Jordan-Gauss Elimination

Unique Solutions

Question 1

Solve the following simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

$$\begin{pmatrix} 1 & 1 & -3 \\ 2 & 1 & 4 \\ 5 & 2 & 16 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix}$$

V , $\boxed{}$, $\boxed{x=-10, \quad y=19, \quad z=1}$

WRITE THE SYSTEM INTO AN AUGMENTED MATRIX

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 6 \\ 2 & 1 & 4 & 3 \\ 5 & 2 & 16 & 4 \end{array}\right] \begin{array}{l} r_{12}(-2) \\ r_{13}(-5) \end{array} = \left[\begin{array}{ccc|c} 1 & 1 & -3 & 6 \\ 0 & -1 & 10 & -9 \\ 0 & -3 & 31 & -26 \end{array}\right] r_2(-1) = \left[\begin{array}{ccc|c} 1 & 1 & -3 & 6 \\ 0 & 1 & -10 & 9 \\ 0 & -3 & 31 & -26 \end{array}\right]$$

$$r_{23}(3) = \left[\begin{array}{ccc|c} 1 & 1 & -3 & 6 \\ 0 & 1 & -10 & 9 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad r_{21}(-1) = \left[\begin{array}{ccc|c} 1 & 0 & 7 & -3 \\ 0 & 1 & -10 & 9 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\begin{array}{l} r_{32}(10) \\ r_{31}(-7) \end{array} = \begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -10 \\ 0 & 1 & 0 & 19 \\ 0 & 0 & 1 & 1 \end{array}\right] \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \\ x & y & z & \end{array} \end{array}$$

$\therefore \ x=-10, \ y=19, \ z=1$

KEY TO THE ROW OPERATIONS

r_{12} = SWAP ROW 1 & 2

$r_3\left(\frac{1}{2}\right)$ = MULTIPLY ROW 3 BY $\frac{1}{2}$

$r_{23}(-4)$ = MULTIPLY ROW 2 BY -4, AND ADD IT INTO ROW 3

Question 2

$$\begin{aligned} x + 3y + 5z &= 6 \\ 6x - 8y + 4z &= -3 \\ 3x + 11y + 13z &= 17 \end{aligned}$$

Solve the above system of simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

V, , $\boxed{x = -\frac{1}{2},\ y = \frac{1}{2},\ z = 1}$

TRANSFER THE SYSTEM INTO MATRIX FORM

$$\left.\begin{aligned} x + 3y + 5z &= 6 \\ 6x - 8y + 4z &= -3 \\ 3x + 11y + 13z &= 17 \end{aligned}\right\} \Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 5 & 6 \\ 6 & -8 & 4 & -3 \\ 3 & 11 & 13 & 17 \end{array}\right]$$

APPLY ROW OPERATIONS

$$\begin{array}{c} r_{12}(-6) \\ r_{13}(-3) \end{array} \left[\begin{array}{ccc|c} 1 & 3 & 5 & 6 \\ 0 & -26 & -26 & -39 \\ 0 & 2 & -2 & -1 \end{array}\right] \underset{=}{r_2\left(-\tfrac{1}{26}\right)} \left[\begin{array}{ccc|c} 1 & 3 & 5 & 6 \\ 0 & 1 & 1 & \tfrac{3}{2} \\ 0 & 2 & -2 & -1 \end{array}\right] \underset{=}{r_{23}(-2)}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 6 \\ 0 & 1 & 1 & 3/2 \\ 0 & 0 & -4 & -4 \end{array}\right] \underset{=}{r_3\left(-\tfrac{1}{4}\right)} \left[\begin{array}{ccc|c} 1 & 3 & 5 & 6 \\ 0 & 1 & 1 & \tfrac{3}{2} \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{c} r_{32}(-1) \\ r_{31}(-5) \end{array} =$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & 1 & 0 & \tfrac{1}{2} \\ 0 & 0 & 1 & 1 \end{array}\right] r_{21}(-3) \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\tfrac{1}{2} \\ 0 & 1 & 0 & \tfrac{1}{2} \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$\therefore\ x = -\frac{1}{2},\ y = \frac{1}{2},\ z = 1$

KEY TO ROW OPERATIONS

r_{12} = SWAP ROW 1 & 2

$r_3\left(\frac{1}{2}\right)$ = MULTIPLY ROW 3 BY $\frac{1}{2}$

$r_{23}(-2)$ = MULTIPLY ROW 2 BY −2 AND ADD IT INTO ROW 3

Question 3

$$\begin{aligned} x + 5y + 7z &= 41 \\ 5x - 4y + 6z &= 2 \\ 7x + 9y - 3z &= 1 \end{aligned}$$

Solve the above system of simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

$\boxed{}$, $\boxed{x = -2, \quad y = 3, \quad z = 4}$

PUT THE SYSTEM INTO A AUGMENT MATRIX

$$\left.\begin{aligned} x + 5y + 7z &= 41 \\ 5x - 4y + 6z &= 2 \\ 7x + 9y - 3z &= 1 \end{aligned}\right\} \Rightarrow \left[\begin{array}{ccc|c} 1 & 5 & 7 & 41 \\ 5 & -4 & 6 & 2 \\ 7 & 9 & -3 & 1 \end{array}\right]$$

APPLY ELEMENTARY ROW OPERATIONS, TO REDUCE THE MATRIX

$$\begin{matrix} r_{12}(-5) \\ r_{13}(-7) \end{matrix} = \left[\begin{array}{ccc|c} 1 & 5 & 7 & 41 \\ 0 & -29 & -29 & -203 \\ 0 & -26 & -52 & -286 \end{array}\right] \quad r_2\left(-\tfrac{1}{29}\right) = \left[\begin{array}{ccc|c} 1 & 5 & 7 & 41 \\ 0 & 1 & 1 & 7 \\ 0 & -26 & -52 & -286 \end{array}\right]$$

$$r_{23}(26) = \left[\begin{array}{ccc|c} 1 & 5 & 7 & 41 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & -26 & -104 \end{array}\right] \quad r_3\left(-\tfrac{1}{26}\right) = \left[\begin{array}{ccc|c} 1 & 5 & 7 & 41 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$$r_{21}(-5) = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 6 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 1 & 4 \end{array}\right] \quad \begin{matrix} r_{31}(-2) \\ r_{32}(-1) \end{matrix} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ x & y & z \end{matrix}$$

$\therefore\ x = -2,\ y = 3,\ z = 4$

KEY TO ROW OPERATIONS

r_{12} = SWAP ROW 1 & 2

$r_3\left(-\tfrac{1}{2}\right)$ = MULTIPLY ROW 3 BY $-\tfrac{1}{2}$

$r_{23}(5)$ = MULTIPLY ROW 2 BY 5, AND ADD IT INTO ROW 3

Question 4

$$\begin{aligned} 4x \ + 2y + 7z &= \ \ 2 \\ 10x - 4y - 5z &= 50 \\ 4x \ + 3y + 9z &= -2 \end{aligned}$$

Solve the above system of simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

$\boxed{x = 4, \ \ y = 0, \ \ z = -2}$

START BY CREATING AN AUGMENTED MATRIX / AS y HAS BETTER COEFFICIENTS REWRITE AS FOLLOWS

$$\left.\begin{aligned} 4x + 2y + 7z &= 2 \\ 10x - 4y - 5z &= 50 \\ 4x + 3y + 9z &= -2 \end{aligned}\right\} \Longrightarrow \left.\begin{aligned} 2y + 4x + 7z &= 2 \\ -4y + 10x - 5z &= 50 \\ 3y + 4x + 9z &= -2 \end{aligned}\right\} \Longrightarrow$$

$$\left.\begin{aligned} y + 2x + \tfrac{7}{2}z &= 1 \\ -4y + 10x - 5z &= 50 \\ 3y + 4x + 9z &= -2 \end{aligned}\right\} \Longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & \frac{7}{2} & 1 \\ -4 & 10 & -5 & 50 \\ 3 & 4 & 9 & -2 \end{array}\right]$$

APPLY ROW OPERATIONS

$$\begin{matrix} r_{12}(4) \\ r_{13}(-3) \end{matrix} = \left[\begin{array}{ccc|c} 1 & 2 & \frac{7}{2} & 1 \\ 0 & 18 & 9 & 54 \\ 0 & -2 & -\frac{3}{2} & -5 \end{array}\right] \quad \begin{matrix} r_{2}(\frac{1}{18}) \\ r_{3}(\frac{1}{2}) \end{matrix} = \left[\begin{array}{ccc|c} 1 & 2 & \frac{7}{2} & 1 \\ 0 & 1 & \frac{1}{2} & 3 \\ 0 & -1 & -\frac{3}{4} & -\frac{5}{2} \end{array}\right]$$

$$r_{23}(1) = \left[\begin{array}{ccc|c} 1 & 2 & \frac{7}{2} & 1 \\ 0 & 1 & \frac{1}{2} & 3 \\ 0 & 0 & -\frac{1}{4} & \frac{1}{2} \end{array}\right] \quad r_{3}(-4) = \left[\begin{array}{ccc|c} 1 & 2 & \frac{7}{2} & 1 \\ 0 & 1 & \frac{1}{2} & 3 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$r_{21}(-2) = \left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{2} & -5 \\ 0 & 1 & \frac{1}{2} & 3 \\ 0 & 0 & 1 & -2 \end{array}\right] \quad \begin{matrix} r_{32}(-\frac{1}{2}) \\ r_{31}(-\frac{5}{2}) \end{matrix} = \begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right] \\ \begin{array}{cccc} y & x & z & \end{array} \end{array}$$

$\therefore \ x = 4, \ y = 0, \ z = -2$

KEY TO ROW OPERATIONS

r_{12} = SWAP ROW 1 & 2

$r_{3}(\frac{1}{2})$ = MULTIPLY ROW 3 BY $\frac{1}{2}$

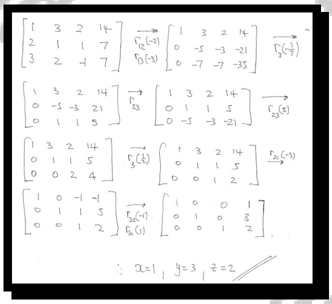
$r_{21}(-3)$ = MULTIPLY ROW 2 BY -3, AND ADD IT INTO ROW 1

Question 5

$$\begin{aligned} x+3y+2z &= 14 \\ 2x+y+z &= 7 \\ 3x+2y-z &= 7 \end{aligned}$$

Solve the above system of simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

$$\boxed{x=1,\ \ y=3,\ \ z=2}$$

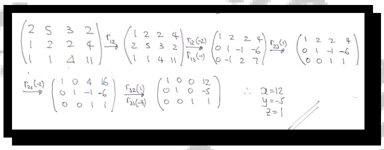


Question 6

$$\begin{aligned} 2x+5y+3z &= 2 \\ x+2y+2z &= 4 \\ x+y+4z &= 11 \end{aligned}$$

Solve the above simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

$$\boxed{x=12,\ \ y=-5,\ \ z=1}$$

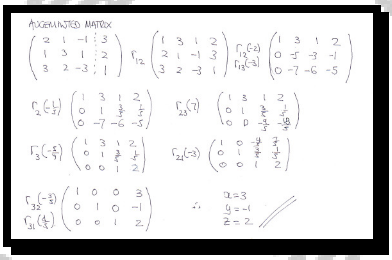


Question 7

$$\begin{aligned} 2x + y - z &= 3 \\ x + 3y + z &= 2 \\ 3x + 2y - 3z &= 1 \end{aligned}$$

Solve the above system of simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

$$\boxed{x = 3,\ \ y = -1,\ \ z = 2}$$

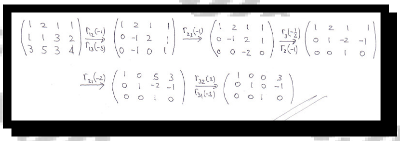


Question 8

Solve the following simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 3 \\ 3 & 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

$$\boxed{x = 3,\ \ y = -1,\ \ z = 0}$$



Question 9

$$\begin{aligned} x + 3y + 2z &= 13 \\ 3x + 2y - z &= 4 \\ 2x + y + z &= 7 \end{aligned}$$

Solve the above system of simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

$$\boxed{x = 1,\ \ y = 2,\ \ z = 3}$$

AUGMENTED MATRIX

$$\begin{pmatrix} 1 & 3 & 2 & 13 \\ 3 & 2 & -1 & 4 \\ 2 & 1 & 1 & 7 \end{pmatrix} \begin{matrix} r_{12}(-3) \\ r_{13}(-2) \end{matrix} \begin{pmatrix} 1 & 3 & 2 & 13 \\ 0 & -7 & -7 & -35 \\ 0 & -5 & -3 & -19 \end{pmatrix} r_2\left(-\tfrac{1}{7}\right) \begin{pmatrix} 1 & 3 & 2 & 13 \\ 0 & 1 & 1 & 5 \\ 0 & -5 & -3 & -19 \end{pmatrix}$$

$$r_{23}(5) \begin{pmatrix} 1 & 3 & 2 & 13 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 2 & 6 \end{pmatrix} r_3\left(\tfrac{1}{2}\right) \begin{pmatrix} 1 & 3 & 2 & 13 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 \end{pmatrix} r_{21}(-3) \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

$$\begin{matrix} r_{32}(-1) \\ r_{31}(1) \end{matrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix} \qquad \therefore \begin{matrix} x = 1 \\ y = 2 \\ z = 3 \end{matrix}$$

Question 10

Solve the following simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ 8 \end{pmatrix}$$

$$\boxed{x = 2,\ \ y = -1,\ \ z = 4}$$

AUGMENTED MATRIX

$$\begin{pmatrix} 1 & 1 & 1 & 5 \\ 2 & 4 & 2 & 8 \\ 1 & 2 & 2 & 8 \end{pmatrix} \begin{matrix} r_{12}(-2) \\ r_{13}(-1) \end{matrix} \begin{pmatrix} 1 & 1 & 1 & 5 \\ 0 & 2 & 0 & -2 \\ 0 & 1 & 1 & 3 \end{pmatrix} r_{23} \begin{pmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 0 & -2 \end{pmatrix} r_{23}(-2)$$

$$\begin{pmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -2 & -8 \end{pmatrix} r_3\left(-\tfrac{1}{2}\right) \begin{pmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{pmatrix} r_{21}(-1) \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

$$r_{32}(-1) \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{pmatrix} \qquad \therefore \begin{matrix} x = 2 \\ y = -1 \\ z = 4 \end{matrix}$$

Question 11

$$\begin{aligned} x + 5y + 7z &= 41 \\ 5x - 4y + 6z &= 2 \\ 7x + 9y - 3z &= k \end{aligned}$$

Use the Jordan Gauss algorithm to determine the solution of the above system of simultaneous equations, giving the answers in terms of the constant k.

$\boxed{}$, $\boxed{x = \dfrac{k-27}{13}, \quad y = \dfrac{k+77}{26}, \quad z = \dfrac{105-k}{26}}$

PROCEED BY THE JORDAN GAUSS ALGORITHM

$$\left[\begin{array}{ccc|c} 1 & 5 & 7 & 41 \\ 5 & -4 & 6 & 2 \\ 7 & 9 & -3 & k \end{array}\right] \xrightarrow[r_{13}(-7)]{r_{12}(-5)} \left[\begin{array}{ccc|c} 1 & 5 & 7 & 41 \\ 0 & -29 & -29 & -203 \\ 0 & -26 & -52 & -287+k \end{array}\right] \xrightarrow{r_2\left(-\frac{1}{29}\right)}$$

$$\left[\begin{array}{ccc|c} 1 & 5 & 7 & 41 \\ 0 & 1 & 1 & 7 \\ 0 & -26 & -52 & k-287 \end{array}\right] \xrightarrow{r_{23}(26)} \left[\begin{array}{ccc|c} 1 & 5 & 7 & 41 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & -26 & k-105 \end{array}\right] \xrightarrow{r_3\left(-\frac{1}{26}\right)}$$

$$\left[\begin{array}{ccc|c} 1 & 5 & 7 & 41 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 1 & \frac{105-k}{26} \end{array}\right] \xrightarrow{r_{21}(-5)} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 6 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 1 & \frac{105-k}{26} \end{array}\right] \xrightarrow[r_{31}(-2)]{r_{32}(-1)}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 6+\frac{k-105}{13} \\ 0 & 1 & 0 & 7+\frac{k-105}{26} \\ 0 & 0 & 1 & \frac{105-k}{26} \end{array}\right]$$

$$\therefore x = 6 + \frac{k-105}{13} = \frac{6\times13 + k - 105}{13} = \frac{k-27}{13}$$

$$y = 7 + \frac{k-105}{26} = \frac{26\times7 + k - 105}{26} = \frac{k+77}{26}$$

$$z = \frac{105-k}{26}$$

Non-Unique Solutions

Question 1

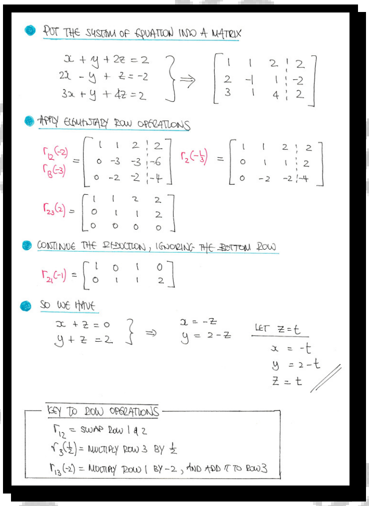
$$
\begin{aligned}
x + y + 2z &= \ \ 2 \\
2x - y \ \ + z &= -2 \\
3x + y \ + 4z &= \ \ 2
\end{aligned}
$$

Show, by reducing the augmented matrix of the above system of equations into row echelon form, that the solution can be written as

$$x = -t\,, \qquad y = 2 - t\,, \qquad z = t$$

where t is a scalar parameter.

[ ] , [proof]



- PUT THE SYSTEM OF EQUATION INTO A MATRIX

$$\left.\begin{aligned} x + y + 2z &= 2 \\ 2x - y + z &= -2 \\ 3x + y + 4z &= 2 \end{aligned}\right\} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 2 & -1 & 1 & -2 \\ 3 & 1 & 4 & 2 \end{array}\right]$$

- APPLY ELEMENTARY ROW OPERATIONS

$$\begin{matrix} r_{12}(-2) \\ r_{13}(-3) \end{matrix} = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & -3 & -3 & -6 \\ 0 & -2 & -2 & -4 \end{array}\right] \quad r_2\left(-\tfrac{1}{3}\right) = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & -2 & -2 & -4 \end{array}\right]$$

$$r_{23}(2) = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- CONTINUE THE REDUCTION, IGNORING THE BOTTOM ROW

$$r_{21}(-1) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

- SO WE HAVE

$$\left.\begin{aligned} x + z &= 0 \\ y + z &= 2 \end{aligned}\right\} \Rightarrow \begin{aligned} x &= -z \\ y &= 2 - z \end{aligned}$$

LET $z = t$

$$x = -t \qquad y = 2 - t \qquad z = t$$

KEY TO ROW OPERATIONS

r_{12} = SWAP ROW 1 & 2

$r_3\left(\frac{1}{2}\right)$ = MULTIPLY ROW 3 BY $\frac{1}{2}$

$r_{13}(-2)$ = MULTIPLY ROW 1 BY -2, AND ADD IT TO ROW 3

Question 2

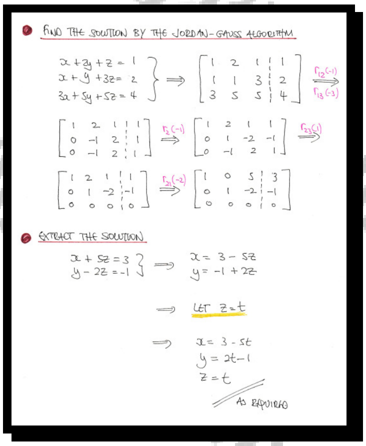
$$\begin{aligned} x+2y+\ \ z &= 1 \\ x+\ \ y+3z &= 2 \\ 3x+5y+5z &= 4 \end{aligned}$$

Show that the solution of the above simultaneous equations is

$$x = 3-5t\,, \qquad y = 2t-1\,, \qquad z = t$$

where t is a parameter.

V, $\boxed{}$, $\boxed{\text{proof}}$



FIND THE SOLUTION BY THE JORDAN–GAUSS ALGORITHM

$$\left.\begin{aligned} x+2y+z&=1\\ x+y+3z&=2\\ 3x+5y+5z&=4\end{aligned}\right\} \Longrightarrow \left[\begin{array}{ccc|c} 1&2&1&1\\ 1&1&3&2\\ 3&5&5&4\end{array}\right] \xrightarrow[r_{13}(-3)]{r_{12}(-1)}$$

$$\left[\begin{array}{ccc|c} 1&2&1&1\\ 0&-1&2&1\\ 0&-1&2&1\end{array}\right] \xrightarrow{r_2(-1)} \left[\begin{array}{ccc|c} 1&2&1&1\\ 0&1&-2&-1\\ 0&-1&2&1\end{array}\right] \xrightarrow{r_{23}(1)}$$

$$\left[\begin{array}{ccc|c} 1&2&1&1\\ 0&1&-2&-1\\ 0&0&0&0\end{array}\right] \xrightarrow{r_{21}(-2)} \left[\begin{array}{ccc|c} 1&0&5&3\\ 0&1&-2&-1\\ 0&0&0&0\end{array}\right]$$

EXTRACT THE SOLUTION

$$\left.\begin{aligned} x+5z&=3\\ y-2z&=-1\end{aligned}\right\} \Longrightarrow \begin{aligned} x&=3-5z\\ y&=-1+2z\end{aligned}$$

$\Longrightarrow$ LET $z=t$

$$\Longrightarrow \begin{aligned} x&=3-5t\\ y&=2t-1\\ z&=t\end{aligned}$$

AS REQUIRED

Question 3

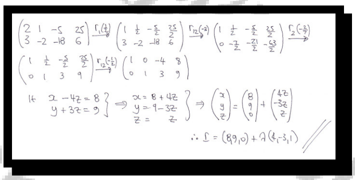
$$\begin{aligned} 3x - 2y - 18z &= 6 \\ 2x + y - 5z &= 25 \end{aligned}$$

Show, by reducing the above system of equations into row echelon form, that the solution can be written as

$$\mathbf{r} = 8\mathbf{i} + 9\mathbf{j} + \lambda(4\mathbf{i} - 3\mathbf{j} + \mathbf{k}),$$

where λ is a scalar parameter.

$\boxed{\text{proof}}$



Question 4

$$\begin{aligned} x + y - 2z &= 2 \\ 3x - y + 6z &= 2 \\ 6x + 5y - 9z &= 11 \end{aligned}$$

Show, by reducing the above equation system into row echelon form, that the consistent solution of the system can be written as

$$x = 1 - t\,, \qquad y = 3t + 1\,, \qquad z = t$$

where t is a scalar parameter.

$\boxed{}$, $\boxed{\text{proof}}$

START BY WRITING THE SYSTEM IN MATRIX FORM

$$\left.\begin{aligned} x + y - 2z &= 2 \\ 3x - y + 6z &= 2 \\ 6x + 5y - 9z &= 11 \end{aligned}\right\} \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -2 & 2 \\ 3 & -1 & 6 & 2 \\ 6 & 5 & -9 & 11 \end{array}\right]$$

APPLY STANDARD JORDAN - GAUSS ELIMINATION BY ROW OPERATIONS

$$\begin{matrix} r_{12}(-3) \\ r_{13}(-6) \end{matrix} = \left[\begin{array}{ccc|c} 1 & 1 & -2 & 2 \\ 0 & -4 & 12 & -4 \\ 0 & -1 & 3 & -1 \end{array}\right] \quad r_2(-\tfrac{1}{4}) = \left[\begin{array}{ccc|c} 1 & 1 & -2 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & -1 & 3 & -1 \end{array}\right]$$

$$r_{23}(1) = \left[\begin{array}{ccc|c} 1 & 1 & -2 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad r_{21}(-1) = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

EXTRACTING THE SOLUTION WE HAVE

$$\left.\begin{aligned} x + z &= 1 \\ y - 3z &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} x &= 1 - z \\ y &= 1 + 3z \end{aligned}$$

$$\Longrightarrow \text{LET } z = t$$

$$x = 1 - t$$
$$y = 1 + 3t$$

AS REQUIRED

KEY TO OPERATIONS

r_{12} : SWAP ROW 1 & 2

$r_2(-\tfrac{1}{3})$: MULTIPLY ROW 2 BY $-\tfrac{1}{3}$

$r_{31}(-2)$: MULTIPLY ROW 3 BY -2, AND ADD IT TO ROW 1

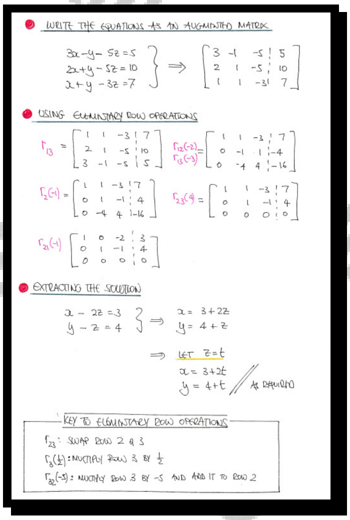
Question 5

$$3x - y - 5z = 5$$
$$2x + y - 5z = 10$$
$$x + y - 3z = 7$$

Show, by reducing the above system into row echelon form, that the consistent solution of the system can be written as

$$x = 2t + 3, \quad y = t + 4, \quad z = t.$$

V, $\boxed{}$, $\boxed{\text{proof}}$



WRITE THE EQUATIONS AS AN AUGMENTED MATRIX

$$\left.\begin{aligned} 3x - y - 5z &= 5 \\ 2x + y - 5z &= 10 \\ x + y - 3z &= 7 \end{aligned}\right\} \Longrightarrow \left[\begin{array}{ccc|c} 3 & -1 & -5 & 5 \\ 2 & 1 & -5 & 10 \\ 1 & 1 & -3 & 7 \end{array}\right]$$

USING ELEMENTARY ROW OPERATIONS

$$r_{13} = \left[\begin{array}{ccc|c} 1 & 1 & -3 & 7 \\ 2 & 1 & -5 & 10 \\ 3 & -1 & -5 & 5 \end{array}\right] \quad \begin{array}{l} r_{12}(-2) \\ r_{13}(-3) \end{array} = \left[\begin{array}{ccc|c} 1 & 1 & -3 & 7 \\ 0 & -1 & 1 & -4 \\ 0 & -4 & 4 & -16 \end{array}\right]$$

$$r_2(-1) = \left[\begin{array}{ccc|c} 1 & 1 & -3 & 7 \\ 0 & 1 & -1 & 4 \\ 0 & -4 & 4 & -16 \end{array}\right] \quad r_{23}(4) = \left[\begin{array}{ccc|c} 1 & 1 & -3 & 7 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$r_{21}(-1) \left[\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

EXTRACTING THE SOLUTION

$$\left.\begin{aligned} x - 2z &= 3 \\ y - z &= 4 \end{aligned}\right\} \Longrightarrow \begin{aligned} x &= 3 + 2z \\ y &= 4 + z \end{aligned}$$

$$\Longrightarrow \text{LET } z = t$$
$$x = 3 + 2t$$
$$y = 4 + t \quad // \text{ AS REQUIRED}$$

KEY TO ELEMENTARY ROW OPERATIONS

r_{23}: SWAP ROW 2 & 3

$r_3(\frac{1}{2})$: MULTIPLY ROW 3 BY $\frac{1}{2}$

$r_{32}(-5)$: MULTIPLY ROW 3 BY -5 AND ADD IT TO ROW 2

Question 6

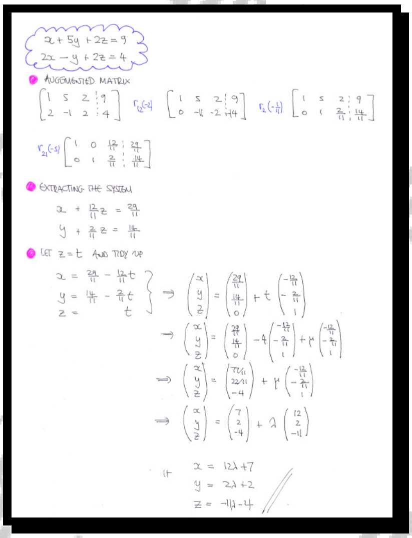
$$\begin{aligned} x+5y+2z &= 9 \\ 2x\ -y\ +2z &= 4 \end{aligned}$$

Show, by reducing the above system of equations into row echelon form, that the solution can be written as

$$x = A\lambda + B\,,\ \ y = C\lambda + D\,,\ \ x = E\lambda + F$$

where A, B, C, D, E and F are integers, and λ is a scalar parameter.

V, $\boxed{x = 12\lambda + 7}$, $\boxed{y = 2\lambda + 2}$, $\boxed{z = -11\lambda - 4}$



$$\begin{aligned} x+5y+2z &= 9 \\ 2x-y+2z &= 4 \end{aligned}$$

AUGMENTED MATRIX

$$\left[\begin{array}{ccc|c} 1 & 5 & 2 & 9 \\ 2 & -1 & 2 & 4 \end{array}\right] \xrightarrow{r_{12}(-2)} \left[\begin{array}{ccc|c} 1 & 5 & 2 & 9 \\ 0 & -11 & -2 & -14 \end{array}\right] \xrightarrow{r_2(-\frac{1}{11})} \left[\begin{array}{ccc|c} 1 & 5 & 2 & 9 \\ 0 & 1 & \frac{2}{11} & \frac{14}{11} \end{array}\right]$$

$$\xrightarrow{r_{21}(-5)} \left[\begin{array}{ccc|c} 1 & 0 & \frac{12}{11} & \frac{29}{11} \\ 0 & 1 & \frac{2}{11} & \frac{14}{11} \end{array}\right]$$

EXTRACTING THE SYSTEM

$$x + \frac{12}{11}z = \frac{29}{11}$$
$$y + \frac{2}{11}z = \frac{14}{11}$$

LET $z = t$ AND TIDY UP

$$\left.\begin{aligned} x &= \tfrac{29}{11} - \tfrac{12}{11}t \\ y &= \tfrac{14}{11} - \tfrac{2}{11}t \\ z &= t \end{aligned}\right\} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{29}{11} \\ \frac{14}{11} \\ 0 \end{pmatrix} + t\begin{pmatrix} -\frac{12}{11} \\ -\frac{2}{11} \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{29}{11} \\ \frac{14}{11} \\ 0 \end{pmatrix} - 4\begin{pmatrix} -\frac{12}{11} \\ -\frac{2}{11} \\ 1 \end{pmatrix} + \mu\begin{pmatrix} -\frac{12}{11} \\ -\frac{2}{11} \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 77/11 \\ 22/11 \\ -4 \end{pmatrix} + \mu\begin{pmatrix} -\frac{12}{11} \\ -\frac{2}{11} \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \\ -4 \end{pmatrix} + \lambda\begin{pmatrix} 12 \\ 2 \\ -11 \end{pmatrix}$$

$$\text{IE} \quad \begin{aligned} x &= 12\lambda + 7 \\ y &= 2\lambda + 2 \\ z &= -11\lambda - 4 \end{aligned} \quad //$$

Question 7

$$\begin{aligned} x+y+z&=0\\ 2x+4z+w&=-1\\ 3x+2y+4z+w&=0 \end{aligned}$$

Find a general solution of the above system of simultaneous equations.

$\boxed{\text{V}}$, $\boxed{\begin{pmatrix} x\\ y\\ z\\ w \end{pmatrix} = \begin{pmatrix} -1\\ 1\\ 0\\ 1 \end{pmatrix} + t\begin{pmatrix} -2\\ 1\\ 1\\ 0 \end{pmatrix}}$

$$\left.\begin{aligned} x+y+z&=0\\ 2x+4z+w&=-1\\ 3x+2y+4z+w&=0 \end{aligned}\right\}\ \text{AUGEMENT MATRIX}\ \left[\begin{array}{cccc|c} 1&1&1&0&0\\ 2&0&4&1&-1\\ 3&2&4&1&0 \end{array}\right]$$

$$\begin{matrix} r_{12}(-2)\\ r_{13}(-3) \end{matrix}\left[\begin{array}{cccc|c} 1&1&1&0&0\\ 0&-2&2&1&-1\\ 0&-1&1&1&0 \end{array}\right] \quad r_{23}\left[\begin{array}{cccc|c} 1&1&1&0&0\\ 0&-1&1&1&-1\\ 0&-2&2&1&0 \end{array}\right]$$

$$r_2(-1)\left[\begin{array}{cccc|c} 1&1&1&0&0\\ 0&1&-1&-1&0\\ 0&-2&2&1&-1 \end{array}\right] \quad r_{23}(2)\left[\begin{array}{cccc|c} 1&1&1&0&0\\ 0&1&-1&-1&0\\ 0&0&0&-1&-1 \end{array}\right]$$

$$r_3(-1)\left[\begin{array}{cccc|c} 1&1&1&0&0\\ 0&1&-1&-1&0\\ 0&0&0&1&1 \end{array}\right] \quad r_{21}(-1)\left[\begin{array}{cccc|c} 1&0&2&1&0\\ 0&1&-1&-1&0\\ 0&0&0&1&1 \end{array}\right]$$

$$\begin{matrix} r_{32}(1)\\ r_{31}(-1) \end{matrix}\left[\begin{array}{cccc|c} 1&0&2&0&-1\\ 0&1&-1&0&1\\ 0&0&0&1&1 \end{array}\right]$$

$$\text{So}\ \left.\begin{aligned} x+2z&=-1\\ y-z&=1\\ w&=1 \end{aligned}\right\} \Rightarrow \begin{aligned} x&=-1-2z\\ y&=1+z\\ z&=z\\ w&=1 \end{aligned}$$

$$\begin{pmatrix} x\\ y\\ z\\ w \end{pmatrix} = \begin{pmatrix} -1\\ 1\\ 0\\ 1 \end{pmatrix} + t\begin{pmatrix} -2\\ 1\\ 1\\ 0 \end{pmatrix}$$

Cramer's Rule

Question 1

Use Cramer's rule to solve the following system of simultaneous equations.

$$\begin{aligned} 3x + y + 2z &= 11 \\ x + y + z &= 4 \\ x - y + 2z &= 9 \end{aligned}$$

No credit will be given for using alternative solution methods.

$\boxed{\text{MTN}}$, $\boxed{x=2}$, $\boxed{y=-1}$, $\boxed{z=3}$

- WRITE THE SYSTEM IN MATRIX NOTATION

$$\left\{\begin{aligned} 3x+y+2z &= 11 \\ x+y+z &= 4 \\ x-y+2z &= 9 \end{aligned}\right\} \Longrightarrow \begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ 4 \\ 9 \end{bmatrix}$$

- WORK OUT THE DETERMINANT OF THE SYSTEM

$$\Delta = \begin{vmatrix} 3 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 3\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + 2\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= 3(+3) - 1 + 2(-2) = 4$$

- CALCULATE $\Delta_x, \Delta_y, \Delta_z$

$$\Delta_x = \begin{vmatrix} 11 & 1 & 2 \\ 4 & 1 & 1 \\ 9 & -1 & 2 \end{vmatrix} = 11\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} - \begin{vmatrix} 4 & 1 \\ 9 & 2 \end{vmatrix} + 2\begin{vmatrix} 4 & 1 \\ 9 & -1 \end{vmatrix}$$

$$= 11(+3) - (-1) + 2(-13)$$

$$= 8$$

$$\Delta_y = \begin{vmatrix} 3 & 11 & 2 \\ 1 & 4 & 1 \\ 1 & 9 & 2 \end{vmatrix} = 3\begin{vmatrix} 4 & 1 \\ 9 & 2 \end{vmatrix} - 11\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + 2\begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix}$$

$$= 3(-1) - 11(+1) + 2(+5)$$

$$= -4$$

$$\Delta_z = \begin{vmatrix} 3 & 1 & 11 \\ 1 & 1 & 4 \\ 1 & -1 & 9 \end{vmatrix} = 3\begin{vmatrix} 1 & 4 \\ -1 & 9 \end{vmatrix} - \begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix} + 11\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= 3(+13) - (+5) + 11(-2)$$

$$= 12$$

- HENCE WE HAVE

$$x = \frac{\Delta_x}{\Delta} = \frac{8}{4} = 2$$

$$y = \frac{\Delta_y}{\Delta} = \frac{-4}{4} = -1$$

$$z = \frac{\Delta_z}{\Delta} = \frac{12}{4} = 3$$

i.e. $(x,y,z) = (2,-1,3)$

Question 2

Use Cramer's rule to solve the following system of simultaneous equations.

$$\begin{aligned} 3x \quad - y \quad + z \ &= 7 \\ x \quad + y \quad + 2z \ &= 7 \\ x \quad + 3y \quad + z \ &= 0 \end{aligned}$$

No credit will be given for using alternative solution methods.

$\boxed{}$, $\boxed{x = \frac{1}{2}}$, $\boxed{y = -\frac{3}{2}}$, $\boxed{z = 4}$

WRITE THE EQUATIONS AS A MATRIX EQUATION

$$\left.\begin{aligned} 3x - y + z &= 7 \\ x + y + 2z &= 7 \\ x + 3y + z &= 0 \end{aligned}\right\} \Rightarrow \begin{bmatrix} 3 & -1 & 1 \\ 1 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 0 \end{bmatrix}$$

FIND THE DETERMINANT OF THE SYSTEM

$$\Delta = \begin{vmatrix} 3 & -1 & 1 \\ 1 & 1 & 2 \\ 1 & 3 & 1 \end{vmatrix} = 3\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix}$$

$$= 3(-5) + (-1) + (+2)$$

$$= -14$$

CALCULATE Δ_x, Δ_y & Δ_z

- $\Delta_x = \begin{vmatrix} 7 & -1 & 1 \\ 7 & 1 & 2 \\ 0 & 3 & 1 \end{vmatrix} = 7\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 7 & 2 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 7 & 1 \\ 0 & 3 \end{vmatrix}$

  $= 7(-5) + 7 + 21 = -7$

- $\Delta_y = \begin{vmatrix} 3 & 7 & 1 \\ 1 & 7 & 2 \\ 1 & 0 & 1 \end{vmatrix} = 3\begin{vmatrix} 7 & 2 \\ 0 & 1 \end{vmatrix} - 7\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 7 \\ 1 & 0 \end{vmatrix}$

  $= 3(+7) - 7(-1) + (-7) = 21$

- $\Delta_z = \begin{vmatrix} 3 & -1 & 7 \\ 1 & 1 & 7 \\ 1 & 3 & 0 \end{vmatrix} = 3\begin{vmatrix} 1 & 7 \\ 3 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 7 \\ 1 & 0 \end{vmatrix} + 7\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix}$

  $= 3(-21) + (-7) + 7($

  $= -56$

HENCE WE CAN OBTAIN THE SOLUTION

$$\left.\begin{aligned} x &= \frac{\Delta_x}{\Delta} = \frac{-7}{-14} = \frac{1}{2} \\ y &= \frac{\Delta_y}{\Delta} = \frac{21}{-14} = \frac{3}{2} \\ z &= \frac{\Delta_z}{\Delta} = \frac{-56}{-14} = 4 \end{aligned}\right\} \Rightarrow (x, y, z) = \left(\tfrac{1}{2}, -\tfrac{3}{2}, 4\right)$$

Question 3

$$
\begin{aligned}
x+2y+3z &= 5\\
3x+\ y+2z &= 18\\
4x-\ y+\ z &= 27
\end{aligned}
$$

Solve the above system of the simultaneous equations ...

a) ... by manipulating their augmented matrix into reduced row echelon form.

b) ... by using Cramer's rule.

$\boxed{}$, $\boxed{x=6,\ y=-2,\ z=1}$

a) WRITE THE SYSTEM AS AN AUGMENTED MATRIX

$$\left.\begin{aligned} x+2y+3z&=5\\ 3x+y+2z&=18\\ 4x-y+z&=27\end{aligned}\right\} \Longrightarrow \left[\begin{array}{ccc|c} 1&2&3&5\\ 3&1&2&18\\ 4&-1&1&27\end{array}\right]$$

$$\begin{matrix} r_{12}(-3)\\ r_{13}(-4)\end{matrix} = \left[\begin{array}{ccc|c} 1&2&3&5\\ 0&-5&-7&3\\ 0&-9&-11&7\end{array}\right] \quad r_2\left(-\tfrac{1}{5}\right) = \left[\begin{array}{ccc|c} 1&2&3&5\\ 0&1&\frac{7}{5}&-\frac{3}{5}\\ 0&-9&-11&7\end{array}\right]$$

$$r_{23}(9) = \left[\begin{array}{ccc|c} 1&2&3&5\\ 0&1&\frac{7}{5}&-\frac{3}{5}\\ 0&0&\frac{8}{5}&\frac{8}{5}\end{array}\right] \quad r_3\left(\tfrac{5}{8}\right) = \left[\begin{array}{ccc|c} 1&2&3&5\\ 0&1&\frac{7}{5}&-\frac{3}{5}\\ 0&0&1&1\end{array}\right]$$

$$r_{21}(-2) = \left[\begin{array}{ccc|c} 1&0&\frac{1}{5}&\frac{31}{5}\\ 0&1&\frac{7}{5}&-\frac{3}{5}\\ 0&0&1&1\end{array}\right] \quad \begin{matrix} r_{32}\left(-\frac{7}{5}\right)\\ r_{31}\left(-\frac{1}{5}\right)\end{matrix} = \left[\begin{array}{ccc|c} 1&0&0&6\\ 0&1&0&-2\\ 0&0&1&1\end{array}\right]$$

THUS THE UNIQUE SOLUTION IS $\begin{pmatrix} x\\ y\\ z\end{pmatrix} = \begin{pmatrix} 6\\ -2\\ 1\end{pmatrix}$

b) PREPARE THE REQUIRED DETERMINANTS OF THE SYSTEM

$$\begin{bmatrix} 1&2&3\\ 3&1&2\\ 4&-1&1\end{bmatrix}\begin{bmatrix} x\\ y\\ z\end{bmatrix} = \begin{bmatrix} 5\\ 18\\ 27\end{bmatrix}$$

$$|\underline{A}| = \begin{vmatrix} 1&2&3\\ 3&1&2\\ 4&-1&1\end{vmatrix} = \begin{vmatrix} 1&2\\ -1&1\end{vmatrix} - 2\begin{vmatrix} 3&2\\ 4&1\end{vmatrix} + 3\begin{vmatrix} 3&1\\ 4&-1\end{vmatrix}$$

$$= 3-2(-5)+3(-7) = 3+10-21 = -8$$

$$|\underline{A}_x| = \begin{vmatrix} 5&2&3\\ 18&1&2\\ 27&-1&1\end{vmatrix} = 5\begin{vmatrix} 1&2\\ -1&1\end{vmatrix} - 2\begin{vmatrix} 18&2\\ 27&1\end{vmatrix} + 3\begin{vmatrix} 18&1\\ 27&-1\end{vmatrix}$$

$$= 5\times 3 - 2(-36) + 3(-45) = 15+72-135 = -48$$

$$|\underline{A}_y| = \begin{vmatrix} 1&5&3\\ 3&18&2\\ 4&27&1\end{vmatrix} = \begin{vmatrix} 18&2\\ 27&1\end{vmatrix} - 5\begin{vmatrix} 3&2\\ 4&1\end{vmatrix} + 3\begin{vmatrix} 3&18\\ 4&27\end{vmatrix}$$

$$= -36-5(-5)+3(9) = -36+25+27 = 16$$

$$|\underline{A}_z| = \begin{vmatrix} 1&2&5\\ 3&1&18\\ 4&-1&27\end{vmatrix} = \begin{vmatrix} 1&18\\ -1&27\end{vmatrix} - 2\begin{vmatrix} 3&18\\ 4&27\end{vmatrix} + 5\begin{vmatrix} 3&1\\ 4&-1\end{vmatrix}$$

$$= 45-2(9)+5(-7) = 45-18-35 = -8$$

HENCE BY CRAMER'S RULE

$$x = \frac{|\underline{A}_x|}{|\underline{A}|} = \frac{-48}{-8} = 6$$

$$y = \frac{|\underline{A}_y|}{|\underline{A}|} = \frac{16}{-8} = -2$$

$$z = \frac{|\underline{A}_z|}{|\underline{A}|} = \frac{-8}{-8} = 1$$

Question 4

$$\begin{aligned} 7x &+2y &-3z &= 30 \\ 3x &+4y &-5z &= 14 \\ 5x &-3y &+4z &= 18 \end{aligned}$$

Solve the above system of the simultaneous equations by using Cramer's rule.

, $\boxed{x = 4, \ y = -2, \ z = -2}$

WRITE THE SYSTEM IN MATRIX FORM

$$\underbrace{\begin{pmatrix} 7 & 2 & -3 \\ 3 & 4 & -5 \\ 5 & -3 & 4 \end{pmatrix}}_{\underline{A}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 30 \\ 14 \\ 18 \end{pmatrix}$$

CALCULATE ALL THE RELEVANT DETERMINANTS

- $\det \underline{A} = \begin{vmatrix} 7 & 2 & -3 \\ 3 & 4 & -5 \\ 5 & -3 & 4 \end{vmatrix} = 7\begin{vmatrix} 4 & -5 \\ -3 & 4 \end{vmatrix} - 2\begin{vmatrix} 3 & -5 \\ 5 & 4 \end{vmatrix} - 3\begin{vmatrix} 3 & 4 \\ 5 & -3 \end{vmatrix}$
  $= 7\times 1 - 2\times 37 - 3(-29) = \underline{20}$
- $\det \underline{A}_x = \begin{vmatrix} 30 & 2 & -3 \\ 14 & 4 & -5 \\ 18 & -3 & 4 \end{vmatrix} = 30\begin{vmatrix} 4 & -5 \\ -3 & 4 \end{vmatrix} - 2\begin{vmatrix} 14 & -5 \\ 18 & 4 \end{vmatrix} - 3\begin{vmatrix} 14 & 4 \\ 18 & -3 \end{vmatrix}$
  $= 30\times 1 - 2\times 146 - 3(-114) = \underline{80}$
- $\det \underline{A}_y = \begin{vmatrix} 7 & 30 & -3 \\ 3 & 14 & -5 \\ 5 & 18 & 4 \end{vmatrix} = 7\begin{vmatrix} 14 & -5 \\ 18 & 4 \end{vmatrix} - 30\begin{vmatrix} 3 & -5 \\ 5 & 4 \end{vmatrix} - 3\begin{vmatrix} 3 & 14 \\ 5 & 18 \end{vmatrix}$
  $= 7\times 146 - 30\times 37 - 3(-16) = \underline{-40}$
- $\det \underline{A}_z \begin{vmatrix} 7 & 2 & 30 \\ 3 & 4 & 14 \\ 5 & -3 & 18 \end{vmatrix} = 7\begin{vmatrix} 4 & 14 \\ -3 & 18 \end{vmatrix} - 2\begin{vmatrix} 3 & 14 \\ 5 & 18 \end{vmatrix} + 30\begin{vmatrix} 3 & 4 \\ 5 & -3 \end{vmatrix}$
  $= 7\times 114 - 2(-16) + 30(-29) = \underline{-40}$

HENCE WE HAVE

- $x = \dfrac{\det \underline{A}_x}{\det \underline{A}} = \dfrac{80}{20} = 4$
- $y = \dfrac{\det \underline{A}_y}{\det \underline{A}} = \dfrac{-40}{20} = -2$
- $z = \dfrac{\det \underline{A}_z}{\det \underline{A}} = \dfrac{-40}{20} = -2$ //

Question 5

$$\begin{aligned} x+\ y+\ z+\ w&=2\\ 2x-\ y+2z-\ w&=1\\ 3x+\ y-\ z-\ w&=1\\ 4x+2y+3z-2w&=0 \end{aligned}$$

Use Cramer's rule to find the value of w in the above system of the simultaneous equations

$\boxed{}$, $\boxed{w=\frac{3}{2}}$

WRITE THE HOMOGENEOUS SYSTEM OF EQUATIONS AS A DETERMINANT

$$\begin{aligned} x+y+z+w&=2\\ 2x-y+2z-w&=1\\ 3x+y-z-w&=1\\ 4x+2y+3z-2w&=0 \end{aligned} \qquad \text{ie} \quad \begin{vmatrix} 1&1&1&1\\ 2&-1&2&-1\\ 3&1&-1&-1\\ 4&2&3&-2 \end{vmatrix}$$

BY ROW OPERATIONS & COLUMN OPERATIONS, FOR SIMPLIFICATION

$$\begin{matrix} r_{12}(1)\\ r_{13}(-1)\\ r_{14}(-2) \end{matrix} = \begin{vmatrix} 1&1&1&1\\ 3&0&3&0\\ 2&0&-2&-2\\ 2&0&1&-4 \end{vmatrix} \begin{matrix} c_{13}(1)\\ c_{14}(1) \end{matrix} = \begin{vmatrix} 1&1&2&2\\ 3&0&6&3\\ 2&0&0&0\\ 2&0&3&-2 \end{vmatrix}$$

EXPAND BY THE 2ND COLUMN, THEN EXPAND BY THE 2ND ROW

$$= -\begin{vmatrix} 3&6&3\\ 2&0&0\\ 2&3&-2 \end{vmatrix} = +2\begin{vmatrix} 6&3\\ 3&-2 \end{vmatrix} = 2(-12-9) = -42$$

REPEAT THE DETERMINANT EVALUATION WITH THE "W COLUMN" REPLACED BY THE NUMBER COLUMN, IN ORDER TO APPLY CRAMERS RULE

$$\begin{vmatrix} 1&1&1&2\\ 2&-1&2&1\\ 3&1&-1&1\\ 4&2&3&0 \end{vmatrix} \begin{matrix} r_{12}(1)\\ r_{13}(-1)\\ r_{14}(-2) \end{matrix} = \begin{vmatrix} 1&1&1&2\\ 3&0&3&3\\ 2&0&-2&-1\\ 2&0&1&-4 \end{vmatrix} \begin{matrix} c_{13}(-1)\\ c_{14}(-1) \end{matrix} = \begin{vmatrix} 1&1&0&1\\ 3&0&0&0\\ 2&0&-4&-3\\ 2&0&-1&-6 \end{vmatrix}$$

EXPAND BY THE 2ND COLUMN AGAIN, THEN EXPAND BY THE FIRST ROW

$$= -\begin{vmatrix} 3&0&0\\ 2&-4&-3\\ 2&-1&-6 \end{vmatrix} = -3(24-3) = -63$$

HENCE BY CRAMER'S RULE WE HAVE

$$w = \frac{\text{DETERMINANT OF THE MATRIX, WITH "W COLUMN" REPLACED BY THE "NUMBER COLUMN"}}{\text{DETERMINANT OF THE MATRIX (OF A HOMOGENEOUS SYSTEM)}}$$

$$w = \frac{-63}{-42} = \frac{3}{2}$$

Question 6

Use Cramer's rule to find the value of x in the following system of simultaneous equations.

$$\begin{aligned} 2x &+ y - z & &+ t &= 9 \\ x &+ y + z &- w &- t &= 0 \\ 2x &- y - z &+ 2w &+ 2t &= 12 \\ x &+ 2y &+ w &+ t &= 8 \\ 3x &+ z &- w & &= 6 \end{aligned}$$

No credit will be given for using alternative solution methods.

$\boxed{}$, $\boxed{x = 3}$

WRITE THE SYSTEM IN MATRIX FORM

$2x+y-z+t=9$
$x+y+z-w-t=0$
$2x-y-z+2w+2t=12$
$x+2y+w+t=8$
$3x+z-w=6$

$$\left[\begin{array}{ccccc|c} 2 & 1 & -1 & 0 & 1 & 9 \\ 1 & 1 & 1 & -1 & -1 & 0 \\ 2 & -1 & -1 & 2 & 2 & 12 \\ 1 & 2 & 0 & 1 & 1 & 8 \\ 3 & 0 & 1 & -1 & 0 & 6 \end{array}\right]$$

CALCULATE THE DETERMINANT OF THE MATRIX OF THE SYSTEM

$$\begin{vmatrix} 2 & 1 & -1 & 0 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 2 & -1 & -1 & 2 & 2 \\ 1 & 2 & 0 & 1 & 1 \\ 3 & 0 & 1 & -1 & 0 \end{vmatrix} \underset{r_{14}}{=} -\begin{vmatrix} 1 & 2 & 0 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 2 & -1 & -1 & 2 & 2 \\ 2 & 1 & -1 & 0 & 1 \\ 3 & 0 & 1 & -1 & 0 \end{vmatrix} \begin{matrix} r_{12}(-1) \\ r_{13}(-2) \\ r_{14}(-2) \\ r_{15}(-3) \end{matrix}$$

$$-\begin{vmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & -1 & 1 & -2 & -2 \\ 0 & -5 & -1 & 0 & 0 \\ 0 & -3 & -1 & -2 & -1 \\ 0 & -6 & 1 & -4 & -3 \end{vmatrix} \underset{\substack{r_{23}(-5) \\ r_{24}(-3) \\ r_{25}(-6)}}{=} -\begin{vmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & -1 & 1 & -2 & -2 \\ 0 & 0 & -6 & 10 & 10 \\ 0 & 0 & -4 & 4 & 5 \\ 0 & 0 & -5 & 8 & 9 \end{vmatrix}$$

EXPANDING THE DETERMINANT BY THE FIRST COLUMN

$$= \cancel{-(1)(-1)} \begin{vmatrix} -6 & 10 & 10 \\ -4 & 4 & 5 \\ -5 & 8 & 9 \end{vmatrix} \underset{\substack{c_{23}(-1) \\ c_{21}(-1)}}{=} \begin{vmatrix} 4 & 10 & 0 \\ 0 & 4 & 1 \\ 3 & 8 & 1 \end{vmatrix}$$

$$= 4\begin{vmatrix} 4 & 1 \\ 8 & 1 \end{vmatrix} - 10\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} = 4(-4) - 10\times(-3) = -16+30 = 14$$

NEXT CALCULATE THE DETERMINANT OF THE MATRIX WITH THE x COLUMN REPLACED

$$\begin{vmatrix} 9 & 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & -1 & -1 \\ 12 & -1 & -1 & 2 & 2 \\ 8 & 2 & 0 & 1 & 1 \\ 6 & 0 & 1 & -1 & 0 \end{vmatrix} \underset{c_{15}}{=} -\begin{vmatrix} 1 & 1 & -1 & 0 & 9 \\ -1 & 1 & 1 & -1 & 0 \\ 2 & -1 & -1 & 2 & 12 \\ 1 & 2 & 0 & 1 & 8 \\ 0 & 0 & 1 & -1 & 6 \end{vmatrix}$$

$$\underset{\substack{r_{12}(1) \\ r_{13}(-2) \\ r_{14}(-1)}}{=} -\begin{vmatrix} 1 & 1 & -1 & 0 & 9 \\ 0 & 2 & 0 & -1 & 9 \\ 0 & -3 & 1 & 2 & -6 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 6 \end{vmatrix} \underset{r_{42}}{=} +\begin{vmatrix} 1 & 1 & -1 & 0 & 9 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & -3 & 1 & 2 & -6 \\ 0 & 2 & 0 & -1 & 9 \\ 0 & 0 & 1 & -1 & 6 \end{vmatrix}$$

$$\underset{\substack{r_{23}(3) \\ r_{24}(-2)}}{=} \begin{vmatrix} 1 & 1 & -1 & 0 & 9 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 4 & 5 & -9 \\ 0 & 0 & -2 & -3 & 11 \\ 0 & 0 & 1 & -1 & 6 \end{vmatrix}$$

EXPANDING BY THE FIRST COLUMN WE OBTAIN

$$= 1\times1\times\begin{vmatrix} 4 & 5 & -9 \\ -2 & -3 & 11 \\ 1 & -1 & 6 \end{vmatrix}$$

$$\underset{c_{12}(1)}{=} \begin{vmatrix} 4 & 9 & -9 \\ -2 & -5 & 11 \\ 1 & 0 & 6 \end{vmatrix}$$

EXPANDING BY THE BOTTOM ROW

$$= +\begin{vmatrix} 9 & -9 \\ -5 & 11 \end{vmatrix} + 6\begin{vmatrix} 4 & 9 \\ -2 & -5 \end{vmatrix}$$

$$= 9\begin{vmatrix} 1 & -1 \\ -5 & 11 \end{vmatrix} - 12\begin{vmatrix} 2 & 9 \\ 1 & 5 \end{vmatrix}$$

$= 9(6) - 12(1)$

$= 54 - 12$

$= 42$

THUS BY CRAMER'S RULE WE HAVE

$$x = \frac{\text{DETERMINANT OF THE SYSTEM MATRIX WITH "}x\text{ COLUMN" REPLACED BY THE "NUMBER COLUMN"}}{\text{DETERMINANT OF THE SYSTEM MATRIX}}$$

$x = \frac{42}{14}$

$x = 3$

Matrix Inverse

Question 1

The 3×3 matrix $\mathbf{C}$ is given below.

$$\mathbf{C} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 4 & 2 \end{pmatrix}.$$

a) Use the standard method for finding the inverse of a 3×3 matrix, to determine the elements of $\mathbf{C}^{-1}$.

b) Verify the answer of part **(a)** by obtaining the elements of $\mathbf{C}^{-1}$, by using a method involving elementary row operations.

V , , $\boxed{\mathbf{C}^{-1} = \begin{pmatrix} 2 & 0 & -1 \\ 3 & -1 & -1 \\ -7 & 2 & 3 \end{pmatrix}}$

a)

$$\underline{C} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 4 & 2 \end{bmatrix}$$

MATRIX OF MINORS $= \begin{bmatrix} -2 & 3 & 7 \\ 0 & 1 & 2 \\ 1 & -1 & -3 \end{bmatrix}$

MATRIX OF COFACTORS $= \begin{bmatrix} -2 & -3 & 7 \\ 0 & 1 & -2 \\ 1 & +1 & -3 \end{bmatrix}$

ADJUGATE MATRIX $= \begin{bmatrix} -2 & 0 & 1 \\ -3 & 1 & 1 \\ 7 & -2 & -3 \end{bmatrix}$

$$|\underline{C}| = 1\times(-2) + 2(-3) + 1\times7 = -2-6+7 = -1$$

$$\underline{C}^{-1} = \frac{1}{|C|}(\text{ADJUGATE}) = \frac{1}{-1}\begin{bmatrix} -2 & 0 & 1 \\ -3 & 1 & 1 \\ 7 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 3 & -1 & -1 \\ -7 & 2 & 3 \end{bmatrix}$$

b) NOW THE INVERSE BY ROW OPERATIONS

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 4 & 2 & 0 & 0 & 1 \end{array}\right] \xRightarrow[r_{13}(-1)]{r_{12}(-2)} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -3 & -1 & -2 & 1 & 0 \\ 0 & 2 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xRightarrow{r_{23}} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & -1 & 0 & 1 \\ 0 & -3 & -1 & -2 & 1 & 0 \end{array}\right] \xRightarrow{r_2(\frac{1}{2})} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -3 & -1 & -2 & 1 & 0 \end{array}\right]$$

$$\xRightarrow{r_{23}(3)} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{7}{2} & 1 & \frac{3}{2} \end{array}\right] \xRightarrow{r_3(2)} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -7 & 2 & 3 \end{array}\right]$$

$$\xRightarrow{r_{21}(-2)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -7 & 2 & 3 \end{array}\right] \xRightarrow{r_{32}(-\frac{1}{2})} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & 3 & -1 & -1 \\ 0 & 0 & 1 & -7 & 2 & 3 \end{array}\right]$$

$\underline{I}$ $\quad$ $\underline{C}^{-1}$

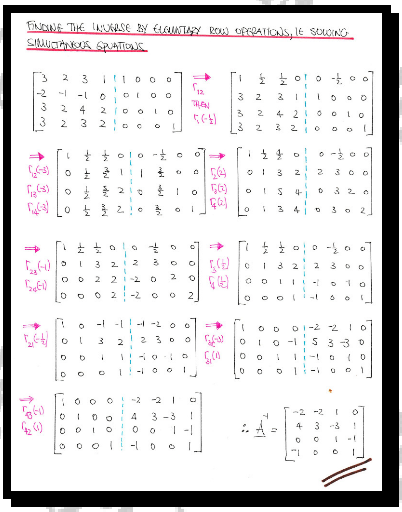
Question 2

The 4×4 matrix $\mathbf{A}$ is given below.

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 3 & 1 \\ -2 & -1 & -1 & 0 \\ 3 & 2 & 4 & 2 \\ 3 & 2 & 3 & 2 \end{pmatrix}.$$

Find $\mathbf{A}^{-1}$, by using a method involving elementary row operations.

$$\boxed{},\ \boxed{\mathbf{A} = \begin{pmatrix} -2 & -2 & 1 & 0 \\ 4 & 3 & -3 & 1 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix}}$$



Various

Question 1

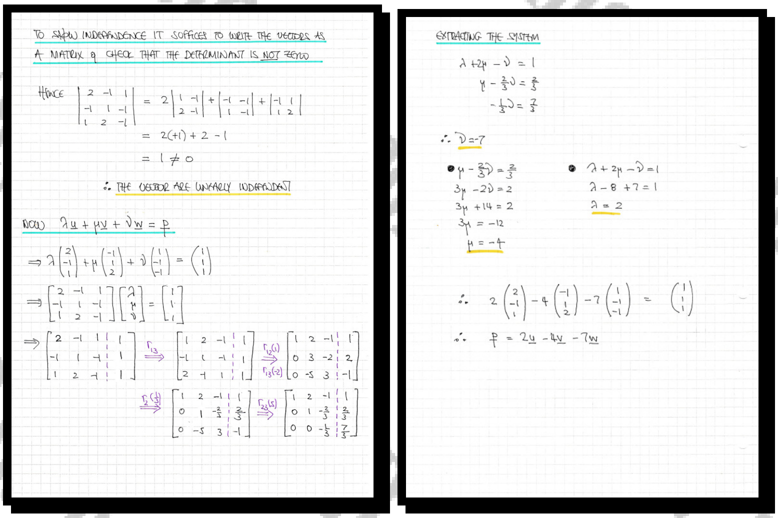
The following four vectors are given.

$$\mathbf{u}=\begin{bmatrix}2\\-1\\1\end{bmatrix},\quad \mathbf{v}=\begin{bmatrix}-1\\1\\2\end{bmatrix},\quad \mathbf{w}=\begin{bmatrix}1\\-1\\-1\end{bmatrix},\quad \mathbf{p}=\begin{bmatrix}1\\1\\1\end{bmatrix}.$$

a) Show that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly independent.

b) Express $\mathbf{p}$ in terms of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

$\boxed{\text{M1}}$, $\boxed{\mathbf{p}=2\mathbf{u}-4\mathbf{v}-7\mathbf{w}}$



TO SHOW INDEPENDENCE IT SUFFICES TO WRITE THE VECTORS AS A MATRIX & CHECK THAT THE DETERMINANT IS NOT ZERO

HENCE $\begin{vmatrix}2&-1&1\\-1&1&-1\\1&2&-1\end{vmatrix} = 2\begin{vmatrix}1&-1\\2&-1\end{vmatrix}+\begin{vmatrix}-1&-1\\1&-1\end{vmatrix}+\begin{vmatrix}-1&1\\1&2\end{vmatrix}$

$= 2(+1)+2-1$

$= 1 \neq 0$

$\therefore$ THE VECTOR ARE LINEARLY INDEPENDENT

NOW $\lambda\underline{u}+\mu\underline{v}+\nu\underline{w}=\underline{p}$

$\Rightarrow \lambda\begin{pmatrix}2\\-1\\1\end{pmatrix}+\mu\begin{pmatrix}-1\\1\\2\end{pmatrix}+\nu\begin{pmatrix}1\\-1\\-1\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}$

$\Rightarrow \begin{bmatrix}2&-1&1\\-1&1&-1\\1&2&-1\end{bmatrix}\begin{bmatrix}\lambda\\\mu\\\nu\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}$

$\Rightarrow \left[\begin{array}{ccc|c}2&-1&1&1\\-1&1&-1&1\\1&2&-1&1\end{array}\right] \xrightarrow{r_{13}} \left[\begin{array}{ccc|c}1&2&-1&1\\-1&1&-1&1\\2&-1&1&1\end{array}\right] \xrightarrow[r_{13}(-2)]{r_{12}(1)} \left[\begin{array}{ccc|c}1&2&-1&1\\0&3&-2&2\\0&-5&3&-1\end{array}\right]$

$\xrightarrow{r_2(\frac{1}{3})} \left[\begin{array}{ccc|c}1&2&-1&1\\0&1&-\frac{2}{3}&\frac{2}{3}\\0&-5&3&-1\end{array}\right] \xrightarrow{r_{23}(5)} \left[\begin{array}{ccc|c}1&2&-1&1\\0&1&-\frac{2}{3}&\frac{2}{3}\\0&0&-\frac{1}{3}&\frac{7}{3}\end{array}\right]$

EXTRACTING THE SYSTEM

$\lambda+2\mu-\nu=1$

$\mu-\frac{2}{3}\nu=\frac{2}{3}$

$-\frac{1}{3}\nu=\frac{7}{3}$

$\therefore \nu=-7$

- $\mu-\frac{2}{3}\nu=\frac{2}{3}$
  $3\mu-2\nu=2$
  $3\mu+14=2$
  $3\mu=-12$
  $\mu=-4$

- $\lambda+2\mu-\nu=1$
  $\lambda-8+7=1$
  $\lambda=2$

$\therefore 2\begin{pmatrix}2\\-1\\1\end{pmatrix}-4\begin{pmatrix}-1\\1\\2\end{pmatrix}-7\begin{pmatrix}1\\-1\\-1\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}$

$\therefore \underline{p}=2\underline{u}-4\underline{v}-7\underline{w}$

Question 2

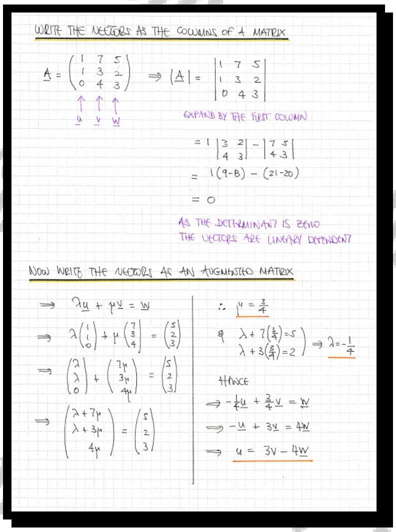
The following three vectors are given.

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 7 \\ 3 \\ 4 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}.$$

a) Show that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent.

b) Find a linear relationship, with integer coefficients, between $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

$\boxed{\mathbf{u} = 3\mathbf{v} - 4\mathbf{w}}$



WRITE THE VECTORS AS THE COLUMNS OF A MATRIX

$\underline{A} = \begin{pmatrix} 1 & 7 & 5 \\ 1 & 3 & 2 \\ 0 & 4 & 3 \end{pmatrix} \Rightarrow |\underline{A}| = \begin{vmatrix} 1 & 7 & 5 \\ 1 & 3 & 2 \\ 0 & 4 & 3 \end{vmatrix}$

$\underline{u}$ $\underline{v}$ $\underline{w}$

EXPAND BY THE FIRST COLUMN

$= 1\begin{vmatrix} 3 & 2 \\ 4 & 3 \end{vmatrix} - \begin{vmatrix} 7 & 5 \\ 4 & 3 \end{vmatrix}$

$= 1(9-8) - (21-20)$

$= 0$

AS THE DETERMINANT IS ZERO THE VECTORS ARE LINEARLY DEPENDENT

NOW WRITE THE VECTORS AS AN AUGMENTED MATRIX

$\Rightarrow \lambda\underline{u} + \mu\underline{v} = \underline{w}$

$\Rightarrow \lambda\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \mu\begin{pmatrix} 7 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} \lambda \\ \lambda \\ 0 \end{pmatrix} + \begin{pmatrix} 7\mu \\ 3\mu \\ 4\mu \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} \lambda + 7\mu \\ \lambda + 3\mu \\ 4\mu \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$

$\therefore \mu = \frac{3}{4}$

& $\left.\begin{array}{l} \lambda + 7\left(\frac{3}{4}\right) = 5 \\ \lambda + 3\left(\frac{3}{4}\right) = 2 \end{array}\right) \Rightarrow \lambda = -\frac{1}{4}$

HENCE

$\Rightarrow -\frac{1}{4}\underline{u} + \frac{3}{4}\underline{v} = \underline{w}$

$\Rightarrow -\underline{u} + 3\underline{v} = 4\underline{w}$

$\Rightarrow \underline{u} = 3\underline{v} - 4\underline{w}$

Question 3

The following four vectors are given.

$$\mathbf{u} = \begin{bmatrix}1\\1\\0\\1\end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix}3\\0\\1\\-1\end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix}1\\-1\\0\\-1\end{bmatrix}, \quad \mathbf{p} = \begin{bmatrix}1\\-1\\-1\\0\end{bmatrix}.$$

a) Show that these four vectors are linearly dependent.

b) Express $\mathbf{p}$ in terms of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

$\boxed{}$, $\boxed{\mathbf{p} = \frac{3}{2}\mathbf{u} - \mathbf{v} + \frac{5}{2}\mathbf{w}}$

$$\underline{u} = \begin{pmatrix}1\\1\\0\\1\end{pmatrix} \quad \underline{v} = \begin{pmatrix}3\\0\\1\\-1\end{pmatrix} \quad \underline{w} = \begin{pmatrix}1\\-1\\0\\-1\end{pmatrix} \quad \underline{p} = \begin{pmatrix}1\\-1\\-1\\0\end{pmatrix}$$

a) FORMING A MATRIX WITH COLUMNS THE 4 VECTORS

$$\begin{vmatrix}1&3&1&1\\1&0&-1&-1\\0&1&0&-1\\1&-1&-1&0\end{vmatrix} \quad \begin{matrix}r_{12}(-1)\\r_{14}(-1)\end{matrix} \quad \begin{vmatrix}1&3&1&1\\0&-3&-2&-2\\0&1&0&-1\\0&-4&-2&-1\end{vmatrix}$$

$$r_{23} \begin{vmatrix}1&3&1&1\\0&1&0&-1\\0&-3&-2&-2\\0&-4&-2&-1\end{vmatrix} \quad \begin{matrix}r_{23}(3)\\r_{24}(4)\end{matrix} \quad \begin{vmatrix}1&3&1&1\\0&1&0&-1\\0&0&-2&-5\\0&0&-2&-5\end{vmatrix}$$

TWO IDENTICAL ROWS IMPLIES ZERO DETERMINANT

HENCE THE VECTORS ARE LINEARLY DEPENDENT

b) $\lambda\underline{u} + \mu\underline{v} + t\underline{w} = \underline{p}$

$$\lambda\begin{pmatrix}1\\1\\0\\1\end{pmatrix} + \mu\begin{pmatrix}3\\0\\1\\-1\end{pmatrix} + t\begin{pmatrix}1\\-1\\0\\-1\end{pmatrix} = \begin{pmatrix}1\\-1\\-1\\0\end{pmatrix}$$

$$\left\{\begin{array}{l}\lambda + 3\mu + t = 1\\ \lambda - t = -1\\ \mu = -1\\ \lambda - \mu - t = 0\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\lambda - 3 + t = 1\\ \lambda - t = -1\\ \mu = -1\\ \lambda + 1 - t = 0\end{array}\right\} \Rightarrow$$

$$\left\{\begin{array}{l}\lambda + t = 4\\ \lambda - t = -1\\ \mu = -1\end{array}\right\} \Rightarrow \begin{array}{l}\lambda = \frac{3}{2}\\ t = \frac{5}{2}\\ \mu = -1\end{array}$$

$$\therefore \underline{p} = \frac{3}{2}\begin{pmatrix}1\\1\\0\\1\end{pmatrix} - 1\begin{pmatrix}3\\0\\1\\-1\end{pmatrix} + \frac{5}{2}\begin{pmatrix}1\\-1\\0\\-1\end{pmatrix}$$

IE $\underline{p} = \frac{3}{2}\underline{u} - \underline{v} + \frac{5}{2}\underline{w}$

ALTERNATIVE

TAKING IT FROM THE ROW REDUCTION OF PART (a) AND IGNORING THE BOTTOM ROW

$$\begin{bmatrix}1&3&1&1\\0&1&0&-1\\0&0&-2&-5\end{bmatrix} \Rightarrow \left\{\begin{array}{l}\lambda + 3\mu + t = 1\\ \mu = -1\\ -2t = -5\end{array}\right\}$$

$$\Rightarrow \left\{\begin{array}{l}\lambda - 3 + \frac{5}{2} = 1\\ \mu = -1\\ t = \frac{5}{2}\end{array}\right\}$$

$$\Rightarrow \begin{array}{l}\lambda = \frac{3}{2}\\ \mu = -1\\ t = \frac{5}{2}\end{array}$$