LIMITS

LIMITS BY STANDARD EXPANSIONS

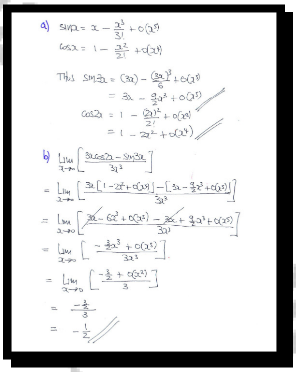
Question 1 (***)

a) Write down the first two non zero terms in the expansions of $\sin 3x$ and $\cos 2x$.

b) Hence find the exact value of

$$\lim_{x\to 0}\left[\frac{3x\cos 2x-\sin 3x}{3x^3}\right]$$

$\boxed{\sin 3x \approx 3x-\frac{9}{2}x^3}$, $\boxed{\cos 2x \approx 1-2x^2}$, $\boxed{-\frac{1}{2}}$



a) $\sin x = x - \frac{x^3}{3!} + O(x^5)$

$\cos x = 1 - \frac{x^2}{2!} + O(x^4)$

Thus $\sin 3x = (3x) - \frac{(3x)^3}{6} + O(x^5)$

$= 3x - \frac{9}{2}x^3 + O(x^5)$

$\cos 2x = 1 - \frac{(2x)^2}{2!} + O(x^4)$

$= 1 - 2x^2 + O(x^4)$

b) $\lim_{x\to 0}\left[\frac{3x\cos 2x - \sin 3x}{3x^3}\right]$

$= \lim_{x\to 0}\left[\frac{3x[1-2x^2+O(x^4)] - [3x - \frac{9}{2}x^3 + O(x^5)]}{3x^3}\right]$

$= \lim_{x\to 0}\left[\frac{\cancel{3x} - 6x^3 + O(x^5) - \cancel{3x} + \frac{9}{2}x^3 + O(x^5)}{3x^3}\right]$

$= \lim_{x\to 0}\left[\frac{-\frac{3}{2}x^3 + O(x^5)}{3x^3}\right]$

$= \lim_{x\to 0}\left[\frac{-\frac{3}{2} + O(x^2)}{3}\right]$

$= \frac{-\frac{3}{2}}{3}$

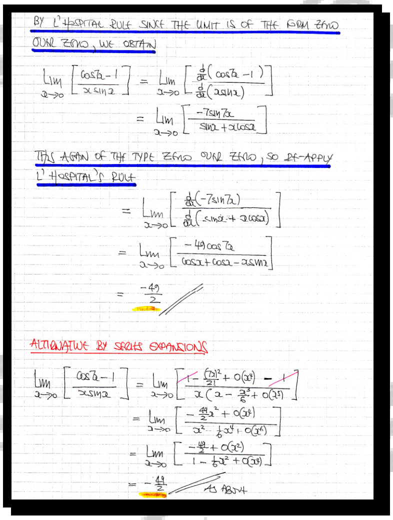
$= -\frac{1}{2}$

Question 2 **(***)**

Use standard expansions of functions to find the value of the following limit.

$$\lim_{x\to 0}\left[\frac{\cos 7x-1}{x\sin x}\right].$$

$\boxed{\text{MP1-A}}$, $\boxed{-\frac{49}{2}}$



BY L'HOSPITAL RULE SINCE THE LIMIT IS OF THE FORM ZERO OVER ZERO, WE OBTAIN

$$\lim_{x\to 0}\left[\frac{\cos 7x-1}{x\sin x}\right]=\lim_{x\to 0}\left[\frac{\frac{d}{dx}(\cos 7x-1)}{\frac{d}{dx}(x\sin x)}\right]$$

$$=\lim_{x\to 0}\left[\frac{-7\sin 7x}{\sin x+x\cos x}\right]$$

THIS AGAIN OF THE TYPE ZERO OVER ZERO, SO RE-APPLY L'HOSPITAL'S RULE

$$=\lim_{x\to 0}\left[\frac{\frac{d}{dx}(-7\sin 7x)}{\frac{d}{dx}(\sin x+x\cos x)}\right]$$

$$=\lim_{x\to 0}\left[\frac{-49\cos 7x}{\cos x+\cos x-x\sin x}\right]$$

$$=\frac{-49}{2}$$

ALTERNATIVE BY SERIES EXPANSIONS

$$\lim_{x\to 0}\left[\frac{\cos 7x-1}{x\sin x}\right]=\lim_{x\to 0}\left[\frac{\cancel{1}-\frac{(7x)^2}{2!}+O(x^4)-\cancel{1}}{x\left(x-\frac{x^3}{6}+O(x^5)\right)}\right]$$

$$=\lim_{x\to 0}\left[\frac{-\frac{49}{2}x^2+O(x^4)}{x^2-\frac{1}{6}x^4+O(x^6)}\right]$$

$$=\lim_{x\to 0}\left[\frac{-\frac{49}{2}+O(x^2)}{1-\frac{1}{6}x^2+O(x^4)}\right]$$

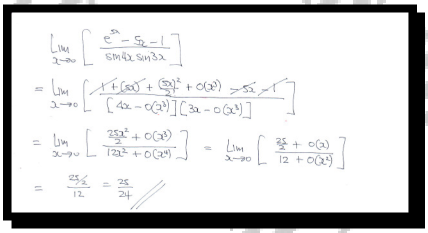
$$=-\frac{49}{2}$$ AS ABOVE

Question 3 **(***)**

Use standard expansions of functions to find the value of the following limit.

$$\lim_{x\to 0}\left[\frac{e^{5x}-5x-1}{\sin 4x\sin 3x}\right].$$

$\boxed{\frac{25}{24}}$



$$\lim_{x\to 0}\left[\frac{e^{5x}-5x-1}{\sin 4x\sin 3x}\right]$$

$$=\lim_{x\to 0}\left[\frac{\cancel{1}+\cancel{(5x)}+\frac{(5x)^2}{2!}+O(x^3)\cancel{-5x}\cancel{-1}}{[4x-O(x^3)][3x-O(x^3)]}\right]$$

$$=\lim_{x\to 0}\left[\frac{\frac{25x^2}{2}+O(x^3)}{12x^2+O(x^4)}\right]=\lim_{x\to 0}\left[\frac{\frac{25}{2}+O(x)}{12+O(x^2)}\right]$$

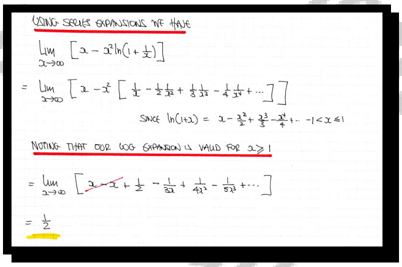
$$=\frac{25/2}{12}=\frac{25}{24}$$

Question 4 (***)

Use standard series expansions to evaluate the following limit.

$$\lim_{x\to\infty}\left[x - x^2\ln\left[x+\frac{1}{x}\right]\right].$$

V , FP3 , $\boxed{\frac{1}{2}}$



USING SERIES EXPANSIONS WE HAVE

$$\lim_{x\to\infty}\left[x - x^2\ln\left(1+\frac{1}{x}\right)\right]$$

$$= \lim_{x\to\infty}\left[x - x^2\left[\frac{1}{x} - \frac{1}{2}\frac{1}{x^2} + \frac{1}{3}\frac{1}{x^3} - \frac{1}{4}\frac{1}{x^4} + \dots\right]\right]$$

SINCE $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad -1 < x \leqslant 1$

NOTING THAT OUR LOG EXPANSION IS VALID FOR $x \geqslant 1$

$$= \lim_{x\to\infty}\left[\cancel{x - x} + \frac{1}{2} - \frac{1}{3x} + \frac{1}{4x^2} - \frac{1}{5x^3} + \dots\right]$$

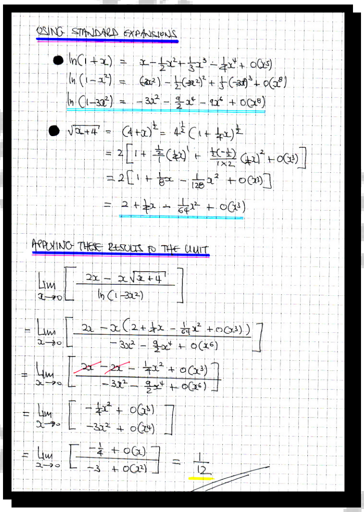
$$= \frac{1}{2}$$

Question 5 (***)

By considering series expansion, determine the value of the following limit.

$$\lim_{x\to 0}\left[\frac{2x - x\sqrt{x+4}}{\ln\left(1-3x^2\right)}\right].$$

$\boxed{}$, $\boxed{\frac{1}{12}}$



USING STANDARD EXPANSIONS

- $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + O(x^5)$

  $\ln(1-x^2) = (-3x^2) - \frac{1}{2}(-3x^2)^2 + \frac{1}{3}(-3x^2)^3 + O(x^8)$

  $\ln(1-3x^2) = -3x^2 - \frac{9}{2}x^4 - 9x^6 + O(x^8)$

- $\sqrt{x+4} = (4+x)^{\frac{1}{2}} = 4^{\frac{1}{2}}\left(1+\frac{1}{4}x\right)^{\frac{1}{2}}$

  $= 2\left[1 + \frac{\frac{1}{2}}{1}\left(\frac{1}{4}x\right)^1 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1\times 2}\left(\frac{1}{4}x\right)^2 + O(x^3)\right]$

  $= 2\left[1 + \frac{1}{8}x - \frac{1}{128}x^2 + O(x^3)\right]$

  $= 2 + \frac{1}{4}x - \frac{1}{64}x^2 + O(x^3)$

APPLYING THESE RESULTS TO THE LIMIT

$$\lim_{x\to 0}\left[\frac{2x - x\sqrt{x+4}}{\ln(1-3x^2)}\right]$$

$$= \lim_{x\to 0}\left[\frac{2x - x\left(2 + \frac{1}{4}x - \frac{1}{64}x^2 + O(x^3)\right)}{-3x^2 - \frac{9}{2}x^4 + O(x^6)}\right]$$

$$= \lim_{x\to 0}\left[\frac{\cancel{2x} - \cancel{2x} - \frac{1}{4}x^2 + O(x^3)}{-3x^2 - \frac{9}{2}x^4 + O(x^6)}\right]$$

$$= \lim_{x\to 0}\left[\frac{-\frac{1}{4}x^2 + O(x^3)}{-3x^2 + O(x^4)}\right]$$

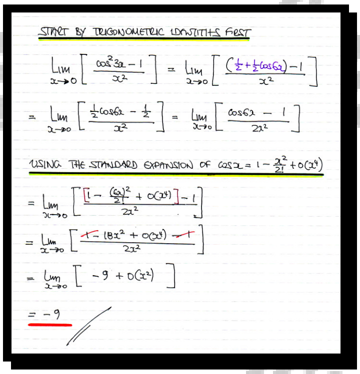
$$= \lim_{x\to 0}\left[\frac{-\frac{1}{4} + O(x)}{-3 + O(x^2)}\right] = \frac{1}{12}$$

Question 6 (***+)

Use standard expansions of functions to find the value of the following limit.

$$\lim_{x\to 0}\left[\frac{\cos^2 3x-1}{x^2}\right].$$

$\boxed{\text{P3-N}}$, $\boxed{-9}$



START BY TRIGONOMETRIC IDENTITIES FIRST

$$\lim_{x\to 0}\left[\frac{\cos^2 3x-1}{x^2}\right]=\lim_{x\to 0}\left[\frac{\left(\frac{1}{2}+\frac{1}{2}\cos 6x\right)-1}{x^2}\right]$$

$$=\lim_{x\to 0}\left[\frac{\frac{1}{2}\cos 6x-\frac{1}{2}}{x^2}\right]=\lim_{x\to 0}\left[\frac{\cos 6x-1}{2x^2}\right]$$

USING THE STANDARD EXPANSION OF $\cos x=1-\frac{x^2}{2!}+O(x^4)$

$$=\lim_{x\to 0}\left[\frac{\left[1-\frac{(6x)^2}{2!}+O(x^4)\right]-1}{2x^2}\right]$$

$$=\lim_{x\to 0}\left[\frac{\cancel{1}-18x^2+O(x^4)-\cancel{1}}{2x^2}\right]$$

$$=\lim_{x\to 0}\left[-9+O(x^2)\right]$$

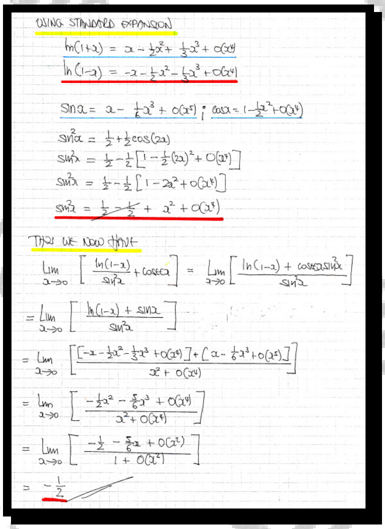
$= -9$

Question 7 (***+)

Use standard expansions of functions to find the value of the following limit.

$$\lim_{x\to 0}\left[\frac{\ln(1-x)}{\sin^2 x}+\operatorname{cosec} x\right].$$

$\boxed{}$, $\boxed{-\frac{1}{2}}$



USING STANDARD EXPANSION

$\ln(1+x)=x-\frac{1}{2}x^2+\frac{1}{3}x^3+O(x^4)$

$\ln(1-x)=-x-\frac{1}{2}x^2-\frac{1}{3}x^3+O(x^4)$

$\sin x=x-\frac{1}{6}x^3+O(x^5)$; $\cos x=1-\frac{1}{2}x^2+O(x^4)$

$\sin^2 x=\frac{1}{2}+\frac{1}{2}\cos(2x)$

$\sin^2 x=\frac{1}{2}-\frac{1}{2}\left[1-\frac{1}{2}(2x)^2+O(x^4)\right]$

$\sin^2 x=\frac{1}{2}-\frac{1}{2}\left[1-2x^2+O(x^4)\right]$

$\sin^2 x=\cancel{\frac{1}{2}-\frac{1}{2}}+x^2+O(x^4)$

THUS WE NOW HAVE

$$\lim_{x\to 0}\left[\frac{\ln(1-x)}{\sin^2 x}+\operatorname{cosec} x\right]=\lim_{x\to 0}\left[\frac{\ln(1-x)+\operatorname{cosec} x\sin^2 x}{\sin^2 x}\right]$$

$$=\lim_{x\to 0}\left[\frac{\ln(1-x)+\sin x}{\sin^2 x}\right]$$

$$=\lim_{x\to 0}\left[\frac{\left[-x-\frac{1}{2}x^2-\frac{1}{3}x^3+O(x^4)\right]+\left[x-\frac{1}{6}x^3+O(x^5)\right]}{x^2+O(x^4)}\right]$$

$$=\lim_{x\to 0}\left[\frac{-\frac{1}{2}x^2-\frac{5}{6}x^3+O(x^4)}{x^2+O(x^4)}\right]$$

$$=\lim_{x\to 0}\left[\frac{-\frac{1}{2}-\frac{5}{6}x+O(x^2)}{1+O(x^2)}\right]$$

$= -\frac{1}{2}$

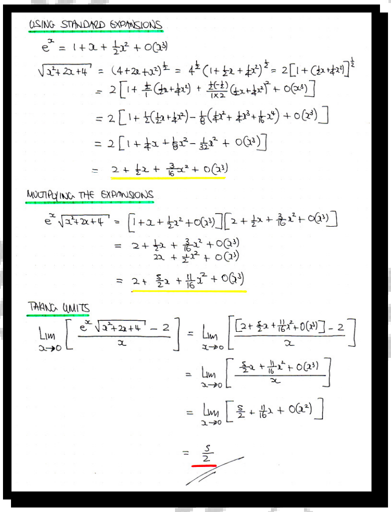
Question 8 (****+)

Use standard expansions of functions to find the value of the following limit.

$$\lim_{x\to0}\left[\frac{\mathrm{e}^{x}\sqrt{x^{2}+2x+4}\ -2}{x}\right].$$

No credit will be given for using alternative methods such as L' Hospital's rule.

$\boxed{}$, $\boxed{\frac{5}{2}}$



USING STANDARD EXPANSIONS

$$e^{x} = 1 + x + \tfrac{1}{2}x^{2} + O(x^{3})$$

$$\sqrt{x^{2}+2x+4} = (4+2x+x^{2})^{\frac{1}{2}} = 4^{\frac{1}{2}}\left(1+\tfrac{1}{2}x+\tfrac{1}{4}x^{2}\right)^{\frac{1}{2}} = 2\left[1+\left(\tfrac{1}{2}x+\tfrac{1}{4}x^{2}\right)\right]^{\frac{1}{2}}$$

$$= 2\left[1+\frac{\frac{1}{2}}{1}\left(\tfrac{1}{2}x+\tfrac{1}{4}x^{2}\right)+\frac{\frac{1}{2}(-\frac{1}{2})}{1\times2}\left(\tfrac{1}{2}x+\tfrac{1}{4}x^{2}\right)^{2}+O(x^{3})\right]$$

$$= 2\left[1+\tfrac{1}{2}\left(\tfrac{1}{2}x+\tfrac{1}{4}x^{2}\right)-\tfrac{1}{8}\left(\tfrac{1}{4}x^{2}+\tfrac{1}{4}x^{3}+\tfrac{1}{16}x^{4}\right)+O(x^{3})\right]$$

$$= 2\left[1+\tfrac{1}{4}x+\tfrac{1}{8}x^{2}-\tfrac{1}{32}x^{2}+O(x^{3})\right]$$

$$= 2+\tfrac{1}{2}x+\tfrac{3}{16}x^{2}+O(x^{3})$$

MULTIPLYING THE EXPANSIONS

$$e^{x}\sqrt{x^{2}+2x+4} = \left[1+x+\tfrac{1}{2}x^{2}+O(x^{3})\right]\left[2+\tfrac{1}{2}x+\tfrac{3}{16}x^{2}+O(x^{3})\right]$$

$$= 2+\tfrac{1}{2}x+\tfrac{3}{16}x^{2}+O(x^{3})$$
$$2x+\tfrac{1}{2}x^{2}+O(x^{3})$$

$$= 2+\tfrac{5}{2}x+\tfrac{11}{16}x^{2}+O(x^{3})$$

TAKING LIMITS

$$\lim_{x\to0}\left[\frac{e^{x}\sqrt{x^{2}+2x+4}-2}{x}\right] = \lim_{x\to0}\left[\frac{\left[2+\frac{5}{2}x+\frac{11}{16}x^{2}+O(x^{3})\right]-2}{x}\right]$$

$$= \lim_{x\to0}\left[\frac{\frac{5}{2}x+\frac{11}{16}x^{2}+O(x^{3})}{x}\right]$$

$$= \lim_{x\to0}\left[\tfrac{5}{2}+\tfrac{11}{16}x+O(x^{2})\right]$$

$$= \frac{5}{2}$$

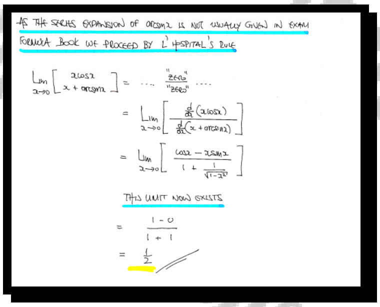
LIMITS BY L’HOSPITAL RULE

Question 1 (**)

Find the value of the following limit

$$\lim_{x\to 0}\left[\frac{x\cos x}{x+\arcsin x}\right].$$

$\boxed{\text{FP3}}$, $\boxed{\frac{1}{2}}$



AS THE SERIES EXPANSION OF $\arcsin x$ IS NOT USUALLY GIVEN IN EXAM FORMULA BOOK WE PROCEED BY L'HOSPITAL'S RULE

$$\lim_{x\to 0}\left[\frac{x\cos x}{x+\arcsin x}\right] = \dots \frac{\text{"zero"}}{\text{"zero"}} \dots$$

$$= \lim_{x\to 0}\left[\frac{\frac{d}{dx}(x\cos x)}{\frac{d}{dx}(x+\arcsin x)}\right]$$

$$= \lim_{x\to 0}\left[\frac{\cos x - x\sin x}{1+\frac{1}{\sqrt{1-x^2}}}\right]$$

THIS LIMIT NOW EXISTS

$$= \frac{1-0}{1+1}$$

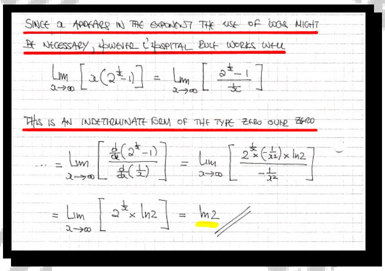
$$= \frac{1}{2}$$

Question 2 (+)**

Find the value of the following limit

$$\lim_{x\to\infty}\left[x\left(2^{\frac{1}{x}}-1\right)\right].$$

$\boxed{}$, $\boxed{\ln 2}$



SINCE x APPEARS IN THE EXPONENT THE USE OF LOGS MIGHT BE NECESSARY, HOWEVER L'HOSPITAL RULE WORKS WELL

$$\lim_{x\to\infty}\left[x\left(2^{\frac{1}{x}}-1\right)\right] = \lim_{x\to\infty}\left[\frac{2^{\frac{1}{x}}-1}{\frac{1}{x}}\right]$$

THIS IS AN INDETERMINATE FORM OF THE TYPE ZERO OVER ZERO

$$\dots = \lim_{x\to\infty}\left[\frac{\frac{d}{dx}\left(2^{\frac{1}{x}}-1\right)}{\frac{d}{dx}\left(\frac{1}{x}\right)}\right] = \lim_{x\to\infty}\left[\frac{2^{\frac{1}{x}}\left(-\frac{1}{x^2}\right)\times\ln 2}{-\frac{1}{x^2}}\right]$$

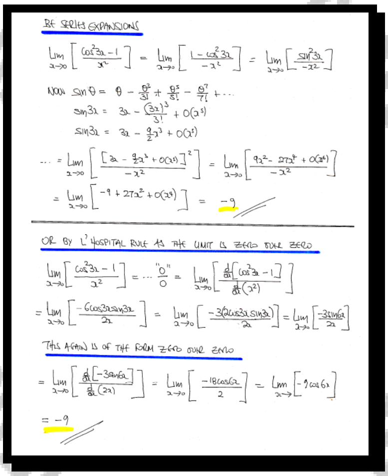
$$= \lim_{x\to\infty}\left[2^{\frac{1}{x}}\times\ln 2\right] = \ln 2$$

Question 3 (***)

Find the value of the following limit

$$\lim_{x \to 0}\left[\frac{\cos^2 3x - 1}{x^2}\right].$$

P3-K , $\boxed{-9}$



BY SERIES EXPANSIONS

$$\lim_{x\to 0}\left[\frac{\cos^2 3x-1}{x^2}\right]=\lim_{x\to 0}\left[\frac{1-\cos^2 3x}{-x^2}\right]=\lim_{x\to 0}\left[\frac{\sin^2 3x}{-x^2}\right]$$

NOW $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots$

$$\sin 3x = 3x - \frac{(3x)^3}{3!} + O(x^5)$$

$$\sin 3x = 3x - \frac{9}{2}x^3 + O(x^5)$$

$$\dots = \lim_{x\to 0}\left[\frac{\left[3x-\frac{9}{2}x^3+O(x^5)\right]^2}{-x^2}\right]=\lim_{x\to 0}\left[\frac{9x^2-27x^4+O(x^6)}{-x^2}\right]$$

$$=\lim_{x\to 0}\left[-9+27x^2+O(x^4)\right] = -9$$

OR BY L'HOSPITAL RULE AS THE LIMIT IS ZERO OVER ZERO

$$\lim_{x\to 0}\left[\frac{\cos^2 3x-1}{x^2}\right]=\dots\text{"}\frac{0}{0}\text{"}=\lim_{x\to 0}\left[\frac{\frac{d}{dx}\left[\cos^2 3x-1\right]}{\frac{d}{dx}(x^2)}\right]$$

$$=\lim_{x\to 0}\left[\frac{-6\cos 3x\sin 3x}{2x}\right]=\lim_{x\to 0}\left[\frac{-3(2\cos 3x\sin 3x)}{2x}\right]=\lim_{x\to 0}\left[\frac{-3\sin 6x}{2x}\right]$$

THIS AGAIN IS OF THE FORM ZERO OVER ZERO

$$=\lim_{x\to 0}\left[\frac{\frac{d}{dx}\left[-3\sin 6x\right]}{\frac{d}{dx}(2x)}\right]=\lim_{x\to 0}\left[\frac{-18\cos 6x}{2}\right]=\lim_{x\to}\left[-9\cos 6x\right]$$

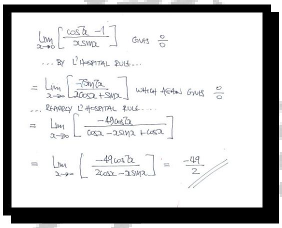
$$= -9$$

Question 4 (***)

Find the value of the following limit

$$\lim_{x \to 0}\left[\frac{\cos 7x - 1}{x\sin x}\right].$$

$\boxed{-\frac{49}{2}}$



$$\lim_{x\to 0}\left[\frac{\cos 7x-1}{x\sin x}\right] \text{ GIVES } \frac{0}{0}$$

... BY L'HOSPITAL RULE ...

$$=\lim_{x\to 0}\left[\frac{-7\sin 7x}{x\cos x+\sin x}\right] \text{ WHICH AGAIN GIVES } \frac{0}{0}$$

... REAPPLY L'HOSPITAL RULE ....

$$=\lim_{x\to 0}\left[\frac{-49\cos 7x}{\cos x - x\sin x + \cos x}\right]$$

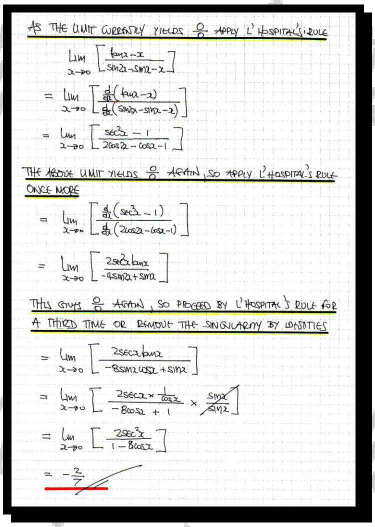
$$=\lim_{x\to 0}\left[\frac{-49\cos 7x}{2\cos x - x\sin x}\right] = \frac{-49}{2}$$

Question 5 (***)

Use L'Hospital's rule to find the value of the following limit

$$\lim_{x\to 0}\left[\frac{\tan x - x}{\sin 2x - \sin x - x}\right].$$

$\boxed{3-0}$, $\boxed{-\frac{2}{7}}$



AS THE LIMIT CURRENTLY YIELDS $\frac{0}{0}$ APPLY L'HOSPITAL'S RULE

$$\lim_{x\to 0}\left[\frac{\tan x - x}{\sin 2x - \sin x - x}\right]$$

$$= \lim_{x\to 0}\left[\frac{\frac{d}{dx}(\tan x - x)}{\frac{d}{dx}(\sin 2x - \sin x - x)}\right]$$

$$= \lim_{x\to 0}\left[\frac{\sec^2 x - 1}{2\cos 2x - \cos x - 1}\right]$$

THE ABOVE LIMIT YIELDS $\frac{0}{0}$ AGAIN, SO APPLY L'HOSPITAL'S RULE ONCE MORE

$$= \lim_{x\to 0}\left[\frac{\frac{d}{dx}(\sec^2 x - 1)}{\frac{d}{dx}(2\cos 2x - \cos x - 1)}\right]$$

$$= \lim_{x\to 0}\left[\frac{2\sec^2 x\tan x}{-4\sin 2x + \sin x}\right]$$

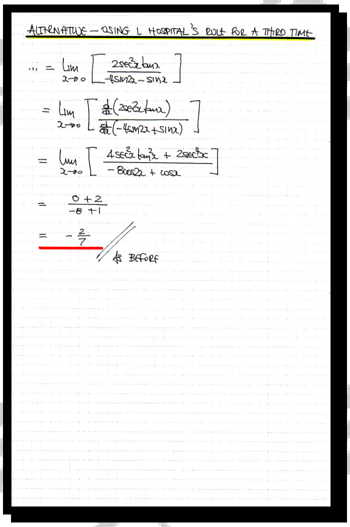
THIS GIVES $\frac{0}{0}$ AGAIN, SO PROCEED BY L'HOSPITAL'S RULE FOR A THIRD TIME OR REMOVE THE SINGULARITY BY IDENTITIES

$$= \lim_{x\to 0}\left[\frac{2\sec x\tan x}{-8\sin x\cos x + \sin x}\right]$$

$$= \lim_{x\to 0}\left[\frac{2\sec x \times \frac{1}{\cos x}}{-8\cos x + 1} \times \frac{\cancel{\sin x}}{\cancel{\sin x}}\right]$$

$$= \lim_{x\to 0}\left[\frac{2\sec^3 x}{1 - 8\cos x}\right]$$

$$= -\frac{2}{7}$$



ALTERNATIVE – USING L HOSPITAL'S RULE FOR A THIRD TIME

$$\dots = \lim_{x\to 0}\left[\frac{2\sec^2 x\tan x}{-4\sin 2x - \sin x}\right]$$

$$= \lim_{x\to 0}\left[\frac{\frac{d}{dx}(2\sec^2 x\tan x)}{\frac{d}{dx}(-4\sin 2x + \sin x)}\right]$$

$$= \lim_{x\to 0}\left[\frac{4\sec^2 x\tan^2 x + 2\sec^4 x}{-8\cos 2x + \cos x}\right]$$

$$= \frac{0+2}{-8+1}$$

$$= -\frac{2}{7}$$

AS BEFORE

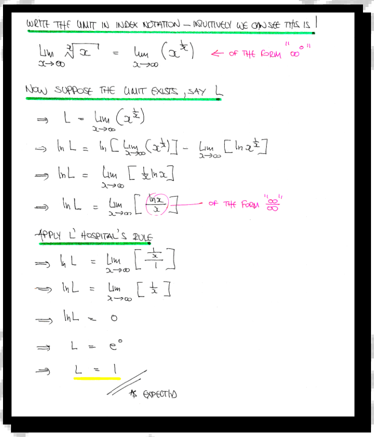
Question 6 **(***+)**

Show clearly that the following limit converges to 1.

$$\lim_{x\to\infty}\left[\sqrt[x]{x}\right]$$

You must justify the evaluation.

$\boxed{}$, $\boxed{\text{proof}}$



WRITE THE LIMIT IN INDEX NOTATION – INTUITIVELY WE CAN SEE THIS IS 1

$\lim\limits_{x\to\infty}\sqrt[x]{x} = \lim\limits_{x\to\infty}\left(x^{\frac{1}{x}}\right)$ ← OF THE FORM "∞^0"

NOW SUPPOSE THE LIMIT EXISTS, SAY L

$\Rightarrow L = \lim\limits_{x\to\infty}\left(x^{\frac{1}{x}}\right)$

$\Rightarrow \ln L = \ln\left[\lim\limits_{x\to\infty}\left(x^{\frac{1}{x}}\right)\right] = \lim\limits_{x\to\infty}\left[\ln x^{\frac{1}{x}}\right]$

$\Rightarrow \ln L = \lim\limits_{x\to\infty}\left[\frac{1}{x}\ln x\right]$

$\Rightarrow \ln L = \lim\limits_{x\to\infty}\left[\frac{\ln x}{x}\right]$ — OF THE FORM "$\frac{\infty}{\infty}$"

APPLY L'HOSPITAL'S RULE

$\Rightarrow \ln L = \lim\limits_{x\to\infty}\left[\frac{\frac{1}{x}}{1}\right]$

$\Rightarrow \ln L = \lim\limits_{x\to\infty}\left[\frac{1}{x}\right]$

$\Rightarrow \ln L = 0$

$\Rightarrow L = e^0$

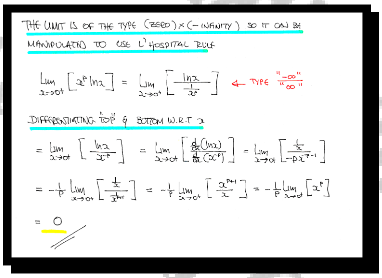
$\Rightarrow L = 1$ // AS EXPECTED

Question 7 **(****)**

If $p\in(0,\infty)$, show that

$$\lim_{x\to 0^+}\left[x^p \ln x\right] = 0\,, \quad x\in(0,\infty).$$

$\boxed{}$, $\boxed{\text{proof}}$



THE LIMIT IS OF THE TYPE (ZERO) × (– INFINITY) SO IT CAN BE MANIPULATED TO USE L'HOSPITAL RULE

$\lim\limits_{x\to 0^+}\left[x^p\ln x\right] = \lim\limits_{x\to 0^+}\left[\frac{\ln x}{\frac{1}{x^p}}\right]$ ← TYPE "$\frac{-\infty}{\infty}$"

DIFFERENTIATING "TOP" & BOTTOM W.R.T x

$= \lim\limits_{x\to 0^+}\left[\frac{\ln x}{x^{-p}}\right] = \lim\limits_{x\to 0^+}\left[\frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x^{-p})}\right] = \lim\limits_{x\to 0^+}\left[\frac{\frac{1}{x}}{-px^{-p-1}}\right]$

$= -\frac{1}{p}\lim\limits_{x\to 0^+}\left[\frac{\frac{1}{x}}{\frac{1}{x^{p+1}}}\right] = -\frac{1}{p}\lim\limits_{x\to 0^+}\left[\frac{x^{p+1}}{x}\right] = -\frac{1}{p}\lim\limits_{x\to 0^+}\left[x^p\right]$

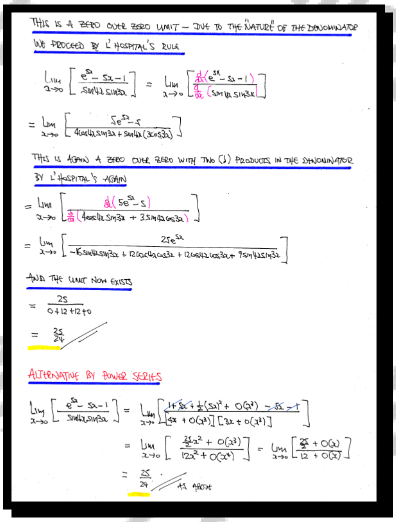
$= 0$ //

Question 8 (****)

Find the value of the following limit

$$\lim_{x \to 0}\left[\frac{e^{5x}-5x-1}{\sin 4x \sin 3x}\right].$$

V , FP3 , $\boxed{\frac{25}{24}}$



THIS IS A ZERO OVER ZERO LIMIT – DUE TO THE "NATURE" OF THE DENOMINATOR

WE PROCEED BY L'HOSPITAL'S RULE

$$\lim_{x \to 0}\left[\frac{e^{5x}-5x-1}{\sin 4x \sin 3x}\right] = \lim_{x \to 0}\left[\frac{\frac{d}{dx}(e^{5x}-5x-1)}{\frac{d}{dx}(\sin 4x \sin 3x)}\right]$$

$$= \lim_{x \to 0}\left[\frac{5e^{5x}-5}{4\cos 4x \sin 3x + \sin 4x(3\cos 3x)}\right]$$

THIS IS AGAIN A ZERO OVER ZERO WITH TWO (2) PRODUCTS IN THE DENOMINATOR

BY L'HOSPITAL'S AGAIN

$$= \lim_{x \to 0}\left[\frac{\frac{d}{dx}(5e^{5x}-5)}{\frac{d}{dx}(4\cos 4x \sin 3x + 3\sin 4x \cos 3x)}\right]$$

$$= \lim_{x \to 0}\left[\frac{25e^{5x}}{-16\sin 4x \sin 3x + 12\cos 4x \cos 3x + 12\cos 4x \cos 3x + 9\sin 4x \sin 3x}\right]$$

AND THE LIMIT NOW EXISTS

$$= \frac{25}{0+12+12+0}$$

$$= \frac{25}{24}$$

ALTERNATIVE BY POWER SERIES

$$\lim_{x \to 0}\left[\frac{e^{5x}-5x-1}{\sin 4x \sin 3x}\right] = \lim_{x \to 0}\left[\frac{\cancel{1}+\cancel{5x}+\frac{1}{2}(5x)^2+O(x^3)\cancel{-5x}\cancel{-1}}{[4x+O(x^3)][3x+O(x^3)]}\right]$$

$$= \lim_{x \to 0}\left[\frac{\frac{25}{2}x^2+O(x^3)}{12x^2+O(x^4)}\right] = \lim_{x \to 0}\left[\frac{\frac{25}{2}+O(x)}{12+O(x)}\right]$$

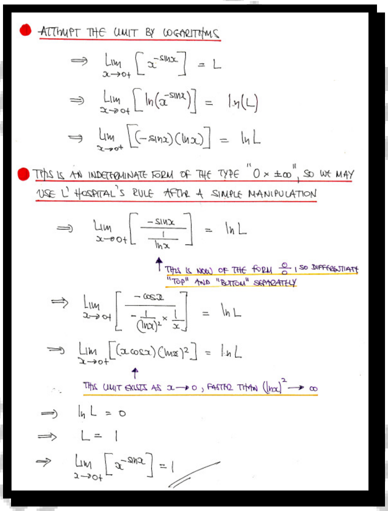
$$= \frac{25}{24}$$ AS ABOVE

Question 9 **(****)**

Find the value of the following limit

$$\lim_{x \to 0+} \left[x^{-\sin x} \right].$$

$\boxed{}$, $\boxed{1}$



● ATTEMPT THE LIMIT BY LOGARITHMS

$$\Rightarrow \lim_{x \to 0+} \left[x^{-\sin x} \right] = L$$

$$\Rightarrow \lim_{x \to 0+} \left[\ln\left(x^{-\sin x}\right) \right] = \ln(L)$$

$$\Rightarrow \lim_{x \to 0+} \left[(-\sin x)(\ln x) \right] = \ln L$$

● THIS IS AN INDETERMINATE FORM OF THE TYPE "$0 \times \pm\infty$", SO WE MAY USE L' HOSPITAL'S RULE AFTER A SIMPLE MANIPULATION

$$\Rightarrow \lim_{x \to 0+} \left[\frac{-\sin x}{\frac{1}{\ln x}} \right] = \ln L$$

↑ THIS IS NOW OF THE FORM $\frac{0}{0}$, SO DIFFERENTIATE "TOP" AND "BOTTOM" SEPARATELY

$$\Rightarrow \lim_{x \to 0+} \left[\frac{-\cos x}{-\frac{1}{(\ln x)^2} \times \frac{1}{x}} \right] = \ln L$$

$$\Rightarrow \lim_{x \to 0+} \left[(x\cos x)(\ln x)^2 \right] = \ln L$$

↑ THIS LIMIT EXISTS AS $x \to 0$, FASTER THAN $(\ln x)^2 \to \infty$

$$\Rightarrow \ln L = 0$$

$$\Rightarrow L = 1$$

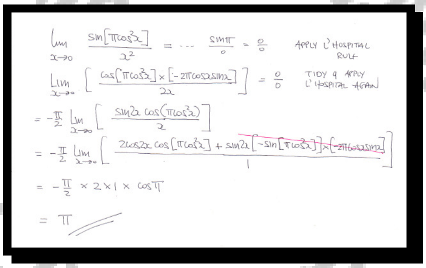
$$\Rightarrow \lim_{x \to 0+} \left[x^{-\sin x} \right] = 1$$

Question 10 **(****+)**

Find the value of the following limit

$$\lim_{x \to 0} \left[\frac{\sin\left(\pi \cos^2 x\right)}{x^2} \right].$$

$\boxed{\pi}$



$$\lim_{x \to 0} \frac{\sin\left[\pi\cos^2 x\right]}{x^2} = \dots \frac{\sin\pi}{0} = \frac{0}{0}$$ APPLY L'HOSPITAL RULE

$$\lim_{x \to 0} \left[\frac{\cos\left[\pi\cos^2 x\right] \times \left[-2\pi\cos x\sin x\right]}{2x} \right] = \frac{0}{0}$$ TIDY & APPLY L'HOSPITAL AGAIN

$$= -\frac{\pi}{2} \lim_{x \to 0} \left[\frac{\sin 2x \cos\left(\pi\cos^2 x\right)}{x} \right]$$

$$= -\frac{\pi}{2} \lim_{x \to 0} \left[\frac{2\cos 2x \cos\left[\pi\cos^2 x\right] + \sin 2x \cancel{\left[-\sin\left[\pi\cos^2 x\right]\right] \times \left[-2\pi\cos x\sin x\right]}}{1} \right]$$

$$= -\frac{\pi}{2} \times 2 \times 1 \times \cos\pi$$

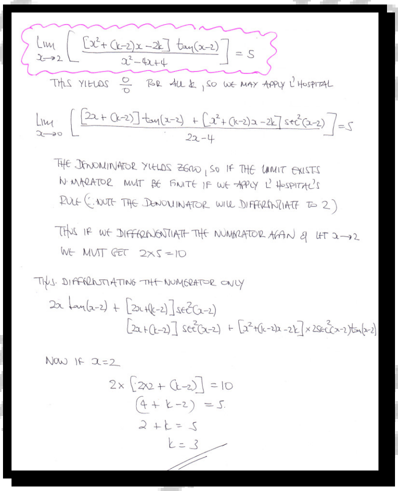
$$= \pi$$

Question 11 (****+)

Find the value of the constant k, given that

$$\lim_{x \to 2} \left\{ \frac{\left[x^2 + (k-2)x - 2k \right] \tan(x-2)}{x^2 - 4x + 4} \right\} = 5 .$$

$\boxed{k = 3}$



$$\lim_{x \to 2} \left[\frac{[x^2+(k-2)x-2k]\tan(x-2)}{x^2-4x+4} \right] = 5$$

THIS YIELDS $\frac{0}{0}$ FOR ALL k, SO WE MAY APPLY L'HOSPITAL

$$\lim_{x \to 0} \left[\frac{[2x+(k-2)]\tan(x-2) + [x^2+(k-2)x-2k]\sec^2(x-2)}{2x-4} \right] = 5$$

THE DENOMINATOR YIELDS ZERO, SO IF THE LIMIT EXISTS NUMARATOR MUST BE FINITE IF WE APPLY L'HOSPITAL'S RULE (∴ NOTE THE DENOMINATOR WILL DIFFERENTIATE TO 2)

THUS IF WE DIFFERENTIATE THE NUMERATOR AGAIN & LET $x \to 2$ WE MUST GET $2 \times 5 = 10$

THUS. DIFFERENTIATING THE NUMERATOR ONLY

$$2x \tan(x-2) + [2x+(k-2)]\sec^2(x-2)$$
$$[2x+(k-2)]\sec^2(x-2) + [x^2+(k-2)x-2k] \times 2\sec^2(x-2)\tan(x-2)$$

NOW IF $x=2$

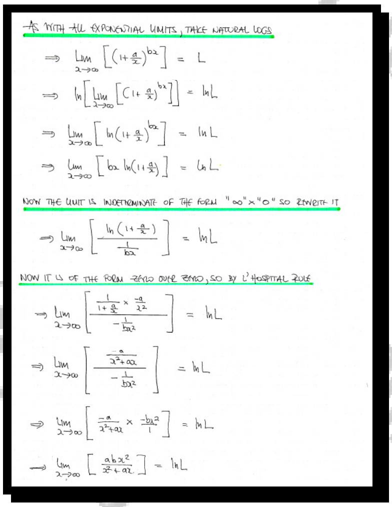
$$2 \times [2 \times 2 + (k-2)] = 10$$
$$(4 + k - 2) = 5.$$
$$2 + k = 5$$
$$k = 3$$

Question 12 **(****+)**

Show with detailed workings that

$$\lim_{x\to\infty}\left[\left(1+\frac{a}{x}\right)^{bx}\right] = \mathrm{e}^{ab}.$$

, $\boxed{\text{proof}}$



AS WITH ALL EXPONENTIAL LIMITS, TAKE NATURAL LOGS

$$\Rightarrow \lim_{x\to\infty}\left[\left(1+\frac{a}{x}\right)^{bx}\right] = L$$

$$\Rightarrow \ln\left[\lim_{x\to\infty}\left[\left(1+\frac{a}{x}\right)^{bx}\right]\right] = \ln L$$

$$\Rightarrow \lim_{x\to\infty}\left[\ln\left(1+\frac{a}{x}\right)^{bx}\right] = \ln L$$

$$\Rightarrow \lim_{x\to\infty}\left[bx\ln\left(1+\frac{a}{x}\right)\right] = \ln L$$

NOW THE LIMIT IS INDETERMINATE OF THE FORM "∞" × "0" SO REWRITE IT

$$\Rightarrow \lim_{x\to\infty}\left[\frac{\ln\left(1+\frac{a}{x}\right)}{\frac{1}{bx}}\right] = \ln L$$

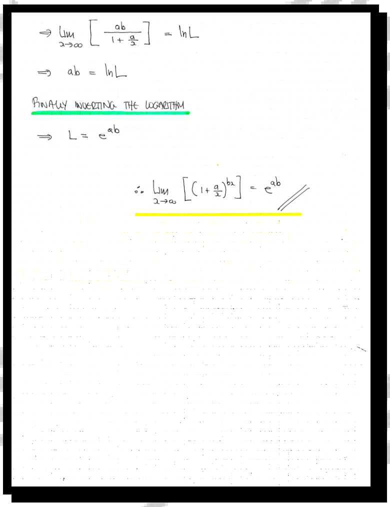
NOW IT IS OF THE FORM ZERO OVER ZERO, SO BY L'HOSPITAL RULE

$$\Rightarrow \lim_{x\to\infty}\left[\frac{\frac{1}{1+\frac{a}{x}}\times\frac{-a}{x^2}}{-\frac{1}{bx^2}}\right] = \ln L$$

$$\Rightarrow \lim_{x\to\infty}\left[\frac{\frac{-a}{x^2+ax}}{-\frac{1}{bx^2}}\right] = \ln L$$

$$\Rightarrow \lim_{x\to\infty}\left[\frac{-a}{x^2+ax}\times\frac{-bx^2}{1}\right] = \ln L$$

$$\Rightarrow \lim_{x\to\infty}\left[\frac{abx^2}{x^2+ax}\right] = \ln L$$



$$\Rightarrow \lim_{x\to\infty}\left[\frac{ab}{1+\frac{a}{x}}\right] = \ln L$$

$$\Rightarrow ab = \ln L$$

FINALLY INVERTING THE LOGARITHM

$$\Rightarrow L = e^{ab}$$

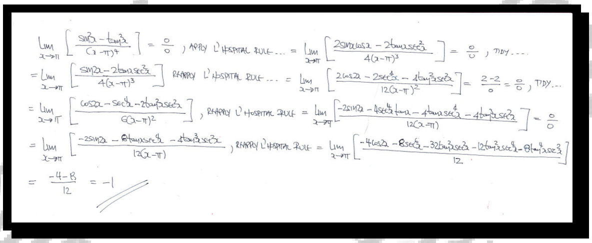
$$\therefore \lim_{x\to\infty}\left[\left(1+\frac{a}{x}\right)^{bx}\right] = e^{ab}$$

Question 13 **(*****)**

Find the value of the following limit

$$\lim_{x\to\pi}\left[\frac{\sin^2 x-\tan^2 x}{(x-\pi)^4}\right].$$

$\boxed{-1}$



$\lim_{x\to\pi}\left[\frac{\sin^2x-\tan^2x}{(x-\pi)^4}\right]=\frac{0}{0}$, APPLY L'HOSPITAL RULE ... $=\lim_{x\to\pi}\left[\frac{2\sin x\cos x-2\tan x\sec^2x}{4(x-\pi)^3}\right]=\frac{0}{0}$, TIDY ...

$=\lim_{x\to\pi}\left[\frac{\sin 2x-2\tan x\sec^2x}{4(x-\pi)^3}\right]$ REAPPLY L'HOSPITAL RULE ... $=\lim_{x\to\pi}\left[\frac{2\cos 2x-2\sec^4x-4\tan^2x\sec^2x}{12(x-\pi)^2}\right]=\frac{2-2}{0}=\frac{0}{0}$, TIDY ...

$=\lim_{x\to\pi}\left[\frac{\cos 2x-\sec^4x-2\tan^2x\sec^2x}{6(x-\pi)^2}\right]$, REAPPLY L'HOSPITAL RULE $=\lim_{x\to\pi}\left[\frac{-2\sin 2x-4\sec^4x\tan x-4\tan x\sec^4x-4\tan^3x\sec^2x}{12(x-\pi)}\right]=\frac{0}{0}$

$=\lim_{x\to\pi}\left[\frac{-2\sin 2x-8\tan x\sec^4x-4\tan^3x\sec^2x}{12(x-\pi)}\right]$, REAPPLY L'HOSPITAL RULE $=\lim_{x\to\pi}\left[\frac{-4\cos 2x-8\sec^6x-32\tan^2x\sec^4x-12\tan^2x\sec^4x-8\tan^4x\sec^2x}{12}\right]$

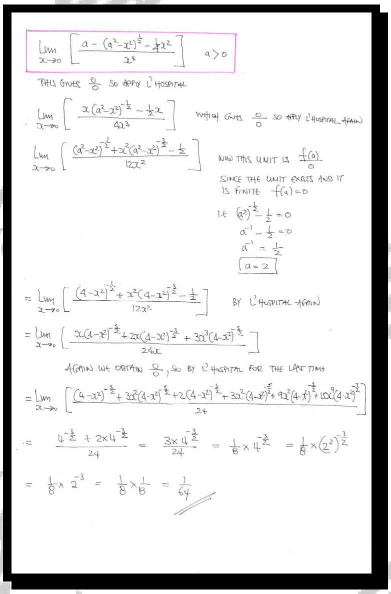
$=\frac{-4-8}{12}=-1$

Question 14 **(*****)**

$$L=\lim_{x\to 0}\left[\frac{a-\sqrt{a^2-x^2}-\frac{1}{4}x^2}{x^4}\right],\ a>0.$$

Given that L is finite, determine its value.

$\boxed{\frac{1}{64}}$



$\boxed{\lim_{x\to 0}\left[\frac{a-(a^2-x^2)^{\frac{1}{2}}-\frac{1}{4}x^2}{x^4}\right]\quad a>0}$

THIS GIVES $\frac{0}{0}$ SO APPLY L'HOSPITAL

$\lim_{x\to 0}\left[\frac{x(a^2-x^2)^{-\frac{1}{2}}-\frac{1}{2}x}{4x^3}\right]$ WHICH GIVES $\frac{0}{0}$ SO APPLY L'HOSPITAL AGAIN

$\lim_{x\to 0}\left[\frac{(a^2-x^2)^{-\frac{1}{2}}+x^2(a^2-x^2)^{-\frac{3}{2}}-\frac{1}{2}}{12x^2}\right]$

NOW THIS LIMIT IS $\frac{f(a)}{0}$

SINCE THE LIMIT EXISTS AND IT IS FINITE $f(a)=0$

I.E $(a^2)^{-\frac{1}{2}}-\frac{1}{2}=0$

$a^{-1}-\frac{1}{2}=0$

$a^{-1}=\frac{1}{2}$

$\boxed{a=2}$

$=\lim_{x\to 0}\left[\frac{(4-x^2)^{-\frac{1}{2}}+x^2(4-x^2)^{-\frac{3}{2}}-\frac{1}{2}}{12x^2}\right]$ BY L'HOSPITAL AGAIN

$=\lim_{x\to 0}\left[\frac{x(4-x^2)^{-\frac{3}{2}}+2x(4-x^2)^{-\frac{3}{2}}+3x^3(4-x^2)^{-\frac{5}{2}}}{24x}\right]$

AGAIN WE OBTAIN $\frac{0}{0}$, SO BY L'HOSPITAL FOR THE LAST TIME

$=\lim_{x\to 0}\left[\frac{(4-x^2)^{-\frac{3}{2}}+3x^2(4-x^2)^{-\frac{5}{2}}+2(4-x^2)^{-\frac{3}{2}}+3x^2(4-x^2)^{-\frac{5}{2}}+9x^2(4-x^2)^{-\frac{5}{2}}+15x^4(4-x^2)^{-\frac{7}{2}}}{24}\right]$

$=\frac{4^{-\frac{3}{2}}+2\times 4^{-\frac{3}{2}}}{24}=\frac{3\times 4^{-\frac{3}{2}}}{24}=\frac{1}{8}\times 4^{-\frac{3}{2}}=\frac{1}{8}\times(2^2)^{-\frac{3}{2}}$

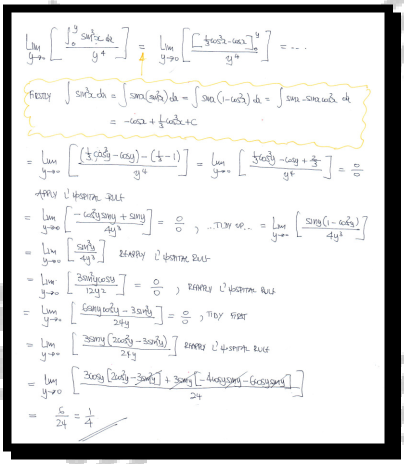
$=\frac{1}{8}\times 2^{-3}=\frac{1}{8}\times\frac{1}{8}=\frac{1}{64}$

Question 15 (*****)

Find the value of the following limit

$$\lim_{y\to 0}\left[\frac{1}{y^4}\int_0^y \sin^3 x \ dx\right].$$

$$\boxed{\frac{1}{4}}$$



$$\lim_{y\to 0}\left[\frac{\int_0^y \sin^3 x\,dx}{y^4}\right] = \lim_{y\to 0}\left[\frac{\left[\frac{1}{3}\cos^3 x-\cos x\right]_0^y}{y^4}\right] = \dots$$

FIRSTLY $\int \sin^3 x\,dx = \int \sin x(\sin^2 x)\,dx = \int \sin x(1-\cos^2 x)\,dx = \int \sin x - \sin x\cos^2 x\,dx$

$= -\cos x + \frac{1}{3}\cos^3 x + C$

$= \lim_{y\to 0}\left[\frac{\left(\frac{1}{3}\cos^3 y-\cos y\right)-\left(\frac{1}{3}-1\right)}{y^4}\right] = \lim_{y\to 0}\left[\frac{\frac{1}{3}\cos^3 y-\cos y+\frac{2}{3}}{y^4}\right] = \frac{0}{0}$

APPLY L'HOSPITAL RULE

$= \lim_{y\to 0}\left[\frac{-\cos^2 y\sin y+\sin y}{4y^3}\right] = \frac{0}{0}$, ...TIDY UP... $= \lim_{y\to 0}\left[\frac{\sin y(1-\cos^2 y)}{4y^3}\right]$

$= \lim_{y\to 0}\left[\frac{\sin^3 y}{4y^3}\right]$ REAPPLY L'HOSPITAL RULE

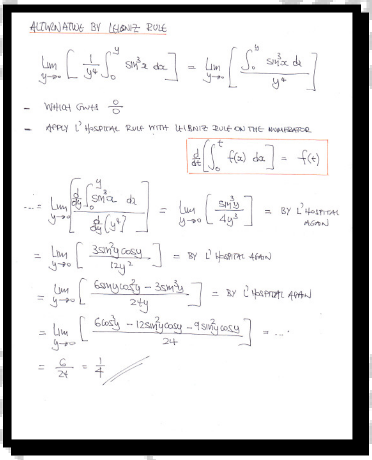
$= \lim_{y\to 0}\left[\frac{3\sin^2 y\cos y}{12y^2}\right] = \frac{0}{0}$, REAPPLY L'HOSPITAL RULE

$= \lim_{y\to 0}\left[\frac{6\sin y\cos^2 y-3\sin^3 y}{24y}\right] = \frac{0}{0}$, TIDY FIRST

$= \lim_{y\to 0}\left[\frac{3\sin y(2\cos^2 y-3\sin^2 y)}{24y}\right]$ REAPPLY L'HOSPITAL RULE

$= \lim_{y\to 0}\left[\frac{3\cos y\left[2\cos^2 y-\cancel{3\sin^2 y}\right]+\cancel{3\sin y\left[-4\cos y\sin y-6\cos y\sin y\right]}}{24}\right]$

$= \frac{6}{24} = \frac{1}{4}$



ALTERNATIVE BY LEIBNIZ RULE

$\lim_{y\to 0}\left[\frac{1}{y^4}\int_0^y \sin^3 x\,dx\right] = \lim_{y\to 0}\left[\frac{\int_0^y \sin^3 x\,dx}{y^4}\right]$

- WHICH GIVES $\frac{0}{0}$
- APPLY L'HOSPITAL RULE WITH LEIBNIZ RULE ON THE NUMERATOR

$\boxed{\frac{d}{dt}\left[\int_0^t f(x)\,dx\right] = f(t)}$

$\dots = \lim_{y\to 0}\left[\frac{\frac{d}{dy}\int_0^y \sin^3 x\,dx}{\frac{d}{dy}(y^4)}\right] = \lim_{y\to 0}\left[\frac{\sin^3 y}{4y^3}\right]$ = BY L'HOSPITAL AGAIN

$= \lim_{y\to 0}\left[\frac{3\sin^2 y\cos y}{12y^2}\right]$ = BY L'HOSPITAL AGAIN

$= \lim_{y\to 0}\left[\frac{6\sin y\cos^2 y-3\sin^3 y}{24y}\right]$ = BY L'HOSPITAL AGAIN

$= \lim_{y\to 0}\left[\frac{6\cos^3 y-12\sin^2 y\cos y-9\sin^2 y\cos y}{24}\right] = \dots$

$= \frac{6}{24} = \frac{1}{4}$

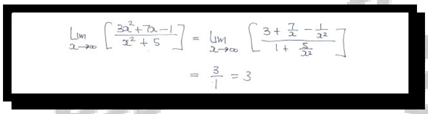
VARIOUS LIMITS

Question 1 (**)

Find the value of the following limit

$$\lim_{x\to\infty}\left[\frac{3x^2+7x-1}{x^2+5}\right].$$

$\boxed{3}$



$$\lim_{x\to\infty}\left[\frac{3x^2+7x-1}{x^2+5}\right]=\lim_{x\to\infty}\left[\frac{3+\frac{7}{x}-\frac{1}{x^2}}{1+\frac{5}{x^2}}\right]$$

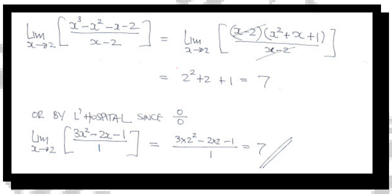
$$=\frac{3}{1}=3$$

Question 2 (**)

Find the value of the following limit

$$\lim_{x\to 2}\left[\frac{x^3-x^2-x-2}{x-2}\right].$$

$\boxed{7}$



$$\lim_{x\to 2}\left[\frac{x^3-x^2-x-2}{x-2}\right]=\lim_{x\to 2}\left[\frac{\cancel{(x-2)}(x^2+x+1)}{\cancel{x-2}}\right]$$

$$=2^2+2+1=7$$

OR BY L'HOSPITAL SINCE $\frac{0}{0}$

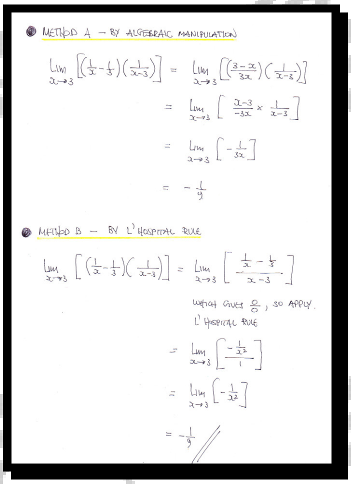
$$\lim_{x\to 2}\left[\frac{3x^2-2x-1}{1}\right]=\frac{3\times2^2-2\times2-1}{1}=7 \,/\!/$$

Question 3 (+)**

Find the value of the following limit

$$\lim_{x\to 3}\left[\left(\frac{1}{x}-\frac{1}{3}\right)\left(\frac{1}{x-3}\right)\right].$$

$$\boxed{-\frac{1}{9}}$$



METHOD A – BY ALGEBRAIC MANIPULATION

$$\lim_{x\to 3}\left[\left(\frac{1}{x}-\frac{1}{3}\right)\left(\frac{1}{x-3}\right)\right] = \lim_{x\to 3}\left[\left(\frac{3-x}{3x}\right)\left(\frac{1}{x-3}\right)\right]$$

$$= \lim_{x\to 3}\left[\frac{x-3}{-3x}\times\frac{1}{x-3}\right]$$

$$= \lim_{x\to 3}\left[-\frac{1}{3x}\right]$$

$$= -\frac{1}{9}$$

METHOD B – BY L'HOSPITAL RULE

$$\lim_{x\to 3}\left[\left(\frac{1}{x}-\frac{1}{3}\right)\left(\frac{1}{x-3}\right)\right] = \lim_{x\to 3}\left[\frac{\frac{1}{x}-\frac{1}{3}}{x-3}\right]$$

WHICH GIVES $\frac{0}{0}$, SO APPLY L'HOSPITAL RULE

$$= \lim_{x\to 3}\left[\frac{-\frac{1}{x^2}}{1}\right]$$

$$= \lim_{x\to 3}\left[-\frac{1}{x^2}\right]$$

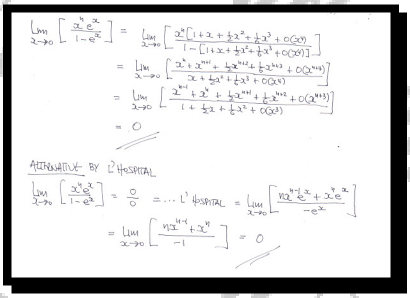
$$= -\frac{1}{9}$$

Question 4 (+)**

Given that n is a positive integer determine

$$\lim_{x\to 0}\left[\frac{x^n\,\mathrm{e}^x}{1-\mathrm{e}^x}\right].$$

$$\boxed{0}$$



$$\lim_{x\to 0}\left[\frac{x^n e^x}{1-e^x}\right] = \lim_{x\to 0}\left[\frac{x^n[1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+O(x^4)}{1-[1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+O(x^4)]}\right]$$

$$= \lim_{x\to 0}\left[\frac{x^n+x^{n+1}+\frac{1}{2}x^{n+2}+\frac{1}{6}x^{n+3}+O(x^{n+4})}{x+\frac{1}{2}x^2+\frac{1}{6}x^3+O(x^4)}\right]$$

$$= \lim_{x\to 0}\left[\frac{x^{n-1}+x^n+\frac{1}{2}x^{n+1}+\frac{1}{6}x^{n+2}+O(x^{n+3})}{1+\frac{1}{2}x+\frac{1}{6}x^2+O(x^3)}\right]$$

$$= 0$$

ALTERNATIVE BY L'HOSPITAL

$$\lim_{x\to 0}\left[\frac{x^n e^x}{1-e^x}\right] = \frac{0}{0} = \dots \text{L'HOSPITAL} = \lim_{x\to 0}\left[\frac{nx^{n-1}e^x+x^n e^x}{-e^x}\right]$$

$$= \lim_{x\to 0}\left[\frac{nx^{n-1}+x^n}{-1}\right] = 0$$

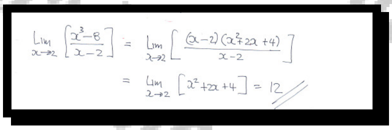
Question 5 **(***)**

Find the value of the following limit

$$\lim_{x \to 2} \left[\frac{x^3 - 8}{x - 2} \right].$$

You may not use the L' Hospital's rule in this question.

$\boxed{12}$



$$\lim_{x \to 2} \left[\frac{x^3-8}{x-2} \right] = \lim_{x \to 2} \left[\frac{(x-2)(x^2+2x+4)}{x-2} \right]$$

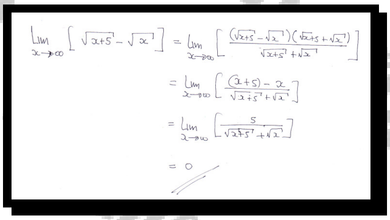
$$= \lim_{x \to 2} \left[x^2+2x+4 \right] = 12$$

Question 6 **(***)**

Find the value of the following limit.

$$\lim_{x \to \infty} \left[\sqrt{x+5} - \sqrt{x} \right].$$

$\boxed{0}$



$$\lim_{x \to \infty} \left[\sqrt{x+5} - \sqrt{x} \right] = \lim_{x \to \infty} \left[\frac{(\sqrt{x+5}-\sqrt{x})(\sqrt{x+5}+\sqrt{x})}{\sqrt{x+5}+\sqrt{x}} \right]$$

$$= \lim_{x \to \infty} \left[\frac{(x+5)-x}{\sqrt{x+5}+\sqrt{x}} \right]$$

$$= \lim_{x \to \infty} \left[\frac{5}{\sqrt{x+5}+\sqrt{x}} \right]$$

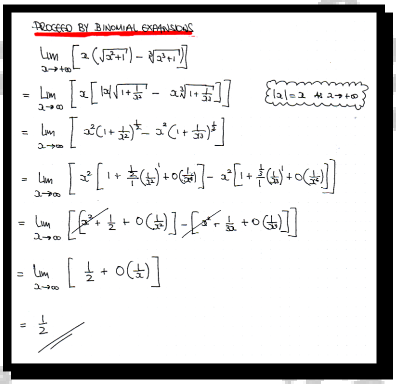
$$= 0$$

Question 7 (***)

Find the value of the following limit.

$$\lim_{x\to\infty}\left[x\sqrt{x^2+1}-\sqrt[3]{x^3+1}\right].$$

[illegible], $\boxed{\frac{1}{2}}$



PROCEED BY BINOMIAL EXPANSIONS

$$\lim_{x\to+\infty}\left[x\left(\sqrt{x^2+1}\right)-\sqrt[3]{x^3+1}\right]$$

$$=\lim_{x\to\infty}\left[x\left[|x|\sqrt{1+\frac{1}{x^2}}-x\sqrt[3]{1+\frac{1}{x^3}}\right]\right] \qquad \left\{|x|=x \text{ AS } x\to+\infty\right\}$$

$$=\lim_{x\to\infty}\left[x^2\left(1+\frac{1}{x^2}\right)^{\frac{1}{2}}-x^2\left(1+\frac{1}{x^3}\right)^{\frac{1}{3}}\right]$$

$$=\lim_{x\to\infty}\left[x^2\left[1+\frac{\frac{1}{2}}{1}\left(\frac{1}{x^2}\right)^1+O\left(\frac{1}{x^4}\right)\right]-x^2\left[1+\frac{\frac{1}{3}}{1}\left(\frac{1}{x^3}\right)^1+O\left(\frac{1}{x^6}\right)\right]\right]$$

$$=\lim_{x\to\infty}\left[\left[\cancel{x^2}+\frac{1}{2}+O\left(\frac{1}{x^2}\right)\right]-\left[\cancel{x^2}+\frac{1}{3x}+O\left(\frac{1}{x^4}\right)\right]\right]$$

$$=\lim_{x\to\infty}\left[\frac{1}{2}+O\left(\frac{1}{x}\right)\right]$$

$$=\frac{1}{2}$$

Question 8 **(***)**

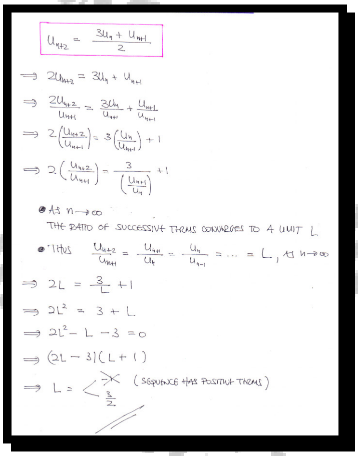
The Fibonacci sequence is given by the recurrence formula

$$u_{n+2} = u_{n+1} + u_n, \quad u_1 = 1,\ u_2 = 1.$$

It is further given that in this sequence **the ratio of consecutive terms** converges to a limit ϕ, known as the *Golden Ratio*.

Show, by using the above recurrence formula, that $\phi = \frac{1}{2}\left(1+\sqrt{5}\right)$.

$\boxed{}$, $\boxed{\text{proof}}$



$$\boxed{u_{n+2} = \frac{3u_n + u_{n+1}}{2}}$$

$\Rightarrow 2u_{n+2} = 3u_n + u_{n+1}$

$\Rightarrow \frac{2u_{n+2}}{u_{n+1}} = \frac{3u_n}{u_{n+1}} + \frac{u_{n+1}}{u_{n+1}}$

$\Rightarrow 2\left(\frac{u_{n+2}}{u_{n+1}}\right) = 3\left(\frac{u_n}{u_{n+1}}\right) + 1$

$\Rightarrow 2\left(\frac{u_{n+2}}{u_{n+1}}\right) = \frac{3}{\left(\frac{u_{n+1}}{u_n}\right)} + 1$

- AS $n \to \infty$
THE RATIO OF SUCCESSIVE TERMS CONVERGES TO A LIMIT L
- THUS $\frac{u_{n+2}}{u_{n+1}} = \frac{u_{n+1}}{u_n} = \frac{u_n}{u_{n-1}} = \dots = L$, AS $n \to \infty$

$\Rightarrow 2L = \frac{3}{L} + 1$

$\Rightarrow 2L^2 = 3 + L$

$\Rightarrow 2L^2 - L - 3 = 0$

$\Rightarrow (2L-3)(L+1)$

$\Rightarrow L = \left\langle \begin{array}{l} \cancel{-1} \\ \frac{3}{2} \end{array} \right.$ (SEQUENCE HAS POSITIVE TERMS)

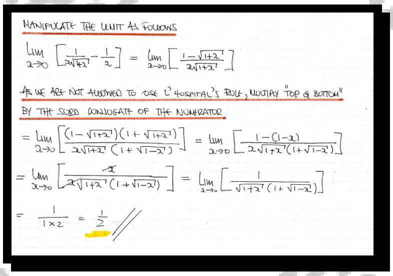
Question 9 (*+) Limits**

Evaluate the following limit.

$$\lim_{x \to 0}\left[\frac{1}{x\sqrt{1+x}} - \frac{1}{x}\right].$$

You may NOT use L'Hospital's rule in this question

$\boxed{\quad\quad}$, $\boxed{x = \frac{1}{2}}$



MANIPULATE THE LIMIT AS FOLLOWS

$$\lim_{x \to 0}\left[\frac{1}{x\sqrt{1+x}} - \frac{1}{x}\right] = \lim_{x \to 0}\left[\frac{1-\sqrt{1+x}}{x\sqrt{1+x}}\right]$$

AS WE ARE NOT ALLOWED TO USE L'HOSPITAL'S RULE, MULTIPLY "TOP & BOTTOM" BY THE SURD CONJUGATE OF THE NUMERATOR

$$= \lim_{x \to 0}\left[\frac{(1-\sqrt{1+x})(1+\sqrt{1+x})}{x\sqrt{1+x}\,(1+\sqrt{1-x})}\right] = \lim_{x \to 0}\left[\frac{1-(1-x)}{x\sqrt{1+x}(1+\sqrt{1-x})}\right]$$

$$= \lim_{x \to 0}\left[\frac{\cancel{x}}{\cancel{x}\sqrt{1+x}(1+\sqrt{1-x})}\right] = \lim_{x \to 0}\left[\frac{1}{\sqrt{1+x}(1+\sqrt{1-x})}\right]$$

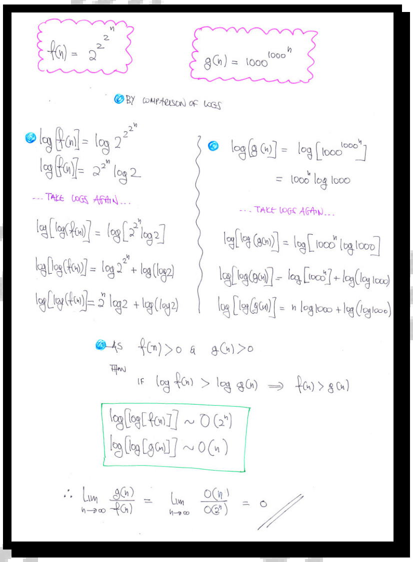
$$= \frac{1}{1 \times 2} = \frac{1}{2} \,//$$

Question 10 **(***+)**

$$f(n) = 2^{2^{2^{n}}} \ , \ n \in \mathbb{R} \qquad \text{and} \qquad f(n) = 1000^{1000^{n}} \ , \ n \in \mathbb{R} \ .$$

Determine whether or not $\lim_{n \to \infty} \left[\frac{g(n)}{f(n)} \right]$ exists.

$$\boxed{\lim_{n \to \infty} \left[\frac{g(n)}{f(n)} \right] = 0}$$



$f(n) = 2^{2^{2^{n}}}$

$g(n) = 1000^{1000^{n}}$

BY COMPARISON OF LOGS

$\log[f(n)] = \log 2^{2^{2^{n}}}$

$\log[f(n)] = 2^{2^{n}} \log 2$

... TAKE LOGS AGAIN ...

$\log[\log(f(n))] = \log[2^{2^{n}} \log 2]$

$\log[\log(f(n))] = \log 2^{2^{n}} + \log(\log 2)$

$\log[\log(f(n))] = 2^{n} \log 2 + \log(\log 2)$

$\log[g(n)] = \log[1000^{1000^{n}}]$

$= 1000^{n} \log 1000$

... TAKE LOGS AGAIN ...

$\log[\log(g(n))] = \log[1000^{n} \log 1000]$

$\log[\log(g(n))] = \log[1000^{n}] + \log(\log 1000)$

$\log[\log(g(n))] = n \log 1000 + \log(\log 1000)$

AS $f(n) > 0$ & $g(n) > 0$

THEN

IF $\log f(n) > \log g(n) \Rightarrow f(n) > g(n)$

$$\boxed{\begin{array}{l} \log[\log[f(n)]] \sim O(2^{n}) \\ \log[\log[g(n)]] \sim O(n) \end{array}}$$

$\therefore \lim_{n \to \infty} \frac{g(n)}{f(n)} = \lim_{n \to \infty} \frac{O(n)}{O(2^{n})} = 0$ //

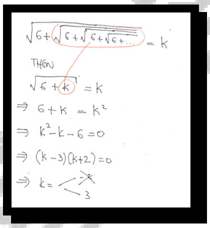
Question 11 **(***+)**

Show clearly without the use of any calculating aid that

$$\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\ldots}}}}=k,$$

where k is an integer to be found.

$\boxed{k=3}$



$\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\ldots}}}}=k$

THEN

$\sqrt{6+k}=k$

$\Rightarrow 6+k=k^2$

$\Rightarrow k^2-k-6=0$

$\Rightarrow (k-3)(k+2)=0$

$\Rightarrow k=\cancel{-2}$, 3

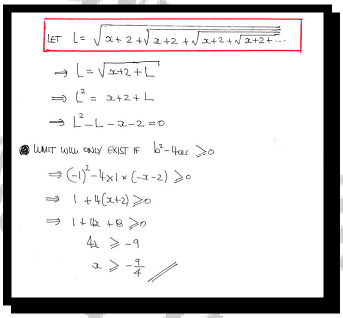
Question 12 **(***+)**

$$\sqrt{x+2+\sqrt{x+2+\sqrt{x+2+\sqrt{x+2+\sqrt{x+2+\ldots}}}}},$$

It is given that the above nested radical converges to a limit L, $L\in\mathbb{R}$.

Determine the range of possible values of x.

$\boxed{}$, $\boxed{x\ge-\frac{9}{4}}$



LET $L=\sqrt{x+2+\sqrt{x+2+\sqrt{x+2+\sqrt{x+2+\ldots}}}}$

$\Rightarrow L=\sqrt{x+2+L}$

$\Rightarrow L^2=x+2+L$

$\Rightarrow L^2-L-x-2=0$

● LIMIT WILL ONLY EXIST IF $b^2-4ac\geqslant0$

$\Rightarrow (-1)^2-4\times1\times(-x-2)\geqslant0$

$\Rightarrow 1+4(x+2)\geqslant0$

$\Rightarrow 1+4x+8\geqslant0$

$4x\geqslant-9$

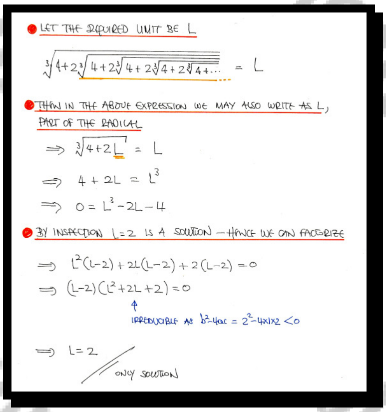
$x\geqslant-\frac{9}{4}$ //

Question 13 (***+)

$$\sqrt[3]{4+2\sqrt[3]{4+2\sqrt[3]{4+2\sqrt[3]{4+2\sqrt[3]{4+\ldots}}}}}$$

Given that the above nested radical converges, determine its limit.

$\boxed{\text{I.Y.G.B.}}$, $\boxed{L=2}$



● LET THE REQUIRED LIMIT BE L

$$\sqrt[3]{4+2\underbrace{\sqrt[3]{4+2\sqrt[3]{4+2\sqrt[3]{4+2\sqrt[3]{4+\ldots}}}}}} = L$$

● THEN IN THE ABOVE EXPRESSION WE MAY ALSO WRITE AS L, PART OF THE RADICAL

$$\Rightarrow \sqrt[3]{4+2\underline{L}} = L$$

$$\Rightarrow 4+2L = L^3$$

$$\Rightarrow 0 = L^3-2L-4$$

● BY INSPECTION $L=2$ IS A SOLUTION — HENCE WE CAN FACTORIZE

$$\Rightarrow L^2(L-2)+2L(L-2)+2(L-2)=0$$

$$\Rightarrow (L-2)(L^2+2L+2)=0$$

IRREDUCIBLE AS $b^2-4ac = 2^2-4\times1\times2<0$

$$\Rightarrow L=2$$

ONLY SOLUTION

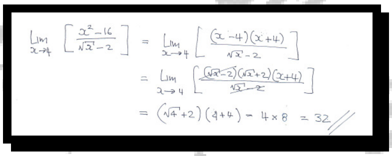
Question 14 (****)

Find the value of the following limit

$$\lim_{x\to4}\left[\frac{x^2-16}{\sqrt{x}-2}\right].$$

You may not use the L' Hospital's rule in this question.

$\boxed{32}$



$$\lim_{x\to4}\left[\frac{x^2-16}{\sqrt{x}-2}\right] = \lim_{x\to4}\left[\frac{(x-4)(x+4)}{\sqrt{x}-2}\right]$$

$$= \lim_{x\to4}\left[\frac{\cancel{(\sqrt{x}-2)}(\sqrt{x}+2)(x+4)}{\cancel{\sqrt{x}-2}}\right]$$

$$= (\sqrt{4}+2)(4+4) = 4\times8 = 32$$

Question 15 **(****)**

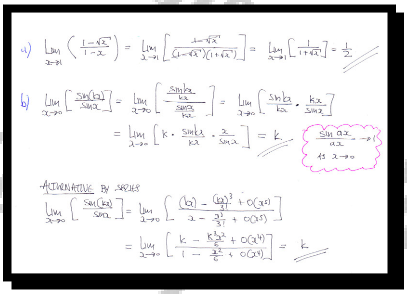
Find the value of each of the following limits.

a) $\lim_{x\to 1}\left[\frac{1-\sqrt{x}}{1-x}\right]$.

b) $\lim_{x\to 0}\left[\frac{\sin(kx)}{\sin x}\right]$.

You may not use the L' Hospital's rule in this question.

$\boxed{32}$



a) $\lim_{x\to 1}\left(\frac{1-\sqrt{x}}{1-x}\right) = \lim_{x\to 1}\left[\frac{\cancel{1-\sqrt{x}}}{\cancel{(1-\sqrt{x})}(1+\sqrt{x})}\right] = \lim_{x\to 1}\left[\frac{1}{1+\sqrt{x}}\right] = \frac{1}{2}$

b) $\lim_{x\to 0}\left[\frac{\sin(kx)}{\sin x}\right] = \lim_{x\to 0}\left[\frac{\frac{\sin kx}{kx}}{\frac{\sin x}{kx}}\right] = \lim_{x\to 0}\left[\frac{\sin kx}{kx}\cdot\frac{kx}{\sin x}\right]$

$= \lim_{x\to 0}\left[k\cdot\frac{\sin kx}{kx}\cdot\frac{x}{\sin x}\right] = k$

$\frac{\sin ax}{ax} \to 1$ as $x\to 0$

ALTERNATIVE BY SERIES

$\lim_{x\to 0}\left[\frac{\sin(kx)}{\sin x}\right] = \lim_{x\to 0}\left[\frac{(kx)-\frac{(kx)^3}{3!}+O(x^5)}{x-\frac{x^3}{3!}+O(x^5)}\right]$

$= \lim_{x\to 0}\left[\frac{k-\frac{k^3x^2}{6}+O(x^4)}{1-\frac{x^2}{6}+O(x^4)}\right] = k$

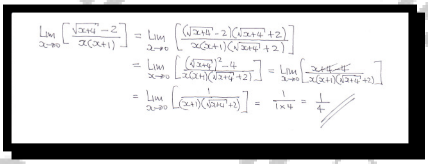
Question 16 **(****)**

Find the value of the following limit

$$\lim_{x\to 0}\left[\frac{\sqrt{x+4}-2}{x(x+1)}\right].$$

You may not use the L' Hospital's rule in this question.

$\boxed{\frac{1}{4}}$



$\lim_{x\to 0}\left[\frac{\sqrt{x+4}-2}{x(x+1)}\right] = \lim_{x\to 0}\left[\frac{(\sqrt{x+4}-2)(\sqrt{x+4}+2)}{x(x+1)(\sqrt{x+4}+2)}\right]$

$= \lim_{x\to 0}\left[\frac{(\sqrt{x+4})^2-4}{x(x+1)(\sqrt{x+4}+2)}\right] = \lim_{x\to 0}\left[\frac{x\cancel{+4}\cancel{-4}}{x(x+1)(\sqrt{x+4}+2)}\right]$

$= \lim_{x\to 0}\left[\frac{1}{(x+1)(\sqrt{x+4}+2)}\right] = \frac{1}{1\times 4} = \frac{1}{4}$

Question 17 **(****)**

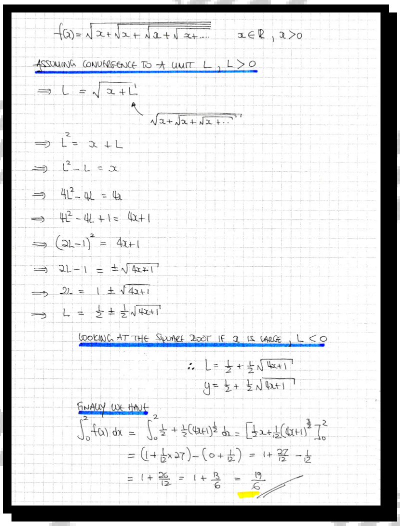
The function f is defined as

$$f(x) \equiv \sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+\ldots}}}}}\ , \quad x \in (0,\infty).$$

Determine the value of

$$\int_0^2 f(x)\ dx\ .$$

$\boxed{}$, $\boxed{\frac{19}{6}}$



$f(x) = \sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+\ldots}}}}$ $\quad x \in \mathbb{R},\ x>0$

ASSUMING CONVERGENCE TO A LIMIT L, $L>0$

$\Rightarrow L = \sqrt{x+L}$

$\sqrt{x+\sqrt{x+\sqrt{x+\ldots}}}$

$\Rightarrow L^2 = x + L$

$\Rightarrow L^2 - L = x$

$\Rightarrow 4L^2 - 4L = 4x$

$\Rightarrow 4L^2 - 4L + 1 = 4x+1$

$\Rightarrow (2L-1)^2 = 4x+1$

$\Rightarrow 2L-1 = \pm\sqrt{4x+1}$

$\Rightarrow 2L = 1 \pm \sqrt{4x+1}$

$\Rightarrow L = \frac{1}{2} \pm \frac{1}{2}\sqrt{4x+1}$

LOOKING AT THE SQUARE ROOT IF x IS LARGE, $L<0$

$\therefore\ L = \frac{1}{2} + \frac{1}{2}\sqrt{4x+1}$

$y = \frac{1}{2} + \frac{1}{2}\sqrt{4x+1}$

FINALLY WE HAVE

$\int_0^2 f(x)\,dx = \int_0^2 \frac{1}{2} + \frac{1}{2}(4x+1)^{\frac{1}{2}}\,dx = \left[\frac{1}{2}x + \frac{1}{12}(4x+1)^{\frac{3}{2}}\right]_0^2$

$= \left(1+\frac{1}{12}\times 27\right) - \left(0+\frac{1}{12}\right) = 1+\frac{27}{12} - \frac{1}{12}$

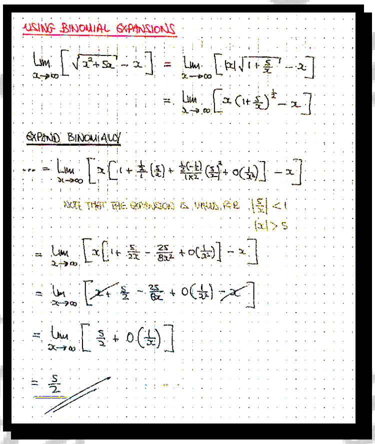
$= 1+\frac{26}{12} = 1+\frac{13}{6} = \frac{19}{6}$

Question 18 (****+)

Find the value of the following limit

$$\lim_{x\to\infty}\left[\sqrt{x^2+5x}-x\right].$$

$\boxed{}$, $\boxed{\frac{5}{2}}$



USING BINOMIAL EXPANSIONS

$$\lim_{x\to\infty}\left[\sqrt{x^2+5x}-x\right]=\lim_{x\to\infty}\left[|x|\sqrt{1+\frac{5}{x}}-x\right]$$

$$=\lim_{x\to\infty}\left[x\left(1+\frac{5}{x}\right)^{\frac{1}{2}}-x\right]$$

EXPAND BINOMIALLY

$$\ldots=\lim_{x\to\infty}\left[x\left[1+\frac{\frac{1}{2}}{1}\left(\frac{5}{x}\right)+\frac{\frac{1}{2}(-\frac{1}{2})}{1\times2}\left(\frac{5}{x}\right)^2+O\left(\frac{1}{x^3}\right)\right]-x\right]$$

NOTE THAT THE EXPANSION IS VALID FOR $\left|\frac{5}{x}\right|<1$

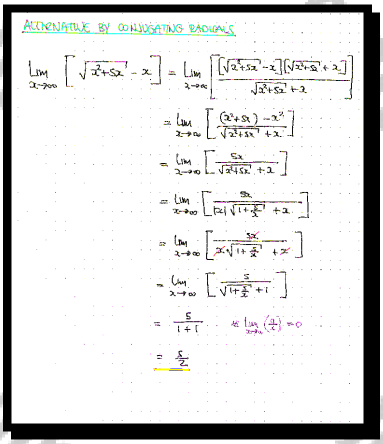
$|x|>5$

$$=\lim_{x\to\infty}\left[x\left[1+\frac{5}{2x}-\frac{25}{8x^2}+O\left(\frac{1}{x^3}\right)\right]-x\right]$$

$$=\lim_{x\to\infty}\left[\cancel{x}+\frac{5}{2}-\frac{25}{8x}+O\left(\frac{1}{x^2}\right)-\cancel{x}\right]$$

$$=\lim_{x\to\infty}\left[\frac{5}{2}+O\left(\frac{1}{x}\right)\right]$$

$$=\frac{5}{2}$$



ALTERNATIVE BY CONJUGATING RADICALS

$$\lim_{x\to\infty}\left[\sqrt{x^2+5x}-x\right]=\lim_{x\to\infty}\left[\frac{\left[\sqrt{x^2+5x}-x\right]\left[\sqrt{x^2+5x}+x\right]}{\sqrt{x^2+5x}+x}\right]$$

$$=\lim_{x\to\infty}\left[\frac{(x^2+5x)-x^2}{\sqrt{x^2+5x}+x}\right]$$

$$=\lim_{x\to\infty}\left[\frac{5x}{\sqrt{x^2+5x}+x}\right]$$

$$=\lim_{x\to\infty}\left[\frac{5x}{|x|\sqrt{1+\frac{5}{x}}+x}\right]$$

$$=\lim_{x\to\infty}\left[\frac{5\cancel{x}}{\cancel{x}\sqrt{1+\frac{5}{x}}+\cancel{x}}\right]$$

$$=\lim_{x\to\infty}\left[\frac{5}{\sqrt{1+\frac{5}{x}}+1}\right]$$

$$=\frac{5}{1+1}\qquad \text{as } \lim_{x\to\infty}\left(\frac{a}{x}\right)=0$$

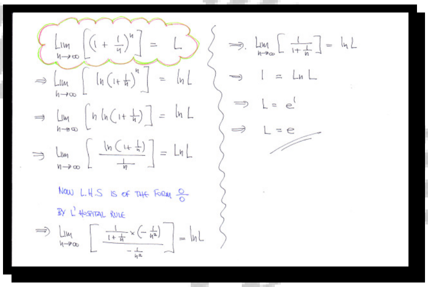
$$=\frac{5}{2}$$

Question 19 (****+)

Find the value of the following limit

$$\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\right].$$

$\boxed{e}$



$$\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\right]=L$$

$$\Rightarrow\lim_{n\to\infty}\left[\ln\left(1+\frac{1}{n}\right)^n\right]=\ln L$$

$$\Rightarrow\lim_{n\to\infty}\left[n\ln\left(1+\frac{1}{n}\right)\right]=\ln L$$

$$\Rightarrow\lim_{n\to\infty}\left[\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}\right]=\ln L$$

NOW L.H.S IS OF THE FORM $\frac{0}{0}$

BY L'HOSPITAL RULE

$$\Rightarrow\lim_{n\to\infty}\left[\frac{\frac{1}{1+\frac{1}{n}}\times\left(-\frac{1}{n^2}\right)}{-\frac{1}{n^2}}\right]=\ln L$$

$$\Rightarrow\lim_{n\to\infty}\left[\frac{1}{1+\frac{1}{n}}\right]=\ln L$$

$$\Rightarrow 1=\ln L$$

$$\Rightarrow L=e^1$$

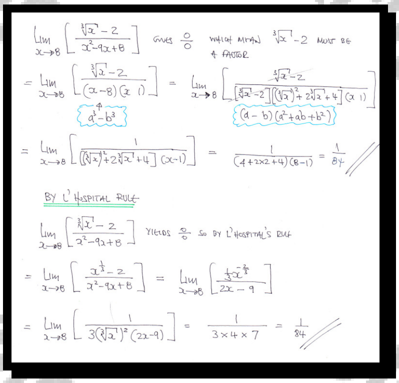
$$\Rightarrow L=e$$

Question 20 **(****+)**

Use two distinct methods to evaluate the following limit

$$\lim_{x \to 8}\left[\frac{\sqrt[3]{x}-2}{x^2-9x+8}\right].$$

$$\boxed{\frac{1}{84}}$$



$\lim_{x\to 8}\left[\frac{\sqrt[3]{x}-2}{x^2-9x+8}\right]$ GIVES $\frac{0}{0}$ WHICH MEAN $\sqrt[3]{x}-2$ MUST BE A FACTOR

$= \lim_{x\to 8}\left[\frac{\sqrt[3]{x}-2}{(x-8)(x-1)}\right] = \lim_{x\to 8}\left[\frac{\cancel{\sqrt[3]{x}-2}}{\cancel{[\sqrt[3]{x}-2]}[(\sqrt[3]{x})^2+2\sqrt[3]{x}+4](x-1)}\right]$

a^3-b^3 $\quad (a-b)(a^2+ab+b^2)$

$= \lim_{x\to 8}\left[\frac{1}{[(\sqrt[3]{x})^2+2\sqrt[3]{x}+4](x-1)}\right] = \frac{1}{(4+2\times2+4)(8-1)} = \frac{1}{84}$

BY L'HOSPITAL RULE

$\lim_{x\to 8}\left[\frac{\sqrt[3]{x}-2}{x^2-9x+8}\right]$ YIELDS $\frac{0}{0}$ SO BY L'HOSPITAL'S RULE

$= \lim_{x\to 8}\left[\frac{x^{\frac{1}{3}}-2}{x^2-9x+8}\right] = \lim_{x\to 8}\left[\frac{\frac{1}{3}x^{-\frac{2}{3}}}{2x-9}\right]$

$= \lim_{x\to 8}\left[\frac{1}{3(\sqrt[3]{x})^2(2x-9)}\right] = \frac{1}{3\times4\times7} = \frac{1}{84}$

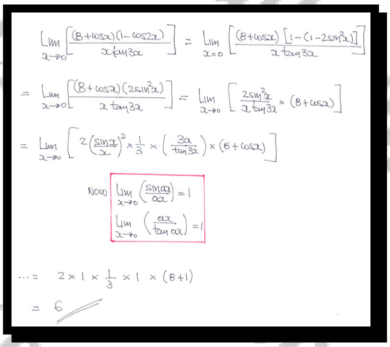
Question 21 **(****+)**

Find the value of the following limit

$$\lim_{x \to 0}\left[\frac{(8+\cos x)(1-\cos 2x)}{x\tan 3x}\right].$$

You may not use the L' Hospital's rule in this question.

$$\boxed{6}$$



$\lim_{x\to 0}\left[\frac{(8+\cos x)(1-\cos 2x)}{x\tan 3x}\right] = \lim_{x=0}\left[\frac{(8+\cos x)[1-(1-2\sin^2 x)]}{x\tan 3x}\right]$

$= \lim_{x\to 0}\left[\frac{(8+\cos x)(2\sin^2 x)}{x\tan 3x}\right] = \lim_{x\to 0}\left[\frac{2\sin^2 x}{x\tan 3x}\times(8+\cos x)\right]$

$= \lim_{x\to 0}\left[2\left(\frac{\sin x}{x}\right)^2\times\frac{1}{3}\times\left(\frac{3x}{\tan 3x}\right)\times(8+\cos x)\right]$

NOW $\boxed{\begin{array}{l}\lim_{x\to 0}\left(\frac{\sin ax}{ax}\right)=1\\ \lim_{x\to 0}\left(\frac{ax}{\tan ax}\right)=1\end{array}}$

$\ldots = 2\times1\times\frac{1}{3}\times1\times(8+1)$

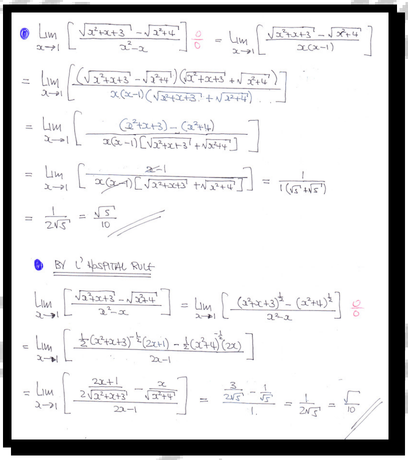
$= 6$

Question 22 (**+)**

Use two distinct methods to evaluate the following limit

$$\lim_{x\to 1}\left[\frac{\sqrt{x^2+x+3}-\sqrt{x^2+4}}{x^2-x}\right].$$

$$\boxed{\frac{\sqrt{10}}{5}}$$



● $\lim_{x\to 1}\left[\frac{\sqrt{x^2+x+3}-\sqrt{x^2+4}}{x^2-x}\right]\frac{0}{0} = \lim_{x\to 1}\left[\frac{\sqrt{x^2+x+3}-\sqrt{x^2+4}}{x(x-1)}\right]$

$= \lim_{x\to 1}\left[\frac{\left(\sqrt{x^2+x+3}-\sqrt{x^2+4}\right)\left(\sqrt{x^2+x+3}+\sqrt{x^2+4}\right)}{x(x-1)\left(\sqrt{x^2+x+3}+\sqrt{x^2+4}\right)}\right]$

$= \lim_{x\to 1}\left[\frac{(x^2+x+3)-(x^2+4)}{x(x-1)\left[\sqrt{x^2+x+3}+\sqrt{x^2+4}\right]}\right]$

$= \lim_{x\to 1}\left[\frac{\cancel{x-1}}{x\cancel{(x-1)}\left[\sqrt{x^2+x+3}+\sqrt{x^2+4}\right]}\right] = \frac{1}{1(\sqrt{5}+\sqrt{5})}$

$= \frac{1}{2\sqrt{5}} = \frac{\sqrt{5}}{10}$

● BY L'HOSPITAL RULE

$\lim_{x\to 1}\left[\frac{\sqrt{x^2+x+3}-\sqrt{x^2+4}}{x^2-x}\right] = \lim_{x\to 1}\left[\frac{(x^2+x+3)^{\frac{1}{2}}-(x^2+4)^{\frac{1}{2}}}{x^2-x}\right]\frac{0}{0}$

$= \lim_{x\to 1}\left[\frac{\frac{1}{2}(x^2+x+3)^{-\frac{1}{2}}(2x+1)-\frac{1}{2}(x^2+4)^{-\frac{1}{2}}(2x)}{2x-1}\right]$

$= \lim_{x\to 1}\left[\frac{\frac{2x+1}{2\sqrt{x^2+x+3}}-\frac{x}{\sqrt{x^2+4}}}{2x-1}\right] = \frac{\frac{3}{2\sqrt{5}}-\frac{1}{\sqrt{5}}}{1.} = \frac{1}{2\sqrt{5}} = \frac{\sqrt{}}{10}$

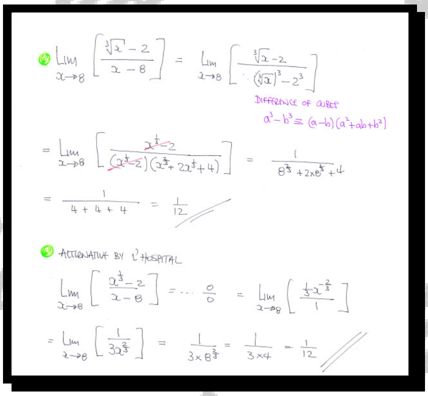
Question 23 (****+)

Find the value of the following limit

$$\lim_{x \to 8}\left[\frac{\sqrt[3]{x}-2}{x-8}\right].$$

You may not use the L' Hospital's rule in this question.

$$\boxed{\frac{1}{12}}$$



$\lim_{x \to 8}\left[\frac{\sqrt[3]{x}-2}{x-8}\right] = \lim_{x \to 8}\left[\frac{\sqrt[3]{x}-2}{(\sqrt[3]{x})^3-2^3}\right]$

DIFFERENCE OF CUBES

$a^3-b^3 \equiv (a-b)(a^2+ab+b^2)$

$= \lim_{x \to 8}\left[\frac{\cancel{x^{\frac{1}{3}}-2}}{\cancel{(x^{\frac{1}{3}}-2)}(x^{\frac{2}{3}}+2x^{\frac{1}{3}}+4)}\right] = \frac{1}{8^{\frac{2}{3}}+2\times8^{\frac{1}{3}}+4}$

$= \frac{1}{4+4+4} = \frac{1}{12}$

ALTERNATIVE BY L' HOSPITAL

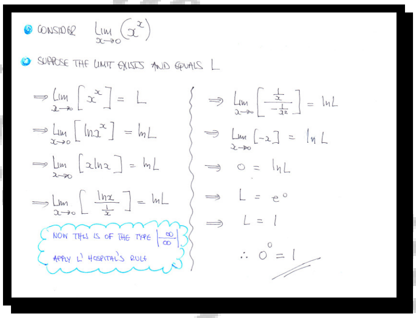
$\lim_{x \to 8}\left[\frac{x^{\frac{1}{3}}-2}{x-8}\right] = \dots \frac{0}{0} = \lim_{x \to 8}\left[\frac{\frac{1}{3}x^{-\frac{2}{3}}}{1}\right]$

$= \lim_{x \to 8}\left[\frac{1}{3x^{\frac{2}{3}}}\right] = \frac{1}{3\times8^{\frac{2}{3}}} = \frac{1}{3\times4} = \frac{1}{12}$

Question 24 (****+)

By considering the limit of an appropriate function show that $0^0 = 1$.

$$\boxed{\text{proof}}$$



CONSIDER $\lim_{x \to 0}(x^x)$

SUPPOSE THE LIMIT EXISTS AND EQUALS L

$\Rightarrow \lim_{x \to 0}\left[x^x\right] = L$

$\Rightarrow \lim_{x \to 0}\left[\ln x^x\right] = \ln L$

$\Rightarrow \lim_{x \to 0}\left[x \ln x\right] = \ln L$

$\Rightarrow \lim_{x \to 0}\left[\frac{\ln x}{\frac{1}{x}}\right] = \ln L$

NOW THIS IS OF THE TYPE $\left|\frac{\infty}{\infty}\right|$

APPLY L' HOSPITAL'S RULE

$\Rightarrow \lim_{x \to 0}\left[\frac{\frac{1}{x}}{-\frac{1}{x^2}}\right] = \ln L$

$\Rightarrow \lim_{x \to 0}\left[-x\right] = \ln L$

$\Rightarrow 0 = \ln L$

$\Rightarrow L = e^0$

$\Rightarrow L = 1$

$\therefore 0^0 = 1$

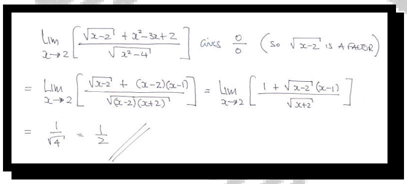
Question 25 **(****+)**

Find the value of the following limit

$$\lim_{x \to 2}\left[\frac{\sqrt{x-2}+x^2-3x+2}{\sqrt{x^2-4}}\right].$$

You may not use the L' Hospital's rule in this question.

$$\boxed{\frac{1}{2}}$$



$$\lim_{x \to 2}\left[\frac{\sqrt{x-2}+x^2-3x+2}{\sqrt{x^2-4}}\right] \text{ GIVES } \frac{0}{0} \quad \left(\text{SO } \sqrt{x-2} \text{ IS A FACTOR}\right)$$

$$= \lim_{x \to 2}\left[\frac{\sqrt{x-2}+(x-2)(x-1)}{\sqrt{(x-2)(x+2)}}\right] = \lim_{x \to 2}\left[\frac{1+\sqrt{x-2}(x-1)}{\sqrt{x+2}}\right]$$

$$= \frac{1}{\sqrt{4}} = \frac{1}{2}$$

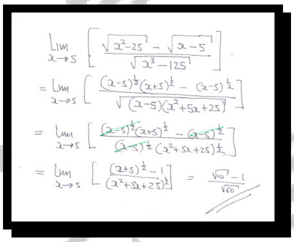
Question 26 **(****+)**

Find the value of the following limit

$$\lim_{x \to 5}\left[\frac{\sqrt{x^2-25}-\sqrt{x-5}}{\sqrt{x^3-125}}\right].$$

You may not use the L' Hospital's rule in this question.

$$\boxed{\frac{\sqrt{10}-1}{\sqrt{60}}}$$



$$\lim_{x \to 5}\left[\frac{\sqrt{x^2-25}-\sqrt{x-5}}{\sqrt{x^3-125}}\right]$$

$$= \lim_{x \to 5}\left[\frac{(x-5)^{\frac{1}{2}}(x+5)^{\frac{1}{2}}-(x-5)^{\frac{1}{2}}}{\sqrt{(x-5)(x^2+5x+25)}}\right]$$

$$= \lim_{x \to 5}\left[\frac{\cancel{(x-5)^{\frac{1}{2}}}(x+5)^{\frac{1}{2}}-\cancel{(x-5)^{\frac{1}{2}}}}{\cancel{(x-5)^{\frac{1}{2}}}(x^2+5x+25)^{\frac{1}{2}}}\right]$$

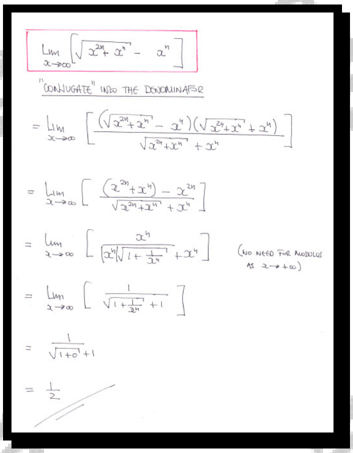
$$= \lim_{x \to 5}\left[\frac{(x+5)^{\frac{1}{2}}-1}{(x^2+5x+25)^{\frac{1}{2}}}\right] = \frac{\sqrt{10}-1}{\sqrt{60}}$$

Question 27 (****+)

Find the value of the following limit

$$\lim_{x \to \infty} \left[\sqrt{x^{2n} - x^{n}} \; - \; x^{n} \right], \; n \in \mathbb{N}.$$

$$\boxed{\frac{1}{2}}$$



$$\boxed{\lim_{x \to \infty} \left[\sqrt{x^{2n} + x^{n}} - x^{n} \right]}$$

"CONJUGATE" INTO THE DENOMINATOR

$$= \lim_{x \to \infty} \left[\frac{\left(\sqrt{x^{2n}+x^{n}} - x^{n}\right)\left(\sqrt{x^{2n}+x^{n}} + x^{n}\right)}{\sqrt{x^{2n}+x^{n}} + x^{n}} \right]$$

$$= \lim_{x \to \infty} \left[\frac{\left(x^{2n}+x^{n}\right) - x^{2n}}{\sqrt{x^{2n}+x^{n}} + x^{n}} \right]$$

$$= \lim_{x \to \infty} \left[\frac{x^{n}}{|x^{n}|\sqrt{1+\frac{1}{x^{n}}} + x^{n}} \right]$$ (NO NEED FOR MODULUS AS $x \to +\infty$)

$$= \lim_{x \to \infty} \left[\frac{1}{\sqrt{1+\frac{1}{x^{n}}} + 1} \right]$$

$$= \frac{1}{\sqrt{1+0}+1}$$

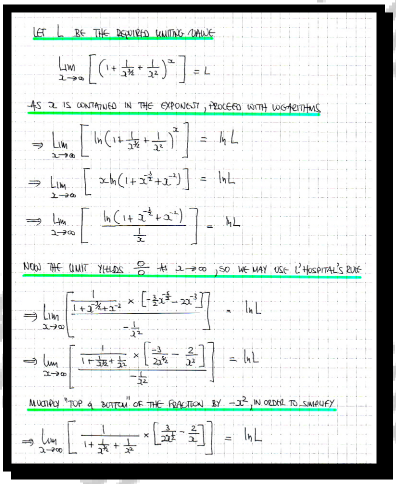
$$= \frac{1}{2}$$

Question 28 (****+)

Find the value of the following limit

$$\lim_{x\to\infty}\left[\left(1+\frac{1}{x^{\frac{3}{2}}}+\frac{1}{x^{2}}\right)^{x}\right].$$

$\boxed{\quad\quad}$, $\boxed{1}$



LET L BE THE REQUIRED LIMITING VALUE

$$\lim_{x\to\infty}\left[\left(1+\frac{1}{x^{3/2}}+\frac{1}{x^2}\right)^{x}\right]=L$$

AS x IS CONTAINED IN THE EXPONENT, PROCEED WITH LOGARITHMS

$$\Rightarrow \lim_{x\to\infty}\left[\ln\left(1+\frac{1}{x^{3/2}}+\frac{1}{x^2}\right)^{x}\right]=\ln L$$

$$\Rightarrow \lim_{x\to\infty}\left[x\ln\left(1+x^{-\frac{3}{2}}+x^{-2}\right)\right]=\ln L$$

$$\Rightarrow \lim_{x\to\infty}\left[\frac{\ln\left(1+x^{-\frac{3}{2}}+x^{-2}\right)}{\frac{1}{x}}\right]=\ln L$$

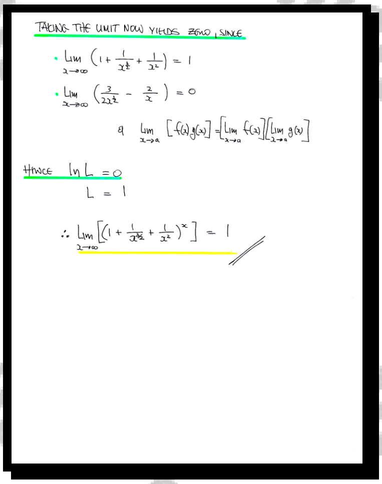
NOW THE LIMIT YIELDS $\frac{0}{0}$ AS $x\to\infty$, SO WE MAY USE L'HOSPITAL'S RULE

$$\Rightarrow \lim_{x\to\infty}\left[\frac{\frac{1}{1+x^{-3/2}+x^{-2}}\times\left[-\frac{3}{2}x^{-\frac{5}{2}}-2x^{-3}\right]}{-\frac{1}{x^2}}\right]=\ln L$$

$$\Rightarrow \lim_{x\to\infty}\left[\frac{\frac{1}{1+\frac{1}{x^{3/2}}+\frac{1}{x^2}}\times\left[\frac{-3}{2x^{5/2}}-\frac{2}{x^3}\right]}{-\frac{1}{x^2}}\right]=\ln L$$

MULTIPLY "TOP & BOTTOM" OF THE FRACTION BY $-x^2$, IN ORDER TO SIMPLIFY

$$\Rightarrow \lim_{x\to\infty}\left[\frac{1}{1+\frac{1}{x^{3/2}}+\frac{1}{x^2}}\times\left[\frac{3}{2x^{\frac{1}{2}}}-\frac{2}{x}\right]\right]=\ln L$$



TAKING THE LIMIT NOW YIELDS ZERO, SINCE

- $\lim_{x\to\infty}\left(1+\frac{1}{x^{\frac{3}{2}}}+\frac{1}{x^2}\right)=1$
- $\lim_{x\to\infty}\left(\frac{3}{2x^{\frac{1}{2}}}-\frac{2}{x}\right)=0$

& $\lim_{x\to a}\left[f(x)g(x)\right]=\left[\lim_{x\to a}f(x)\right]\left[\lim_{x\to a}g(x)\right]$

HENCE $\ln L=0$

$L=1$

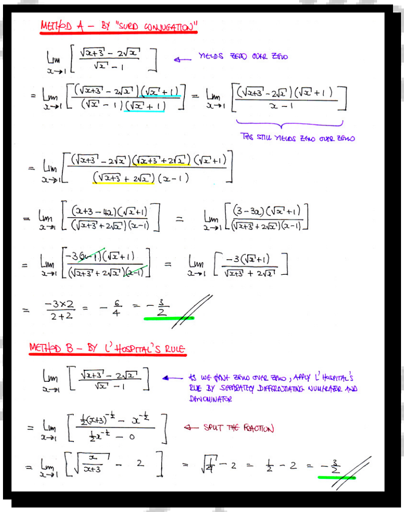
$$\therefore \lim_{x\to\infty}\left[\left(1+\frac{1}{x^{3/2}}+\frac{1}{x^2}\right)^{x}\right]=1$$

Question 29 (****+)

Use two distinct methods to evaluate the following limit.

$$\lim_{x \to 1}\left[\frac{\sqrt{x+3}-2\sqrt{x}}{\sqrt{x}-1}\right].$$

$\boxed{}$, $\boxed{-\frac{3}{2}}$



METHOD A – BY "SURD CONJUGATION"

$$\lim_{x \to 1}\left[\frac{\sqrt{x+3}-2\sqrt{x}}{\sqrt{x}-1}\right]$$ ← YIELDS ZERO OVER ZERO

$$= \lim_{x \to 1}\left[\frac{(\sqrt{x+3}-2\sqrt{x})(\sqrt{x}+1)}{(\sqrt{x}-1)(\sqrt{x}+1)}\right] = \underbrace{\lim_{x \to 1}\left[\frac{(\sqrt{x+3}-2\sqrt{x})(\sqrt{x}+1)}{x-1}\right]}_{\text{THIS STILL YIELDS ZERO OVER ZERO}}$$

$$= \lim_{x \to 1}\left[\frac{(\sqrt{x+3}-2\sqrt{x})(\sqrt{x+3}+2\sqrt{x})(\sqrt{x}+1)}{(\sqrt{x+3}+2\sqrt{x})(x-1)}\right]$$

$$= \lim_{x \to 1}\left[\frac{(x+3-4x)(\sqrt{x}+1)}{(\sqrt{x+3}+2\sqrt{x})(x-1)}\right] = \lim_{x \to 1}\left[\frac{(3-3x)(\sqrt{x}+1)}{(\sqrt{x+3}+2\sqrt{x})(x-1)}\right]$$

$$= \lim_{x \to 1}\left[\frac{-3\cancel{(x-1)}(\sqrt{x}+1)}{(\sqrt{x+3}+2\sqrt{x})\cancel{(x-1)}}\right] = \lim_{x \to 1}\left[\frac{-3(\sqrt{x}+1)}{\sqrt{x+3}+2\sqrt{x}}\right]$$

$$= \frac{-3 \times 2}{2+2} = -\frac{6}{4} = -\frac{3}{2}$$

METHOD B – BY L'HOSPITAL'S RULE

$$\lim_{x \to 1}\left[\frac{\sqrt{x+3}-2\sqrt{x}}{\sqrt{x}-1}\right]$$ ← AS WE HAVE ZERO OVER ZERO, APPLY L'HOSPITAL'S RULE BY SEPARATELY DIFFERENTIATING NUMERATOR AND DENOMINATOR

$$= \lim_{x \to 1}\left[\frac{\frac{1}{2}(x+3)^{-\frac{1}{2}} - x^{-\frac{1}{2}}}{\frac{1}{2}x^{-\frac{1}{2}} - 0}\right]$$ ← SPLIT THE FRACTION

$$= \lim_{x \to 1}\left[\sqrt{\frac{x}{x+3}} - 2\right] = \sqrt{\frac{1}{4}} - 2 = \frac{1}{2} - 2 = -\frac{3}{2}$$

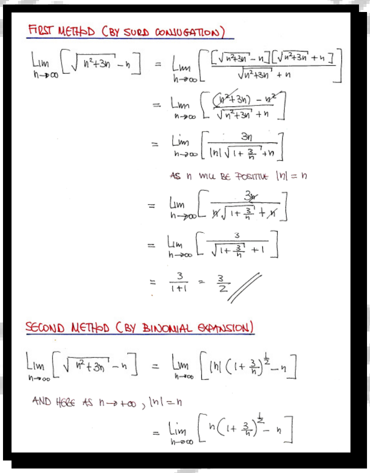
Question 30 (****+)

Use two distinct methods to evaluate the following limit

$$\lim_{n\to\infty}\left[\sqrt{n^2+3n}-n\right].$$

You may not use the L' Hospital's rule in this question.

$\boxed{}$, $\boxed{\frac{3}{2}}$



FIRST METHOD (BY SURD CONJUGATION)

$$\lim_{n\to\infty}\left[\sqrt{n^2+3n}-n\right] = \lim_{n\to\infty}\left[\frac{\left[\sqrt{n^2+3n}-n\right]\left[\sqrt{n^2+3n}+n\right]}{\sqrt{n^2+3n}+n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{(\cancel{n^2}+3n)-\cancel{n^2}}{\sqrt{n^2+3n}+n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{3n}{|n|\sqrt{1+\frac{3}{n}}+n}\right]$$

AS n WILL BE POSITIVE $|n| = n$

$$= \lim_{n\to\infty}\left[\frac{3\cancel{n}}{\cancel{n}\sqrt{1+\frac{3}{n}}+\cancel{n}}\right]$$

$$= \lim_{n\to\infty}\left[\frac{3}{\sqrt{1+\frac{3}{n}}+1}\right]$$

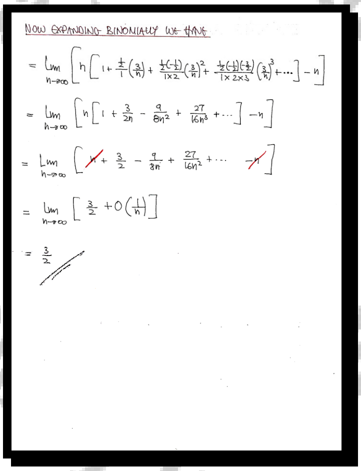
$$= \frac{3}{1+1} = \frac{3}{2}$$

SECOND METHOD (BY BINOMIAL EXPANSION)

$$\lim_{n\to\infty}\left[\sqrt{n^2+3n}-n\right] = \lim_{n\to\infty}\left[|n|\left(1+\frac{3}{n}\right)^{\frac{1}{2}}-n\right]$$

AND HERE AS $n\to+\infty$, $|n| = n$

$$= \lim_{n\to\infty}\left[n\left(1+\frac{3}{n}\right)^{\frac{1}{2}}-n\right]$$



NOW EXPANDING BINOMIALLY WE HAVE

$$= \lim_{n\to\infty}\left[n\left[1+\frac{\frac{1}{2}}{1}\left(\frac{3}{n}\right)+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1\times2}\left(\frac{3}{n}\right)^2+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1\times2\times3}\left(\frac{3}{n}\right)^3+\dots\right]-n\right]$$

$$= \lim_{n\to\infty}\left[n\left[1+\frac{3}{2n}-\frac{9}{8n^2}+\frac{27}{16n^3}+\dots\right]-n\right]$$

$$= \lim_{n\to\infty}\left[\cancel{n}+\frac{3}{2}-\frac{9}{8n}+\frac{27}{16n^2}+\dots\ \cancel{-n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{3}{2}+O\left(\frac{1}{n}\right)\right]$$

$$= \frac{3}{2}$$

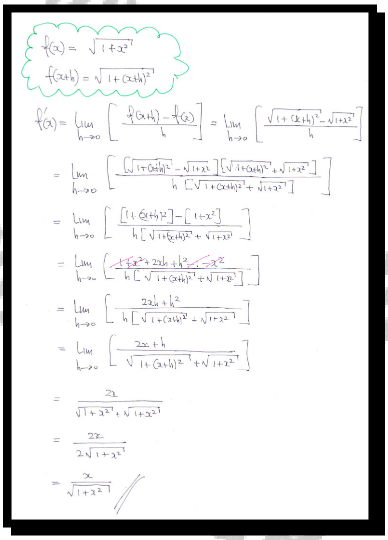
Question 31 **(****+)**

$$f(x) = \sqrt{1+x^2}\,,\ x \in \mathbb{R}\,.$$

Use the formal definition of the derivative as a limit, to show that

$$f'(x) = \frac{x}{\sqrt{1+x^2}}\,.$$

proof



$$f(x) = \sqrt{1+x^2}$$

$$f(x+h) = \sqrt{1+(x+h)^2}$$

$$f'(x) = \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right] = \lim_{h\to 0}\left[\frac{\sqrt{1+(x+h)^2}-\sqrt{1+x^2}}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{\left[\sqrt{1+(x+h)^2}-\sqrt{1+x^2}\right]\left[\sqrt{1+(x+h)^2}+\sqrt{1+x^2}\right]}{h\left[\sqrt{1+(x+h)^2}+\sqrt{1+x^2}\right]}\right]$$

$$= \lim_{h\to 0}\left[\frac{\left[1+(x+h)^2\right]-\left[1+x^2\right]}{h\left[\sqrt{1+(x+h)^2}+\sqrt{1+x^2}\right]}\right]$$

$$= \lim_{h\to 0}\left[\frac{\cancel{1}+\cancel{x^2}+2xh+h^2\cancel{-1}\cancel{-x^2}}{h\left[\sqrt{1+(x+h)^2}+\sqrt{1+x^2}\right]}\right]$$

$$= \lim_{h\to 0}\left[\frac{2xh+h^2}{h\left[\sqrt{1+(x+h)^2}+\sqrt{1+x^2}\right]}\right]$$

$$= \lim_{h\to 0}\left[\frac{2x+h}{\sqrt{1+(x+h)^2}+\sqrt{1+x^2}}\right]$$

$$= \frac{2x}{\sqrt{1+x^2}+\sqrt{1+x^2}}$$

$$= \frac{2x}{2\sqrt{1+x^2}}$$

$$= \frac{x}{\sqrt{1+x^2}} \quad //$$

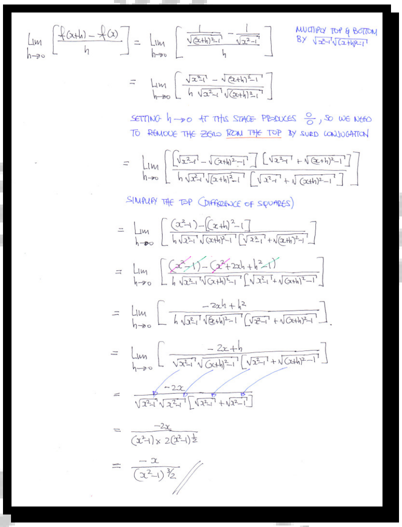
Question 32 (****+)

$$f(x)=\frac{1}{\sqrt{x^2-1}}, \ x\in\mathbb{R}, \ |x|>1 .$$

Use the formal definition of the derivative as a limit, to show that

$$f'(x)=-\frac{x}{\left(x^2-1\right)^{\frac{3}{2}}} .$$

proof



$$\lim_{h\to0}\left[\frac{f(x+h)-f(x)}{h}\right]=\lim_{h\to0}\left[\frac{\frac{1}{\sqrt{(x+h)^2-1}}-\frac{1}{\sqrt{x^2-1}}}{h}\right]$$

MULTIPLY TOP & BOTTOM BY $\sqrt{x^2-1}\sqrt{(x+h)^2-1}$

$$=\lim_{h\to0}\left[\frac{\sqrt{x^2-1}-\sqrt{(x+h)^2-1}}{h\sqrt{x^2-1}\sqrt{(x+h)^2-1}}\right]$$

SETTING $h\to0$ AT THIS STAGE PRODUCES $\frac{0}{0}$, SO WE NEED TO REMOVE THE ZERO FROM THE TOP BY SURD CONJUGATION

$$=\lim_{h\to0}\left[\frac{\left[\sqrt{x^2-1}-\sqrt{(x+h)^2-1}\right]\left[\sqrt{x^2-1}+\sqrt{(x+h)^2-1}\right]}{h\sqrt{x^2-1}\sqrt{(x+h)^2-1}\left[\sqrt{x^2-1}+\sqrt{(x+h)^2-1}\right]}\right]$$

SIMPLIFY THE TOP (DIFFERENCE OF SQUARES)

$$=\lim_{h\to0}\left[\frac{(x^2-1)-\left[(x+h)^2-1\right]}{h\sqrt{x^2-1}\sqrt{(x+h)^2-1}\left[\sqrt{x^2-1}+\sqrt{(x+h)^2-1}\right]}\right]$$

$$=\lim_{h\to0}\left[\frac{(\cancel{x^2}\cancel{-1})-(\cancel{x^2}+2xh+h^2\cancel{-1})}{h\sqrt{x^2-1}\sqrt{(x+h)^2-1}\left[\sqrt{x^2-1}+\sqrt{(x+h)^2-1}\right]}\right]$$

$$=\lim_{h\to0}\left[\frac{-2xh+h^2}{h\sqrt{x^2-1}\sqrt{(x+h)^2-1}\left(\sqrt{x^2-1}+\sqrt{(x+h)^2-1}\right)}\right].$$

$$=\lim_{h\to0}\left[\frac{-2x+h}{\sqrt{x^2-1}\sqrt{(x+h)^2-1}\left[\sqrt{x^2-1}+\sqrt{(x+h)^2-1}\right]}\right]$$

$$=\frac{-2x}{\sqrt{x^2-1}\sqrt{x^2-1}\left[\sqrt{x^2-1}+\sqrt{x^2-1}\right]}$$

$$=\frac{-2x}{(x^2-1)\times2(x^2-1)^{\frac{1}{2}}}$$

$$=\frac{-x}{(x^2-1)^{3/2}} \ //$$

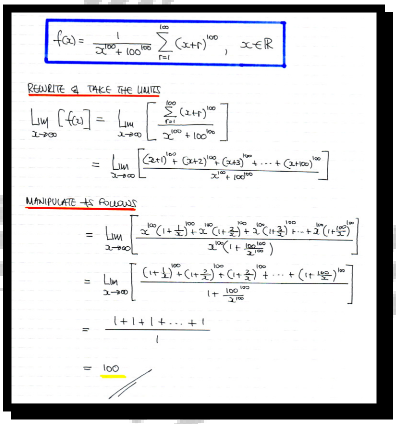
Question 33 **(****+)**

$$f(x) \equiv \frac{1}{x^{100}+100^{100}} \sum_{r=1}^{100} (x+r)^{100}, \ x \in \mathbb{R}.$$

Use a formal method to find

$$\lim_{x \to \infty} f(x).$$

$\boxed{\quad\quad}$, $\boxed{100}$



$$\boxed{f(x) = \frac{1}{x^{100}+100^{100}} \sum_{r=1}^{100} (x+r)^{100}, \quad x \in \mathbb{R}}$$

REWRITE & TAKE THE LIMITS

$$\lim_{x \to \infty} [f(x)] = \lim_{x \to \infty} \left[\frac{\sum_{r=1}^{100} (x+r)^{100}}{x^{100}+100^{100}} \right]$$

$$= \lim_{x \to \infty} \left[\frac{(x+1)^{100} + (x+2)^{100} + (x+3)^{100} + \dots + (x+100)^{100}}{x^{100}+100^{100}} \right]$$

MANIPULATE AS FOLLOWS

$$= \lim_{x \to \infty} \left[\frac{x^{100}\left(1+\frac{1}{x}\right)^{100} + x^{100}\left(1+\frac{2}{x}\right)^{100} + x^{100}\left(1+\frac{3}{x}\right)^{100} + \dots + x^{100}\left(1+\frac{100}{x}\right)^{100}}{x^{100}\left(1+\frac{100^{100}}{x^{100}}\right)} \right]$$

$$= \lim_{x \to \infty} \left[\frac{\left(1+\frac{1}{x}\right)^{100} + \left(1+\frac{2}{x}\right)^{100} + \left(1+\frac{3}{x}\right)^{100} + \dots + \left(1+\frac{100}{x}\right)^{100}}{1+\frac{100^{100}}{x^{100}}} \right]$$

$$= \frac{1+1+1+\dots+1}{1}$$

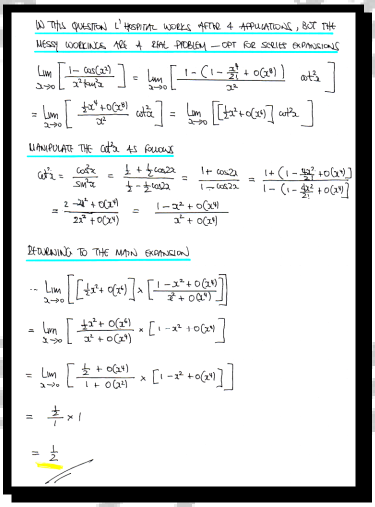
$$= 100$$

Question 34 (****+)

Find the value of the following limit

$$\lim_{x\to 0}\left[\frac{1-\cos\left(x^2\right)}{x^2\tan^2 x}\right].$$

$\boxed{\text{SP}}$, $\boxed{\frac{1}{2}}$



IN THIS QUESTION L'HOSPITAL WORKS AFTER 4 APPLICATIONS, BUT THE MESSY WORKINGS ARE A REAL PROBLEM — OPT FOR SERIES EXPANSIONS

$$\lim_{x\to 0}\left[\frac{1-\cos(x^2)}{x^2\tan^2 x}\right] = \lim_{x\to 0}\left[\frac{1-\left(1-\frac{x^4}{2!}+O(x^8)\right)}{x^2}\cot^2 x\right]$$

$$= \lim_{x\to 0}\left[\frac{\frac{1}{2}x^4+O(x^8)}{x^2}\cot^2 x\right] = \lim_{x\to 0}\left[\left[\tfrac{1}{2}x^2+O(x^6)\right]\cot^2 x\right]$$

MANIPULATE THE $\cot^2 x$ AS FOLLOWS

$$\cot^2 x = \frac{\cos^2 x}{\sin^2 x} = \frac{\frac{1}{2}+\frac{1}{2}\cos 2x}{\frac{1}{2}-\frac{1}{2}\cos 2x} = \frac{1+\cos 2x}{1-\cos 2x} = \frac{1+\left(1-\frac{4x^2}{2!}+O(x^4)\right]}{1-\left(1-\frac{4x^2}{2!}+O(x^4)\right]}$$

$$= \frac{2-2x^2+O(x^4)}{2x^2+O(x^4)} = \frac{1-x^2+O(x^4)}{x^2+O(x^4)}$$

RETURNING TO THE MAIN EXPANSION

$$\dots \lim_{x\to 0}\left[\left[\tfrac{1}{2}x^2+O(x^6)\right]\times\left[\frac{1-x^2+O(x^4)}{x^2+O(x^4)}\right]\right]$$

$$= \lim_{x\to 0}\left[\frac{\frac{1}{2}x^2+O(x^6)}{x^2+O(x^4)}\times\left[1-x^2+O(x^4)\right]\right]$$

$$= \lim_{x\to 0}\left[\frac{\frac{1}{2}+O(x^4)}{1+O(x^2)}\times\left[1-x^2+O(x^4)\right]\right]$$

$$= \frac{\frac{1}{2}}{1}\times 1$$

$$= \frac{1}{2}$$

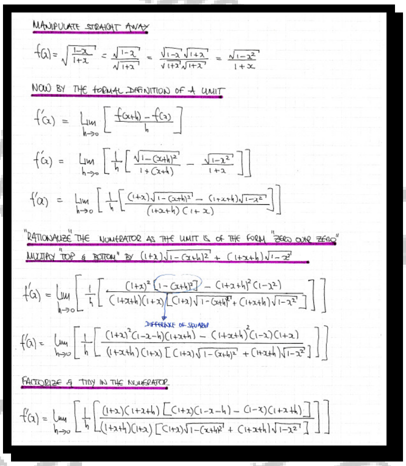
Question 35 (*****)

$$f(x)=\sqrt{\frac{1-x}{1+x}},\ x\in\mathbb{R},\ |x|<1.$$

Use the formal definition of the derivative as a limit, to show that

$$f'(x)=-\frac{1}{(1+x)\sqrt{1-x^2}}.$$

☐ , proof



MANIPULATE STRAIGHT AWAY

$$f(x)=\sqrt{\frac{1-x}{1+x}}=\frac{\sqrt{1-x}}{\sqrt{1+x}}=\frac{\sqrt{1-x}\sqrt{1+x}}{\sqrt{1+x}\sqrt{1+x}}=\frac{\sqrt{1-x^2}}{1+x}$$

NOW BY THE FORMAL DEFINITION OF A LIMIT

$$f'(x)=\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]$$

$$f'(x)=\lim_{h\to 0}\left[\frac{1}{h}\left[\frac{\sqrt{1-(x+h)^2}}{1+(x+h)}-\frac{\sqrt{1-x^2}}{1+x}\right]\right]$$

$$f'(x)=\lim_{h\to 0}\left[\frac{1}{h}\left[\frac{(1+x)\sqrt{1-(x+h)^2}-(1+x+h)\sqrt{1-x^2}}{(1+x+h)(1+x)}\right]\right]$$

"RATIONALIZE" THE NUMERATOR AS THE LIMIT IS OF THE FORM "ZERO OVER ZERO"

MULTIPLY "TOP & BOTTOM" BY $(1+x)\sqrt{1-(x+h)^2}+(1+x+h)\sqrt{1-x^2}$

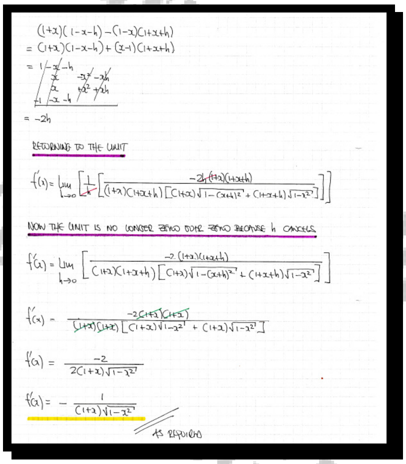
$$f'(x)=\lim_{h\to 0}\left[\frac{1}{h}\left[\frac{(1+x)^2\left[1-(x+h)^2\right]-(1+x+h)^2(1-x^2)}{(1+x+h)(1+x)\left[(1+x)\sqrt{1-(x+h)^2}+(1+x+h)\sqrt{1-x^2}\right]}\right]\right]$$

DIFFERENCE OF SQUARES

$$f'(x)=\lim_{h\to 0}\left[\frac{1}{h}\left[\frac{(1+x)^2(1-x-h)(1+x+h)-(1+x+h)^2(1-x)(1+x)}{(1+x+h)(1+x)\left[(1+x)\sqrt{1-(x+h)^2}+(1+x+h)\sqrt{1-x^2}\right]}\right]\right]$$

FACTORIZE & TIDY IN THE NUMERATOR

$$f'(x)=\lim_{h\to 0}\left[\frac{1}{h}\left[\frac{(1+x)(1+x+h)\left[(1+x)(1-x-h)-(1-x)(1+x+h)\right]}{(1+x+h)(1+x)\left[(1+x)\sqrt{1-(x+h)^2}+(1+x+h)\sqrt{1-x^2}\right]}\right]\right]$$



$$(1+x)(1-x-h)-(1-x)(1+x+h)$$

$$=(1+x)(1-x-h)+(x-1)(1+x+h)$$

$$\begin{array}{llll} = \cancel{1} & -\cancel{x} & -h & \\ & \cancel{x} & -\cancel{x^2} & -\cancel{xh} \\ & \cancel{x} & +\cancel{x^2} & +\cancel{xh} \\ \cancel{-1} & -x & -h & \end{array}$$

$$=-2h$$

RETURNING TO THE LIMIT

$$f'(x)=\lim_{h\to 0}\left[\frac{1}{\cancel{h}}\left[\frac{-2\cancel{h}(1+x)(1+x+h)}{(1+x)(1+x+h)\left[(1+x)\sqrt{1-(x+h)^2}+(1+x+h)\sqrt{1-x^2}\right]}\right]\right]$$

NOW THE LIMIT IS NO LONGER ZERO OVER ZERO BECAUSE h CANCELS

$$f'(x)=\lim_{h\to 0}\left[\frac{-2(1+x)(1+x+h)}{(1+x)(1+x+h)\left[(1+x)\sqrt{1-(x+h)^2}+(1+x+h)\sqrt{1-x^2}\right]}\right]$$

$$f'(x)=\frac{-2\cancel{(1+x)}\cancel{(1+x)}}{\cancel{(1+x)}\cancel{(1+x)}\left[(1+x)\sqrt{1-x^2}+(1+x)\sqrt{1-x^2}\right]}$$

$$f'(x)=\frac{-2}{2(1+x)\sqrt{1-x^2}}$$

$$f'(x)=-\frac{1}{(1+x)\sqrt{1-x^2}}$$

AS REQUIRED

Question 36 (*****)

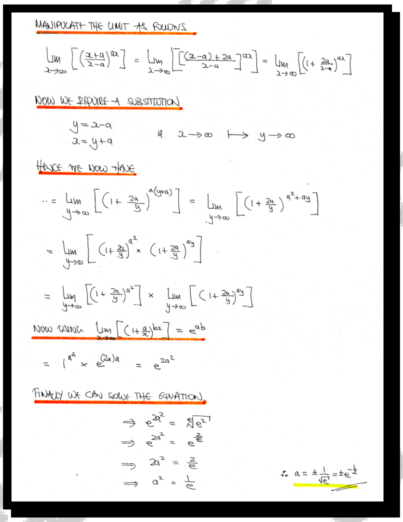
Solve the following equation over the set of real numbers.

$$\lim_{x\to\infty}\left[\left(\frac{x+a}{x-a}\right)^{ax}\right] = \sqrt[e]{e^2}\,.$$

You may assume that the limit in the left hand side of the equation exists.

You must clearly state any results used in the solution.

$\boxed{}$, $\boxed{a = \pm e^{-\frac{1}{2}}}$



MANIPULATE THE LIMIT AS FOLLOWS

$$\lim_{x\to\infty}\left[\left(\frac{x+a}{x-a}\right)^{ax}\right] = \lim_{x\to\infty}\left[\left[\frac{(x-a)+2a}{x-a}\right]^{ax}\right] = \lim_{x\to\infty}\left[\left(1+\frac{2a}{x-a}\right)^{ax}\right]$$

NOW WE REQUIRE A SUBSTITUTION

$$y = x-a \qquad \& \quad x\to\infty \longmapsto y\to\infty$$
$$x = y+a$$

HENCE WE NOW HAVE

$$\dots = \lim_{y\to\infty}\left[\left(1+\frac{2a}{y}\right)^{a(y+a)}\right] = \lim_{y\to\infty}\left[\left(1+\frac{2a}{y}\right)^{a^2+ay}\right]$$

$$= \lim_{y\to\infty}\left[\left(1+\frac{2a}{y}\right)^{a^2}\times\left(1+\frac{2a}{y}\right)^{ay}\right]$$

$$= \lim_{y\to\infty}\left[\left(1+\frac{2a}{y}\right)^{a^2}\right]\times\lim_{y\to\infty}\left[\left(1+\frac{2a}{y}\right)^{ay}\right]$$

NOW USING $\lim_{x\to\infty}\left[\left(1+\frac{a}{x}\right)^{bx}\right] = e^{ab}$

$$= 1^{a^2}\times e^{(2a)a} = e^{2a^2}$$

FINALLY WE CAN SOLVE THE EQUATION

$$\Rightarrow e^{2a^2} = \sqrt[e]{e^2}$$
$$\Rightarrow e^{2a^2} = e^{\frac{2}{e}}$$
$$\Rightarrow 2a^2 = \frac{2}{e}$$
$$\Rightarrow a^2 = \frac{1}{e}$$

$$\therefore\ a = \pm\frac{1}{\sqrt{e}} = \pm e^{-\frac{1}{2}}$$

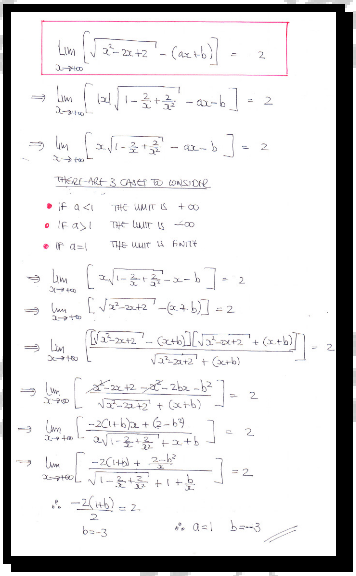
Question 37 (*****)

It is given that for some real constants a and b,

$$\lim_{x \to +\infty} \left[\sqrt{x^2 - 2x + 2} - (ax + b) \right] = 2, \; x \in \mathbb{R}, \; x > 0.$$

Determine the value of a and the value of b.

$\boxed{a = 1}$, $\boxed{b = -3}$



$$\boxed{\lim_{x \to +\infty} \left[\sqrt{x^2 - 2x + 2} - (ax + b) \right] = 2}$$

$$\Rightarrow \lim_{x \to +\infty} \left[|x| \sqrt{1 - \frac{2}{x} + \frac{2}{x^2}} - ax - b \right] = 2$$

$$\Rightarrow \lim_{x \to +\infty} \left[x \sqrt{1 - \frac{2}{x} + \frac{2}{x^2}} - ax - b \right] = 2$$

THERE ARE 3 CASES TO CONSIDER

- IF $a < 1$ THE LIMIT IS $+\infty$
- IF $a > 1$ THE LIMIT IS $-\infty$
- IF $a = 1$ THE LIMIT IS FINITE

$$\Rightarrow \lim_{x \to +\infty} \left[x \sqrt{1 - \frac{2}{x} + \frac{2}{x^2}} - x - b \right] = 2$$

$$\Rightarrow \lim_{x \to +\infty} \left[\sqrt{x^2 - 2x + 2} - (x + b) \right] = 2$$

$$\Rightarrow \lim_{x \to +\infty} \left[\frac{\left[\sqrt{x^2 - 2x + 2} - (x + b)\right]\left[\sqrt{x^2 - 2x + 2} + (x + b)\right]}{\sqrt{x^2 - 2x + 2} + (x + b)} \right] = 2$$

$$\Rightarrow \lim_{x \to +\infty} \left[\frac{\cancel{x^2} - 2x + 2 - \cancel{x^2} - 2bx - b^2}{\sqrt{x^2 - 2x + 2} + (x + b)} \right] = 2$$

$$\Rightarrow \lim_{x \to +\infty} \left[\frac{-2(1 + b)x + (2 - b^2)}{x\sqrt{1 - \frac{2}{x} + \frac{2}{x^2}} + x + b} \right] = 2$$

$$\Rightarrow \lim_{x \to +\infty} \left[\frac{-2(1 + b) + \frac{2 - b^2}{x}}{\sqrt{1 - \frac{2}{x} + \frac{2}{x^2}} + 1 + \frac{b}{x}} \right] = 2$$

$\therefore \frac{-2(1+b)}{2} = 2$

$b = -3$

$\therefore a = 1 \quad b = -3$

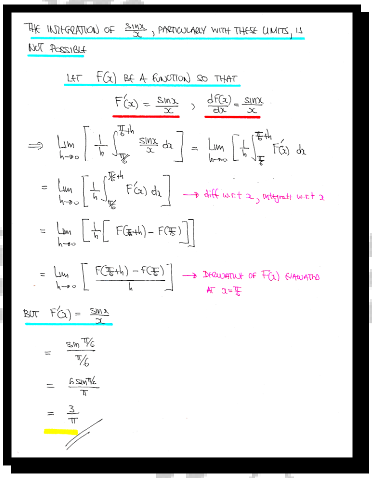
Question 38 (*****)

Determine the exact value of the following limit.

$$\lim_{h\to 0}\left[\frac{1}{h}\left[\int_{\frac{1}{6}\pi}^{\frac{1}{6}\pi+h}\frac{\sin x}{x}\,dx\right]\right]$$

You must justify the evaluation.

$\boxed{\qquad}$, $\boxed{\frac{3}{\pi}}$



THE INTEGRATION OF $\frac{\sin x}{x}$, PARTICULARLY WITH THESE LIMITS, IS NOT POSSIBLE

LET $F(x)$ BE A FUNCTION SO THAT

$$F'(x)=\frac{\sin x}{x}\ ,\quad \frac{dF(x)}{dx}=\frac{\sin x}{x}$$

$$\Rightarrow \lim_{h\to 0}\left[\frac{1}{h}\int_{\frac{\pi}{6}}^{\frac{\pi}{6}+h}\frac{\sin x}{x}\,dx\right]=\lim_{h\to 0}\left[\frac{1}{h}\int_{\frac{\pi}{6}}^{\frac{\pi}{6}+h}F'(x)\,dx\right]$$

$$=\lim_{h\to 0}\left[\frac{1}{h}\int_{\pi/6}^{\pi/6+h}F'(x)\,dx\right]$$ → diff w.r.t x, integrate w.r.t x

$$=\lim_{h\to 0}\left[\frac{1}{h}\left[F\left(\tfrac{\pi}{6}+h\right)-F\left(\tfrac{\pi}{6}\right)\right]\right]$$

$$=\lim_{h\to 0}\left[\frac{F\left(\frac{\pi}{6}+h\right)-F\left(\frac{\pi}{6}\right)}{h}\right]$$ → DERIVATIVE OF $F(x)$ EVALUATED AT $x=\frac{\pi}{6}$

BUT $F'(x)=\frac{\sin x}{x}$

$$=\frac{\sin \pi/6}{\pi/6}$$

$$=\frac{6\sin \pi/6}{\pi}$$

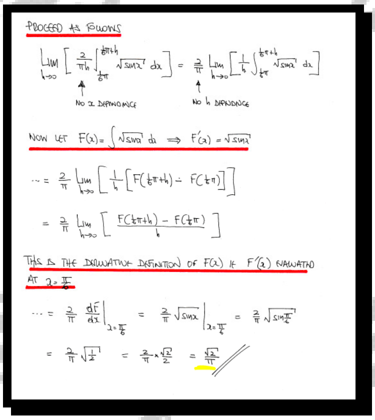
$$=\frac{3}{\pi}$$

Question 39 (*****)

Evaluate the following limit.

$$\lim_{h\to 0}\left[\int_{\frac{1}{6}\pi}^{\frac{1}{6}\pi+h} \frac{2\sqrt{\sin x}}{\pi h}\, dx\right].$$

$\boxed{\quad\quad}$, $\boxed{\dfrac{\sqrt{2}}{\pi}}$



PROCEED AS FOLLOWS

$$\lim_{h\to 0}\left[\frac{2}{\pi h}\int_{\frac{1}{6}\pi}^{\frac{1}{6}\pi+h}\sqrt{\sin x}\,dx\right] = \frac{2}{\pi}\lim_{h\to 0}\left[\frac{1}{h}\int_{\frac{1}{6}\pi}^{\frac{1}{6}\pi+h}\sqrt{\sin x}\,dx\right]$$

NO x DEPENDENCE

NO h DEPENDENCE

NOW LET $F(x)=\int\sqrt{\sin x}\,dx \Rightarrow F'(x)=\sqrt{\sin x}$

$$\dots = \frac{2}{\pi}\lim_{h\to 0}\left[\frac{1}{h}\left[F\left(\tfrac{1}{6}\pi+h\right)-F\left(\tfrac{1}{6}\pi\right)\right]\right]$$

$$= \frac{2}{\pi}\lim_{h\to 0}\left[\frac{F\left(\frac{1}{6}\pi+h\right)-F\left(\frac{1}{6}\pi\right)}{h}\right]$$

THIS IS THE DERIVATIVE DEFINITION OF $F(x)$ IE $F'(x)$ EVALUATED AT $x=\frac{\pi}{6}$

$$\dots = \frac{2}{\pi}\left.\frac{dF}{dx}\right|_{x=\frac{\pi}{6}} = \frac{2}{\pi}\left.\sqrt{\sin x}\right|_{x=\frac{\pi}{6}} = \frac{2}{\pi}\sqrt{\sin\frac{\pi}{6}}$$

$$= \frac{2}{\pi}\sqrt{\frac{1}{2}} = \frac{2}{\pi}\times\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{\pi}$$

Question 40 (*****)

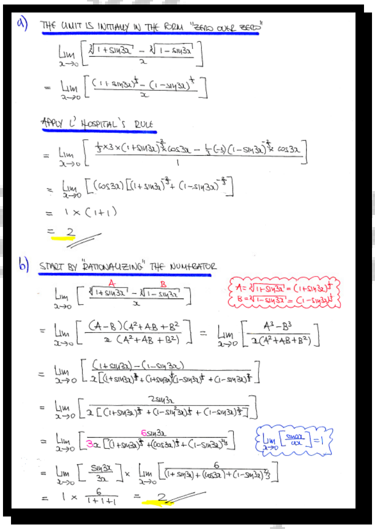
a) Use L'Hospital's rule to evaluate

$$\lim_{x\to 0}\left[\frac{\sqrt[3]{1+\sin 3x}-\sqrt[3]{1-\sin 3x}}{x}\right].$$

b) Verify the answer to part **(a)** by an alternative method.

You must state clearly any additional results used.

$\boxed{}$, $\boxed{2}$



a) THE LIMIT IS INITIALLY IN THE FORM "ZERO OVER ZERO"

$$\lim_{x\to0}\left[\frac{\sqrt[3]{1+\sin3x}-\sqrt[3]{1-\sin3x}}{x}\right]$$

$$=\lim_{x\to0}\left[\frac{(1+\sin3x)^{\frac13}-(1-\sin3x)^{\frac13}}{x}\right]$$

APPLY L'HOSPITAL'S RULE

$$=\lim_{x\to0}\left[\frac{\frac13\times3\times(1+\sin3x)^{-\frac23}\times\cos3x-\frac13(-3)(1-\sin3x)^{-\frac23}\times\cos3x}{1}\right]$$

$$=\lim_{x\to0}\left[(\cos3x)\left[(1+\sin3x)^{-\frac23}+(1-\sin3x)^{-\frac23}\right]\right]$$

$$=1\times(1+1)$$

$$=2$$

b) START BY "RATIONALIZING" THE NUMERATOR

$$\lim_{x\to0}\left[\frac{\overset{A}{\sqrt[3]{1+\sin3x}}-\overset{B}{\sqrt[3]{1-\sin3x}}}{x}\right]$$

$A=\sqrt[3]{1+\sin3x}=(1+\sin3x)^{\frac13}$
$B=\sqrt[3]{1-\sin3x}=(1-\sin3x)^{\frac13}$

$$=\lim_{x\to0}\left[\frac{(A-B)(4^2+AB+B^2)}{x(A^2+AB+B^2)}\right]=\lim_{x\to0}\left[\frac{A^3-B^3}{x(A^2+AB+B^2)}\right]$$

$$=\lim_{x\to0}\left[\frac{(1+\sin3x)-(1-\sin3x)}{x\left[(1+\sin3x)^{\frac23}+(1+\sin3x)^{\frac13}(1-\sin3x)^{\frac13}+(1-\sin3x)^{\frac23}\right]}\right]$$

$$=\lim_{x\to0}\left[\frac{2\sin3x}{x\left[(1+\sin3x)^{\frac23}+(1-\sin^23x)^{\frac13}+(1-\sin3x)^{\frac23}\right]}\right]$$

$$=\lim_{x\to0}\left[\frac{6\sin3x}{3x\left[(1+\sin3x)^{\frac23}+(\cos3x)^{\frac23}+(1-\sin3x)^{\frac23}\right]}\right]$$

$\lim_{x\to0}\left[\frac{\sin ax}{ax}\right]=1$

$$=\lim_{x\to0}\left[\frac{\sin3x}{3x}\right]\times\lim_{x\to0}\left[\frac{6}{(1+\sin3x)+(\cos3x)+(1-\sin3x)^{\frac23}}\right]$$

$$=1\times\frac{6}{1+1+1}=2$$

Question 41 (*****)

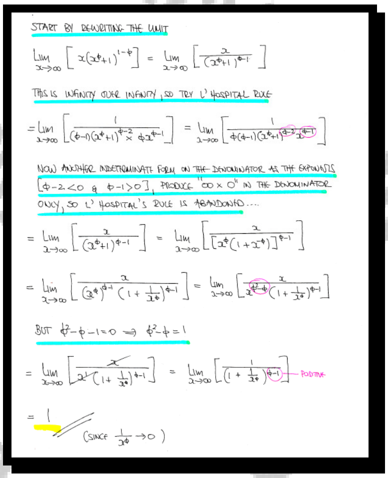
The positive solution of the quadratic equation $x^2 - x - 1 = 0$ is denoted by ϕ, and is commonly known as the golden section or golden number.

This implies that $\phi^2 - \phi - 1 = 0$, $\phi = \frac{1}{2}\left(1+\sqrt{5}\right) \approx 1.62$.

Show, with full justification, that

$$\lim_{x\to\infty}\left[x\left(x^{\phi}+1\right)^{1-\phi}\right] = 1.$$

☐, $\boxed{\text{proof}}$



START BY REWRITING THE LIMIT

$$\lim_{x\to\infty}\left[x\left(x^{\phi}+1\right)^{1-\phi}\right] = \lim_{x\to\infty}\left[\frac{x}{\left(x^{\phi}+1\right)^{\phi-1}}\right]$$

THIS IS INFINITY OVER INFINITY, SO TRY L'HOSPITAL RULE

$$= \lim_{x\to\infty}\left[\frac{1}{(\phi-1)\left(x^{\phi}+1\right)^{\phi-2}\times\phi x^{\phi-1}}\right] = \lim_{x\to\infty}\left[\frac{1}{\phi(\phi-1)\left(x^{\phi}+1\right)^{\boxed{\phi-2}}x^{\boxed{\phi-1}}}\right]$$

NOW ANOTHER INDETERMINATE FORM ON THE DENOMINATOR AS THE EXPONENTS $[\phi-2<0$ & $\phi-1>0]$, PRODUCE "$\infty \times 0$" IN THE DENOMINATOR ONLY, SO L'HOSPITAL'S RULE IS ABANDONED....

$$= \lim_{x\to\infty}\left[\frac{x}{\left(x^{\phi}+1\right)^{\phi-1}}\right] = \lim_{x\to\infty}\left[\frac{x}{\left[x^{\phi}\left(1+x^{-\phi}\right)\right]^{\phi-1}}\right]$$

$$= \lim_{x\to\infty}\left[\frac{x}{\left(x^{\phi}\right)^{\phi-1}\left(1+\frac{1}{x^{\phi}}\right)^{\phi-1}}\right] = \lim_{x\to\infty}\left[\frac{x}{x^{\boxed{\phi^2-\phi}}\left(1+\frac{1}{x^{\phi}}\right)^{\phi-1}}\right]$$

BUT $\phi^2-\phi-1=0 \Rightarrow \phi^2-\phi=1$

$$= \lim_{x\to\infty}\left[\frac{\cancel{x}}{\cancel{x^1}\left(1+\frac{1}{x^{\phi}}\right)^{\phi-1}}\right] = \lim_{x\to\infty}\left[\frac{1}{\left(1+\frac{1}{x^{\phi}}\right)^{\boxed{\phi-1}}}\right] \text{— POSITIVE}$$

$= 1$ // (SINCE $\frac{1}{x^{\phi}} \to 0$)

Question 42 (*****)

A curve has equation $y = f(x)$.

The finite region R is bounded by the curve, the x axis and the straight lines with equations $x = a$ and $x = b$, and hence the area of R is given by

$$I(a,b) = \int_a^b f(x)\ dx .$$

The area of R is also given by the limiting value of the sum of the areas of rectangles of width δx and height $f(x_i)$, known as a "right (upper) Riemann sum"

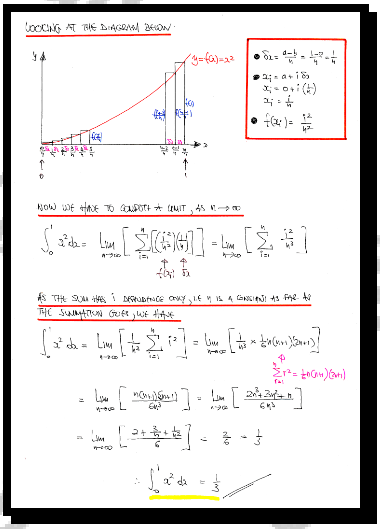
$$I(a,b) = \lim_{n \to \infty} \left[\sum_{i=1}^{n} \left[f(x_i)\, \delta x \right] \right],$$

where $\delta x = \dfrac{b-a}{n}$ and $x_i = a + i\,\delta x$.

Using the "right (upper) Riemann sum" definition, and with the aid of a diagram where appropriate, show clearly that

$$\int_0^1 x^2\ dx = \frac{1}{3}.$$

$\boxed{}$, $\boxed{\text{proof}}$



LOOKING AT THE DIAGRAM BELOW

- $\delta x = \frac{a-b}{n} = \frac{1-0}{n} = \frac{1}{n}$
- $x_i = a + i\,\delta x$
  $x_i = 0 + i\left(\frac{1}{n}\right)$
  $x_i = \frac{i}{n}$
- $f(x_i) = \frac{i^2}{n^2}$

NOW WE HAVE TO COMPUTE A LIMIT, AS $n \to \infty$

$$\int_0^1 x^2\,dx = \lim_{n\to\infty}\left[\sum_{i=1}^{n}\left[\underbrace{\left(\frac{i^2}{n^2}\right)}_{f(x_i)}\underbrace{\left(\frac{1}{n}\right)}_{\delta x}\right]\right] = \lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{i^2}{n^3}\right]$$

AS THE SUM HAS i DEPENDENCE ONLY, I.E n IS A CONSTANT AS FAR AS THE SUMMATION GOES, WE HAVE

$$\int_0^1 x^2\,dx = \lim_{n\to\infty}\left[\frac{1}{n^3}\sum_{i=1}^{n} i^2\right] = \lim_{n\to\infty}\left[\frac{1}{n^3}\times\underbrace{\frac{1}{6}n(n+1)(2n+1)}_{\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)}\right]$$

$$= \lim_{n\to\infty}\left[\frac{n(n+1)(2n+1)}{6n^3}\right] = \lim_{n\to\infty}\left[\frac{2n^3+3n^2+n}{6n^3}\right]$$

$$= \lim_{n\to\infty}\left[\frac{2+\frac{3}{n}+\frac{1}{n^2}}{6}\right] = \frac{2}{6} = \frac{1}{3}$$

$$\therefore \int_0^1 x^2\,dx = \frac{1}{3}$$

Question 43 (*****)

The Lambert W function, also called the omega function or product logarithm, is a multivalued function which has the property

$$W\left(xe^{x}\right) \equiv x,$$

and hence if $xe^{x} = y$ then $x = W(y)$.

For example

$$-xe^{-x} = 2 \;\Rightarrow\; -x = W(2), \quad (x+\pi)e^{x+\pi} = \tfrac{1}{2} \;\Rightarrow\; x+\pi = W\left(\tfrac{1}{2}\right) \text{ and so on.}$$

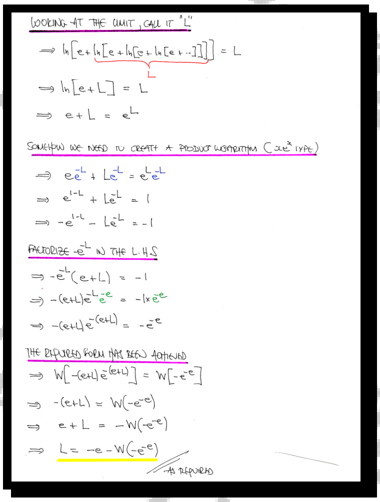
Use this result to show that the limit of

$$\ln\left(e + \ln\left(e + \ln\left(e + \ln\left(e + \ldots\right)\right)\right)\right)$$

is given by

$$-e - W\left[-e^{-e}\right].$$

$\boxed{}$, $\boxed{\text{proof}}$



LOOKING AT THE LIMIT, CALL IT "L"

$$\Rightarrow \ln\Big[e + \underbrace{\ln\big[e + \ln[e + \ln[e + \ldots]]\big]}_{L}\Big] = L$$

$$\Rightarrow \ln[e + L] = L$$

$$\Rightarrow e + L = e^{L}$$

SOMEHOW WE NEED TO CREATE A PRODUCT LOGARITHM (xe^{x} TYPE)

$$\Rightarrow ee^{-L} + Le^{-L} = e^{L}e^{-L}$$

$$\Rightarrow e^{1-L} + Le^{-L} = 1$$

$$\Rightarrow -e^{1-L} - Le^{-L} = -1$$

FACTORIZE $-e^{-L}$ IN THE L.H.S

$$\Rightarrow -e^{-L}(e + L) = -1$$

$$\Rightarrow -(e+L)e^{-L}e^{-e} = -1 \times e^{-e}$$

$$\Rightarrow -(e+L)e^{-(e+L)} = -e^{-e}$$

THE REQUIRED FORM HAS BEEN ACHIEVED

$$\Rightarrow W\left[-(e+L)e^{-(e+L)}\right] = W\left[-e^{-e}\right]$$

$$\Rightarrow -(e+L) = W(-e^{-e})$$

$$\Rightarrow e + L = -W(-e^{-e})$$

$$\Rightarrow L = -e - W(-e^{-e})$$

AS REQUIRED

Question 44 (*****)

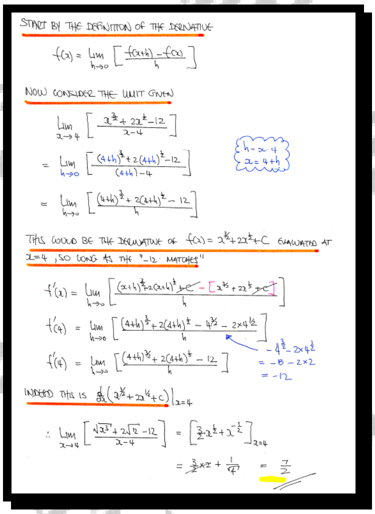
No credit will be given for using L'Hospital's rule in this question.

a) Use the formal definition of the derivative of a suitable expression, to find the value for the following limit

$$\lim_{x\to 4}\left[\frac{\sqrt{x^3}+2\sqrt{x}-12}{x-4}\right].$$

b) Verify the answer to part **(a)** by an alternative method.

$\boxed{}$, $\boxed{\frac{7}{2}}$



START BY THE DEFINITION OF THE DERIVATIVE

$$f(x) = \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]$$

NOW CONSIDER THE LIMIT GIVEN

$$\lim_{x\to 4}\left[\frac{x^{\frac{3}{2}}+2x^{\frac{1}{2}}-12}{x-4}\right]$$

$$= \lim_{h\to 0}\left[\frac{(4+h)^{\frac{3}{2}}+2(4+h)^{\frac{1}{2}}-12}{(4+h)-4}\right]$$

$$= \lim_{h\to 0}\left[\frac{(4+h)^{\frac{3}{2}}+2(4+h)^{\frac{1}{2}}-12}{h}\right]$$

$h = x-4$
$x = 4+h$

THIS WOULD BE THE DERIVATIVE OF $f(x) = x^{3/2}+2x^{1/2}+C$ EVALUATED AT $x=4$, SO LONG AS THE "-12 MATCHES"

$$f'(x) = \lim_{h\to 0}\left[\frac{(x+h)^{\frac{3}{2}}+2(x+h)^{\frac{1}{2}}\cancel{+C}-\left[x^{3/2}+2x^{1/2}\cancel{+C}\right]}{h}\right]$$

$$f'(4) = \lim_{h\to 0}\left[\frac{(4+h)^{\frac{3}{2}}+2(4+h)^{\frac{1}{2}}-4^{3/2}-2\times 4^{1/2}}{h}\right]$$

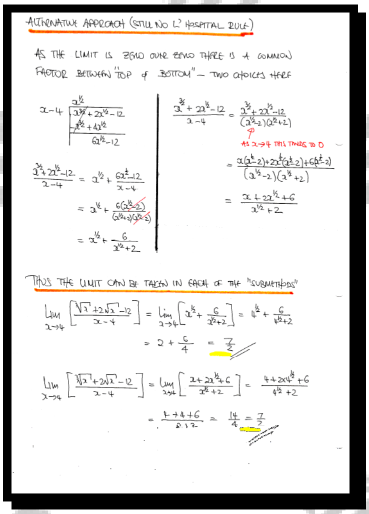
$-4^{\frac{3}{2}}-2\times 4^{\frac{1}{2}}$
$= -8-2\times 2$
$= -12$

$$f'(4) = \lim_{h\to 0}\left[\frac{(4+h)^{3/2}+2(4+h)^{\frac{1}{2}}-12}{h}\right]$$

INDEED THIS IS $\frac{d}{dx}\left(x^{3/2}+2x^{1/2}+C\right)\Big|_{x=4}$

$$\therefore \lim_{x\to 4}\left[\frac{\sqrt{x^3}+2\sqrt{x}-12}{x-4}\right] = \left[\frac{3}{2}x^{\frac{1}{2}}+x^{-\frac{1}{2}}\right]_{x=4}$$

$$= \cancel{\tfrac{3}{2}}\times\cancel{2} + \frac{1}{\sqrt{4}} = \frac{7}{2}$$



ALTERNATIVE APPROACH (STILL NO L' HOSPITAL RULE)

AS THE LIMIT IS ZERO OVER ZERO THERE IS A COMMON FACTOR BETWEEN "TOP & BOTTOM" – TWO CHOICES HERE

$$\begin{array}{r|l} & x^{1/2} \\ x-4 & \cancel{x^{3/2}}+2x^{1/2}-12 \\ & -\cancel{x^{3/2}}+4x^{1/2} \\ \hline & 6x^{1/2}-12 \end{array}$$

$$\frac{x^{3/2}+2x^{1/2}-12}{x-4} = x^{1/2}+\frac{6x^{\frac{1}{2}}-12}{x-4}$$

$$= x^{1/2}+\frac{6\cancel{(x^{1/2}-2)}}{(x^{1/2}+2)\cancel{(x^{1/2}-2)}}$$

$$= x^{1/2}+\frac{6}{x^{1/2}+2}$$

$$\frac{x^{3/2}+2x^{1/2}-12}{x-4} = \frac{x^{3/2}+2x^{1/2}-12}{(x^{1/2}-2)(x^{1/2}+2)}$$

AS $x\to 4$ THIS TENDS TO 0

$$= \frac{x(x^{\frac{1}{2}}-2)+2x^{\frac{1}{2}}(x^{\frac{1}{2}}-2)+6(x^{\frac{1}{2}}-2)}{(x^{1/2}-2)(x^{1/2}+2)}$$

$$= \frac{x+2x^{1/2}+6}{x^{1/2}+2}$$

THUS THE LIMIT CAN BE TAKEN IN EACH OF THE "SUBMETHODS"

$$\lim_{x\to 4}\left[\frac{\sqrt[3]{x}+2\sqrt{x}-12}{x-4}\right] = \lim_{x\to 4}\left[x^{1/2}+\frac{6}{x^{1/2}+2}\right] = 4^{1/2}+\frac{6}{4^{1/2}+2}$$

$$= 2+\frac{6}{4} = \frac{7}{2}$$

$$\lim_{x\to 4}\left[\frac{\sqrt[3]{x}+2\sqrt{x}-12}{x-4}\right] = \lim_{x\to 4}\left[\frac{x+2x^{1/2}+6}{x^{1/2}+2}\right] = \frac{4+2\times 4^{1/2}+6}{4^{1/2}+2}$$

$$= \frac{4+4+6}{2+2} = \frac{14}{4} = \frac{7}{2}$$

Question 45 (*****)

Use the formal definition of the derivative to prove that if

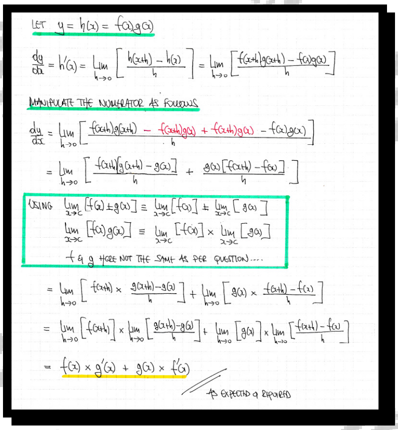
$$y = f(x)\,g(x),$$

then $\dfrac{dy}{dx} = f'(x)\,g(x) + f(x)\,g'(x)$

You may assume that

- $\lim\limits_{x\to c}\left[f(x)+g(x)\right] = \lim\limits_{x\to c}\left[f(x)\right] + \lim\limits_{x\to c}\left[g(x)\right]$
- $\lim\limits_{x\to c}\left[f(x)\times g(x)\right] = \lim\limits_{x\to c}\left[f(x)\right] \times \lim\limits_{x\to c}\left[g(x)\right]$

V , X , proof



LET $y = h(x) = f(x)g(x)$

$$\frac{dy}{dx} = h'(x) = \lim_{h\to 0}\left[\frac{h(x+h) - h(x)}{h}\right] = \lim_{h\to 0}\left[\frac{f(x+h)g(x+h) - f(x)g(x)}{h}\right]$$

MANIPULATE THE NUMERATOR AS FOLLOWS

$$\frac{dy}{dx} = \lim_{h\to 0}\left[\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{f(x+h)\left[g(x+h) - g(x)\right]}{h} + \frac{g(x)\left[f(x+h) - f(x)\right]}{h}\right]$$

USING $\lim\limits_{x\to c}\left[f(x) \pm g(x)\right] \equiv \lim\limits_{x\to c}\left[f(x)\right] \pm \lim\limits_{x\to c}\left[g(x)\right]$

$\lim\limits_{x\to c}\left[f(x)g(x)\right] \equiv \lim\limits_{x\to c}\left[f(x)\right] \times \lim\limits_{x\to c}\left[g(x)\right]$

f & g HERE NOT THE SAME AS PER QUESTION....

$$= \lim_{h\to 0}\left[f(x+h) \times \frac{g(x+h)-g(x)}{h}\right] + \lim_{h\to 0}\left[g(x) \times \frac{f(x+h) - f(x)}{h}\right]$$

$$= \lim_{h\to 0}\left[f(x+h)\right] \times \lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right] + \lim_{h\to 0}\left[g(x)\right] \times \lim_{h\to 0}\left[\frac{f(x+h) - f(x)}{h}\right]$$

$$= f(x) \times g'(x) + g(x) \times f'(x)$$

AS EXPECTED & REQUIRED

Question 46 (*****)

A curve has equation $y = f(x)$.

The finite region R is bounded by the curve, the x axis and the straight lines with equations $x = a$ and $x = b$, and hence the area of R is given by

$$I(a,b) = \int_a^b f(x)\ dx.$$

The area of R is also given by the limiting value of the sum of the areas of rectangles of width δx and height $f(x_i)$, known as a "right (upper) Riemann sum"

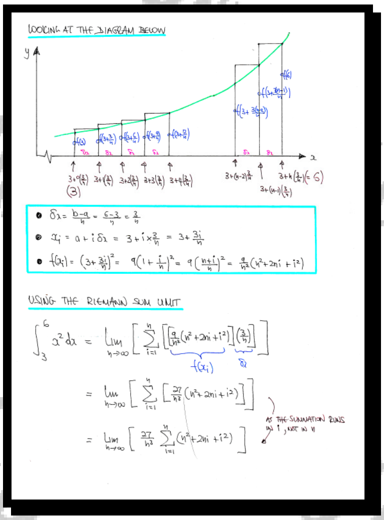
$$I(a,b) = \lim_{n\to\infty}\left[\sum_{i=1}^{n}\left[f(x_i)\,\delta x\right]\right],$$

where $\delta x = \dfrac{b-a}{n}$ and $x_i = a + i\,\delta x$.

Using the "right (upper) Riemann sum" definition, and with the aid of a diagram where appropriate, show clearly that

$$\int_3^6 x^2\ dx = 63.$$

$\boxed{}$, $\boxed{\text{proof}}$



LOOKING AT THE DIAGRAM BELOW

- $\delta x = \frac{b-a}{n} = \frac{6-3}{n} = \frac{3}{n}$
- $x_i = a + i\delta x = 3 + i \times \frac{3}{n} = 3 + \frac{3i}{n}$
- $f(x_i) = \left(3 + \frac{3i}{n}\right)^2 = 9\left(1 + \frac{i}{n}\right)^2 = 9\left(\frac{n+i}{n}\right)^2 = \frac{9}{n^2}(n^2 + 2ni + i^2)$

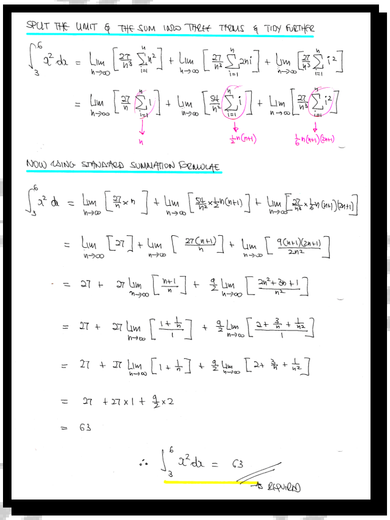
USING THE RIEMANN SUM LIMIT

$$\int_3^6 x^2\,dx = \lim_{n\to\infty}\left[\sum_{i=1}^{n}\left[\underbrace{\left[\frac{9}{n^2}(n^2+2ni+i^2)\right]}_{f(x_i)}\underbrace{\left(\frac{3}{n}\right)}_{\delta x}\right]\right]$$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n}\left[\frac{27}{n^3}(n^2+2ni+i^2)\right]\right]$$

$$= \lim_{n\to\infty}\left[\frac{27}{n^3}\sum_{i=1}^{n}(n^2+2ni+i^2)\right]$$

AS THE SUMMATION RUNS IN i, NOT IN n



SPLIT THE LIMIT & THE SUM INTO THREE TERMS & TIDY FURTHER

$$\int_3^6 x^2\,dx = \lim_{n\to\infty}\left[\frac{27}{n^3}\sum_{i=1}^{n}n^2\right] + \lim_{n\to\infty}\left[\frac{27}{n^3}\sum_{i=1}^{n}2ni\right] + \lim_{n\to\infty}\left[\frac{27}{n^3}\sum_{i=1}^{n}i^2\right]$$

$$= \lim_{n\to\infty}\left[\frac{27}{n}\underbrace{\sum_{i=1}^{n}1}_{n}\right] + \lim_{n\to\infty}\left[\frac{54}{n^2}\underbrace{\sum_{i=1}^{n}i}_{\frac{1}{2}n(n+1)}\right] + \lim_{n\to\infty}\left[\frac{27}{n^3}\underbrace{\sum_{i=1}^{n}i^2}_{\frac{1}{6}n(n+1)(2n+1)}\right]$$

NOW USING STANDARD SUMMATION FORMULAE

$$\int_3^6 x^2\,dx = \lim_{n\to\infty}\left[\frac{27}{n}\times n\right] + \lim_{n\to\infty}\left[\frac{54}{n^2}\times\frac{1}{2}n(n+1)\right] + \lim_{n\to\infty}\left[\frac{27}{n^3}\times\frac{1}{6}n(n+1)(2n+1)\right]$$

$$= \lim_{n\to\infty}[27] + \lim_{n\to\infty}\left[\frac{27(n+1)}{n}\right] + \lim_{n\to\infty}\left[\frac{9(n+1)(2n+1)}{2n^2}\right]$$

$$= 27 + 27\lim_{n\to\infty}\left[\frac{n+1}{n}\right] + \frac{9}{2}\lim_{n\to\infty}\left[\frac{2n^2+3n+1}{n^2}\right]$$

$$= 27 + 27\lim_{n\to\infty}\left[\frac{1+\frac{1}{n}}{1}\right] + \frac{9}{2}\lim_{n\to\infty}\left[\frac{2+\frac{3}{n}+\frac{1}{n^2}}{1}\right]$$

$$= 27 + 27\lim_{n\to\infty}\left[1+\frac{1}{n}\right] + \frac{9}{2}\lim_{n\to\infty}\left[2+\frac{3}{n}+\frac{1}{n^2}\right]$$

$$= 27 + 27\times 1 + \frac{9}{2}\times 2$$

$$= 63$$

$$\therefore \int_3^6 x^2\,dx = 63$$

AS REQUIRED

Question 47 (*****)

A curve has equation $y = f(x)$.

The finite region R is bounded by the curve, the x axis and the straight lines with equations $x = a$ and $x = b$, and hence the area of R is given by

$$I(a,b) = \int_a^b f(x)\ dx .$$

The area of R is also given by the limiting value of the sum of the areas of rectangles of width δx and height $f(x_i)$, known as a “right (upper) Riemann sum”

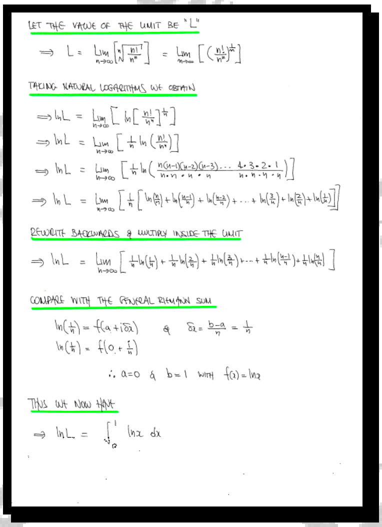
$$I(a,b) = \lim_{n\to\infty}\left[\sum_{i=1}^{n}\left[f(x_i)\,\delta x\right]\right],$$

where $\delta x = \dfrac{b-a}{n}$ and $x_i = a + i\,\delta x$.

Using the “right (upper) Riemann sum” definition, and with the aid of a diagram where appropriate, show clearly that

$$\lim_{n\to\infty}\left[\sqrt[n]{\frac{n!}{n^n}}\right] = \frac{1}{\mathrm{e}}.$$

☐, $\boxed{\text{proof}}$



LET THE VALUE OF THE LIMIT BE "L"

$$\Rightarrow L = \lim_{n\to\infty}\left[\sqrt[n]{\frac{n!}{n^n}}\right] = \lim_{n\to\infty}\left[\left(\frac{n!}{n^n}\right)^{\frac{1}{n}}\right]$$

TAKING NATURAL LOGARITHMS WE OBTAIN

$$\Rightarrow \ln L = \lim_{n\to\infty}\left[\ln\left[\frac{n!}{n^n}\right]^{\frac{1}{n}}\right]$$

$$\Rightarrow \ln L = \lim_{n\to\infty}\left[\frac{1}{n}\ln\left(\frac{n!}{n^n}\right)\right]$$

$$\Rightarrow \ln L = \lim_{n\to\infty}\left[\frac{1}{n}\ln\left(\frac{n(n-1)(n-2)(n-3)\dots}{n\cdot n\cdot n\cdot n}\ \frac{4\cdot3\cdot2\cdot1}{n\cdot n\cdot n\cdot n}\right)\right]$$

$$\Rightarrow \ln L = \lim_{n\to\infty}\left[\frac{1}{n}\left[\ln\left(\frac{n}{n}\right)+\ln\left(\frac{n-1}{n}\right)+\ln\left(\frac{n-2}{n}\right)+\dots+\ln\left(\frac{3}{n}\right)+\ln\left(\frac{2}{n}\right)+\ln\left(\frac{1}{n}\right)\right]\right]$$

REWRITE BACKWARDS & MULTIPLY INSIDE THE LIMIT

$$\Rightarrow \ln L = \lim_{n\to\infty}\left[\frac{1}{n}\ln\left(\frac{1}{n}\right)+\frac{1}{n}\ln\left(\frac{2}{n}\right)+\frac{1}{n}\ln\left(\frac{3}{n}\right)+\dots+\frac{1}{n}\ln\left(\frac{n-1}{n}\right)+\frac{1}{n}\ln\left(\frac{n}{n}\right)\right]$$

COMPARE WITH THE GENERAL RIEMANN SUM

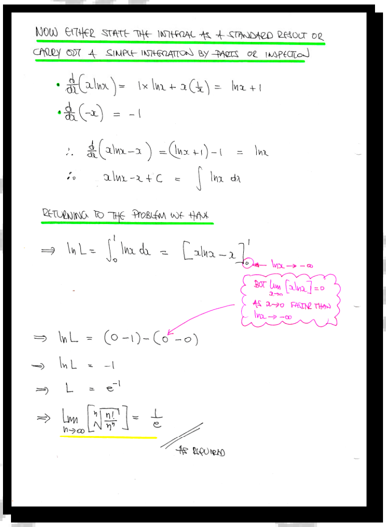
$$\ln\left(\frac{i}{n}\right) = f(a+i\delta x) \qquad \& \qquad \delta x = \frac{b-a}{n} = \frac{1}{n}$$

$$\ln\left(\frac{i}{n}\right) = f\left(0+\frac{i}{n}\right)$$

$$\therefore a=0 \ \& \ b=1 \text{ WITH } f(x) = \ln x$$

THUS WE NOW HAVE

$$\Rightarrow \ln L = \int_0^1 \ln x\ dx$$



NOW EITHER STATE THE INTEGRAL AS A STANDARD RESULT OR CARRY OUT A SIMPLE INTEGRATION BY PARTS OR INSPECTION

- $\frac{d}{dx}(x\ln x) = 1\times\ln x + x\left(\frac{1}{x}\right) = \ln x + 1$
- $\frac{d}{dx}(-x) = -1$

$$\therefore \frac{d}{dx}(x\ln x - x) = (\ln x + 1) - 1 = \ln x$$

$$\therefore x\ln x - x + C = \int \ln x\ dx$$

RETURNING TO THE PROBLEM WE HAVE

$$\Rightarrow \ln L = \int_0^1 \ln x\ dx = \left[x\ln x - x\right]_0^1$$

$\ln x \to -\infty$

BUT $\lim_{x\to0}\left[x\ln x\right] = 0$ AS $x\to0$ FASTER THAN $\ln x\to-\infty$

$$\Rightarrow \ln L = (0-1)-(0-0)$$

$$\Rightarrow \ln L = -1$$

$$\Rightarrow L = e^{-1}$$

$$\Rightarrow \lim_{n\to\infty}\left[\sqrt[n]{\frac{n!}{n^n}}\right] = \frac{1}{e}$$

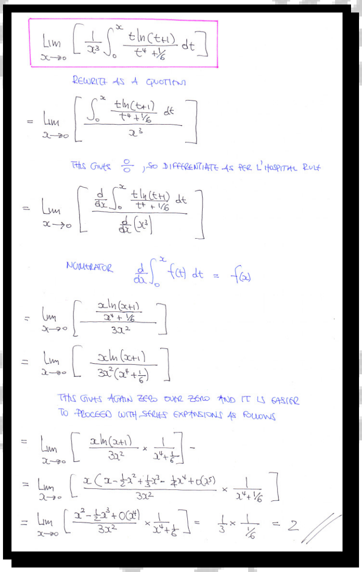
AS REQUIRED

Question 48 (*****)

Use Leibniz rule and standard series expansions to evaluate the following limit

$$\lim_{x \to 0}\left[\frac{1}{x^3}\int_0^x \frac{t\ln(t+1)}{t^4+\frac{1}{6}}\ dt\right].$$

$\boxed{}$, $\boxed{2}$



$$\boxed{\lim_{x \to 0}\left[\frac{1}{x^3}\int_0^x \frac{t\ln(t+1)}{t^4+1/6}\,dt\right]}$$

REWRITE AS A QUOTIENT

$$= \lim_{x \to 0}\left[\frac{\int_0^x \frac{t\ln(t+1)}{t^4+1/6}\,dt}{x^3}\right]$$

THIS GIVES $\frac{0}{0}$, SO DIFFERENTIATE AS PER L'HOSPITAL RULE

$$= \lim_{x \to 0}\left[\frac{\frac{d}{dx}\int_0^x \frac{t\ln(t+1)}{t^4+1/6}\,dt}{\frac{d}{dx}(x^3)}\right]$$

NUMERATOR $\frac{d}{dx}\int_0^x f(t)\,dt = f(x)$

$$= \lim_{x \to 0}\left[\frac{\frac{x\ln(x+1)}{x^4+1/6}}{3x^2}\right]$$

$$= \lim_{x \to 0}\left[\frac{x\ln(x+1)}{3x^2\left(x^4+\frac{1}{6}\right)}\right]$$

THIS GIVES AGAIN ZERO OVER ZERO AND IT IS EASIER TO PROCEED WITH SERIES EXPANSIONS AS FOLLOWS

$$= \lim_{x \to 0}\left[\frac{x\ln(x+1)}{3x^2}\times\frac{1}{x^4+\frac{1}{6}}\right] -$$

$$= \lim_{x \to 0}\left[\frac{x\left(x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+O(x^5)\right.}{3x^2}\times\frac{1}{x^4+1/6}\right]$$

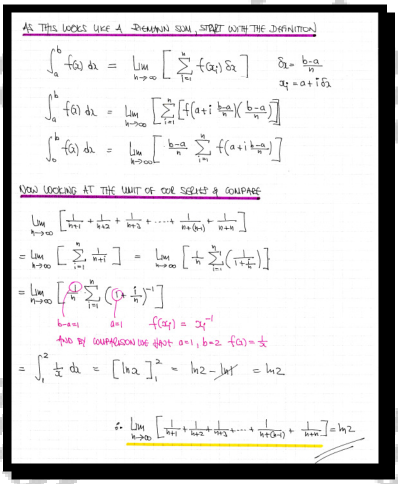
$$= \lim_{x \to 0}\left[\frac{x^2-\frac{1}{2}x^3+O(x^4)}{3x^2}\times\frac{1}{x^4+\frac{1}{6}}\right] = \frac{1}{3}\times\frac{1}{1/6} = 2$$

Question 49 (*****)

Determine the limit of the following series.

$$\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\frac{1}{n+4}+\ldots+\frac{1}{n+n-2}+\frac{1}{n+n-1}+\frac{1}{n+n}+\right].$$

$\boxed{\text{V}}$, $\boxed{\text{S X V}}$, $\boxed{\ln 2}$



AS THIS LOOKS LIKE A RIEMANN SUM, START WITH THE DEFINITION

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left[\sum_{i=1}^{n} f(x_i)\,\delta x\right] \qquad \delta x = \frac{b-a}{n}$$

$$x_i = a + i\delta x$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left[\sum_{i=1}^{n}\left[f\left(a+i\frac{b-a}{n}\right)\left(\frac{b-a}{n}\right)\right]\right]$$

$$\int_0^b f(x)\,dx = \lim_{n\to\infty}\left[\frac{b-a}{n}\sum_{i=1}^{n} f\left(a+i\frac{b-a}{n}\right)\right]$$

NOW LOOKING AT THE LIMIT OF OUR SERIES & COMPARE

$$\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\dots+\frac{1}{n+(n-1)}+\frac{1}{n+n}\right]$$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{1}{n+i}\right] = \lim_{n\to\infty}\left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{1+\frac{i}{n}}\right)\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}\sum_{i=1}^{n}\left(1+\frac{i}{n}\right)^{-1}\right]$$

$b-a=1 \qquad a=1 \qquad f(x_i) = x_i^{-1}$

AND BY COMPARISON WE HAVE $a=1,\ b=2\ f(x)=\frac{1}{x}$

$$= \int_1^2 \frac{1}{x}\,dx = \left[\ln x\right]_1^2 = \ln 2 - \cancel{\ln 1} = \ln 2$$

$$\therefore \lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\dots+\frac{1}{n+(n-1)}+\frac{1}{n+n}\right] = \ln 2$$

Question 50 (*****)

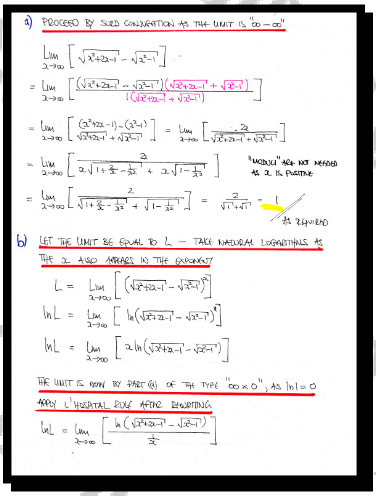
a) Show with detailed workings that

$$\lim_{x\to\infty}\left[\sqrt{x^2+2x-1}-\sqrt{x^2-1}\right] = 1.$$

b) Hence determine in exact simplified form the value of

$$\lim_{x\to\infty}\left[\left(\sqrt{x^2+2x-1}-\sqrt{x^2-1}\right)^x\right].$$

$\boxed{\text{SPX}}$, $\boxed{e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}}$



a) PROCEED BY SURD CONJUGATION AS THE LIMIT IS "$\infty - \infty$"

$$\lim_{x\to\infty}\left[\sqrt{x^2+2x-1}-\sqrt{x^2-1}\right]$$

$$= \lim_{x\to\infty}\left[\frac{\left(\sqrt{x^2+2x-1}-\sqrt{x^2-1}\right)\left(\sqrt{x^2+2x-1}+\sqrt{x^2-1}\right)}{1\left(\sqrt{x^2+2x-1}+\sqrt{x^2-1}\right)}\right]$$

$$= \lim_{x\to\infty}\left[\frac{(x^2+2x-1)-(x^2-1)}{\sqrt{x^2+2x-1}+\sqrt{x^2-1}}\right] = \lim_{x\to\infty}\left[\frac{2x}{\sqrt{x^2+2x-1}+\sqrt{x^2-1}}\right]$$

$$= \lim_{x\to\infty}\left[\frac{2x}{x\sqrt{1+\frac{2}{x}-\frac{1}{x^2}}+x\sqrt{1-\frac{1}{x^2}}}\right]$$ "MODULI" ARE NOT NEEDED AS x IS POSITIVE

$$= \lim_{x\to\infty}\left[\frac{2}{\sqrt{1+\frac{2}{x}-\frac{1}{x^2}}+\sqrt{1-\frac{1}{x^2}}}\right] = \frac{2}{\sqrt{1}+\sqrt{1}} = 1$$

AS REQUIRED

b) LET THE LIMIT BE EQUAL TO L – TAKE NATURAL LOGARITHMS AS THE x ALSO APPEARS IN THE EXPONENT

$$L = \lim_{x\to\infty}\left[\left(\sqrt{x^2+2x-1}-\sqrt{x^2-1}\right)^x\right]$$

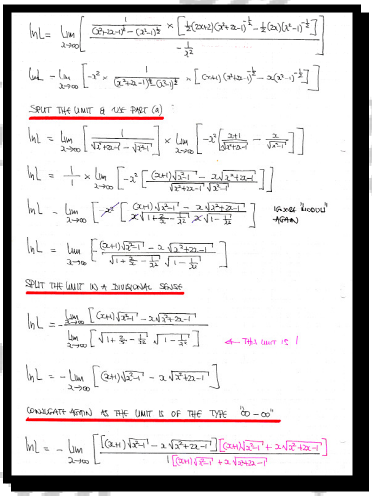
$$\ln L = \lim_{x\to\infty}\left[\ln\left(\sqrt{x^2+2x-1}-\sqrt{x^2-1}\right)^x\right]$$

$$\ln L = \lim_{x\to\infty}\left[x\ln\left(\sqrt{x^2+2x-1}-\sqrt{x^2-1}\right)\right]$$

THE LIMIT IS NOW BY PART (a) OF THE TYPE "$\infty \times 0$", AS $\ln 1 = 0$

APPLY L'HOSPITAL RULE AFTER REWRITING

$$\ln L = \lim_{x\to\infty}\left[\frac{\ln\left(\sqrt{x^2+2x-1}-\sqrt{x^2-1}\right)}{\frac{1}{x}}\right]$$



$$\ln L = \lim_{x\to\infty}\left[\frac{\frac{1}{(x^2+2x-1)^{\frac{1}{2}}-(x^2-1)^{\frac{1}{2}}}\times\left[\frac{1}{2}(2x+2)(x^2+2x-1)^{-\frac{1}{2}}-\frac{1}{2}(2x)(x^2-1)^{-\frac{1}{2}}\right]}{-\frac{1}{x^2}}\right]$$

$$\ln L = \lim_{x\to\infty}\left[-x^2\times\frac{1}{(x^2+2x-1)^{\frac{1}{2}}-(x^2-1)^{\frac{1}{2}}}\times\left[(x+1)(x^2+2x-1)^{-\frac{1}{2}}-x(x^2-1)^{-\frac{1}{2}}\right]\right]$$

SPLIT THE LIMIT & USE PART (a)

$$\ln L = \lim_{x\to\infty}\left[\frac{1}{\sqrt{x^2+2x-1}-\sqrt{x^2-1}}\right]\times\lim_{x\to\infty}\left[-x^2\left[\frac{x+1}{\sqrt{x^2+2x-1}}-\frac{x}{\sqrt{x^2-1}}\right]\right]$$

$$\ln L = \frac{1}{1}\times\lim_{x\to\infty}\left[-x^2\left[\frac{(x+1)\sqrt{x^2-1}-x\sqrt{x^2+2x-1}}{\sqrt{x^2+2x-1}\sqrt{x^2-1}}\right]\right]$$

$$\ln L = \lim_{x\to\infty}\left[-\cancel{x^2}\left[\frac{(x+1)\sqrt{x^2-1}-x\sqrt{x^2+2x-1}}{\cancel{x}\sqrt{1+\frac{2}{x}-\frac{1}{x^2}}\,\cancel{x}\sqrt{1-\frac{1}{x^2}}}\right]\right]$$ IGNORE "MODULI" AGAIN

$$\ln L = \lim_{x\to\infty}\left[-\frac{(x+1)\sqrt{x^2-1}-x\sqrt{x^2+2x-1}}{\sqrt{1+\frac{2}{x}-\frac{1}{x^2}}\sqrt{1-\frac{1}{x^2}}}\right]$$

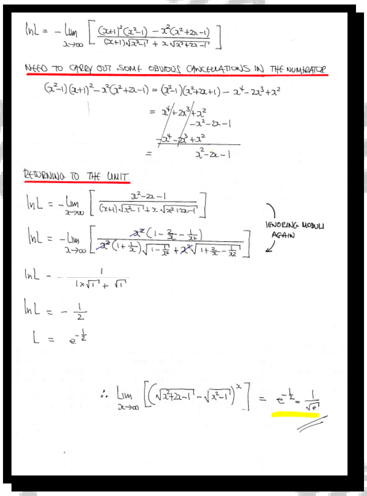
SPLIT THE LIMIT IN A DIVISIONAL SENSE

$$\ln L = -\frac{\lim_{x\to\infty}\left[(x+1)\sqrt{x^2-1}-x\sqrt{x^2+2x-1}\right]}{\lim_{x\to\infty}\left[\sqrt{1+\frac{2}{x}-\frac{1}{x^2}}\sqrt{1-\frac{1}{x^2}}\right]}$$ ← THIS LIMIT IS 1

$$\ln L = -\lim_{x\to\infty}\left[(x+1)\sqrt{x^2-1}-x\sqrt{x^2+2x-1}\right]$$

CONJUGATE AGAIN AS THE LIMIT IS OF THE TYPE "$\infty - \infty$"

$$\ln L = -\lim_{x\to\infty}\left[\frac{\left[(x+1)\sqrt{x^2-1}-x\sqrt{x^2+2x-1}\right]\left[(x+1)\sqrt{x^2-1}+x\sqrt{x^2+2x-1}\right]}{1\left[(x+1)\sqrt{x^2-1}+x\sqrt{x^2+2x-1}\right]}\right]$$



$$\ln L = -\lim_{x\to\infty}\left[\frac{(x+1)^2(x^2-1)-x^2(x^2+2x-1)}{(x+1)\sqrt{x^2-1}+x\sqrt{x^2+2x-1}}\right]$$

NEED TO CARRY OUT SOME OBVIOUS CANCELLATIONS IN THE NUMERATOR

$$(x^2-1)(x+1)^2-x^2(x^2+2x-1) = (x^2-1)(x^2+2x+1)-x^4-2x^3+x^2$$

$$= \cancel{x^4}+\cancel{2x^3}+x^2 - x^2-2x-1 \cancel{-x^4}\cancel{-2x^3}+x^2$$

$$= x^2-2x-1$$

RETURNING TO THE LIMIT

$$\ln L = -\lim_{x\to\infty}\left[\frac{x^2-2x-1}{(x+1)\sqrt{x^2-1}+x\sqrt{x^2+2x-1}}\right]$$

$$\ln L = -\lim_{x\to\infty}\left[\frac{\cancel{x^2}\left(1-\frac{2}{x}-\frac{1}{x^2}\right)}{\cancel{x^2}\left(1+\frac{1}{x}\right)\sqrt{1-\frac{1}{x^2}}+\cancel{x^2}\sqrt{1+\frac{2}{x}-\frac{1}{x^2}}}\right]$$ IGNORING MODULI AGAIN

$$\ln L = -\frac{1}{1\times\sqrt{1}+\sqrt{1}}$$

$$\ln L = -\frac{1}{2}$$

$$L = e^{-\frac{1}{2}}$$

$$\therefore \lim_{x\to\infty}\left[\left(\sqrt{x^2+2x-1}-\sqrt{x^2-1}\right)^x\right] = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$$