LAPLACE TRANSFORMS INTRODUCTION

SUMMARY OF THE LAPLACE TRANFORM

The Laplace Transform of a function $f(t)$, $t \geq 0$ is defined as

$$\mathcal{L}\left[f(t)\right] \equiv \bar{f}(s) \equiv \int_0^\infty \mathrm{e}^{-st} f(t)\ dt,$$

where $s \in \mathbb{C}$, with $\mathrm{Re}(s)$ sufficiently large for the integral to converge.

The Laplace Transform is a linear operation

$$\mathcal{L}\left[a\,f(t) + b\,g(t)\right] \equiv a\mathcal{L}\left[f(t)\right] + b\mathcal{L}\left[g(t)\right].$$

Laplace Transforms of Common Functions

- $\mathcal{L}\left(t^n\right) = \dfrac{n}{s^{n+1}}$

$$\mathcal{L}(1) = \frac{1}{s},\quad \mathcal{L}(a) = \frac{a}{s},\quad \mathcal{L}(t) = \frac{1}{s^2},\quad \mathcal{L}\left(t^2\right) = \frac{2}{s^3},\quad \mathcal{L}\left(t^3\right) = \frac{3}{s^4},\ \ldots.$$

- $\mathcal{L}\left(\mathrm{e}^{at}\right) = \dfrac{1}{s-a}$, $\mathcal{L}\left(\mathrm{e}^{-at}\right) = \dfrac{1}{s+a}$
- $\mathcal{L}(\cos at) = \dfrac{s}{s^2+a^2}$, $\mathcal{L}(\sin at) = \dfrac{a}{s^2+a^2}$
- $\mathcal{L}(\cosh at) = \dfrac{s}{s^2-a^2}$, $\mathcal{L}(\sinh at) = \dfrac{a}{s^2-a^2}$

Laplace Transforms of Derivatives

- $\mathcal{L}\left[x(t)\right] = \bar{x}(t)$
- $\mathcal{L}\left[\dot{x}(t)\right] = s\bar{x}(t) - x(0)$
- $\mathcal{L}\left[\ddot{x}(t)\right] = s^2\bar{x}(t) - sx(0) - \dot{x}(0)$
- $\mathcal{L}\left[\dddot{x}(t)\right] = s^3\bar{x}(t) - s^2x(0) - s\dot{x}(0) - \ddot{x}(0)$

Laplace Transforms Theorems

- 1st Shift Theorem

$$\mathcal{L}\left[e^{-at} f(t)\right] = \bar{f}(s+a) \quad \text{or} \quad \mathcal{L}\left[e^{at} F(t)\right] = \bar{f}(s-a)$$

- 2nd Shift Theorem

$$\mathcal{L}\left[f(t-a)\right] = e^{-as}\,\bar{f}(s),\ t > a \quad \text{or} \quad \mathcal{L}\left[f(t+a)\right] = e^{as}\,\bar{f}(s),\ t > -a.$$

$$\mathcal{L}\left[\mathrm{H}(t-a)\,f(t-a)\right] = e^{-as}\,\bar{f}(s) \quad \text{or} \quad \mathcal{L}\left[\mathrm{H}(t+a)\,f(t+a)\right] = e^{as}\,\bar{f}(s)$$

- Multiplication by t^n

$$\mathcal{L}\left[t^n f(t)\right] = \left(-\frac{d}{ds}\right)^n \left[\bar{f}(s)\right] \quad \text{or} \quad \mathcal{L}\left[t\,f(t)\right] = -\frac{d}{ds}\left[\bar{f}(s)\right]$$

- Division by t

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^{\infty} \bar{f}(\sigma)\ d\sigma$$

provided that $\lim_{t\to 0}\left(\frac{f(t)}{t}\right)$ exists and the integral converges.

- Initial/Final value theorem

$$\lim_{t\to 0}\left[f(t)\right] = \lim_{s\to\infty}\left[s\,\bar{f}(s)\right] \quad \text{and} \quad \lim_{t\to\infty}\left[f(t)\right] = \lim_{s\to 0}\left[s\,\bar{f}(s)\right]$$

The Impulse Function / The Dirac Function

1. $\delta(t-c) = \begin{cases} \infty & t = c \\ 0 & t \neq c \end{cases}$, $\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$

2. $\displaystyle\int_a^b \delta(t-c)\ dt = \begin{cases} 1 & a \le c \le b \\ 0 & \text{otherwise} \end{cases}$

3. $\displaystyle\int_a^b f(t)\delta(t-c)\ dt = \begin{cases} f(a) & a \le c \le b \\ 0 & \text{otherwise} \end{cases}$

4. $\mathcal{L}\left[\delta(t-c)\right] = \mathrm{e}^{-cs}$

5. $\mathcal{L}\left[f(t)\delta(t-c)\right] = f(c)\mathrm{e}^{-cs}$

6. $\dfrac{d}{dt}\left[\mathrm{H}(t-c)\right] = \delta(t-c)$

LAPLACE TRANSFORMS FROM FIRST PRINCIPLES

Question 1

Find, from first principles, the Laplace Transform of

$$f(t) = k \,, \; t \geq 0$$

where k is non zero constant.

$$\boxed{\bar{f}(s) = \frac{a}{s}}$$

$$\mathcal{L}[k] = \int_0^\infty k e^{-st}\,dt = -\frac{k}{s}\left[e^{-st}\right]_0^\infty = \frac{k}{s}\left[e^{-st}\right]_\infty^0$$
$$= \frac{k}{s}[1-0] = \frac{k}{s}$$

Question 2

Use integration to find the Laplace Transform of

$$f(t) = e^{at} \,, \; t \geq 0$$

where a is non zero constant.

$$\boxed{\bar{f}(s) = \frac{1}{s-a}}$$

$$\mathcal{L}\left[e^{at}\right] = \int_0^\infty e^{at} e^{-st}\,dt = \int_0^\infty e^{(a-s)t}\,dt = \left[\frac{1}{a-s}e^{(a-s)t}\right]_0^\infty$$
$$= \left[\frac{1}{s-a}e^{(a-s)t}\right]_\infty^0 = \frac{1}{s-a}(1-0) = \frac{1}{s-a}$$

(NOTE THAT s IS SUFFICIENTLY LARGE FOR THE INTEGRAL TO CONVERGE)

Question 3

Find, from first principles, the Laplace Transform of

$$f(t) = \cos(at), \quad t \geq 0$$

$$g(t) = \sin(at), \quad t \geq 0$$

where a is non zero constant.

$$\boxed{\bar{f}(s) = \frac{s}{s^2 + a^2}}, \quad \boxed{\bar{g}(s) = \frac{a}{s^2 + a^2}}$$

$$\mathcal{L}[\cos at] + i\mathcal{L}[\sin at] = \mathcal{L}[e^{iat}] = \int_0^\infty e^{iat} e^{-st}\, dt$$

$$= \int_0^\infty e^{(ia-s)t}\, dt = \frac{1}{ia-s}\left[e^{(ia-s)t}\right]_0^\infty = \frac{1}{s-ia}\left[e^{(ia-s)t}\right]_\infty^0$$

NOW $s > |ia|$ SO THE INTEGRAL CONVERGES

$$= \frac{1}{s-ia}(1-0) = \frac{1}{s-ia} = \frac{s+ia}{(s-ia)(s+ia)} = \frac{s+ia}{s^2+a^2} = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$$

$$\therefore \mathcal{L}[\cos at] = \frac{s}{s^2+a^2}$$

$$\mathcal{L}[\sin at] = \frac{a}{s^2+a^2} \quad //$$

IT WILL BE FAR <u>HARDER</u> TO DO THEM BY PARTS e.g. $\int_0^\infty e^{-st} \cos at\, dt$

Question 4

Use integration to find the Laplace Transform of

$$f(t) = \cosh(at), \ t \geq 0$$

where a is non zero constant.

$\boxed{}$, $\boxed{\bar{f}(s) = \dfrac{s}{s^2 - a^2}}$

STARTING FROM THE DEFINITION OF THE LAPLACE TRANSFORM

$$\mathcal{L}[f(t)] \equiv \int_0^\infty f(t) e^{-st}\, dt$$

LET $f(t) = \cosh at$

$$\mathcal{L}[\cosh at] = \int_0^\infty (\cosh at) e^{-st}\, dt = \int_0^\infty \left(\tfrac{1}{2}e^{at} + \tfrac{1}{2}e^{-at}\right) e^{-st}\, dt$$

$$= \int_0^\infty \tfrac{1}{2}e^{(a-s)t} + \tfrac{1}{2}e^{-(a+s)t}\, dt$$

$$= \left[\tfrac{1}{2} \times \tfrac{1}{a-s} e^{(a-s)t} + \tfrac{1}{2} \times \tfrac{1}{-(a+s)} e^{-(a+s)t} \right]_0^\infty$$

$$= \tfrac{1}{2}\left[\frac{e^{(a-s)t}}{a-s} - \frac{e^{-(a+s)t}}{a+s} \right]_0^\infty$$

s IS SUFFICIENTLY LARGE FOR THE INTEGRAL TO CONVERGE

$$= \tfrac{1}{2}\left[(0-0) - \left(\tfrac{1}{a-s} - \tfrac{1}{a+s} \right) \right]$$

$$= \tfrac{1}{2}\left[\tfrac{1}{s-a} + \tfrac{1}{s+a} \right] = \tfrac{1}{2}\left[\frac{s+a+s-a}{(s-a)(s+a)} \right]$$

$$= \tfrac{1}{2} \times \frac{2s}{s^2-a^2}$$

$$= \frac{s}{s^2-a^2}$$

NOTE THAT $\mathcal{L}[\cosh at] = \mathcal{L}[\cos(iat)] = \ldots$ STANDARD RESULTS

$$= \frac{s}{s^2+(ia)^2} = \frac{s}{s^2-a^2}$$

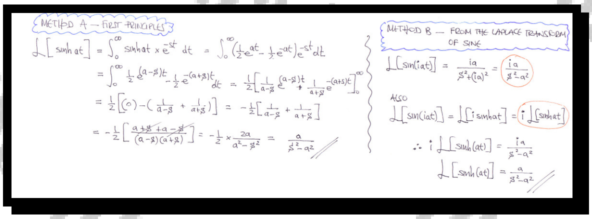
Question 5

Find, from first principles, the Laplace Transform of

$$f(t) = \sinh(at), \quad t \geq 0$$

where a is non zero constant.

$$\boxed{\bar{f}(s) = \frac{a}{s^2 - a^2}}$$



METHOD A – FIRST PRINCIPLES

$\mathcal{L}[\sinh at] = \int_0^\infty \sinh at \times e^{-st}\,dt = \int_0^\infty \left(\frac{1}{2}e^{at} - \frac{1}{2}e^{-at}\right)e^{-st}\,dt$

$= \int_0^\infty \frac{1}{2}e^{(a-s)t} - \frac{1}{2}e^{-(a+s)t}\,dt = \frac{1}{2}\left[\frac{1}{a-s}e^{(a-s)t} + \frac{1}{a+s}e^{-(a+s)t}\right]_0^\infty$

$= \frac{1}{2}\left[(0) - \left(\frac{1}{a-s} + \frac{1}{a+s}\right)\right] = -\frac{1}{2}\left[\frac{1}{a-s} + \frac{1}{a+s}\right]$

$= -\frac{1}{2}\left[\frac{a+\cancel{s}+a-\cancel{s}}{(a-s)(a+s)}\right] = -\frac{1}{2} \times \frac{2a}{a^2-s^2} = \frac{a}{s^2-a^2}$

METHOD B – FROM THE LAPLACE TRANSFORM OF SINE

$\mathcal{L}[\sin(iat)] = \frac{ia}{s^2+(ia)^2} = \frac{ia}{s^2-a^2}$

ALSO

$\mathcal{L}[\sin(iat)] = \mathcal{L}[i\sinh at] = i\mathcal{L}[\sinh at]$

$\therefore i\mathcal{L}[\sinh(at)] = \frac{ia}{s^2-a^2}$

$\mathcal{L}[\sinh(at)] = \frac{a}{s^2-a^2}$

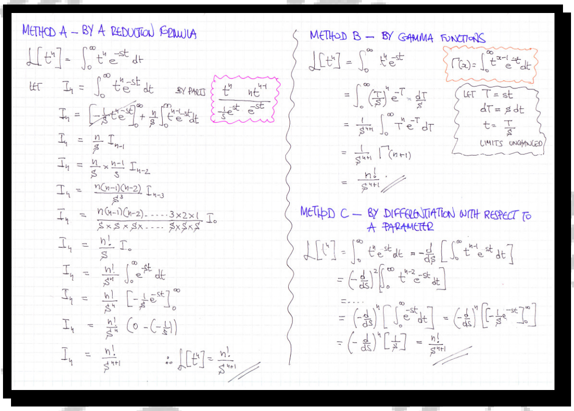
Question 6

Use integration to find the Laplace Transform of

$$f(t) = t^n, \ t \geq 0$$

where $n \neq \ldots -4, -3, -2, -1, 0$.

$$\boxed{\bar{f}(s) = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}}$$



METHOD A – BY A REDUCTION FORMULA

$\mathcal{L}[t^n] = \int_0^\infty t^n e^{-st}\,dt$

LET $I_n = \int_0^\infty t^n e^{-st}\,dt$ BY PARTS $\boxed{\begin{array}{cc} t^n & nt^{n-1} \\ -\frac{1}{s}e^{-st} & e^{-st} \end{array}}$

$I_n = \cancel{\left[-\frac{1}{s}t^n e^{-st}\right]_0^\infty} + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\,dt$

$I_n = \frac{n}{s}I_{n-1}$

$I_n = \frac{n}{s} \times \frac{n-1}{s} I_{n-2}$

$I_n = \frac{n(n-1)(n-2)}{s^3} I_{n-3}$

$I_n = \frac{n(n-1)(n-2)\ldots\ldots 3\times2\times1}{s\times s\times s\times\ldots\ldots s\times s\times s} I_0$

$I_n = \frac{n!}{s} I_0$

$I_n = \frac{n!}{s^n}\int_0^\infty e^{-st}\,dt$

$I_n = \frac{n!}{s^n}\left[-\frac{1}{s}e^{-st}\right]_0^\infty$

$I_n = \frac{n!}{s^n}\left(0 - \left(-\frac{1}{s}\right)\right)$

$I_n = \frac{n!}{s^{n+1}}$ $\quad \therefore \mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$

METHOD B – BY GAMMA FUNCTIONS

$\boxed{\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt}$

$\mathcal{L}[t^n] = \int_0^\infty t^n e^{-st}$

$= \int_0^\infty \left(\frac{T}{s}\right)^n e^{-T} \times \frac{dT}{s}$

$= \frac{1}{s^{n+1}}\int_0^\infty T^n e^{-T}\,dT$

$= \frac{1}{s^{n+1}}\Gamma(n+1)$

$= \frac{n!}{s^{n+1}}$

LET $T = st$
$dT = s\,dt$
$t = \frac{T}{s}$
LIMITS UNCHANGED

METHOD C – BY DIFFERENTIATION WITH RESPECT TO A PARAMETER

$\mathcal{L}[t^n] = \int_0^\infty t^n e^{-st}\,dt = -\frac{d}{ds}\left[\int_0^\infty t^{n-1}e^{-st}\,dt\right]$

$= \left(-\frac{d}{ds}\right)^2\left[\int_0^\infty t^{n-2}e^{-st}\,dt\right]$

$= \ldots$

$= \left(-\frac{d}{ds}\right)^n\left[\int_0^\infty e^{-st}\,dt\right] = \left(-\frac{d}{ds}\right)^n\left[\left[-\frac{1}{s}e^{-st}\right]_0^\infty\right]$

$= \left(-\frac{d}{ds}\right)^n\left[\frac{1}{s}\right] = \frac{n!}{s^{n+1}}$

Question 7

The Heaviside function $\mathrm{H}(t)$ is defined as

$$\mathrm{H}(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Determine the Laplace transform of $\mathrm{H}(t-c)$.

$$\boxed{\mathcal{L}\big(\mathrm{H}(t-c)\big) = \frac{\mathrm{e}^{-cs}}{s}}$$

$$\boxed{\mathrm{H}(t-c) = \begin{cases} 1 & t \geqslant c \\ 0 & t < c \end{cases}}$$

$$\mathcal{L}\left[\mathrm{H}(t-c)\right] = \int_0^\infty \mathrm{H}(t-c)\, e^{-st}\, dt$$

$$= \cancel{\int_0^c 0 \times e^{-st}\, dt} + \int_c^\infty 1 \times e^{-st}\, dt$$

$$= \left[-\frac{1}{s}e^{-st}\right]_c^\infty = \left[\frac{e^{-st}}{s}\right]_\infty^c = \frac{e^{-cs}}{s} - 0$$

$$= \frac{e^{-cs}}{s}$$

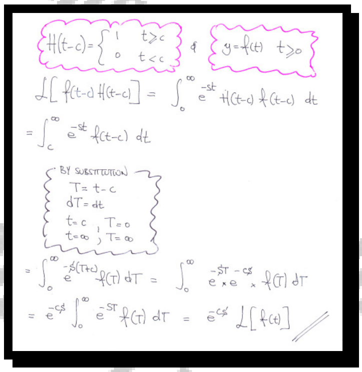
Question 8

The Heaviside step function $\mathrm{H}(t)$ is defined as

$$\mathrm{H}(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Determine the Laplace transform of $\mathrm{H}(t-c)\,f(t-c)$, where $f(t)$ is a continuous or piecewise continuous function defined for $t \geq 0$.

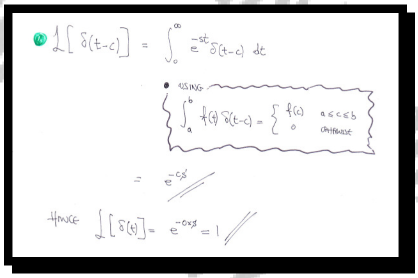
$$\boxed{\mathcal{L}\big(\mathrm{H}(t-c)\,f(t-c)\big) = \mathrm{e}^{-cs}\,\mathcal{L}\big(f(t)\big)}$$



Question 9

Find the Laplace transform of $\delta(t-c)$, where c is a positive constant, and hence state the Laplace transform of $\delta(t)$.

$$\boxed{\mathcal{L}\big[\delta(t-c)\big] = \mathrm{e}^{-cs}}, \quad \boxed{\mathcal{L}\big[\delta(t)\big] = 1}$$



Question 10

Given that $F(t)$ is a piecewise continuous function defined for $t \geq 0$, find the Laplace transform of $F(t)\,\delta(t-c)$, where c is a positive constant.

$$\boxed{\mathcal{L}\left[F(t)\,\delta(t-c)\right] = F(c)\mathrm{e}^{-cs}}$$

$$\mathcal{L}\left[F(t)\,\delta(t-c)\right] = \int_0^\infty e^{-st} F(t)\,\delta(t-c)\,dt$$

$$= \int_0^\infty G(t)\,\delta(t-c)\,dt$$

where $G(t) = e^{-st}F(t)$

$$= G(c)$$

$$= F(c)\,e^{-cs}$$

LAPLACE TRANSFORM GENERAL PRACTICE

Question 1

Find each of the following Laplace transforms or inverse Laplace transforms, showing, where appropriate, the techniques used.

a) $\mathcal{L}\left(t^3+2\mathrm{e}^{-2t}\right)$

b) $\mathcal{L}\left(\mathrm{e}^{-2t}\cosh 3t\right)$

c) $\mathcal{L}\left(t^2 \sin t\right)$

d) $\mathcal{L}\left(\dfrac{\mathrm{e}^t-1}{t}\right)$

e) $\mathcal{L}^{-1}\left(\dfrac{2}{2s-3}\right)$

f) $\mathcal{L}^{-1}\left(\dfrac{6s-17}{s^2-6s+9}\right)$

$\boxed{}$, $\boxed{\dfrac{6}{s^4}+\dfrac{2}{s+2}}$, $\boxed{\dfrac{s+2}{s^2+4s-5}}$, $\boxed{\dfrac{6s^2-2}{\left(s^2+1\right)^3}}$, $\boxed{\ln\left(\dfrac{s}{s-1}\right)}$, $\boxed{\mathrm{e}^{\frac{3}{2}t}}$, $\boxed{6\mathrm{e}^{3t}+t\,\mathrm{e}^{3t}}$

a) BY STANDARD RESULTS

$$\mathcal{L}\left[t^3+2e^{-2t}\right] = \frac{3!}{s^{3+1}} + 2\times\frac{1}{s+2} = \frac{6}{s^4}+\frac{2}{s+2}$$

b) OBTAIN THE TRANSFORM OF $\cosh 3t$ FIRST

$$\mathcal{L}\left[\cosh 3t\right] = \frac{s}{s^2-3^2} = \frac{s}{s^2-9}$$

NOW USING A "SHIFT" THEOREM

$$\mathcal{L}\left[e^{-2t}\cosh 3t\right] = \frac{(s+2)}{(s+2)^2-9} = \frac{s+2}{s^2+4s-5}$$

ALTERNATIVE IN EXPONENTIALS

$$\mathcal{L}\left[e^{-2t}\cosh 3t\right] = \mathcal{L}\left[e^{-2t}\times\tfrac{1}{2}\left(e^{3t}+e^{-3t}\right)\right] = \tfrac{1}{2}\mathcal{L}\left[e^{t}+e^{-5t}\right]$$

$$= \tfrac{1}{2}\left[\frac{1}{s-1}+\frac{1}{s+5}\right] = \tfrac{1}{2}\left[\frac{s+5+s-1}{(s-1)(s+5)}\right]$$

$$= \tfrac{1}{2}\times\frac{2s+4}{s^2+4s-5} = \frac{s+2}{s^2+4s-5}\text{, AS ABOVE}$$

c) START WITH THE TRANSFORM OF $\sin t$

$$\mathcal{L}\left[\sin t\right] = \frac{1}{s^2+1^2} = \frac{1}{s^2+1}$$

USING THE RESULT OF MULTIPLYING BY t^2, OR BY t TWICE

$$\mathcal{L}\left[t\sin t\right] = -\frac{d}{ds}\left[\mathcal{L}\left[\sin t\right]\right] = -\frac{d}{ds}\left[\frac{1}{s^2+1}\right] = -\frac{d}{ds}\left[(s^2+1)^{-1}\right]$$

$$= -\left[-(s^2+1)^{-2}\times(2s)\right] = \frac{2s}{(s^2+1)^2}$$

$$\mathcal{L}\left[t^2\sin t\right] = -\frac{d}{ds}\left[\mathcal{L}\left[t\sin t\right]\right] = -\frac{d}{ds}\left[\frac{2s}{(s^2+1)^2}\right] \leftarrow \text{QUOTIENT RULE}$$

$$= -\frac{(s^2+1)^2\times 2 - 4s(s^2+1)\times 2s}{(s^2+1)^4} = \frac{8s^2(s^2+1)-2(s^2+1)^2}{(s^2+1)^4}$$

$$= \frac{8s^2-2(s^2+1)}{(s^2+1)^3} = \frac{6s^2-2}{(s^2+1)^3}$$

d) FIRSTLY WE CHECK THE EXISTENCE OF THIS LIMIT

$$\lim_{t\to 0}\left[\frac{e^t-1}{t}\right] = \dots\text{ L'HOSPITAL }\dots \lim_{t\to 0}\left[\frac{e^t-0}{1}\right] = 1$$

AS THE LIMIT EXISTS, WE USE THE THEOREM OF DIVISION BY t

$$\mathcal{L}\left[\frac{e^t-1}{t}\right] = \int_s^\infty \mathcal{L}\left[e^t-1\right]\,ds = \int_s^\infty \frac{1}{s-1}-\frac{1}{s}\,ds$$

$$= \Big[\ln|s-1|-\ln|s|\Big]_s^\infty = \left[\ln\frac{s-1}{s}\right]_s^\infty$$

$$= \cancel{\ln 1} - \ln\left|\frac{s-1}{s}\right| = -\ln\left(\frac{s-1}{s}\right) = \ln\left(\frac{s}{s-1}\right)$$

e) STANDARD RESULT ON INVERSION

$$\mathcal{L}^{-1}\left[\frac{2}{2s-3}\right] = \mathcal{L}^{-1}\left[\frac{1}{s-\frac{3}{2}}\right] = e^{\frac{3}{2}t}$$

f) GETTING IT TO A DESIRABLE FORM TO BE RECOGNIZED

$$\mathcal{L}^{-1}\left[\frac{6s-17}{s^2-6s+9}\right] = \mathcal{L}^{-1}\left[\frac{6s-17}{(s-3)^2}\right] = \mathcal{L}^{-1}\left[\frac{6(s-3)+1}{(s-3)^2}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{6}{s-3}+\frac{1}{(s-3)^2}\right] = 6e^{3t}+te^{3t}$$

NOTE FOR $\mathcal{L}^{-1}\left[\frac{1}{(s-3)^2}\right]$

EITHER

$$\mathcal{L}[t] = \frac{1!}{s^2} = \frac{1}{s^2}$$

$$\mathcal{L}\left[te^{3t}\right] = \frac{1}{(s-3)^2}$$

OR

$$\mathcal{L}\left[e^{3t}\right] = \frac{1}{s-3}$$

$$\mathcal{L}\left[t\times e^{3t}\right] = -\frac{d}{ds}\left(\frac{1}{s-3}\right) = \frac{1}{(s-3)^2}$$

Question 2

Find each of the following Laplace transforms or inverse Laplace transforms, showing, where appropriate, the techniques used.

a) $\mathcal{L}\left(3\cos 2t - 2\sinh 3t\right)$

b) $\mathcal{L}\left(2\mathrm{e}^{-3t}\cosh 4t\right)$

c) $\mathcal{L}\left(4t\,\mathrm{e}^{-t}\right)$

d) $\mathcal{L}\left(\dfrac{\sin t}{t}\right)$

e) $\mathcal{L}^{-1}\left[\dfrac{6}{(s-4)^3}\right]$

f) $\mathcal{L}^{-1}\left(\dfrac{s+2}{s^2+4s+13}\right)$

$\boxed{\dfrac{3s}{s^2+4}-\dfrac{6}{s^2-9}}$, $\boxed{\dfrac{2s+6}{s^2+6s-7}}$, $\boxed{\dfrac{4}{(s+1)^2}}$, $\boxed{\arctan\left(\dfrac{1}{s}\right)}$, $\boxed{3t^2\mathrm{e}^{4t}}$, $\boxed{\mathrm{e}^{-2t}\cos 3t}$

a) $\mathcal{L}[3\cos 2t - 2\sinh 3t] = 3\times\frac{s}{s^2+2^2} - 2\times\frac{3}{s^2-3^2} = \frac{3s}{s^2+4} - \frac{6}{s^2-9}$

b) $\mathcal{L}[2\cosh 4t] = 2\times\frac{s}{s^2-4^2} = \frac{2s}{s^2-16}$

$\therefore \mathcal{L}[2e^{-3t}\cosh 4t] = \frac{2(s+3)}{(s+3)^2-16} = \frac{2s+6}{s^2+6s-7}$

c) $\mathcal{L}[4t] = 4\times\frac{1!}{s^2} = \frac{4}{s^2}$

$\therefore \mathcal{L}[4t\times e^{-t}] = \frac{4}{(s+1)^2}$

ALTERNATIVE

$\mathcal{L}[4e^{-t}] = \frac{4}{s-1}$

$\therefore \mathcal{L}[t\times 4e^{-t}] = -\frac{d}{ds}\left(\frac{4}{s-1}\right) = \frac{4}{(s+1)^2}$

d) FIRSTLY $\lim_{t\to 0}\frac{\sin t}{t} = \dots$ L'HOSPITAL $= \lim_{t\to 0}\frac{\cos t}{1} = 1$

SO THE LIMIT EXISTS

THUS $\mathcal{L}\left[\frac{\sin t}{t}\right] = \int_s^\infty \mathcal{L}(\sin t)\,dt = \int_s^\infty \frac{1}{s^2+1}\,ds = \left[\arctan s\right]_s^\infty$

$= \frac{\pi}{2} - \arctan s = \operatorname{arccot} s = \arctan\frac{1}{s}$

e) $\mathcal{L}^{-1}\left(\frac{6}{(s-4)^3}\right) = \mathcal{L}^{-1}\left(3\times\frac{2!}{(s-4)^3}\right) = 3t^2e^{4t}$

SINCE $\mathcal{L}(t^2) = \frac{2!}{s^3}$

f) $\mathcal{L}^{-1}\left(\frac{s+2}{s^2+4s+13}\right) = \mathcal{L}^{-1}\left[\frac{s+2}{(s+2)^2+9}\right] = e^{-2t}\cos 3t$

(DOES NOT FACTORIZE)

SINCE $\mathcal{L}[\cos 3t] = \frac{s}{s^2+9}$

Question 3

Find each of the following Laplace transforms or inverse Laplace transforms, showing, where appropriate, the techniques used.

a) $\mathcal{L}\left(e^{3t}+3\sin 2t\right)$

b) $\mathcal{L}\left(3e^{3t}\sin 2t\right)$

c) $\mathcal{L}\left(t\cosh 2t\right)$

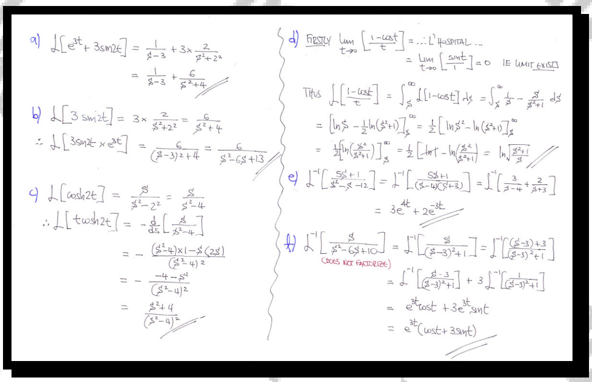
d) $\mathcal{L}\left(\dfrac{1-\cos t}{t}\right)$

e) $\mathcal{L}^{-1}\left(\dfrac{5s+1}{s^2-s-12}\right)$

f) $\mathcal{L}^{-1}\left(\dfrac{s}{s^2-6s+10}\right)$

$$\boxed{\frac{1}{s-3}+\frac{6}{s^2+4}},\ \boxed{\frac{6}{s^2-6s+13}},\ \boxed{\frac{s^2+4}{\left(s^2-4\right)^2}},\ \boxed{\ln\sqrt{\frac{s^2+1}{s^2}}},\ \boxed{3e^{4t}+2e^{-3t}},$$

$$\boxed{e^{3t}\left(\cos t+3\sin t\right)}$$



a) $\mathcal{L}[e^{3t}+3\sin 2t] = \frac{1}{s-3}+3\times\frac{2}{s^2+2^2} = \frac{1}{s-3}+\frac{6}{s^2+4}$

b) $\mathcal{L}[3\sin 2t] = 3\times\frac{2}{s^2+2^2} = \frac{6}{s^2+4}$

$\therefore \mathcal{L}[3\sin 2t\times e^{3t}] = \frac{6}{(s-3)^2+4} = \frac{6}{s^2-6s+13}$

c) $\mathcal{L}[\cosh 2t] = \frac{s}{s^2-2^2} = \frac{s}{s^2-4}$

$\therefore \mathcal{L}[t\cosh 2t] = -\frac{d}{ds}\left[\frac{s}{s^2-4}\right] = -\frac{(s^2-4)\times 1-s(2s)}{(s^2-4)^2} = -\frac{-4-s^2}{(s^2-4)^2} = \frac{s^2+4}{(s^2-4)^2}$

d) FIRSTLY $\lim_{t\to 0}\left[\frac{1-\cos t}{t}\right] = \dots$ L'HOSPITAL $\dots = \lim_{t\to 0}\left[\frac{\sin t}{1}\right]=0$ IE LIMIT EXISTS

THUS $\mathcal{L}\left[\frac{1-\cos t}{t}\right] = \int_s^\infty \mathcal{L}[1-\cos t]\,ds = \int_s^\infty \frac{1}{s}-\frac{s}{s^2+1}\,ds$

$= \left[\ln s-\frac{1}{2}\ln(s^2+1)\right]_s^\infty = \frac{1}{2}\left[\ln s^2-\ln(s^2+1)\right]_s^\infty$

$= \frac{1}{2}\left[\ln\left(\frac{s^2}{s^2+1}\right)\right]_s^\infty = \frac{1}{2}\left[\cancel{\ln 1}-\ln\left(\frac{s^2}{s^2+1}\right)\right] = \ln\sqrt{\frac{s^2+1}{s^2}}$

e) $\mathcal{L}^{-1}\left[\frac{5s+1}{s^2-s-12}\right] = \mathcal{L}^{-1}\left[\frac{5s+1}{(s-4)(s+3)}\right] = \mathcal{L}^{-1}\left[\frac{3}{s-4}+\frac{2}{s+3}\right]$

$= 3e^{4t}+2e^{-3t}$

f) $\mathcal{L}^{-1}\left[\frac{s}{s^2-6s+10}\right] = \mathcal{L}^{-1}\left[\frac{s}{(s-3)^2+1}\right] = \mathcal{L}^{-1}\left[\frac{(s-3)+3}{(s-3)^2+1}\right]$

(DOES NOT FACTORIZE)

$= \mathcal{L}^{-1}\left[\frac{s-3}{(s-3)^2+1}\right]+3\mathcal{L}^{-1}\left[\frac{1}{(s-3)^2+1}\right]$

$= e^{3t}\cos t+3e^{3t}\sin t$

$= e^{3t}(\cos t+3\sin t)$

Question 4

Find each of the following Laplace transforms or inverse Laplace transforms, showing, where appropriate, the techniques used.

a) $\mathcal{L}\left(\dfrac{2t^4+5t^2}{t}\right)$

b) $\mathcal{L}\left(e^{2t}\cos t\right)$

c) $\mathcal{L}\left(4t\sinh 3t\right)$

d) $\mathcal{L}\left(\dfrac{e^{-t}-1}{t}\right)$

e) $\mathcal{L}^{-1}\left(\dfrac{9s-8}{s^2-2s}\right)$

f) $\mathcal{L}^{-1}\left(\dfrac{2s-10}{s^2+2s+17}\right)$

$\boxed{\dfrac{12}{s^4}+\dfrac{5}{s^2}}$, $\boxed{\dfrac{2s+6}{s^2+6s-7}}$, $\boxed{\dfrac{24s}{s^2+9}}$, $\boxed{\ln\left(\dfrac{s}{s-1}\right)}$, $\boxed{4+5e^{2t}}$, $\boxed{e^{-t}\left(2\cos 4t-3\sin 4t\right)}$

a) $\mathcal{L}\left[\frac{2t^4+5t^2}{t}\right]=\mathcal{L}\left[2t^3+5t\right]=2\times\frac{3!}{s^4}+5\times\frac{1!}{s^2}$
$=\frac{12}{s^4}+\frac{5}{s^2}$

b) $\mathcal{L}[\cos t]=\frac{s}{s^2+1^2}=\frac{s}{s^2+1}$
$\therefore \mathcal{L}\left[e^{2t}\cos t\right]=\frac{s-2}{(s-2)^2+1}=\frac{s-2}{s^2-4s+5}$

c) $\mathcal{L}\left[4\sinh 3t\right]=4\times\frac{3}{s^2-3^2}=\frac{12}{s^2-9}$
$\therefore \mathcal{L}\left[4t\sinh 3t\right]=-\frac{d}{ds}\left(\frac{12}{s^2-9}\right)=-\frac{d}{ds}\left[12(s^2-9)^{-1}\right]$
$=-12\times(-2s)(s^2-9)^{-2}=\frac{24s}{s^2+9}$

d) FIRSTLY
$\lim_{t\to 0}\left[\frac{e^{-t}-1}{t}\right]=\ldots$ BY L'HOSPITAL... $\lim_{t\to 0}\left[\frac{-e^{-t}}{1}\right]=-1$

IE THE LIMIT EXISTS

THUS
$\mathcal{L}\left[\frac{e^{-t}-1}{t}\right]=\int_s^{\infty}\mathcal{L}\left[e^{-t}-1\right]ds=\int_s^{\infty}\frac{1}{s+1}-\frac{1}{s}\,ds$
$=\left[\ln(s+1)-\ln s\right]_s^{\infty}=\left[\ln\left(\frac{s+1}{s}\right)\right]_s^{\infty}$
$=\cancel{\ln 1}-\ln\left(\frac{s+1}{s}\right)=\ln\left(\frac{s}{s+1}\right)$

e) $\mathcal{L}^{-1}\left[\frac{9s-8}{s^2-2s}\right]=\mathcal{L}^{-1}\left[\frac{9s-8}{s(s-2)}\right]=\mathcal{L}^{-1}\left[\frac{4}{s}+\frac{5}{s-2}\right]$
$=4+5e^{2t}$

f) $\mathcal{L}^{-1}\left[\frac{2s-10}{s^2+2s+17}\right]=\mathcal{L}^{-1}\left[\frac{2s-10}{(s+1)^2+16}\right]$
$=\mathcal{L}^{-1}\left[\frac{2(s+1)-12}{(s+1)^2+16}\right]$
$=2\mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+4^2}\right]-3\mathcal{L}^{-1}\left[\frac{4}{(s+1)^2+4^2}\right]$
$=2\cos 4t\times e^{-t}-3\sin 4t\times e^{-t}$
$=e^{-t}\left(2\cos 4t-3\sin 4t\right)$

Question 5

Find each of the following Laplace transforms or inverse Laplace transforms, showing, where appropriate, the techniques used.

a) $\mathcal{L}\left(2t^2-5\right)$

b) $\mathcal{L}\left(\mathrm{e}^t \sinh 2t\right)$

c) $\mathcal{L}\left(t^3\,\mathrm{e}^{2t}\right)$

d) $\mathcal{L}\left(\dfrac{\sin 2t}{t}\right)$

e) $\mathcal{L}^{-1}\left(\dfrac{3s+4}{s^2+9}\right)$

f) $\mathcal{L}^{-1}\left(\dfrac{2-s}{s^2+4s-12}\right)$

$\boxed{\dfrac{4}{s^3}-\dfrac{5}{s}}$, $\boxed{\dfrac{2}{s^2-2s-3}}$, $\boxed{\dfrac{6}{(s-2)^4}}$, $\boxed{\arctan\left(\dfrac{2}{s}\right)}$, $\boxed{3\cos 3t+\dfrac{4}{3}\sin 3t}$, $\boxed{-\mathrm{e}^{-6t}}$

a) $\mathcal{L}\left[2t^2-5\right] = 2\times\frac{2!}{s^3}-\frac{5}{s} = \frac{4}{s^3}-\frac{5}{s}$

b) $\mathcal{L}\left[\sinh 2t\right] = \frac{2}{s^2-2^2} = \frac{2}{s^2-4}$

$\therefore \mathcal{L}\left[e^t\sinh 2t\right] = \frac{2}{(s-1)^2-4} = \frac{2}{s^2-2s-3}$

c) $\mathcal{L}\left[t^3\right] = \frac{3!}{s^4} = \frac{6}{s^4}$

$\therefore \mathcal{L}\left[t^3e^{2t}\right] = \frac{6}{(s-2)^4}$

ALTERNATIVE

$\mathcal{L}\left[e^{2t}\right] = \frac{1}{s-2}$

$\therefore \mathcal{L}\left[t^3e^{2t}\right] = \left(-\frac{d}{ds}\right)^3\left[\frac{1}{s-2}\right] = -\frac{d^3}{ds^3}\left[(s-2)^{-1}\right]$

$= -\frac{d^2}{ds^2}\left[-(s-2)^{-2}\right] = -\frac{d}{ds}\left[2(s-2)^{-3}\right]$

$= -\left[-6(s-2)^{-4}\right] = \frac{6}{(s-2)^4}$

d) FIRSTLY $\lim_{t\to 0}\left[\frac{\sin 2t}{t}\right] = \lim_{t\to 0}\left[\frac{2t+O(t^3)}{t}\right] = \lim_{t\to 0}\left[2+O(t^2)\right]$

$= 2$ IE THE LIMIT EXISTS

THUS

$\mathcal{L}\left[\frac{\sin 2t}{t}\right] = \int_s^\infty \mathcal{L}\left[\sin 2t\right]\,ds = \int_s^\infty \frac{2}{s^2+4}\,ds$

$= \left[\arctan\frac{s}{2}\right]_s^\infty = \frac{\pi}{2}-\arctan\frac{s}{2}$

$= \operatorname{arccot}\frac{s}{2} = \arctan\left(\frac{2}{s}\right)$

e) $\mathcal{L}^{-1}\left[\frac{3s+4}{s^2+9}\right] = 3\mathcal{L}^{-1}\left[\frac{s}{s^2+3^2}\right]+\mathcal{L}^{-1}\left[\frac{4}{s^2+3^2}\right]$

$= 3\mathcal{L}^{-1}\left[\frac{s}{s^2+9}\right]+\frac{4}{3}\mathcal{L}^{-1}\left[\frac{3}{s^2+9}\right]$

$= 3\cos 3t+\frac{4}{3}\sin 3t$

f) $\mathcal{L}^{-1}\left[\frac{2-s}{s^2+4s-12}\right] = \mathcal{L}^{-1}\left[\frac{2-s}{(s-2)(s+6)}\right]$

$= \mathcal{L}^{-1}\left[-\frac{s-2}{s-2}\times\frac{1}{s+6}\right]$

$= \mathcal{L}^{-1}\left[-\frac{1}{s+6}\right]$

$= -e^{-6t}$

Question 6

Find each of the following Laplace transforms or inverse Laplace transforms, showing, where appropriate, the techniques used.

a) $\mathcal{L}\big[(t+2)(t+3)\big]$

b) $\mathcal{L}\left(e^{4t}\sin 2t\right)$

c) $\mathcal{L}\left[8t\cosh\left(\frac{1}{2}t\right)\right]$

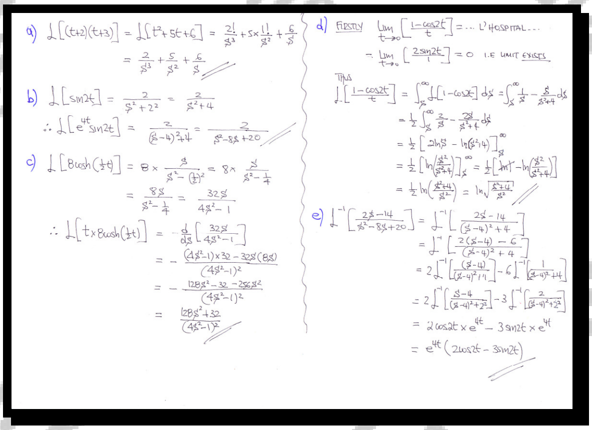
d) $\mathcal{L}\left(\frac{1-\cos 2t}{t}\right)$

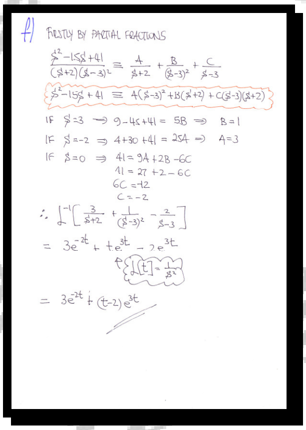
e) $\mathcal{L}^{-1}\left(\frac{2s-14}{s^2-8s+20}\right)$

f) $\mathcal{L}^{-1}\left[\frac{s^2-15s+41}{(s+2)(s-3)^2}\right]$

$\boxed{\frac{2}{s^3}+\frac{5}{s^2}+\frac{6}{s}}$, $\boxed{\frac{6}{s^2-6s+13}}$, $\boxed{\frac{128s^2+32}{\left(4s^2-1\right)^2}}$, $\boxed{\ln\sqrt{\frac{s^2+4}{s^2}}}$, $\boxed{e^{4t}\left(2\cos 2t-3\sin 2t\right)}$,

$\boxed{3e^{-2t}+(t-2)e^{3t}}$





Question 7

Determine each of the following inverse Laplace transforms, showing, if appropriate, the techniques used.

a) $\mathcal{L}^{-1}\left[\dfrac{4s^2-5s+6}{(s+1)(s^2+4)}\right]$

b) $\mathcal{L}^{-1}\left[\dfrac{3(s^2+3)}{s^4-81}\right]$

c) $\mathcal{L}^{-1}\left[\dfrac{s^2+4}{(s^2-4)^2}\right]$

d) $\mathcal{L}^{-1}\left[\dfrac{6s^2-2}{(s^2+1)^3}\right]$

$\boxed{3e^{-t}+\cos 2t-3\sin 2t}$, $\boxed{\frac{1}{3}(\sin 3t+2\sinh 3t)}$, $\boxed{t\cosh 2t}$, $\boxed{t^2\sin t}$

a) $\dfrac{4s^2-5s+6}{(s+1)(s^2+4)} \equiv \dfrac{A}{s+1}+\dfrac{Bs+C}{s^2+4}$

$4s^2-5s+6 \equiv A(s^2+4)+(s+1)(Bs+C)$

- If $s=-1 \Rightarrow 4+5+6=5A \Rightarrow 5A=15 \Rightarrow \boxed{A=3}$
- If $s=0 \Rightarrow 6=4A+C$, $6=12+C$, $\boxed{C=-6}$
- If $s=1 \Rightarrow 4-5+6=5A+2(B+C) \Rightarrow 5=15+2(B-6) \Rightarrow -10=2(B-6) \Rightarrow -5=B-6 \Rightarrow \boxed{B=1}$

$\therefore \mathcal{L}^{-1}\left[\dfrac{3}{s+1}+\dfrac{s-6}{s^2+4}\right]$

$=\mathcal{L}^{-1}\left[\dfrac{3}{s+1}+\dfrac{s}{s^2+4}-3\times\dfrac{2}{s^2+4}\right]$

$=3e^{-t}+\cos 2t-3\sin 2t$

b) $\mathcal{L}^{-1}\left[\dfrac{3(s^2+3)}{s^4-81}\right]=\mathcal{L}^{-1}\left[\dfrac{3(s^2+3)}{(s^2-9)(s^2+9)}\right]$

$=\mathcal{L}^{-1}\left[\dfrac{3(s^2+3)}{(s+3)(s-3)(s^2+9)}\right]$

$\dfrac{3(s^2+3)}{(s+3)(s-3)(s^2+9)} \equiv \dfrac{A}{s-3}+\dfrac{B}{s+3}+\dfrac{Cs+D}{s^2+9}$

$3(s^2+3) \equiv A(s+3)(s^2+9)+B(s-3)(s^2+9)+(Cs+D)(s^2-9)$

- If $s=3 \Rightarrow 36=108A \Rightarrow \boxed{A=\frac{1}{3}}$
- If $s=-3 \Rightarrow 36=108A \Rightarrow \boxed{B=-\frac{1}{3}}$
- If $s=0 \Rightarrow 9=27A-27B-9D$, $9=9+9-9D$, $9D=9$, $\boxed{D=1}$
- If $s=1$: $12=40A-20B-8(C+D)$, $12=\frac{40}{3}+\frac{20}{3}-8(C+1)$, $8(C+1)=8$, $C+1=1$, $\boxed{C=0}$

$\therefore \mathcal{L}^{-1}\left[\dfrac{\frac{1}{3}}{s-3}-\dfrac{\frac{1}{3}}{s+3}+\dfrac{1}{s^2+9}\right]$

$=\frac{1}{3}e^{3t}-\frac{1}{3}e^{-3t}+\frac{1}{3}\times\dfrac{3}{s^2+9}$

$=\frac{1}{3}e^{3t}-\frac{1}{3}e^{-3t}+\frac{1}{3}\sin 3t$

OR $\frac{2}{3}\left[\frac{1}{2}e^{3t}-\frac{1}{2}e^{-3t}\right]+\frac{1}{3}\sin 3t$

$=\frac{2}{3}\sinh 3t+\frac{1}{3}\sin 3t$

$=\frac{1}{3}\left[2\sinh 3t+\sin 3t\right]$

c) $\dfrac{s^2+4}{(s^2-4)^2}=\dfrac{s^2+4}{(s-2)^2(s+2)^2}=\dfrac{A}{s-2}+\dfrac{B}{s+2}+\dfrac{C}{(s-2)^2}+\dfrac{D}{(s+2)^2}$

$(s^2+4) \equiv A(s-2)(s+2)^2+B(s+2)(s-2)^2+C(s+2)^2+D(s-2)^2$

- If $s=2 \Rightarrow 8=16C \Rightarrow \boxed{C=\frac{1}{2}}$
- If $s=-2 \Rightarrow 8=16D \Rightarrow \boxed{D=\frac{1}{2}}$
- If $s=3 \Rightarrow 13=25A+5B+25C+D$, $13=25A+5B+\frac{25}{2}+\frac{1}{2}$, $0=25A+5B$, $B=-5A$
- If $s=1 \Rightarrow 5=-9A+3B+9C+D$, $5=-9A+3(-5A)+\frac{9}{2}+\frac{1}{2}$, $5=-24A+5$, $\boxed{A=0}$ & $\boxed{B=0}$

$\therefore \mathcal{L}^{-1}\left[\dfrac{s^2+4}{(s^2-4)^2}\right]=\mathcal{L}^{-1}\left[\dfrac{\frac{1}{2}}{(s-2)^2}+\dfrac{\frac{1}{2}}{(s+2)^2}\right]$

$=\frac{1}{2}\mathcal{L}^{-1}\left[\dfrac{1}{(s-2)^2}+\dfrac{1}{(s+2)^2}\right]$

$=\frac{1}{2}\left[te^{2t}+te^{-2t}\right]$

$=t\cosh 2t$

d) $\dfrac{6s^2-2}{(s^2+1)^3} \equiv \dfrac{As+B}{s^2+1}+\dfrac{Cs+D}{(s^2+1)^2}+\dfrac{Es+F}{(s^2+1)^3}$

$6s^2-2 \equiv (As+B)(s^2+1)^2+(Cs+D)(s^2+1)+Es+F$

$6s^2-2 \equiv (As+B)(s^4+2s^2+1)+Cs^3+Ds^2+Cs+D+Es+F$

$6s^2-2 \equiv As^5+Bs^4+2As^3+2Bs^2+As+B$
$\qquad\qquad Cs^3+Ds^2+Cs+D$
$\qquad\qquad Es+F$

$\therefore A=0$, $B=0$, $2A+C=0$, $C=0$, $2B+D=6$, $D=6$, $A+C+E=0$, $E=0$, $B+D+F=-2$, $6+F=-2$, $F=-8$

$\mathcal{L}^{-1}\left[\dfrac{6s^2-2}{(s^2+1)^3}\right]=\mathcal{L}^{-1}\left[\dfrac{6}{(s^2+1)^2}\right]-\mathcal{L}^{-1}\left[\dfrac{8}{(s^2+1)^3}\right]=\ldots$ WHICH IS VERY HARD TO INVERT

NOW THIS LOOKS LIKE: $\mathcal{L}\left[t^2(P\cos t+Q\sin t)\right]=\left(-\dfrac{d}{ds}\right)^2\left[\dfrac{Ps}{s^2+1}+\dfrac{Q}{s^2+1}\right]=\dfrac{d^2}{ds^2}\left[\dfrac{Ps}{s^2+1}+\dfrac{Q}{s^2+1}\right]$

$=\dfrac{d}{ds}\left(\dfrac{d}{ds}\left(\dfrac{Ps+Q}{s^2+1}\right)\right)=\dfrac{d}{ds}\left[\dfrac{P(s^2+1)-2s(Ps+Q)}{(s^2+1)^2}\right]=\dfrac{d}{ds}\left[\dfrac{-Ps^2-2Qs+P}{(s^2+1)^2}\right]$

$=\dfrac{(-2Ps-2Q)(s^2+1)^2-4s(s^2+1)(-Ps^2-2Qs+P)}{(s^2+1)^4}=-2\left[\dfrac{(Ps+Q)(s^2+1)+2s(-Ps^2-2Qs+P)}{(s^2+1)^3}\right]$

$=-2\left[\dfrac{Ps^3+Qs^2+Ps+Q-2Ps^3-4Qs^2+2Ps}{(s^2+1)^3}\right]$ IF THIS IS TO WORK $P=0$, SO THIS REDUCES TO

$=-2\left[\dfrac{Qs^2+Q-4Qs^2}{(s^2+1)^3}\right]=\dfrac{6Qs^2-2Q}{(s^2+1)^3}$ WHICH WORKS IF $Q=1$ $\quad\therefore \mathcal{L}^{-1}\left(\dfrac{6s^2-2}{(s^2+1)^3}\right)=t^2\sin t$

Question 8

Find each of the following Laplace transforms or inverse Laplace transforms, showing, where appropriate, the techniques used.

a) $\mathcal{L}(\cos 6t)$

b) $\mathcal{L}\left(t^5 \mathrm{e}^{2t}\right)$

c) $\mathcal{L}^{-1}\left(\dfrac{6}{s^2+6s+18}\right)$

d) $\mathcal{L}\left[(t-3)^3 \mathrm{H}(t-2)\right]$

e) $\mathcal{L}\left[4\delta(t-2)\right]$

f) $\mathcal{L}^{-1}\left(\dfrac{5\mathrm{e}^{-s}}{s}\right)$

$\boxed{\dfrac{s}{s^2+36}}$, $\boxed{\dfrac{120}{(s-2)^4}}$, $\boxed{2\mathrm{e}^{-3t}\sin 3t}$, $\boxed{\dfrac{6\mathrm{e}^{-5s}}{s^4}}$, $\boxed{4\mathrm{e}^{-2s}}$, $\boxed{5\mathrm{H}(t-1)}$

a) $\mathcal{L}[\cos 6t] = \dfrac{s}{s^2+6^2} = \dfrac{s}{s^2+36}$ //

b) $\mathcal{L}[t^5 e^{2t}] = \dfrac{5!}{(s-2)^4} = \dfrac{120}{(s-2)^4}$ //

c) $\mathcal{L}^{-1}\left[\dfrac{6}{s^2+6s+18}\right] = \mathcal{L}^{-1}\left[\dfrac{2 \times 3}{(s+3)^2+9}\right] = 2e^{-3t}\sin 3t$ //

d) $\mathcal{L}\left[(t-3)^3 H(t-2)\right] = e^{-2s} \times \dfrac{3!}{s^4} = \dfrac{6e^{-2s}}{s^4}$ //

e) $\mathcal{L}[4\delta(t-2)] = 4 \times e^{-2s} = 4e^{-2s}$ //

f) $\mathcal{L}^{-1}\left[\dfrac{5e^{-s}}{s}\right] = \mathcal{L}^{-1}\left[5e^{-s} \times \dfrac{1}{s}\right] = 5\,H(t-1) \times 1 = 5H(t-1)$ //

Question 9

Find each of the following Laplace transforms or inverse Laplace transforms, showing, where appropriate, the techniques used.

a) $\mathcal{L}\left(e^{3t}\cosh 4t\right)$

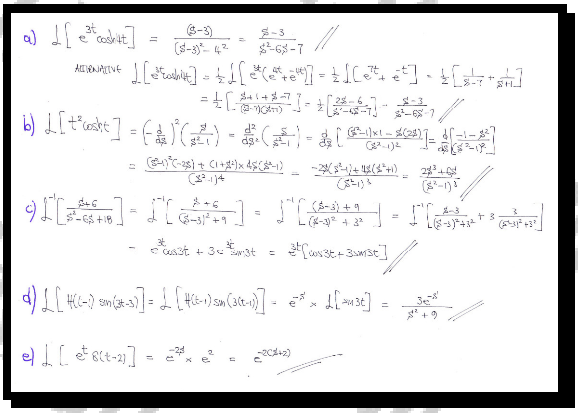
b) $\mathcal{L}\left(t^2\cosh t\right)$

c) $\mathcal{L}^{-1}\left(\dfrac{s+6}{s^2-6s+18}\right)$

d) $\mathcal{L}\left[\mathrm{H}(t-1)\sin(3t-3)\right]$

e) $\mathcal{L}\left[e^{t}\,\delta(t-2)\right]$

$$\boxed{\frac{s-3}{s^2-6s-7}},\ \boxed{\frac{2s^3+6s}{\left(s^2-1\right)^3}},\ \boxed{e^{3t}\left(\cos 3t+3\sin 3t\right)},\ \boxed{\frac{3e^{-s}}{s^2+9}},\ \boxed{e^{-2(s+2)}}$$



a) $\mathcal{L}\left[e^{3t}\cosh 4t\right] = \dfrac{(s-3)}{(s-3)^2-4^2} = \dfrac{s-3}{s^2-6s-7}$ //

ALTERNATIVE $\mathcal{L}\left[e^{3t}\cosh 4t\right] = \frac{1}{2}\mathcal{L}\left[e^{3t}\left(e^{4t}+e^{-4t}\right)\right] = \frac{1}{2}\mathcal{L}\left[e^{7t}+e^{-t}\right] = \frac{1}{2}\left[\frac{1}{s-7}+\frac{1}{s+1}\right]$

$= \frac{1}{2}\left[\dfrac{s+1+s-7}{(s-7)(s+1)}\right] = \frac{1}{2}\left[\dfrac{2s-6}{s^2-6s-7}\right] = \dfrac{s-3}{s^2-6s-7}$ //

b) $\mathcal{L}\left[t^2\cosh t\right] = \left(-\frac{d}{ds}\right)^2\left(\dfrac{s}{s^2-1}\right) = \dfrac{d^2}{ds^2}\left(\dfrac{s}{s^2-1}\right) = \dfrac{d}{ds}\left[\dfrac{(s^2-1)\times 1 - s(2s)}{(s^2-1)^2}\right] = \dfrac{d}{ds}\left[\dfrac{-1-s^2}{(s^2-1)^2}\right]$

$= \dfrac{(s^2-1)^2(-2s)+(1+s^2)\times 4s(s^2-1)}{(s^2-1)^4} = \dfrac{-2s(s^2-1)+4s(s^2+1)}{(s^2-1)^3} = \dfrac{2s^3+6s}{(s^2-1)^3}$ //

c) $\mathcal{L}^{-1}\left[\dfrac{s+6}{s^2-6s+18}\right] = \mathcal{L}^{-1}\left[\dfrac{s+6}{(s-3)^2+9}\right] = \mathcal{L}^{-1}\left[\dfrac{(s-3)+9}{(s-3)^2+3^2}\right] = \mathcal{L}^{-1}\left[\dfrac{s-3}{(s-3)^2+3^2} + 3\,\dfrac{3}{(s-3)^2+3^2}\right]$

$= e^{3t}\cos 3t + 3e^{3t}\sin 3t = e^{3t}\left[\cos 3t + 3\sin 3t\right]$ //

d) $\mathcal{L}\left[\mathrm{H}(t-1)\sin(3t-3)\right] = \mathcal{L}\left[\mathrm{H}(t-1)\sin(3(t-1))\right] = e^{-s}\times\mathcal{L}\left[\sin 3t\right] = \dfrac{3e^{-s}}{s^2+9}$ //

e) $\mathcal{L}\left[e^{t}\,\delta(t-2)\right] = e^{-2s}\times e^{2} = e^{-2(s+2)}$ //

Question 10

Find each of the following Laplace transforms or inverse Laplace transforms, showing, where appropriate, the techniques used.

a) $\mathcal{L}\left(t^2 \mathrm{e}^{-\frac{1}{2}t}\right)$

b) $\mathcal{L}^{-1}\left(\dfrac{6s+1}{9s^2+1}\right)$

c) $\mathcal{L}\left[\mathrm{e}^{t-5}\,\mathrm{H}(t-5)\right]$

d) $\mathcal{L}^{-1}\left(\dfrac{8\mathrm{e}^{-4s}}{s^2+4}\right)$

e) $\mathcal{L}\left[t^3 \mathrm{e}^{\frac{1}{3}t}\,\delta(t-3)\right]$

f) $\mathcal{L}\left[\mathrm{e}^{t}\,\mathrm{H}(t-2)\right]$

$\boxed{\dfrac{16}{(2s+1)^3}}$, $\boxed{\dfrac{2}{3}\cosh\left(\dfrac{1}{3}t\right)+\sinh\left(\dfrac{1}{3}t\right)}$, $\boxed{\dfrac{\mathrm{e}^{-5s}}{s-1}}$, $\boxed{4\mathrm{H}(t-4)\sin(2t-8)}$, $\boxed{4\mathrm{e}^{-2s}}$, $\boxed{\dfrac{\mathrm{e}^{2-2s}}{s-1}}$

a) $\mathcal{L}\left[t^2 e^{-\frac{1}{2}t}\right] = \dfrac{2!}{(s+\frac{1}{2})^3} = \dfrac{2}{\frac{1}{8}(2s+1)^3} = \dfrac{16}{(2s+1)^3}$

b) $\mathcal{L}^{-1}\left[\dfrac{6s+1}{9s^2+1}\right] = \mathcal{L}^{-1}\left[\dfrac{\frac{2}{3}s+\frac{1}{9}}{s^2+\frac{1}{9}}\right] = \mathcal{L}^{-1}\left[\dfrac{2}{3}\times\dfrac{s}{s^2+(\frac{1}{3})^2} + \dfrac{(\frac{1}{3})^2}{s^2+(\frac{1}{3})^2}\right] = \frac{2}{3}\cosh\frac{1}{3}t + \sinh\frac{1}{3}t$

c) $\mathcal{L}\left[H(t-5)e^{t-5}\right] = \dfrac{e^{-5s}}{s-1}$ $\leftarrow \mathcal{L}[e^t]$

d) $\mathcal{L}^{-1}\left[\dfrac{8e^{-4s}}{s^2+4}\right] = \mathcal{L}^{-1}\left[4e^{-4s}\times\dfrac{2}{s^2+4}\right] = 4H(t-4)\sin(2(t-4)) = 4H(t-4)\sin(2t-8)$

e) $\mathcal{L}\left[t^3 e^{\frac{1}{3}t}\delta(t-3)\right] = e^{-3s}\times 3^3\times e^1 = 27e^{1-3s}$

f) $\mathcal{L}\left[H(t-2)e^t\right] = \mathcal{L}\left[H(t-2)e^{t-2}\times e^2\right] = e^2\mathcal{L}\left[H(t-2)e^{t-2}\right] = e^2\times\dfrac{e^{-2s}}{s-1} = \dfrac{e^{2-2s}}{s-1}$

Question 11

Find each of the following Laplace transforms or inverse Laplace transforms, showing, where appropriate, the techniques used.

a) $\mathcal{L}\left[t\sin\left(\frac{1}{2}t\right)\right]$

b) $\mathcal{L}^{-1}\left[\frac{1}{(s-2)^6}\right]$

c) $\mathcal{L}\left[(t-5)\mathrm{H}(t-5)\right]$

d) $\mathcal{L}^{-1}\left[\frac{3\mathrm{e}^{-2s}}{s^2-1}\right]$

e) $\mathcal{L}\left[t^2\,\delta(t-2)\right]$

f) $\mathcal{L}\left(2^t\right)$

$$\boxed{\frac{16s}{\left(4s^2+1\right)^2}},\ \boxed{\frac{t^5\,\mathrm{e}^{-2t}}{120}},\ \boxed{\frac{\mathrm{e}^{-5s}}{s}},\ \boxed{3\mathrm{H}(t-2)\sinh(t-2)},\ \boxed{9\mathrm{e}^{-3s}},\ \boxed{\frac{1}{s-\ln 2}}$$

a) $\mathcal{L}\left[t\sin\tfrac{1}{2}t\right] = -\frac{d}{ds}\left[\mathcal{L}(\sin\tfrac{1}{2}t)\right] = -\frac{d}{ds}\left[\frac{1/2}{s^2+1/4}\right] = -\frac{d}{ds}\left[\frac{2}{4s^2+1}\right] = -\frac{d}{ds}\left[2(4s^2+1)^{-1}\right]$

$= -\left[-16s(4s^2+1)^{-2}\right] = \frac{16s}{(4s^2+1)^2}$ //

b) $\mathcal{L}^{-1}\left[\frac{1}{(s-2)^6}\right] = e^{-2t} \times \frac{1}{120} \times \mathcal{L}^{-1}\left[\frac{120}{s^6}\right] = \frac{1}{120}t^5e^{-2t}$ //

5!

c) $\mathcal{L}\left[(t-5)H(t-5)\right] = e^{-5s} \times \mathcal{L}[t] = \frac{e^{-5s}}{s^2}$ //

d) $\mathcal{L}^{-1}\left[\frac{3e^{-2s}}{s^2-1}\right] = 3H(t-2) \times \sinh(t-2) = 3H(t-2)\sinh(t-2)$ //

e) $\mathcal{L}\left[t^2\delta(t-3)\right] = e^{-3s} \times 3^2 = 9e^{-3s}$ //

f) $\mathcal{L}\left[2^t\right] = \mathcal{L}\left[e^{t\ln 2}\right] = \mathcal{L}\left[1 \times e^{t\ln 2}\right] = \frac{1}{s-\ln 2}$ //

Question 12

Find each of the following Laplace transforms or inverse Laplace transforms, showing, where appropriate, the techniques used.

a) $\mathcal{L}\left[\mathrm{H}(t-2)\sin\left(\frac{1}{2}t-1\right)\right]$

b) $\mathcal{L}^{-1}\left[\frac{\mathrm{e}^{-4s}}{s^2}\right]$

c) $\mathcal{L}\left[2t\sin t\ \delta\left(t-\frac{\pi}{2}\right)\right]$

d) $\mathcal{L}\left[t^2\mathrm{e}^{-\frac{1}{2}t}\mathrm{H}(t-2)\right]$

$$\boxed{\frac{2\mathrm{e}^{-2s}}{4s^2+1}},\ \boxed{(t-4)\,\mathrm{H}(t-4)},\ \boxed{\pi\mathrm{e}^{-\frac{1}{2}\pi s}},\ \boxed{\frac{8\mathrm{e}^{-2(s+2)}}{(2s+1)^3}\left[4s^2+8s+5\right]}$$

a) $\mathcal{L}\left[\mathrm{H}(t-2)\sin\left(\frac{1}{2}t-1\right)\right]=\mathcal{L}\left[\mathrm{H}(t-2)\sin\left(\frac{1}{2}(t-2)\right)\right]=\frac{\mathrm{e}^{-2s}\times\frac{1}{2}}{s^2+\left(\frac{1}{2}\right)^2}=\frac{2\mathrm{e}^{-2s}}{4s^2+1}$

b) $\mathcal{L}^{-1}\left[\frac{\mathrm{e}^{-4s}}{s^2}\right]=(t-4)\mathrm{H}(t-4)$

c) $\mathcal{L}\left[2t\sin t\,\delta\left(t-\frac{\pi}{2}\right)\right]=2\mathrm{e}^{-\frac{\pi s}{2}}\times\frac{\pi}{2}\times\sin\frac{\pi}{2}=\pi\mathrm{e}^{-\frac{\pi s}{2}}$

d) $\mathcal{L}\left[t^2\mathrm{e}^{-\frac{1}{2}t}\mathrm{H}(t-2)\right]=\mathcal{L}\left[t^2\mathrm{e}^{-\frac{1}{2}t}\mathrm{e}^{4}\mathrm{e}^{-4}\mathrm{H}(t-2)\right]=\mathrm{e}^{-4}\mathcal{L}\left[t^2\mathrm{e}^{-\frac{1}{2}(t-2)}\mathrm{H}(t-2)\right]$

$=\mathrm{e}^{-4}\mathcal{L}\left[\left\{(t-2)^2+4(t-2)+4\right\}\mathrm{e}^{-\frac{1}{2}(t-2)}\mathrm{H}(t-2)\right]$

$=\mathrm{e}^{-4}\mathcal{L}\left[(t-2)^2\mathrm{e}^{-\frac{1}{2}(t-2)}\mathrm{H}(t-2)+4(t-2)\mathrm{e}^{-\frac{1}{2}(t-2)}\mathrm{H}(t-2)+4\mathrm{e}^{-\frac{1}{2}(t-2)}\mathrm{H}(t-2)\right]$

$=\mathrm{e}^{-4}\left[\frac{2!}{(s+\frac{1}{2})^3}\mathrm{e}^{-2s}+4\frac{1!}{(s+\frac{1}{2})^2}\mathrm{e}^{-2s}+\frac{4}{s+\frac{1}{2}}\mathrm{e}^{-2s}\right]$

$=\mathrm{e}^{-4}\left[\frac{16\mathrm{e}^{-2s}}{(2s+1)^3}+\frac{16\mathrm{e}^{-2s}}{(2s+1)^2}+\frac{8\mathrm{e}^{-2s}}{2s+1}\right]=\frac{8\mathrm{e}^{-2(s+2)}}{(2s+1)^3}\left[2+2(2s+1)+(2s+1)^2\right]$

$=\frac{8\mathrm{e}^{-2(s+2)}}{(2s+1)^3}\left(4s^2+8s+5\right)$

HEAVISIDE FUNCTION

Question 1

The Heaviside function $\mathrm{H}(t)$ is defined as

$$\mathrm{H}(t)=\begin{cases}1 & t\geq 0\\ 0 & t<0\end{cases}$$

Determine the Laplace transform of $\mathrm{H}(t-c)$.

$$\boxed{\mathcal{L}\big(\mathrm{H}(t-c)\big)=\frac{\mathrm{e}^{-cs}}{s}}$$

$$\boxed{\mathrm{H}(t-c)=\begin{cases}1 & t\geq c\\ 0 & t<c\end{cases}}$$

$$\mathcal{L}\left[\mathrm{H}(t-c)\right]=\int_0^\infty \mathrm{H}(t-c)\,e^{-st}\,dt$$

$$=\cancel{\int_0^c 0\times e^{-st}\,dt}+\int_c^\infty 1\times e^{-st}\,dt$$

$$=\left[-\frac{1}{s}e^{-st}\right]_c^\infty=\left[\frac{e^{-st}}{s}\right]_\infty^c=\frac{e^{-cs}}{s}-0$$

$$=\frac{e^{-cs}}{s}$$

Question 2

The Heaviside step function $\mathrm{H}(t)$ is defined as

$$\mathrm{H}(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Determine the Laplace transform of $\mathrm{H}(t-c)\,f(t-c)$, where $f(t)$ is a continuous or piecewise continuous function defined for $t \geq 0$.

$$\boxed{\mathcal{L}\big(\mathrm{H}(t-c)\,f(t-c)\big) = \mathrm{e}^{-cs}\,\mathcal{L}\big(f(t)\big)}$$

$$\mathrm{H}(t-c) = \begin{cases} 1 & t \geqslant c \\ 0 & t < c \end{cases} \quad \& \quad y = f(t) \quad t \geqslant 0$$

$$\mathcal{L}\left[f(t-c)\,\mathrm{H}(t-c)\right] = \int_0^{\infty} e^{-st}\,\mathrm{H}(t-c)\,f(t-c)\,dt$$

$$= \int_c^{\infty} e^{-st} f(t-c)\,dt$$

BY SUBSTITUTION

$T = t - c$

$dT = dt$

$t = c \;,\; T = 0$

$t = \infty \;;\; T = \infty$

$$= \int_0^{\infty} e^{-s(T+c)} f(T)\,dT = \int_0^{\infty} e^{-sT} \times e^{-cs} \times f(T)\,dT$$

$$= e^{-cs} \int_0^{\infty} e^{-sT} f(T)\,dT = e^{-cs}\,\mathcal{L}\left[f(t)\right]$$

Question 3

The piecewise continuous function $f(t)$ is defined as

$$f(t) = \begin{cases} 4 & 0 \le t \le 2 \\ 12 - 4t & 2 < t \le 4 \\ t - 8 & t > 4 \end{cases}$$

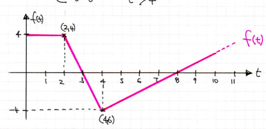
a) Sketch the graph of $f(t)$.

b) Express $f(t)$ in terms of the Heaviside step function, and hence find the Laplace transform of $f(t)$.

$\boxed{}$,

$\boxed{f(t) = 4\mathrm{H}(t) - 4(t-2)\mathrm{H}(t-2) + 5(t-4)\mathrm{H}(t-4)}$, $\boxed{\mathcal{L}(f(t)) = \frac{8}{s} - \frac{4e^{-2s}}{s^2} + \frac{5e^{-4s}}{s^2}}$

a) $f(t) = \begin{cases} 4 & 0 \le t \le 2 \\ 12-4t & 2 < t \le 4 \\ t-8 & t > 4 \end{cases}$



b) EXPRESSING THE GRAPH IN TERMS OF "HEAVISIDES"

$\Rightarrow f(t) = 4H(t) - 4H(t-2) + (12-4t)H(t-2) - (12-4t)H(t-4) + (t-8)H(t-4)$ "ON" "OFF"

$\Rightarrow f(t) = 4H(t) + (8-4t)H(t-2) + (5t-20)H(t-4)$

$\Rightarrow f(t) = 4H(t) - 4(t-2)H(t-2) + 5(t-4)H(t-4)$

$\Rightarrow \bar{f}(s) = \frac{4e^{0s}}{s} - 4\left(\frac{e^{-2s}}{s^2}\right) + 5\left(\frac{e^{-4s}}{s^2}\right)$

$\Rightarrow \bar{f}(s) = \frac{4}{s} - \frac{4}{s^2}e^{-2s} + \frac{5}{s^2}e^{-4s}$

NOTE THAT $\mathcal{L}[f(t-a)H(t-a)] = e^{-as}\mathcal{L}[f(t)]$

Question 4

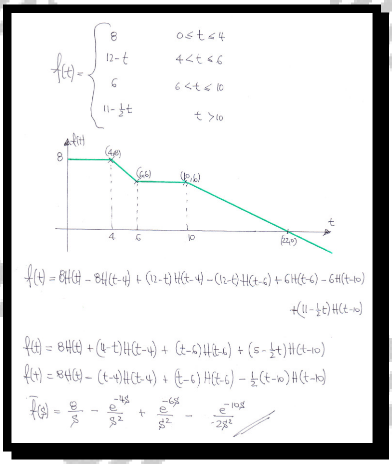
The piecewise continuous function $f(t)$ is defined as

$$f(t)=\begin{cases} 8 & 0 \le t \le 4 \\ 12-t & 4 < t \le 6 \\ 6 & 6 < t \le 10 \\ 11-\frac{1}{2}t & t > 10 \end{cases}$$

Express $f(t)$ in terms of the Heaviside step function, and hence find the Laplace transform of $f(t)$.

$$\boxed{f(t)=8\text{H}(t)-(t-4)\text{H}(t-4)+(t-4)\text{H}(t-6)-\frac{1}{2}(t-10)\text{H}(t-10)},$$

$$\boxed{\mathcal{L}\big(f(t)\big)=\frac{8}{s}-\frac{e^{-4s}}{s^2}+\frac{e^{-6s}}{s^2}-\frac{e^{-10s}}{2s^2}}$$



Question 5

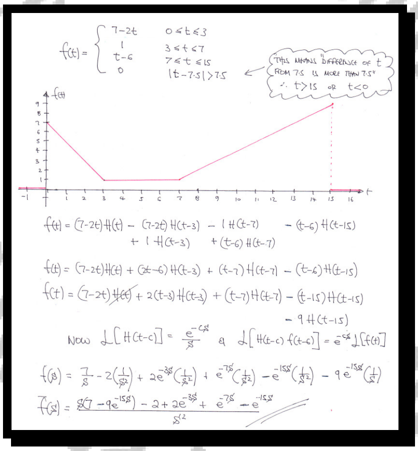
The piecewise continuous function $f(t)$ is defined as

$$f(t)=\begin{cases} 7-2t & 0<t\le 3 \\ 1 & 3<t\le 7 \\ t-6 & 7<t\le 15 \\ 0 & |t-7.5|>7.5 \end{cases}$$

Express $f(t)$ in terms of the Heaviside step function, and hence find the Laplace transform of $f(t)$.

$$\boxed{f(t)=(7-2t)\mathrm{H}(t)+2(t-3)\mathrm{H}(t-3)+(t-7)\mathrm{H}(t-7)-(t-15)\mathrm{H}(t-15)-9\mathrm{H}(t-15)},$$

$$\boxed{\mathcal{L}(f(t))=\frac{s\left(7-9\mathrm{e}^{-15s}\right)-2+2\mathrm{e}^{-3s}+\mathrm{e}^{-7s}-\mathrm{e}^{-15s}}{s^2}}$$



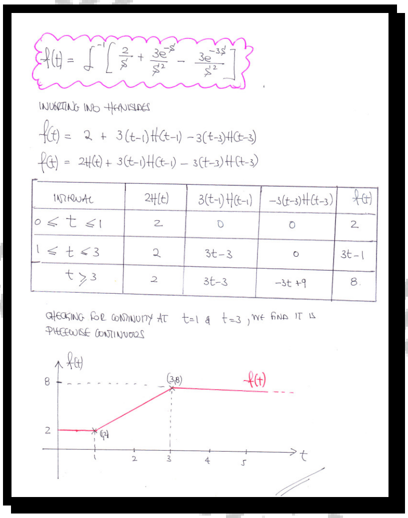
Question 6

The piecewise continuous function $f(t)$ is defined as

$$f(t) = \mathcal{L}^{-1}\left(\frac{2}{s} + \frac{3e^{-s}}{s^2} - \frac{3e^{-3s}}{s^2}\right).$$

Sketch the graph of $f(t)$.

graph



$$f(t) = \mathcal{L}^{-1}\left[\frac{2}{s} + \frac{3e^{-s}}{s^2} - \frac{3e^{-3s}}{s^2}\right]$$

INVERTING INTO HEAVISIDES

$f(t) = 2 + 3(t-1)H(t-1) - 3(t-3)H(t-3)$

$f(t) = 2H(t) + 3(t-1)H(t-1) - 3(t-3)H(t-3)$

INTERVAL	$2H(t)$	$3(t-1)H(t-1)$	$-3(t-3)H(t-3)$	$f(t)$
$0 \leqslant t \leqslant 1$	2	0	0	2
$1 \leqslant t \leqslant 3$	2	$3t-3$	0	$3t-1$
$t \geqslant 3$	2	$3t-3$	$-3t+9$	8.

CHECKING FOR CONTINUITY AT $t=1$ & $t=3$, WE FIND IT IS PIECEWISE CONTINUOUS

Question 7

The piecewise continuous function $f(t)$ is defined as

$$f(t) = \mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2}\left(3s\,e^{2s} - 4\,e^{s} + 5\right)\right].$$

a) Determine an expression for $f(t)$.

b) Sketch the graph of $f(t)$.

$$\boxed{f(t) = 3\mathrm{H}(t) - 4(t-1)\mathrm{H}(t-1) + 5(t-2)\mathrm{H}(t-2)}$$

a) $f(t) = \mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2}\left(3se^{2s} - 4e^{s} + 5\right)\right] = \mathcal{L}^{-1}\left[\frac{3}{s} - \frac{4e^{-s}}{s^2} + \frac{5e^{-2s}}{s^2}\right]$

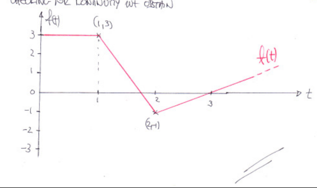
$= 3 - 4(t-1)H(t-1) + 5(t-2)H(t-2)$

$= 3H(t) - 4(t-1)H(t-1) + 5(t-2)H(t-2)$ //

b) TO SKETCH

$0 \le t \le 1$:	3	0	0	=	3
$1 \le t \le 2$:	3	$-4(t-1)$	0	=	$7-4t$
$t \ge 2$:	3	$-4(t-1)$	$5(t-2)$	=	$t-3$
	$\uparrow$ $H(t)$	$\uparrow$ $-4(t-1)H(t-1)$	$\uparrow$ $5(t-2)H(t-2)$		$\uparrow$ $f(t)$

CHECKING FOR CONTINUITY WE OBTAIN



Question 8

The piecewise continuous function $f(t)$ is defined as

$$f(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\left(2 - 2e^{-4s} - e^{-6s}\right)\right].$$

Sketch the graph of $f(t)$.

$\boxed{}$, $\boxed{\text{graph}}$

$$\boxed{f(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\left(2 - 2e^{-4s} - e^{-6s}\right)\right]}$$

- INVERTING INTO "HEAVISIDES"

$$f(t) = \mathcal{L}^{-1}\left[\frac{2}{s^2} - \frac{2e^{-4s}}{s^2} - \frac{e^{-6s}}{s^2}\right]$$

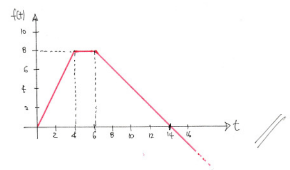
$$f(t) = 2t - 2(t-4)H(t-4) - (t-6)H(t-6)$$

$$f(t) = 2tH(t) - 2(t-4)H(t-4) - (t-6)H(t-6)$$

- USE A TABLE TO EXTRACT THE FUNCTION

INTERVAL	$2tH(t)$	$-2(t-4)H(t-4)$	$-(t-6)H(t-6)$	$f(t)$
$0 \leqslant t \leqslant 4$	$2t$	0	0	$2t$
$4 < t \leqslant 6$	$2t$	$8-2t$	0	8
$t > 6$	$2t$	$8-2t$	$6-t$	$14-t$

- FINALLY BEFORE SKETCHING CHECK FOR CONTINUITY AT $x=4$, $x=6$



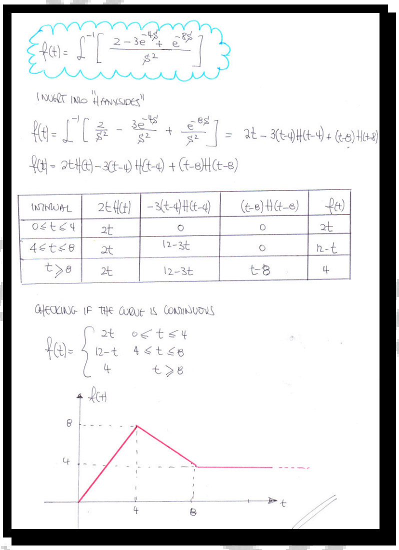
Question 9

The piecewise continuous function $f(t)$ is defined as

$$f(t) = \mathcal{L}^{-1}\left[\frac{2-3e^{-4s}+e^{-8s}}{s^2}\right].$$

Sketch the graph of $f(t)$.

graph



$f(t) = \mathcal{L}^{-1}\left[\frac{2-3e^{-4s}+e^{-8s}}{s^2}\right]$

INVERT INTO "HEAVISIDES"

$f(t) = \mathcal{L}^{-1}\left[\frac{2}{s^2} - \frac{3e^{-4s}}{s^2} + \frac{e^{-8s}}{s^2}\right] = 2t - 3(t-4)H(t-4) + (t-8)H(t-8)$

$f(t) = 2tH(t) - 3(t-4)H(t-4) + (t-8)H(t-8)$

INTERVAL	$2tH(t)$	$-3(t-4)H(t-4)$	$(t-8)H(t-8)$	$f(t)$
$0 \leqslant t \leqslant 4$	$2t$	0	0	$2t$
$4 \leqslant t \leqslant 8$	$2t$	$12-3t$	0	$12-t$
$t \geqslant 8$	$2t$	$12-3t$	$t-8$	4

CHECKING IF THE CURVE IS CONTINUOUS

$$f(t) = \begin{cases} 2t & 0 \leqslant t \leqslant 4 \\ 12-t & 4 \leqslant t \leqslant 8 \\ 4 & t \geqslant 8 \end{cases}$$

Question 10

The piecewise continuous function $f(t)$ is defined as

$$f(t) = \mathcal{L}^{-1}\left[\frac{s\left(7-9e^{-15s}\right)-2+2e^{-3s}+e^{-7s}-e^{-15s}}{s^2}\right].$$

Sketch the graph of $f(t)$.

graph

$$\boxed{f(t) = \mathcal{L}^{-1}\left[\frac{7s-2+2e^{-3s}+e^{-7s}-(9s+1)e^{-15s}}{s^2}\right]}$$

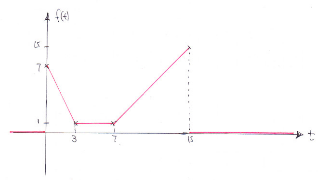
$$f(t) = \mathcal{L}^{-1}\left[\frac{7}{s}-\frac{2}{s^2}+\frac{2}{s^2}e^{-3s}+\frac{1}{s^2}e^{-7s}-\frac{9}{s}e^{-15s}-\frac{1}{s^2}e^{-15s}\right]$$

INVERT INTO "HEAVISIDES" BY INSPECTION

$$f(t) = 7-2t+2(t-3)H(t-3)+(t-7)H(t-7)-9H(t-15)-(t-15)H(t-15)$$

$$f(t) = (7-2t)+2(t-3)H(t-3)+(t-7)H(t-7)-(t-6)H(t-15)$$

INTERVAL	$7-2t$	$2(t-3)H(t-3)$	$(t-7)H(t-7)$	$(t-6)H(t-15)$	$f(t)$
$0 \le t \le 3$	$7-2t$	0	0	0	$7-2t$
$3 \le t \le 7$	$7-2t$	$2t-6$	0	0	1
$7 \le t \le 15$	$7-2t$	$2t-6$	$t-7$	0	$t-6$
$t \ge 15$	$7-2t$	$2t-6$	$t-7$	$-t-6$	0



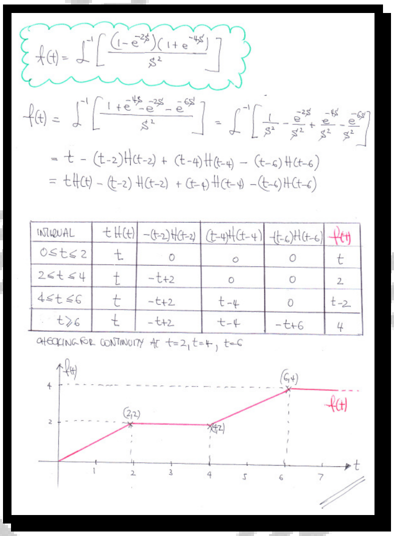
Question 11

The piecewise continuous function $f(t)$ is defined as

$$f(t) = \mathcal{L}^{-1}\left[\frac{\left(1-e^{-2s}\right)\left(1+e^{-4s}\right)}{s^2}\right].$$

Sketch the graph of $f(t)$.

graph



$$f(t) = \mathcal{L}^{-1}\left[\frac{(1-e^{-2s})(1+e^{-4s})}{s^2}\right]$$

$$f(t) = \mathcal{L}^{-1}\left[\frac{1+e^{-4s}-e^{-2s}-e^{-6s}}{s^2}\right] = \mathcal{L}^{-1}\left[\frac{1}{s^2}-\frac{e^{-2s}}{s^2}+\frac{e^{-4s}}{s^2}-\frac{e^{-6s}}{s^2}\right]$$

$$= t-(t-2)H(t-2)+(t-4)H(t-4)-(t-6)H(t-6)$$

$$= tH(t)-(t-2)H(t-2)+(t-4)H(t-4)-(t-6)H(t-6)$$

INTERVAL	$tH(t)$	$-(t-2)H(t-2)$	$(t-4)H(t-4)$	$-(t-6)H(t-6)$	$f(t)$
$0 \le t \le 2$	t	0	0	0	t
$2 \le t \le 4$	t	$-t+2$	0	0	2
$4 \le t \le 6$	t	$-t+2$	$t-4$	0	$t-2$
$t \ge 6$	t	$-t+2$	$t-4$	$-t+6$	4

CHECKING FOR CONTINUITY AT $t=2$, $t=4$, $t=6$

PERIODIC FUNCTIONS

Question 1

The piecewise continuous function $f(t)$ is defined for $t \geq 0$ and further satisfies $f(t+\omega) = f(t)$.

Show from the definition of a Laplace transform, that

$$\mathcal{L}\left[f(t)\right] = \frac{1}{1-\mathrm{e}^{-\omega s}} \int_0^{\omega} \mathrm{e}^{-st} f(t)\ dt .$$

$\boxed{\text{proof}}$

$f(t+\omega) = f(t)$

- $\mathcal{L}[f(t)] = \int_0^{\infty} e^{-st} f(t)\, dt$

$\mathcal{L}[f(t)] = \int_0^{\infty} e^{-st} f(t+\omega)\, dt$

$\mathcal{L}[f(t)] = \int_{\omega}^{\infty} e^{-s(T-\omega)} f(T)\, dT$

$\mathcal{L}[f(t)] = \int_{\omega}^{\infty} e^{\omega s} e^{-sT} f(T)\, dT$

$\mathcal{L}[f(t)] = e^{\omega s} \int_{\omega}^{\infty} e^{-sT} f(T)\, dT$

$e^{-\omega s} \mathcal{L}[f(t)] = \int_{\omega}^{\infty} e^{-st} f(t)\, dt$ (AS T IS A DUMMY VARIABLE)

BY SUBSTITUTION
$T = t+\omega$
$dT = dt$
$t=0 \quad T=\omega$
$t=\infty \quad T=\infty$

- THUS

$\mathcal{L}[f(t)] = \int_0^{\infty} e^{-st} f(t)\, dt$

$-e^{-\omega s} \mathcal{L}[f(t)] = -\int_{\omega}^{\infty} e^{-st} f(t)\, dt$

- ADD

$(1 - e^{-\omega s}) \mathcal{L}[f(t)] = \int_0^{\infty} e^{-st} f(t)\, dt - \int_{\omega}^{\infty} e^{-st} f(t)\, dt$

$(1 - e^{-\omega s}) \mathcal{L}[f(t)] = \int_0^{\omega} e^{-st} f(t)\, dt$

$\mathcal{L}[f(t)] = \frac{1}{1-e^{-\omega s}} \int_0^{\omega} e^{-st} f(t)\, dt$

Question 2

$$f(t) = \begin{cases} 1 & 0 \le t \le 1 \\ -1 & 1 < t < 2 \end{cases} \qquad \text{and} \qquad f(t+2) = f(t),\ t \ge 0 .$$

Determine the Laplace transform of $f(t)$.

$$\boxed{\mathcal{L}\big(f(t)\big) = \frac{1 - 2\mathrm{e}^{-s} + \mathrm{e}^{-2s}}{s\left(1 + \mathrm{e}^{-s}\right)}}$$

$$f(t) = \begin{cases} 1 & 0 \le t \le 1 \\ -1 & 1 < t < 2 \end{cases} \qquad f(t+2) = f(t)$$

(PERIOD 2)

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2s}} \int_0^1 e^{-st}\, dt - \frac{1}{1-e^{-2s}} \int_1^2 e^{-st}\, dt$$

$$= \frac{1}{1-e^{-2s}} \left[\frac{1}{-s} e^{-st}\right]_0^1 - \frac{1}{1-e^{-2s}} \left[-\frac{1}{s} e^{-st}\right]_1^2$$

$$= \frac{1}{1-e^{-2s}} \left[\frac{1}{s} e^{-st}\right]_1^0 + \frac{1}{1-e^{-2s}} \left[\frac{1}{s} e^{-st}\right]_1^2$$

$$= \frac{1}{1-e^{-2s}} \left[\frac{1}{s} - \frac{1}{s} e^{-s}\right] + \frac{1}{1-e^{-2s}} \left[\frac{1}{s} e^{-2s} - \frac{1}{s} e^{-s}\right]$$

$$= \frac{1}{s(1-e^{-2s})} \left[1 - e^{-s} + e^{-2s} - e^{-s}\right]$$

$$= \frac{1 - 2e^{-s} + e^{-2s}}{s(1 - e^{-2s})}$$

Question 3

$$f(t)=\begin{cases}1 & 0\le t\le 1\\ 0 & 1<t<2\end{cases} \qquad \text{and} \qquad f(t+2)=f(t),\ t\ge 0.$$

Determine the Laplace transform of $f(t)$.

$$\boxed{\mathcal{L}\big(f(t)\big)=\frac{1}{s\left(1+\mathrm{e}^{-s}\right)}}$$

$$f(t)=\begin{cases}1 & 0\le t\le 1\\ 0 & 1<t<2\end{cases} \qquad f(t)=f(t+2)$$

(PERIOD = 2)

$$\mathcal{L}[f(t)]=\bar{f}(s)=\frac{1}{1-e^{-2s}}\int_0^2 e^{-st}f(t)\,dt=\frac{1}{1-e^{-2s}}\int_0^1 e^{-st}\times 1\,dt$$

$$=\frac{1}{1-e^{-2s}}\left[-\frac{1}{s}e^{-st}\right]_0^1=\frac{1}{s(1-e^{-2s})}\left[e^{-st}\right]_1^0$$

$$=\frac{1}{s(1-e^{-2s})}\left[1-e^{-s}\right]=\frac{1-e^{-s}}{s(1-e^{-s})(1+e^{-s})}$$

$$=\frac{1}{s(1+e^{-s})}$$

Question 4

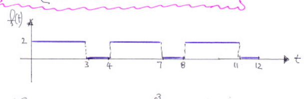
$$f(t)=\begin{cases}2 & 0\le t\le 3\\ 0 & 3<t<4\end{cases} \qquad \text{and} \qquad f(t+4)=f(t),\ t\ge 0.$$

Determine the Laplace transform of $f(t)$.

$$\boxed{\mathcal{L}\big(f(t)\big)=\frac{2\left(1-\mathrm{e}^{-3s}\right)}{s\left(1-\mathrm{e}^{-4s}\right)}}$$

$$f(t)=\begin{cases}2 & 0\le t\le 3\\ 0 & 3<t<4\\ f(t)=f(t+4)\end{cases}$$

PERIOD 4



$$\mathcal{L}[f(t)]=\bar{f}(s)=\frac{1}{1-e^{-4s}}\int_0^3 2e^{-st}\,dt=\frac{1}{1-e^{-4s}}\left[-\frac{2}{s}e^{-st}\right]_0^3$$

$$=\frac{1}{1-e^{-4s}}\left[\frac{2}{s}e^{-st}\right]_3^0=\frac{2}{s}\frac{1}{1-e^{-4s}}\left(1-e^{-3s}\right)$$

$$=\frac{2(1-e^{-3s})}{s(1-e^{-4s})}$$

Question 5

$$f(t)=\begin{cases}2 & 0\le t\le 1\\ 0 & 1<t<3\end{cases} \qquad \text{and} \qquad f(t+4)=f(t),\ t\ge 0.$$

Determine the Laplace transform of $f(t)$.

$$\boxed{\mathcal{L}\big(f(t)\big)=\frac{2\left(1-\mathrm{e}^{-s}\right)}{s\left(1-\mathrm{e}^{-3s}\right)}}$$

$$f(t)=\begin{cases}2 & 0\leqslant t\leqslant 1\\ 0 & 1<t<3\end{cases} \qquad f(t+3)=f(t)$$

(PERIOD 3)

$$\mathcal{L}\left[f(t)\right]=\bar{f}(s)=\frac{1}{1-e^{-3s}}\int_0^3 e^{-st}f(t)\,dt$$

$$=\frac{1}{1-e^{-3s}}\int_0^1 2e^{-st}\,dt$$

$$=\frac{1}{1-e^{-3s}}\left[-\frac{2}{s}e^{-st}\right]_0^1$$

$$=\frac{2}{s(1-e^{-3s})}\left[e^{-st}\right]_1^0$$

$$=\frac{2}{s(1-e^{-3s})}\left[1-e^{-s}\right]$$

$$=\frac{2(1-e^{-s})}{s(1-e^{-3s})}$$

Question 6

$$f(t) = \mathrm{e}^t, \ t \geq 0 \qquad \text{and} \qquad f(t+2) = f(t).$$

Determine the Laplace transform of $f(t)$.

$$\boxed{\mathcal{L}(f(t)) = \frac{\mathrm{e}^{2(1-s)} - 1}{(1-s)\left(1 - \mathrm{e}^{-2s}\right)} = \frac{\mathrm{e}^{2s} - \mathrm{e}^2}{(s-1)\left(\mathrm{e}^{2s} - 1\right)}}$$

$f(t) = e^t \quad 0 \leq t \leq 2 \qquad f(t+2) = f(t) \quad$ PERIOD 2

$$\mathcal{L}[f(t)] = \bar{f}(s) = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} f(t) \, dt$$

$$= \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} e^{t} \, dt$$

$$= \frac{1}{1-e^{-2s}} \int_0^2 e^{t(1-s)} \, dt$$

$$= \frac{1}{1-e^{-2s}} \times \frac{1}{1-s} \left[e^{t(1-s)} \right]_0^2$$

$$= \frac{1}{(1-e^{-2s})(1-s)} \left[e^{2(1-s)} - 1 \right]$$

$$= \frac{e^{2(1-s)} - 1}{(1-e^{-2s})(1-s)}$$

$$\text{OR} \quad \frac{e^2 - e^{2s}}{(e^{2s}-1)(1-s)} = \frac{e^{2s} - e^2}{(s-1)(e^{2s}-1)}$$

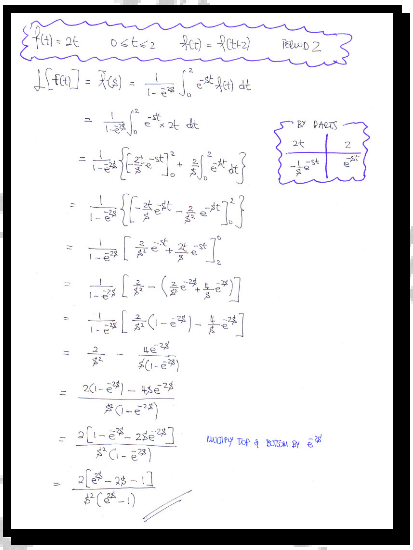
Question 7

$$f(t) = 2t \,,\; t \geq 0 \qquad \text{and} \qquad f(t+2) = f(t)\,.$$

Show that the Laplace transform of $f(t)$ is

$$\frac{2\left(e^{2s} - 2s - 1\right)}{s^2\left(e^{2s} - 1\right)}$$

$\boxed{\text{proof}}$



$f(t) = 2t \quad 0 \leq t \leq 2 \quad f(t) = f(t+2) \quad$ PERIOD 2

$$\mathcal{L}[f(t)] = \bar{f}(s) = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} \times 2t \, dt$$

$$= \frac{1}{1-e^{-2s}} \left\{ \left[-\frac{2t}{s} e^{-st} \right]_0^2 + \frac{2}{s} \int_0^2 e^{-st}\, dt \right\}$$

BY PARTS

$2t$	2
$-\frac{1}{s}e^{-st}$	e^{-st}

$$= \frac{1}{1-e^{-2s}} \left\{ \left[-\frac{2t}{s} e^{-st} - \frac{2}{s^2} e^{-st} \right]_0^2 \right\}$$

$$= \frac{1}{1-e^{-2s}} \left[\frac{2}{s^2} e^{-st} + \frac{2t}{s} e^{-st} \right]_2^0$$

$$= \frac{1}{1-e^{-2s}} \left[\frac{2}{s^2} - \left(\frac{2}{s^2} e^{-2s} + \frac{4}{s} e^{-2s} \right) \right]$$

$$= \frac{1}{1-e^{-2s}} \left[\frac{2}{s^2}\left(1 - e^{-2s}\right) - \frac{4}{s} e^{-2s} \right]$$

$$= \frac{2}{s^2} - \frac{4e^{-2s}}{s\left(1-e^{-2s}\right)}$$

$$= \frac{2\left(1-e^{-2s}\right) - 4se^{-2s}}{s^2\left(1-e^{-2s}\right)}$$

$$= \frac{2\left[1 - e^{-2s} - 2se^{-2s}\right]}{s^2\left(1-e^{-2s}\right)}$$

MULTIPLY TOP & BOTTOM BY e^{2s}

$$= \frac{2\left[e^{2s} - 2s - 1\right]}{s^2\left(e^{2s} - 1\right)}$$

Question 8

$$f(t) = \begin{cases} \sin t & 0 \le t \le \pi \\ 0 & \pi < t < 2\pi \end{cases} \quad \text{and} \quad f(t+2\pi) = f(t),\ t \ge 0.$$

Show that the Laplace transform of $f(t)$ is

$$\frac{1}{\left(s^2+1\right)\left(1+e^{-\pi s}\right)}$$

proof

$$f(t) = \begin{cases} \sin t & 0 \le t \le \pi \\ 0 & \pi < t < 2\pi \end{cases} \qquad f(t+2\pi) = f(t)$$

PERIOD $= 2\pi$

$$\mathcal{L}[f(t)] = \bar{f}(s) = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} f(t) e^{-st}\, dt = \frac{1}{1-e^{-2\pi s}} \int_0^{\pi} (\sin t) e^{-st}\, dt$$

COMPLETE THE INTEGRATION IN INDEFINITE FORM WITH COMPLEX NUMBERS

$$\int e^{-st} \sin t\, dt = \operatorname{Im} \int e^{-st} e^{it}\, dt = \operatorname{Im} \int e^{t(-s+i)}\, dt$$

$$= \operatorname{Im}\left\{ \frac{1}{-s+i} e^{t(-s+i)} \right\} = \operatorname{Im}\left\{ \frac{-s-i}{s^2+1} e^{-st} (\cos t + i \sin t) \right\}$$

$$= \frac{e^{-st}}{s^2+1} \operatorname{Im}\left[(-s-i)(\cos t + i \sin t) \right] = \frac{e^{-st}}{s^2+1}\left[-s \sin t - \cos t \right]$$

$$= -\frac{e^{-st}(\cos t + s \sin t)}{s^2+1}$$

$$\dots = \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{-st}(\cos t + s \sin t)}{s^2+1} \right]_{\pi}^{0}$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{1}{s^2+1} + \frac{e^{-s\pi}}{s^2+1} \right]$$

$$= \frac{1+e^{-\pi s}}{1-e^{-2\pi s}} \times \frac{1}{s^2+1}$$

$$= \frac{1+e^{-\pi s}}{\left(1-e^{-\pi s}\right)\left(1+e^{-\pi s}\right)} \times \frac{1}{s^2+1}$$

$$= \frac{1}{\left(s^2+1\right)\left(1+e^{-\pi s}\right)}$$

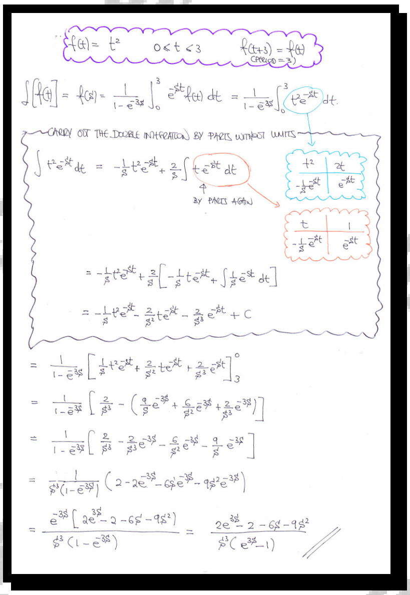
Question 9

$$f(t) = t^2 \,,\; t \geq 0 \qquad \text{and} \qquad f(t+3) = f(t) \,.$$

Show that the Laplace transform of $f(t)$ is

$$\frac{2e^{3s} - 2 - 6s - 9s^2}{s^3\left(e^{3s} - 1\right)}$$

proof



$f(t) = t^2 \qquad 0 \leqslant t \leqslant 3 \qquad f(t+3) = f(t)$ (PERIOD = 3)

$$\mathcal{L}[f(t)] = \bar{f}(s) = \frac{1}{1-e^{-3s}} \int_0^3 e^{-st} f(t)\,dt = \frac{1}{1-e^{-3s}} \int_0^3 t^2 e^{-st}\,dt.$$

CARRY OUT THE DOUBLE INTEGRATION BY PARTS WITHOUT LIMITS

$$\int t^2 e^{-st}\,dt = -\frac{1}{s}t^2 e^{-st} + \frac{2}{s}\int t e^{-st}\,dt$$

BY PARTS AGAIN

$$\begin{array}{c|c} t^2 & 2t \\ \hline -\frac{1}{s}e^{-st} & e^{-st} \end{array} \qquad \begin{array}{c|c} t & 1 \\ \hline -\frac{1}{s}e^{-st} & e^{-st} \end{array}$$

$$= -\frac{1}{s}t^2 e^{-st} + \frac{2}{s}\left[-\frac{1}{s}te^{-st} + \int \frac{1}{s}e^{-st}\,dt\right]$$

$$= -\frac{1}{s}t^2 e^{-st} - \frac{2}{s^2}te^{-st} - \frac{2}{s^3}e^{-st} + C$$

$$= \frac{1}{1-e^{-3s}}\left[\frac{1}{s}t^2 e^{-st} + \frac{2}{s^2}te^{-st} + \frac{2}{s^3}e^{-st}\right]_3^0$$

$$= \frac{1}{1-e^{-3s}}\left[\frac{2}{s^3} - \left(\frac{9}{s}e^{-3s} + \frac{6}{s^2}e^{-3s} + \frac{2}{s^3}e^{-3s}\right)\right]$$

$$= \frac{1}{1-e^{-3s}}\left[\frac{2}{s^3} - \frac{2}{s^3}e^{-3s} - \frac{6}{s^2}e^{-3s} - \frac{9}{s}e^{-3s}\right]$$

$$= \frac{1}{s^3(1-e^{-3s})}\left(2 - 2e^{-3s} - 6se^{-3s} - 9s^2e^{-3s}\right)$$

$$= \frac{e^{-3s}\left[2e^{3s} - 2 - 6s - 9s^2\right]}{s^3(1-e^{-3s})} = \frac{2e^{3s} - 2 - 6s - 9s^2}{s^3(e^{3s}-1)}$$

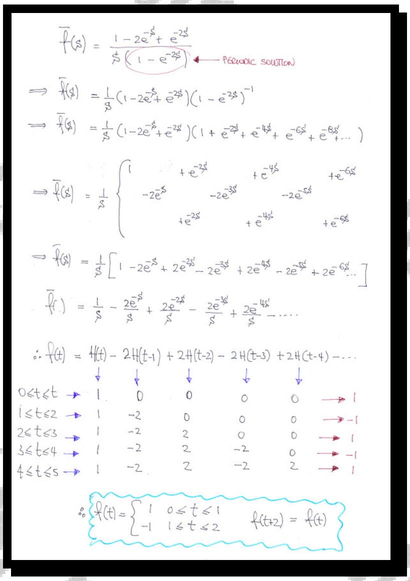
Question 10

The piecewise continuous function $f(t)$ is defined as

$$f(t) = \mathcal{L}^{-1}\left[\frac{1-2\mathrm{e}^{-s}+\mathrm{e}^{-2s}}{s\left(1-\mathrm{e}^{-2s}\right)}\right].$$

Find an expression for $f(t)$.

$$\boxed{f(t) = \begin{cases} 1 & 0 \le t \le 1 \\ -1 & 1 < t < 2 \end{cases} \qquad f(t+2) = f(t)}$$



$$\bar{f}(s) = \frac{1-2e^{-s}+e^{-2s}}{s\left(1-e^{-2s}\right)}$$ ← PERIODIC SOLUTION

$$\Rightarrow \bar{f}(s) = \frac{1}{s}\left(1-2e^{-s}+e^{-2s}\right)\left(1-e^{-2s}\right)^{-1}$$

$$\Rightarrow \bar{f}(s) = \frac{1}{s}\left(1-2e^{-s}+e^{-2s}\right)\left(1+e^{-2s}+e^{-4s}+e^{-6s}+e^{-8s}+\dots\right)$$

$$\Rightarrow \bar{f}(s) = \frac{1}{s}\begin{cases} 1 \quad +e^{-2s} \quad +e^{-4s} \quad +e^{-6s} \\ -2e^{-s} \quad -2e^{-3s} \quad -2e^{-5s} \\ +e^{-2s} \quad +e^{-4s} \quad +e^{-6s} \end{cases}$$

$$\Rightarrow \bar{f}(s) = \frac{1}{s}\left[1-2e^{-s}+2e^{-2s}-2e^{-3s}+2e^{-4s}-2e^{-5s}+2e^{-6s}\dots\right]$$

$$\bar{f}(\ ) = \frac{1}{s}-\frac{2e^{-s}}{s}+\frac{2e^{-2s}}{s}-\frac{2e^{-3s}}{s}+\frac{2e^{-4s}}{s}-\dots$$

$$\therefore f(t) = H(t)-2H(t-1)+2H(t-2)-2H(t-3)+2H(t-4)-\dots$$

$0 \le t \le t$ →	1	0	0	0	0	→ 1
$1 \le t \le 2$ →	1	−2	0	0	0	→ −1
$2 \le t \le 3$ →	1	−2	2	0	0	→ 1
$3 \le t \le 4$ →	1	−2	2	−2	0	→ −1
$4 \le t \le 5$ →	1	−2	2	−2	2	→ 1

$$\therefore \boxed{f(t) = \begin{cases} 1 & 0 \le t \le 1 \\ -1 & 1 \le t \le 2 \end{cases} \qquad f(t+2) = f(t)}$$

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Question 11

The piecewise continuous function $f(t)$ is defined as

$$f(t) = \mathcal{L}^{-1}\left[\frac{2\left(1-e^{-s}\right)}{s\left(1-e^{-3s}\right)}\right].$$

Find an expression for $f(t)$.

$$\boxed{f(t) = \begin{cases} 2 & 0 \le t \le 1 \\ 0 & 1 < t < 3 \end{cases} \qquad f(t+3) = f(t)}$$

$$\bar{f}(s) = \frac{2(1-e^{-s})}{s(1-e^{-3s})} \leftarrow \text{PERIODIC TERM}$$

$$\Rightarrow \bar{f}(s) = \frac{2}{s}(1-e^{-s})(1-e^{-3s})^{-1}$$

$$\Rightarrow \bar{f}(s) = \frac{2}{s}(1-e^{-s})(1+e^{-3s}+e^{-6s}+e^{-9s}+e^{-12s}+\dots)$$

$$\Rightarrow \bar{f}(s) = \frac{2}{s}\left(1 - e^{-s} + e^{-3s} - e^{-4s} + e^{-6s} - e^{-7s} + e^{-9s} - e^{-10s} + \dots\right)$$

$$\Rightarrow \bar{f}(s) = \frac{2}{s} - \frac{2e^{-s}}{s} + \frac{2e^{-3s}}{s} - \frac{2e^{-4s}}{s} + \frac{2e^{-6s}}{s} - \frac{2e^{-7s}}{s} + \frac{2e^{-9s}}{s} - \frac{2e^{-10s}}{s} + \dots$$

$$\therefore f(t) = 2H(t) - 2H(t-1) + 2H(t-3) - 2H(t-4) + 2H(t-6) - 2H(t-7) + \dots$$

2	2	0	0	0	0	0	$\leftarrow 0 \le t \le 1$
0	2	−2	0	0	0	0	$\leftarrow 1 \le t \le 3$
2	2	−2	2	0	0	0	$\leftarrow 3 \le t \le 4$
0	2	−2	2	−2	0	0	$\leftarrow 4 \le t \le 6$
2	2	−2	2	−2	2	0	$\leftarrow 6 \le t \le 7$
0	2	−2	2	−2	2	−2	$\leftarrow 7 \le t \le 9$

$$\therefore \boxed{f(t) = \begin{cases} 2 & 0 \le t \le 1 \\ 0 & 1 \le t \le 3 \end{cases} \qquad f(t+3) = f(t)}$$

Question 12

The piecewise continuous function $f(t)$ is defined as

$$f(t) = \mathcal{L}^{-1}\left[\frac{2e^{3s} - 2 - 6s - 9s^2}{s^3\left(e^{3s} - 1\right)}\right].$$

Find an expression for $f(t)$.

$$\boxed{f(t) = t^2 \quad 0 \le t \le 3 \quad f(t+3) = f(t)}$$

$\bar{f}(s) = \dfrac{2e^{3s} - 2 - 6s - 9s^2}{s^3\left(e^{3s} - 1\right)}$ → PERIODIC TERM — NEEDS MANIPULATION (MULTIPLY TOP/BOTTOM BY e^{-3s})

$\Rightarrow \bar{f}(s) = \dfrac{2 - 2e^{-3s} - 6se^{-3s} - 9s^2e^{-3s}}{s^3\left(1 - e^{-3s}\right)}$

$\Rightarrow \bar{f}(s) = \dfrac{2 - 2e^{-3s} - 6se^{-3s} - 9s^2e^{-3s}}{s^3}\left(1 - e^{-3s}\right)^{-1}$

$\Rightarrow \bar{f}(s) = \dfrac{2e^{3s} - 2 - 6s - 9s^2}{s^3} \times e^{-3s} \times \left(1 + e^{-3s} + e^{-6s} + e^{-9s} + \dots\right)$

$\Rightarrow \bar{f}(s) = \dfrac{2e^{3s} - 2 - 6s - 9s^2}{s^3}\left(e^{-3s} + e^{-6s} + e^{-9s} + e^{-12s} + \dots\right)$

$\Rightarrow \bar{f}(s) = \left[\dfrac{2e^{3s}}{s^3} - \dfrac{2 + 6s + 9s^2}{s^3}\right]\left[e^{-3s} + e^{-6s} + e^{-9s} + e^{-12s} + \dots\right]$

EXPAND $\bar{f}(s) = \dfrac{2}{s^3} \quad + \cancel{\dfrac{2e^{-3s}}{s^3}} \quad + \cancel{\dfrac{2e^{-6s}}{s^3}} \quad + \cancel{\dfrac{2e^{-9s}}{s^3}}$

$\qquad - \cancel{\dfrac{2e^{-3s}}{s^3}} \quad - \cancel{\dfrac{2e^{-6s}}{s^3}} \quad - 2e^{-9s}$

$\qquad - \dfrac{6}{s^2}e^{-3s} \quad - \dfrac{6}{s^2}e^{-6s} \quad - \dfrac{6}{s^2}e^{-9s}$

$\qquad - \dfrac{9}{s}e^{-3s} \quad - \dfrac{9}{s}e^{-6s} \quad - \dfrac{9}{s}e^{-9s}$

$\bar{f}(s) = \dfrac{2}{s^3} + \dfrac{6}{s^2}e^{-3s} - \dfrac{9}{s}e^{-3s} - \dfrac{6}{s^2}e^{-6s} - \dfrac{9}{s}e^{-6s} - \dfrac{6}{s^2}e^{-9s} - \dfrac{9}{s}e^{-9s} \dots$

$$\therefore \boxed{\begin{aligned} f(t) = \ & t^2H(t) - 6(t-3)H(t-3) - 6(t-6)H(t-6) - 6(t-9)H(t-9) \dots \\ & - 9H(t-3) \quad - 9H(t-6) \quad - 9H(t-9) \ \dots \end{aligned}}$$

THUS

$0 \le t \le 3$	t^2	0	0	0
$3 \le t \le 6$	t^2	$-6t + 18 - 9$	0	0
$6 \le t \le 9$	t^2	$-6t + 18 - 9$	$-6t + 36 - 9$	0
$9 \le t \le 12$	t^2	$-6t + 18 - 9$	$-6t + 36 - 9$	$-6t + 54 - 9$
$\vdots$				

TIDY FURTHER

$f(t) = t^2 \quad \dots\dots\dots \quad 0 \le t \le 3$

$f(t) = t^2 - 6t + 9 = (t-3)^2 \ \dots \quad 3 \le t \le 6$

$f(t) = t^2 - 12t + 36 = (t-6)^2 \ \dots \quad 6 \le t \le 9$

$f(t) = t^2 - 18t + 81 = (t-9)^2 \ \dots \quad 9 \le t \le 12$

HENCE

$$\boxed{f(t) = t^2, \quad 0 \le t \le 3, \quad f(t+3) = f(t)}$$

SOLVING SIMPLE O.D.E.s

Question 1

Use Laplace transforms to solve the differential equation

$$\frac{dx}{dt} - 2x = 4 \,, \; t \geq 0 \,,$$

subject to the initial condition $x = 1$ at $t = 0$.

$\boxed{}$, $\boxed{x = 3e^{2t} - 2}$

TAKING THE LAPLACE TRANSFORM OF THE O.D.E, W.R.T t

$\Rightarrow \frac{dx}{dt} - 2x = 4 \qquad [t=0,\ x=1]$

$\Rightarrow \mathcal{L}\left[\frac{dx}{dt}\right] - \mathcal{L}[2x] = \mathcal{L}[4]$

$\Rightarrow s\bar{x} - x_0 - 2\bar{x} = \frac{4}{s}$

$\Rightarrow s\bar{x} - 1 - 2\bar{x} = \frac{4}{s}$

$\Rightarrow (s-2)\bar{x} = \frac{4}{s} + 1$

$\Rightarrow (s-2)\bar{x} = \frac{4+s}{s}$

$\Rightarrow \bar{x} = \frac{s+4}{s(s-2)}$

INVERT BY PARTIAL FRACTIONS (COVER UP)

$\Rightarrow \bar{x} = \frac{3}{s-2} - \frac{2}{s}$

$\Rightarrow x = \mathcal{L}^{-1}\left[\frac{3}{s-2} - \frac{2}{s}\right]$

THESE ARE SIMPLE STANDARD RESULTS

$\Rightarrow x(t) = 3e^{2t} - 2$

Question 2

Use Laplace transforms to solve the differential equation

$$\frac{dy}{dx} + 2y = 10e^{3x} \,, \; x \geq 0 \,,$$

subject to the boundary condition $y = 6$ at $x = 0$.

$\boxed{y = 2e^{3x} + 4e^{-2x}}$

$\frac{dy}{dx} + 2y = 10e^{3x}$, SUBJECT TO $x=0,\ y=6$

$\Rightarrow y' + 2y = 10e^{3x}$

$\Rightarrow s\bar{y} - y_0 + 2\bar{y} = \frac{10}{s-3}$ $\quad y_0 = 6$

$\Rightarrow s\bar{y} - 6 + 2\bar{y} = \frac{10}{s-3}$

$\Rightarrow (s+2)\bar{y} = \frac{10}{s-3} + 6$

$\Rightarrow (s+2)\bar{y} = \frac{6s-8}{s-3}$

$\Rightarrow \bar{y} = \frac{6s-8}{(s-3)(s+2)}$

$\Rightarrow \bar{y} = \frac{2}{s-3} + \frac{4}{s+2}$ (BY COVER UP)

$\Rightarrow y = \mathcal{L}^{-1}\left[\frac{2}{s-3} + \frac{4}{s+2}\right]$

$\Rightarrow y = 2e^{3x} + 4e^{-2x}$

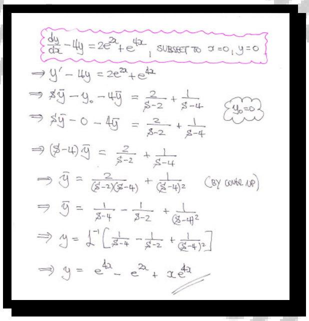
Question 3

Use Laplace transforms to solve the differential equation

$$\frac{dy}{dx} - 4y = 2e^{2x} + e^{4x}, \quad x \ge 0,$$

subject to the boundary condition $y = 0$ at $x = 0$.

$$\boxed{y = xe^{4x} + e^{4x} - 2e^{2x}}$$



$\frac{dy}{dx} - 4y = 2e^{2x} + e^{4x}$, SUBJECT TO $x = 0, y = 0$

$\Rightarrow y' - 4y = 2e^{2x} + e^{4x}$

$\Rightarrow s\bar{y} - y_0 - 4\bar{y} = \frac{2}{s-2} + \frac{1}{s-4}$

$y_0 = 0$

$\Rightarrow s\bar{y} - 0 - 4\bar{y} = \frac{2}{s-2} + \frac{1}{s-4}$

$\Rightarrow (s-4)\bar{y} = \frac{2}{s-2} + \frac{1}{s-4}$

$\Rightarrow \bar{y} = \frac{2}{(s-2)(s-4)} + \frac{1}{(s-4)^2}$ (BY COVER UP)

$\Rightarrow \bar{y} = \frac{1}{s-4} - \frac{1}{s-2} + \frac{1}{(s-4)^2}$

$\Rightarrow y = \mathcal{L}^{-1}\left[\frac{1}{s-4} - \frac{1}{s-2} + \frac{1}{(s-4)^2}\right]$

$\Rightarrow y = e^{4x} - e^{2x} + xe^{4x}$

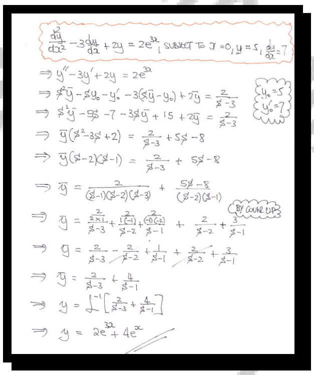
Question 4

Use Laplace transforms to solve the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2e^{3x}, \quad x \ge 0,$$

subject to the boundary conditions $y = 5$, $\frac{dy}{dx} = 7$ at $x = 0$.

$$\boxed{y = 2e^{3x} + 4e^{x}}$$



$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2e^{3x}$, SUBJECT TO $x = 0, y = 5, \frac{dy}{dx} = 7$

$\Rightarrow y'' - 3y' + 2y = 2e^{3x}$

$\Rightarrow s^2\bar{y} - sy_0 - y'_0 - 3(s\bar{y} - y_0) + 2\bar{y} = \frac{2}{s-3}$

$y_0 = 5$, $y'_0 = 7$

$\Rightarrow s^2\bar{y} - 5s - 7 - 3s\bar{y} + 15 + 2\bar{y} = \frac{2}{s-3}$

$\Rightarrow \bar{y}(s^2 - 3s + 2) = \frac{2}{s-3} + 5s - 8$

$\Rightarrow \bar{y}(s-2)(s-1) = \frac{2}{s-3} + 5s - 8$

$\Rightarrow \bar{y} = \frac{2}{(s-1)(s-2)(s-3)} + \frac{5s-8}{(s-2)(s-1)}$

BY COVER UP

$\Rightarrow \bar{y} = \frac{\frac{2}{2\times1}}{s-3} + \frac{\frac{2}{1(-1)}}{s-2} + \frac{\frac{2}{(-1)(-2)}}{s-1} + \frac{2}{s-2} + \frac{3}{s-1}$

$\Rightarrow \bar{y} = \frac{2}{s-3} - \cancel{\frac{2}{s-2}} + \frac{1}{s-1} + \cancel{\frac{2}{s-2}} + \frac{3}{s-1}$

$\Rightarrow \bar{y} = \frac{2}{s-3} + \frac{4}{s-1}$

$\Rightarrow y = \mathcal{L}^{-1}\left[\frac{2}{s-3} + \frac{4}{s-1}\right]$

$\Rightarrow y = 2e^{3x} + 4e^{x}$

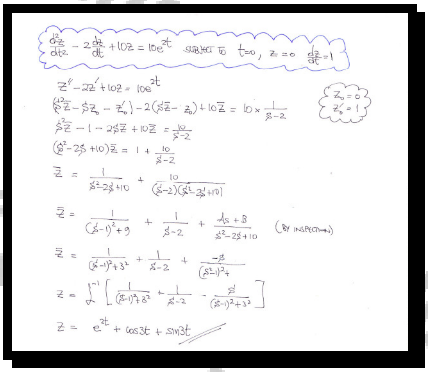
Question 5

Use Laplace transforms to solve the differential equation

$$\frac{d^2z}{dt^2} - 2\frac{dz}{dt} + 10z = 10e^{2t},$$

subject to the initial conditions $z = 0$, $\frac{dz}{dt} = 1$ at $t = 0$.

$$\boxed{y = e^{2t} + \cos 3t + \sin 3t}$$



$\frac{d^2z}{dt^2} - 2\frac{dz}{dt} + 10z = 10e^{2t}$ SUBJECT TO $t=0$, $z=0$ $\frac{dz}{dt}=1$

$z'' - 2z' + 10z = 10e^{2t}$

$z_0 = 0$
$z_0' = 1$

$(s^2\bar{z} - sz_0 - z_0') - 2(s\bar{z} - z_0) + 10\bar{z} = 10 \times \frac{1}{s-2}$

$s^2\bar{z} - 1 - 2s\bar{z} + 10\bar{z} = \frac{10}{s-2}$

$(s^2 - 2s + 10)\bar{z} = 1 + \frac{10}{s-2}$

$\bar{z} = \frac{1}{s^2-2s+10} + \frac{10}{(s-2)(s^2-2s+10)}$

$\bar{z} = \frac{1}{(s-1)^2+9} + \frac{1}{s-2} + \frac{As+B}{s^2-2s+10}$ (BY INSPECTION)

$\bar{z} = \frac{1}{(s-1)^2+3^2} + \frac{1}{s-2} + \frac{-s}{(s-1)^2+}$

$z = \mathcal{L}^{-1}\left[\frac{1}{(s-1)^2+3^2} + \frac{1}{s-2} - \frac{s}{(s-1)^2+3^2}\right]$

$z = e^{2t} + \cos 3t + \sin 3t$

Question 6

Use Laplace transforms to solve the differential equation

$$\frac{d^2y}{dx^2} - 4y = 24\cos 2x, \quad x \geq 0,$$

subject to the boundary conditions $y = 3$, $\dfrac{dy}{dx} = 4$ at $x = 0$.

$$\boxed{y = 4\mathrm{e}^{2x} + 2\mathrm{e}^{-2x} - 3\cos 2x}$$

$$\boxed{\frac{d^2y}{dx^2} - 4y = 24\cos 2x, \quad x \geqslant 0, \quad x=0,\ y=3,\ \frac{dy}{dx}=4}$$

WRITE THE O.D.E IN COMPACT FORM, & TAKE LAPLACE TRANSFORMS IN x

$$\Rightarrow y'' - 4y = 24\cos 2x$$

$$\Rightarrow s^2\bar{y} - sy_0 - y_0' - 4\bar{y} = \mathcal{L}\left[24\cos 2x\right]$$

$$\Rightarrow s^2\bar{y} - 3s - 4 - 4\bar{y} = 24 \times \frac{s}{s^2+4}$$

$$\Rightarrow (s^2-4)\bar{y} = 3s + 4 + \frac{24s}{s^2+4}$$

$$\Rightarrow \bar{y} = \frac{3s+4}{s^2-4} + \frac{24s}{(s^2-4)(s^2+4)}$$

$$\Rightarrow \bar{y} = \frac{3s+4}{(s-2)(s+2)} + \frac{24s}{(s-2)(s+2)(s^2+4)}$$

PARTIAL FRACTIONS MAINLY BY INSPECTION (COVER UP)

$$\Rightarrow \bar{y} = \frac{\frac{10}{4}}{s-2} + \frac{\frac{-2}{-4}}{s+2} + \frac{\frac{48}{4\times 8}}{s-2} + \frac{\frac{-48}{-4\times 8}}{s+2} + \frac{As+B}{s^2+4}$$

$$\Rightarrow \bar{y} = \frac{\frac{5}{2}}{s-2} + \frac{\frac{1}{2}}{s+2} + \underbrace{\frac{\frac{3}{2}}{s-2} + \frac{\frac{3}{2}}{s+2} + \frac{As+B}{s^2+4}}$$

$$24s \equiv \tfrac{3}{2}(s+2)(s^2+4) + \tfrac{3}{2}(s-2)(s^2+4) + (s^2-4)(As+B)$$

- IF $s=0 \Rightarrow 0 = 12 - 12 - 4B$
  $\Rightarrow 4B = 0$
  $\Rightarrow B = 0$
- IF $s=1 \Rightarrow 24 = \frac{45}{2} - \frac{15}{2} - 3(A)$
  $\Rightarrow 24 = 15 - 3A$
  $\Rightarrow 3A = -9$
  $\Rightarrow A = -3$

COLLECTING ALL RESULTS

$$\bar{y} = \frac{\frac{5}{2}}{s-2} + \frac{\frac{1}{2}}{s+2} + \frac{\frac{3}{2}}{s-2} + \frac{\frac{3}{2}}{s+2} - \frac{3s}{s^2+4}$$

$$\bar{y} = \frac{4}{s-2} + \frac{2}{s+2} - 3\left(\frac{s}{s^2+4}\right)$$

INVERTING (ALL ONLY SIMPLE STANDARD RESULTS)

$$y = 4e^{2x} + 2e^{-2x} - 3\cos 2x$$

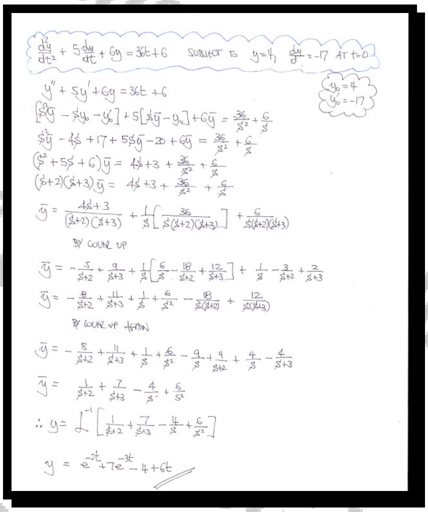
Question 7

Use Laplace transforms to solve the differential equation

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 36t + 6,$$

subject to the initial conditions $y = 4$, $\frac{dy}{dt} = -17$ at $t = 0$.

$$\boxed{y = e^{-2t} + 7e^{-3t} + 6t - 4}$$



$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 36t + 6$ SUBJECT TO $y = 4$, $\frac{dy}{dt} = -17$ AT $t = 0$

$y_0 = 4$
$y'_0 = -17$

$y'' + 5y' + 6y = 36t + 6$

$[s^2\bar{y} - sy_0 - y'_0] + 5[s\bar{y} - y_0] + 6\bar{y} = \frac{36}{s^2} + \frac{6}{s}$

$s^2\bar{y} - 4s + 17 + 5s\bar{y} - 20 + 6\bar{y} = \frac{36}{s^2} + \frac{6}{s}$

$(s^2 + 5s + 6)\bar{y} = 4s + 3 + \frac{36}{s^2} + \frac{6}{s}$

$(s+2)(s+3)\bar{y} = 4s + 3 + \frac{36}{s^2} + \frac{6}{s}$

$\bar{y} = \frac{4s+3}{(s+2)(s+3)} + \frac{1}{s}\left[\frac{36}{s(s+2)(s+3)}\right] + \frac{6}{s(s+2)(s+3)}$

BY COVER UP

$\bar{y} = -\frac{5}{s+2} + \frac{9}{s+3} + \frac{1}{s}\left[\frac{6}{s} - \frac{18}{s+2} + \frac{12}{s+3}\right] + \frac{1}{s} - \frac{3}{s+2} + \frac{2}{s+3}$

$\bar{y} = -\frac{8}{s+2} + \frac{11}{s+3} + \frac{1}{s} + \frac{6}{s^2} - \frac{18}{s(s+2)} + \frac{12}{s(s+3)}$

BY COVER UP AGAIN

$\bar{y} = -\frac{8}{s+2} + \frac{11}{s+3} + \frac{1}{s} + \frac{6}{s^2} - \frac{9}{s} + \frac{9}{s+2} + \frac{4}{s} - \frac{4}{s+3}$

$\bar{y} = \frac{1}{s+2} + \frac{7}{s+3} - \frac{4}{s} + \frac{6}{s^2}$

$\therefore y = \mathcal{L}^{-1}\left[\frac{1}{s+2} + \frac{7}{s+3} - \frac{4}{s} + \frac{6}{s^2}\right]$

$y = e^{-2t} + 7e^{-3t} - 4 + 6t$

Question 8

$$\frac{dx}{dt} + y = \mathrm{e}^{-t} \qquad \text{and} \qquad \frac{dy}{dt} - x = \mathrm{e}^{t}.$$

Use Laplace transformations to solve the above simultaneous differential equations, subject to the initial conditions $x = 0$, $y = 0$ at $t = 0$.

$\boxed{}$, $\boxed{x = -\cosh t + \sin t + \cos t, \quad y = \cosh t + \sin t - \cos t}$

$$\boxed{\begin{array}{l} \frac{dy}{dt} - x = e^{t} \\ \frac{dx}{dt} + y = e^{-t} \end{array} \quad \text{, SUBJECT TO } t=0, x=0, y=0}$$

- WRITE IN COMPACT NOTATION & TAKE LAPLACE TRANSFORMS IN t

$$\left.\begin{array}{l} \dot{y} - x = e^{t} \\ \dot{x} + y = e^{-t} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} s\bar{y} - y_0 - \bar{x} = \frac{1}{s-1} \\ s\bar{x} - x_0 + \bar{y} = \frac{1}{s+1} \end{array}\right\} \qquad y_0 = x_0 = 0$$

$$\Rightarrow \left\{\begin{array}{l} s\bar{y} - \bar{x} = \frac{1}{s-1} \\ s\bar{x} + \bar{y} = \frac{1}{s+1} \end{array}\right\} \leftarrow (\times s)$$

$$\Rightarrow \left\{\begin{array}{l} s^2\bar{y} - s\bar{x} = \frac{s}{s-1} \\ s\bar{x} + \bar{y} = \frac{1}{s+1} \end{array}\right\} \text{ ADDING}$$

$$\Rightarrow (s^2+1)\bar{y} = \frac{s}{s-1} + \frac{1}{s+1}$$

$$\Rightarrow (s^2+1)\bar{y} = \frac{s^2+s+s-1}{(s-1)(s+1)}$$

$$\Rightarrow \bar{y} = \frac{s^2+2s-1}{(s^2+1)(s-1)(s+1)}$$

- SPLIT BY PARTIAL FRACTIONS IN ORDER TO INVERT

$$\frac{s^2+2s-1}{(s^2+1)(s+1)(s-1)} \equiv \frac{A}{s+1} + \frac{B}{s-1} + \frac{Cs+D}{s^2+1}$$

$$s^2+2s-1 \equiv A(s-1)(s^2+1) + B(s+1)(s^2+1) + (s^2-1)(Cs+D)$$

- If $s=1$, $2 = 4B \Rightarrow B = \frac{1}{2}$
- If $s=-1$, $-2 = -4A \Rightarrow A = \frac{1}{2}$
- IF $s=0$, $-1 = -A+B-D$
  $D = 1-A+B = 1-\frac{1}{2}+\frac{1}{2}$
  $D = 1$
- IF $s=2$ $\quad 4+4-1 = 5A + 15B + 3(2C+D)$
  $7 = \frac{5}{2} + \frac{15}{2} + 3(2C+1)$
  $7 = 10 + 3(2C+1)$
  $-3 = 3(2C+1)$
  $-1 = 2C+1$
  $-2 = 2C$
  $C = -1$

- INVERTING THE TRANSFORM, USING STANDARD RESULTS

$$\Rightarrow \bar{y} = \frac{\frac{1}{2}}{s-1} + \frac{\frac{1}{2}}{s+1} - \frac{s-1}{s^2+1}$$

$$\Rightarrow \bar{y} = \frac{1}{2}\left(\frac{1}{s-1}\right) + \frac{1}{2}\left(\frac{1}{s-1}\right) - \left(\frac{s}{s^2+1}\right) + \left(\frac{1}{s^2+1}\right)$$

$$\Rightarrow y = \frac{1}{2}e^{t} + \frac{1}{2}e^{-t} - \cos t + \sin t$$

$$\Rightarrow y = \cosh t - \cos t + \sin t$$

- TO FIND THE OTHER SOLUTION, USE THE FIRST O.D.E

$$\Rightarrow x = \frac{dy}{dt} - e^{t}$$

$$\Rightarrow x = \sinh t + \sin t + \cos t - e^{t}$$

$$\Rightarrow x = \frac{1}{2}e^{t} - \frac{1}{2}e^{-t} - e^{t} + \sin t + \cos t$$

$$\Rightarrow x = -\frac{1}{2}e^{t} - \frac{1}{2}e^{-t} + \sin t + \cos t$$

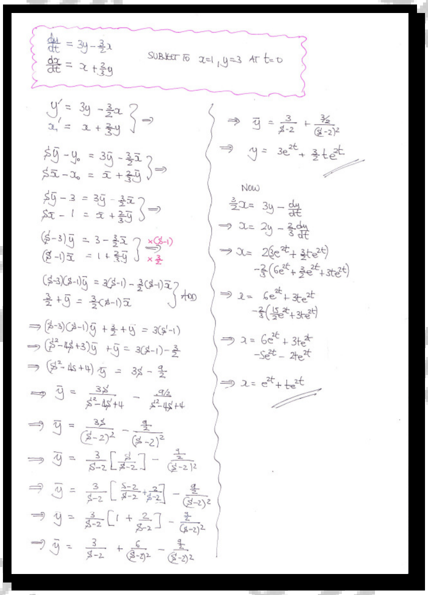
$$\Rightarrow x = -\cosh t + \cos t + \sin t$$

Question 9

$$\frac{dx}{dt} = x + \frac{2}{3}y \qquad \text{and} \qquad \frac{dy}{dt} = 3y - \frac{3}{2}x \, .$$

Use Laplace transformations to solve the above simultaneous differential equations, subject to the initial conditions $x = 1$, $y = 3$ at $t = 0$.

$$\boxed{x = e^{2t} + t\,e^{2t}, \quad y = 3e^{2t} + \frac{3}{2}t\,e^{2t}}$$



$\frac{dy}{dt} = 3y - \frac{3}{2}x$

$\frac{dx}{dt} = x + \frac{2}{3}y$

SUBJECT TO $x=1, y=3$ AT $t=0$

$\left.\begin{array}{l} y' = 3y - \frac{3}{2}x \\ x' = x + \frac{2}{3}y \end{array}\right\} \Rightarrow$

$\left.\begin{array}{l} s\bar{y} - y_0 = 3\bar{y} - \frac{3}{2}\bar{x} \\ s\bar{x} - x_0 = \bar{x} + \frac{2}{3}\bar{y} \end{array}\right\} \Rightarrow$

$\left.\begin{array}{l} s\bar{y} - 3 = 3\bar{y} - \frac{3}{2}\bar{x} \\ s\bar{x} - 1 = \bar{x} + \frac{2}{3}\bar{y} \end{array}\right\} \Rightarrow$

$\left.\begin{array}{l} (s-3)\bar{y} = 3 - \frac{3}{2}\bar{x} \\ (s-1)\bar{x} = 1 + \frac{2}{3}\bar{y} \end{array}\right\} \begin{array}{l} \times(s-1) \\ \Rightarrow \\ \times\frac{3}{2} \end{array}$

$\left.\begin{array}{l} (s-3)(s-1)\bar{y} = 3(s-1) - \frac{3}{2}(s-1)\bar{x} \\ \frac{3}{2} + \bar{y} = \frac{3}{2}(s-1)\bar{x} \end{array}\right\}$ ADD

$\Rightarrow (s-3)(s-1)\bar{y} + \frac{3}{2} + \bar{y} = 3(s-1)$

$\Rightarrow (s^2-4s+3)\bar{y} + \bar{y} = 3(s-1) - \frac{3}{2}$

$\Rightarrow (s^2-4s+4)\bar{y} = 3s - \frac{9}{2}$

$\Rightarrow \bar{y} = \frac{3s}{s^2-4s+4} - \frac{9/2}{s^2-4s+4}$

$\Rightarrow \bar{y} = \frac{3s}{(s-2)^2} - \frac{\frac{9}{2}}{(s-2)^2}$

$\Rightarrow \bar{y} = \frac{3}{s-2}\left[\frac{s}{s-2}\right] - \frac{\frac{9}{2}}{(s-2)^2}$

$\Rightarrow \bar{y} = \frac{3}{s-2}\left[\frac{s-2}{s-2} + \frac{2}{s-2}\right] - \frac{\frac{9}{2}}{(s-2)^2}$

$\Rightarrow \bar{y} = \frac{3}{s-2}\left[1 + \frac{2}{s-2}\right] - \frac{\frac{9}{2}}{(s-2)^2}$

$\Rightarrow \bar{y} = \frac{3}{s-2} + \frac{6}{(s-2)^2} - \frac{\frac{9}{2}}{(s-2)^2}$

$\Rightarrow \bar{y} = \frac{3}{s-2} + \frac{3/2}{(s-2)^2}$

$\Rightarrow y = 3e^{2t} + \frac{3}{2}te^{2t}$

NOW

$\frac{3}{2}x = 3y - \frac{dy}{dt}$

$\Rightarrow x = 2y - \frac{2}{3}\frac{dy}{dt}$

$\Rightarrow x = 2(3e^{2t} + \frac{3}{2}te^{2t}) - \frac{2}{3}(6e^{2t} + \frac{3}{2}e^{2t} + 3te^{2t})$

$\Rightarrow x = 6e^{2t} + 3te^{2t} - \frac{2}{3}(\frac{15}{2}e^{2t} + 3te^{2t})$

$\Rightarrow x = 6e^{2t} + 3te^{2t} - 5e^{2t} - 2te^{2t}$

$\Rightarrow x = e^{2t} + te^{2t}$

Question 10

$$\frac{d^2x}{dt^2} = 15\frac{dy}{dt} - 9y + 22e^t \qquad \text{and} \qquad \frac{d^2y}{dt^2} = 2x + e^{3t}.$$

The functions $x = f(t)$ and $y = g(t)$ satisfy the above simultaneous differential equations, subject to the initial conditions

$$x = 2, \quad y = -3, \quad \frac{dx}{dt} = 10, \quad \frac{dy}{dt} = -1 \quad \text{at } t = 0.$$

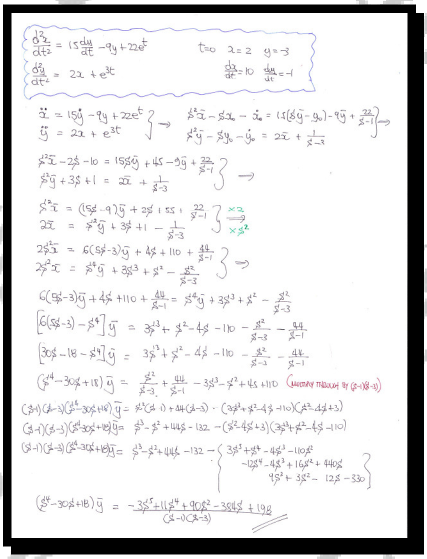
a) By using Laplace transforms, show that

$$\left(s^4 - 30s + 18\right)\bar{y} = \frac{-3s^5 + 11s^4 + 90s^2 - 384s + 198}{(s-1)(s-3)},$$

where $\bar{y} = \mathcal{L}\left[g(t)\right]$.

b) Given further that $s^4 - 30s + 18$ is a factor of $-3s^5 + 11s^4 + 90s^2 - 384s + 198$, find expressions for x and y.

$$\boxed{x = 4e^{3t} - 2e^t, \quad y = e^{3t} - 4e^t}$$



$$\frac{d^2x}{dt^2} = 15\frac{dy}{dt} - 9y + 22e^t \qquad t=0 \quad x=2 \quad y=-3$$
$$\frac{d^2y}{dt^2} = 2x + e^{3t} \qquad \frac{dx}{dt}=10 \quad \frac{dy}{dt}=-1$$

$$\left.\begin{array}{l}\ddot{x} = 15\dot{y} - 9y + 22e^t \\ \ddot{y} = 2x + e^{3t}\end{array}\right\} \Rightarrow \left.\begin{array}{l} s^2\bar{x} - sx_0 - \dot{x}_0 = 15(s\bar{y} - y_0) - 9\bar{y} + \frac{22}{s-1} \\ s^2\bar{y} - sy_0 - \dot{y}_0 = 2\bar{x} + \frac{1}{s-3}\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l} s^2\bar{x} - 2s - 10 = 15s\bar{y} + 45 - 9\bar{y} + \frac{22}{s-1} \\ s^2\bar{y} + 3s + 1 = 2\bar{x} + \frac{1}{s-3}\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l} s^2\bar{x} = (15s - 9)\bar{y} + 2s + 55 + \frac{22}{s-1} \\ 2\bar{x} = s^2\bar{y} + 3s + 1 - \frac{1}{s-3}\end{array}\right\} \xrightarrow[\times s^2]{\times 2}$$

$$\left.\begin{array}{l} 2s^2\bar{x} = 6(5s-3)\bar{y} + 4s + 110 + \frac{44}{s-1} \\ 2s^2\bar{x} = s^4\bar{y} + 3s^3 + s^2 - \frac{s^2}{s-3}\end{array}\right\} \Rightarrow$$

$$6(5s-3)\bar{y} + 4s + 110 + \frac{44}{s-1} = s^4\bar{y} + 3s^3 + s^2 - \frac{s^2}{s-3}$$

$$\left[6(5s-3) - s^4\right]\bar{y} = 3s^3 + s^2 - 4s - 110 - \frac{s^2}{s-3} - \frac{44}{s-1}$$

$$\left[30s - 18 - s^4\right]\bar{y} = 3s^3 + s^2 - 4s - 110 - \frac{s^2}{s-3} - \frac{44}{s-1}$$

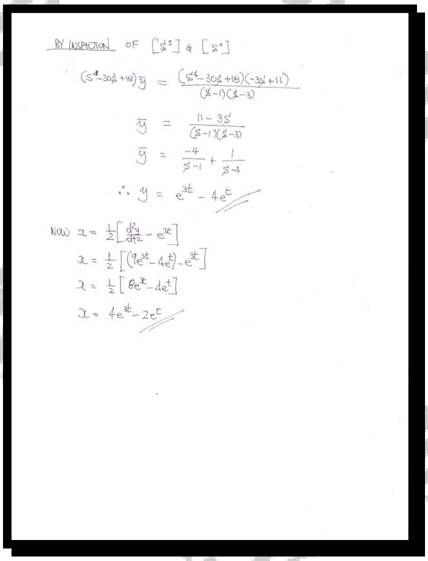
$$(s^4 - 30s + 18)\bar{y} = \frac{s^2}{s-3} + \frac{44}{s-1} - 3s^3 - s^2 + 4s + 110 \quad \text{(MULTIPLY THROUGH BY } (s-1)(s-3))$$

$$(s-1)(s-3)(s^4 - 30s + 18)\bar{y} = s^2(s-1) + 44(s-3) - (3s^3 + s^2 - 4s - 110)(s^2 - 4s + 3)$$

$$(s-1)(s-3)(s^4 - 30s + 18)\bar{y} = s^3 - s^2 + 44s - 132 - (s^2 - 4s + 3)(3s^3 + s^2 - 4s - 110)$$

$$(s-1)(s-3)(s^4 - 30s + 18)\bar{y} = s^3 - s^2 + 44s - 132 - \left\{\begin{array}{r} 3s^5 + s^4 - 4s^3 - 110s^2 \\ -12s^4 - 4s^3 + 16s^2 + 440s \\ 9s^3 + 3s^2 - 12s - 330 \end{array}\right\}$$

$$(s^4 - 30s + 18)\bar{y} = \frac{-3s^5 + 11s^4 + 90s^2 - 384s + 198}{(s-1)(s-3)}$$



BY INSPECTION OF $[s^5]$ & $[s^0]$

$$(s^4 - 30s + 18)\bar{y} = \frac{(s^4 - 30s + 18)(-3s + 11)}{(s-1)(s-3)}$$

$$\bar{y} = \frac{11 - 3s}{(s-1)(s-3)}$$

$$\bar{y} = \frac{-4}{s-1} + \frac{1}{s-3}$$

$$\therefore y = e^{3t} - 4e^t$$

NOW $x = \frac{1}{2}\left[\frac{d^2y}{dt^2} - e^{3t}\right]$

$$x = \frac{1}{2}\left[(9e^{3t} - 4e^t) - e^{3t}\right]$$

$$x = \frac{1}{2}\left[8e^{3t} - 4e^t\right]$$

$$x = 4e^{3t} - 2e^t$$

Question 11

Use Laplace transforms to solve the differential equation

$$\frac{d^2x}{dt^2} + x = f(t),$$

given further that $x = 1$, $\frac{dx}{dt} = 1$ at $t = 0$, and $f(t) = \begin{cases} 0 & t < 0 \\ t & 0 \le t \le \pi \\ \pi & t > \pi \end{cases}$

$$\boxed{x = t + \cos t - (t-\pi)\mathrm{H}(t-\pi) + \sin(t-\pi)\mathrm{H}(t-\pi)}$$

$$\boxed{\frac{d^2x}{dt^2} + x = f(t) \quad \text{WHERE } f(t) = \begin{cases} t & 0 \le t \le \pi \\ \pi & t > \pi \\ 0 & t < 0 \end{cases} \qquad \text{SUBJECT TO } x = 1,\ \frac{dx}{dt} = 1 \text{ AT } t = 0}$$

$$\ddot{x} + x = f(t)$$

TAKING LAPLACE TRANSFORMS

$$\Rightarrow \mathcal{L}[\ddot{x}] + \mathcal{L}[x] = \mathcal{L}[f(t)]$$

$$\Rightarrow s^2\bar{x} - sx_0 - \dot{x}_0 + \bar{x} = \int_0^\infty f(t)e^{-st}\,dt$$

$$\Rightarrow s^2\bar{x} - s - 1 + \bar{x} = \int_0^\pi te^{-st}\,dt + \int_\pi^\infty \pi e^{-st}\,dt$$

$$\Rightarrow (1+s^2)\bar{x} - (1+s) = \int_0^\pi -\frac{d}{ds}\left[e^{-st}\right]dt + \frac{\pi}{s}\left[e^{-st}\right]_\infty^\pi$$

$$\Rightarrow (1+s^2)\bar{x} = (s+1) - \frac{d}{ds}\int_0^\pi e^{-st}\,dt + \frac{\pi}{s}\left[e^{-s\pi} - 0\right]$$

$$\Rightarrow (1+s^2)\bar{x} = s + 1 - \frac{d}{ds}\left[-\frac{1}{s}\left[e^{-st}\right]_0^\pi\right] + \frac{\pi}{s}\left[e^{-s\pi}\right]$$

$$\Rightarrow (1+s^2)\bar{x} = s + 1 - \frac{d}{ds}\left[\frac{1}{s} - \frac{1}{s}e^{-s\pi}\right] + \frac{\pi e^{-s\pi}}{s}$$

$$\Rightarrow (1+s^2)\bar{x} = s + 1 - \left[-\frac{1}{s^2} + \frac{1}{s^2}e^{-s\pi} + \frac{\pi}{s}e^{-s\pi}\right] + \frac{\pi}{s}e^{-s\pi}$$

$$\Rightarrow (1+s^2)\bar{x} = s + 1 + \frac{1}{s^2} - \frac{1}{s^2}e^{-s\pi} \cancel{- \frac{\pi}{s}e^{-s\pi}} \cancel{+ \frac{\pi}{s}e^{-s\pi}}$$

$$\Rightarrow \bar{x} = \frac{s + 1 + \frac{1}{s^2}(1 - e^{-s\pi})}{s^2+1}$$

$$\Rightarrow \bar{x} = \frac{s+1}{s^2+1} + \frac{1}{s^2(s^2+1)}(1 - e^{-s\pi})$$

PARTIAL FRACTIONS BY COVER UP SINCE IT CAN BE TREATED AS $\frac{1}{A(A+1)}$

$$\Rightarrow \bar{x} = \frac{s}{s^2+1} + \frac{1}{s^2+1} + \frac{1}{s^2}(1 - e^{-s\pi}) - \frac{1}{s^2+1}(1 - e^{-s\pi})$$

$$\Rightarrow \bar{x} = \frac{s}{s^2+1} + \cancel{\frac{1}{s^2+1}} + \frac{1}{s^2} - \frac{e^{-s\pi}}{s^2} - \cancel{\frac{1}{s^2+1}} + \frac{e^{-s\pi}}{s^2+1}$$

$$\Rightarrow \bar{x} = \frac{s}{s^2+1} + \frac{1}{s^2} - \frac{e^{-s\pi}}{s^2} + \frac{e^{-s\pi}}{s^2+1}$$

$\sin t$ "HEAVISIDED"

STANDARD $\cos t$ — STANDARD t — t "HEAVISIDED"

INVERTING

$$x(t) = \cos t + t - (t-\pi)\mathrm{H}(t-\pi) + \sin(t-\pi)\mathrm{H}(t-\pi)$$

$$\boxed{\text{SINCE } \mathcal{L}\left[f(t-c)\mathrm{H}(t-c)\right] = e^{-cs}\bar{f}(s)}$$

WHICH CAN ALSO BE WRITTEN AS

$$x(t) = \begin{cases} \cos t + t & 0 \le t \le \pi \\ \pi + \cos t - \sin t & t \ge \pi \end{cases}$$

$\sin(t-\pi) = -\sin t$

Question 12

Use Laplace transforms to solve the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = \delta(t-2),$$

given further that $x = 0$, $\frac{dx}{dt} = 1$ at $t = 0$.

$$\boxed{x = e^{-t}\left[\sin 2t - e^4 \sin(2t-4)\text{H}(t-2)\right]}$$

$$\ddot{x} + 2\dot{x} + 5x = \delta(t-2) \qquad t=0,\ x_0 = 0,\ \dot{x}_0 = 1$$

TAKING LAPLACE TRANSFORMS

$$\Rightarrow \left[s^2\bar{x} - sx_0 - \dot{x}_0\right] + 2\left[s\bar{x} - x_0\right] + 5\bar{x} = \mathcal{L}\left[\delta(t-2)\right]$$

$$\Rightarrow s^2\bar{x} - 1 + 2s\bar{x} + 5\bar{x} = e^{-2s}$$

$$\Rightarrow \bar{x}(s^2+2s+5) = 1 - e^{-2s}$$

$$\Rightarrow \bar{x} = \frac{1-e^{-2s}}{s^2+2s+5}$$

$$\Rightarrow \bar{x} = \frac{1-e^{-2s}}{(s+1)^2+4}$$

$$\Rightarrow \bar{x} = \frac{1}{(s+1)^2+4} - \frac{e^{-2s}}{(s+1)^2+4}$$

INVERTING

$$\Rightarrow x = e^{-t}\sin 2t - e^{-(t-2)}\sin 2(t-2)\,\text{H}(t-2)$$

$$\Rightarrow x = e^{-t}\sin 2t - e^{-t}e^{4}\sin(2t-4)\,\text{H}(t-2) \quad //$$

Question 13

Use Laplace transforms to solve the differential equation

$$\frac{d^2x}{dt^2}+4\frac{dx}{dt}+3x=2\delta(t-6),$$

given further that $x=0$, $\frac{dx}{dt}=2$ at $t=0$.

$$\boxed{x=e^{-3t}\left[e^{2t}-1\right]+e^{-3t}e^{6}\left[e^{12}-e^{2t}\right]H(t-6)}$$

$\ddot{x}+4\dot{x}+3x=2\delta(t-6)$ SUBJECT TO $t=0$, $x_0=0$, $\dot{x}_0=2$

TAKING LAPLACE TRANSFORMS

$\Rightarrow \left[s^2\bar{x}-sx_0-\dot{x}_0\right]+4\left[s\bar{x}-x_0\right]+3\bar{x}=\mathcal{L}\left[2\delta(t-6)\right]$

$\Rightarrow s^2\bar{x}-2+4s\bar{x}+3\bar{x}=2e^{-6s}$

$\Rightarrow \bar{x}(s^2+4s+3)=2-2e^{-6s}$

$\Rightarrow \bar{x}=\frac{2(1-e^{-6s})}{s^2+4s+3}$

$\Rightarrow \bar{x}=2(1-e^{-6s})\times\frac{1}{(s+1)(s+3)}$ ← PARTIAL FRACTIONS

$\Rightarrow \bar{x}=2(1-e^{-6s})\times\left[\frac{\frac{1}{2}}{s+1}-\frac{\frac{1}{2}}{s+3}\right]$

$\Rightarrow \bar{x}=\frac{1-e^{-6s}}{s+1}-\frac{1-e^{-6s}}{s+3}$

$\Rightarrow \bar{x}=\frac{1}{s+1}-\frac{e^{-6s}}{s+1}-\frac{1}{s+3}+\frac{e^{-6s}}{s+3}$

INVERTING...

$x(t)=e^{-t}-e^{-(t-6)}H(t-6)-e^{-3t}+e^{-3(t-6)}H(t-6)$

$x(t)=e^{-t}-e^{-3t}+e^{-3t}e^{18}H(t-6)-e^{-t}e^{6}H(t-6)$

$x(t)=e^{-3t}\left[e^{2t}-1\right]+e^{-3t}e^{6}H(t-6)\left[e^{12}-e^{2t}\right]$ //

Question 14

Use Laplace transforms to solve the differential equation

$$\frac{d^2y}{dt^2} + y = f(t),$$

given further that $y = 0$, $\frac{dy}{dt} = 1$ at $t = 0$, and $f(t)$ is a known function which has a Laplace transform.

You may leave the final answer containing a convolution type integral.

$$\boxed{y = \sin t + \int_0^t f(u)\sin(t-u)\, du}$$

$$\boxed{\frac{d^2y}{dt^2} + y = f(t) \quad \text{SUBJECT TO } t=0,\ y=0,\ \frac{dy}{dt}=1}$$

TAKING THE LAPLACE TRANSFORM OF THE O.D.E IN t

$\Rightarrow \mathcal{L}\left[\frac{d^2y}{dt^2}\right] + \mathcal{L}[y] = \mathcal{L}[f(t)]$

$\Rightarrow s^2\bar{y} - \cancel{sy_0} - \dot{y}_0 + \bar{y} = \bar{f}(s)$

$\Rightarrow s^2\bar{y} - 1 + \bar{y} = \bar{f}(s)$

$\Rightarrow (s^2+1)\bar{y} - 1 = \bar{f}(s)$

$\Rightarrow \bar{y} = \frac{1+\bar{f}(s)}{s^2+1} = \frac{1}{s^2+1} + \frac{\bar{f}(s)}{s^2+1} = \frac{1}{s^2+1} + \bar{f}(s) \times \frac{1}{s^2+1}$

INVERTING

$\Rightarrow y = \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] + \mathcal{L}^{-1}\left[\bar{f}(s) \times \frac{1}{s^2+1}\right]$

$\Rightarrow y = \sin t + \mathcal{L}^{-1}\left[\bar{f}(s) \times \frac{1}{s^2+1}\right]$

BY THE CONVOLUTION THEOREM

$\mathcal{L}[f*g] = \mathcal{L}[f]\,\mathcal{L}[g]$

$\overline{f*g} = \bar{f}\,\bar{g}$

$f*g = \mathcal{L}^{-1}[\bar{f}\,\bar{g}]$

$\mathcal{L}^{-1}[\bar{f}\,\bar{g}] = \int_0^t f(u)\,g(t-u)\,du$

HERE $f(t) \mapsto \bar{f}(s)$

$g(t) = \sin t \mapsto \bar{g}(s) = \frac{1}{s^2+1}$

$\therefore y = \sin t + \int_0^t f(u)\sin(t-u)\,du$

Question 15

$$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+2x=f(t).$$

a) Use Laplace transforms to solve the above differential equation, given further that $x=0$, $\frac{dx}{dt}=0$ at $t=0$, and $f(t)$ is a known function which has a Laplace transform.

You may leave the answer containing a convolution type integral.

b) If $f(t)=e^{2t}$ find $x=x(t)$ explicitly.

$$\boxed{x=\int_0^t f(t-u)\,e^{-u}\sin u\;du}\,,\quad \boxed{x=-\frac{1}{10}e^{-t}\left[3\sin t+\cos t\right]+\frac{1}{10}e^{2t}}$$

a) $\boxed{\frac{d^2x}{dt^2}+2\frac{dx}{dt}+2x=f(t) \text{ SUBJECT TO } t=0,\ x=0,\ \dot{x}=0}$

TAKING THE LAPLACE TRANSFORM OF THE EQUATION

$\Rightarrow \mathcal{L}\left[\frac{d^2x}{dt^2}\right]+2\mathcal{L}\left[\frac{dx}{dt}\right]+2\mathcal{L}[x]=\mathcal{L}[f(t)]$

$\Rightarrow s^2\bar{x}-\cancel{sx_0}-\cancel{\dot{x}_0}+2[s\bar{x}-\cancel{x_0}]+2\bar{x}=\bar{f}(s)$

$\Rightarrow (s^2+2s+2)\bar{x}=\bar{f}(s)$

$\Rightarrow \bar{x}=\frac{\bar{f}(s)}{s^2+2s+2}$

$\Rightarrow \bar{x}=\bar{f}(s)\times\frac{1}{s^2+2s+2}$

$\Rightarrow \bar{x}=\bar{f}(s)\times\frac{1}{(s+1)^2+1}$

$\bar{f}(s)$ $\quad \bar{g}(s) \rightarrow g(t)=e^{-t}\sin t$ (BY INSPECTION)

BY THE CONVOLUTION THEOREM

$\Rightarrow \mathcal{L}[f*g]=\mathcal{L}[f]\,\mathcal{L}[g]$

$\Rightarrow \overline{f*g}=\bar{f}\,\bar{g}$

$\Rightarrow \mathcal{L}^{-1}[\overline{f*g}]=\mathcal{L}^{-1}[\bar{f}\,\bar{g}]$

$\Rightarrow f*g=\mathcal{L}^{-1}[\bar{f}\,\bar{g}]$

$\Rightarrow f*g \quad \mathcal{L}^{-1}(\bar{x})$

$\Rightarrow \int_0^t f(u)g(t-u)\,du = x$

$\Rightarrow x=\int_0^t f(t-u)g(u)\,du$ ("COMMUTATIVE")

$\Rightarrow x=\int_0^t f(t-u)e^{-u}\sin u\,du$

b) NOW $f(t)=e^{2t}$ SO $f(t-u)=e^{2(t-u)}$

$\bar{x}=\int_0^t e^{2t-2u}e^{-u}\sin u\,du = e^{2t}\int_0^t e^{-3u}\sin u\,du$

$=e^{2t}\,\mathrm{Im}\left[\int_0^t e^{-3u}e^{iu}\,du\right]=e^{2t}\,\mathrm{Im}\left[\int_0^t e^{u(-3+i)}\,du\right]$

$=e^{2t}\,\mathrm{Im}\left[\frac{1}{-3+i}e^{u(-3+i)}\right]_0^t=e^{2t}\,\mathrm{Im}\left[\frac{-3-i}{10}e^{-3u}e^{iu}\right]_0^t$

$=e^{2t}\,\mathrm{Im}\left[\frac{1}{10}(-3-i)e^{-3u}(\cos u+i\sin u)\right]_0^t$

SIMPLIFYING

$=e^{2t}\,\mathrm{Im}\left[\frac{1}{10}e^{-3t}(-3-i)(\cos t+i\sin t)+\frac{1}{10}(-3-i)\right]$

$=\frac{1}{10}e^{2t}\left[e^{-3t}(-3\sin t-\cos t)+1\right]$

$=\frac{1}{10}\left[e^{2t}+e^{-t}(-3\sin t-\cos t)\right]$

$=-\frac{1}{10}e^{-t}(3\sin t+\cos t)+\frac{1}{10}e^{2t}$