LEGENDRE'S EQUATION

including Legendre's functions and Legendre's polynomials

Summary on Legendre Functions/Polynomials

Legendre's Differential Equation

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n\left(n-1\right)y=0\,,\quad n\in\mathbb{R}\,.$$

General Solution of Legendre's Differential Equation

$$y=A\left[1-\frac{(n+1)n}{2!}x^2+\frac{(n+3)(n+1)n(n-2)}{4!}x^4-\frac{(n+5)(n+3)(n+1)n(n-2)(n-4)}{6!}x^6+\ldots\right]$$

$$+$$

$$B\left[x-\frac{(n+2)(n-1)}{3!}x^3+\frac{(n+4)(n+2)(n-1)(n-3)}{3!}x^5-\frac{(n+6)(n+4)(n+2)(n-1)(n-3)(n-5)}{7!}x^7+\ldots\right]$$

- If n is an even integer, the first solution terminates after a finite number of terms, while the second one produces an infinite series.
- If n is an odd integer, the second solution terminates after a finite number of terms, while the first solution produces an infinite series.
- The finite solutions are the Legendre Polynomials, also known as solutions of the first kind, denoted by $P_n(x)$.
- The infinite series solutions are known as solutions of the second kind, denoted by $Q_n(x)$.

The second solution $Q_n(x)$ can be written in terms of $P_n(x)$ by

$$Q_n(x)=P_n(x)\int\frac{1}{\left(1-x^2\right)\left(P_n(x)\right)^2}\,dx$$

The infinite series form for the Legendre's polynomial $P_n(x)$ is given by

$$P_n(x) = \sum_{k=0}^{N} \left[\frac{(2n-2k)!}{2^n \; k!(n-k)!(n-2k)!} (-1)^k \, x^{n-2k} \right],$$

where N is the floor function

$$N = \begin{cases} \frac{1}{2}n & \text{if } n \text{ is even} \\ \frac{1}{2}(n-1) & \text{if } n \text{ is odd} \end{cases}$$

The generating function for the Legendre's polynomial $P_n(x)$ is given by

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \left[t^n \, P_n(x) \right]$$

Question 1

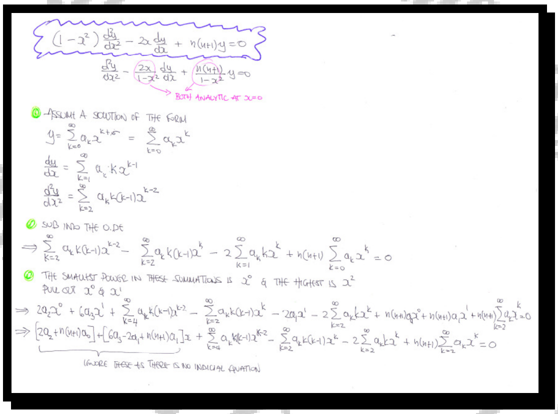
Find the two independent solutions of Legendre's equation

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0\,,\ \ n\in\mathbb{R}\,.$$

$$y=A\left[1-\frac{(n+1)n}{2!}x^2+\frac{(n+3)(n+1)n(n-2)}{4!}x^4-\frac{(n+5)(n+3)(n+1)n(n-2)(n-4)}{6!}x^6+\dots\right]$$

$$+$$

$$B\left[x-\frac{(n+2)(n-1)}{3!}x^3+\frac{(n+4)(n+2)(n-1)(n-3)}{3!}x^5-\frac{(n+6)(n+4)(n+2)(n-1)(n-3)(n-5)}{7!}x^7+\dots\right]$$



$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0$

$\frac{d^2y}{dx^2}-\frac{2x}{1-x^2}\frac{dy}{dx}+\frac{n(n+1)}{1-x^2}y=0$

BOTH ANALYTIC AT $x=0$

ASSUME A SOLUTION OF THE FORM

$y=\sum_{k=0}^{\infty}a_kx^{k+\sigma}=\sum_{k=0}^{\infty}a_kx^k$

$\frac{dy}{dx}=\sum_{k=1}^{\infty}a_kkx^{k-1}$

$\frac{d^2y}{dx^2}=\sum_{k=2}^{\infty}a_kk(k-1)x^{k-2}$

SUB INTO THE O.D.E

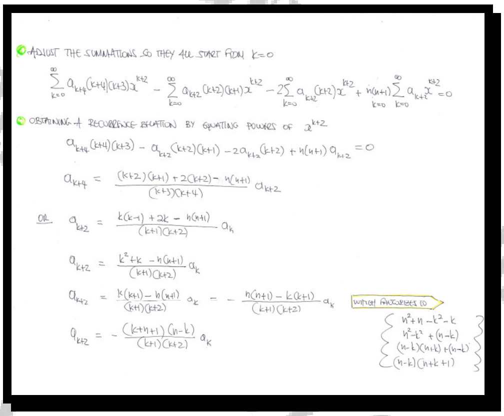
$\Rightarrow \sum_{k=2}^{\infty}a_kk(k-1)x^{k-2}-\sum_{k=2}^{\infty}a_kk(k-1)x^k-2\sum_{k=1}^{\infty}a_kkx^k+n(n+1)\sum_{k=0}^{\infty}a_kx^k=0$

THE SMALLEST POWER IN THESE SUMMATIONS IS x^0 & THE HIGHEST IS x^2
PULL OUT x^0 & x^1

$\Rightarrow 2a_2x^0+6a_3x^1+\sum_{k=4}^{\infty}a_kk(k-1)x^{k-2}-\sum_{k=2}^{\infty}a_kk(k-1)x^k-2a_1x^1-2\sum_{k=2}^{\infty}a_kkx^k+n(n+1)a_0x^0+n(n+1)a_1x^1+n(n+1)\sum_{k=2}^{\infty}a_kx^k=0$

$\Rightarrow [2a_2+n(n+1)a_0]+[6a_3-2a_1+n(n+1)a_1]x+\sum_{k=4}^{\infty}a_kk(k-1)x^{k-2}-\sum_{k=2}^{\infty}a_kk(k-1)x^k-2\sum_{k=2}^{\infty}a_kkx^k+n(n+1)\sum_{k=2}^{\infty}a_kx^k=0$

(IGNORE THESE AS THERE IS NO INDICIAL EQUATION)



ADJUST THE SUMMATIONS SO THEY ALL START FROM $k=0$

$\sum_{k=0}^{\infty}a_{k+4}(k+4)(k+3)x^{k+2}-\sum_{k=0}^{\infty}a_{k+2}(k+2)(k+1)x^{k+2}-2\sum_{k=0}^{\infty}a_{k+2}(k+2)x^{k+2}+n(n+1)\sum_{k=0}^{\infty}a_{k+2}x^{k+2}=0$

OBTAINING A RECURRENCE RELATION BY EQUATING POWERS OF x^{k+2}

$a_{k+4}(k+4)(k+3)-a_{k+2}(k+2)(k+1)-2a_{k+2}(k+2)+n(n+1)a_{k+2}=0$

$a_{k+4}=\frac{(k+2)(k+1)+2(k+2)-n(n+1)}{(k+3)(k+4)}a_{k+2}$

OR $a_{k+2}=\frac{k(k-1)+2k-n(n+1)}{(k+1)(k+2)}a_k$

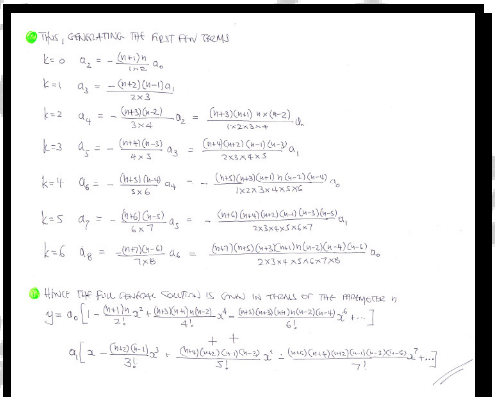
$a_{k+2}=\frac{k^2+k-n(n+1)}{(k+1)(k+2)}a_k$

$a_{k+2}=\frac{k(k+1)-n(n+1)}{(k+1)(k+2)}a_k=-\frac{n(n+1)-k(k+1)}{(k+1)(k+2)}a_k$

$a_{k+2}=-\frac{(k+n+1)(n-k)}{(k+1)(k+2)}a_k$

WHICH FACTORIZES TO

n^2+n-k^2-k
$n^2-k^2+(n-k)$
$(n-k)(n+k)+(n-k)$
$(n-k)(n+k+1)$



THUS, GENERATING THE FIRST FEW TERMS

$k=0\quad a_2=-\frac{(n+1)n}{1\times2}a_0$

$k=1\quad a_3=\frac{-(n+2)(n-1)a_1}{2\times3}$

$k=2\quad a_4=-\frac{(n+3)(n-2)}{3\times4}a_2=\frac{(n+3)(n+1)n\times(n-2)}{1\times2\times3\times4}a_0$

$k=3\quad a_5=-\frac{(n+4)(n-3)}{4\times5}a_3=\frac{(n+4)(n+2)(n-1)(n-3)}{2\times3\times4\times5}a_1$

$k=4\quad a_6=-\frac{(n+5)(n-4)}{5\times6}a_4=-\frac{(n+5)(n+3)(n+1)n(n-2)(n-4)}{1\times2\times3\times4\times5\times6}a_0$

$k=5\quad a_7=-\frac{(n+6)(n-5)}{6\times7}a_5=-\frac{(n+6)(n+4)(n+2)(n-1)(n-3)(n-5)}{2\times3\times4\times5\times6\times7}a_1$

$k=6\quad a_8=\frac{-(n+7)(n-6)}{7\times8}a_6=\frac{(n+7)(n+5)(n+3)(n+1)n(n-2)(n-4)(n-6)}{2\times3\times4\times5\times6\times7\times8}a_0$

HENCE THE FULL GENERAL SOLUTION IS GIVEN IN TERMS OF THE PARAMETER n

$y=a_0\left[1-\frac{(n+1)n}{2!}x^2+\frac{(n+3)(n+1)n(n-2)}{4!}x^4-\frac{(n+5)(n+3)(n+1)n(n-2)(n-4)}{6!}x^6+\dots\right]$

$+$

$a_1\left[x-\frac{(n+2)(n-1)}{3!}x^3+\frac{(n+4)(n+2)(n-1)(n-3)}{5!}x^5-\frac{(n+6)(n+4)(n+2)(n-1)(n-3)(n-5)}{7!}x^7+\dots\right]$

Question 2

Legendre's equation is given below

$$\left(1-t^2\right)\frac{d^2w}{dt^2}-2t\frac{dw}{dt}+n(n+1)w=0\,,\ n\in\mathbb{N}\,.$$

a) By assuming a series solution of the form

$$w(t)=\sum_{r=0}^{\infty} a_r\, t^r\,,\ a_0\neq 0\,,$$

show by a detailed method that

$$a_{r+2}=-\frac{(n-r)(n+r+1)}{(r+2)(r+1)}a_r\,.$$

b) By rewriting the recurrence relation of part **(a)** backwards, and taking the value of a_n as

$$a_n=\prod_{m=1}^{n}\frac{(2n-2m+1)}{n!}\,,$$

show further that the Legendre's polynomials $P_n(t)$ can be written as

$$P_n(t)=\sum_{k=0}^{N}\left[\frac{(2n-2k)!}{2^n\ k!(n-k)!(n-2k)!}(-1)^k\,t^{n-2k}\right],$$

where N is the floor function

$$N=\begin{cases}\frac{1}{2}n & \text{if } n \text{ is even}\\ \frac{1}{2}(n-1) & \text{if } n \text{ is odd}\end{cases}$$

proof

[solution overleaf]

a) $(1-t^2)\frac{d^2w}{dt^2} - 2t\frac{dw}{dt} + n(n+1)w = 0$

- ON DIVISION THIS IS ANALYTIC AT $t=0$, SO WE CAN EXPAND AS POWERS OF t^r

LET $w = \sum_{r=0}^{\infty} a_r t^r$ & $\frac{dw}{dt} = \sum_{r=1}^{\infty} a_r r t^{r-1}$ & $\frac{d^2w}{dt^2} = \sum_{r=2}^{\infty} a_r r(r-1) t^{r-2}$

- SUBSTITUTE INTO THE O.D.E

$$(1-t^2)\sum_{r=2}^{\infty} a_r r(r-1) t^{r-2} - 2t\sum_{r=1}^{\infty} a_r r t^{r-1} + n(n+1)\sum_{r=0}^{\infty} a_r t^r = 0$$

$$\sum_{r=2}^{\infty} a_r r(r-1) t^{r-2} - \sum_{r=2}^{\infty} a_r r(r-1) t^{r} - \sum_{r=1}^{\infty} 2a_r r t^{r} + n(n+1)\sum_{r=0}^{\infty} a_r t^r = 0$$

- THE LOWEST POWER OF t IN THESE SUMMATIONS IS t^0 & THE HIGHEST IS t^2 – PULL OUT OF THE SUMMATIONS THE POWERS t^0 & t^1

$$\Rightarrow 2a_2t^0 + 6a_3t^1 + \sum_{r=4}^{\infty} a_r r(r-1) t^{r-2} - \sum_{r=2}^{\infty} a_r r(r-1) t^{r} - 2a_1t^1 - \sum_{r=2}^{\infty} 2a_r r t^{r} + n(n+1)a_0t^0 + n(n+1)a_1t^1 + n(n+1)\sum_{r=2}^{\infty} a_r t^r = 0$$

$$\Rightarrow [2a_2 + n(n+1)a_0] + [6a_3 - 2a_1 + n(n+1)a_1]t + \sum_{r=4}^{\infty} a_r r(r-1) t^{r-2} - \sum_{r=2}^{\infty} a_r r(r-1) t^{r} - \sum_{r=2}^{\infty} 2a_r r t^{r} + n(n+1)\sum_{r=2}^{\infty} a_r t^r = 0$$

IGNORE THESE ARE THERE IS NO INDICIAL EQUATION HERE

- ADJUST THE SUMMATIONS SO THEY ALL START FROM $r=0$

$$\sum_{r=0}^{\infty} a_{r+4}(r+4)(r+3)t^{r+2} - \sum_{r=0}^{\infty} a_{r+2}(r+2)(r+1)t^{r+2} - \sum_{r=0}^{\infty} 2a_{r+2}(r+2)t^{r+2} + n(n+1)\sum_{r=0}^{\infty} a_{r+2}t^{r+2} = 0$$

- EQUATE POWER OF t IN THESE SUMMATIONS, SAY t^{r+2}, TO OBTAIN A RECURRENCE RELATION

$$\Longrightarrow a_{r+4}(r+4)(r+3) - a_{r+2}(r+2)(r+1) - 2a_{r+2}(r+2) + n(n+1)a_{r+2} = 0$$

- "SHIFT DOWN BY 2"

$$\Rightarrow a_{r+2}(r+2)(r+1) - a_r r(r-1) - 2a_r r + n(n+1)a_r = 0$$

$$\Rightarrow a_{r+2}(r+2)(r+1) = [r(r-1) + 2r - n(n+1)]a_r$$

$$\Rightarrow a_{r+2} = \frac{r^2 + r - n^2 - n}{(r+2)(r+1)}a_r$$

$r^2 - n^2 + (r-n) = (r-n)(r+n) + (r-n) = (r-n)(r+n+1)$

$$\Rightarrow a_{r+2} = \frac{(r-n)(r+n+1)}{(r+2)(r+1)}a_r$$

$$\Rightarrow a_{r+2} = -\frac{(n-r)(n+r+1)}{(r+2)(r+1)}a_r$$

AS REQUIRED

b) - REWRITE THE RECURRENCE RELATION BACKWARDS

$$a_r = -\frac{(r+2)(r+1)}{(n-r)(n+r+1)}a_{r+2}$$

NOTE THAT IF $r=n$, $a_{n+2} = a_{n+4} = a_{n+6} = \dots = 0$

- HENCE

$r = n-2$: $a_{n-2} = -\frac{n(n-1)}{2(2n-1)}a_n$

$r = n-4$: $a_{n-4} = -\frac{(n-2)(n-3)}{4(2n-3)}a_{n-2} = \frac{n(n-1)(n-2)(n-3)}{2\times4(2n-1)(2n-3)}a_n$

$r = n-6$: $a_{n-6} = -\frac{(n-4)(n-5)}{6(2n-5)}a_{n-4} = \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{2\times4\times6(2n-1)(2n-3)(2n-5)}a_n$ ETC.

- THEN THE POLYNOMIAL FORM OF THE SOLUTION OF LEGENDRE'S EQUATION BECOMES

$$w = \sum_{r=0}^{\infty} a_r t^n = a_n t^n - \frac{n(n-1)}{2(2n-1)}a_n t^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\times4\times(2n-1)(2n-3)}a_n t^{n-4} - \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{2\times4\times6\times(2n-1)(2n-3)(2n-5)}a_n t^{n-6} + \dots$$

$$= a_n\left[t^n - \frac{n(n-1)}{2(2n-1)}t^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\times4(2n-1)(2n-3)}t^{n-4} - \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{2\times4\times6\times(2n-1)(2n-3)(2n-5)}t^{n-6} + \dots\right]$$

- TO NORMALIZE THESE POLYNOMIALS WE CHOOSE a_n, SO THAT $P_n(1) = 1$

THE VALUE OF a_n TO PRODUCE THIS IS $a_n = \frac{(2n-1)(2n-3)\dots5\times3\times1}{n!}$

$$P_n(t) = \frac{(2n-1)(2n-3)\dots5\times3\times1}{n!}\left[t^n - \frac{n(n-1)}{2(2n-1)}t^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\times4\times(2n-1)(2n-3)}t^{n-4} - \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{2\times4\times6\times(2n-1)(2n-3)(2n-5)}t^{n-6} + \dots\right]$$

$k=0$ $\quad k=1$ $\quad k=2$ $\quad k=3$

THE GENERAL TERM OF THIS SERIES STARTING FROM $k=0$

$$(-1)^k\frac{(2n-1)(2n-3)\dots5\times3\times1}{n!}\left[t^{n-2k}\frac{n(n-1)(n-2)\dots(n-2k+1)}{[2\times4\times6\times\dots2k][(2n-1)(2n-3)(2n-5)\dots(2n-2k+1)]}\right]$$

- NOW IT IS BEST TO LOOK AT SOME MANIPULATIONS WITHIN THE ABOVE TERM SEPARATELY

- $(2n-1)(2n-3)(2n-5)\dots5\times3\times1 = \frac{2n(2n-1)(2n-2)(2n-3)\dots\times5\times4\times3\times2\times1}{(2n)(2n-2)\dots\times4\times2} = \frac{(2n)!}{2^n n(n-1)\dots\times2\times1} = \frac{(2n)!}{2^n n!}$
- $n(n-1)(n-2)\dots(n-2k+1) = \frac{n(n-1)(n-2)\dots(n-2k+1)(n-2k)(n-2k-1)\dots3\times2\times1}{(n-2k)(n-2k-1)\dots3\times2\times1} = \frac{n!}{(n-2k)!}$
- $2\times4\times6\times\dots2k = 2^k(1\times2\times3\dots\times k) = 2^k\times k!$

- $(2n-1)(2n-3)(2n-5)\dots(2n-2k+1) = \frac{2n(2n-1)(2n-2)(2n-3)(2n-4)(2n-5)\dots(2n-2k+1)}{2n(2n-2)(2n-4)\dots(2n-2k+2)} \times \frac{(2n-2k)(2n-2k-1)(2n-2k-2)\dots3\times2\times1}{(2n-2k)(2n-2k-1)(2n-2k-2)\dots\times3\times2\times1}$

$$= \frac{(2n)!}{[2^k n(n-1)(n-2)\dots(n-k+1)](2n-2k)!}$$

ATTENTION: $n-(n-k) = k$ TERM TERMS TERMS

$$= \frac{(2n)!}{2^k(2n-2k)![n(n-1)(n-2)\dots(n-k+1)]}$$

$$= \frac{(2n)!}{2^k(2n-2k)!\,n(n-1)(n-2)(n-3)\dots(n-k+1)} \times \frac{(n-k)(n-k-1)\dots3\times2\times1}{(n-k)(n-k-1)\dots3\times2\times1}$$

$$= \frac{(2n!)(n-k)!}{2^k(2n-2k)!\,n!}$$

- COLLECTING ALL THESE RESULTS INTO THE GENERAL TERM FROM EARLIER GIVES

$$\dots = t^{n-2k}(-1)^k \times \frac{\frac{2n!}{2^n n!}}{n!} \times \frac{\frac{n!}{(n-2k)!}}{2^k\times k! \times \frac{(2n)!(n-k)!}{2^k(2n-2k)!\,n!}} = \frac{(-1)^k(2n-2k)!}{2^n k!(n-2k)!(n-k)!}$$

THUS
$$P_n(t) = \sum_{k=0}^{N}(-1)^k\frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!}t^{n-2k}$$

N IS FLOOR FUNCTION $N = \begin{cases} n/2 & \text{IF } n \text{ IS EVEN} \\ n-1/2 & \text{IF } n \text{ IS ODD}\end{cases}$

SINCE THE POLYNOMIAL IS OF DEGREE n $\Longrightarrow$ $n - 2k \geq 0$

$-2k \geq n$

$k \leq \frac{n}{2}$

SO IF n IS EVEN THE SUMMATION THE SUMMATION GOES TO $n/2$

IF n IS ODD, THE SUMMATION HALVES & ROUNDS DOWN

Question 3

It can be shown that the Legendre's polynomials $P_n(x)$ can be written as

$$P_n(x) = \sum_{k=0}^{N} \left[\frac{(2n-2k)!}{2^n \; k!(n-k)!(n-2k)!} (-1)^k \; x^{n-2k} \right],$$

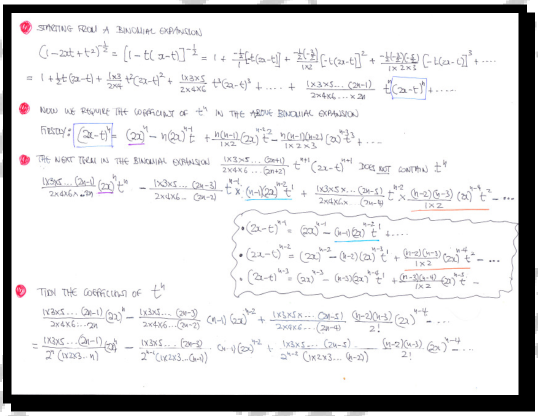
where N is the floor function

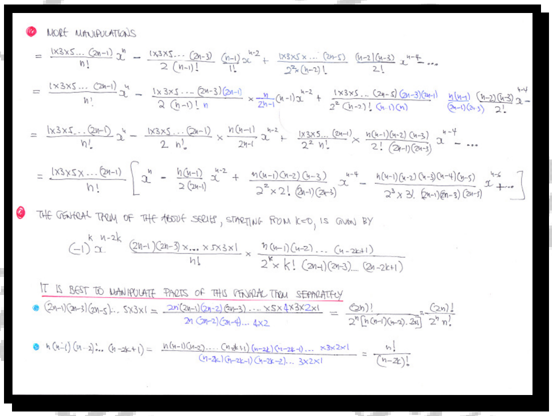
$$N = \begin{cases} \frac{1}{2}n & \text{if } n \text{ is even} \\ \frac{1}{2}(n-1) & \text{if } n \text{ is odd} \end{cases}$$

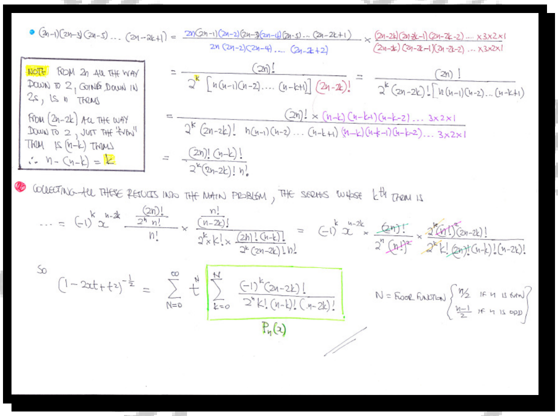
Show that the generating function for $P_n(x)$ satisfies

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \left[t^n P_n(x) \right].$$

proof







Question 4

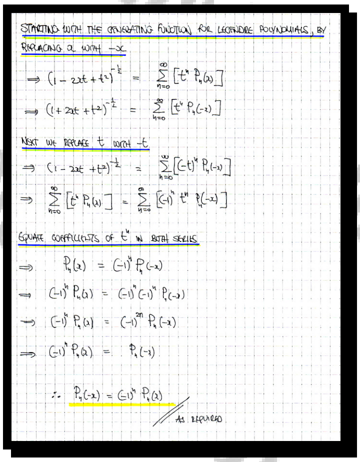
The generating function for the Legendre's polynomials $P_n(x)$, satisfies

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right].$$

Use this relationship to prove that

$$P_n(-x) = (-1)^n P_n(x).$$

[ ], proof


STARTING WITH THE GENERATING FUNCTION FOR LEGENDRE POLYNOMIALS, BY REPLACING x WITH $-x$

$$\Rightarrow \left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right]$$

$$\Rightarrow \left(1+2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(-x)\right]$$

NEXT WE REPLACE t WITH $-t$

$$\Rightarrow \left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[(-t)^n P_n(-x)\right]$$

$$\Rightarrow \sum_{n=0}^{\infty}\left[t^n P_n(x)\right] = \sum_{n=0}^{\infty}\left[(-1)^n t^n P_n(-x)\right]$$

EQUATE COEFFICIENTS OF t^n IN BOTH SERIES

$$\Rightarrow P_n(x) = (-1)^n P_n(-x)$$

$$\Rightarrow (-1)^n P_n(x) = (-1)^n(-1)^n P_n(-x)$$

$$\Rightarrow (-1)^n P_n(x) = (-1)^{2n} P_n(-x)$$

$$\Rightarrow (-1)^n P_n(x) = P_n(-x)$$

$$\therefore P_n(-x) = (-1)^n P_n(x)$$

AS REQUIRED

Question 5

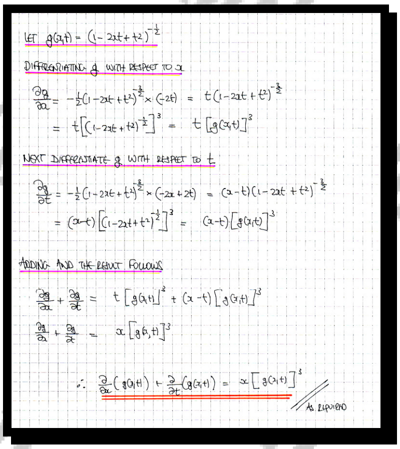
The generating function $g(x,t)$ for the Legendre's polynomials $P_n(x)$, satisfies

$$g(x,t) = \left(1 - 2xt + t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right].$$

Use this relationship to prove that

$$\frac{\partial}{\partial x}\left[g(x,t)\right] + \frac{\partial}{\partial t}\left[g(x,t)\right] = x\left[g(x,t)\right]^3.$$

[ ], [proof]



LET $g(x,t) = (1 - 2xt + t^2)^{-\frac{1}{2}}$

DIFFERENTIATING g WITH RESPECT TO x

$$\frac{\partial g}{\partial x} = -\frac{1}{2}(1-2xt+t^2)^{-\frac{3}{2}} \times (-2t) = t(1-2xt+t^2)^{-\frac{3}{2}}$$
$$= t\left[(1-2xt+t^2)^{-\frac{1}{2}}\right]^3 = t\left[g(x,t)\right]^3$$

NEXT DIFFERENTIATE g WITH RESPECT TO t

$$\frac{\partial g}{\partial t} = -\frac{1}{2}(1-2xt+t^2)^{-\frac{3}{2}} \times (-2x+2t) = (x-t)(1-2xt+t^2)^{-\frac{3}{2}}$$
$$= (x-t)\left[(1-2xt+t^2)^{-\frac{1}{2}}\right]^3 = (x-t)\left[g(x,t)\right]^3$$

ADDING AND THE RESULT FOLLOWS

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial t} = t\left[g(x,t)\right]^3 + (x-t)\left[g(x,t)\right]^3$$
$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial t} = x\left[g(x,t)\right]^3$$

$$\therefore \frac{\partial}{\partial x}\left(g(x,t)\right) + \frac{\partial}{\partial t}\left(g(x,t)\right) = x\left[g(x,t)\right]^3$$

AS REQUIRED

Question 6

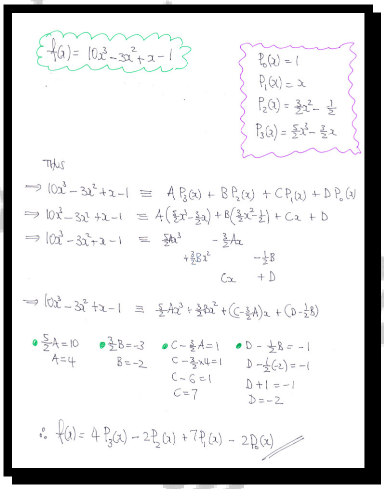
$$f(x) \equiv 10x^3 - 3x^2 + x - 1.$$

Express $f(x)$ as a linear combination of Legendre's polynomials, $P_n(x)$.

You may assume

- $P_0(x) = 1$
- $P_1(x) = x$
- $P_2(x) = \frac{1}{2}(3x^2 - 1)$
- $P_3(x) = \frac{1}{2}(5x^3 - 3x)$,
- $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

$$\boxed{f(x) = 4P_3(x) - 2P_2(x) + 7P_1(x) - 2P_0(x)}$$



$f(x) = 10x^3 - 3x^2 + x - 1$

$P_0(x) = 1$
$P_1(x) = x$
$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$
$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$

Thus

$\Rightarrow 10x^3 - 3x^2 + x - 1 \equiv AP_3(x) + BP_2(x) + CP_1(x) + DP_0(x)$

$\Rightarrow 10x^3 - 3x^2 + x - 1 \equiv A\left(\frac{5}{2}x^3 - \frac{3}{2}x\right) + B\left(\frac{3}{2}x^2 - \frac{1}{2}\right) + Cx + D$

$\Rightarrow 10x^3 - 3x^2 + x - 1 \equiv \frac{5}{2}Ax^3 - \frac{3}{2}Ax + \frac{3}{2}Bx^2 - \frac{1}{2}B + Cx + D$

$\Rightarrow 10x^3 - 3x^2 + x - 1 \equiv \frac{5}{2}Ax^3 + \frac{3}{2}Bx^2 + \left(C - \frac{3}{2}A\right)x + \left(D - \frac{1}{2}B\right)$

- $\frac{5}{2}A = 10$, $A = 4$
- $\frac{3}{2}B = -3$, $B = -2$
- $C - \frac{3}{2}A = 1$, $C - \frac{3}{2}\times 4 = 1$, $C - 6 = 1$, $C = 7$
- $D - \frac{1}{2}B = -1$, $D - \frac{1}{2}(-2) = -1$, $D + 1 = -1$, $D = -2$

$\therefore f(x) = 4P_3(x) - 2P_2(x) + 7P_1(x) - 2P_0(x)$

Question 7

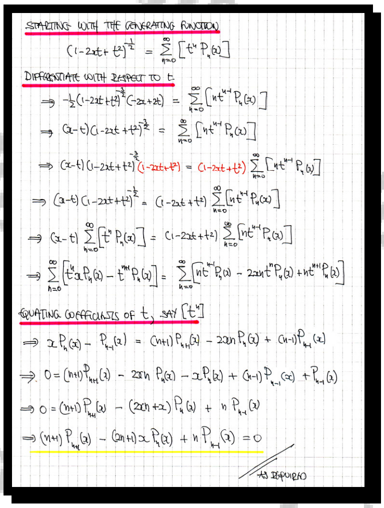
The generating function for the Legendre's polynomials $P_n(x)$, satisfies

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right].$$

By differentiating the above relationship with respect to t, prove that

$$(2n+1)\,x\,P_n(x)-(n+1)\,P_{n+1}(x)+n\,P_{n-1}(x)=0\,.$$

[ ] , [proof]



Question 8

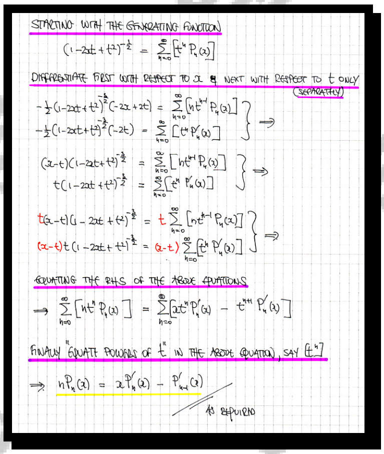
The generating function for the Legendre's polynomials $P_n(x)$, satisfies

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right].$$

By separately differentiating the above relationship once with respect to t and once with respect to x, prove that

$$n\,P_n(x) = x\,P_n'(x) - P_{n-1}'(x).$$

MM, proof



STARTING WITH THE GENERATING FUNCTION

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right]$$

DIFFERENTIATE FIRST WITH RESPECT TO x & NEXT WITH RESPECT TO t ONLY (SEPARATELY)

$$-\tfrac{1}{2}(1-2xt+t^2)^{-\frac{3}{2}}(-2x+2t) = \sum_{n=0}^{\infty}\left[nt^{n-1}P_n(x)\right]$$
$$-\tfrac{1}{2}(1-2xt+t^2)^{-\frac{3}{2}}(-2t) = \sum_{n=0}^{\infty}\left[t^n P_n'(x)\right] \implies$$

$$(x-t)(1-2xt+t^2)^{-\frac{3}{2}} = \sum_{n=0}^{\infty}\left[nt^{n-1}P_n(x)\right]$$
$$t(1-2xt+t^2)^{-\frac{3}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n'(x)\right] \implies$$

$$t(x-t)(1-2xt+t^2)^{-\frac{3}{2}} = t\sum_{n=0}^{\infty}\left[nt^{n-1}P_n(x)\right]$$
$$(x-t)t(1-2xt+t^2)^{-\frac{3}{2}} = (x-t)\sum_{n=0}^{\infty}\left[t^n P_n'(x)\right] \implies$$

EQUATING THE RHS OF THE ABOVE EQUATIONS

$$\implies \sum_{n=0}^{\infty}\left[nt^n P_n(x)\right] = \sum_{n=0}^{\infty}\left[xt^n P_n'(x) - t^{n+1}P_n'(x)\right]$$

FINALLY "EQUATE POWERS OF t" IN THE ABOVE EQUATION, SAY $[t^n]$

$$\implies nP_n(x) = xP_n'(x) - P_{n-1}'(x)$$

AS REQUIRED

Question 9

The generating function for the Legendre's polynomials $P_n(x)$, satisfies

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right].$$

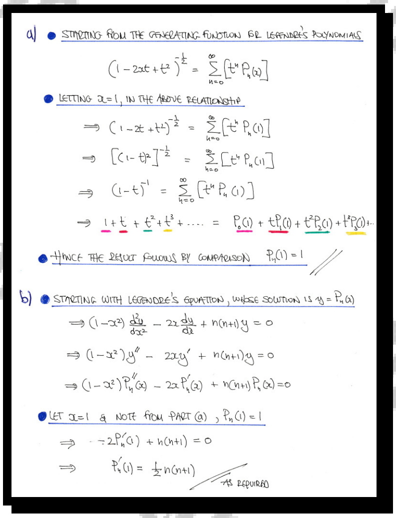
a) Use this result to show that

$$P_n(1)=1.$$

b) By using the result of part **(a)** and Legendre's equation, deduce that

$$P_n'(1)=\frac{1}{2}n(n+1).$$

[ ], proof



a) • STARTING FROM THE GENERATING FUNCTION FOR LEGENDRE'S POLYNOMIALS

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right]$$

• LETTING $x=1$, IN THE ABOVE RELATIONSHIP

$$\Rightarrow \left(1-2t+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(1)\right]$$

$$\Rightarrow \left[(1-t)^2\right]^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(1)\right]$$

$$\Rightarrow (1-t)^{-1} = \sum_{n=0}^{\infty}\left[t^n P_n(1)\right]$$

$$\Rightarrow 1+t+t^2+t^3+\dots = P_0(1)+tP_1(1)+t^2P_2(1)+t^3P_3(1)+\dots$$

• HENCE THE RESULT FOLLOWS BY COMPARISON $P_n(1)=1$

b) • STARTING WITH LEGENDRE'S EQUATION, WHOSE SOLUTION IS $y=P_n(x)$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

$$\Rightarrow (1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$$\Rightarrow (1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0$$

• LET $x=1$ & NOTE FROM PART (a), $P_n(1)=1$

$$\Rightarrow -2P_n'(1) + n(n+1) = 0$$

$$\Rightarrow P_n'(1) = \frac{1}{2}n(n+1)$$

AS REQUIRED

Question 10

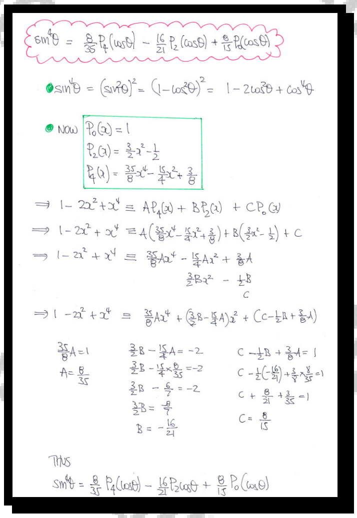
Use trigonometric identities to show that

$$\sin^2\theta = \frac{8}{35}P_4(\cos\theta) - \frac{16}{21}P_2(\cos\theta) + \frac{8}{15}P_0(\cos\theta)$$

You may assume

- $P_0(x) = 1$
- $P_1(x) = x$
- $P_2(x) = \frac{1}{2}(3x^2 - 1)$
- $P_3(x) = \frac{1}{2}(5x^3 - 3x)$,
- $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

proof



Question 11

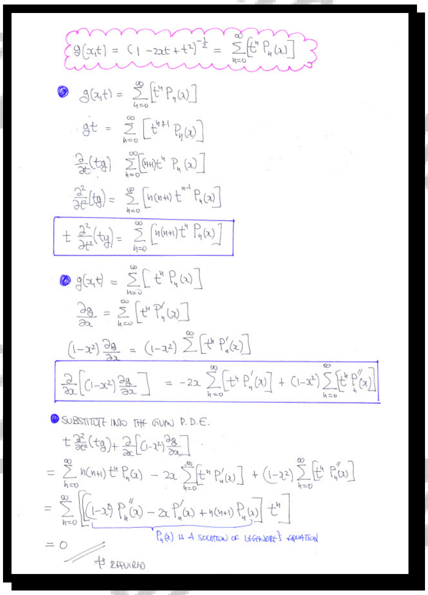
The generating function $g(x,t)$ for the Legendre's polynomials $P_n(x)$, satisfies

$$g(x,t) = \left(1 - 2xt + t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right].$$

Verify that $g = g(x,t)$ is a solution of the differential equation

$$t\frac{\partial^2}{\partial t^2}[t\,g] + \frac{\partial}{\partial x}\left[\left(1-x^2\right)\frac{\partial g}{\partial x}\right] = 0.$$

proof


$$g(x,t) = (1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right]$$

- $g(x,t) = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right]$

$gt = \sum_{n=0}^{\infty}\left[t^{n+1} P_n(x)\right]$

$\frac{\partial}{\partial t}(tg) \ \sum_{n=0}^{\infty}\left[(n+1)t^n P_n(x)\right]$

$\frac{\partial^2}{\partial t^2}(tg) = \sum_{n=0}^{\infty}\left[n(n+1)t^{n-1} P_n(x)\right]$

$$t\frac{\partial^2}{\partial t^2}(tg) = \sum_{n=0}^{\infty}\left[n(n+1)t^n P_n(x)\right]$$

- $g(x,t) = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right]$

$\frac{\partial g}{\partial x} = \sum_{n=0}^{\infty}\left[t^n P_n'(x)\right]$

$(1-x^2)\frac{\partial g}{\partial x} = (1-x^2)\sum\left[t^n P_n'(x)\right]$

$$\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial g}{\partial x}\right] = -2x\sum_{n=0}^{\infty}\left[t^n P_n'(x)\right] + (1-x^2)\sum_{n=0}^{\infty}\left[t^n P_n''(x)\right]$$

- SUBSTITUTE INTO THE GIVEN P.D.E.

$t\frac{\partial^2}{\partial t^2}(tg) + \frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial g}{\partial x}\right]$

$= \sum_{n=0}^{\infty} n(n+1)t^n P_n(x) - 2x\sum_{n=0}^{\infty}\left[t^n P_n'(x)\right] + (1-x^2)\sum_{n=0}^{\infty}\left[t^n P_n''(x)\right]$

$= \sum_{n=0}^{\infty}\left[\left[(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x)\right]t^n\right]$

$P_n(x)$ IS A SOLUTION OF LEGENDRE'S EQUATION

$= 0$

AS REQUIRED

Question 12

The generating function for the Legendre's polynomials $P_n(x)$, satisfies

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right].$$

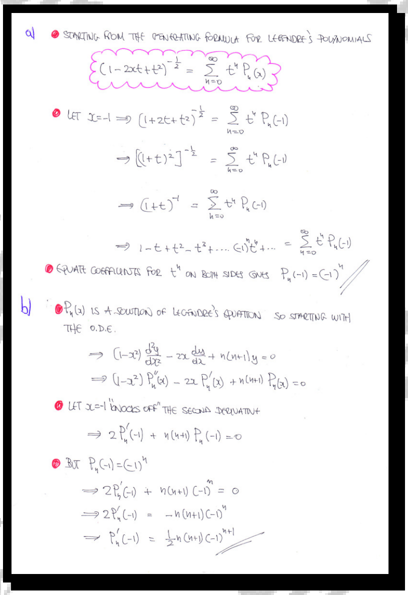
a) Use this result to show that

$$P_n(-1) = (-1)^n .$$

b) By using the result of part **(a)** and Legendre's equation, deduce that

$$P_n'(-1) = \tfrac{1}{2}n(n+1)(-1)^{n+1} .$$

[ ] , [proof]



a) ● STARTING FROM THE GENERATING FORMULA FOR LEGENDRE'S POLYNOMIALS

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(x)$$

● LET $x=-1 \Rightarrow \left(1+2t+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(-1)$

$$\Rightarrow \left[(1+t)^2\right]^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(-1)$$

$$\Rightarrow (1+t)^{-1} = \sum_{n=0}^{\infty} t^n P_n(-1)$$

$$\Rightarrow 1-t+t^2-t^3+\dots(-1)^n t^n+\dots = \sum_{n=0}^{\infty} t^n P_n(-1)$$

● EQUATE COEFFICIENTS FOR t^n ON BOTH SIDES GIVES $P_n(-1)=(-1)^n$

b) ● $P_n(x)$ IS A SOLUTION OF LEGENDRE'S EQUATION SO STARTING WITH THE O.D.E.

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

$$\Rightarrow (1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0$$

● LET $x=-1$ "KNOCKS OFF" THE SECOND DERIVATIVE

$$\Rightarrow 2P_n'(-1) + n(n+1)P_n(-1) = 0$$

● BUT $P_n(-1)=(-1)^n$

$$\Rightarrow 2P_n'(-1) + n(n+1)(-1)^n = 0$$

$$\Rightarrow 2P_n'(-1) = -n(n+1)(-1)^n$$

$$\Rightarrow P_n'(-1) = \tfrac{1}{2}n(n+1)(-1)^{n+1}$$

Question 13

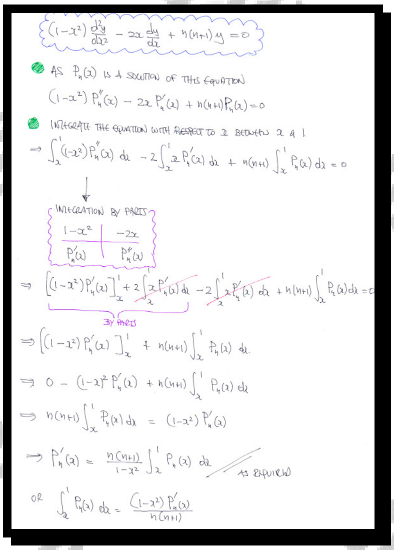
The Legendre's polynomial $P_n(x)$ is a solution of the differential equation

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)\,y=0\,,\quad n\in\mathbb{R}\,.$$

Show that

$$P_n'(x)=\frac{n(n+1)}{1-x^2}\int_1^x P_n(x)\ dx\,.$$

proof



$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0$$

AS $P_n(x)$ IS A SOLUTION OF THIS EQUATION

$$(1-x^2)P_n''(x)-2xP_n'(x)+n(n+1)P_n(x)=0$$

INTEGRATE THE EQUATION WITH RESPECT TO x BETWEEN x & 1

$$\Rightarrow \int_x^1 (1-x^2)P_n''(x)\,dx-2\int_x^1 xP_n'(x)\,dx+n(n+1)\int_x^1 P_n(x)\,dx=0$$

INTEGRATION BY PARTS

$1-x^2$	$-2x$
$P_n'(x)$	$P_n''(x)$

$$\Rightarrow \left[(1-x^2)P_n'(x)\right]_x^1+2\int_x^1 xP_n'(x)\,dx-2\int_x^1 xP_n'(x)\,dx+n(n+1)\int_x^1 P_n(x)\,dx=0$$

BY PARTS

$$\Rightarrow \left[(1-x^2)P_n'(x)\right]_x^1+n(n+1)\int_x^1 P_n(x)\,dx$$

$$\Rightarrow 0-(1-x)^2P_n'(x)+n(n+1)\int_x^1 P_n(x)\,dx$$

$$\Rightarrow n(n+1)\int_x^1 P_n(x)\,dx=(1-x^2)P_n'(x)$$

$$\Rightarrow P_n'(x)=\frac{n(n+1)}{1-x^2}\int_x^1 P_n(x)\,dx$$

AS REQUIRED

OR $\displaystyle\int_x^1 P_n(x)\,dx=\frac{(1-x^2)P_n'(x)}{n(n+1)}$

Question 14

The generating function for the Legendre's polynomials $P_n(x)$, satisfies

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right].$$

a) By differentiating the above relationship with respect to t, prove that

$$(2n+1)\,x\,P_n(x) = (n+1)\,P_{n+1}(x) + n\,P_{n-1}(x).$$

b) By separately differentiating the generating function for the Legendre's polynomials once with respect to t and once with respect to x, prove that

$$n\,P_n(x) = x\,P_n'(x) - P_{n-1}'(x).$$

c) Use parts **(a)** and **(b)** to show that

$$(2n+1)\,P_n(x) = P_{n+1}'(x) + P_{n-1}'(x).$$

d) Use parts **(b)** and **(c)** to deduce that

$$(n+1)\,P_n(x) = P_{n+1}'(x) - x\,P_n'(x).$$

proof

a) • STARTING FROM THE GENERATING FUNCTION

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right]$$

• DIFFERENTIATE WITH RESPECT TO t.

$$\Rightarrow -\tfrac{1}{2}\left(1-2xt+t^2\right)^{-\frac{3}{2}}(-2x+2t) = \sum_{n=0}^{\infty}\left[nt^{n-1}P_n(x)\right]$$

$$\Rightarrow (x-t)\left(1-2xt+t^2\right)^{-\frac{3}{2}} = \sum_{n=0}^{\infty}\left[nt^{n-1}P_n(x)\right]$$

$$\Rightarrow (x-t)\left(1-2xt+t^2\right)^{-\frac{3}{2}}\left(1-2xt+t^2\right) = \left(1-2xt+t^2\right)\sum_{n=0}^{\infty}\left[nt^{n-1}P_n(x)\right]$$

$$\Rightarrow (x-t)\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \left(1-2xt+t^2\right)\sum_{n=0}^{\infty}\left[nt^{n-1}P_n(x)\right]$$

$$\Rightarrow (x-t)\sum_{n=0}^{\infty}\left[t^nP_n(x)\right] = \left(1-2xt+t^2\right)\sum_{n=0}^{\infty}\left[nt^{n-1}P_n(x)\right]$$

$$\Rightarrow \sum_{n=0}^{\infty}\left[t^nxP_n(x) - t^{n+1}P_n(x)\right] = \sum_{n=0}^{\infty}\left[nt^{n-1}P_n(x) - 2xnt^nP_n(x) + nt^{n+1}P_n(x)\right]$$

• EQUATE THE COEFFICIENTS OF POWER OF t, SAY $[t^n]$

$$\Rightarrow xP_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x)$$

$$\Rightarrow 0 = (n+1)P_{n+1}(x) - x(2n+1)P_n(x) + [(n-1)+1]P_{n-1}(x)$$

$$\Rightarrow (n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$$

AS REQUIRED

b) • STARTING FROM THE GENERATING FUNCTION

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right]$$

• DIFFERENTIATE WITH RESPECT TO t, AND WITH RESPECT TO x

$$-\tfrac{1}{2}\left(1-2xt+t^2\right)^{-\frac{3}{2}}(-2x+2t) = \sum_{n=0}^{\infty}\left[nt^{n-1}P_n(x)\right]$$

$$-\tfrac{1}{2}\left(1-2xt+t^2\right)^{-\frac{3}{2}}(-2t) = \sum_{n=0}^{\infty}\left[t^nP_n'(x)\right] \quad \Rightarrow$$

$$(x-t)\left(1-2xt+t^2\right)^{-\frac{3}{2}} = \sum_{n=0}^{\infty}\left[nt^{n-1}P_n(x)\right]$$

$$t\left(1-2xt+t^2\right)^{-\frac{3}{2}} = \sum_{n=0}^{\infty}\left[t^nP_n'(x)\right] \quad \Rightarrow$$

• MULTIPLY THE FIRST BY t, AND THE SECOND BY $(x-t)$ & EQUATE THE RHS.

$$t\sum_{n=0}^{\infty}\left[nt^{n-1}P_n(x)\right] = (x-t)\sum_{n=0}^{\infty}\left[t^nP_n'(x)\right]$$

$$\sum_{n=0}^{\infty}\left[nt^nP_n(x)\right] = \sum_{n=0}^{\infty}\left[xt^nP_n'(x) - t^{n+1}P_n'(x)\right]$$

• EQUATE COEFFICIENTS OF POWERS OF t, SAY $[t^n]$

$$nP_n(x) = xP_n'(x) - P_{n-1}'(x)$$

c) • FROM (a) $(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$

DIFFERENTIATE W.R.T x

$$(n+1)P_{n+1}'(x) - (2n+1)P_n(x) - (2n+1)xP_n'(x) + nP_{n-1}'(x) = 0$$

• FROM (b) $xP_n'(x) = nP_n(x) + P_{n-1}'(x)$

$$\Rightarrow (n+1)P_{n+1}'(x) - (2n+1)P_n(x) - (2n+1)\left[nP_n(x) + P_{n-1}'(x)\right] + nP_{n-1}'(x) = 0$$

$$\Rightarrow (n+1)P_{n+1}'(x) - (2n+1)P_n(x) - n(2n+1)P_n(x) - (2n+1)P_{n-1}'(x) + nP_{n-1}'(x) = 0$$

$$\Rightarrow (n+1)P_{n+1}'(x) - (2n+1)(n+1)P_n(x) - (n+1)P_{n-1}'(x) = 0$$

$$\Rightarrow P_{n+1}'(x) - (2n+1)P_n(x) - P_{n-1}'(x) = 0$$

$$\Rightarrow (2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)$$

AS REQUIRED

d) • FROM (b): $nP_n(x) = xP_n'(x) - P_{n-1}'(x)$

• FROM (c): $(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)$

• SUBTRACTING THE EQUATIONS "UPWARDS"

$$(n+1)P_n(x) = P_{n+1}'(x) - xP_n'(x)$$

AS REQUIRED

Question 15

The generating function for the Legendre's polynomials $P_n(x)$, satisfies

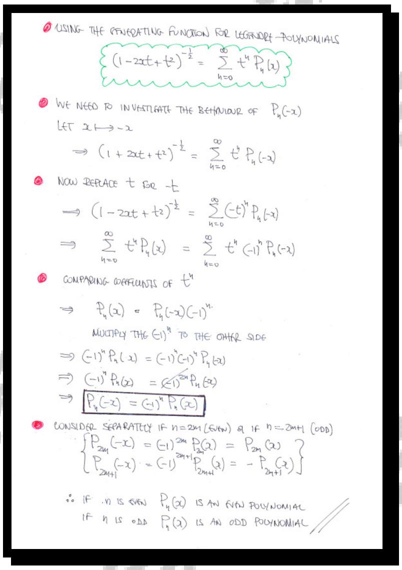
$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right].$$

Use this result to show that …

… if n is even, $P_n(x)$ is an even polynomial in x.

… if n is odd, $P_n(x)$ is an odd polynomial in x.

proof



USING THE GENERATING FUNCTION FOR LEGENDRE POLYNOMIALS

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(x)$$

WE NEED TO INVESTIGATE THE BEHAVIOUR OF $P_n(-x)$

LET $x \mapsto -x$

$$\Rightarrow \left(1+2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(-x)$$

NOW REPLACE t FOR $-t$

$$\Rightarrow \left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} (-t)^n P_n(-x)$$

$$\Rightarrow \sum_{n=0}^{\infty} t^n P_n(x) = \sum_{n=0}^{\infty} t^n (-1)^n P_n(-x)$$

COMPARING COEFFICIENTS OF t^n

$$\Rightarrow P_n(x) = P_n(-x)(-1)^n.$$

MULTIPLY THE $(-1)^n$ TO THE OTHER SIDE

$$\Rightarrow (-1)^n P_n(x) = (-1)^n(-1)^n P_n(-x)$$

$$\Rightarrow (-1)^n P_n(x) = (-1)^{2n} P_n(-x)$$

$$\Rightarrow \boxed{P_n(-x) = (-1)^n P_n(x)}$$

CONSIDER SEPARATELY IF $n=2m$ (EVEN) & IF $n=2m+1$ (ODD)

$$\begin{cases} P_{2m}(-x) = (-1)^{2m} P_{2m}(x) = P_{2m}(x) \\ P_{2m+1}(-x) = (-1)^{2m+1} P_{2m+1}(x) = -P_{2m+1}(x) \end{cases}$$

$\therefore$ IF n IS EVEN $P_n(x)$ IS AN EVEN POLYNOMIAL
IF n IS ODD $P_n(x)$ IS AN ODD POLYNOMIAL

Question 16

The generating function for the Legendre's Polynomials $P_n(x)$, satisfies

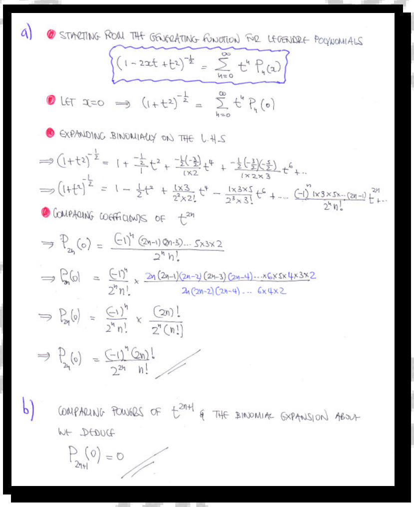
$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right].$$

a) Use this result to show that

$$P_{2n}(0) = \frac{(-1)^n (2n)!}{2^{2n}\ n!}.$$

b) Deduce the value of $P_{2n+1}(0)$.

$$\boxed{P_{2n+1}(0) = 0}$$



a) STARTING FROM THE GENERATING FUNCTION FOR LEGENDRE POLYNOMIALS

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(x)$$

LET $x=0 \implies (1+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(0)$

EXPANDING BINOMIALLY ON THE L.H.S

$$\implies (1+t^2)^{-\frac{1}{2}} = 1 + \frac{-\frac{1}{2}}{1}t^2 + \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)}{1\times 2}t^4 + \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{1\times 2\times 3}t^6 + \dots$$

$$\implies (1+t^2)^{-\frac{1}{2}} = 1 - \frac{1}{2}t^2 + \frac{1\times 3}{2^2\times 2!}t^4 - \frac{1\times 3\times 5}{2^3\times 3!}t^6 + \dots \frac{(-1)^n\, 1\times 3\times 5\times\dots(2n-1)}{2^n n!}t^{2n} + \dots$$

COMPARING COEFFICIENTS OF t^{2n}

$$\implies P_{2n}(0) = \frac{(-1)^n (2n-1)(2n-3)\dots 5\times 3\times 2}{2^n n!}$$

$$\implies P_{2n}(0) = \frac{(-1)^n}{2^n n!} \times \frac{2n(2n-1)(2n-2)(2n-3)(2n-4)\dots\times 6\times 5\times 4\times 3\times 2}{2n(2n-2)(2n-4)\dots 6\times 4\times 2}$$

$$\implies P_{2n}(0) = \frac{(-1)^n}{2^n n!} \times \frac{(2n)!}{2^n (n!)}$$

$$\implies P_{2n}(0) = \frac{(-1)^n (2n)!}{2^{2n}\ n!}$$

b) COMPARING POWERS OF t^{2n+1} & THE BINOMIAL EXPANSION ABOUT WE DEDUCE

$$P_{2n+1}(0) = 0$$

Question 17

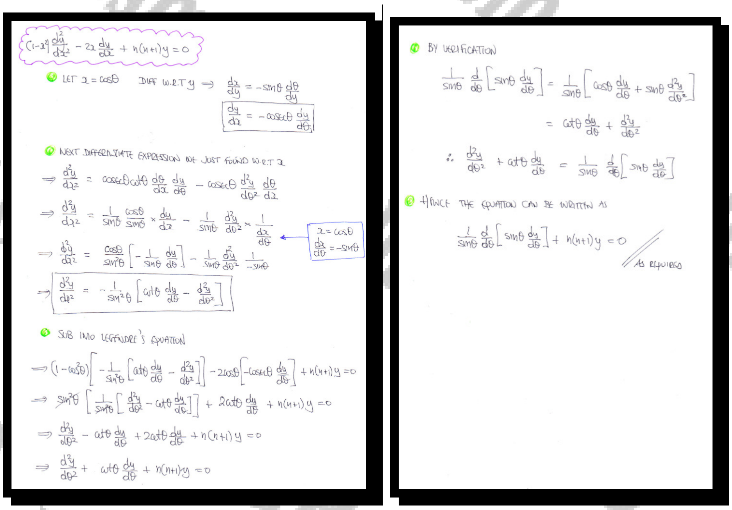
Legendre's equation is given below

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n\left(n+1\right)y=0\,,\quad n\in\mathbb{R}\,.$$

Use the substitution $x=\cos\theta$ to show that

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\frac{dy}{d\theta}\sin\theta\right]+n\left(n+1\right)y=0\,.$$

proof



$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0$$

LET $x=\cos\theta$ DIFF W.R.T y $\Rightarrow$ $\frac{dx}{dy}=-\sin\theta\frac{d\theta}{dy}$

$$\frac{dy}{dx}=-\operatorname{cosec}\theta\frac{dy}{d\theta}$$

NEXT DIFFERENTIATE EXPRESSION WE JUST FOUND W.R.T x

$$\Rightarrow \frac{d^2y}{dx^2}=\operatorname{cosec}\theta\cot\theta\frac{d\theta}{dx}\frac{dy}{d\theta}-\operatorname{cosec}\theta\frac{d^2y}{d\theta^2}\frac{d\theta}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2}=\frac{1}{\sin\theta}\frac{\cos\theta}{\sin\theta}\times\frac{dy}{dx}-\frac{1}{\sin\theta}\frac{d^2y}{d\theta^2}\times\frac{1}{\frac{dx}{d\theta}}$$

$x=\cos\theta$

$\frac{dx}{d\theta}=-\sin\theta$

$$\Rightarrow \frac{d^2y}{dx^2}=\frac{\cos\theta}{\sin^2\theta}\left[-\frac{1}{\sin\theta}\frac{dy}{d\theta}\right]-\frac{1}{\sin\theta}\frac{d^2y}{d\theta^2}\frac{1}{-\sin\theta}$$

$$\Rightarrow \frac{d^2y}{d\theta^2}=-\frac{1}{\sin^2\theta}\left[\cot\theta\frac{dy}{d\theta}-\frac{d^2y}{d\theta^2}\right]$$

SUB INTO LEGENDRE'S EQUATION

$$\Rightarrow (1-\cos^2\theta)\left[-\frac{1}{\sin^2\theta}\left[\cot\theta\frac{dy}{d\theta}-\frac{d^2y}{d\theta^2}\right]\right]-2\cos\theta\left[-\operatorname{cosec}\theta\frac{dy}{d\theta}\right]+n(n+1)y=0$$

$$\Rightarrow \sin^2\theta\left[\frac{1}{\sin^2\theta}\left[\frac{d^2y}{d\theta^2}-\cot\theta\frac{dy}{d\theta}\right]\right]+2\cot\theta\frac{dy}{d\theta}+n(n+1)y=0$$

$$\Rightarrow \frac{d^2y}{d\theta^2}-\cot\theta\frac{dy}{d\theta}+2\cot\theta\frac{dy}{d\theta}+n(n+1)y=0$$

$$\Rightarrow \frac{d^2y}{d\theta^2}+\cot\theta\frac{dy}{d\theta}+n(n+1)y=0$$

BY VERIFICATION

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{dy}{d\theta}\right]=\frac{1}{\sin\theta}\left[\cos\theta\frac{dy}{d\theta}+\sin\theta\frac{d^2y}{d\theta^2}\right]$$

$$=\cot\theta\frac{dy}{d\theta}+\frac{d^2y}{d\theta^2}$$

$$\therefore \frac{d^2y}{d\theta^2}+\cot\theta\frac{dy}{d\theta}=\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{dy}{d\theta}\right]$$

HENCE THE EQUATION CAN BE WRITTEN AS

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{dy}{d\theta}\right]+n(n+1)y=0$$

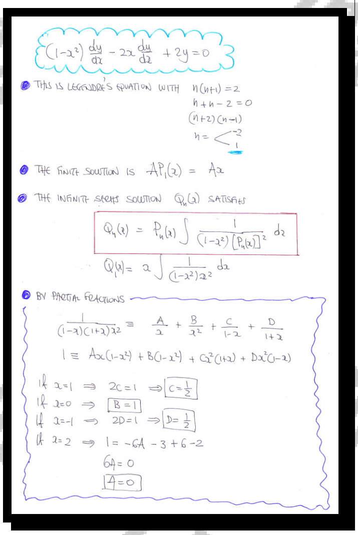
AS REQUIRED

Question 18

Find the two independent solutions of Legendre's equation

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=0.$$

$$\boxed{y=Ax+B\left[\frac{1}{x}\ln\left(\frac{1+x}{1-x}\right)-1\right]}$$



$\left(1-x^2\right)\frac{dy}{dx}-2x\frac{dy}{dx}+2y=0$

- THIS IS LEGENDRE'S EQUATION WITH $n(n+1)=2$

$n+n-2=0$

$(n+2)(n-1)$

$n = \begin{cases}-2\\1\end{cases}$

- THE FINITE SOLUTION IS $AP_1(x) = Ax$
- THE INFINITE SERIES SOLUTION $Q_n(x)$ SATISFIES

$$Q_n(x) = P_n(x)\int\frac{1}{(1-x^2)\left[P_n(x)\right]^2}\,dx$$

$$Q_1(x) = x\int\frac{1}{(1-x^2)x^2}\,dx$$

- BY PARTIAL FRACTIONS

$$\frac{1}{(1-x)(1+x)x^2} \equiv \frac{A}{x}+\frac{B}{x^2}+\frac{C}{1-x}+\frac{D}{1+x}$$

$$1 \equiv Ax(1-x^2)+B(1-x^2)+Cx^2(1+x)+Dx^2(1-x)$$

If $x=1 \Rightarrow 2C=1 \Rightarrow \boxed{C=\frac{1}{2}}$

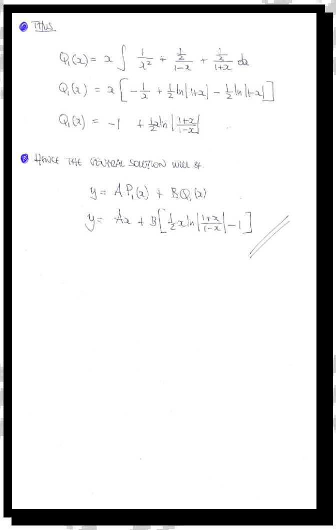
If $x=0 \Rightarrow \boxed{B=1}$

If $x=-1 \Rightarrow 2D=1 \Rightarrow \boxed{D=\frac{1}{2}}$

If $x=2 \Rightarrow 1=-6A-3+6-2$

$6A=0$

$\boxed{A=0}$



- THUS

$$Q_1(x) = x\int\frac{1}{x^2}+\frac{\frac{1}{2}}{1-x}+\frac{\frac{1}{2}}{1+x}\,dx$$

$$Q_1(x) = x\left[-\frac{1}{x}+\frac{1}{2}\ln|1+x|-\frac{1}{2}\ln|1-x|\right]$$

$$Q_1(x) = -1+\frac{1}{2}x\ln\left|\frac{1+x}{1-x}\right|$$

- HENCE THE GENERAL SOLUTION WILL BE

$$y = AP_1(x)+BQ_1(x)$$

$$y = Ax+B\left[\frac{1}{2}x\ln\left|\frac{1+x}{1-x}\right|-1\right]$$

Question 19

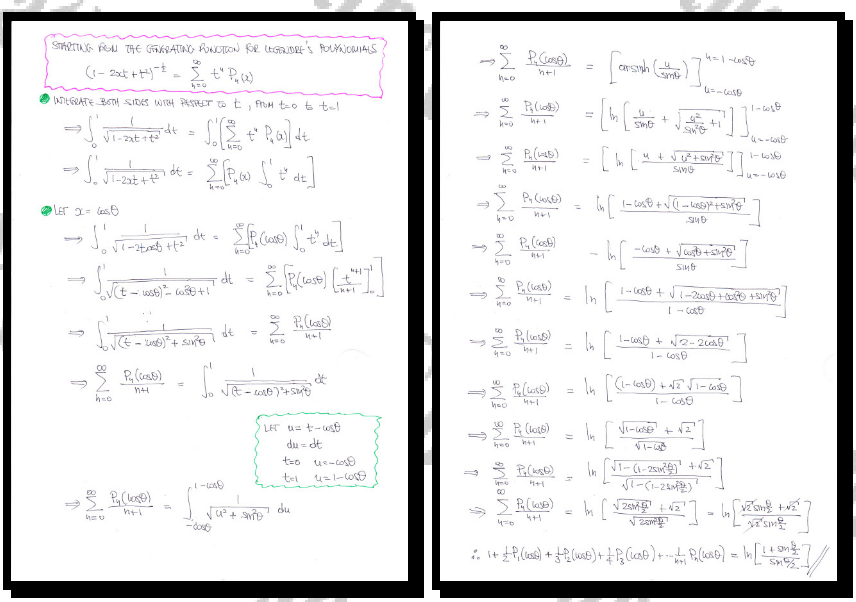
The generating function for the Legendre's Polynomials $P_n(x)$, satisfies

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right].$$

Using this result, and integrating both sides with respect to t, from 0 to 1, show that

$$\sum_{n=0}^{\infty}\left[\frac{P_n(\cos\theta)}{n+1}\right] = \ln\left[1+\text{cosec}\left(\tfrac{1}{2}\theta\right)\right].$$

proof



STARTING FROM THE GENERATING FUNCTION FOR LEGENDRE'S POLYNOMIALS

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(x)$$

INTEGRATE BOTH SIDES WITH RESPECT TO t, FROM $t=0$ TO $t=1$

$$\Rightarrow \int_0^1 \frac{1}{\sqrt{1-2xt+t^2}}\,dt = \int_0^1\left[\sum_{n=0}^{\infty} t^n P_n(x)\right]dt$$

$$\Rightarrow \int_0^1 \frac{1}{\sqrt{1-2xt+t^2}}\,dt = \sum_{n=0}^{\infty}\left[P_n(x)\int_0^1 t^n\,dt\right]$$

LET $x=\cos\theta$

$$\Rightarrow \int_0^1 \frac{1}{\sqrt{1-2t\cos\theta+t^2}}\,dt = \sum_{n=0}^{\infty}\left[P_n(\cos\theta)\int_0^1 t^n\,dt\right]$$

$$\Rightarrow \int_0^1 \frac{1}{\sqrt{(t-\cos\theta)^2-\cos^2\theta+1}}\,dt = \sum_{n=0}^{\infty}\left[P_n(\cos\theta)\left[\frac{t^{n+1}}{n+1}\right]_0^1\right]$$

$$\Rightarrow \int_0^1 \frac{1}{\sqrt{(t-\cos\theta)^2+\sin^2\theta}}\,dt = \sum_{n=0}^{\infty}\frac{P_n(\cos\theta)}{n+1}$$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{P_n(\cos\theta)}{n+1} = \int_0^1 \frac{1}{\sqrt{(t-\cos\theta)^2+\sin^2\theta}}\,dt$$

LET $u=t-\cos\theta$
$du=dt$
$t=0 \quad u=-\cos\theta$
$t=1 \quad u=1-\cos\theta$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{P_n(\cos\theta)}{n+1} = \int_{-\cos\theta}^{1-\cos\theta}\frac{1}{\sqrt{u^2+\sin^2\theta}}\,du$$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{P_n(\cos\theta)}{n+1} = \left[\text{arsinh}\left(\frac{u}{\sin\theta}\right)\right]_{u=-\cos\theta}^{u=1-\cos\theta}$$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{P_n(\cos\theta)}{n+1} = \left[\ln\left[\frac{u}{\sin\theta}+\sqrt{\frac{u^2}{\sin^2\theta}+1}\right]\right]_{u=-\cos\theta}^{1-\cos\theta}$$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{P_n(\cos\theta)}{n+1} = \left[\ln\left[\frac{u+\sqrt{u^2+\sin^2\theta}}{\sin\theta}\right]\right]_{u=-\cos\theta}^{1-\cos\theta}$$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{P_n(\cos\theta)}{n+1} = \ln\left[\frac{1-\cos\theta+\sqrt{(1-\cos\theta)^2+\sin^2\theta}}{\sin\theta}\right]$$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{P_n(\cos\theta)}{n+1} \quad - \ln\left[\frac{-\cos\theta+\sqrt{\cos^2\theta+\sin^2\theta}}{\sin\theta}\right]$$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{P_n(\cos\theta)}{n+1} = \ln\left[\frac{1-\cos\theta+\sqrt{1-2\cos\theta+\cos^2\theta+\sin^2\theta}}{1-\cos\theta}\right]$$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{P_n(\cos\theta)}{n+1} = \ln\left[\frac{1-\cos\theta+\sqrt{2-2\cos\theta}}{1-\cos\theta}\right]$$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{P_n(\cos\theta)}{n+1} = \ln\left[\frac{(1-\cos\theta)+\sqrt{2}\sqrt{1-\cos\theta}}{1-\cos\theta}\right]$$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{P_n(\cos\theta)}{n+1} = \ln\left[\frac{\sqrt{1-\cos\theta}+\sqrt{2}}{\sqrt{1-\cos\theta}}\right]$$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{P_n(\cos\theta)}{n+1} = \ln\left[\frac{\sqrt{1-(1-2\sin^2\frac{\theta}{2})}+\sqrt{2}}{\sqrt{1-(1-2\sin^2\frac{\theta}{2})}}\right]$$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{P_n(\cos\theta)}{n+1} = \ln\left[\frac{\sqrt{2\sin^2\frac{\theta}{2}}+\sqrt{2}}{\sqrt{2\sin^2\frac{\theta}{2}}}\right] = \ln\left[\frac{\sqrt{2}\sin\frac{\theta}{2}+\sqrt{2}}{\sqrt{2}\sin\frac{\theta}{2}}\right]$$

$$\therefore\ 1+\tfrac{1}{2}P_1(\cos\theta)+\tfrac{1}{3}P_2(\cos\theta)+\tfrac{1}{4}P_3(\cos\theta)+\cdots\tfrac{1}{n+1}P_n(\cos\theta) = \ln\left[\frac{1+\sin\frac{\theta}{2}}{\sin\theta/2}\right]$$

Question 20

The generating function g for the Legendre's polynomials $P_n(x)$, satisfies

$$g(x,t) = \left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right].$$

a) By differentiating g with respect to t, prove that

$$(2n+1)\,x\,P_n(x) = (n+1)\,P_{n+1}(x) + n\,P_{n-1}(x).$$

b) By differentiating g once with respect to t and once with respect to x, prove that

$$n\,P_n(x) = x\,P_n'(x) - P_{n-1}'(x).$$

c) Use parts **(a)** and **(b)** to show that

$$(2n+1)\,P_n(x) = P_{n+1}'(x) + P_{n-1}'(x).$$

d) Use parts **(b)** and **(c)** to deduce that

$$(n+1)\,P_n(x) = P_{n+1}'(x) - x\,P_n'(x).$$

e) Use parts **(b)** and **(d)** to show that

$$\left(1-x^2\right)P_n'(x) = n\left[P_{n-1}(x) - x\,P_n(x)\right].$$

f) Use parts **(a)** and **(e)** to show that

$$\left(1-x^2\right)P_n'(x) = (n+1)\left[xP_n(x) - P_{n+1}(x)\right]$$

proof

[solution overleaf]

a) STARTING FROM THE GENERATING FUNCTION

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right]$$

DIFFERENTIATE WITH RESPECT TO t.

$$\Rightarrow -\tfrac{1}{2}(1-2xt+t^2)^{-\frac{3}{2}}(-2x+2t) = \sum_{n=0}^{\infty}\left[nt^{n-1}P_n(x)\right]$$

$$\Rightarrow (x-t)(1-2xt+t^2)^{-\frac{3}{2}} = \sum_{n=0}^{\infty}\left[nt^{n-1}P_n(x)\right]$$

$$\Rightarrow (x-t)(1-2xt+t^2)^{-\frac{3}{2}}(1-2xt+t^2) = (1-2xt+t^2)\sum_{n=0}^{\infty}\left[nt^{n-1}P_n(x)\right]$$

$$\Rightarrow (x-t)(1-2xt+t^2)^{-\frac{1}{2}} = (1-2xt+t^2)\sum_{n=0}^{\infty}\left[nt^{n-1}P_n(x)\right]$$

$$\Rightarrow (x-t)\sum_{n=0}^{\infty}\left[t^n P_n(x)\right] = (1-2xt+t^2)\sum_{n=0}^{\infty}\left[nt^{n-1}P_n(x)\right]$$

$$\Rightarrow \sum_{n=0}^{\infty}\left[t^n xP_n(x) - t^{n+1}P_n(x)\right] = \sum_{n=0}^{\infty}\left[nt^{n-1}P_n(x) - 2xnt^nP_n(x) + nt^{n+1}P_n(x)\right]$$

EQUATE THE COEFFICIENTS OF POWER OF t, SAY $[t^n]$

$$\Rightarrow xP_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x)$$

$$\Rightarrow 0 = (n+1)P_{n+1}(x) - x(2n+1)P_n(x) + [(n-1)+1]P_{n-1}(x)$$

$$\Rightarrow (n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$$

AS REQUIRED

b) STARTING FROM THE GENERATING FUNCTION

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right]$$

DIFFERENTIATE WITH RESPECT TO t, AND WITH RESPECT TO x

$$\left.\begin{aligned} -\tfrac{1}{2}(1-2xt+t^2)^{-\frac{3}{2}}(-2x+2t) &= \sum_{n=0}^{\infty}\left[nt^{n-1}P_n(x)\right] \\ -\tfrac{1}{2}(1-2xt+t^2)^{-\frac{3}{2}}(-2t) &= \sum_{n=0}^{\infty}\left[t^n P_n'(x)\right] \end{aligned}\right\} \Rightarrow$$

$$\left.\begin{aligned} (x-t)(1-2xt+t^2)^{-\frac{3}{2}} &= \sum_{n=0}^{\infty}\left[nt^{n-1}P_n(x)\right] \\ t(1-2xt+t^2)^{-\frac{3}{2}} &= \sum_{n=0}^{\infty}\left[t^n P_n'(x)\right] \end{aligned}\right\} \Rightarrow$$

MULTIPLY THE FIRST BY t, AND THE SECOND BY $(x-t)$ & EQUATE THE RHS.

$$t\sum_{n=0}^{\infty}\left[nt^{n-1}P_n(x)\right] = (x-t)\sum_{n=0}^{\infty}\left[t^n P_n'(x)\right]$$

$$\sum_{n=0}^{\infty}\left[nt^n P_n(x)\right] = \sum_{n=0}^{\infty}\left[xt^n P_n'(x) - t^{n+1}P_n'(x)\right]$$

EQUATE COEFFICIENTS OF POWERS OF t, SAY $[t^n]$

$$nP_n(x) = xP_n'(x) - P_{n-1}'(x)$$

c) FROM (a) $(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$

DIFFERENTIATE W.R.T x

$$(n+1)P_{n+1}'(x) - (2n+1)P_n(x) - (2n+1)xP_n'(x) + nP_{n-1}'(x) = 0$$

FROM (b) $\boxed{xP_n'(x) = nP_n(x) + P_{n-1}'(x)}$

$$\Rightarrow (n+1)P_{n+1}'(x) - (2n+1)P_n(x) - (2n+1)\left[nP_n(x) + P_{n-1}'(x)\right] + nP_{n-1}'(x) = 0$$

$$\Rightarrow (n+1)P_{n+1}'(x) - (2n+1)P_n(x) - n(2n+1)P_n(x) - (2n+1)P_{n-1}'(x) + nP_{n-1}'(x) = 0$$

$$\Rightarrow (n+1)P_{n+1}'(x) - (2n+1)(n+1)P_n(x) - (n+1)P_{n-1}'(x) = 0$$

$$\Rightarrow P_{n+1}'(x) - (2n+1)P_n(x) - P_{n-1}'(x) = 0$$

$$\Rightarrow (2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)$$

AS REQUIRED

d) FROM (b): $nP_n(x) = xP_n'(x) - P_{n-1}'(x)$

FROM (c): $(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)$

SUBTRACTING THE EQUATIONS "UPWARDS"

$$(n+1)P_n(x) = P_{n+1}'(x) - xP_n'(x)$$

AS REQUIRED

e) FROM (b): $nP_n(x) = xP_n'(x) - P_{n-1}'(x)$

FROM (d): $(n+1)P_n(x) = P_{n+1}'(x) - xP_n'(x)$ $\Rightarrow$

REPLACE n BY $n-1$ IN THE SECOND EQUATION

$$P_n(x) = xP_n'(x) - P_n'(x)$$

$$nP_{n-1}(x) = P_n'(x) - xP_{n-1}'(x)$$

MULTIPLY THE THE FIRST EQUATION BY x

$$nxP_n(x) = x^2P_n'(x) - xP_{n-1}'(x)$$

$$nP_{n-1}(x) = P_n'(x) - xP_{n-1}'(x)$$

SUBTRACT "UPWARDS"

$$nP_{n-1}(x) - nxP_n(x) = (1-x^2)P_n'(x)$$

$$\text{ie } (1-x^2)P_n'(x) = n\left[P_{n-1}(x) - xP_n(x)\right]$$

AS REQUIRED

f) FROM (a): $(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$

FROM (e): $(1-x^2)P_n'(x) = n\left[P_{n-1}(x) - xP_n(x)\right]$ $\Rightarrow$

REWRITE AS

$$\left.\begin{aligned} (n+1)xP_n(x) + nxP_n(x) &= (n+1)P_{n+1}(x) + nP_{n-1}(x) \\ (1-x^2)P_n'(x) &= nP_{n-1}(x) - nxP_n(x) \end{aligned}\right\} \Rightarrow$$

TIDY SO THE EQUATIONS HAVE IDENTICAL R.H.S

$$\left.\begin{aligned} (n+1)xP_n(x) - (n+1)P_{n+1}(x) &= nP_{n-1}(x) - nxP_n(x) \\ (1-x^2)P_n'(x) &= nP_{n-1}(x) - nxP_n(x) \end{aligned}\right\} \Rightarrow$$

$$\therefore (1-x^2)P_n'(x) = (n+1)xP_n(x) - (n+1)P_{n+1}(x)$$

$$(1-x^2)P_n'(x) = (n+1)\left[xP_n(x) - P_{n+1}(x)\right]$$

Question 21

Find one series solution for the Legendre's equation

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n\left(n+1\right)y=0\,,\;\; n\in\mathbb{R}\,,$$

about $x=1$.

$$\boxed{y=A\sum_{r=0}^{\infty}\left[\frac{(n+r)!}{(n-r)!}\times\frac{1}{(r!)^2}\times\left(\frac{x-1}{2}\right)^2\right]}$$