LINE INTEGRALS

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IN 2 DIMENSIONAL CARTESIAN COORDINATES

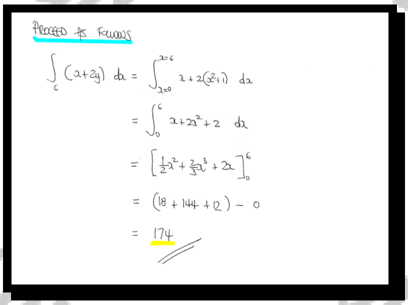
Question 1

Evaluate the integral

$$\int_C (x+2y)\, dx\,,$$

where C is the path along the curve with equation $y = x^2 + 1$, from $(0,1)$ to $(6,37)$.

$\boxed{174}$



Question 2

It is given that the vector function $\mathbf{F}$ satisfies

$$\mathbf{F} = \left(x^2 - y^2\right)\mathbf{i} + (2xy)\,\mathbf{j}\,.$$

Evaluate the line integral

$$\int_{(-2,-1)}^{(4,2)} \mathbf{F} \cdot \mathbf{dr}\,,$$

along a path joining directly the points with Cartesian coordinates $(-2,-1)$ and $(4,2)$.

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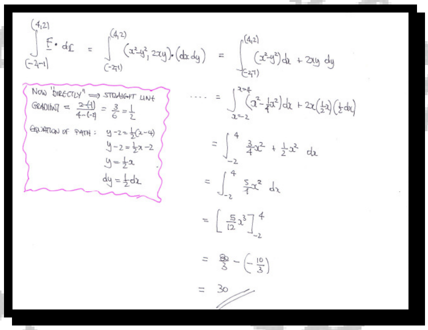


Question 3

The path along the straight line with equation $y = x+2$, from $A(0,2)$ to $B(3,5)$, is denoted by C.

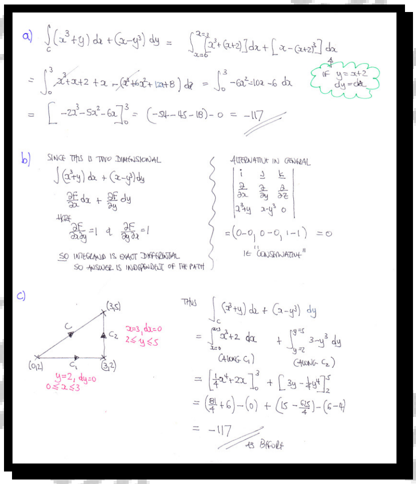
a) Evaluate the integral

$$\int_C \left(x^3 + y\right)\,dx + \left(x - y^3\right)\,dy\,.$$

b) Show that the integral is independent of the path chosen from A to B.

c) Verify the independence of the path by evaluating the integral of part **(a)** along a different path from A to B.

$\boxed{-117}$



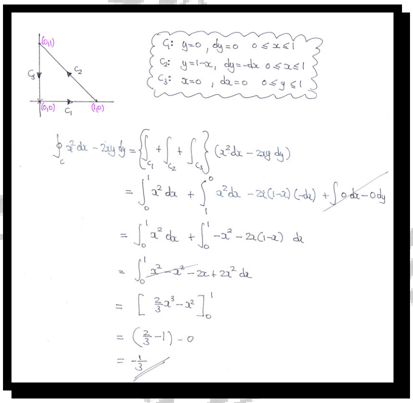
Question 4

The path along the perimeter of the triangle with vertices at $(0,0)$, $(1,0)$ and $(0,1)$, is denoted by C.

Evaluate the integral

$$\oint_C x^2 \ dx - 2xy \ dy .$$

$\boxed{-\frac{1}{3}}$



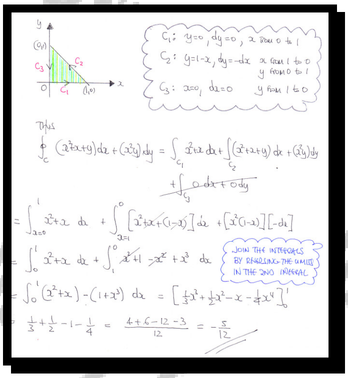
Question 5

The path along the perimeter of the triangle with vertices at $(0,0)$, $(1,0)$ and $(0,1)$, is denoted by C.

Evaluate the integral

$$\oint_C \left(x^2+x+y\right)\,dx+\left(x^2y\right)\,dy .$$

$$\boxed{-\frac{5}{12}}$$



Question 6

The functions F and G are defined as

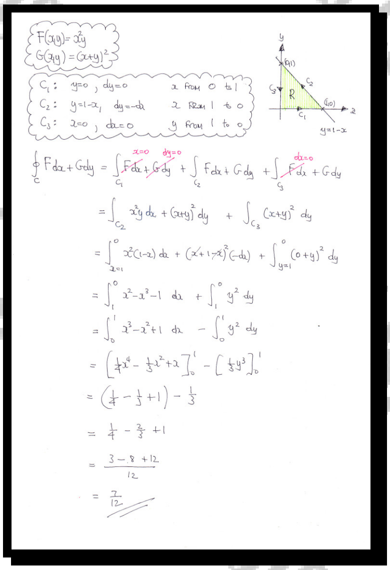
$$F(x,y) = x^2 y \quad \text{and} \quad G(x,y) = (x+y)^2$$

The anticlockwise path along the perimeter of the triangle whose vertices are located at $(0,0)$, $(1,0)$ and $(0,1)$, is denoted by C.

Evaluate the line integral

$$\int_C F\,dx + G\,dy .$$

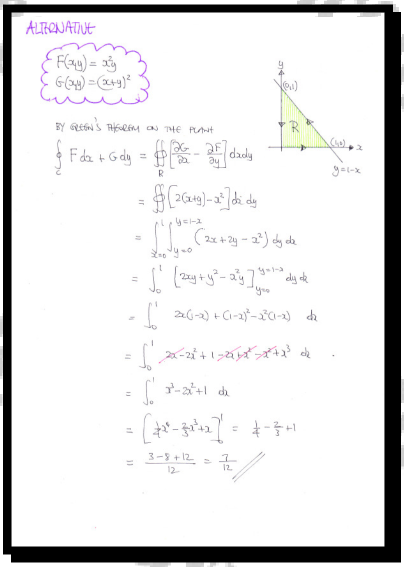
$\boxed{\frac{7}{12}}$



$F(x,y) = x^2 y$
$G(x,y) = (x+y)^2$

C_1: $y=0$, $dy=0$, x from 0 to 1
C_2: $y=1-x$, $dy=-dx$, x from 1 to 0
C_3: $x=0$, $dx=0$, y from 1 to 0

$$\oint_C F\,dx + G\,dy = \int_{C_1} F\,dx + G\,dy + \int_{C_2} F\,dx + G\,dy + \int_{C_3} F\,dx + G\,dy$$
$$= \int_{C_2} x^2 y\,dx + (x+y)^2\,dy + \int_{C_3} (x+y)^2\,dy$$
$$= \int_{x=1}^{0} x^2(1-x)\,dx + (x+1-x)^2(-dx) + \int_{y=1}^{0} (0+y)^2\,dy$$
$$= \int_1^0 x^2 - x^3 - 1\,dx + \int_1^0 y^2\,dy$$
$$= \int_0^1 x^3 - x^2 + 1\,dx - \int_0^1 y^2\,dy$$
$$= \left[\tfrac{1}{4}x^4 - \tfrac{1}{3}x^3 + x\right]_0^1 - \left[\tfrac{1}{3}y^3\right]_0^1$$
$$= \left(\tfrac{1}{4} - \tfrac{1}{3} + 1\right) - \tfrac{1}{3}$$
$$= \tfrac{1}{4} - \tfrac{2}{3} + 1$$
$$= \frac{3-8+12}{12}$$
$$= \frac{7}{12}$$



ALTERNATIVE

$F(x,y) = x^2 y$
$G(x,y) = (x+y)^2$

BY GREEN'S THEOREM ON THE PLANE

$$\oint_C F\,dx + G\,dy = \iint_R \left[\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\right] dx\,dy$$
$$= \iint \left[2(x+y) - x^2\right] dx\,dy$$
$$= \int_{x=0}^{1}\int_{y=0}^{y=1-x} \left(2x + 2y - x^2\right) dy\,dx$$
$$= \int_0^1 \left[2xy + y^2 - x^2 y\right]_{y=0}^{y=1-x} dx$$
$$= \int_0^1 2x(1-x) + (1-x)^2 - x^2(1-x)\,dx$$
$$= \int_0^1 2x - 2x^2 + 1 - 2x + x^2 - x^2 + x^3\,dx$$
$$= \int_0^1 x^3 - 2x^2 + 1\,dx$$
$$= \left[\tfrac{1}{4}x^4 - \tfrac{2}{3}x^3 + x\right]_0^1 = \tfrac{1}{4} - \tfrac{2}{3} + 1$$
$$= \frac{3-8+12}{12} = \frac{7}{12}$$

Question 7

The anticlockwise path along the perimeter of the square whose vertices are located at the points $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$, is denoted by C.

Evaluate the line integral

$$\int_C \left(x^2 + xy\right)dx + \left(x+y\right)^3 dy \,.$$

You may not use Green's theorem in this question.

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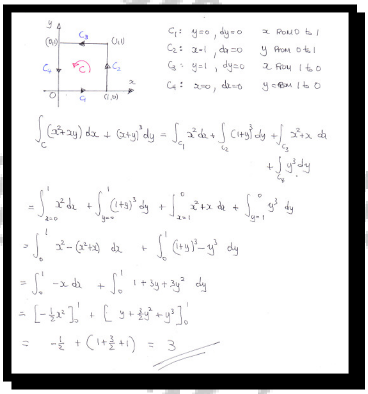


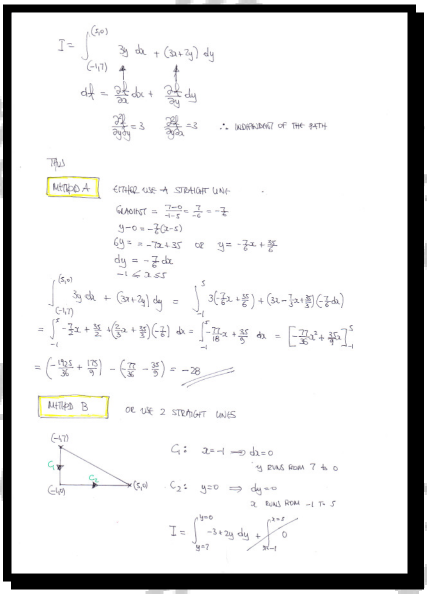
Question 8

Evaluate the integral

$$\int_{(-1,7)}^{(5,0)} (3y)\,dx + (3x+2y)\,dy\,,$$

along a path joining the points with Cartesian coordinates $(-1,7)$ and $(5,0)$.

$\boxed{-28}$



$I = \int_{(-1,7)}^{(5,0)} 3y\,dx + (3x+2y)\,dy$

$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$

$\frac{\partial^2 f}{\partial y \partial y} = 3 \quad \frac{\partial^2 f}{\partial y \partial x} = 3 \quad \therefore$ INDEPENDENT OF THE PATH

THUS

METHOD A

EITHER USE A STRAIGHT LINE

GRADIENT $= \frac{7-0}{-1-5} = \frac{7}{-6} = -\frac{7}{6}$

$y - 0 = -\frac{7}{6}(x-5)$

$6y = -7x + 35$ OR $y = -\frac{7}{6}x + \frac{35}{6}$

$dy = -\frac{7}{6}dx$

$-1 \leqslant x \leqslant 5$

$\int_{(-1,7)}^{(5,0)} 3y\,dx + (3x+2y)\,dy = \int_{-1}^{5} 3\left(-\frac{7}{6}x + \frac{35}{6}\right) + \left(3x - \frac{7}{3}x + \frac{35}{3}\right)\left(-\frac{7}{6}dx\right)$

$= \int_{-1}^{5} -\frac{7}{2}x + \frac{35}{2} + \left(\frac{2}{3}x + \frac{35}{3}\right)\left(-\frac{7}{6}\right)dx = \int_{-1}^{5} -\frac{77}{18}x + \frac{35}{9}\,dx = \left[-\frac{77}{36}x^2 + \frac{35}{9}x\right]_{-1}^{5}$

$= \left(-\frac{1925}{36} + \frac{175}{9}\right) - \left(-\frac{77}{36} - \frac{35}{9}\right) = -28$

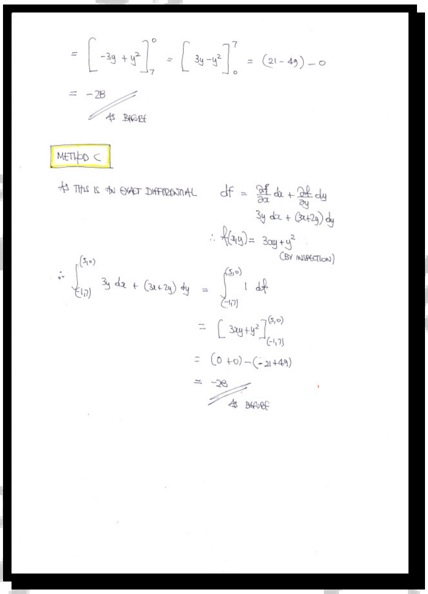
METHOD B

OR USE 2 STRAIGHT LINES

C_1: $x = -1 \Rightarrow dx = 0$, y RUNS FROM 7 to 0

C_2: $y = 0 \Rightarrow dy = 0$, x RUNS FROM -1 TO 5

$I = \int_{y=7}^{y=0} -3 + 2y\,dy + \int_{x=-1}^{x=5} 0$



$= \left[-3y + y^2\right]_7^0 = \left[3y - y^2\right]_0^7 = (21-49) - 0$

$= -28$ AS BEFORE

METHOD C

AS THIS IS AN EXACT DIFFERENTIAL $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$

$3y\,dx + (3x+2y)\,dy$

$\therefore f(x,y) = 3xy + y^2$ (BY INSPECTION)

$\therefore \int_{(-1,7)}^{(5,0)} 3y\,dx + (3x+2y)\,dy = \int_{(-1,7)}^{(5,0)} 1\,df$

$= \left[3xy + y^2\right]_{(-1,7)}^{(5,0)}$

$= (0+0) - (-21+49)$

$= -28$ AS BEFORE

Question 9

Evaluate the integral

$$\int_{(1,1)}^{(3,4)} \left(3x^2y^2\right)\,dx + \left(2x^3y\right)\,dy,$$

along a path joining the points with Cartesian coordinates $(1,1)$ and $(3,4)$.

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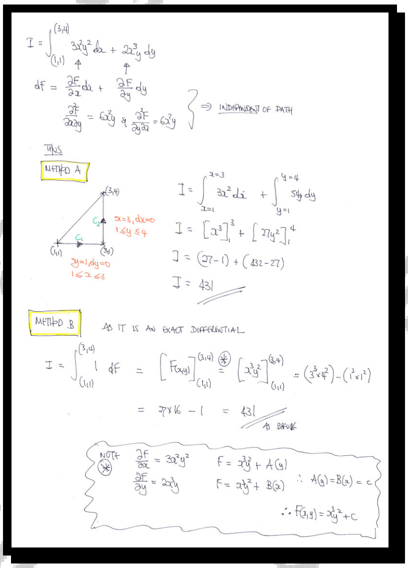

Question 10

$$\mathbf{F}(x, y) \equiv \left(2xy^2 + \cos x\right)\mathbf{i} + \left(2x^2 y - \sin y\right)\mathbf{j}.$$

Show that the vector field $\mathbf{F}$ is conservative, and hence evaluate the integral

$$\int_C \mathbf{F} \cdot \mathbf{dr},$$

where C is the arc of the circle with equation

$$x^2 + y^2 = \frac{\pi^2}{4}, \quad y \geq 0,$$

from $A\left(-\frac{\pi}{2}, 0\right)$ to $B\left(\frac{\pi}{2}, 0\right)$.

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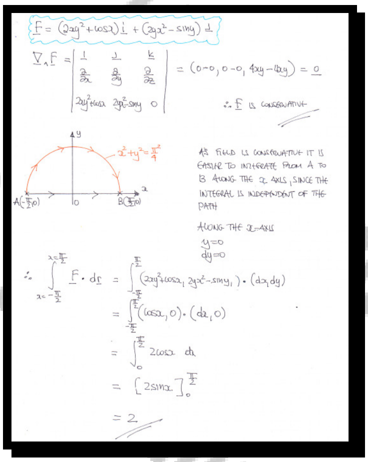


Question 11

In this question α, β *and* γ *are positive constants.*

$$\mathbf{F} = -\alpha y^2 \mathbf{e_x} + \beta x^2 \mathbf{e_y}.$$

A particle of mass m is moving on the x-y plane, under the action of $\mathbf{F}$.

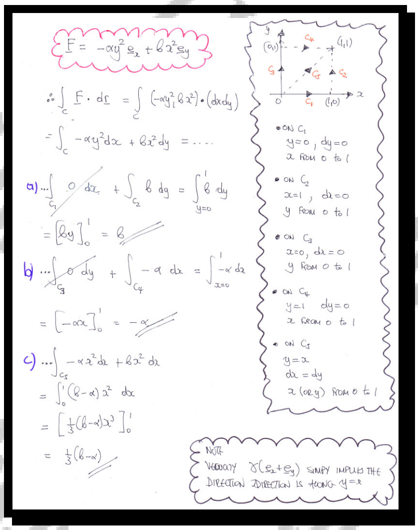
Find the work done by $\mathbf{F}$ on the particle in moving it from the Cartesian origin O to the point $(1,1)$, in each of the following cases.

a) Directly from O to $(1,0)$, then directly from $(1,0)$ to $(1,1)$.

b) Directly from O to $(0,1)$, then directly from $(0,1)$ to $(1,1)$.

c) Moving the particle with velocity $\mathbf{v} = \gamma(\mathbf{e_x} + \mathbf{e_y})$.

$\boxed{W_1 = \beta}$, $\boxed{W_2 = -\alpha}$, $\boxed{W_3 = \frac{1}{3}(\beta - \alpha)}$



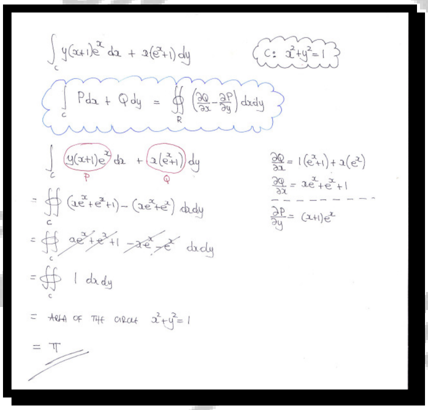
Question 12

Evaluate the line integral

$$\oint_C \left[y(x+1)e^x \, dx \;+\; x\left(e^x+1\right) \, dy \right],$$

where C is a circle of radius 1, centre at the origin O, traced anticlockwise.

$\boxed{\pi}$



$\int_C y(x+1)e^x \, dx + x(e^x+1) \, dy$ $\quad C: x^2+y^2=1$

$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

$\int_C y(x+1)e^x \, dx + x(e^x+1) \, dy$ (P = $y(x+1)e^x$, Q = $x(e^x+1)$)

$\frac{\partial Q}{\partial x} = 1(e^x+1) + x(e^x)$

$\frac{\partial Q}{\partial x} = xe^x + e^x + 1$

$\frac{\partial P}{\partial y} = (x+1)e^x$

$= \iint_C (xe^x + e^x + 1) - (xe^x + e^x) \, dx\,dy$

$= \iint_C xe^x + e^x + 1 - xe^x - e^x \, dx\,dy$

$= \iint_C 1 \, dx\,dy$

= AREA OF THE CIRCLE $x^2+y^2=1$

$= \pi$

Question 13

It is given that the vector function $\mathbf{F}$ satisfies

$$\mathbf{F} = \left(\sin x^3 - xy\right)\mathbf{i} + \left(x + y^3 \sin y\right)\mathbf{j}.$$

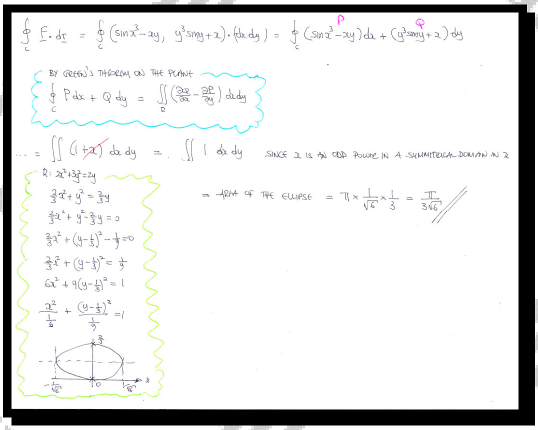
Evaluate the line integral

$$\oint_C \mathbf{F} \cdot \mathbf{dr},$$

where C is the ellipse with Cartesian equation

$$2x^2 + 3y^2 = 2y.$$

$$\boxed{\frac{\pi}{3\sqrt{6}}}$$



Question 14

It is given that the vector function $\mathbf{F}$ satisfies

$$\mathbf{F} = [x \cos x]\mathbf{i} + \left[15xy + \ln\left(1 + y^3\right)\right]\mathbf{j}.$$

Evaluate the line integral

$$\oint_C \mathbf{F} \cdot \mathbf{dr},$$

where C is the curve

$$\{(x, y) : y = 3, \ -2 \le x \le 2\} \cup \left\{(x, y) : y = x^2 - 1, \ -2 \le x \le 2\right\},$$

traced in an anticlockwise direction.

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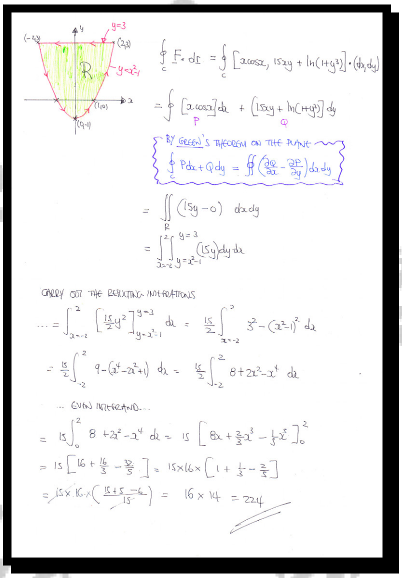


$\oint_C \underline{F} \cdot d\underline{r} = \oint_C [x\cos x, 15xy + \ln(1+y^3)] \cdot (dx, dy)$

$= \oint [x\cos x] dx + [15xy + \ln(1+y^3)] dy$

P Q

BY GREEN'S THEOREM ON THE PLANE

$\oint_C P dx + Q dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

$= \iint_R (15y - 0)\ dx\,dy$

$= \int_{x=-2}^{2} \int_{y=x^2-1}^{y=3} (15y)\,dy\,dx$

CARRY OUT THE RESULTING INTEGRATIONS

$\ldots = \int_{x=-2}^{2} \left[\frac{15}{2}y^2\right]_{y=x^2-1}^{y=3} dx = \frac{15}{2}\int_{x=-2}^{2} 3^2 - (x^2-1)^2\, dx$

$= \frac{15}{2}\int_{-2}^{2} 9 - (x^4 - 2x^2 + 1)\, dx = \frac{15}{2}\int_{-2}^{2} 8 + 2x^2 - x^4\, dx$

... EVEN INTEGRAND ...

$= 15\int_0^2 8 + 2x^2 - x^4\, dx = 15\left[8x + \frac{2}{3}x^3 - \frac{1}{5}x^5\right]_0^2$

$= 15\left[16 + \frac{16}{3} - \frac{32}{5}\right] = 15 \times 16 \times \left[1 + \frac{1}{3} - \frac{2}{5}\right]$

$= 15 \times 16 \times \left(\frac{15 + 5 - 6}{15}\right) = 16 \times 14 = 224$

LINE INTEGRALS

2 DIMENSIONAL PARAMETERIZATIONS

Question 1

The path along the semicircle with equation

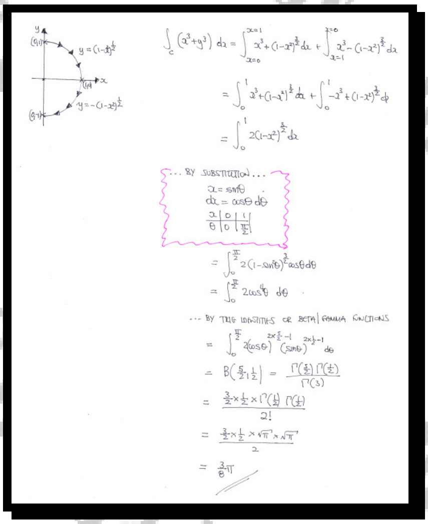
$$x^2 + y^2 = 1, \ x \geq 0$$

from $A(0,1)$ to $B(0,-1)$, is denoted by C.

Evaluate the integral

$$\int_C \left(x^3 + y^3\right) \, dx .$$

$\boxed{\frac{3}{8}\pi}$



$(0,1)$, $y=(1-x^2)^{\frac{1}{2}}$, $(1,0)$, $y=-(1-x^2)^{\frac{1}{2}}$, $(0,-1)$

$$\int_C (x^3+y^3)\,dx = \int_{x=0}^{x=1} x^3+(1-x^2)^{\frac{3}{2}}\,dx + \int_{x=1}^{x=0} x^3-(1-x^2)^{\frac{3}{2}}\,dx$$

$$= \int_0^1 x^3+(1-x^2)^{\frac{3}{2}}\,dx + \int_0^1 -x^3+(1-x^2)^{\frac{3}{2}}\,dx$$

$$= \int_0^1 2(1-x^2)^{\frac{3}{2}}\,dx$$

... BY SUBSTITUTION ...

$x = \sin\theta$

$dx = \cos\theta\,d\theta$

x	0	1
θ	0	$\frac{\pi}{2}$

$$= \int_0^{\frac{\pi}{2}} 2(1-\sin^2\theta)^{\frac{3}{2}}\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} 2\cos^4\theta\,d\theta$$

... BY TRIG IDENTITIES OR BETA|GAMMA FUNCTIONS

$$= \int_0^{\frac{\pi}{2}} 2(\cos\theta)^{2\times\frac{5}{2}-1}(\sin\theta)^{2\times\frac{1}{2}-1}\,d\theta$$

$$= B\left(\tfrac{5}{2},\tfrac{1}{2}\right) = \frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})}{\Gamma(3)}$$

$$= \frac{\frac{3}{2}\times\frac{1}{2}\times\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})}{2!}$$

$$= \frac{\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}\times\sqrt{\pi}}{2}$$

$$= \frac{3}{8}\pi$$

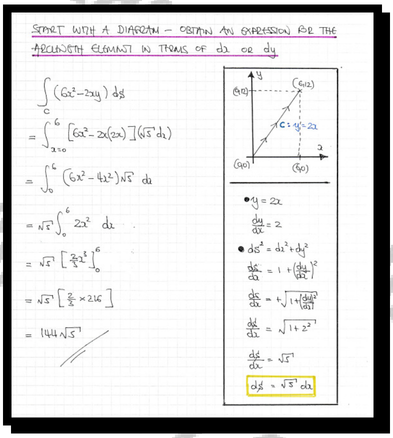
Question 2

Evaluate the integral

$$\int_{(0,0)}^{(6,12)} \left(6x^2 - 2xy\right) \, ds \, ,$$

where s is the arclength along the straight line segment from $(0,0)$ to $(6,12)$.

$\boxed{}$, $\boxed{144\sqrt{5}}$



Question 3

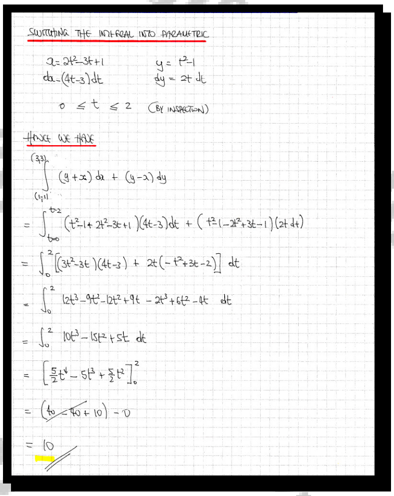
Evaluate the integral

$$\int_{(1,-1)}^{(3,3)} (y+x)\,dx + (y-x)\,dy\,,$$

along the curve with parametric equations

$$x = 2t^2 - 3t + 1 \quad \text{and} \quad y = t^2 - 1.$$

$\boxed{}$, $\boxed{10}$


SWITCHING THE INTEGRAL INTO PARAMETRIC

$x = 2t^2 - 3t + 1$ $\quad$ $y = t^2 - 1$

$dx = (4t-3)\,dt$ $\quad$ $dy = 2t\,dt$

$0 \leq t \leq 2$ (BY INSPECTION)

HENCE WE HAVE

$$\int_{(1,-1)}^{(3,3)} (y+x)\,dx + (y-x)\,dy$$

$$= \int_{t=0}^{t=2} (t^2-1+2t^2-3t+1)(4t-3)\,dt + (t^2-1-2t^2+3t-1)(2t\,dt)$$

$$= \int_0^2 \left[(3t^2-3t)(4t-3) + 2t(-t^2+3t-2)\right] dt$$

$$= \int_0^2 12t^3 - 9t^2 - 12t^2 + 9t - 2t^3 + 6t^2 - 4t \; dt$$

$$= \int_0^2 10t^3 - 15t^2 + 5t \; dt$$

$$= \left[\tfrac{5}{2}t^4 - 5t^3 + \tfrac{5}{2}t^2\right]_0^2$$

$$= (40 - 40 + 10) - 0$$

$$= 10$$

Question 4

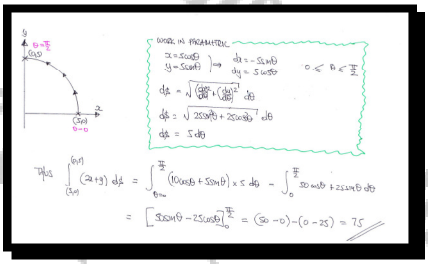
Evaluate the line integral

$$\int_{(5,0)}^{(0,5)} (2x+y)\, ds \,,$$

where s is the arclength along the quarter circle with equation

$$x^2 + y^2 = 25\,.$$

$\boxed{75}$



Question 5

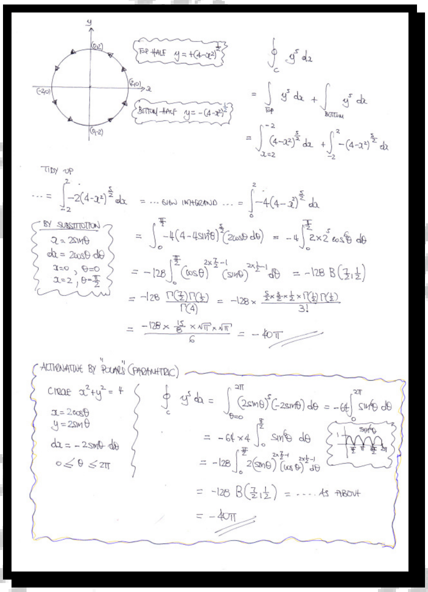
Evaluate the line integral

$$\oint_C y^5 \, dx \,,$$

where C is a circle of radius 2, centre at the origin O, traced anticlockwise.

You may not use Green's theorem in this question.

$\boxed{-40\pi}$



Question 6

Evaluate the line integral

$$\oint_C \left[y^3 dx + (xy)\, dy \right] ,$$

where C is a circle of radius 1, centre at the origin O, traced anticlockwise.

You may not use Green's theorem in this question.

$\boxed{}$, $\boxed{-\frac{3\pi}{4}}$

PROCEED BY PARAMETERIZING THE CIRCULAR PATH

UNIT CIRCLE: $x = \cos\theta$, $y = \sin\theta$, $0 \leqslant \theta < 2\pi$; $dx = -\sin\theta\, d\theta$, $dy = \cos\theta\, d\theta$

$$\oint_C \left[y^3\, dx + xy\, dy \right]$$

$$= \int_0^{2\pi} \sin^3\theta(-\sin\theta\, d\theta) + (\cos\theta\sin\theta)(\cos\theta\, d\theta)$$

$$= \int_0^{2\pi} (-\sin^4\theta + \cos^2\theta\sin\theta)\, d\theta = \int -(\sin^2\theta)^2$$

NO CONTRIBUTION OVER THESE LIMITS

$$= \int_0^{2\pi} -\left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta\right)^2 d\theta = \int_0^{2\pi} -\left(\tfrac{1}{4}\cos^2 2\theta + \tfrac{1}{4} - \tfrac{1}{2}\cos 2\theta\right) d\theta$$

NO CONTRIBUTION OVER THESE LIMITS

$$= \int_0^{2\pi} \left(-\tfrac{1}{4} - \tfrac{1}{4}\cos^2 2\theta\right) d\theta = \int_0^{2\pi} \left[-\tfrac{1}{4} - \tfrac{1}{4}\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 4\theta\right)\right] d\theta$$

$$= \int_0^{2\pi} \left(-\tfrac{3}{8} - \tfrac{1}{8}\cos 4\theta\right) d\theta = \int_0^{2\pi} -\tfrac{3}{8}\, d\theta$$

NO CONTRIBUTION OVER THESE LIMITS

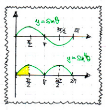
$$= -\tfrac{3}{8} \times 2\pi = -\frac{3\pi}{4}$$

ALTERNATIVE EVALUATION FROM $\int_0^{2\pi} -\sin^4\theta\, d\theta$

$$\int_0^{2\pi} -\sin^4\theta\, d\theta = -\int_0^{\pi/2} 4\sin^4\theta\, d\theta$$

$$= -2\int_0^{\frac{\pi}{2}} 2(\sin\theta)^{2\times\frac{5}{2}-1}(\cos\theta)^{2\times\frac{1}{2}-1}\, d\theta$$

$$= -2\, B\left(\tfrac{5}{2}, \tfrac{1}{2}\right)$$



TO GAMMA FUNCTIONS

$$= -2 \times \frac{\Gamma(\frac{5}{2})\,\Gamma(\frac{1}{2})}{\Gamma(3)}$$

$$= -2 \times \frac{\left[\frac{3}{2}\times\frac{1}{2}\times\Gamma(\frac{1}{2})\right]\Gamma(\frac{1}{2})}{2!}$$

$$= -2 \times \frac{\frac{3}{4}\times\sqrt{\pi}\times\sqrt{\pi}}{2}$$

$$= -\frac{3\pi}{4}$$

AS BEFORE

Question 7

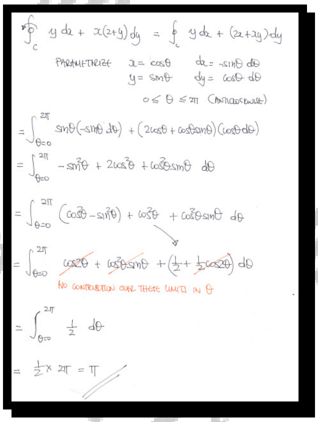
Evaluate the line integral

$$\oint_C \left[y\,dx + x(2+y)\,dy \right] ,$$

where C is a circle of radius 1, centre at the origin O, traced anticlockwise.

You may not use Green's theorem in this question.

$\boxed{\pi}$



$\oint_C y\,dx + x(2+y)\,dy = \oint_C y\,dx + (2x+xy)\,dy$

PARAMETRIZE $x = \cos\theta$, $y = \sin\theta$, $dx = -\sin\theta\,d\theta$, $dy = \cos\theta\,d\theta$

$0 \le \theta \le 2\pi$ (ANTICLOCKWISE)

$= \int_{\theta=0}^{2\pi} \sin\theta(-\sin\theta\,d\theta) + (2\cos\theta + \cos\theta\sin\theta)(\cos\theta\,d\theta)$

$= \int_{\theta=0}^{2\pi} -\sin^2\theta + 2\cos^2\theta + \cos^2\theta\sin\theta\,d\theta$

$= \int_{\theta=0}^{2\pi} (\cos^2\theta - \sin^2\theta) + \cos^2\theta + \cos^2\theta\sin\theta\,d\theta$

$= \int_{\theta=0}^{2\pi} \cos 2\theta + \cos^2\theta\sin\theta + \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$

NO CONTRIBUTION OVER THESE LIMITS IN θ

$= \int_{\theta=0}^{2\pi} \frac{1}{2}\,d\theta$

$= \frac{1}{2} \times 2\pi = \pi$

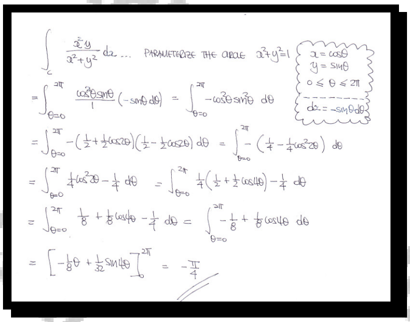
Question 8

Evaluate the line integral

$$\oint_C \frac{x^2 y}{x^2+y^2}\, dx,$$

where C is a circle of radius 1, centre at the origin O, traced anticlockwise.

$$\boxed{-\frac{\pi}{4}}$$


$\int_C \frac{x^2y}{x^2+y^2}\,dx \ldots$ PARAMETRIZE THE CIRCLE $x^2+y^2=1$

$x=\cos\theta$, $y=\sin\theta$, $0 \leq \theta \leq 2\pi$, $dx=-\sin\theta\,d\theta$

$= \int_{\theta=0}^{2\pi} \frac{\cos^2\theta\sin\theta}{1}(-\sin\theta\,d\theta) = \int_{\theta=0}^{2\pi} -\cos^2\theta\sin^2\theta\,d\theta$

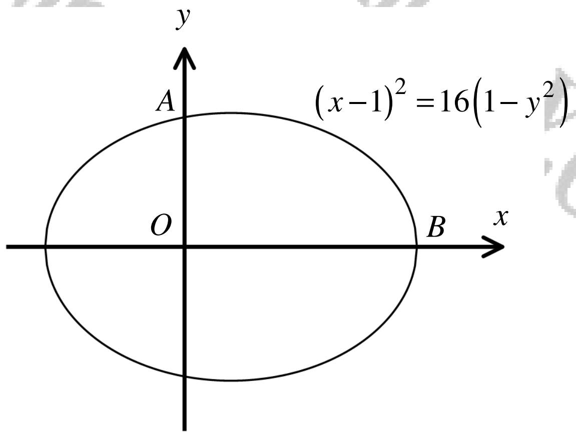
$= \int_{\theta=0}^{2\pi} -\left(\frac{1}{2}+\frac{1}{2}\cos2\theta\right)\left(\frac{1}{2}-\frac{1}{2}\cos2\theta\right)d\theta = \int_{\theta=0}^{2\pi} -\left(\frac{1}{4}-\frac{1}{4}\cos^22\theta\right)d\theta$

$= \int_{\theta=0}^{2\pi} \frac{1}{4}\cos^22\theta-\frac{1}{4}\,d\theta = \int_{\theta=0}^{2\pi} \frac{1}{4}\left(\frac{1}{2}+\frac{1}{2}\cos4\theta\right)-\frac{1}{4}\,d\theta$

$= \int_{\theta=0}^{2\pi} \frac{1}{8}+\frac{1}{8}\cos4\theta-\frac{1}{4}\,d\theta = \int_{\theta=0}^{2\pi} -\frac{1}{8}+\frac{1}{8}\cos4\theta\,d\theta$

$= \left[-\frac{1}{8}\theta+\frac{1}{32}\sin4\theta\right]_0^{2\pi} = -\frac{\pi}{4}$

Question 9



The figure above shows the ellipse with equation

$$(x-1)^2 = 16\left(1-y^2\right).$$

The ellipse meets the positive y and x axes at the points A and B, respectively, as shown in the figure.

The elliptic path C is the clockwise section from A to B.

Determine the value of each of the following line integrals.

a) $\displaystyle\int_C \left[\left(x^2+xy\right)\,dx + \left(y^2+\tfrac{1}{2}x^2\right)\,dy\right].$

b) $\displaystyle\int_C \left[y^3\,dx + \tfrac{1}{16}(x-1)^3\,dy\right].$

$\boxed{}$, $\boxed{\dfrac{125}{3}-\dfrac{5}{64}\sqrt{15}}$, $\boxed{\tfrac{1}{4}\sqrt{15}}$

[solution overleaf]

a) FIRSTLY DETERMINE THE CO-ORDINATES OF A & B

- $x=0$

$1 = 16(1-y^2)$

$\frac{1}{16} = 1-y^2$

$y^2 = \frac{15}{16}$

$y = +\frac{1}{4}\sqrt{15}$

$A\left(0, \frac{1}{4}\sqrt{15}\right)$

- $y=0$

$(x-1)^2 = 16$

$x-1 = \pm 4$ (4 or -4)

$x = 5$

$B(5,0)$

NEXT WE TEST WHETHER THE INTEGRAL IS INDEPENDENT OF THE PATH.

$$\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = d\phi$$

$$x^2+xy \qquad y^2+\frac{1}{2}x^2$$

$$\frac{\partial}{\partial y}\left(x^2+xy\right) = x \qquad \frac{\partial}{\partial x}\left(y^2+\frac{1}{2}x^2\right) = x \qquad \text{IT EXACT}$$

- $\frac{\partial \phi}{\partial x} = x^2+xy \Rightarrow \phi(x,y) = \frac{1}{3}x^3 + \frac{1}{2}x^2y + f(y)$
- $\frac{\partial \phi}{\partial y} = y^2+\frac{1}{2}x^2 \Rightarrow \phi(x,y) = \frac{1}{3}y^3 + \frac{1}{2}x^2y + g(x)$

$$\therefore \phi(x,y) = \frac{1}{3}x^3 + \frac{1}{3}y^3 + \frac{1}{2}x^2y + C$$

HENCE WE OBTAIN

$$\int_A^B \left[(x^2+xy)\,dx + \left(y^2+\frac{1}{2}x^2\right)dy\right] = \int_A^B d\phi$$

$$= \left[\frac{1}{3}x^3+\frac{1}{3}y^3+\frac{1}{2}x^2y\right]_{(0,\frac{1}{4}\sqrt{15})}^{(5,0)}$$

$$= \frac{1}{3}\times 5^3 - \frac{1}{3}\left(\frac{1}{4}\sqrt{15}\right)^3$$

$$= \frac{125}{3} - \frac{1}{3}\,\frac{15}{64}\sqrt{15} = \frac{125}{3} - \frac{5}{64}\sqrt{15}$$

b) CHECKING FOR PATH INDEPENDENCE FOR (b)

$$\frac{\partial}{\partial y}\left(y^3\right) = 3y^2 \qquad \frac{\partial}{\partial x}\left(\frac{1}{16}(x-1)^3\right) = \frac{3}{16}(x-1)^2$$

DIFFERENTIAL IS NOT EXACT SO IT DEPENDS ON THE PATH

PARAMETRIZE THE ELLIPSE

$\Rightarrow (x-1)^2 = 16(1-y^2)$

$\Rightarrow (x-1)^2 = 16 - 16y^2$

$\Rightarrow (x-1)^2 + 16y^2 = 16$

$\Rightarrow \frac{(x-1)^2}{16} + y^2 = 1$

$\left[\cos^2\theta + \sin^2\theta = 1\right]$

$\therefore \cos\theta = \frac{x-1}{4}$ & $\sin\theta = y$

$\boxed{x = 1+4\cos\theta}$ & $\boxed{y = \sin\theta}$

$dx = -4\sin\theta$ & $dy = \cos\theta\,d\theta$

TRANSFORMING THE LIMITS

$A\left(0, \frac{1}{4}\sqrt{15}\right) \longmapsto \theta = \arcsin\frac{\sqrt{15}}{4} = \arccos\left(\frac{1}{4}\right)$

$B(5,0) \longmapsto \theta = 0$

TRANSFORMING THE INTEGRAL INTO PARAMETRIC

$$\int_C y^3\,dx + \frac{1}{16}(x-1)^3\,dy$$

$$= \int_{\arccos(\frac{1}{4})\,/\,\arcsin\frac{\sqrt{15}}{4}}^{0} \left[\sin^3\theta(-4\sin\theta) + \frac{1}{16}(1+4\cos\theta-1)^3\cos\theta\right]d\theta$$

$$= \int_{\arcsin\frac{\sqrt{15}}{4}\,/\,\arccos(\frac{1}{4})}^{0} -4\sin^4\theta + 4\cos^4\theta\,d\theta$$

$$= \int_{\arcsin\frac{\sqrt{15}}{4}\,/\,\arccos(\frac{1}{4})}^{0} 4\left(\cos^4\theta - \sin^4\theta\right)d\theta$$

$$= \int_{\arcsin\frac{\sqrt{15}}{4}\,/\,\arccos(\frac{1}{4})}^{0} 4\left(\cos^2\theta-\sin^2\theta\right)\left(\cos^2\theta+\sin^2\theta\right)d\theta$$

$$= \int_{\arcsin\frac{\sqrt{15}}{4}\,/\,\arccos(\frac{1}{4})}^{0} 4\cos 2\theta\,d\theta$$

$$= \Big[2\sin 2\theta\Big]_{\theta=\arcsin\frac{\sqrt{15}}{4}\,/\,\theta=\arccos(\frac{1}{4})}^{0}$$

$$= \Big[4\sin\theta\cos\theta\Big]_{\theta=\arcsin\frac{\sqrt{15}}{4}\,/\,\theta=\arccos(\frac{1}{4})}^{0}$$

$$= 0 - 4\times\frac{\sqrt{15}}{4}\times\left(\frac{1}{4}\right)$$

$$= \frac{1}{4}\sqrt{15}$$

Question 10

The closed curve C bounds the finite region R in the x-y plane defined as

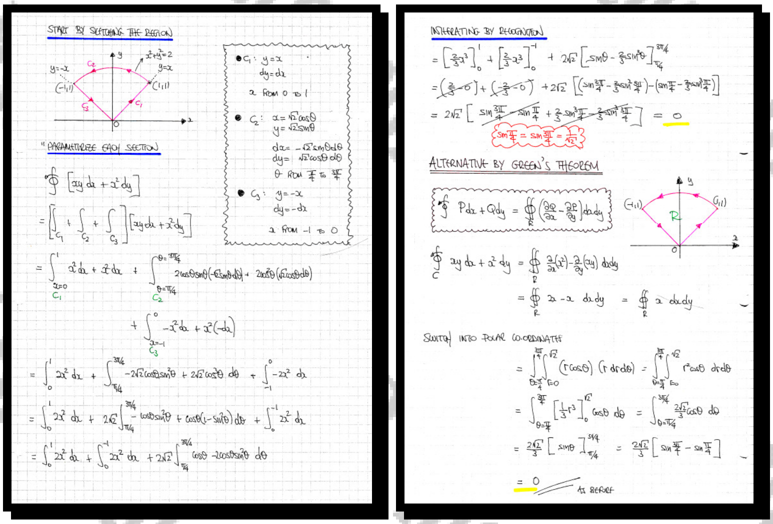
$$R(x,y) = \{x + y \geq 0 \cap x - y \leq 0 \cap x^2 + y^2 \leq 2\}.$$

Evaluate the line integral

$$\oint_C \left(xy\,dx + x^2\,dy \right),$$

where C is traced anticlockwise.

[ ], [0]



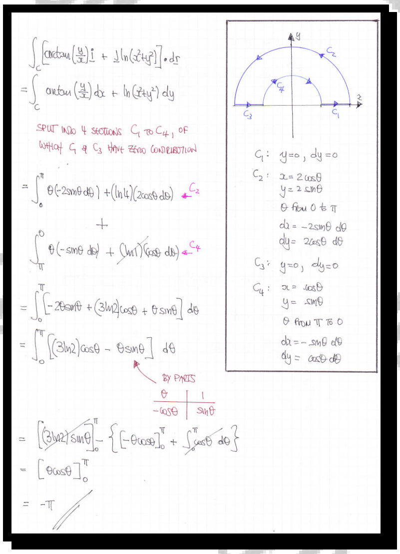
Question 11

Evaluate the line integral

$$\oint_C \left[\arctan\left(\frac{y}{x}\right) dx + \ln\left(x^2+y^2\right) dy \right],$$

where C is the polar rectangle such that $1 \le r \le 2$, $0 \le \theta \le \pi$, traced anticlockwise.

$\boxed{-\pi}$



Question 12

Evaluate the line integral

$$\oint_C \left[(2x-y)\,dx + (2y-x)\,dy\right],$$

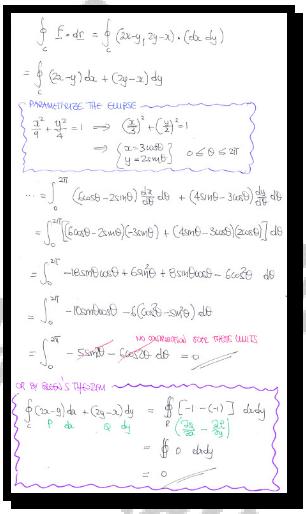
where C is an ellipse with Cartesian equation

$$\frac{x^2}{9}+\frac{y^2}{4}=1,$$

traced anticlockwise.

You may not use Green's theorem in this question.

$\boxed{0}$



Question 13

It is given that the vector function $\mathbf{F}$ satisfies

$$\mathbf{F} = (x-3y)\mathbf{i} + (y-2x)\mathbf{j}.$$

Evaluate the line integral

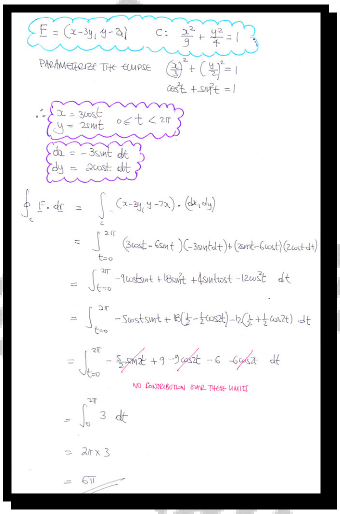
$$\oint_C \mathbf{F} \cdot \mathbf{dr},$$

where C is the ellipse with cartesian equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

You may not use Green's theorem in this question.

$\boxed{6\pi}$



Question 14

$$\mathbf{F}(x,y) \equiv \left(-\frac{y}{x^2+y^2}\right)\mathbf{i} + \left(\frac{x}{x^2+y^2}\right)\mathbf{j}.$$

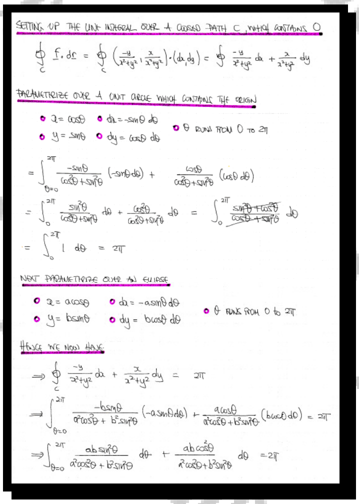
By considering the line integral of $\mathbf{F}$ over two different suitably parameterized closed paths, show that

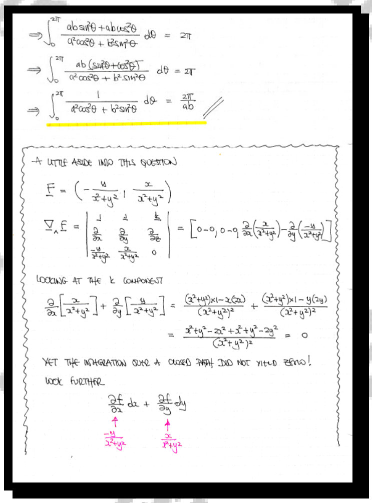
$$\int_0^{2\pi} \frac{1}{a^2\cos^2\theta + b^2\sin^2\theta}\, d\theta = \frac{2\pi}{ab},$$

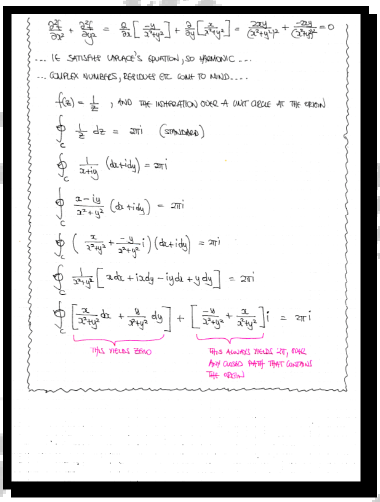
where a and b are real constants.

You may assume without proof that the line integral of $\mathbf{F}$ yields the same value over any simple closed curve which contains the origin.

[   ], [proof]







LINE INTEGRALS

IN 3 DIMENSIONS

Question 1

It is given that the vector function $\mathbf{F}$ satisfies

$$\mathbf{F} = \left(x^2 y\right)\mathbf{i} + \left(4xy^2\right)\mathbf{j} + \left(-6xz\right)\mathbf{k}.$$

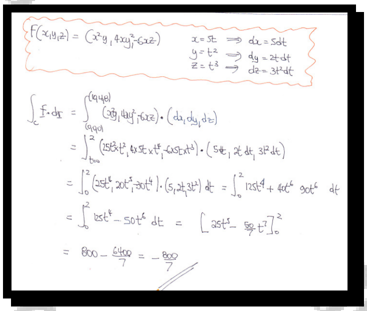
Evaluate the line integral

$$\int_{(0,0,0)}^{(10,4,8)} \mathbf{F} \cdot d\mathbf{r}, \qquad \text{where } \mathbf{r} = \left(dx, dy, dz\right)^{\mathrm{T}},$$

along a path given by the parametric equations

$$x = 5t, \quad y = t^2, \quad z = t^3.$$

$$\boxed{-\frac{800}{7}}$$



Question 2

It is given that the vector function $\mathbf{F}$ satisfies

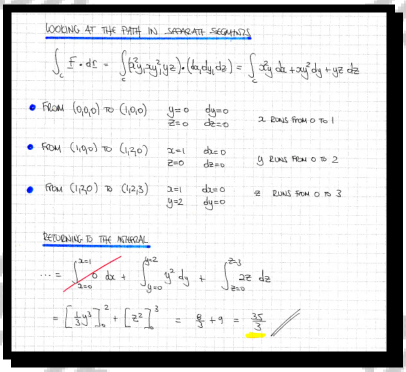
$$\mathbf{F} = \left(x^2 y\right)\mathbf{i} + \left(xy^2\right)\mathbf{j} + \left(yz\right)\mathbf{k}.$$

Evaluate the line integral

$$\int_{(0,0,0)}^{(1,2,3)} \mathbf{F}\cdot d\mathbf{r}, \quad \text{where} \quad d\mathbf{r} = \left(dx, dy, dz\right)^{\mathrm{T}},$$

along a path of three straight line segments joining $(0,0,0)$ to $(1,0,0)$, $(1,0,0)$ to $(1,2,0)$ and $(1,2,0)$ to $(1,2,3)$.

$\boxed{}$, $\boxed{\frac{35}{3}}$



Question 3

It is given that

$$\mathbf{F}(x, y, z) \equiv \mathbf{j} \wedge \mathbf{r} \,,$$

where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

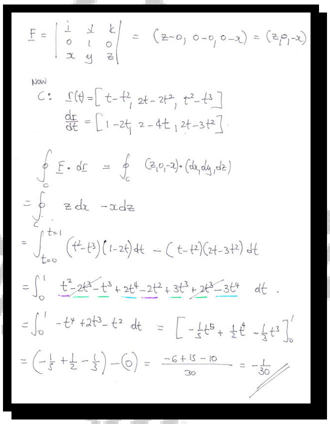
Evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r} \,,$$

where C is the closed curve given parametrically by

$$\mathbf{R}(t) = \left(t - t^2\right)\mathbf{i} + \left(2t - 2t^2\right)\mathbf{j} + \left(t^2 - t^3\right)\mathbf{k} \,,\ 0 \le t \le 1 \,.$$

$$\boxed{-\frac{1}{30}}$$



$$\underline{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ x & y & z \end{vmatrix} = (z-0,\ 0-0,\ 0-x) = (z, 0, -x)$$

Now

$$C: \ \underline{r}(t) = \left[t - t^2,\ 2t - 2t^2,\ t^2 - t^3\right]$$

$$\frac{d\underline{r}}{dt} = \left[1 - 2t,\ 2 - 4t,\ 2t - 3t^2\right].$$

$$\oint_C \underline{F} \cdot d\underline{r} = \oint_C (z, 0, -x) \cdot (dx, dy, dz)$$

$$= \oint_C z\,dx - x\,dz$$

$$= \int_{t=0}^{t=1} (t^2 - t^3)(1 - 2t)\,dt - (t - t^2)(2t - 3t^2)\,dt$$

$$= \int_0^1 t^2 - 2t^3 - t^3 + 2t^4 - 2t^2 + 3t^3 + 2t^3 - 3t^4 \ dt\,.$$

$$= \int_0^1 -t^4 + 2t^3 - t^2\,dt = \left[-\tfrac{1}{5}t^5 + \tfrac{1}{2}t^4 - \tfrac{1}{3}t^3\right]_0^1$$

$$= \left(-\tfrac{1}{5} + \tfrac{1}{2} - \tfrac{1}{3}\right) - (0) = \frac{-6 + 15 - 10}{30} = -\frac{1}{30}$$

Question 4

The simple closed curve C has Cartesian equation

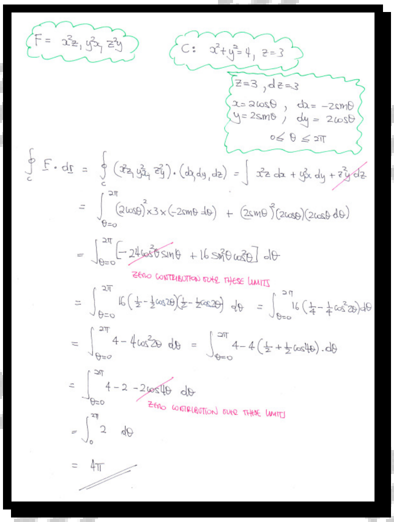
$$x^2 + y^2 = 4, \ z = 3.$$

Given that $\mathbf{F} = x^2 z\,\mathbf{i} + y^2 x\,\mathbf{j} + z^2 y\,\mathbf{k}$, evaluate the integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r}.$$

You may not use Green's theorem in this question.

$\boxed{4\pi}$



$\mathbf{F} = x^2z,\ y^2x,\ z^2y$

$C:\ x^2+y^2=4,\ z=3$

$z=3,\ dz=3$
$x=2\cos\theta,\ dx=-2\sin\theta$
$y=2\sin\theta,\ dy=2\cos\theta$
$0 \leq \theta \leq 2\pi$

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C (x^2z, y^2x, z^2y)\cdot(dx,dy,dz) = \int x^2z\,dx + y^2x\,dy + z^2y\,dz$$

$$= \int_{\theta=0}^{2\pi} (2\cos\theta)^2 \times 3 \times (-2\sin\theta\,d\theta) + (2\sin\theta)^2(2\cos\theta)(2\cos\theta\,d\theta)$$

$$= \int_{\theta=0}^{2\pi} \left[-24\cos^2\theta\sin\theta + 16\sin^2\theta\cos^2\theta\right] d\theta$$

ZERO CONTRIBUTION OVER THESE LIMITS

$$= \int_{\theta=0}^{2\pi} 16\left(\tfrac{1}{2}-\tfrac{1}{2}\cos2\theta\right)\left(\tfrac{1}{2}-\tfrac{1}{2}\cos2\theta\right) d\theta = \int_{\theta=0}^{2\pi} 16\left(\tfrac{1}{4}-\tfrac{1}{4}\cos^2 2\theta\right) d\theta$$

$$= \int_{\theta=0}^{2\pi} 4-4\cos^2 2\theta\,d\theta = \int_{\theta=0}^{2\pi} 4-4\left(\tfrac{1}{2}+\tfrac{1}{2}\cos4\theta\right) d\theta$$

$$= \int_{\theta=0}^{2\pi} 4-2-2\cos4\theta\,d\theta$$

ZERO CONTRIBUTION OVER THESE LIMITS

$$= \int_0^{2\pi} 2\,d\theta$$

$$= 4\pi$$

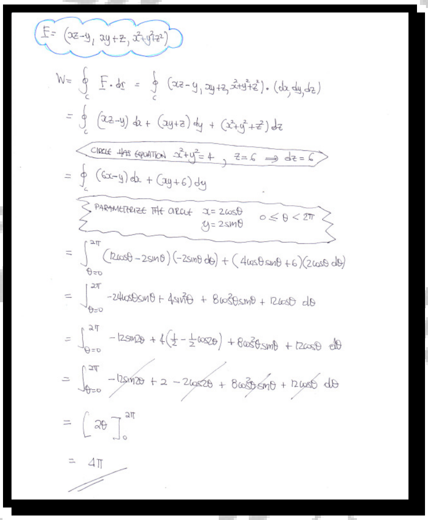
Question 5

$$\mathbf{F} = (xz - y)\mathbf{i} + (xy + z)\mathbf{j} + (x^2 + y^2 + z^2)\mathbf{k}.$$

Determine the work done by $\mathbf{F}$, when it moves in a complete revolution in a circular path of radius 2 around the z axis, at the level of the plane with equation $z = 6$.

You may not use Green's theorem in this question.

$\boxed{4\pi}$



$\mathbf{F} = (xz - y, xy + z, x^2 + y^2 + z^2)$

$W = \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C (xz - y, xy + z, x^2 + y^2 + z^2) \cdot (dx, dy, dz)$

$= \oint_C (xz - y)\,dx + (xy + z)\,dy + (x^2 + y^2 + z^2)\,dz$

CIRCLE HAS EQUATION $x^2 + y^2 = 4$, $z = 6 \Rightarrow dz = 0$

$= \oint_C (6x - y)\,dx + (xy + 6)\,dy$

PARAMETRIZE THE CIRCLE $x = 2\cos\theta$, $y = 2\sin\theta$, $0 \leqslant \theta < 2\pi$

$= \int_{\theta=0}^{2\pi} (12\cos\theta - 2\sin\theta)(-2\sin\theta\,d\theta) + (4\cos\theta\sin\theta + 6)(2\cos\theta\,d\theta)$

$= \int_{\theta=0}^{2\pi} -24\cos\theta\sin\theta + 4\sin^2\theta + 8\cos^2\theta\sin\theta + 12\cos\theta\,d\theta$

$= \int_{\theta=0}^{2\pi} -12\sin 2\theta + 4\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right) + 8\cos^2\theta\sin\theta + 12\cos\theta\,d\theta$

$= \int_{\theta=0}^{2\pi} -12\sin 2\theta + 2 - 2\cos 2\theta + 8\cos^2\theta\sin\theta + 12\cos\theta\,d\theta$

$= \left[2\theta\right]_0^{2\pi}$

$= 4\pi$

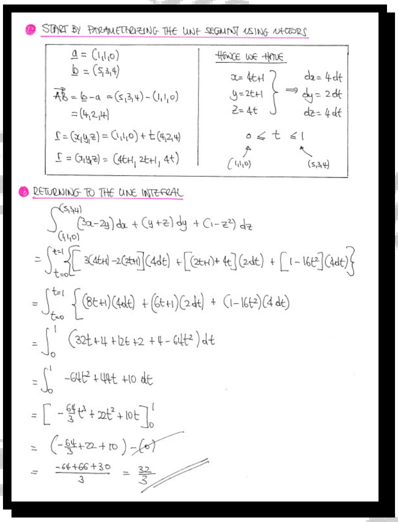
Question 6

Evaluate the integral

$$\int_{(1,1,0)}^{(5,3,4)} (3x-2y)\,dx + (y+z)\,dy + \left(1-z^2\right)\,dz,$$

along the straight line segment joining the points with Cartesian coordinates $(1,1,0)$ and $(5,3,4)$.

$\boxed{}$, $\boxed{\dfrac{32}{3}}$



START BY PARAMETRIZING THE LINE SEGMENT USING VECTORS

$\underline{a} = (1,1,0)$

$\underline{b} = (5,3,4)$

$\overrightarrow{AB} = \underline{b} - \underline{a} = (5,3,4) - (1,1,0) = (4,2,4)$

$\underline{r} = (x,y,z) = (1,1,0) + t(4,2,4)$

$\underline{r} = (x,y,z) = (4t+1,\ 2t+1,\ 4t)$

HENCE WE HAVE

$x = 4t+1,\ y = 2t+1,\ z = 4t \implies dx = 4\,dt,\ dy = 2\,dt,\ dz = 4\,dt$

$0 \leqslant t \leqslant 1$, with $t=0$ at $(1,1,0)$ and $t=1$ at $(5,3,4)$

RETURNING TO THE LINE INTEGRAL

$$\int_{(1,1,0)}^{(5,3,4)} (3x-2y)\,dx + (y+z)\,dy + (1-z^2)\,dz$$

$$= \int_{t=0}^{t=1} \left\{ \left[3(4t+1) - 2(2t+1)\right](4\,dt) + \left[(2t+1) + 4t\right](2\,dt) + \left[1 - 16t^2\right](4\,dt) \right\}$$

$$= \int_{t=0}^{t=1} \left\{ (8t+1)(4\,dt) + (6t+1)(2\,dt) + (1-16t^2)(4\,dt) \right\}$$

$$= \int_0^1 \left(32t + 4 + 12t + 2 + 4 - 64t^2\right)\,dt$$

$$= \int_0^1 -64t^2 + 44t + 10\,dt$$

$$= \left[-\frac{64}{3}t^3 + 22t^2 + 10t\right]_0^1$$

$$= \left(-\frac{64}{3} + 22 + 10\right) - (0)$$

$$= \frac{-64 + 66 + 30}{3} = \frac{32}{3}$$

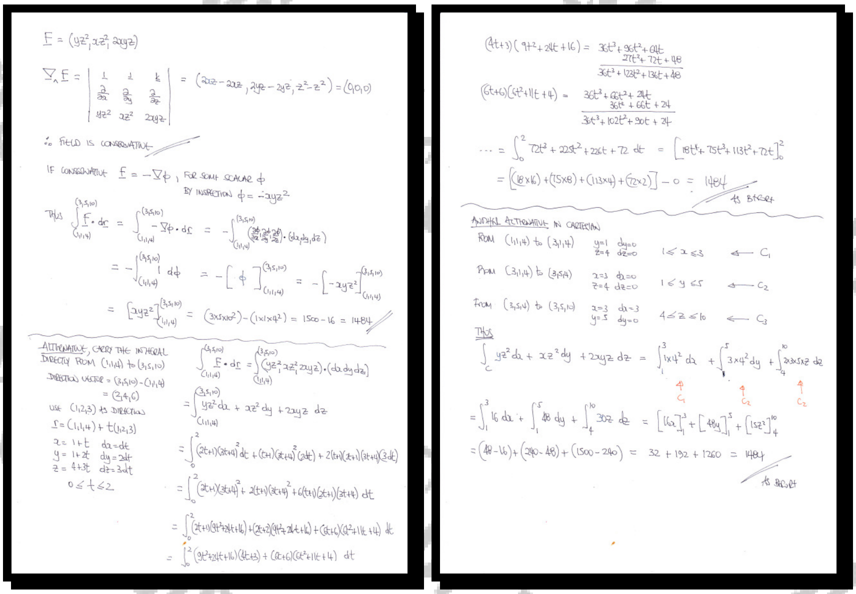
Question 7

$$\mathbf{F}(x, y, z) \equiv yz^2\mathbf{i} + xz^2\mathbf{j} + 2xyz\mathbf{k} .$$

Show that the vector field $\mathbf{F}$ is conservative, and hence evaluate the integral

$$\int_{(1,1,4)}^{(3,5,10)} \mathbf{F} \cdot \mathbf{dr} .$$

$\boxed{1484}$



Question 8

A vector field $\mathbf{F}$ is defined as

$$\mathbf{F}(x,y,z) \equiv [x+yz]\mathbf{i} + [y+xz]\mathbf{j} + \left[x(y+1)+z^2\right]\mathbf{k} .$$

The closed path C joins $(0,0,0)$ to $(1,1,1)$, $(1,1,1)$ to $(1,1,0)$, $(1,1,0)$ to $(0,0,0)$, in that order.

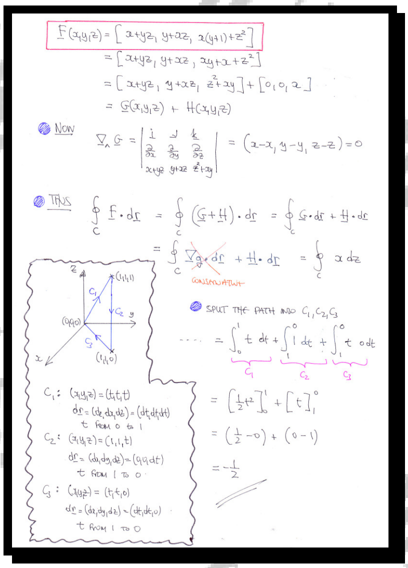
By writing

$$\mathbf{F}(x,y,z) = \mathbf{G}(x,y,z) + \mathbf{H}(x,y,z),$$

for some vector functions $\mathbf{G}$ and $\mathbf{H}$, where $\nabla g(x,y,z) = \mathbf{G}(x,y,z)$ for some smooth scalar function $g(x,y,z)$, evaluate the line integral

$$\oint_C \mathbf{F} \cdot \mathbf{dr} .$$

$$\boxed{-\frac{1}{2}}$$



Question 9

A vector field $\mathbf{F}$ is defined as

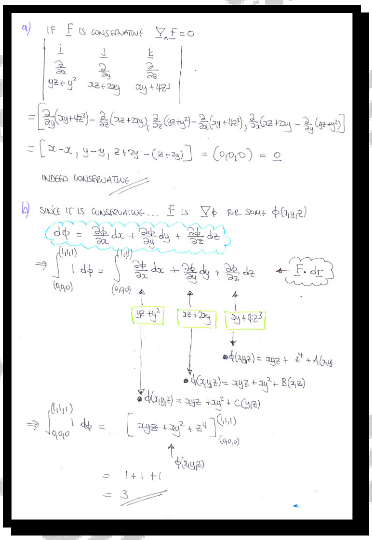
$$\mathbf{F}(x,y,z) \equiv \left(yz + y^2\right)\mathbf{i} + \left(xz + 2xy\right)\mathbf{j} + \left(xy + 4z^3\right)\mathbf{k}\,.$$

a) Show that $\mathbf{F}$ is conservative.

b) Hence evaluate the integral

$$\int_{(0,0,0)}^{(1,1,1)} \mathbf{F}\cdot\mathbf{dr}\,.$$

$\boxed{3}$



Question 10

A curve C is defined as

$$(x, y, z) = (\cos 3t, \sin 3t, t), \; 0 \le t \le 2\pi .$$

a) Sketch the graph of C.

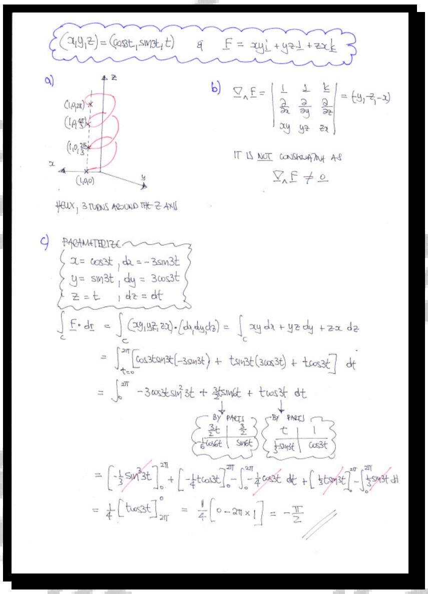
$$\mathbf{F}(x, y, z) \equiv xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k} .$$

b) Determine whether the vector field $\mathbf{F}$ is conservative.

c) Evaluate the integral

$$\int_C \mathbf{F} \cdot \mathbf{dr} .$$

$$\boxed{-\frac{\pi}{2}}$$



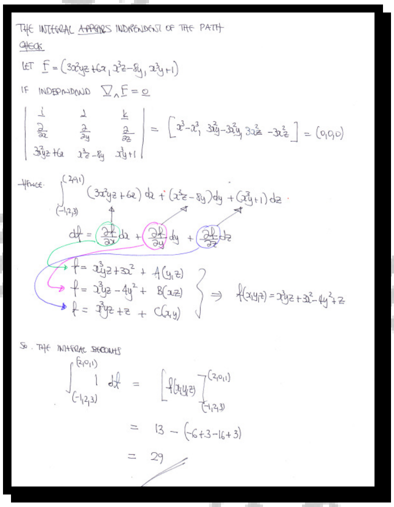
Question 11

Evaluate the integral

$$\int_{(-1,2,3)}^{(2,0,1)} \left(3x^2yz+6x\right)\,dx+\left(x^3z-8y\right)\,dy+\left(x^3y+1\right)\,dz\,,$$

along a path joining the points with Cartesian coordinates $(-1,2,3)$ and $(2,0,1)$.

$\boxed{29}$



Question 12

A curve C is defined by $\mathbf{r} = \mathbf{r}(t)$, $0 \le t \le 2\pi$ as

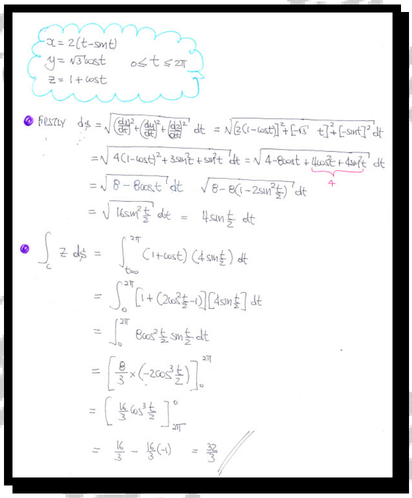
$$\mathbf{r}(t) = (x, y, z) = \left[2(t - \sin t),\ \sqrt{3}\cos t\ ,\ 1 + \cos t \right].$$

Evaluate the integral

$$\int_C z\ ds,$$

where s is the arclength along C.

$$\boxed{\frac{32}{3}}$$



$x = 2(t - \sin t)$
$y = \sqrt{3}\cos t \qquad 0 \le t \le 2\pi$
$z = 1 + \cos t$

FIRSTLY $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt = \sqrt{[2(1-\cos t)]^2 + [-\sqrt{3}\sin t]^2 + [-\sin t]^2}\ dt$

$= \sqrt{4(1-\cos t)^2 + 3\sin^2 t + \sin^2 t}\ dt = \sqrt{4 - 8\cos t + 4\cos^2 t + 4\sin^2 t}\ dt$

$= \sqrt{8 - 8\cos t}\ dt \qquad \sqrt{8 - 8(1 - 2\sin^2 \frac{t}{2})}\ dt$

$= \sqrt{16\sin^2 \frac{t}{2}}\ dt = 4\sin\frac{t}{2}\ dt$

$\int_C z\ ds = \int_{t=0}^{2\pi} (1 + \cos t)(4\sin\frac{t}{2})\ dt$

$= \int_0^{2\pi} \left[1 + (2\cos^2\frac{t}{2} - 1)\right]\left[4\sin\frac{t}{2}\right] dt$

$= \int_0^{2\pi} 8\cos^2\frac{t}{2}\sin\frac{t}{2}\ dt$

$= \left[\frac{8}{3} \times \left(-2\cos^3\frac{t}{2}\right)\right]_0^{2\pi}$

$= \left[\frac{16}{3}\cos^3\frac{t}{2}\right]_{2\pi}^{0}$

$= \frac{16}{3} - \frac{16}{3}(-1) = \frac{32}{3}$

Question 13

A vector field $\mathbf{F}$ and a scalar field ψ are given.

$$\mathbf{F} = \left(3x^3 y\right)\mathbf{i} + \left(15\sqrt{z}\right)\mathbf{j} - \left(\frac{13}{96}xz\right)\mathbf{k} \quad \text{and} \quad \psi(x,y,z) = x\mathrm{e}^{\frac{2y}{\sqrt{z}}}.$$

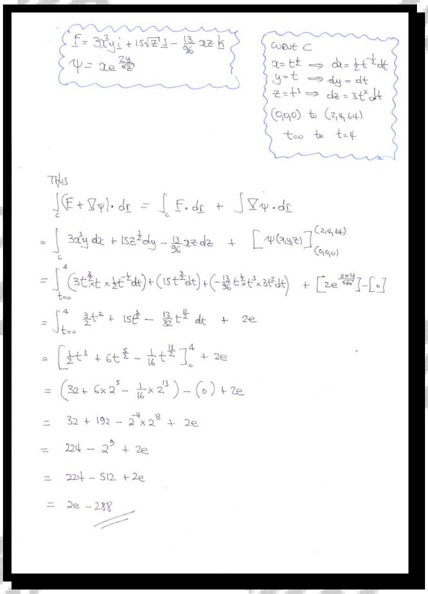
Evaluate the integral

$$\int_{(0,0,0)}^{(2,4,64)} \left[\mathbf{F} + \nabla\psi\right] \cdot \mathbf{dr},$$

along the curve with parametric equations

$$x = \sqrt{t}, \quad y = t \quad \text{and} \quad z = t^3.$$

$\boxed{2\mathrm{e} - 288}$



Question 14

It is given that the vector function $\mathbf{F}$ satisfies

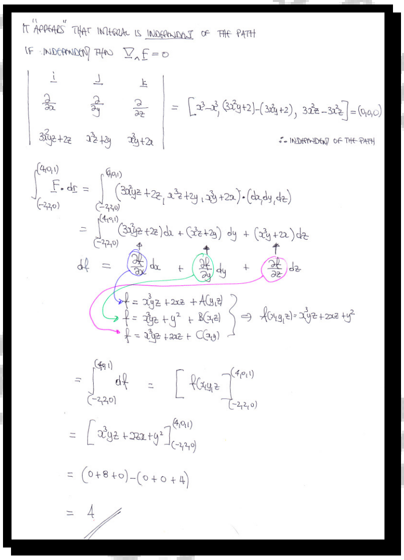
$$\mathbf{F} = \left(3x^2yz + 2z\right)\mathbf{i} + \left(x^3z + 2y\right)\mathbf{j} + \left(x^3y + 2x\right)\mathbf{k}.$$

Evaluate the line integral

$$\int_{(-2,2,0)}^{(4,0,1)} \mathbf{F}\cdot\mathbf{dr},$$

along a path joining the points with Cartesian coordinates $(4,0,1)$ and $(-2,2,0)$.

4


IT "APPEARS" THAT INTEGRAL IS INDEPENDENT OF THE PATH

IF INDEPENDENT THEN $\nabla \wedge \underline{F} = 0$

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2yz+2z & x^3z+2y & x^3y+2x \end{vmatrix} = \left[x^3-x^3, (3x^2y+2)-(3x^2y+2), 3x^2z-3x^2z\right] = (0,0,0)$$

∴ INDEPENDENT OF THE PATH

$$\int_{(-2,2,0)}^{(4,0,1)} \underline{F}\cdot d\underline{r} = \int_{(-2,2,0)}^{(4,0,1)} \left(3x^2yz+2z, x^3z+2y, x^3y+2x\right)\cdot(dx,dy,dz)$$

$$= \int_{(-2,2,0)}^{(4,0,1)} (3x^2yz+2z)\,dx + (x^3z+2y)\,dy + (x^3y+2x)\,dz$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$f = x^3yz + 2xz + A(y,z)$

$f = x^3yz + y^2 + B(x,z)$

$f = x^3yz + 2xz + C(x,y)$

$\Rightarrow f(x,y,z) = x^3yz + 2xz + y^2$

$$= \int_{(-2,2,0)}^{(4,0,1)} df = \Big[f(x,y,z)\Big]_{(-2,2,0)}^{(4,0,1)}$$

$$= \Big[x^3yz + 2xz + y^2\Big]_{(-2,2,0)}^{(4,0,1)}$$

$$= (0+8+0)-(0+0+4)$$

$$= 4$$

Question 15

It is given that the vector function $\mathbf{F}$ satisfies

$$\mathbf{F} = \left(1 + xy^2\right)\mathbf{i} + \left(x + xyz\right)\mathbf{j} + \left(y \sin z\right)\mathbf{k} \,.$$

Evaluate the line integral

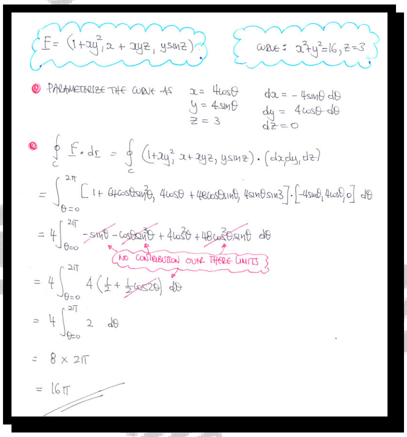
$$\oint_C \mathbf{F} \cdot \mathbf{dr} \,,$$

where C is the anticlockwise cartesian path

$$x^2 + y^2 = 16 \,, \quad z = 3 \,.$$

You may not use Green's theorem in this question.

$\boxed{16\pi}$



Question 16

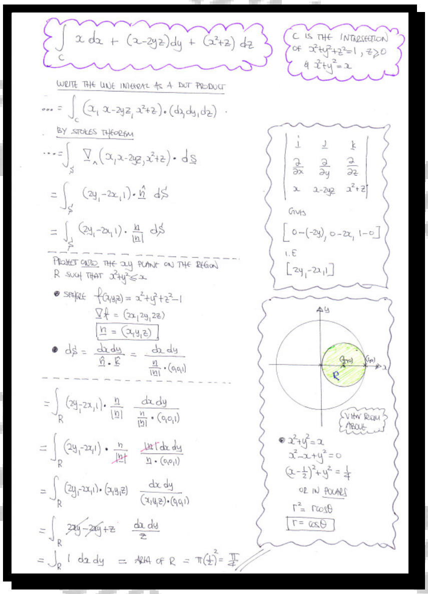
Evaluate the line integral

$$\oint_C \left[x\,dx + (x-2yz)\,dy + \left(x^2+z\right)dz \right],$$

where C is the intersection of the surfaces with respective Cartesian equations

$$x^2+y^2+z^2=1, \quad z \geq 0 \qquad \text{and} \qquad x^2+y^2=x, \quad z \geq 0 .$$

$$\boxed{\frac{\pi}{4}}$$


$\int_C x\,dx + (x-2yz)\,dy + (x^2+z)\,dz$

C IS THE INTERSECTION OF $x^2+y^2+z^2=1$, $z \geq 0$ & $x^2+y^2=x$

WRITE THE LINE INTEGRAL AS A DOT PRODUCT

$\dots = \int_C (x, x-2yz, x^2+z)\cdot(dx, dy, dz)$

BY STOKES THEOREM

$\dots = \int_S \nabla \wedge (x, x-2yz, x^2+z)\cdot d\underline{S}$

$= \int_S (2y, -2x, 1)\cdot \hat{n}\, dS$

$= \int_S (2y, -2x, 1)\cdot \frac{\underline{n}}{|\underline{n}|}\, dS$

$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & x-2yz & x^2+z \end{vmatrix}$

GIVES

$[0-(-2y), 0-2x, 1-0]$

I.E

$[2y, -2x, 1]$

PROJECT ONTO THE xy PLANE ON THE REGION R SUCH THAT $x^2+y^2 \leq x$

• SPHERE $f(x,y,z) = x^2+y^2+z^2-1$

$\nabla f = (2x, 2y, 2z)$

$\boxed{\underline{n} = (x,y,z)}$

• $dS = \frac{dx\,dy}{\hat{n}\cdot\underline{k}} = \frac{dx\,dy}{\frac{\underline{n}}{|\underline{n}|}\cdot(0,0,1)}$

$= \int_R (2y, -2x, 1)\cdot \frac{\underline{n}}{|\underline{n}|}\, \frac{dx\,dy}{\frac{\underline{n}}{|\underline{n}|}\cdot(0,0,1)}$

$= \int_R (2y, -2x, 1)\cdot \frac{\underline{n}}{|\underline{n}|}\, \frac{|\underline{n}|\,dx\,dy}{\underline{n}\cdot(0,0,1)}$

$= \int_R (2y, -2x, 1)\cdot(x,y,z)\, \frac{dx\,dy}{(x,y,z)\cdot(0,0,1)}$

$= \int_R 2xy - 2xy + z\, \frac{dx\,dy}{z}$

$= \int_R 1\, dx\,dy = \text{AREA OF } R = \pi\left(\frac{1}{2}\right)^2 = \frac{\pi}{4}$

• $x^2+y^2=x$

$x^2-x+y^2=0$

$\left(x-\frac{1}{2}\right)^2+y^2=\frac{1}{4}$

OR IN POLARS

$r^2 = r\cos\theta$

$\boxed{r=\cos\theta}$

Question 17

It is given that the vector field $\mathbf{F}$ satisfies

$$\mathbf{F} = 8z\,\mathbf{i} + 4x\,\mathbf{j} + y\,\mathbf{k}\,.$$

Evaluate the line integral

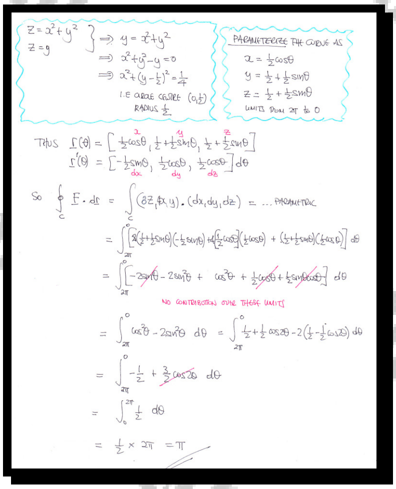
$$\oint_C \mathbf{F} \cdot \mathbf{dr}\,,$$

where C is the intersection of the surfaces with respective Cartesian equations

$$z = x^2 + y^2 \qquad \text{and} \qquad z = y\,.$$

*You may **not** use Stokes' Theorem in this question.*

$\boxed{\pi}$



Question 18

It is given that the vector field $\mathbf{F}$ satisfies

$$\mathbf{F} = y^2\,\mathbf{i} + z^2\,\mathbf{j} + x^2\,\mathbf{k}\,.$$

Evaluate the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r}\,,$$

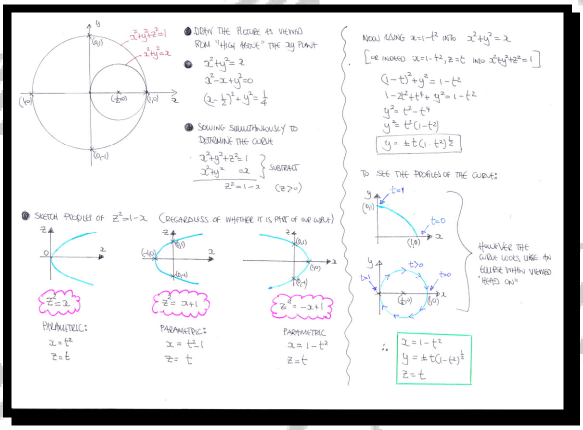
where C is the intersection of the surfaces with respective Cartesian equations

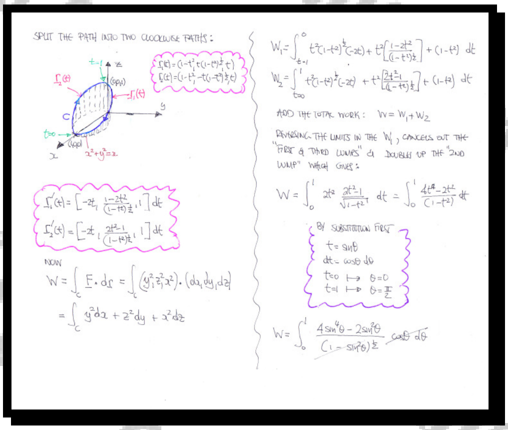
$$x^2 + y^2 + z^2 = 1,\quad z \geq 0 \qquad \text{and} \qquad x^2 + y^2 = x,\quad z \geq 0\,.$$

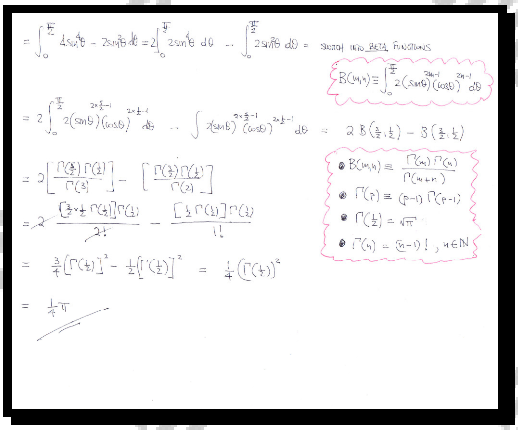
*You may **not** use Stokes' Theorem in this question.*

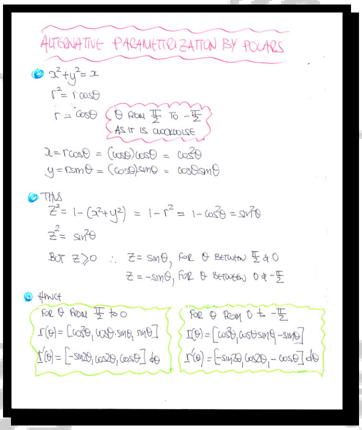
$$\boxed{\frac{\pi}{4}}$$

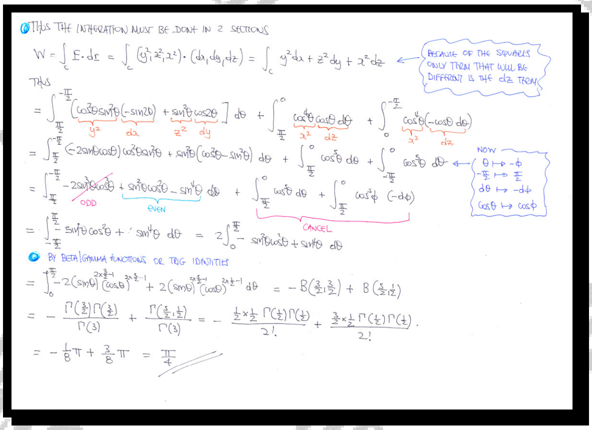
[solution overleaf]







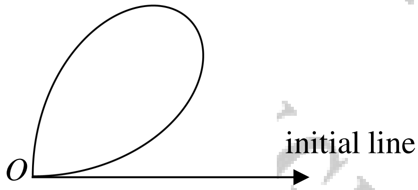

ALTERNATIVE PARAMETRIZATION BY POLARS



LINE INTEGRALS

IN POLAR COORDINATES

Question 1



The figure above shows the closed curve C with polar equation

$$r = \sin 2\theta \, , \; 0 \le \theta \le \frac{\pi}{2} .$$

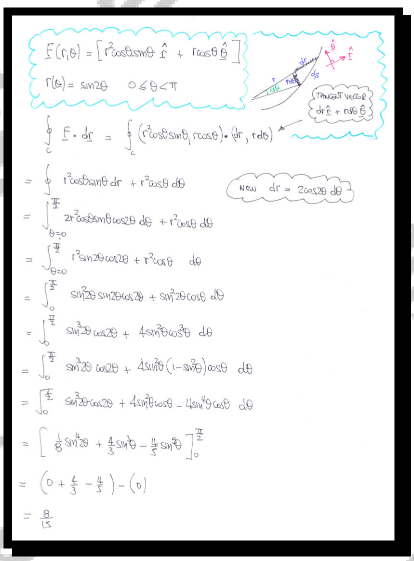
The vector field $\mathbf{F}$ is given in plane polar coordinates (r, θ) by

$$\mathbf{F}(r,\theta) = \left(r^2 \cos\theta \sin\theta\right)\hat{\mathbf{r}} + (r\cos\theta)\,\hat{\boldsymbol{\theta}} .$$

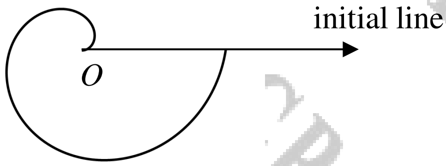
Evaluate the line integral

$$\oint_C \mathbf{F} \cdot \mathbf{dr} .$$

$\boxed{\frac{8}{15}}$



Question 2



The figure above shows the curve C with polar equation

$$r = \theta\,,\ 0 \le \theta \le 2\pi\,.$$

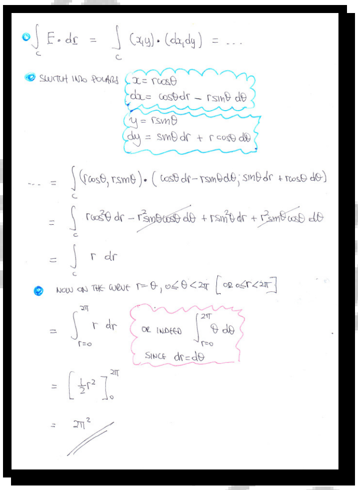
The vector field $\mathbf{F}$ is given in Cartesian coordinates by

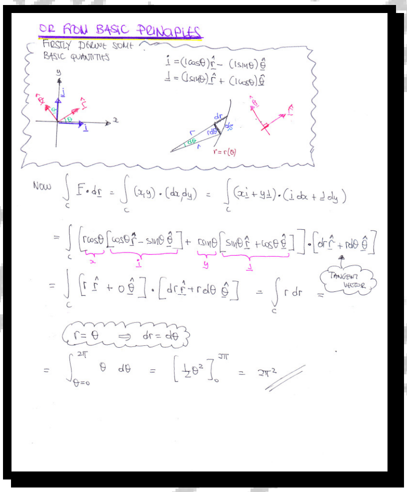
$$\mathbf{F}(x, y) = x\,\mathbf{i} + y\,\mathbf{j}\,.$$

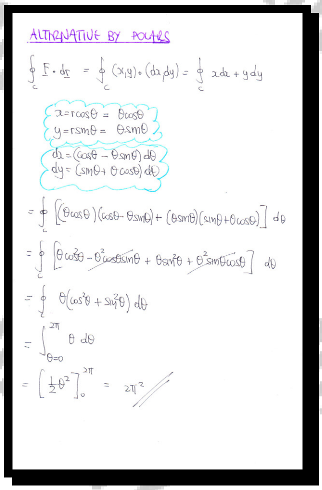
Evaluate the line integral

$$\int_C \mathbf{F} \cdot \mathbf{dr}\,.$$

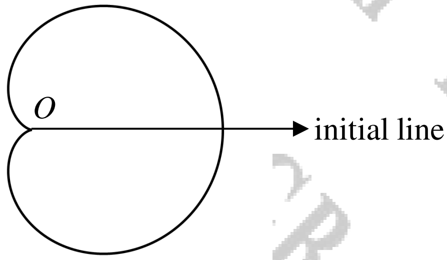
$\boxed{2\pi^2}$







Question 3



The figure above shows the closed curve C with polar equation

$$r = 1 + \cos\theta, \quad 0 \le \theta \le 2\pi .$$

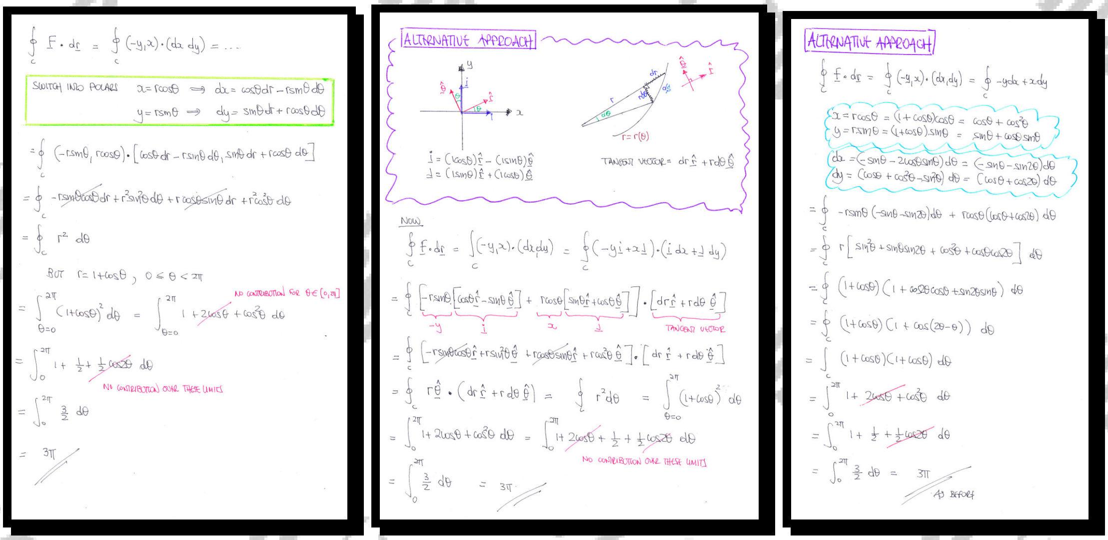
The vector field $\mathbf{F}$ is given in Cartesian coordinates by

$$\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j} .$$

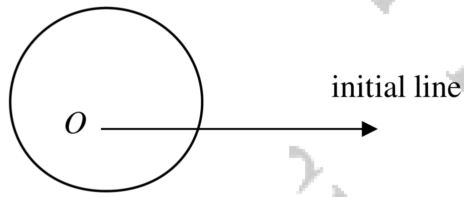
Evaluate the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r} .$$

$\boxed{3\pi}$



Question 4



The figure above shows the closed curve C with polar equation

$$r = 3 + \sin\theta, \ 0 \le \theta \le 2\pi .$$

The vector field $\mathbf{F}$ is given in Cartesian coordinates by

$$\mathbf{F}(x, y) = (x + y)\mathbf{i} + (-x + y)\mathbf{j} .$$

Evaluate the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r} .$$

$\boxed{}$, $\boxed{19\pi}$