos(t-\pi) + \frac{1}{2}\sin(t-\pi)$$

& SINCE THIS HOLDS FOR $t>\pi$

$$f(t) = \frac{1}{2}H(t-\pi)\sin(t-\pi) - \frac{1}{2}H(t-\pi)(t-\pi)\cos(t-\pi)$$

$$f(t) = \frac{1}{2}H(t-\pi)\left[\sin(t-\pi) - (t-\pi)\cos(t-\pi)\right]$$

Question 8

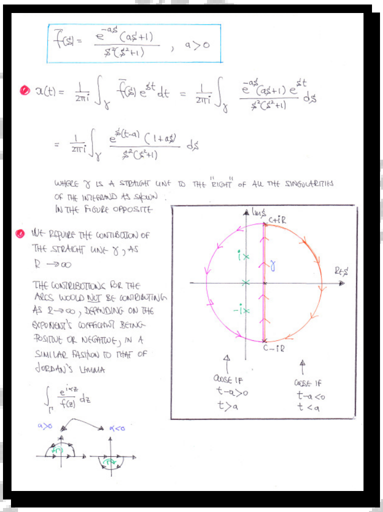
$$\bar{f}(s) = \frac{(as+1)\mathrm{e}^{-as}}{s^2\left(s^2+1\right)}, \quad a > 0 .$$

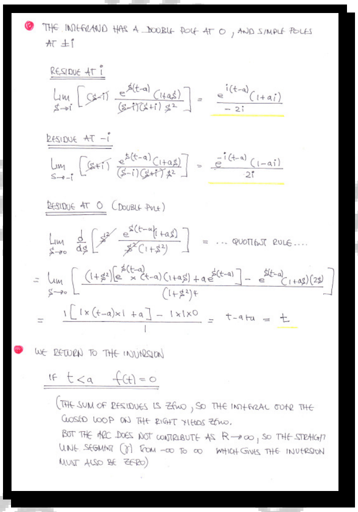
Use complex variable methods to invert the above Laplace transform.

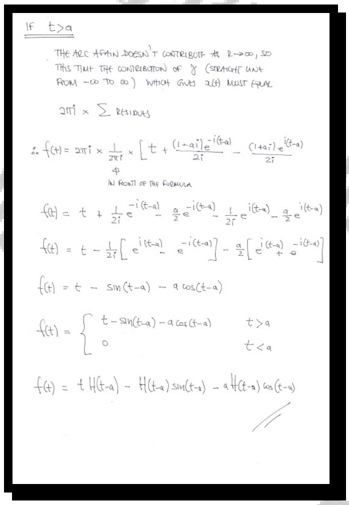
$$\mathcal{L}\left[\bar{f}(s)\right] = t\,\mathrm{H}(t-a) - \mathrm{H}(t-a)\sin(t-a) + a\,\mathrm{H}(t-a)\cos(t-a)$$

Use a detailed method, describing briefly each stage in the workings.

proof







Question 9

$$\bar{f}(s)=\frac{s^3+s^2+1-\mathrm{e}^{-s\pi}}{s^2\left(s^2+1\right)}.$$

Use complex variable methods to invert the above Laplace transform.

Use a detailed method, describing briefly each stage in the workings.

$$\boxed{f(t)=\begin{cases}0 & t<0\\ t+\cos t & 0\le t\le\pi\\ \pi+\cos t-\sin t & t>\pi\end{cases}}$$

$$\boxed{\bar{f}(s)=\frac{s^3+s^2+1-e^{-s\pi}}{s^2(s^2+1)}=\frac{s^3+s^2+1}{s^2(s^2+1)}-\frac{e^{-s\pi}}{(s^2+1)s^2}}$$

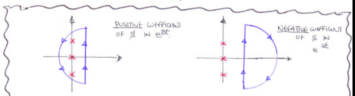
BY CONTOUR INTEGRATION

- $f(t)=\frac{1}{2\pi i}\int_{\gamma_1}\frac{s^3+s^2+1}{s^2(s^2+1)}e^{st}\,ds-\frac{1}{2\pi i}\int_{\gamma_2}\frac{e^{-s\pi}}{s^2(s^2+1)}e^{st}\,ds$
- BOTH INTEGRANDS HAVE A DOUBLE POLE AT $s=0$ & SIMPLE POLES AT $\pm i$

PICK A STRAIGHT LINE TO THE "RIGHT" OF ALL SINGULARITIES SAY $c=1$ IN THE BROMWICH FORMULA

$$\boxed{f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{st}\bar{f}(s)\,ds}$$

- THE WAY WE CLOSE THE CONTOUR TO THE LEFT OR RIGHT DEPENDS ON t SO THAT THE CONTRIBUTION OF THE ARC IS ZERO (IN A JORDAN LEMMA LIKE FASHION)



- IF $t<0$ WE CLOSE TO THE RIGHT (1ST INTEGRAL)
  WE CLOSE TO THE RIGHT (2ND INTEGRAL).
  $e^{-s\pi}e^{st}=e^{(t-\pi)s}$
  $t-\pi$ WILL BE LESS THAN ZERO
- IF $0<t<\pi$ WE CLOSE TO THE LEFT (1ST INTEGRAL)
  WE CLOSE TO THE RIGHT (2ND INTEGRAL)
  $t-\pi$ STILL NEGATIVE
- IF $t>0$ WE CLOSE BOTH TO THE LEFT

- CALCULATE THE RESIDUES AT EACH POLE FOR EACH INTEGRAND.

$$\boxed{f(s)=\frac{e^{st}(1+s^2+s^3)}{s^2(s+i)(s-i)}}$$

- AT $s=0$: $\lim_{s\to0}\frac{d}{ds}\left[\cancel{s^2}\frac{e^{st}(1+s^2+s^3)}{\cancel{s^2}(1+s^2)}\right]=\lim_{s\to0}\frac{d}{ds}\left[\frac{e^{st}(1+s^2+s^3)}{1+s^2}\right]$

$=\lim_{s\to0}\left[\frac{(1+s^2)\left[te^{st}(1+s^2+s^3)+e^{st}(2s+3s^2)\right]-e^{st}(1+s^2+s^3)(2s)}{(1+s^2)^2}\right]$

$=t$

- AT $s=i$: $\lim_{s\to i}\left[\cancel{(s-i)}\frac{e^{st}(1+s^2+s^3)}{s^2\cancel{(s-i)}(s+i)}\right]=\frac{e^{it}(1-1-i)}{-1(2i)}=\frac{-ie^{it}}{-2i}$

$=\frac{1}{2}e^{it}$

- AT $s=-i$: $\lim_{s\to -i}\left[\cancel{(s+i)}\frac{e^{st}(1+s^2+s^3)}{s^2(s-i)\cancel{(s+i)}}\right]=\frac{e^{-it}(1-1+i)}{-(-2i)}=\frac{ie^{-it}}{2i}$

$=\frac{1}{2}e^{-it}.$

$$\boxed{f(s)=\frac{e^{st}e^{-s\pi}}{s^2(s^2+1)}=\frac{e^{(t-\pi)s}}{s^2(s^2+1)}}$$

- AT $s=0$: $\lim_{s\to0}\frac{d}{ds}\left[\cancel{s^2}\frac{e^{(t-\pi)s}}{\cancel{s^2}(s^2+1)}\right]=\lim_{s\to0}\frac{d}{ds}\left[\frac{e^{(t-\pi)s}}{s^2+1}\right]$

$\lim_{s\to0}\left[\frac{(t-\pi)e^{(t-\pi)s}(s^2+1)-e^{(t-\pi)s}(2s)}{(s^2+1)^2}\right]=t-\pi$

- AT $s=i$: $\lim_{s\to i}\left[\cancel{(s-i)}\frac{e^{(t-\pi)s}}{s^2\cancel{(s-i)}(s+i)}\right]=\frac{e^{i(t-\pi)}}{-2i}=-\frac{1}{2i}e^{i(t-\pi)}$

- AT $s=-i$: $\lim_{s\to -i}\left[\cancel{(s+i)}\frac{e^{(t-\pi)s}}{s^2(s-i)\cancel{(s+i)}}\right]=\frac{e^{-i(t-\pi)}}{-(-2i)}=\frac{1}{2i}e^{-i(t-\pi)}$

- NOW THE INVERSION FORMULA

$$f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\bar{f}(s)e^{st}$$

- IF $t<0$ WE CLOSE BOTH TO THE RIGHT SO NO SINGULARITIES INSIDE

$f(t)=\frac{1}{2\pi i}\times0$ BY CAUCHY THEOREM

- IF $0<t<\pi$ FIRST INTEGRAL CLOSES TO THE LEFT & SECOND TO THE RIGHT (NO CONTRIBUTION)

$\therefore f(t)=\frac{1}{2\pi i}\times2\pi i\sum(\text{RESIDUES INSIDE})$

$=\frac{1}{2}e^{it}+\frac{1}{2}e^{-it}+t=\cosh(it)+t=\cos t+t$

- IF $t>\pi$ BOTH INTEGRAL PATHS CONTRIBUTE AS BOTH ARE "CLOSED TO THE LEFT"

$f(t)=2\pi i\left[\underbrace{\frac{1}{2\pi i}\sum\text{RESIDUES}}_{\text{FIRST INTEGRAL}}-\underbrace{\frac{1}{2\pi i}\sum\text{RESIDUES}}_{\text{SECOND INTEGRAL}}\right.$

$\Rightarrow f(t)=\sum\text{RESIDUES OF 1ST INTEGRAL}-\sum\text{RESIDUES OF 2ND INTEGRAL}$

$\Rightarrow f(t)=\underbrace{\left[\cos t+t\right]}_{\text{FROM BEFORE}}-\left[(t-\pi)-\frac{1}{2i}e^{i(t-\pi)}+\frac{1}{2i}e^{-i(t-\pi)}\right]$

$\Rightarrow f(t)=\cos t+\pi+\frac{1}{2i}\left[e^{i(t-\pi)}-e^{-i(t-\pi)}\right]$

$\Rightarrow f(t)=\cos t+\pi+\frac{1}{i}\sinh\left[i(t-\pi)\right]=\pi+\cos t+\sin\left[t-\pi\right]$

$\Rightarrow f(t)=\pi+\cos t-\sin t$

$$\therefore f(t)=\begin{cases}0 & t<0\\ t+\cos t & 0\leqslant t\leqslant\pi\\ \pi+\cos t-\sin t & t>\pi\end{cases}$$

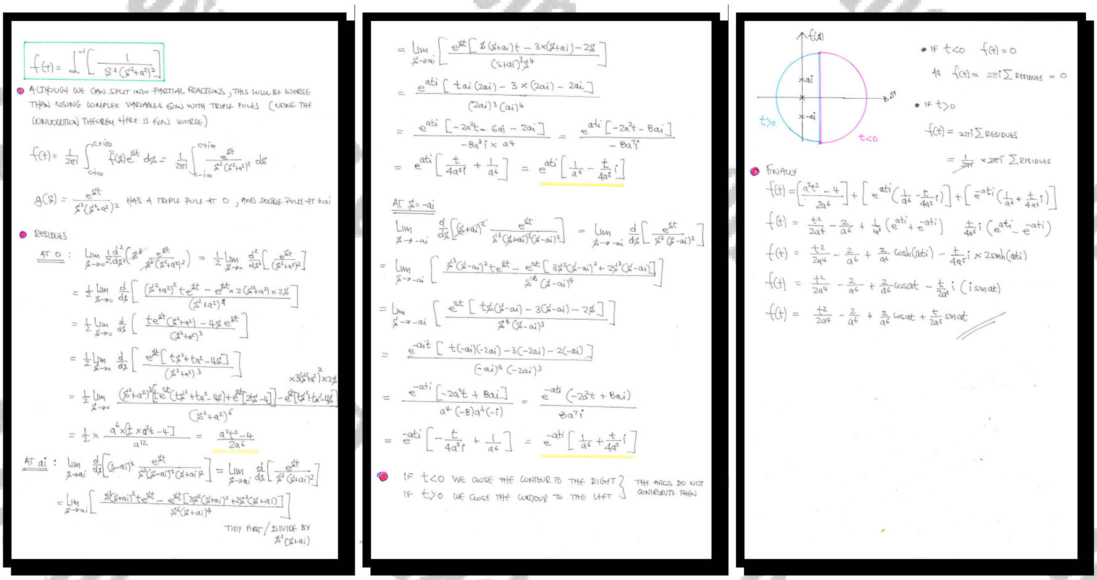
Question 10

Given that a is a positive constant, use complex variable methods to find the following inverse Laplace transform.

$$\mathcal{L}^{-1}\left[\frac{1}{s^3\left(s^2+a^2\right)^2}\right].$$

Use a detailed method, describing briefly each stage in the workings.

$$\boxed{f(t)=\frac{t^2}{2a^4}-\frac{2}{a^6}+\frac{2}{a^6}\cos at+\frac{t}{2a^5}\sin at}$$



Question 11

Use complex variable methods to find the following inverse Laplace transform.

$$\mathcal{L}^{-1}\left[\ln\left[\frac{1+s^2}{s(s+1)}\right]\right].$$

Use a detailed method, describing briefly each stage in the workings.

$$\boxed{f(t)=\frac{1}{t}\left[1+\mathrm{e}^{-t}-2\cos t\right]}$$

$$\boxed{f(t)=\mathcal{L}^{-1}\left[\ln\left(\frac{1+s^2}{s(s+1)}\right)\right]}$$

- BY COMPLEX VARIABLE

$$f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\ln\left(\frac{1+s^2}{s(s+1)}\right)\,ds$$

- RATHER THAN CLOSING THE CONTOUR & START COMPUTING RESIDUES, WE MAY PROCEED BY PARTS

$\ln\left(\frac{1+s^2}{s(s+1)}\right)=\ln(s^2+1)-\ln s-\ln(s+1)$	$\frac{2s}{s^2+1}-\frac{1}{s}-\frac{1}{s+1}$
$\frac{1}{t}e^{st}$	e^{st}

$$\Rightarrow f(t)=\frac{1}{2\pi i}\frac{1}{t}\left[e^{st}\ln\left(\frac{1+s^2}{s(s+1)}\right)\right]_{c-i\infty}^{c+i\infty}-\frac{1}{2\pi i}\times\frac{1}{t}\int_{c-i\infty}^{c+i\infty}e^{st}\left[\frac{2s}{s^2+1}-\frac{1}{s}-\frac{1}{s+1}\right]ds$$

$$\Rightarrow f(t)=\frac{1}{2\pi i}\times\frac{1}{t}\left[e^{st}\ln\left(\frac{1+s^2}{s(s+1)}\right)\right]_{c-i\infty}^{c+i\infty}-\frac{1}{t}\mathcal{L}^{-1}\left[\frac{2s}{s^2+1}-\frac{1}{s}-\frac{1}{s+1}\right]$$

$$\Rightarrow f(t)=\frac{1}{2\pi i}\times\frac{1}{t}\left[e^{st}\ln\left(\frac{1+s^2}{s(s+1)}\right)\right]_{c-i\infty}^{c+i\infty}-\frac{1}{t}\left[2\cos t-1-e^{-t}\right]$$

- NOW LOOKING AT THE CONTRIBUTION OF THE "SQUARE" BRACKET
LET $s=c\pm iR$ AND LET $R\to\infty$

$$e^{t(c+iR)}\ln\left[\frac{1+(c+iR)^2}{(c+iR)^2+(c+iR)}\right]-e^{t(c-iR)}\ln\left[\frac{1+(c-iR)^2}{(c-iR)^2+(c-iR)}\right]$$

$s(s+1)=s^2+s$

$$=e^{t}\left[e^{iR}\ln\left[\frac{1+c^2+2cRi-R^2}{c^2+2cRi-R^2+c+iR}\right]-e^{-iR}\ln\left[\frac{1+c^2-2cRi-R^2}{c^2-2cRi-R+c-iR}\right]\right]$$

- NOW IF WE LOOK AT THE MODULUS OF THIS EXPRESSION AS $R\to\infty$
  - FIRSTLY $\left|e^{\pm iR}\right|=1$ FOR REAL R
  - THE LEADING POWERS INSIDE THE ARGUMENTS OF THE LOGS IS R^2 (TOP|BOTTOM)
    I.E $\ln\left[\frac{O(R^2)}{O(R^2)}\right]\longrightarrow\ln 1\longrightarrow 0$
    $\ln\left(\frac{-R^2+\dots}{-R^2+\dots}\right)$
- SO AS $R\to\infty$ THIS VANISHES AND THUS

$$f(t)=\frac{1}{t}\left[1+e^{-t}-2\cos t\right]$$

Question 12

The function $y = f(t)$, $t \geq 0$ satisfies

$$\mathcal{L}\left[f(t)\right] = \frac{s}{s^4+1}.$$

Use complex variable methods to show that

$$f(t) = \sin\left(\frac{t}{\sqrt{2}}\right)\sinh\left(\frac{t}{\sqrt{2}}\right).$$

Use a detailed method, describing briefly each stage in the workings.

proof

$$\boxed{\mathcal{L}\left[f(t)\right] = \frac{s}{s^4+1}}$$

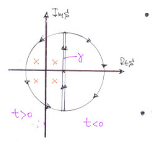
- WE SHALL USE THE STANDARD INVERSION FORMULA

$$\boxed{f(t) = \frac{1}{2\pi i}\int_{\gamma} \bar{f}(s)\,e^{st}\,ds},$$ WHERE γ IS AN INFINITE VERTICAL LINE TO THE "RIGHT" OF ALL THE SINGULARITIES OF $\bar{f}(s)$

- $(s) = \frac{se^{st}}{s^4+1}$ HAS SIMPLE POLES AT $s^4 = -1 = e^{i(\pi + 2k\pi)} = e^{i\pi(2k+1)}$

$$s = e^{i\frac{\pi}{4}(2k+1)}$$

$$s = e^{i\frac{\pi}{4}},\ e^{i\frac{3\pi}{4}},\ e^{-i\frac{\pi}{4}},\ e^{-i\frac{3\pi}{4}}$$



- FOR $t<0$ WE CLOSE THE CONTOUR TO THE "RIGHT" OF γ & FOR $t>0$ WE CLOSE THE CONTOUR TO THE LEFT OF γ. THIS ENSURES THAT AS THE RADIUS OF THE SEMICIRCLE GETS INFINITE, THE ARC DOES <u>NOT</u> CONTRIBUTE (LIKE IN JORDAN'S LEMMA)
- FOR $t<0$, γ PRODUCES ZERO BY CAUCHY'S THEOREM, AS THERE ARE NO POLES INSIDE AND THE ARC DOES NOT CONTRIBUTE

- FOR $t>0$

$$f(t) = \frac{1}{2\pi i}\int_{\gamma}\left(\frac{s}{s^4+1}\right)e^{st}\,ds = \frac{1}{2\pi i}\times\left[2\pi i \sum \text{RESIDUES OF } \frac{se^{st}}{1+s^4}\right].$$

- CALCULATE THE RESIDUES AT THESE POLES USING A GENERAL METHOD

<u>LET A POLE BE AT $s = s_0$</u>

$$\lim_{s\to s_0}\left[\frac{(s-s_0)se^{st}}{1+s^4}\right]$$ WHICH GIVES $\frac{0}{0}$ AS $s-s_0$ IS A FACTOR OF THE DENOMINATOR

<u>BY L'HOSPITAL'S RULE</u>

$$\ldots = \lim_{s\to s_0}\left[\frac{\frac{d}{ds}\left[(s-s_0)se^{st}\right]}{\frac{d}{ds}\left[1+s^4\right]}\right] = \lim_{s\to s_0}\left[\frac{se^{st} + (s-s_0)e^{st} + (s-s_0)tse^{st}}{4s^3}\right]$$

$$= \frac{s_0e^{s_0t}}{4s_0^3} = \boxed{\frac{e^{s_0t}}{4s_0^2}}$$

- OBTAINING THE RESIDUES AT EACH OF THE FOUR POLES

$$s_0 = e^{i\frac{\pi}{4}},\quad \frac{e^{te^{i\frac{\pi}{4}}}}{4e^{i\frac{\pi}{2}}} = \frac{e^{t\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)}}{4i}$$

$$s_0 = e^{-i\frac{\pi}{4}},\quad \frac{e^{te^{-i\frac{\pi}{4}}}}{4e^{-i\frac{\pi}{2}}} = \frac{e^{t\left(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)}}{-4i}$$

$$s_0 = e^{i\frac{3\pi}{4}},\quad \frac{e^{te^{\frac{3\pi}{4}i}}}{4e^{\frac{3\pi}{2}i}} = \frac{e^{t\left(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)}}{-4i}$$

$$s_0 = e^{-i\frac{3\pi}{4}},\quad \frac{e^{te^{-\frac{3\pi}{4}i}}}{4e^{-\frac{3\pi}{2}i}} = \frac{e^{t\left(-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)}}{4i}$$

- $f(t) = \sum$ (RESIDUES AT THE 4 POLES)

$$= \frac{1}{4i}\left[e^{t\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)} - e^{t\left(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)} - e^{t\left(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)} + e^{t\left(-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)}\right]$$

$$= \frac{1}{4i}\left[\left(e^{at}e^{iat} - e^{at}e^{-iat}\right) + \left(e^{-at}e^{-iat} - e^{-at}e^{iat}\right)\right]$$

$$\boxed{\text{WHERE } a = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}}$$

$$= \frac{1}{4i}\left[e^{at}\left(e^{iat} - e^{-iat}\right) - e^{-at}\left(e^{ati} - e^{-ati}\right)\right]$$

$$= \frac{1}{4i}\left(e^{ati} - e^{-ati}\right)\left(e^{at} - e^{-at}\right)$$

$$= \frac{1}{4i}\left(2\sinh(ati)\right)\left(2\sinh(at)\right)$$

$$= \frac{1}{4i}\times 2i\sin(at)\times 2\sinh at$$

$$= \sin at\,\sinh at$$

$$= \sin\left(\frac{t}{\sqrt{2}}\right)\sinh\left(\frac{t}{\sqrt{2}}\right)$$

$$\therefore f(t) = \sinh\left(\frac{t}{\sqrt{2}}\right)\sinh\left(\frac{t}{\sqrt{2}}\right) \quad //$$

Question 13

The Bromwich integral for inverting a Laplace transform $\bar{f}(s)$ is given by

$$f(t) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma} \mathrm{e}^{st}\, \bar{f}(s)\, ds .$$

a) Describe briefly the contour used in this integral and the general method used to invert the transform.

b) Given that a is a positive constant, show that

$$\mathcal{L}^{-1}\left[\mathrm{e}^{-a\sqrt{s}}\right] = \frac{a}{2t^{\frac{3}{2}}\sqrt{\pi}} \exp\left(-\frac{a^2}{4t}\right).$$

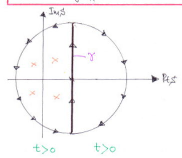
c) Hence find in a simplified form of a convolution integral the following inverse Laplace transform

$$\mathcal{L}^{-1}\left[\frac{\mathrm{e}^{-a\sqrt{s}}}{\sqrt{s}}\right].$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{\mathrm{e}^{-a\sqrt{s}}}{\sqrt{s}}\right] = \frac{a}{2\pi} \int_0^{\infty} \left[\frac{1}{u^{\frac{3}{2}}\sqrt{t=u}}\right] \exp\left(-\frac{a^2}{4u}\right) du}$$

[solution overleaf]

a) $\boxed{f(t) = \frac{1}{2\pi i}\int_{\gamma - iR}^{\gamma + iR} \bar{f}(s)\, e^{st}\, ds \quad \text{AS } R \to \infty}$



- WE CHOSE γ, A STRAIGHT LINE PARALLEL TO THE y AXIS SO THAT IT LIES TO THE "RIGHT" OF ALL THE POLES OF $\bar{f}(s)$
- THE JORDAN'S LEMMA "WORKS" ON THE "CLOSING ARC", IE THE ARC HAS NO CONTRIBUTION AS $R \to \infty$ FOR $t>0$ BY CLOSING THE ARC TO THE "LEFT" OF γ AND FOR $t<0$ BY CLOSING THE ARC TO THE "RIGHT" OF γ

IF THE INTEGRAND IS OF THE FORM $g(s)e^{s(t-\alpha)}$
THEN WE CLOSE TO THE "LEFT" FOR $t-\alpha>0$, $t>\alpha$
WE CLOSE TO THE "RIGHT" FOR $t-\alpha<0$, $t<\alpha$

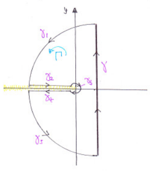
- $\frac{1}{2\pi i}\left[\int_{\gamma} + \cancel{\int_{\text{ARC}}}\right] = \frac{1}{2\pi i} 2\pi i \sum \text{RESIDUES INSIDE}$

$\therefore f(t) = \sum \text{RESIDUES INSIDE}$

(IN THE SIMPLE/STANDARD CASES)

b) $\boxed{\text{WE NEED } \mathcal{L}^{-1}\left[e^{-a\sqrt{s}}\right]}$

- $f(t) = \frac{1}{2\pi i}\int_{\gamma} e^{st}e^{-a\sqrt{s}}\, ds$
- THE INTEGRAND HAS NO POLES BUT IT HAS A BRANCH POINT AT THE ORIGIN, DUE TO THE SQUARE ROOT
- PICK γ AT ANY POSITION TO THE "RIGHT" OF THE y-AXIS & TAKE THE BRANCH CUT ON THE NEGATIVE x-AXIS
- CLOSING THE CONTOUR TO THE RIGHT $(t<0)$ YIELDS $f(t)=0$, $t<0$
- CLOSING THE CONTOUR TO THE LEFT $t>0$ AS SHOWN BELOW



- THE CONTRIBUTION OF THE LARGER ARC, ALTHOUGH IT IS SPLIT INTO γ_1 & γ_5 IS ZERO AS ITS RADIUS TENDS TO INFINITY (BY JORDAN'S LEMMA)
- NEXT LOOK AT THE CONTRIBUTION OF γ_3 AS $\varepsilon \to 0$

$s = \varepsilon e^{i\theta}$

θ RUNS FROM π TO $-\pi$

$ds = i\varepsilon e^{i\theta}\, d\theta$

$$\left|\frac{1}{2\pi i}\int_{\gamma} e^{st}e^{-a\sqrt{s}}\, ds\right| \leqslant \frac{1}{2\pi}\left|\int_{\pi}^{-\pi} e^{t\varepsilon e^{i\theta}} e^{-a(\varepsilon e^{i\theta})^{\frac{1}{2}}}\, i\varepsilon e^{i\theta}\, d\theta\right|$$

$$= \frac{1}{2\pi}\left|\int_{\pi}^{-\pi} e^{\varepsilon t\cos\theta} \cdot e^{\varepsilon t i\sin\theta} \cdot e^{-a\varepsilon^{\frac{1}{2}}\cos\frac{\theta}{2}} \cdot e^{-ia\varepsilon^{\frac{1}{2}}\sin\frac{\theta}{2}}\, i\varepsilon e^{i\theta}\, d\theta\right|$$

$$\leqslant \frac{1}{2\pi}\int_{\pi}^{-\pi} \left|e^{\varepsilon t\cos\theta}\right|\left|e^{\varepsilon t i\sin\theta}\right|\left|e^{-a\varepsilon^{\frac{1}{2}}\cos\frac{\theta}{2}}\right|\left|e^{-ia\varepsilon^{\frac{1}{2}}\sin\frac{\theta}{2}}\right|\left|i\varepsilon e^{i\theta}\right| d\theta$$

$$\leqslant \frac{1}{2\pi}\int_{\pi}^{-\pi} e^{\varepsilon t} \cdot 1 \cdot e^{+a\varepsilon^{\frac{1}{2}}} \cdot 1 \cdot \varepsilon\, d\theta$$

SINCE $-1 \leqslant \cos\theta \leqslant 1$
$-1 \leqslant \cos\frac{\theta}{2} \leqslant 1$

$$= \frac{1}{2\pi} e^{\varepsilon t}e^{a\varepsilon^{\frac{1}{2}}}\varepsilon\int_{\pi}^{-\pi} d\theta = -\varepsilon e^{\varepsilon t}e^{a\varepsilon^{\frac{1}{2}}} \to 0$$

AS $\varepsilon \to 0$

- HENCE SO FAR WE HAVE

- PARAMETRIZE γ_2 & γ_4

γ_2: $s = ue^{i\pi}$ OR $s=-u$, $ds = -du$, NEED IT IN THIS FORM FOR THE SQUARE ROOT, $\int_{-\infty}^{0} \mapsto \int_{\infty}^{0}$	γ_4: $s = ue^{-i\pi}$ OR $s=-u$, $ds=-du$, FOR THE SQUARE ROOT, $\int_{0}^{-\infty} \mapsto \int_{0}^{\infty}$

HENCE

$$f(t) = \frac{1}{2\pi i}\left[\int_{\gamma_2} + \int_{\gamma_4}\right]$$

$$f(t) = \frac{1}{2\pi i}\int_{u=0}^{\infty} e^{-ut}e^{-a(ue^{i\pi})^{\frac{1}{2}}}(-du) + \frac{1}{2\pi i}\int_{u=\infty}^{0} e^{-ut}e^{-a(ue^{-i\pi})^{\frac{1}{2}}}(-du)$$

$$f(t) = \frac{1}{2\pi i}\int_{0}^{\infty} e^{-ut}e^{-a^{\frac{1}{2}}e^{i\frac{\pi}{2}}}(-du) + \frac{1}{2\pi i}\int_{u=0}^{\infty} e^{-ut}e^{-au^{\frac{1}{2}}e^{-i\frac{\pi}{2}}}\, du$$

$$\Rightarrow f(t) = \frac{1}{2\pi i}\int_{0}^{\infty} e^{-ut}e^{-au^{\frac{1}{2}}e^{-\frac{i\pi}{2}}} - e^{-ut}e^{-au^{\frac{1}{2}}e^{i\frac{\pi}{2}}}\, du$$

$$\Rightarrow f(t) = \frac{1}{2\pi i}\int_{0}^{\infty} e^{-ut}\left[e^{-au^{\frac{1}{2}}(-i)} - e^{-au^{\frac{1}{2}}(+i)}\right] du$$

$$\Rightarrow f(t) = \frac{1}{2\pi i}\int_{0}^{\infty} e^{-ut}\left[e^{aiu^{\frac{1}{2}}} - e^{-aiu^{\frac{1}{2}}}\right] du$$

$$\Rightarrow f(t) = \frac{1}{2\pi i}\int_{0}^{\infty} e^{-ut}2\sinh\left(aiu^{\frac{1}{2}}\right) du$$

$$\Rightarrow f(t) = \frac{1}{2\pi i}\int_{0}^{\infty} e^{-ut}\, 2i\sin\left(au^{\frac{1}{2}}\right) du$$

$$\Rightarrow f(t) = \frac{1}{\pi}\int_{0}^{\infty} e^{-ut}\sin\left(au^{\frac{1}{2}}\right) du$$

- LET $p = u^{\frac{1}{2}}$
$p^2 = u$
$du = 2p\, dp$
LIMITS UNCHANGED

$$\Rightarrow f(t) = \frac{1}{\pi}\int_{0}^{\infty} e^{-p^2t}\sin(ap)\; 2p\, dp$$

$$\Rightarrow f(t) = \frac{1}{\pi}\int_{0}^{\infty} 2pe^{-p^2t}\sin(ap)\, dp$$

- INTEGRATION BY PARTS

$\sin ap$	$a\cos ap$
$-\frac{1}{t}e^{-p^2t}$	$2pe^{-p^2t}$

$$\Rightarrow f(t) = \frac{1}{\pi}\left\{\cancel{\left[-\frac{1}{t}e^{-p^2t}\sin ap\right]_{p=0}^{p=\infty}} + \frac{a}{t}\int_{0}^{\infty} e^{-p^2t}\cos ap\, dp\right.$$

$$\Rightarrow f(t) = \frac{a}{\pi t}\int_{0}^{\infty} e^{-p^2t}\cos ap\, dp$$

- NOW CONSIDER THE INTEGRAL AS

$$\Rightarrow I = \int_{0}^{\infty} e^{-p^2t}\cos ap\, dp$$

$$\Rightarrow \frac{\partial I}{\partial a} = \int_{0}^{\infty} e^{-p^2t}\frac{\partial}{\partial a}\cos(ap)\, dp$$

$$\Rightarrow \frac{\partial I}{\partial a} = \int_{0}^{\infty} -pe^{-p^2t}\sin(ap)\, dp$$

- BY PARTS AGAIN

$\sin ap$	$a\cos ap$
$\frac{1}{2t}e^{-p^2t}$	$-pe^{-p^2t}$

$$\Rightarrow \frac{\partial I}{\partial a} = \cancel{\left[\frac{1}{2t}e^{-p^2t}\sin ap\right]_{0}^{\infty}} - \frac{a}{2t}\int_{0}^{\infty} e^{-p^2t}\cos ap\, dp$$

$$\Rightarrow \frac{\partial I}{\partial a} = -\frac{a}{2t}I$$

$$\Rightarrow \frac{1}{I}\partial I = -\frac{a}{2t}\partial a$$

$$\Rightarrow \ln I = -\frac{a^2}{4t} + C$$

$$\Rightarrow I = Ae^{-\frac{a^2}{4t}} \quad (A = e^C)$$

$$\Rightarrow \int_{0}^{\infty} e^{-p^2t}\cos ap\, dp = Ae^{-\frac{a^2}{4t}}$$

- LET $a=0$ IN ORDER TO FIND THE CONSTANT A

$$\Rightarrow \int_{0}^{\infty} e^{-p^2t}\, dp = A$$

$$\Rightarrow A = \int_{0}^{\infty} e^{-p^2t}\, dp$$

ANOTHER SUBSTITUTION
$v^2 = p^2t$
$v = pt^{\frac{1}{2}}$
$dv = t^{\frac{1}{2}}dp$
$dp = \frac{dv}{t^{1/2}}$
LIMITS UNCHANGED

$$\Rightarrow A = \int_{0}^{\infty} e^{-v^2}\frac{dv}{t^{1/2}}$$

$$\Rightarrow A = \frac{1}{t^{1/2}}\int_{0}^{\infty} e^{-v^2}\, dv$$

$$\Rightarrow A = \frac{1}{t^{1/2}}\frac{\sqrt{\pi}}{2} \quad \text{(STANDARD RESULT)}$$

- THUS $I = \int_{0}^{\infty} e^{-p^2t}\cos(ap)\, dp = \frac{1}{2}\sqrt{\frac{\pi}{t}}\, e^{-\frac{a^2}{4t}}$

$$\Rightarrow f(t) = \frac{a}{\pi t}\int_{0}^{\infty} e^{-p^2t}\cos(ap)\, dp = \frac{a}{2\pi t}\frac{\sqrt{\pi}}{\sqrt{t}}e^{-\frac{a^2}{4t}}$$

$$\mathcal{L}^{-1}\left[e^{-a\sqrt{s}}\right] = \frac{a}{2t^{\frac{3}{2}}\sqrt{\pi}}e^{-\frac{a^2}{4t}}$$

c) $\boxed{\mathcal{L}\left[(f*g)(t)\right] = \mathcal{L}\left[f(t)\right]\mathcal{L}\left[g(t)\right]}$

$$(f*g)t = \mathcal{L}^{-1}\left[\mathcal{L}[f(t)]\,\mathcal{L}[g(t)]\right]$$

$\frac{1}{\sqrt{s}}$ $\qquad e^{-a\sqrt{s}}$

WE NEED TO FIND $f(t)$ (given that $\bar{f}(s) = \frac{1}{\sqrt{s}}$

$$\mathcal{L}\left[t^n\right] = \frac{n!}{\,^{n+1}}$$

THUS $\mathcal{L}\left[t^{-\frac{1}{2}}\right] = \frac{\left(-\frac{1}{2}\right)!}{s^{\frac{1}{2}}}$

$$\mathcal{L}\left[\frac{1}{t^{\frac{1}{2}}}\right] = \frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}}$$

$$\mathcal{L}\left[\frac{1}{t^{\frac{1}{2}}}\right] = \frac{\sqrt{\pi}}{s^{\frac{1}{2}}}$$

$$\mathcal{L}\left[\frac{1}{\sqrt{\pi t}}\right] = \frac{1}{s^{\frac{1}{2}}}$$

$$\therefore \mathcal{L}^{-1}\left[\frac{e^{-a\sqrt{s}}}{\sqrt{s}}\right] = \int_{0}^{\infty}\underbrace{\frac{1}{\sqrt{\pi(t-u)}}}_{f(t-u)}\;\underbrace{\frac{a}{2u^{\frac{3}{2}}\sqrt{\pi}}e^{-\frac{a^2}{4u}}}_{g(u)}\, du$$

$$\int_{0}^{\infty} f(t-u)\, g(u)\, du$$

OR INDEED

$$= \boxed{\int_{0}^{\infty}\frac{1}{\sqrt{\pi t}}\frac{a}{2(t-u)^{\frac{3}{2}}\sqrt{\pi}}e^{-\frac{a^2}{4(t-u)}}\, du}$$

$$\mathcal{L}^{-1}\left[\frac{e^{-a\sqrt{s}}}{\sqrt{s}}\right] = \frac{a}{2\pi}\int_{0}^{\infty}\frac{e^{-\frac{a^2}{4u}}}{u^{\frac{3}{2}}(t-u)^{\frac{1}{2}}}\, du$$