O.D.E.s

SERIES SOLUTIONS

LEIBNIZ
METHOD

Leibniz Theorem

If $y = u(x)v(x)$ then

$$y_n = \sum_{r=1}^{n} \binom{n}{r} u_r v_{n-r} = u_n + n u_{n-1} v_1 + \frac{n(n-1)}{2!} u_{n-2} v_2 + \frac{n(n-1)(n-2)}{3!} u_{n-3} v_3 + \dots \quad ,$$

where $u_m = \dfrac{d^m u}{dx^m}$ and $v_m = \dfrac{d^m v}{dx^m}$.

n$^{\text{th}}$ order differential coefficients

$$\frac{d^n}{dx^n}\left(x^a\right) = y_n = \frac{a!}{(a-n)!} a^{a-n}$$

$$\frac{d^n}{dx^n}\left(e^{ax}\right) = y_n = a^n e^{ax}$$

$$\frac{d^n}{dx^n}(\sin ax) = y_n = a^n \sin\left[ax + \frac{n\pi}{2}\right]$$

$$\frac{d^n}{dx^n}(\cos ax) = y_n = a^n \cos\left[ax + \frac{n\pi}{2}\right]$$

$$\frac{d^n}{dx^n}(\sinh ax) = y_n = \frac{1}{2} a^n \left[\left[1-(-1)^n\right]\sinh ax + \left[1+(-1)^n\right]\cosh ax\right]$$

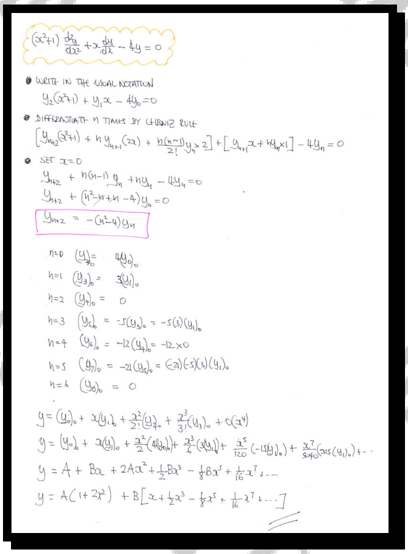
$$\frac{d^n}{dx^n}(\cosh ax) = y_n = \frac{1}{2} a^n \left[\left[1+(-1)^n\right]\sinh ax + \left[1-(-1)^n\right]\cosh ax\right]$$

Question 1

Use the Leibniz rule to find a general solution, as an infinite series, for the following differential equation

$$\left(1+x^2\right)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-4y=0.$$

$$\boxed{y=A\left(1+2x^2\right)+Bx\left(1+\frac{1}{2}x^2-\frac{1}{8}x^4+\frac{1}{16}x^6+\ldots\right)}$$



$$(x^2+1)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-4y=0$$

- WRITE IN THE USUAL NOTATION

$$y_2(x^2+1)+y_1x-4y_0=0$$

- DIFFERENTIATE n TIMES BY LEIBNIZ RULE

$$\left[y_{n+2}(x^2+1)+ny_{n+1}(2x)+\frac{n(n-1)}{2!}y_n\times 2\right]+\left[y_{n+1}x+ny_n\times 1\right]-4y_n=0$$

- SET $x=0$

$$y_{n+2}+n(n-1)y_n+ny_n-4y_n=0$$

$$y_{n+2}+(n^2-\cancel{n}+\cancel{n}-4)y_n=0$$

$$\boxed{y_{n+2}=-(n^2-4)y_n}$$

$n=0$ $(y_2)_0=4(y_0)_0$

$n=1$ $(y_3)_0=3(y_1)_0$

$n=2$ $(y_4)_0=0$

$n=3$ $(y_5)_0=-5(y_3)_0=-5(3)(y_1)_0$

$n=4$ $(y_6)_0=-12(y_4)_0=-12\times 0$

$n=5$ $(y_7)_0=-21(y_5)_0=(-21)(-5)(3)(y_1)_0$

$n=6$ $(y_8)_0=0$

$$y=(y_0)_0+x(y_1)_0+\frac{x^2}{2!}(y_2)_0+\frac{x^3}{3!}(y_3)_0+O(x^4)$$

$$y=(y_0)_0+x(y_1)_0+\frac{x^2}{2}(4(y_0)_0)+\frac{x^3}{6}(3(y_1)_0)+\frac{x^5}{120}(-15(y_1)_0)+\frac{x^7}{5040}(315(y_1)_0)+\ldots$$

$$y=A+Bx+2Ax^2+\frac{1}{2}Bx^3-\frac{1}{8}Bx^5+\frac{1}{16}x^7+\ldots$$

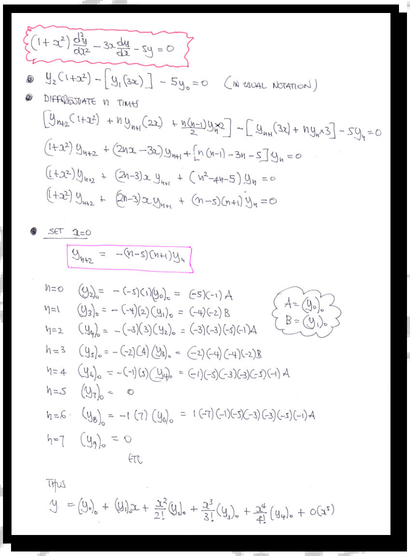
$$y=A(1+2x^2)+B\left[x+\frac{1}{2}x^3-\frac{1}{8}x^5+\frac{1}{16}x^7+\ldots\right]$$

Question 2

Use the Leibniz rule to find a general solution, as an infinite series, for the following differential equation

$$\left(1+x^2\right)\frac{d^2y}{dx^2}-3\frac{dy}{dx}-5y=0.$$

$$\boxed{y=A\left(1+\frac{5}{2}x^2+\frac{15}{8}x^4+\frac{5}{16}x^6+\frac{5}{128}x^8+\frac{3}{256}x^{10}+\ldots\right)+Bx\left(1+\frac{4}{3}x^2+\frac{8}{15}x^4\right)}$$



$$\left(1+x^2\right)\frac{d^2y}{dx^2}-3x\frac{dy}{dx}-5y=0$$

- $y_2(1+x^2)-\left[y_1(3x)\right]-5y_0=0$ (IN USUAL NOTATION)
- DIFFERENTIATE n TIMES

$$\left[y_{n+2}(1+x^2)+ny_{n+1}(2x)+\frac{n(n-1)}{2}y_n\times 2\right]-\left[y_{n+1}(3x)+ny_n\times 3\right]-5y_n=0$$

$$(1+x^2)y_{n+2}+(2nx-3x)y_{n+1}+\left[n(n-1)-3n-5\right]y_n=0$$

$$(1+x^2)y_{n+2}+(2n-3)x\,y_{n+1}+(n^2-4n-5)y_n=0$$

$$(1+x^2)y_{n+2}+(2n-3)x\,y_{n+1}+(n-5)(n+1)y_n=0$$

- SET $x=0$

$$\boxed{y_{n+2}=-(n-5)(n+1)y_n}$$

$n=0$ $(y_2)_0=-(-5)(1)(y_0)_0=(-5)(-1)A$

$n=1$ $(y_3)_0=-(-4)(2)(y_1)_0=(-4)(-2)B$

$n=2$ $(y_4)_0=-(-3)(3)(y_2)_0=(-3)(-3)(-5)(-1)A$

$n=3$ $(y_5)_0=-(-2)(4)(y_3)_0=(-2)(-4)(-4)(-2)B$

$n=4$ $(y_6)_0=-(-1)(5)(y_4)_0=(-1)(-5)(-3)(-3)(-5)(-1)A$

$n=5$ $(y_7)_0=0$

$n=6$ $(y_8)_0=-1(7)(y_6)_0=1(-7)(-1)(-5)(-3)(-3)(-5)(-1)A$

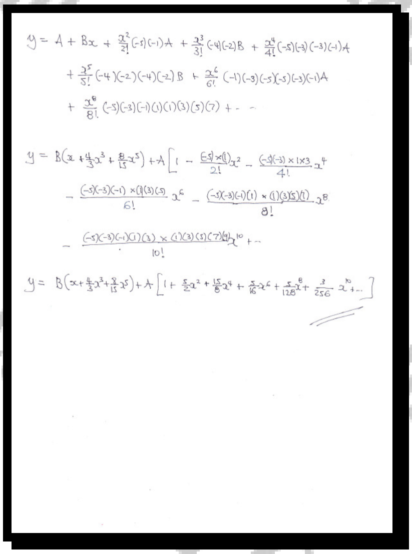
$n=7$ $(y_9)_0=0$

ETC

$A=(y_0)_0$
$B=(y_1)_0$

THUS

$$y=(y_0)_0+(y_1)_0x+\frac{x^2}{2!}(y_2)_0+\frac{x^3}{3!}(y_3)_0+\frac{x^4}{4!}(y_4)_0+O(x^5)$$



$$y=A+Bx+\frac{x^2}{2!}(-5)(-1)A+\frac{x^3}{3!}(-4)(-2)B+\frac{x^4}{4!}(-5)(-3)(-3)(-1)A$$
$$+\frac{x^5}{5!}(-4)(-2)(-4)(-2)B+\frac{x^6}{6!}(-1)(-3)(-5)(-5)(-3)(-1)A$$
$$+\frac{x^8}{8!}(-5)(-3)(-1)(1)(1)(3)(5)(7)+\ldots$$

$$y=B\left(x+\frac{4}{3}x^3+\frac{8}{15}x^5\right)+A\left[1-\frac{(-5)\times(1)}{2!}x^2-\frac{(-5)(-3)\times 1\times 3}{4!}x^4\right.$$
$$-\frac{(-5)(-3)(-1)\times(1)(3)(5)}{6!}x^6-\frac{(-5)(-3)(-1)(1)\times(1)(3)(5)(7)}{8!}x^8$$
$$\left.-\frac{(-5)(-3)(-1)(1)(3)\times(1)(3)(5)(7)(9)}{10!}x^{10}+\ldots\right.$$

$$y=B\left(x+\frac{4}{3}x^3+\frac{8}{15}x^5\right)+A\left[1+\frac{5}{2}x^2+\frac{15}{8}x^4+\frac{5}{16}x^6+\frac{5}{128}x^8+\frac{3}{256}x^{10}+\ldots\right]$$

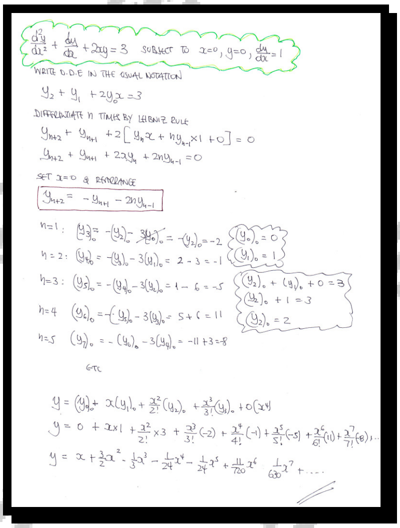
Question 3

Use the Leibniz rule to find a general solution, as an infinite series, for the following differential equation

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}+2xy=3,$$

subject to the boundary conditions $y=0$, $\dfrac{dy}{dx}=1$ at $x=0$.

$$\boxed{y=x+\frac{3}{2}x^2-\frac{1}{3}x^3-\frac{1}{24}x^4-\frac{1}{24}x^5+\frac{11}{720}x^6-\frac{1}{630}x^7...}$$



$\frac{d^2y}{dx^2}+\frac{dy}{dx}+2xy=3$ SUBJECT TO $x=0, y=0, \frac{dy}{dx}=1$

WRITE O.D.E IN THE USUAL NOTATION

$$y_2+y_1+2y_0x=3$$

DIFFERENTIATE n TIMES BY LEIBNIZ RULE

$$y_{n+2}+y_{n+1}+2\left[y_nx+ny_{n-1}\times1+0\right]=0$$

$$y_{n+2}+y_{n+1}+2xy_n+2ny_{n-1}=0$$

SET $x=0$ & REARRANGE

$$\boxed{y_{n+2}=-y_{n+1}-2ny_{n-1}}$$

$(y_0)_0=0$

$(y_1)_0=1$

$(y_2)_0+(y_1)_0+0=3$

$(y_2)_0+1=3$

$(y_2)_0=2$

$n=1$: $(y_3)_0=-(y_2)_0-\cancel{3(y_0)_0}=-(y_2)_0=-2$

$n=2$: $(y_4)_0=-(y_3)_0-3(y_1)_0=2-3=-1$

$n=3$: $(y_5)_0=-(y_4)_0-3(y_2)_0=1-6=-5$

$n=4$ $(y_6)_0=-(y_5)_0-3(y_3)_0=5+6=11$

$n=5$ $(y_7)_0=-(y_6)_0-3(y_4)_0=-11+3=-8$

ETC

$$y=(y_0)_0+x(y_1)_0+\frac{x^2}{2!}(y_2)_0+\frac{x^3}{3!}(y_3)_0+O(x^4)$$

$$y=0+x\times1+\frac{x^2}{2!}\times3+\frac{x^3}{3!}(-2)+\frac{x^4}{4!}(-1)+\frac{x^5}{5!}(-5)+\frac{x^6}{6!}(11)+\frac{x^7}{7!}(-8),\dots$$

$$y=x+\frac{3}{2}x^2-\frac{1}{3}x^3-\frac{1}{24}x^4-\frac{1}{24}x^5+\frac{11}{720}x^6\quad\frac{1}{630}x^7+\dots$$

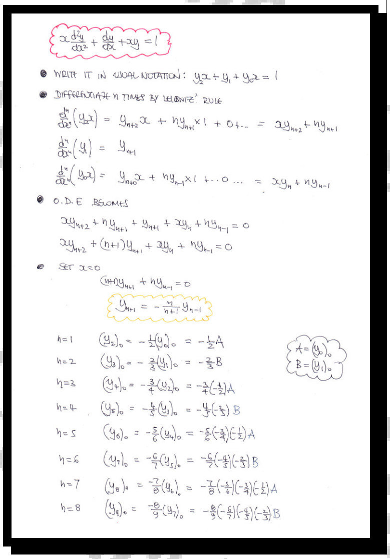
Question 4

Use the Leibniz rule to find a general solution, as an infinite series, for the following differential equation

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+xy=1\,.$$

Give the final answer in simplified Sigma notation.

$$\boxed{y=A\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{2^{2n}(n!)^2}x^{2n}\right]+B\sum_{n=0}^{\infty}\left[\frac{(-1)^n\,2^{2n}(n!)^2}{\left[(2n+1)!\right]^2}x^{2n+1}\right]}$$



$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+xy=1$$

- WRITE IT IN USUAL NOTATION: $y_2x+y_1+y_0x=1$
- DIFFERENTIATE n TIMES BY LEIBNIZ' RULE

$$\frac{d^n}{dx^n}(y_2x)=y_{n+2}x+ny_{n+1}\times1+0+\ldots=xy_{n+2}+ny_{n+1}$$

$$\frac{d^n}{dx^n}(y_1)=y_{n+1}$$

$$\frac{d^n}{dx^n}(y_0x)=y_{n+0}x+ny_{n-1}\times1+\ldots0\ldots=xy_n+ny_{n-1}$$

- O.D.E BECOMES

$$xy_{n+2}+ny_{n+1}+y_{n+1}+xy_n+ny_{n-1}=0$$

$$xy_{n+2}+(n+1)y_{n+1}+xy_n+ny_{n-1}=0$$

- SET $x=0$

$$(n+1)y_{n+1}+ny_{n-1}=0$$

$$y_{n+1}=-\frac{n}{n+1}y_{n-1}$$

$n=1$ $(y_2)_0=-\frac{1}{2}(y_0)_0=-\frac{1}{2}A$

$n=2$ $(y_3)_0=-\frac{2}{3}(y_1)_0=-\frac{2}{3}B$

$n=3$ $(y_4)_0=-\frac{3}{4}(y_2)_0=-\frac{3}{4}(-\frac{1}{2})A$

$n=4$ $(y_5)_0=-\frac{4}{5}(y_3)_0=-\frac{4}{5}(-\frac{2}{3})B$

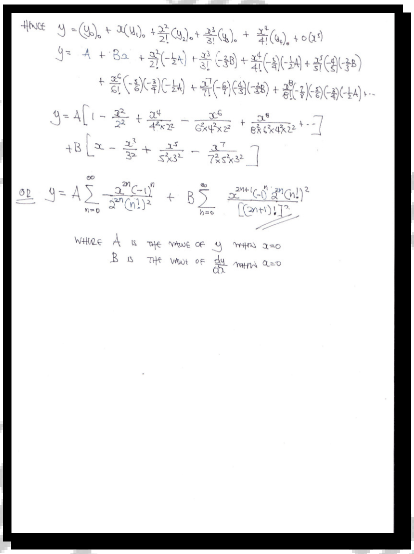
$n=5$ $(y_6)_0=-\frac{5}{6}(y_4)_0=-\frac{5}{6}(-\frac{3}{4})(-\frac{1}{2})A$

$n=6$ $(y_7)_0=-\frac{6}{7}(y_5)_0=-\frac{6}{7}(-\frac{4}{5})(-\frac{2}{3})B$

$n=7$ $(y_8)_0=-\frac{7}{8}(y_6)_0=-\frac{7}{8}(-\frac{5}{6})(-\frac{3}{4})(-\frac{1}{2})A$

$n=8$ $(y_9)_0=-\frac{8}{9}(y_7)_0=-\frac{8}{9}(-\frac{6}{7})(-\frac{4}{5})(-\frac{2}{3})B$

$A=(y_0)_0$
$B=(y_1)_0$



HENCE $y=(y_0)_0+x(y_1)_0+\frac{x^2}{2!}(y_2)_0+\frac{x^3}{3!}(y_3)_0+\frac{x^4}{4!}(y_4)_0+O(x^5)$

$$y=A+Bx+\frac{x^2}{2!}(-\tfrac{1}{2}A)+\frac{x^3}{3!}(-\tfrac{2}{3}B)+\frac{x^4}{4!}(-\tfrac{3}{4})(-\tfrac{1}{2}A)+\frac{x^5}{5!}(-\tfrac{4}{5})(-\tfrac{2}{3}B)$$
$$+\frac{x^6}{6!}(-\tfrac{5}{6})(-\tfrac{3}{4})(-\tfrac{1}{2}A)+\frac{x^7}{7!}(-\tfrac{6}{7})(-\tfrac{4}{5})(-\tfrac{2}{3}B)+\frac{x^8}{8!}(-\tfrac{7}{8})(-\tfrac{5}{6})(-\tfrac{3}{4})(-\tfrac{1}{2}A)+\ldots$$

$$y=A\left[1-\frac{x^2}{2^2}+\frac{x^4}{4^2\times2^2}-\frac{x^6}{6^2\times4^2\times2^2}+\frac{x^8}{8^2\times6^2\times4^2\times2^2}+\ldots\right]$$
$$+B\left[x-\frac{x^3}{3^2}+\frac{x^5}{5^2\times3^2}-\frac{x^7}{7^2\times5^2\times3^2}\right]$$

OR $$y=A\sum_{n=0}^{\infty}\frac{x^{2n}(-1)^n}{2^{2n}(n!)^2}+B\sum_{n=0}^{\infty}\frac{x^{2n+1}(-1)^n2^{2n}(n!)^2}{[(2n+1)!]^2}$$

WHERE A IS THE VALUE OF y WHEN $x=0$
B IS THE VALUE OF $\frac{dy}{dx}$ WHEN $x=0$

Question 5

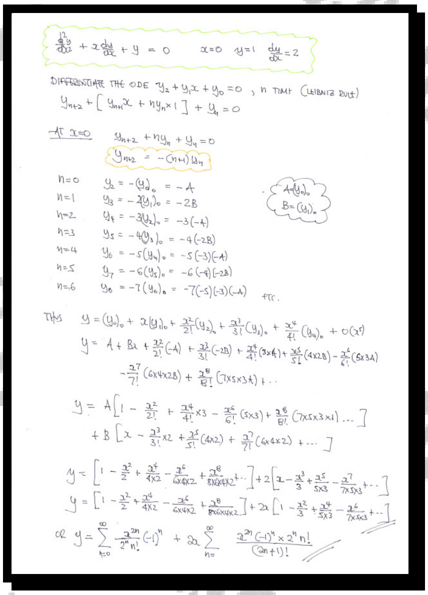
Use the Leibniz rule to find a general solution, as an infinite series, for the following differential equation

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+xy=1,$$

subject to the boundary conditions $y=1$, $\frac{dy}{dx}=2$ at $x=0$

Give the final answer in simplified Sigma notation.

$$\boxed{y=\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{2^n n!}x^{2n}\right]+2x\sum_{n=0}^{\infty}\left[\frac{(-2)^n n!}{(2n+1)!}x^{2n}\right]}$$



$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$ $\quad x=0 \quad y=1 \quad \frac{dy}{dx}=2$

DIFFERENTIATE THE ODE $y_2 + y_1 x + y_0 = 0$, n TIMES (LEIBNIZ RULE)

$y_{n+2} + \left[y_{n+1}x + ny_n \times 1\right] + y_n = 0$

AT $x=0$ $\quad y_{n+2} + ny_n + y_n = 0$

$y_{n+2} = -(n+1)y_n$

n=0 $\quad y_2 = -(y_0)_0 = -A$

n=1 $\quad y_3 = -2(y_1)_0 = -2B$

n=2 $\quad y_4 = -3(y_2)_0 = -3(-A)$

n=3 $\quad y_5 = -4(y_3)_0 = -4(-2B)$

n=4 $\quad y_6 = -5(y_4)_0 = -5(-3)(-A)$

n=5 $\quad y_7 = -6(y_5)_0 = -6(-4)(-2B)$

n=6 $\quad y_8 = -7(y_6)_0 = -7(-5)(-3)(-A)$ ETC.

$A=(y_0)_0$

$B=(y_1)_0$

THUS $y = (y_0)_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \frac{x^4}{4!}(y_4)_0 + O(x^5)$

$y = A + Bx + \frac{x^2}{2!}(-A) + \frac{x^3}{3!}(-2B) + \frac{x^4}{4!}(3\times A) + \frac{x^5}{5!}(4\times2B) - \frac{x^6}{6!}(5\times3A) - \frac{x^7}{7!}(6\times4\times2B) + \frac{x^8}{8!}(7\times5\times3A) + \dots$

$y = A\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!}\times3 - \frac{x^6}{6!}(5\times3) + \frac{x^8}{8!}(7\times5\times3\times1)\dots\right] + B\left[x - \frac{x^3}{3!}\times2 + \frac{x^5}{5!}(4\times2) + \frac{x^7}{7!}(6\times4\times2) + \dots\right]$

$y = \left[1 - \frac{x^2}{2} + \frac{x^4}{4\times2} - \frac{x^6}{6\times4\times2} + \frac{x^8}{8\times6\times4\times2} + \dots\right] + 2\left[x - \frac{x^3}{3} + \frac{x^5}{5\times3} - \frac{x^7}{7\times5\times3} + \dots\right]$

$y = \left[1 - \frac{x^2}{2} + \frac{x^4}{4\times2} - \frac{x^6}{6\times4\times2} + \frac{x^8}{8\times6\times4\times2}\right] + 2x\left[1 - \frac{x^2}{3} + \frac{x^4}{5\times3} - \frac{x^6}{7\times5\times3} + \dots\right]$

OR $y = \sum_{n=0}^{\infty}\frac{x^{2n}}{2^n n!}(-1)^n + 2x\sum_{n=}^{\infty}\frac{x^{2n}(-1)^n\times2^n n!}{(2n+1)!}$

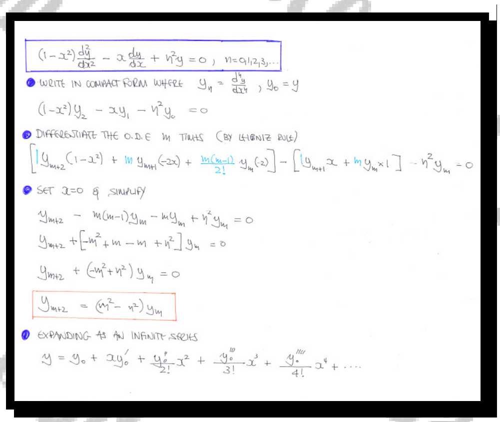
Question 6

Chebyshev's equation is shown below

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+n^2y=0\,,\ \ n=0,1,2,3,\ldots$$

Find a series solution for Chebyshev's equation, by using the Leibniz method

$$\boxed{\begin{array}{l} y= \\ A\left[x-\frac{\left(1-n^2\right)}{3!}x^3-\frac{\left(1-n^2\right)\left(9-n^2\right)}{5!}x^5-\frac{\left(1-n^2\right)\left(9-n^2\right)\left(25-n^2\right)}{7!}x^7-\ldots\right] \\ \qquad + \\ B\left[1-\frac{n^2}{2!}x^2-\frac{n^2\left(4-n^2\right)}{4!}x^4-\frac{n^2\left(4-n^2\right)\left(16-n^2\right)}{6!}x^6-\frac{n^2\left(4-n^2\right)\left(16-n^2\right)\left(36-n^2\right)}{8!}x^8-\ldots\right] \end{array}}$$



$$\boxed{(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2y = 0\,,\ n=0,1,2,3,\ldots}$$

- WRITE IN COMPACT FORM WHERE $y_n = \frac{d^ny}{dx^n}$, $y_0 = y$

$$(1-x^2)y_2 - xy_1 - n^2y_0 = 0$$

- DIFFERENTIATE THE O.D.E m TIMES (BY LEIBNIZ RULE)

$$\left[1y_{m+2}(1-x^2) + m\,y_{m+1}(-2x) + \frac{m(m-1)}{2!}y_m(-2)\right] - \left[1y_{m+1}x + m\,y_m\times 1\right] - n^2y_m = 0$$

- SET $x=0$ & SIMPLIFY

$$y_{m+2} - m(m-1)y_m - my_m + n^2y_m = 0$$

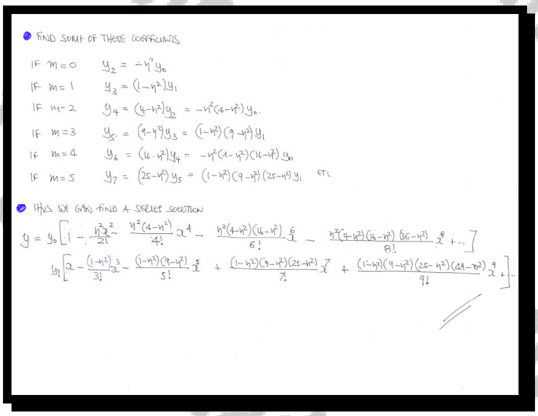
$$y_{m+2} + \left[-m^2 + m - m + n^2\right]y_m = 0$$

$$y_{m+2} + (-m^2+n^2)y_m = 0$$

$$\boxed{y_{m+2} = (m^2-n^2)y_m}$$

- EXPANDING AS AN INFINITE SERIES

$$y = y_0 + xy_0' + \frac{y_0''}{2!}x^2 + \frac{y_0'''}{3!}x^3 + \frac{y_0''''}{4!}x^4 + \ldots$$



- FIND SOME OF THESE COEFFICIENTS

IF $m=0$ $\quad y_2 = -n^2y_0$

IF $m=1$ $\quad y_3 = (1-n^2)y_1$

IF $m=2$ $\quad y_4 = (4-n^2)y_2 = -n^2(4-n^2)y_0$.

IF $m=3$ $\quad y_5 = (9-n^2)y_3 = (1-n^2)(9-n^2)y_1$

IF $m=4$ $\quad y_6 = (16-n^2)y_4 = -n^2(4-n^2)(16-n^2)y_0$

IF $m=5$ $\quad y_7 = (25-n^2)y_5 = (1-n^2)(9-n^2)(25-n^2)y_1$ ETC

- THUS WE CAN FIND A SERIES SOLUTION

$$y = y_0\left[1 - \frac{n^2x^2}{2!} - \frac{n^2(4-n^2)}{4!}x^4 - \frac{n^2(4-n^2)(16-n^2)}{6!}x^6 - \frac{n^2(4-n^2)(16-n^2)(36-n^2)}{8!}x^8 + \ldots\right]$$

$$y_1\left[x - \frac{(1-n^2)}{3!}x^3 - \frac{(1-n^2)(9-n^2)}{5!}x^5 + \frac{(1-n^2)(9-n^2)(25-n^2)}{7!}x^7 + \frac{(1-n^2)(9-n^2)(25-n^2)(49-n^2)}{9!}x^9 + \ldots\right]$$

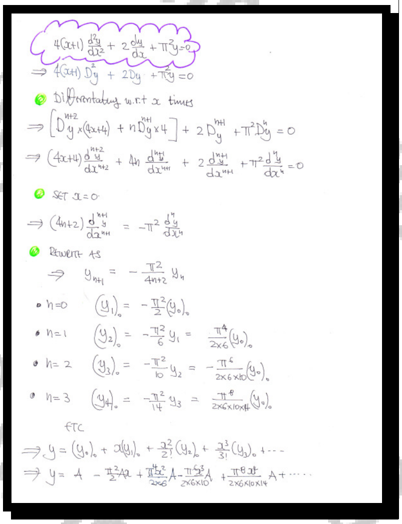
Question 7

Use Leibniz rule to find a solution, as an infinite series, for the following differential equation

$$4(x+1)\frac{d^2y}{dx^2}+2\frac{dy}{dx}+\pi^2y=0.$$

Give the final answer in simplified Sigma notation.

$$\boxed{y=A\sum_{r=0}^{\infty}\left[\frac{(-1)^r\,r!\,\pi^{2r}x^r}{(2r)!}\right]}$$



$$4(x+1)\frac{d^2y}{dx^2}+2\frac{dy}{dx}+\pi^2y=0$$

$\Rightarrow 4(x+1)D^2y+2Dy+\pi^2y=0$

Differentating w.r.t x times

$\Rightarrow \left[D^{n+2}_y\times(4x+4)+nD^{n+1}_y\times4\right]+2D^{n+1}_y+\pi^2D^n_y=0$

$\Rightarrow (4x+4)\frac{d^{n+2}y}{dx^{n+2}}+4n\frac{d^{n+1}y}{dx^{n+1}}+2\frac{d^{n+1}y}{dx^{n+1}}+\pi^2\frac{d^ny}{dx^n}=0$

SET $x=0$

$\Rightarrow (4n+2)\frac{d^{n+1}y}{dx^{n+1}}=-\pi^2\frac{d^ny}{dx^n}$

REWRITE AS

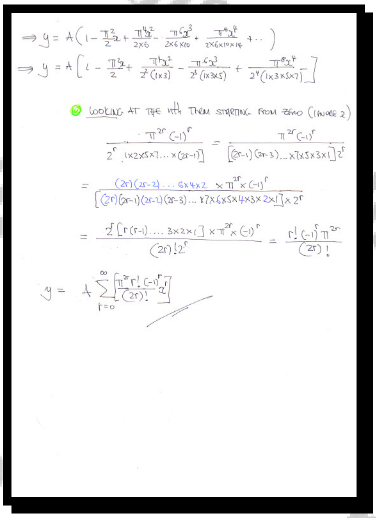
$\Rightarrow y_{n+1}=-\frac{\pi^2}{4n+2}y_n$

- $n=0$ $(y_1)_0=-\frac{\pi^2}{2}(y_0)_0$
- $n=1$ $(y_2)_0=-\frac{\pi^2}{6}y_1=\frac{\pi^4}{2\times6}(y_0)_0$
- $n=2$ $(y_3)_0=-\frac{\pi^2}{10}y_2=-\frac{\pi^6}{2\times6\times10}(y_0)_0$
- $n=3$ $(y_4)_0=-\frac{\pi^2}{14}y_3=\frac{\pi^8}{2\times6\times10\times14}(y_0)_0$

ETC

$\Rightarrow y=(y_0)_0+x(y_1)_0+\frac{x^2}{2!}(y_2)_0+\frac{x^3}{3!}(y_3)_0+\cdots$

$\Rightarrow y=A-\frac{\pi^2}{2}Ax+\frac{\pi^4x^2}{2\times6}A-\frac{\pi^6x^3}{2\times6\times10}A+\frac{\pi^8x^4}{2\times6\times10\times14}A+\cdots$



$\Rightarrow y=A\left(1-\frac{\pi^2}{2}x+\frac{\pi^4x^2}{2\times6}-\frac{\pi^6x^3}{2\times6\times10}+\frac{\pi^8x^4}{2\times6\times10\times14}+\cdots\right)$

$\Rightarrow y=A\left[1-\frac{\pi^2x}{2}+\frac{\pi^4x^2}{2^2(1\times3)}-\frac{\pi^6x^3}{2^3(1\times3\times5)}+\frac{\pi^8x^4}{2^4(1\times3\times5\times7)}-\right]$

LOOKING AT THE nth TERM STARTING FROM ZERO (IGNORE x)

$$\frac{\pi^{2r}(-1)^r}{2^r\,1\times2\times5\times7\ldots\times(2r-1)]}=\frac{\pi^{2r}(-1)^r}{[(2r-1)(2r-3)\ldots\times7\times5\times3\times1]2^r}$$

$$=\frac{(2r)(2r-2)\ldots6\times4\times2\times\pi^{2r}\times(-1)^r}{[(2r)(2r-1)(2r-2)(2r-3)\ldots\times7\times6\times5\times4\times3\times2\times1]\times2^r}$$

$$=\frac{2^r[r(r-1)\ldots3\times2\times1]\times\pi^{2r}\times(-1)^r}{(2r)!\,2^r}=\frac{r!(-1)^r\pi^{2r}}{(2r)!}$$

$$y=A\sum_{r=0}^{\infty}\left[\frac{\pi^{2r}r!(-1)^r}{(2r)!}x^r\right]$$

FROBENIUS METHOD

[analytic at $x = 0$]

Question 1

$$(x+1)\frac{dy}{dx}-(x+2)y=0,\ \ y(0)=1.$$

a) Find the solution of the above differential equation, by separation of variables.

b) Show that the solution can be written as

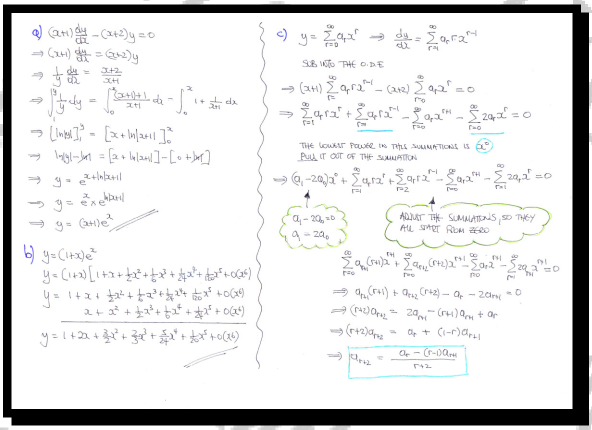
$$y=1+2x+\frac{3}{2}x^2+\frac{2}{3}x^3+\frac{5}{24}x^4+\frac{1}{20}x^5+O\left(x^6\right).$$

c) Assuming a solution of the form

$$y=\sum_{r=1}^{\infty}a_r x^r,$$

use the Frobenius method to verify the answer of part **(b).**

$$\boxed{y=(x+1)\mathrm{e}^x}$$



a) $(x+1)\frac{dy}{dx}-(x+2)y=0$

$\Rightarrow (x+1)\frac{dy}{dx}=(x+2)y$

$\Rightarrow \frac{1}{y}\frac{dy}{dx}=\frac{x+2}{x+1}$

$\Rightarrow \int_1^y \frac{1}{y}\,dy=\int_0^x \frac{(x+1)+1}{x+1}\,dx=\int_0^x 1+\frac{1}{x+1}\,dx$

$\Rightarrow \left[\ln|y|\right]_1^y=\left[x+\ln|x+1|\right]_0^x$

$\Rightarrow \ln|y|-\cancel{\ln 1}=\left[x+\ln|x+1|\right]-\left[0+\cancel{\ln 1}\right]$

$\Rightarrow y=e^{x+\ln|x+1|}$

$\Rightarrow y=e^x\times e^{\ln|x+1|}$

$\Rightarrow y=(x+1)e^x$

b) $y=(1+x)e^x$

$y=(1+x)\left[1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{120}x^5+O(x^6)\right.$

$y=1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{120}x^5+O(x^6)$

$\quad\ \ x+x^2+\frac{1}{2}x^3+\frac{1}{6}x^4+\frac{1}{24}x^5+O(x^6)$

$y=1+2x+\frac{3}{2}x^2+\frac{2}{3}x^3+\frac{5}{24}x^4+\frac{1}{20}x^5+O(x^6)$

c) $y=\sum_{r=0}^{\infty}a_r x^r \Rightarrow \frac{dy}{dx}=\sum_{r=1}^{\infty}a_r r x^{r-1}$

SUB INTO THE O.D.E

$\Rightarrow (x+1)\sum_{r=}^{\infty}a_r r x^{r-1}-(x+2)\sum_{r=0}^{\infty}a_r x^r=0$

$\Rightarrow \sum_{r=1}^{\infty}a_r r x^r+\underline{\sum_{r=1}^{\infty}a_r r x^{r-1}}-\sum_{r=0}^{\infty}a_r x^{r+1}-\underline{\sum_{r=0}^{\infty}2a_r x^r}=0$

THE LOWEST POWER IN THIS SUMMATIONS IS x^0

PULL IT OUT OF THE SUMMATION

$\Rightarrow (a_1-2a_0)x^0+\sum_{r=1}^{\infty}a_r r x^r+\sum_{r=2}^{\infty}a_r r x^{r-1}-\sum_{r=0}^{\infty}a_r x^{r+1}-\sum_{r=1}^{\infty}2a_r x^r=0$

$a_1-2a_0=0$

$a_1=2a_0$

ADJUST THE SUMMATIONS, SO THEY ALL START FROM ZERO

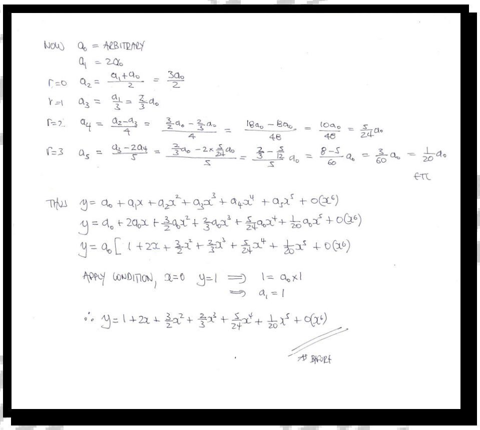
$\sum_{r=0}^{\infty}a_{r+1}(r+1)x^{r+1}+\sum_{r=0}^{\infty}a_{r+2}(r+2)x^{r+1}-\sum_{r=0}^{\infty}a_r x^{r+1}-\sum_{r=1}^{\infty}2a_{r+1}x^{r+1}=0$

$\Rightarrow a_{r+1}(r+1)+a_{r+2}(r+2)-a_r-2a_{r+1}=0$

$\Rightarrow (r+2)a_{r+2}=2a_{r+1}-(r+1)a_{r+1}+a_r$

$\Rightarrow (r+2)a_{r+2}=a_r+(1-r)a_{r+1}$

$\Rightarrow \boxed{a_{r+2}=\frac{a_r-(r-1)a_{r+1}}{r+2}}$



NOW $a_0=$ ARBITRARY

$a_1=2a_0$

$r=0$ $\quad a_2=\frac{a_1+a_0}{2}=\frac{3a_0}{2}$

$r=1$ $\quad a_3=\frac{a_1}{3}=\frac{2}{3}a_0$

$r=2$ $\quad a_4=\frac{a_2-a_3}{4}=\frac{\frac{3}{2}a_0-\frac{2}{3}a_0}{4}=\frac{18a_0-8a_0}{48}=\frac{10a_0}{48}=\frac{5}{24}a_0$

$r=3$ $\quad a_5=\frac{a_3-2a_4}{5}=\frac{\frac{2}{3}a_0-2\times\frac{5}{24}a_0}{5}=\frac{\frac{2}{3}-\frac{5}{12}}{5}a_0=\frac{8-5}{60}a_0=\frac{3}{60}a_0=\frac{1}{20}a_0$

ETC

THUS $y=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+O(x^6)$

$y=a_0+2a_0x+\frac{3}{2}a_0x^2+\frac{2}{3}a_0x^3+\frac{5}{24}a_0x^4+\frac{1}{20}a_0x^5+O(x^6)$

$y=a_0\left[1+2x+\frac{3}{2}x^2+\frac{2}{3}x^3+\frac{5}{24}x^4+\frac{1}{20}x^5+O(x^6)\right.$

APPLY CONDITION, $x=0$ $y=1 \Rightarrow 1=a_0\times 1$

$\Rightarrow a_1=1$

$\therefore y=1+2x+\frac{3}{2}x^2+\frac{2}{3}x^3+\frac{5}{24}x^4+\frac{1}{20}x^5+O(x^6)$

AS BEFORE

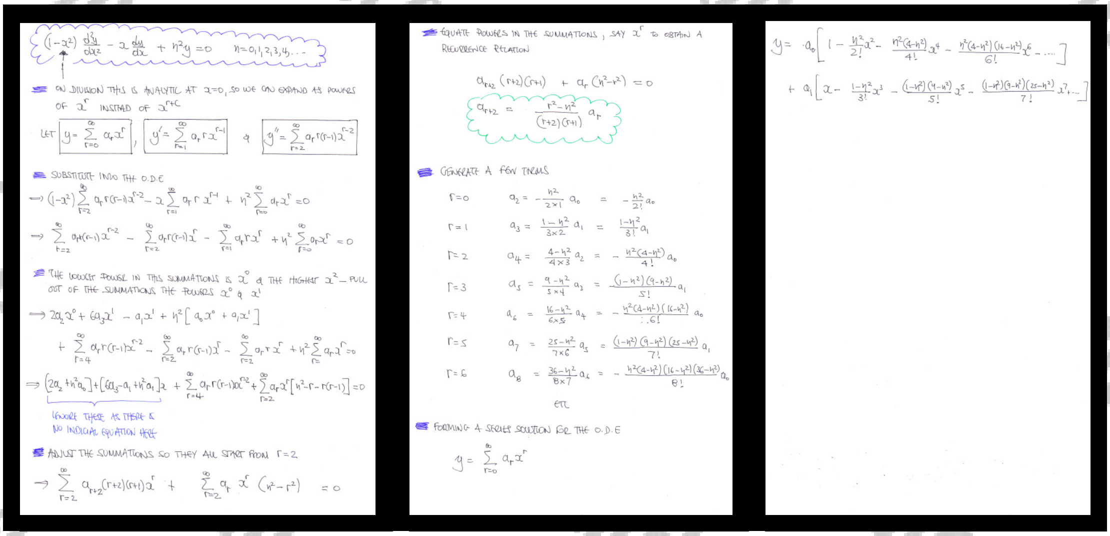
Question 2

Chebyshev's equation is shown below

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+n^2y=0\,,\ \ n=0,1,2,3,\ldots$$

Find a series solution for Chebyshev's equation, by using the Frobenius method.

$$y=$$
$$A\left[x-\frac{\left(1-n^2\right)}{3!}x^3-\frac{\left(1-n^2\right)\left(9-n^2\right)}{5!}x^5-\frac{\left(1-n^2\right)\left(9-n^2\right)\left(25-n^2\right)}{7!}x^7-\ldots\right]$$
$$+$$
$$B\left[1-\frac{n^2}{2!}x^2-\frac{n^2\left(4-n^2\right)}{4!}x^4-\frac{n^2\left(4-n^2\right)\left(16-n^2\right)}{6!}x^6-\frac{n^2\left(4-n^2\right)\left(16-n^2\right)\left(36-n^2\right)}{8!}x^8-\ldots\right]$$



$$\boxed{(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+n^2y=0 \qquad n=0,1,2,3,4,\ldots}$$

ON DIVISION THIS IS ANALYTIC AT $x=0$, SO WE CAN EXPAND AS POWERS OF x^r INSTEAD OF x^{r+c}

LET $\boxed{y=\sum_{r=0}^{\infty}a_rx^r}$, $\boxed{y'=\sum_{r=1}^{\infty}a_rrx^{r-1}}$ & $\boxed{y''=\sum_{r=2}^{\infty}a_rr(r-1)x^{r-2}}$

SUBSTITUTE INTO THE O.D.E

$$\Rightarrow (1-x^2)\sum_{r=2}^{\infty}a_rr(r-1)x^{r-2}-x\sum_{r=1}^{\infty}a_rrx^{r-1}+n^2\sum_{r=0}^{\infty}a_rx^r=0$$

$$\Rightarrow \sum_{r=2}^{\infty}a_rr(r-1)x^{r-2}-\sum_{r=2}^{\infty}a_rr(r-1)x^r-\sum_{r=1}^{\infty}a_rrx^r+n^2\sum_{r=0}^{\infty}a_rx^r=0$$

THE LOWEST POWER IN THIS SUMMATIONS IS x^0 & THE HIGHEST x^2 – PULL OUT OF THE SUMMATIONS THE POWERS x^0 & x^1

$$\Rightarrow 2a_2x^0+6a_3x^1-a_1x^1+n^2\left[a_0x^0+a_1x^1\right]$$
$$+\sum_{r=4}^{\infty}a_rr(r-1)x^{r-2}-\sum_{r=2}^{\infty}a_rr(r-1)x^r-\sum_{r=2}^{\infty}a_rrx^r+n^2\sum_{r=}^{\infty}a_rx^r=0$$

$$\Rightarrow \underbrace{\left[2a_2+n^2a_0\right]+\left[6a_3-a_1+n^2a_1\right]x}_{}+\sum_{r=4}^{\infty}a_rr(r-1)x^{r-2}+\sum_{r=2}^{\infty}a_rx^r\left[n^2-r-r(r-1)\right]=0$$

IGNORE THESE AS THERE IS NO INDICIAL EQUATION HERE

ADJUST THE SUMMATIONS SO THEY ALL START FROM $r=2$

$$\Rightarrow \sum_{r=2}^{\infty}a_{r+2}(r+2)(r+1)x^r+\sum_{r=2}^{\infty}a_rx^r(n^2-r^2)=0$$

EQUATE POWERS IN THE SUMMATIONS, SAY x^r TO OBTAIN A RECURRENCE RELATION

$$a_{r+2}(r+2)(r+1)+a_r(n^2-r^2)=0$$

$$\boxed{a_{r+2}=\frac{r^2-n^2}{(r+2)(r+1)}a_r}$$

GENERATE A FEW TERMS

$r=0 \qquad a_2=-\frac{n^2}{2\times1}a_0=-\frac{n^2}{2!}a_0$

$r=1 \qquad a_3=\frac{1-n^2}{3\times2}a_1=\frac{1-n^2}{3!}a_1$

$r=2 \qquad a_4=\frac{4-n^2}{4\times3}a_2=-\frac{n^2(4-n^2)}{4!}a_0$

$r=3 \qquad a_5=\frac{9-n^2}{5\times4}a_3=\frac{(1-n^2)(9-n^2)}{5!}a_1$

$r=4 \qquad a_6=\frac{16-n^2}{6\times5}a_4=-\frac{n^2(4-n^2)(16-n^2)}{6!}a_0$

$r=5 \qquad a_7=\frac{25-n^2}{7\times6}a_5=\frac{(1-n^2)(9-n^2)(25-n^2)}{7!}a_1$

$r=6 \qquad a_8=\frac{36-n^2}{8\times7}a_6=-\frac{n^2(4-n^2)(16-n^2)(36-n^2)}{8!}a_0$

ETC

FORMING A SERIES SOLUTION FOR THE O.D.E

$$y=\sum_{r=0}^{\infty}a_rx^r$$

$$y=a_0\left[1-\frac{n^2}{2!}x^2-\frac{n^2(4-n^2)}{4!}x^4-\frac{n^2(4-n^2)(16-n^2)}{6!}x^6-\ldots\right]$$
$$+a_1\left[x-\frac{1-n^2}{3!}x^3-\frac{(1-n^2)(4-n^2)}{5!}x^5-\frac{(1-n^2)(9-n^2)(25-n^2)}{7!}x^7+\ldots\right]$$

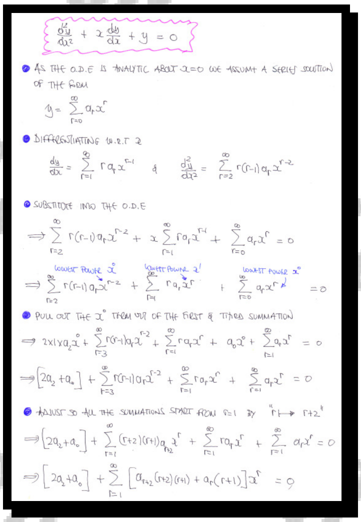
Question 3

Find the two independent solutions of the following differential equation

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0\,.$$

Give the final answer in simplified Sigma notation.

$$\boxed{y = A\sum_{n=0}^{\infty}\left[\frac{(-1)^n\,x^{2n}}{2^n\;n!}\right] + B\sum_{n=0}^{\infty}\left[\frac{(-1)^n\;n!\,x^{2n+1}}{(2n+1)!}\right]}$$



$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

- AS THE O.D.E IS ANALYTIC ABOUT $x=0$ WE ASSUME A SERIES SOLUTION OF THE FORM

$$y = \sum_{r=0}^{\infty} a_r x^r$$

- DIFFERENTIATING W.R.T x

$$\frac{dy}{dx} = \sum_{r=1}^{\infty} r a_r x^{r-1} \quad \& \quad \frac{d^2y}{dx^2} = \sum_{r=2}^{\infty} r(r-1)a_r x^{r-2}$$

- SUBSTITUTE INTO THE O.D.E

$$\Rightarrow \sum_{r=2}^{\infty} r(r-1)a_r x^{r-2} + x\sum_{r=1}^{\infty} r a_r x^{r-1} + \sum_{r=0}^{\infty} a_r x^r = 0$$

$$\Rightarrow \sum_{r=2}^{\infty} r(r-1)a_r \overset{\text{LOWEST POWER } x^0}{x^{r-2}} + \sum_{r=1}^{\infty} r a_r \overset{\text{LOWEST POWER } x^1}{x^r} + \sum_{r=0}^{\infty} a_r \overset{\text{LOWEST POWER } x^0}{x^r} = 0$$

- PULL OUT THE x^0 TERM OUT OF THE FIRST & THIRD SUMMATION

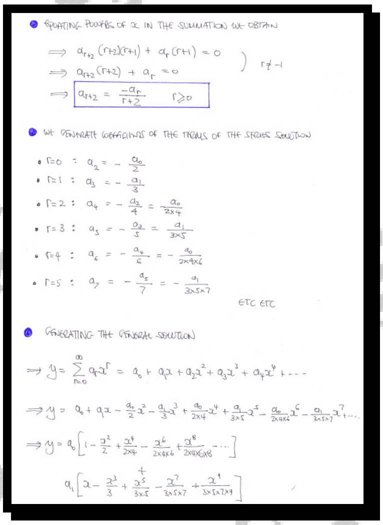
$$\Rightarrow 2\times1\times a_2x^0 + \sum_{r=3}^{\infty} r(r-1)a_r x^{r-2} + \sum_{r=1}^{\infty} r a_r x^r + a_0x^0 + \sum_{r=1}^{\infty} a_r x^r = 0$$

$$\Rightarrow \left[2a_2 + a_0\right] + \sum_{r=3}^{\infty} r(r-1)a_r x^{r-2} + \sum_{r=1}^{\infty} r a_r x^r + \sum_{r=1}^{\infty} a_r x^r = 0$$

- ADJUST SO ALL THE SUMMATIONS START FROM $r=1$ BY "$r \mapsto r+2$"

$$\Rightarrow \left[2a_2 + a_0\right] + \sum_{r=1}^{\infty} (r+2)(r+1)a_{r+2}x^r + \sum_{r=1}^{\infty} r a_r x^r + \sum_{r=1}^{\infty} a_r x^r = 0$$

$$\Rightarrow \left[2a_2 + a_0\right] + \sum_{r=1}^{\infty}\left[a_{r+2}(r+2)(r+1) + a_r(r+1)\right]x^r = 0$$



- EQUATING POWERS OF x IN THE SUMMATION WE OBTAIN

$$\left.\begin{array}{l}\Rightarrow a_{r+2}(r+2)(r+1) + a_r(r+1) = 0 \\ \Rightarrow a_{r+2}(r+2) + a_r = 0\end{array}\right)\ r \neq -1$$

$$\Rightarrow \boxed{a_{r+2} = \frac{-a_r}{r+2} \qquad r \geqslant 0}$$

- WE GENERATE COEFFICIENTS OF THE TERMS OF THE SERIES SOLUTION

- $r=0$: $a_2 = -\frac{a_0}{2}$
- $r=1$: $a_3 = -\frac{a_1}{3}$
- $r=2$: $a_4 = -\frac{a_2}{4} = \frac{a_0}{2\times4}$
- $r=3$: $a_5 = -\frac{a_3}{5} = \frac{a_1}{3\times5}$
- $r=4$: $a_6 = -\frac{a_4}{6} = -\frac{a_0}{2\times4\times6}$
- $r=5$: $a_7 = -\frac{a_5}{7} = -\frac{a_1}{3\times5\times7}$

ETC ETC

- GENERATING THE GENERAL SOLUTION

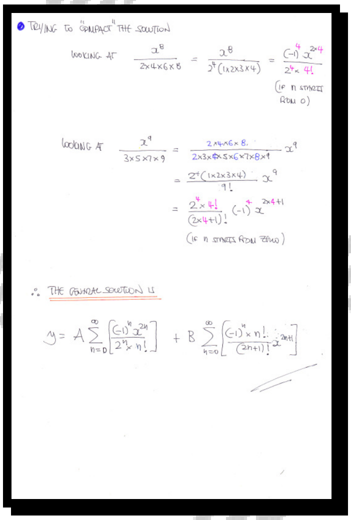
$$\Rightarrow y = \sum_{r=0}^{\infty} a_r x^r = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots$$

$$\Rightarrow y = a_0 + a_1x - \frac{a_0}{2}x^2 - \frac{a_1}{3}x^3 + \frac{a_0}{2\times4}x^4 + \frac{a_1}{3\times5}x^5 - \frac{a_0}{2\times4\times6}x^6 - \frac{a_1}{3\times5\times7}x^7 + \dots$$

$$\Rightarrow y = a_0\left[1 - \frac{x^2}{2} + \frac{x^4}{2\times4} - \frac{x^6}{2\times4\times6} + \frac{x^8}{2\times4\times6\times8} - \dots\right]$$

$$+$$

$$a_1\left[x - \frac{x^3}{3} + \frac{x^5}{3\times5} - \frac{x^7}{3\times5\times7} + \frac{x^9}{3\times5\times7\times9}\right]$$



- TRYING TO "COMPACT" THE SOLUTION

LOOKING AT $\frac{x^8}{2\times4\times6\times8} = \frac{x^8}{2^4(1\times2\times3\times4)} = \frac{(-1)^4x^{2\times4}}{2^4\times4!}$ (IF n STARTS FROM 0)

LOOKING AT $\frac{x^9}{3\times5\times7\times9} = \frac{2\times4\times6\times8}{2\times3\times4\times5\times6\times7\times8\times9}x^9$

$$= \frac{2^4(1\times2\times3\times4)}{9!}x^9$$

$$= \frac{2^4\times4!}{(2\times4+1)!}(-1)^4x^{2\times4+1}$$

(IF n STARTS FROM ZERO)

∴ THE GENERAL SOLUTION IS

$$y = A\sum_{n=0}^{\infty}\left[\frac{(-1)^nx^{2n}}{2^n\times n!}\right] + B\sum_{n=0}^{\infty}\left[\frac{(-1)^n\times n!}{(2n+1)!}x^{2n+1}\right]$$

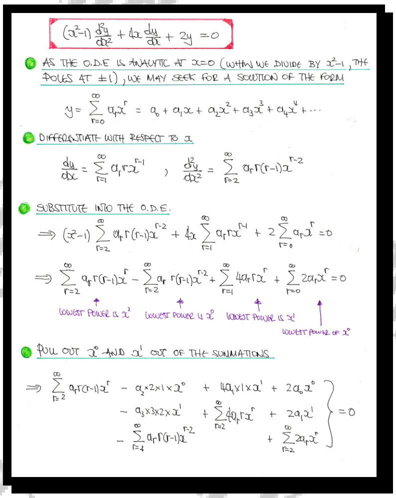
Question 4

Find the two independent solutions of the following differential equation

$$\left(x^2-1\right)\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=0\ ,\quad |x|<1.$$

Give the final answer in simplified form without involving infinite sums.

$\boxed{}$, $\boxed{y=\frac{A+Bx}{1-x^2}}$



$\boxed{(x^2-1)\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=0}$

AS THE O.D.E IS ANALYTIC AT $x=0$ (WHEN WE DIVIDE BY x^2-1, THE POLES AT ±1), WE MAY SEEK FOR A SOLUTION OF THE FORM

$$y=\sum_{r=0}^{\infty}a_rx^r=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\dots$$

DIFFERENTIATE WITH RESPECT TO x

$$\frac{dy}{dx}=\sum_{r=1}^{\infty}a_rrx^{r-1}\ ,\quad \frac{d^2y}{dx^2}=\sum_{r=2}^{\infty}a_rr(r-1)x^{r-2}$$

SUBSTITUTE INTO THE O.D.E.

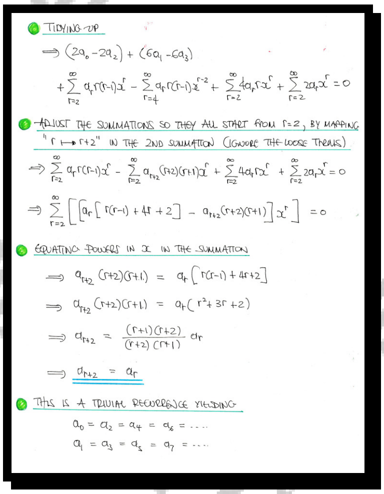
$$\Rightarrow (x^2-1)\sum_{r=2}^{\infty}a_rr(r-1)x^{r-2}+4x\sum_{r=1}^{\infty}a_rrx^{r-1}+2\sum_{r=0}^{\infty}a_rx^r=0$$

$$\Rightarrow \sum_{r=2}^{\infty}a_rr(r-1)x^r-\sum_{r=2}^{\infty}a_rr(r-1)x^{r-2}+\sum_{r=1}^{\infty}4a_rrx^r+\sum_{r=0}^{\infty}2a_rx^r=0$$

LOWEST POWER IS x^2 LOWEST POWER IS x^0 LOWEST POWER IS x^1 LOWEST POWER OF x^0

PULL OUT x^0 AND x^1 OUT OF THE SUMMATIONS

$$\Rightarrow \left.\begin{array}{l}\sum_{r=2}^{\infty}a_rr(r-1)x^r-a_2\times2\times1\times x^0+4a_1\times1\times x^1+2a_0x^0\\ \qquad -a_3\times3\times2\times x^1+\sum_{r=2}^{\infty}4a_rrx^r+2a_1x^1\\ \qquad -\sum_{r=4}^{\infty}a_rr(r-1)x^{r-2}+\sum_{r=2}^{\infty}2a_rx^r\end{array}\right\}=0$$



TIDYING UP

$$\Rightarrow (2a_0-2a_2)+(6a_1-6a_3)$$
$$+\sum_{r=2}^{\infty}a_rr(r-1)x^r-\sum_{r=4}^{\infty}a_rr(r-1)x^{r-2}+\sum_{r=2}^{\infty}4a_rrx^r+\sum_{r=2}^{\infty}2a_rx^r=0$$

ADJUST THE SUMMATIONS SO THEY ALL START FROM $r=2$, BY MAPPING "$r\mapsto r+2$" IN THE 2ND SUMMATION (IGNORE THE LOOSE TERMS)

$$\Rightarrow \sum_{r=2}^{\infty}a_rr(r-1)x^r-\sum_{r=2}^{\infty}a_{r+2}(r+2)(r+1)x^r+\sum_{r=2}^{\infty}4a_rrx^r+\sum_{r=2}^{\infty}2a_rx^r=0$$

$$\Rightarrow \sum_{r=2}^{\infty}\Big[\big[a_r[r(r-1)+4r+2]-a_{r+2}(r+2)(r+1)\big]x^r\Big]=0$$

EQUATING POWERS IN x IN THE SUMMATION

$$\Rightarrow a_{r+2}(r+2)(r+1)=a_r[r(r-1)+4r+2]$$

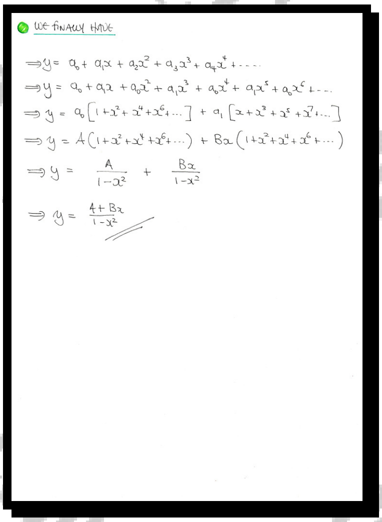
$$\Rightarrow a_{r+2}(r+2)(r+1)=a_r(r^2+3r+2)$$

$$\Rightarrow a_{r+2}=\frac{(r+1)(r+2)}{(r+2)(r+1)}a_r$$

$$\Rightarrow \underline{\underline{a_{r+2}=a_r}}$$

THIS IS A TRIVIAL RECURRENCE YIELDING

$$a_0=a_2=a_4=a_6=\dots$$
$$a_1=a_3=a_5=a_7=\dots$$



WE FINALLY HAVE

$$\Rightarrow y=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\dots$$

$$\Rightarrow y=a_0+a_1x+a_0x^2+a_1x^3+a_0x^4+a_1x^5+a_0x^6+\dots$$

$$\Rightarrow y=a_0\left[1+x^2+x^4+x^6+\dots\right]+a_1\left[x+x^3+x^5+x^7+\dots\right]$$

$$\Rightarrow y=A(1+x^2+x^4+x^6+\dots)+Bx(1+x^2+x^4+x^6+\dots)$$

$$\Rightarrow y=\frac{A}{1-x^2}+\frac{Bx}{1-x^2}$$

$$\Rightarrow y=\frac{A+Bx}{1-x^2}$$

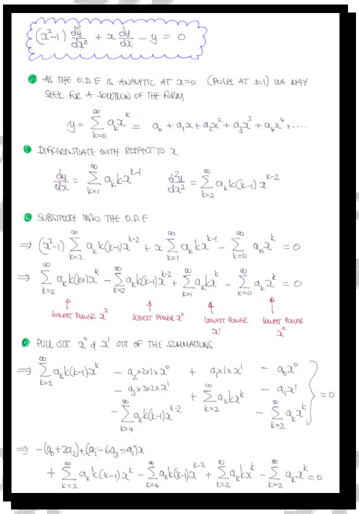
Question 5

Find the two independent solutions of the following differential equation

$$\left(x^2-1\right)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0\,.$$

Give the final answer in simplified Sigma notation.

$$\boxed{y=Ax+B\left[1-\sum_{n=0}^{\infty}\left[\frac{(2n)!\,x^{2n+2}}{2^{2n+1}\;n!\,(n+1)!}\right]\right]}$$



$$\boxed{(x^2-1)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0}$$

- AS THE O.D.E IS ANALYTIC AT $x=0$ (POLES AT ±1) WE MAY SEEK FOR A SOLUTION OF THE FORM

$$y=\sum_{k=0}^{\infty}a_kx^k=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\dots$$

- DIFFERENTIATE WITH RESPECT TO x

$$\frac{dy}{dx}=\sum_{k=1}^{\infty}a_kkx^{k-1}\qquad\frac{d^2y}{dx^2}=\sum_{k=2}^{\infty}a_kk(k-1)x^{k-2}$$

- SUBSTITUTE INTO THE O.D.E

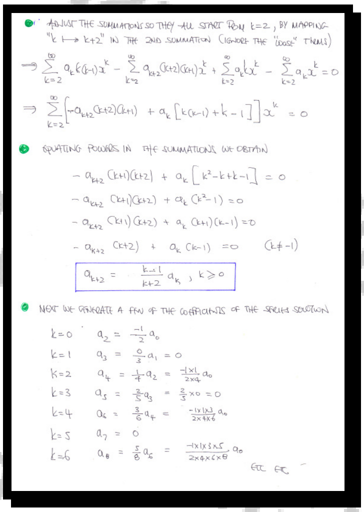
$$\Rightarrow (x^2-1)\sum_{k=2}^{\infty}a_kk(k-1)x^{k-2}+x\sum_{k=1}^{\infty}a_kkx^{k-1}-\sum_{k=0}^{\infty}a_kx^k=0$$

$$\Rightarrow \underset{\uparrow\ \text{LOWEST POWER } x^2}{\sum_{k=2}^{\infty}a_kk(k-1)x^k}-\underset{\uparrow\ \text{LOWEST POWER } x^0}{\sum_{k=2}^{\infty}a_kk(k-1)x^{k-2}}+\underset{\uparrow\ \text{LOWEST POWER } x^1}{\sum_{k=1}^{\infty}a_kkx^k}-\underset{\uparrow\ \text{LOWEST POWER } x^0}{\sum_{k=0}^{\infty}a_kx^k}=0$$

- PULL OUT x^0 & x^1 OUT OF THE SUMMATIONS

$$\Rightarrow \left.\begin{matrix}\displaystyle\sum_{k=2}^{\infty}a_kk(k-1)x^k & -a_2\times2\times1\times x^0 & +\ a_1\times1\times x^1 & -a_0x^0\\ & -a_3\times3\times2\times x^1 & +\displaystyle\sum_{k=2}^{\infty}a_kkx^k & -a_1x^1\\ & -\displaystyle\sum_{k=4}^{\infty}a_kk(k-1)x^{k-2} & & -\displaystyle\sum_{k=2}^{\infty}a_kx^k\end{matrix}\right\}=0$$

$$\Rightarrow -(a_0+2a_2)+(a_1-6a_3-a_1)x+\sum_{k=2}^{\infty}a_kk(k-1)x^k-\sum_{k=4}^{\infty}a_kk(k-1)x^{k-2}+\sum_{k=2}^{\infty}a_kkx^k-\sum_{k=2}^{\infty}a_kx^k=0$$



- ADJUST THE SUMMATIONS SO THEY ALL START FROM $k=2$, BY MAPPING "$k\mapsto k+2$" IN THE 2ND SUMMATION (IGNORE THE "LOOSE" TERMS)

$$\Rightarrow \sum_{k=2}^{\infty}a_kk(k-1)x^k-\sum_{k=2}^{\infty}a_{k+2}(k+2)(k+1)x^k+\sum_{k=2}^{\infty}a_kkx^k-\sum_{k=2}^{\infty}a_kx^k=0$$

$$\Rightarrow \sum_{k=2}^{\infty}\Big[-a_{k+2}(k+2)(k+1)+a_k\left[k(k-1)+k-1\right]\Big]x^k=0$$

- EQUATING POWERS IN THE SUMMATIONS WE OBTAIN

$$-a_{k+2}(k+1)(k+2)+a_k\left[k^2-k+k-1\right]=0$$

$$-a_{k+2}(k+1)(k+2)+a_k(k^2-1)=0$$

$$-a_{k+2}(k+1)(k+2)+a_k(k+1)(k-1)=0$$

$$-a_{k+2}(k+2)+a_k(k-1)=0\qquad(k\neq-1)$$

$$\boxed{a_{k+2}=\frac{k-1}{k+2}a_k\ ,\ k\geqslant0}$$

- NEXT WE GENERATE A FEW OF THE COEFFICIENTS OF THE SERIES SOLUTION

$k=0\qquad a_2=\frac{-1}{2}a_0$

$k=1\qquad a_3=\frac{0}{3}a_1=0$

$k=2\qquad a_4=\frac{1}{4}a_2=\frac{-1\times1}{2\times4}a_0$

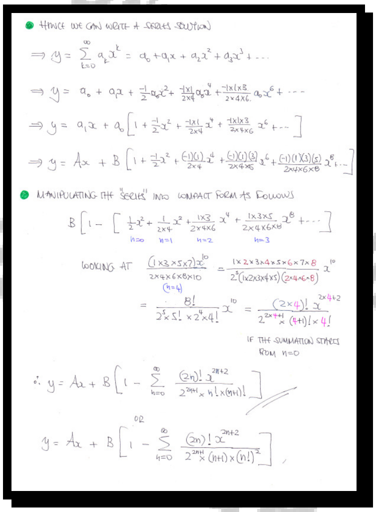
$k=3\qquad a_5=\frac{2}{5}a_3=\frac{2}{5}\times0=0$

$k=4\qquad a_6=\frac{3}{6}a_4=\frac{-1\times1\times3}{2\times4\times6}a_0$

$k=5\qquad a_7=0$

$k=6\qquad a_8=\frac{5}{8}a_6=\frac{-1\times1\times3\times5}{2\times4\times6\times8}a_0$

ETC ETC



- HENCE WE CAN WRITE A SERIES SOLUTION

$$\Rightarrow y=\sum_{k=0}^{\infty}a_kx^k=a_0+a_1x+a_2x^2+a_3x^3+\dots$$

$$\Rightarrow y=a_0+a_1x+\frac{-1}{2}a_0x^2+\frac{-1\times1}{2\times4}a_0x^4+\frac{-1\times1\times3}{2\times4\times6}a_0x^6+\dots$$

$$\Rightarrow y=a_1x+a_0\left[1+\frac{-1}{2}x^2+\frac{-1\times1}{2\times4}x^4+\frac{-1\times1\times3}{2\times4\times6}x^6+\dots\right]$$

$$\Rightarrow y=Ax+B\left[1+\frac{-1}{2}x^2+\frac{(-1)(1)}{2\times4}x^4+\frac{(-1)(1)(3)}{2\times4\times6}x^6+\frac{(-1)(1)(3)(5)}{2\times4\times6\times8}x^8+\dots\right]$$

- MANIPULATING THE "SERIES" INTO COMPACT FORM AS FOLLOWS

$$B\Big[1-\Big[\underset{n=0}{\frac{1}{2}x^2}+\underset{n=1}{\frac{1}{2\times4}x^2}+\underset{n=2}{\frac{1\times3}{2\times4\times6}x^4}+\underset{n=3}{\frac{1\times3\times5}{2\times4\times6\times8}x^8}+\dots\Big]$$

LOOKING AT $\underset{(n=4)}{\frac{(1\times3\times5\times7)x^{10}}{2\times4\times6\times8\times10}}=\frac{1\times2\times3\times4\times5\times6\times7\times8}{2^5(1\times2\times3\times4\times5)(2\times4\times6\times8)}x^{10}$

$$=\frac{8!}{2^5\times5!\times2^4\times4!}x^{10}=\frac{(2\times4)!\,x^{2\times4+2}}{2^{2\times4+1}\times(4+1)!\times4!}$$

IF THE SUMMATION STARTS FROM $n=0$

$$\therefore\ y=Ax+B\left[1-\sum_{n=0}^{\infty}\frac{(2n)!\,x^{2n+2}}{2^{2n+1}\times n!\times(n+1)!}\right]$$

OR

$$y=Ax+B\left[1-\sum_{n=0}^{\infty}\frac{(2n)!\,x^{2n+2}}{2^{2n+1}\times(n+1)\times(n!)^2}\right]$$

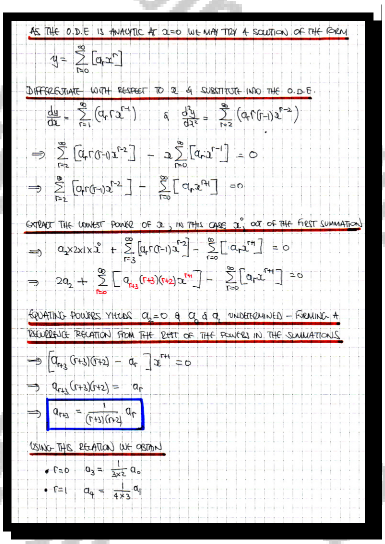
Question 6

Use the Frobenius method to find a general solution, as an infinite series, for Airy's differential equation

$$\frac{d^2y}{dx^2} - xy = 0\,.$$

Give the final answer in simplified Sigma notation.

$$, \quad \boxed{y = \sum_{r=0}^{\infty}\left[\frac{x^{3r}}{9^r \times r!}\left[\frac{A}{\Gamma\left(\frac{3r+2}{3}\right)} + \frac{Bx}{\Gamma\left(\frac{3r+4}{3}\right)}\right]\right]}$$



AS THE O.D.E IS ANALYTIC AT $x=0$ WE MAY TRY A SOLUTION OF THE FORM

$$y = \sum_{r=0}^{\infty}\left[a_r x^r\right]$$

DIFFERENTIATE WITH RESPECT TO x & SUBSTITUTE INTO THE O.D.E.

$$\frac{dy}{dx} = \sum_{r=1}^{\infty}\left(a_r r x^{r-1}\right) \quad \& \quad \frac{d^2y}{dx^2} = \sum_{r=2}^{\infty}\left(a_r r(r-1)x^{r-2}\right)$$

$$\Rightarrow \sum_{r=2}^{\infty}\left[a_r r(r-1)x^{r-2}\right] - x\sum_{r=0}^{\infty}\left[a_r x^{r-1}\right] = 0$$

$$\Rightarrow \sum_{r=2}^{\infty}\left[a_r r(r-1)x^{r-2}\right] - \sum_{r=0}^{\infty}\left[a_r x^{r+1}\right] = 0$$

EXTRACT THE LOWEST POWER OF x, IN THIS CASE x^0, OUT OF THE FIRST SUMMATION

$$\Rightarrow a_2 \times 2 \times 1 \times x^0 + \sum_{r=3}^{\infty}\left[a_r r(r-1)x^{r-2}\right] - \sum_{r=0}^{\infty}\left[a_r x^{r+1}\right] = 0$$

$$\Rightarrow 2a_2 + \sum_{r=0}^{\infty}\left[a_{r+3}(r+3)(r+2)x^{r+1}\right] - \sum_{r=0}^{\infty}\left[a_r x^{r+1}\right] = 0$$

EQUATING POWERS YIELDS $a_2=0$ & a_0 & a_1 UNDETERMINED – FORMING A RECURRENCE RELATION FROM THE REST OF THE POWERS IN THE SUMMATIONS

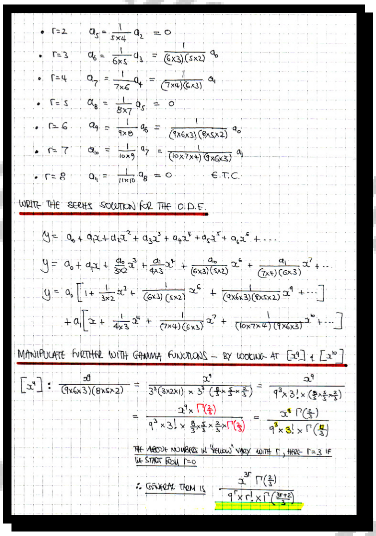
$$\Rightarrow \left[a_{r+3}(r+3)(r+2) - a_r\right]x^{r+1} = 0$$

$$\Rightarrow a_{r+3}(r+3)(r+2) = a_r$$

$$\Rightarrow \boxed{a_{r+3} = \frac{1}{(r+3)(r+2)}a_r}$$

USING THIS RELATION WE OBTAIN

- $r=0 \quad a_3 = \frac{1}{3\times2}a_0$
- $r=1 \quad a_4 = \frac{1}{4\times3}a_1$



- $r=2 \quad a_5 = \frac{1}{5\times4}a_2 = 0$
- $r=3 \quad a_6 = \frac{1}{6\times5}a_3 = \frac{1}{(6\times3)(5\times2)}a_0$
- $r=4 \quad a_7 = \frac{1}{7\times6}a_4 = \frac{1}{(7\times4)(6\times3)}a_1$
- $r=5 \quad a_8 = \frac{1}{8\times7}a_5 = 0$
- $r=6 \quad a_9 = \frac{1}{9\times8}a_6 = \frac{1}{(9\times6\times3)(8\times5\times2)}a_0$
- $r=7 \quad a_{10} = \frac{1}{10\times9}a_7 = \frac{1}{(10\times7\times4)(9\times6\times3)}a_1$
- $r=8 \quad a_{11} = \frac{1}{11\times10}a_8 = 0 \qquad$ E.T.C.

WRITE THE SERIES SOLUTION FOR THE O.D.E.

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \dots$$

$$y = a_0 + a_1x + \frac{a_0}{3\times2}x^3 + \frac{a_1}{4\times3}x^4 + \frac{a_0}{(6\times3)(5\times2)}x^6 + \frac{a_1}{(7\times4)(6\times3)}x^7 + \dots$$

$$y = a_0\left[1 + \frac{1}{3\times2}x^3 + \frac{1}{(6\times3)(5\times2)}x^6 + \frac{1}{(9\times6\times3)(8\times5\times2)}x^9 + \dots\right]$$

$$+ a_1\left[x + \frac{1}{4\times3}x^4 + \frac{1}{(7\times4)(6\times3)}x^7 + \frac{1}{(10\times7\times4)(9\times6\times3)}x^{10} + \dots\right]$$

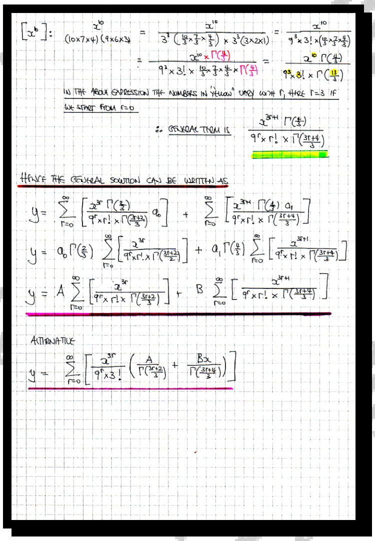
MANIPULATE FURTHER WITH GAMMA FUNCTIONS – BY LOOKING AT $[x^9]$ & $[x^{10}]$

$$[x^9]: \frac{x^9}{(9\times6\times3)(8\times5\times2)} = \frac{x^9}{3^3(3\times2\times1)\times3^3\left(\frac{8}{3}\times\frac{5}{3}\times\frac{2}{3}\right)} = \frac{x^9}{9^3\times3!\times\left(\frac{8}{3}\times\frac{5}{3}\times\frac{2}{3}\right)}$$

$$= \frac{x^9\times\Gamma\left(\frac{2}{3}\right)}{9^3\times3!\times\frac{8}{3}\times\frac{5}{3}\times\frac{2}{3}\times\Gamma\left(\frac{2}{3}\right)} = \frac{x^9\,\Gamma\left(\frac{2}{3}\right)}{9^3\times3!\times\Gamma\left(\frac{11}{3}\right)}$$

THE ABOVE NUMBERS IN "YELLOW" VARY WITH r, HERE $r=3$ IF WE START FROM $r=0$

$\therefore$ GENERAL TERM IS $\dfrac{x^{3r}\,\Gamma\left(\frac{2}{3}\right)}{9^r\times r!\times\Gamma\left(\frac{3r+2}{3}\right)}$



$$[x^{10}]: \frac{x^{10}}{(10\times7\times4)(9\times6\times3)} = \frac{x^{10}}{3^3\left(\frac{10}{3}\times\frac{7}{3}\times\frac{4}{3}\right)\times3^3(3\times2\times1)} = \frac{x^{10}}{9^3\times3!\times\left(\frac{10}{3}\times\frac{7}{3}\times\frac{4}{3}\right)}$$

$$= \frac{x^{10}\times\Gamma\left(\frac{4}{3}\right)}{9^3\times3!\times\frac{10}{3}\times\frac{7}{3}\times\frac{4}{3}\times\Gamma\left(\frac{4}{3}\right)} = \frac{x^{10}\,\Gamma\left(\frac{4}{3}\right)}{9^3\times3!\times\Gamma\left(\frac{13}{3}\right)}$$

IN THE ABOVE EXPRESSION THE NUMBERS IN "YELLOW" VARY WITH r, HERE $r=3$ IF WE START FROM $r=0$

$\therefore$ GENERAL TERM IS $\dfrac{x^{3r+1}\,\Gamma\left(\frac{4}{3}\right)}{9^r\times r!\times\Gamma\left(\frac{3r+4}{3}\right)}$

HENCE THE GENERAL SOLUTION CAN BE WRITTEN AS

$$y = \sum_{r=0}^{\infty}\left[\frac{x^{3r}\,\Gamma\left(\frac{2}{3}\right)}{9^r\times r!\times\Gamma\left(\frac{3r+2}{3}\right)}a_0\right] + \sum_{r=0}^{\infty}\left[\frac{x^{3r+1}\,\Gamma\left(\frac{4}{3}\right)a_1}{9^r\times r!\times\Gamma\left(\frac{3r+4}{3}\right)}\right]$$

$$y = a_0\Gamma\left(\tfrac{2}{3}\right)\sum_{r=0}^{\infty}\left[\frac{x^{3r}}{9^r\times r!\times\Gamma\left(\frac{3r+2}{2}\right)}\right] + a_1\Gamma\left(\tfrac{4}{3}\right)\sum_{r=0}^{\infty}\left[\frac{x^{3r+1}}{9^r\times r!\times\Gamma\left(\frac{3r+4}{3}\right)}\right]$$

$$y = A\sum_{r=0}^{\infty}\left[\frac{x^{3r}}{9^r\times r!\times\Gamma\left(\frac{3r+2}{3}\right)}\right] + B\sum_{r=0}^{\infty}\left[\frac{x^{3r+1}}{9^r\times r!\times\Gamma\left(\frac{3r+4}{3}\right)}\right]$$

ALTERNATIVE

$$y = \sum_{r=0}^{\infty}\left[\frac{x^{3r}}{9^r\times3!}\left(\frac{A}{\Gamma\left(\frac{3r+2}{3}\right)} + \frac{Bx}{\Gamma\left(\frac{3r+4}{3}\right)}\right)\right]$$

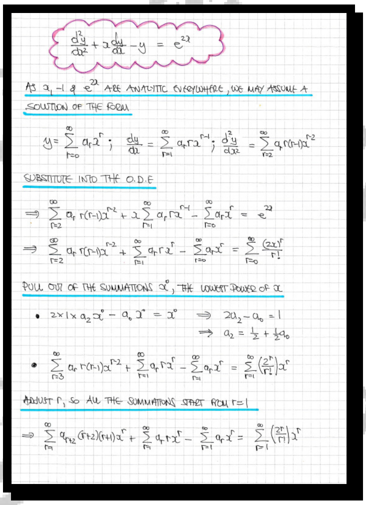
Question 7

Find, as a series, a solution of the following differential equation

$$\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=\mathrm{e}^{2x} \ .$$

Give the final answer in simplified form up and including the term in x^8.

$\boxed{}$, $\boxed{\begin{aligned}&y=Ax\\&+B\left[1+\frac{1}{2}x^2-\frac{1}{24}x^4+\frac{1}{240}x^6-\frac{1}{2688}x^8+\ldots\right]\\&+x^2\left[\frac{1}{2}+\frac{1}{3}x+\frac{1}{8}x^2+\frac{1}{30}x^3+\frac{7}{720}x^4+\frac{1}{315}x^5+\frac{29}{40320}x^6+\ldots\ldots\right]\end{aligned}}$



$$\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=e^{2x}$$

AS x, -1 & e^{2x} ARE ANALYTIC EVERYWHERE, WE MAY ASSUME A SOLUTION OF THE FORM

$$y=\sum_{r=0}^{\infty}a_rx^r;\quad \frac{dy}{dx}=\sum_{r=1}^{\infty}a_rrx^{r-1};\quad \frac{d^2y}{dx^2}=\sum_{r=2}^{\infty}a_rr(r-1)x^{r-2}$$

SUBSTITUTE INTO THE O.D.E

$$\Rightarrow \sum_{r=2}^{\infty}a_rr(r-1)x^{r-2}+x\sum_{r=1}^{\infty}a_rrx^{r-1}-\sum_{r=0}^{\infty}a_rx^r=e^{2x}$$

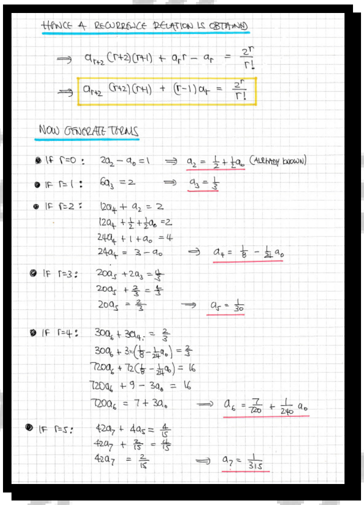
$$\Rightarrow \sum_{r=2}^{\infty}a_rr(r-1)x^{r-2}+\sum_{r=1}^{\infty}a_rrx^r-\sum_{r=0}^{\infty}a_rx^r=\sum_{r=0}^{\infty}\frac{(2x)^r}{r!}$$

PULL OUT OF THE SUMMATIONS x^0, THE LOWEST POWER OF x

- $2\times1\times a_2x^0-a_0x^0=x^0 \Rightarrow 2a_2-a_0=1$
  $\Rightarrow a_2=\frac{1}{2}+\frac{1}{2}a_0$
- $\sum_{r=3}^{\infty}a_rr(r-1)x^{r-2}+\sum_{r=1}^{\infty}a_rrx^r-\sum_{r=1}^{\infty}a_rx^r=\sum_{r=1}^{\infty}\left(\frac{2^r}{r!}\right)x^r$

ADJUST r, SO ALL THE SUMMATIONS START FROM $r=1$

$$\Rightarrow \sum_{r=1}^{\infty}a_{r+2}(r+2)(r+1)x^r+\sum_{r=1}^{\infty}a_rrx^r-\sum_{r=1}^{\infty}a_rx^r=\sum_{r=1}^{\infty}\left(\frac{2^r}{r!}\right)x^r$$



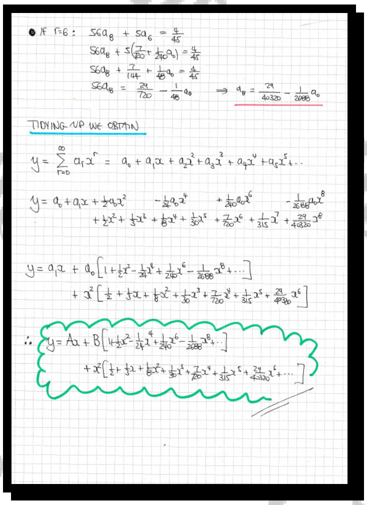
HENCE A RECURRENCE RELATION IS OBTAINED

$$\Rightarrow a_{r+2}(r+2)(r+1)+a_rr-a_r=\frac{2^r}{r!}$$

$$\Rightarrow \boxed{a_{r+2}(r+2)(r+1)+(r-1)a_r=\frac{2^r}{r!}}$$

NOW GENERATE TERMS

- IF $r=0$: $2a_2-a_0=1 \Rightarrow \underline{a_2=\frac{1}{2}+\frac{1}{2}a_0}$ (ALREADY KNOWN)
- IF $r=1$: $6a_3=2 \Rightarrow \underline{a_3=\frac{1}{3}}$
- IF $r=2$: $12a_4+a_2=2$
  $12a_4+\frac{1}{2}+\frac{1}{2}a_0=2$
  $24a_4+1+a_0=4$
  $24a_4=3-a_0 \Rightarrow \underline{a_4=\frac{1}{8}-\frac{1}{24}a_0}$
- IF $r=3$: $20a_5+2a_3=\frac{4}{3}$
  $20a_5+\frac{2}{3}=\frac{4}{3}$
  $20a_5=\frac{2}{3} \Rightarrow \underline{a_5=\frac{1}{30}}$
- IF $r=4$: $30a_6+3a_4=\frac{2}{3}$
  $30a_6+3\times\left(\frac{1}{8}-\frac{1}{24}a_0\right)=\frac{2}{3}$
  $720a_6+72\left(\frac{1}{8}-\frac{1}{24}a_0\right)=16$
  $720a_6+9-3a_0=16$
  $720a_6=7+3a_0 \Rightarrow \underline{a_6=\frac{7}{720}+\frac{1}{240}a_0}$
- IF $r=5$: $42a_7+4a_5=\frac{4}{15}$
  $42a_7+\frac{2}{15}=\frac{4}{15}$
  $42a_7=\frac{2}{15} \Rightarrow \underline{a_7=\frac{1}{315}}$



- IF $r=6$: $56a_8+5a_6=\frac{4}{45}$
  $56a_8+5\left(\frac{7}{720}+\frac{1}{240}a_0\right)=\frac{4}{45}$
  $56a_8+\frac{7}{144}+\frac{1}{48}a_0=\frac{4}{45}$
  $56a_8=\frac{29}{720}-\frac{1}{48}a_0 \Rightarrow \underline{a_8=\frac{29}{40320}-\frac{1}{2688}a_0}$

TIDYING UP WE OBTAIN

$$y=\sum_{r=0}^{\infty}a_rx^r=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+\ldots$$

$$\begin{aligned}y=a_0+a_1x+\tfrac{1}{2}a_0x^2\quad-\tfrac{1}{24}a_0x^4\quad+\tfrac{1}{240}a_0x^6\quad-\tfrac{1}{2688}a_0x^8\\+\tfrac{1}{2}x^2+\tfrac{1}{3}x^3+\tfrac{1}{8}x^4+\tfrac{1}{30}x^5+\tfrac{7}{720}x^6+\tfrac{1}{315}x^7+\tfrac{29}{40320}x^8\end{aligned}$$

$$\begin{aligned}y=a_1x+a_0\left[1+\tfrac{1}{2}x^2-\tfrac{1}{24}x^4+\tfrac{1}{240}x^6-\tfrac{1}{2688}x^8+\ldots\right]\\+x^2\left[\tfrac{1}{2}+\tfrac{1}{3}x+\tfrac{1}{8}x^2+\tfrac{1}{30}x^3+\tfrac{7}{720}x^4+\tfrac{1}{315}x^5+\tfrac{29}{40320}x^6\right]\end{aligned}$$

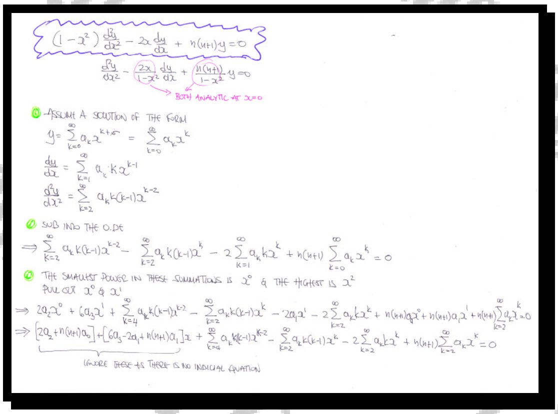
$$\therefore\ \begin{aligned}y=Ax+B\left[1+\tfrac{1}{2}x^2-\tfrac{1}{24}x^4+\tfrac{1}{240}x^6-\tfrac{1}{2688}x^8+\ldots\right]\\+x^2\left[\tfrac{1}{2}+\tfrac{1}{3}x+\tfrac{1}{8}x^2+\tfrac{1}{30}x^3+\tfrac{7}{720}x^4+\tfrac{1}{315}x^5+\tfrac{29}{40320}x^6+\ldots\right]\end{aligned}$$

Question 8

Find the two independent solutions of Legendre's equation

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n\left(n+1\right)y=0\,,\ n\in\mathbb{R}\,.$$

$$\boxed{\begin{gathered}y=A\left[1-\frac{(n+1)n}{2!}x^2+\frac{(n+3)(n+1)n(n-2)}{4!}x^4-\frac{(n+5)(n+3)(n+1)n(n-2)(n-4)}{6!}x^6+\ldots\right]\\ +\\ B\left[x-\frac{(n+2)(n-1)}{3!}x^3+\frac{(n+4)(n+2)(n-1)(n-3)}{3!}x^5-\frac{(n+6)(n+4)(n+2)(n-1)(n-3)(n-5)}{7!}x^7+\ldots\right]\end{gathered}}$$



$$\boxed{(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0}$$

$$\frac{d^2y}{dx^2}-\boxed{\frac{2x}{1-x^2}}\frac{dy}{dx}+\boxed{\frac{n(n+1)}{1-x^2}}y=0$$

BOTH ANALYTIC AT $x=0$

- ASSUME A SOLUTION OF THE FORM

$$y=\sum_{k=0}^{\infty}a_kx^{k+\sigma}=\sum_{k=0}^{\infty}a_kx^k$$

$$\frac{dy}{dx}=\sum_{k=1}^{\infty}a_kkx^{k-1}$$

$$\frac{d^2y}{dx^2}=\sum_{k=2}^{\infty}a_kk(k-1)x^{k-2}$$

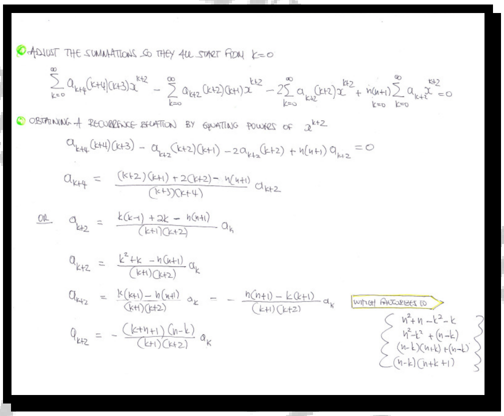
- SUB INTO THE O.D.E

$$\Rightarrow\sum_{k=2}^{\infty}a_kk(k-1)x^{k-2}-\sum_{k=2}^{\infty}a_kk(k-1)x^k-2\sum_{k=1}^{\infty}a_kkx^k+n(n+1)\sum_{k=0}^{\infty}a_kx^k=0$$

- THE SMALLEST POWER IN THESE SUMMATIONS IS x^0 & THE HIGHEST IS x^2
PULL OUT x^0 & x^1

$$\Rightarrow 2a_2x^0+6a_3x^1+\sum_{k=4}^{\infty}a_kk(k-1)x^{k-2}-\sum_{k=2}^{\infty}a_kk(k-1)x^k-2a_1x^1-2\sum_{k=2}^{\infty}a_kkx^k+n(n+1)a_0x^0+n(n+1)a_1x^1+n(n+1)\sum_{k=2}^{\infty}a_kx^k=0$$

$$\Rightarrow\underbrace{[2a_2+n(n+1)a_0]+[6a_3-2a_1+n(n+1)a_1]x}_{\text{IGNORE THESE AS THERE IS NO INDICIAL EQUATION}}+\sum_{k=4}^{\infty}a_kk(k-1)x^{k-2}-\sum_{k=2}^{\infty}a_kk(k-1)x^k-2\sum_{k=2}^{\infty}a_kkx^k+n(n+1)\sum_{k=2}^{\infty}a_kx^k=0$$



- ADJUST THE SUMMATIONS SO THEY ALL START FROM $k=0$

$$\sum_{k=0}^{\infty}a_{k+4}(k+4)(k+3)x^{k+2}-\sum_{k=0}^{\infty}a_{k+2}(k+2)(k+1)x^{k+2}-2\sum_{k=0}^{\infty}a_{k+2}(k+2)x^{k+2}+n(n+1)\sum_{k=0}^{\infty}a_{k+2}x^{k+2}=0$$

- OBTAINING A RECURRENCE RELATION BY EQUATING POWERS OF x^{k+2}

$$a_{k+4}(k+4)(k+3)-a_{k+2}(k+2)(k+1)-2a_{k+2}(k+2)+n(n+1)a_{k+2}=0$$

$$a_{k+4}=\frac{(k+2)(k+1)+2(k+2)-n(n+1)}{(k+3)(k+4)}a_{k+2}$$

OR $$a_{k+2}=\frac{k(k-1)+2k-n(n+1)}{(k+1)(k+2)}a_k$$

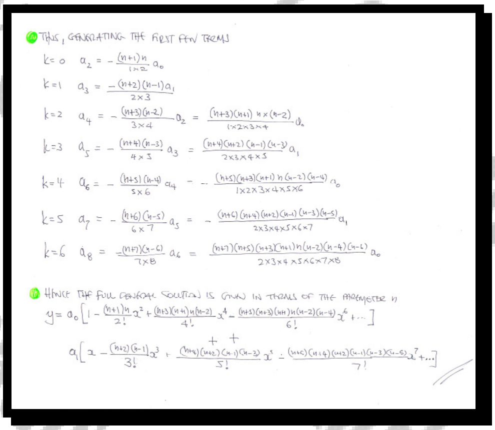
$$a_{k+2}=\frac{k^2+k-n(n+1)}{(k+1)(k+2)}a_k$$

$$a_{k+2}=\frac{k(k+1)-n(n+1)}{(k+1)(k+2)}a_k=-\frac{n(n+1)-k(k+1)}{(k+1)(k+2)}a_k$$

WHICH FACTORIZES TO

$$\left\{\begin{gathered}n^2+n-k^2-k\\ n^2-k^2+(n-k)\\ (n-k)(n+k)+(n-k)\\ (n-k)(n+k+1)\end{gathered}\right\}$$

$$a_{k+2}=-\frac{(k+n+1)(n-k)}{(k+1)(k+2)}a_k$$



- THUS, GENERATING THE FIRST FEW TERMS

$k=0$ $$a_2=-\frac{(n+1)n}{1\times2}a_0$$

$k=1$ $$a_3=\frac{-(n+2)(n-1)a_1}{2\times3}$$

$k=2$ $$a_4=-\frac{(n+3)(n-2)}{3\times4}a_2=\frac{(n+3)(n+1)n\times(n-2)}{1\times2\times3\times4}a_0$$

$k=3$ $$a_5=-\frac{(n+4)(n-3)}{4\times5}a_3=\frac{(n+4)(n+2)(n-1)(n-3)}{2\times3\times4\times5}a_1$$

$k=4$ $$a_6=-\frac{(n+5)(n-4)}{5\times6}a_4=-\frac{(n+5)(n+3)(n+1)n(n-2)(n-4)}{1\times2\times3\times4\times5\times6}a_0$$

$k=5$ $$a_7=-\frac{(n+6)(n-5)}{6\times7}a_5=-\frac{(n+6)(n+4)(n+2)(n-1)(n-3)(n-5)}{2\times3\times4\times5\times6\times7}a_1$$

$k=6$ $$a_8=\frac{-(n+7)(n-6)}{7\times8}a_6=\frac{(n+7)(n+5)(n+3)(n+1)n(n-2)(n-4)(n-6)}{2\times3\times4\times5\times6\times7\times8}a_0$$

- HENCE THE FULL GENERAL SOLUTION IS GIVEN IN TERMS OF THE PARAMETER n

$$y=a_0\left[1-\frac{(n+1)n}{2!}x^2+\frac{(n+3)(n+1)n(n-2)}{4!}x^4-\frac{(n+5)(n+3)(n+1)n(n-2)(n-4)}{6!}x^6+\ldots\right]$$

$$+$$

$$a_1\left[x-\frac{(n+2)(n-1)}{3!}x^3+\frac{(n+4)(n+2)(n-1)(n-3)}{5!}x^5-\frac{(n+6)(n+4)(n+2)(n-1)(n-3)(n-5)}{7!}x^7+\ldots\right]$$

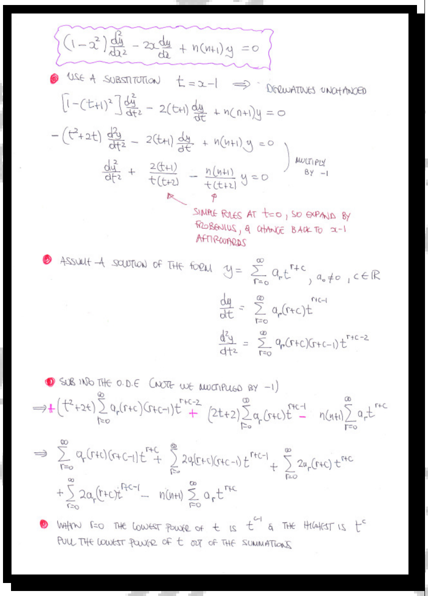
Question 9

Find one series solution for the Legendre's equation

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n\left(n+1\right)y=0\,,\ n\in\mathbb{R}\,,$$

about $x=1$.

$$\boxed{y=A\sum_{r=0}^{\infty}\left[\frac{(n+r)!}{(n-r)!}\times\frac{1}{(r!)^2}\times\left(\frac{x-1}{2}\right)^2\right]}$$



$$\boxed{(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0}$$

- USE A SUBSTITUTION $t=x-1 \Rightarrow$ DERIVATIVES UNCHANGED

$$\left[1-(t+1)^2\right]\frac{d^2y}{dt^2}-2(t+1)\frac{dy}{dt}+n(n+1)y=0$$

$$-(t^2+2t)\frac{d^2y}{dt^2}-2(t+1)\frac{dy}{dt}+n(n+1)y=0$$

$$\frac{d^2y}{dt^2}+\frac{2(t+1)}{t(t+2)}-\frac{n(n+1)}{t(t+2)}y=0$$

MULTIPLY BY -1

SIMPLE POLES AT $t=0$, SO EXPAND BY FROBENIUS, & CHANGE BACK TO $x-1$ AFTERWARDS

- ASSUME A SOLUTION OF THE FORM $y=\sum_{r=0}^{\infty}a_rt^{r+c}$, $a_0\neq0$, $c\in\mathbb{R}$

$$\frac{dy}{dt}=\sum_{r=0}^{\infty}a_r(r+c)t^{r+c-1}$$

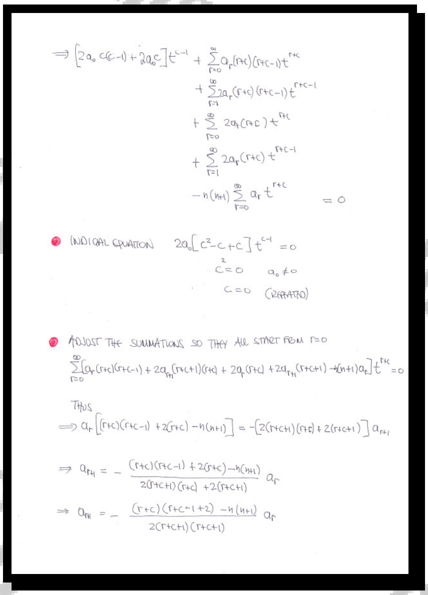
$$\frac{d^2y}{dt^2}=\sum_{r=0}^{\infty}a_r(r+c)(r+c-1)t^{r+c-2}$$

- SUB INTO THE O.D.E (NOTE WE MULTIPLIED BY -1)

$$\Rightarrow +(t^2+2t)\sum_{r=0}^{\infty}a_r(r+c)(r+c-1)t^{r+c-2}+(2t+2)\sum_{r=0}^{\infty}a_r(r+c)t^{r+c-1}-n(n+1)\sum_{r=0}^{\infty}a_rt^{r+c}$$

$$\Rightarrow \sum_{r=0}^{\infty}a_r(r+c)(r+c-1)t^{r+c}+\sum_{r=0}^{\infty}2a_r(r+c)(r+c-1)t^{r+c-1}+\sum_{r=0}^{\infty}2a_r(r+c)t^{r+c}$$
$$+\sum_{r=0}^{\infty}2a_r(r+c)t^{r+c-1}-n(n+1)\sum_{r=0}^{\infty}a_rt^{r+c}$$

- WHEN $r=0$ THE LOWEST POWER OF t IS t^{c-1} & THE HIGHEST IS t^c PULL THE LOWEST POWER OF t OUT OF THE SUMMATIONS



$$\Rightarrow\left[2a_0c(c-1)+2a_0c\right]t^{c-1}+\sum_{r=0}^{\infty}a_r(r+c)(r+c-1)t^{r+c}$$
$$+\sum_{r=1}^{\infty}2a_r(r+c)(r+c-1)t^{r+c-1}$$
$$+\sum_{r=0}^{\infty}2a_r(r+c)t^{r+c}$$
$$+\sum_{r=1}^{\infty}2a_r(r+c)t^{r+c-1}$$
$$-n(n+1)\sum_{r=0}^{\infty}a_rt^{r+c}=0$$

- INDICIAL EQUATION $2a_0\left[c^2-c+c\right]t^{c-1}=0$

$$c^2=0 \quad a_0\neq0$$
$$c=0 \quad \text{(REPEATED)}$$

- ADJUST THE SUMMATIONS SO THEY ALL START FROM $r=0$

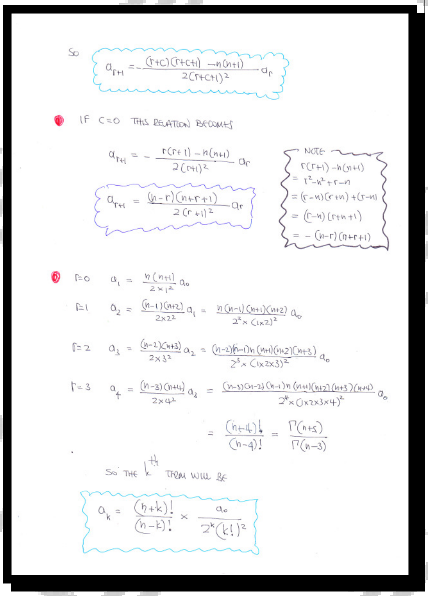
$$\sum_{r=0}^{\infty}\left[a_r(r+c)(r+c-1)+2a_{r+1}(r+c+1)(r+c)+2a_r(r+c)+2a_{r+1}(r+c+1)-n(n+1)a_r\right]t^{r+c}=0$$

THUS

$$\Rightarrow a_r\left[(r+c)(r+c-1)+2(r+c)-n(n+1)\right]=-\left[2(r+c+1)(r+c)+2(r+c+1)\right]a_{r+1}$$

$$\Rightarrow a_{r+1}=-\frac{(r+c)(r+c-1)+2(r+c)-n(n+1)}{2(r+c+1)(r+c)+2(r+c+1)}a_r$$

$$\Rightarrow a_{r+1}=-\frac{(r+c)(r+c-1+2)-n(n+1)}{2(r+c+1)(r+c+1)}a_r$$



SO

$$\boxed{a_{r+1}=-\frac{(r+c)(r+c+1)-n(n+1)}{2(r+c+1)^2}a_r}$$

- IF $c=0$ THIS RELATION BECOMES

$$a_{r+1}=-\frac{r(r+1)-n(n+1)}{2(r+1)^2}a_r$$

$$\boxed{a_{r+1}=\frac{(n-r)(n+r+1)}{2(r+1)^2}a_r}$$

NOTE

$$r(r+1)-n(n+1)$$
$$=r^2-n^2+r-n$$
$$=(r-n)(r+n)+(r-n)$$
$$=(r-n)(r+n+1)$$
$$=-(n-r)(n+r+1)$$

- $r=0$ $\quad a_1=\frac{n(n+1)}{2\times1^2}a_0$

$r=1$ $\quad a_2=\frac{(n-1)(n+2)}{2\times2^2}a_1=\frac{n(n-1)(n+1)(n+2)}{2^2\times(1\times2)^2}a_0$

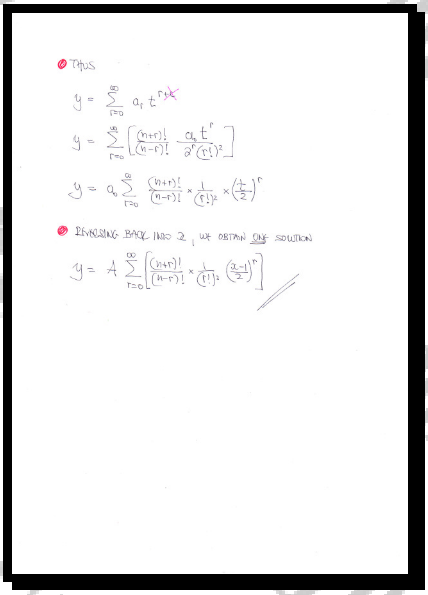
$r=2$ $\quad a_3=\frac{(n-2)(n+3)}{2\times3^2}a_2=\frac{(n-2)(n-1)n(n+1)(n+2)(n+3)}{2^3\times(1\times2\times3)^2}a_0$

$r=3$ $\quad a_4=\frac{(n-3)(n+4)}{2\times4^2}a_3=\frac{(n-3)(n-2)(n-1)n(n+1)(n+2)(n+3)(n+4)}{2^4\times(1\times2\times3\times4)^2}a_0$

$$=\frac{(n+4)!}{(n-4)!}=\frac{\Gamma(n+5)}{\Gamma(n-3)}$$

SO THE k^{th} TERM WILL BE

$$\boxed{a_k=\frac{(n+k)!}{(n-k)!}\times\frac{a_0}{2^k(k!)^2}}$$



- THUS

$$y=\sum_{r=0}^{\infty}a_rt^{r+\cancel{c}}$$

$$y=\sum_{r=0}^{\infty}\left[\frac{(n+r)!}{(n-r)!}\frac{a_0t^r}{2^r(r!)^2}\right]$$

$$y=a_0\sum_{r=0}^{\infty}\frac{(n+r)!}{(n-r)!}\times\frac{1}{(r!)^2}\times\left(\frac{t}{2}\right)^r$$

- REVERSING BACK INTO x, WE OBTAIN ONE SOLUTION

$$y=A\sum_{r=0}^{\infty}\left[\frac{(n+r)!}{(n-r)!}\times\frac{1}{(r!)^2}\left(\frac{x-1}{2}\right)^r\right]$$

FROBENIUS METHOD

[2^{nd} order O.D.E.s, where the roots of the indicial equation do not differ by an integer]

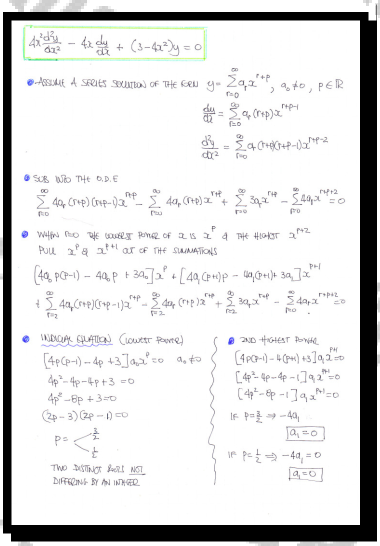
Question 1

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$4x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + \left(3 - 4x^2\right)y = 0.$$

Give the final answer in terms of elementary function.

$$\boxed{y = \sqrt{x}\left(A\cosh x + B\sinh x\right)}$$



$$\boxed{4x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (3-4x^2)y = 0}$$

- ASSUME A SERIES SOLUTION OF THE FORM $y = \sum_{r=0}^{\infty} a_r x^{r+p}$, $a_0 \neq 0$, $p \in \mathbb{R}$

$$\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r(r+p)x^{r+p-1}$$

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r(r+p)(r+p-1)x^{r+p-2}$$

- SUB INTO THE O.D.E

$$\sum_{r=0}^{\infty} 4a_r(r+p)(r+p-1)x^{r+p} - \sum_{r=0}^{\infty} 4a_r(r+p)x^{r+p} + \sum_{r=0}^{\infty} 3a_r x^{r+p} - \sum_{r=0}^{\infty} 4a_r x^{r+p+2} = 0$$

- WHEN $r=0$ THE LOWEST POWER OF x IS x^p & THE HIGHEST x^{p+2}
PULL x^p & x^{p+1} OUT OF THE SUMMATIONS

$$\left[4a_0p(p-1) - 4a_0p + 3a_0\right]x^p + \left[4a_1(p+1)p - 4a_1(p+1) + 3a_1\right]x^{p+1}$$

$$+ \sum_{r=2}^{\infty} 4a_r(r+p)(r+p-1)x^{r+p} - \sum_{r=2}^{\infty} 4a_r(r+p)x^{r+p} + \sum_{r=2}^{\infty} 3a_r x^{r+p} - \sum_{r=0}^{\infty} 4a_r x^{r+p+2} = 0.$$

- INDICIAL EQUATION (LOWEST POWER)

$$\left[4p(p-1) - 4p + 3\right]a_0x^p = 0 \qquad a_0 \neq 0$$

$$4p^2 - 4p - 4p + 3 = 0$$

$$4p^2 - 8p + 3 = 0$$

$$(2p-3)(2p-1) = 0$$

$$p = \begin{cases} \frac{3}{2} \\ \frac{1}{2} \end{cases}$$

TWO DISTINCT ROOTS NOT DIFFERING BY AN INTEGER

- 2ND HIGHEST POWER

$$\left[4p(p-1) - 4(p+1) + 3\right]a_1x^{p+1} = 0$$

$$\left[4p^2 - 4p - 4p - 1\right]a_1x^{p+1} = 0$$

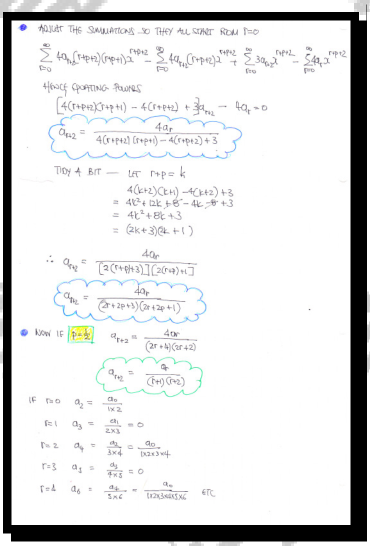
$$\left[4p^2 - 8p - 1\right]a_1x^{p+1} = 0$$

IF $p = \frac{3}{2} \Rightarrow -4a_1$

$$\boxed{a_1 = 0}$$

IF $p = \frac{1}{2} \Rightarrow -4a_1 = 0$

$$\boxed{a_1 = 0}$$



- ADJUST THE SUMMATIONS SO THEY ALL START FROM $r=0$

$$\sum_{r=0}^{\infty} 4a_{r+2}(r+p+2)(r+p+1)x^{r+p+2} - \sum_{r=0}^{\infty} 4a_{r+2}(r+p+2)x^{r+p+2} + \sum_{r=0}^{\infty} 3a_{r+2}x^{r+p+2} - \sum_{r=0}^{\infty} 4a_r x^{r+p+2}$$

HENCE EQUATING POWERS

$$\left[4(r+p+2)(r+p+1) - 4(r+p+2) + 3\right]a_{r+2} - 4a_r = 0$$

$$a_{r+2} = \frac{4a_r}{4(r+p+2)(r+p+1) - 4(r+p+2) + 3}$$

TIDY A BIT — LET $r+p = k$

$$4(k+2)(k+1) - 4(k+2) + 3$$

$$= 4k^2 + 12k \cancel{+8} - 4k \cancel{-8} + 3$$

$$= 4k^2 + 8k + 3$$

$$= (2k+3)(2k+1)$$

$$\therefore a_{r+2} = \frac{4a_r}{\left[2(r+p)+3\right]\left[2(r+p)+1\right]}$$

$$a_{r+2} = \frac{4a_r}{(2r+2p+3)(2r+2p+1)}$$

- NOW IF $\boxed{p = \frac{1}{2}}$ $\quad a_{r+2} = \frac{4a_r}{(2r+4)(2r+2)}$

$$a_{r+2} = \frac{a_r}{(r+1)(r+2)}$$

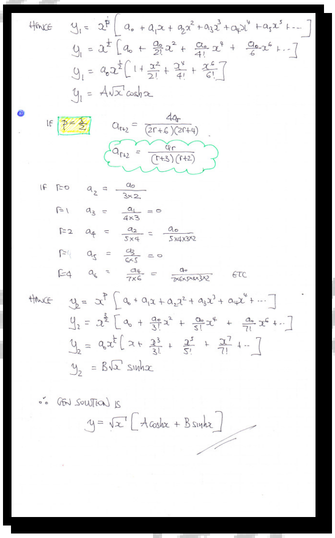
IF $r=0$ $\quad a_2 = \frac{a_0}{1\times 2}$

$r=1$ $\quad a_3 = \frac{a_1}{2\times 3} = 0$

$r=2$ $\quad a_4 = \frac{a_2}{3\times 4} = \frac{a_0}{1\times 2\times 3\times 4}$

$r=3$ $\quad a_5 = \frac{a_3}{4\times 5} = 0$

$r=4$ $\quad a_6 = \frac{a_4}{5\times 6} = \frac{a_0}{1\times 2\times 3\times 4\times 5\times 6}$ ETC



HENCE $\quad y_1 = x^p\left[a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \dots\right]$

$$y_1 = x^{\frac{1}{2}}\left[a_0 + \frac{a_0}{2!}x^2 + \frac{a_0}{4!}x^4 + \frac{a_0}{6}x^6 + \dots\right]$$

$$y_1 = a_0x^{\frac{1}{2}}\left[1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!}\right]$$

$$y_1 = A\sqrt{x}\cosh x$$

- IF $\boxed{p = \frac{3}{2}}$ $\quad a_{r+2} = \frac{4a_r}{(2r+6)(2r+4)}$

$$a_{r+2} = \frac{a_r}{(r+3)(r+2)}$$

IF $r=0$ $\quad a_2 = \frac{a_0}{3\times 2}$

$r=1$ $\quad a_3 = \frac{a_1}{4\times 3} = 0$

$r=2$ $\quad a_4 = \frac{a_2}{5\times 4} = \frac{a_0}{5\times 4\times 3\times 2}$

$r=$ [illegible] $\quad a_5 = \frac{a_3}{6\times 5} = 0$

$r=4$ $\quad a_6 = \frac{a_4}{7\times 6} = \frac{a_0}{7\times 6\times 5\times 4\times 3\times 2}$ ETC

HENCE $\quad y_2 = x^p\left[a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots\right]$

$$y_2 = x^{\frac{3}{2}}\left[a_0 + \frac{a_0}{3!}x^2 + \frac{a_0}{5!}x^4 + \frac{a_0}{7!}x^6 + \dots\right]$$

$$y_2 = a_0x^{\frac{1}{2}}\left[x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots\right]$$

$$y_2 = B\sqrt{x}\sinh x$$

$\therefore$ GEN SOLUTION IS

$$y = \sqrt{x}\left[A\cosh x + B\sinh x\right]$$

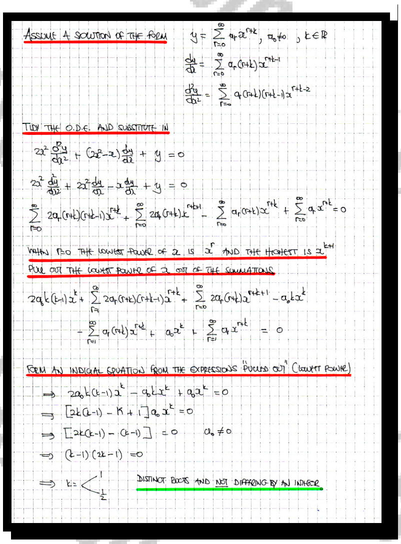
Question 2

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$\frac{d^2y}{dx^2}+\left[1-\frac{1}{2x}\right]\frac{dy}{dx}+\frac{y}{2x^2}=0.$$

Give the final answer in simplified Sigma notation.

$$,\quad \boxed{y=Ax\sum_{r=0}^{\infty}\left[\frac{r!}{(2r+1)!}(-4x)^r\right]+Bx^{\frac{1}{2}}\mathrm{e}^{-x}}$$



ASSUME A SOLUTION OF THE FORM $y=\sum_{r=0}^{\infty}a_rx^{r+k},\ a_0\neq0,\ k\in\mathbb{R}$

$$\frac{dy}{dx}=\sum_{r=0}^{\infty}a_r(r+k)x^{r+k-1}$$

$$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty}a_r(r+k)(r+k-1)x^{r+k-2}$$

TIDY THE O.D.E. AND SUBSTITUTE IN

$$2x^2\frac{d^2y}{dx^2}+(2x^2-x)\frac{dy}{dx}+y=0$$

$$2x^2\frac{d^2y}{dx^2}+2x^2\frac{dy}{dx}-x\frac{dy}{dx}+y=0$$

$$\sum_{r=0}^{\infty}2a_r(r+k)(r+k-1)x^{r+k}+\sum_{r=0}^{\infty}2a_r(r+k)x^{r+k+1}-\sum_{r=0}^{\infty}a_r(r+k)x^{r+k}+\sum_{r=0}^{\infty}a_rx^{r+k}=0$$

WHEN $r=0$ THE LOWEST POWER OF x IS x^r AND THE HIGHEST IS x^{k+1}

PULL OUT THE LOWEST POWER OF x OUT OF THE SUMMATIONS

$$2a_0k(k-1)x^k+\sum_{r=1}^{\infty}2a_r(r+k)(r+k-1)x^{r+k}+\sum_{r=0}^{\infty}2a_r(r+k)x^{r+k+1}-a_0kx^k$$

$$-\sum_{r=1}^{\infty}a_r(r+k)x^{r+k}+a_0x^k+\sum_{r=1}^{\infty}a_rx^{r+k}=0$$

FORM AN INDICIAL EQUATION FROM THE EXPRESSIONS "PULLED OUT" (LOWEST POWER)

$$\Rightarrow 2a_0k(k-1)x^k-a_0kx^k+a_0x^k=0$$

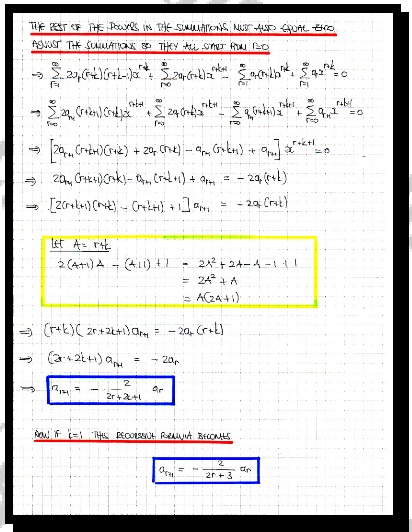
$$\Rightarrow [2k(k-1)-k+1]a_0x^k=0$$

$$\Rightarrow [2k(k-1)-(k-1)]=0 \qquad a_0\neq0$$

$$\Rightarrow (k-1)(2k-1)=0$$

$$\Rightarrow k=\begin{cases}1\\ \frac{1}{2}\end{cases}$$

DISTINCT ROOTS AND NOT DIFFERING BY AN INTEGER



THE REST OF THE POWERS IN THE SUMMATIONS MUST ALSO EQUAL ZERO.

ADJUST THE SUMMATIONS SO THEY ALL START FROM $r=0$

$$\Rightarrow \sum_{r=1}^{\infty}2a_r(r+k)(r+k-1)x^{r+k}+\sum_{r=0}^{\infty}2a_r(r+k)x^{r+k+1}-\sum_{r=1}^{\infty}a_r(r+k)x^{r+k}+\sum_{r=1}^{\infty}a_rx^{r+k}=0$$

$$\Rightarrow \sum_{r=0}^{\infty}2a_{r+1}(r+k+1)(r+k)x^{r+k+1}+\sum_{r=0}^{\infty}2a_r(r+k)x^{r+k+1}-\sum_{r=0}^{\infty}a_{r+1}(r+k+1)x^{r+k+1}+\sum_{r=0}^{\infty}a_{r+1}x^{r+k+1}=0$$

$$\Rightarrow \left[2a_{r+1}(r+k+1)(r+k)+2a_r(r+k)-a_{r+1}(r+k+1)+a_{r+1}\right]x^{r+k+1}=0$$

$$\Rightarrow 2a_{r+1}(r+k+1)(r+k)-a_{r+1}(r+k+1)+a_{r+1}=-2a_r(r+k)$$

$$\Rightarrow [2(r+k+1)(r+k)-(r+k+1)+1]a_{r+1}=-2a_r(r+k)$$

LET $A=r+k$

$$\begin{aligned}2(A+1)A-(A+1)+1&=2A^2+2A-A-1+1\\&=2A^2+A\\&=A(2A+1)\end{aligned}$$

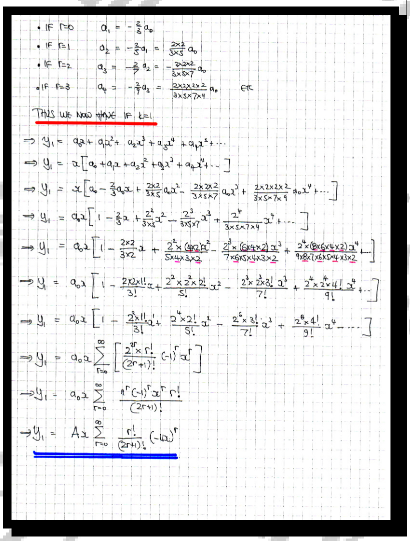
$$\Rightarrow (r+k)(2r+2k+1)a_{r+1}=-2a_r(r+k)$$

$$\Rightarrow (2r+2k+1)a_{r+1}=-2a_r$$

$$\Rightarrow \boxed{a_{r+1}=-\frac{2}{2r+2k+1}a_r}$$

NOW IF $k=1$ THIS RECURSIVE FORMULA BECOMES

$$\boxed{a_{r+1}=-\frac{2}{2r+3}a_r}$$



- IF $r=0$ $\quad a_1=-\frac{2}{3}a_0$
- IF $r=1$ $\quad a_2=-\frac{2}{5}a_1=\frac{2\times2}{3\times5}a_0$
- IF $r=2$ $\quad a_3=-\frac{2}{7}a_2=-\frac{2\times2\times2}{3\times5\times7}a_0$
- IF $r=3$ $\quad a_4=-\frac{2}{9}a_3=\frac{2\times2\times2\times2}{3\times5\times7\times9}a_0$ ETC

THUS WE NOW HAVE IF $k=1$

$$\Rightarrow y_1=a_0x+a_1x^2+a_2x^3+a_3x^4+a_4x^5+\dots$$

$$\Rightarrow y_1=x\left[a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\dots\right]$$

$$\Rightarrow y_1=x\left[a_0-\frac{2}{3}a_0x+\frac{2\times2}{3\times5}a_0x^2-\frac{2\times2\times2}{3\times5\times7}a_0x^3+\frac{2\times2\times2\times2}{3\times5\times7\times9}a_0x^4+\dots\right]$$

$$\Rightarrow y_1=a_0x\left[1-\frac{2}{3}x+\frac{2^2}{3\times5}x^2-\frac{2^3}{3\times5\times7}x^3+\frac{2^4}{3\times5\times7\times9}x^4+\dots\right]$$

$$\Rightarrow y_1=a_0x\left[1-\frac{2\times2}{3\times2}x+\frac{2^2\times(4\times2)x^2}{5\times4\times3\times2}-\frac{2^3\times(6\times4\times2)x^3}{7\times6\times5\times4\times3\times2}+\frac{2^4\times(8\times6\times4\times2)x^4}{9\times8\times7\times6\times5\times4\times3\times2}+\dots\right]$$

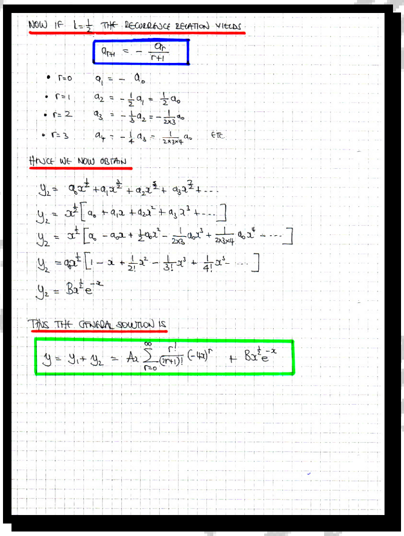
$$\Rightarrow y_1=a_0x\left[1-\frac{2\times2\times1!}{3!}x+\frac{2^2\times2^2\times2!}{5!}x^2-\frac{2^3\times2^3\times3!\,x^3}{7!}+\frac{2^4\times2^4\times4!\,x^4}{9!}+\dots\right]$$

$$\Rightarrow y_1=a_0x\left[1-\frac{2^2\times1!}{3!}x^1+\frac{2^4\times2!}{5!}x^2-\frac{2^6\times3!}{7!}x^3+\frac{2^8\times4!}{9!}x^4-\dots\right]$$

$$\Rightarrow y_1=a_0x\sum_{r=0}^{\infty}\left[\frac{2^{2r}\times r!}{(2r+1)!}(-1)^rx^r\right]$$

$$\Rightarrow y_1=a_0x\sum_{r=0}^{\infty}\frac{4^r(-1)^rx^rr!}{(2r+1)!}$$

$$\Rightarrow y_1=Ax\sum_{r=0}^{\infty}\frac{r!}{(2r+1)!}(-4x)^r$$



NOW IF $k=\frac{1}{2}$ THE RECURRENCE RELATION YIELDS

$$\boxed{a_{r+1}=-\frac{a_r}{r+1}}$$

- $r=0$ $\quad a_1=-a_0$
- $r=1$ $\quad a_2=-\frac{1}{2}a_1=\frac{1}{2}a_0$
- $r=2$ $\quad a_3=-\frac{1}{3}a_2=-\frac{1}{2\times3}a_0$
- $r=3$ $\quad a_4=-\frac{1}{4}a_3=\frac{1}{2\times3\times4}a_0$ ETC.

HENCE WE NOW OBTAIN

$$y_2=a_0x^{\frac{1}{2}}+a_1x^{\frac{3}{2}}+a_2x^{\frac{5}{2}}+a_3x^{\frac{7}{2}}+\dots$$

$$y_2=x^{\frac{1}{2}}\left[a_0+a_1x+a_2x^2+a_3x^3+\dots\right]$$

$$y_2=x^{\frac{1}{2}}\left[a_0-a_0x+\frac{1}{2}a_0x^2-\frac{1}{2\times3}a_0x^3+\frac{1}{2\times3\times4}a_0x^4-\dots\right]$$

$$y_2=a_0x^{\frac{1}{2}}\left[1-x+\frac{1}{2!}x^2-\frac{1}{3!}x^3+\frac{1}{4!}x^5-\dots\right]$$

$$y_2=Bx^{\frac{1}{2}}e^{-x}$$

THUS THE GENERAL SOLUTION IS

$$\boxed{y=y_1+y_2=Ax\sum_{r=0}^{\infty}\frac{r!}{(2r+1)!}(-4x)^r+Bx^{\frac{1}{2}}e^{-x}}$$

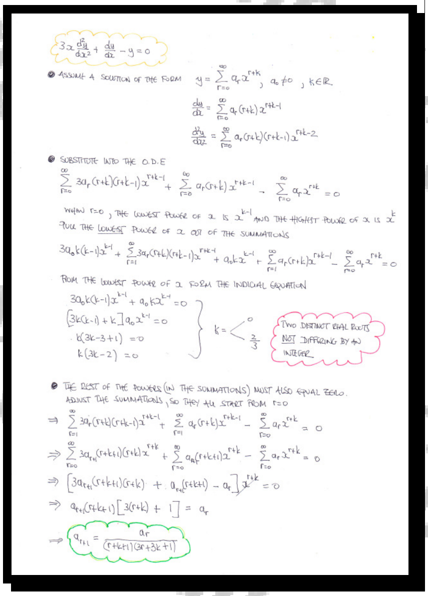
Question 3

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$3x\frac{d^2y}{dx^2}+\frac{dy}{dx}-y=0.$$

Give the final answer in simplified Sigma notation.

$$\boxed{y=A\times\Gamma\left(\tfrac{1}{3}\right)\sum_{r=0}^{\infty}\left[\frac{x^r}{r!\times3^r\times\Gamma\left(\frac{3r+1}{3}\right)}\right]+B\times x^{\frac{2}{3}}\times\Gamma\left(\tfrac{5}{3}\right)\sum_{r=0}^{\infty}\left[\frac{x^r}{r!\times3^r\times\Gamma\left(\frac{3r+5}{3}\right)}\right]}$$



$3x\frac{d^2y}{dx^2}+\frac{dy}{dx}-y=0$

● ASSUME A SOLUTION OF THE FORM $y=\sum_{r=0}^{\infty}a_rx^{r+k}$, $a_0\neq0$, $k\in\mathbb{R}$

$\frac{dy}{dx}=\sum_{r=0}^{\infty}a_r(r+k)x^{r+k-1}$

$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty}a_r(r+k)(r+k-1)x^{r+k-2}$

● SUBSTITUTE INTO THE O.D.E

$\sum_{r=0}^{\infty}3a_r(r+k)(r+k-1)x^{r+k-1}+\sum_{r=0}^{\infty}a_r(r+k)x^{r+k-1}-\sum_{r=0}^{\infty}a_rx^{r+k}=0$

WHEN $r=0$, THE LOWEST POWER OF x IS x^{k-1} AND THE HIGHEST POWER OF x IS x^k

PULL THE LOWEST POWER OF x OUT OF THE SUMMATIONS

$3a_0k(k-1)x^{k-1}+\sum_{r=1}^{\infty}3a_r(r+k)(r+k-1)x^{r+k-1}+a_0kx^{k-1}+\sum_{r=1}^{\infty}a_r(r+k)x^{r+k-1}-\sum_{r=0}^{\infty}a_rx^{r+k}=0$

FROM THE LOWEST POWER OF x FORM THE INDICIAL EQUATION

$$\left.\begin{array}{l}3a_0k(k-1)x^{k-1}+a_0kx^{k-1}=0\\ \left[3k(k-1)+k\right]a_0x^{k-1}=0\\ k(3k-3+1)=0\\ k(3k-2)=0\end{array}\right\}\quad k=\begin{cases}0\\ \frac{2}{3}\end{cases}$$

TWO DISTINCT REAL ROOTS NOT DIFFERING BY AN INTEGER

● THE REST OF THE POWERS (IN THE SUMMATIONS) MUST ALSO EQUAL ZERO. ADJUST THE SUMMATIONS, SO THEY ALL START FROM $r=0$

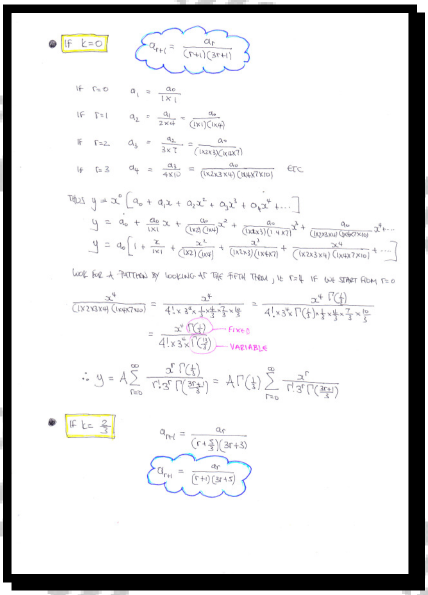
$\Rightarrow \sum_{r=1}^{\infty}3a_r(r+k)(r+k-1)x^{r+k-1}+\sum_{r=1}^{\infty}a_r(r+k)x^{r+k-1}-\sum_{r=0}^{\infty}a_rx^{r+k}=0$

$\Rightarrow \sum_{r=0}^{\infty}3a_{r+1}(r+k+1)(r+k)x^{r+k}+\sum_{r=0}^{\infty}a_{r+1}(r+k+1)x^{r+k}-\sum_{r=0}^{\infty}a_rx^{r+k}=0$

$\Rightarrow \left[3a_{r+1}(r+k+1)(r+k)+a_{r+1}(r+k+1)-a_r\right]\cancel{x^{r+k}}=0$

$\Rightarrow a_{r+1}(r+k+1)\left[3(r+k)+1\right]=a_r$

$\Rightarrow a_{r+1}=\frac{a_r}{(r+k+1)(3r+3k+1)}$



● IF $k=0$ $\quad a_{r+1}=\frac{a_r}{(r+1)(3r+1)}$

IF $r=0$ $\quad a_1=\frac{a_0}{1\times1}$

IF $r=1$ $\quad a_2=\frac{a_1}{2\times4}=\frac{a_0}{(1\times1)(1\times4)}$

IF $r=2$ $\quad a_3=\frac{a_2}{3\times7}=\frac{a_0}{(1\times2\times3)(1\times4\times7)}$

IF $r=3$ $\quad a_4=\frac{a_3}{4\times10}=\frac{a_0}{(1\times2\times3\times4)(1\times4\times7\times10)}$ ETC

THUS $y=x^0\left[a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\dots\right]$

$y=a_0+\frac{a_0}{1\times1}x+\frac{a_0}{(1\times2)(1\times4)}x^2+\frac{a_0}{(1\times2\times3)(1\times4\times7)}x^3+\frac{a_0}{(1\times2\times3\times4)(1\times4\times7\times10)}x^4+\dots$

$y=a_0\left[1+\frac{x}{1\times1}+\frac{x^2}{(1\times2)(1\times4)}+\frac{x^3}{(1\times2\times3)(1\times4\times7)}+\frac{x^4}{(1\times2\times3\times4)(1\times4\times7\times10)}+\dots\right]$

LOOK FOR A PATTERN BY LOOKING AT THE FIFTH TERM, IE $r=4$ IF WE START FROM $r=0$

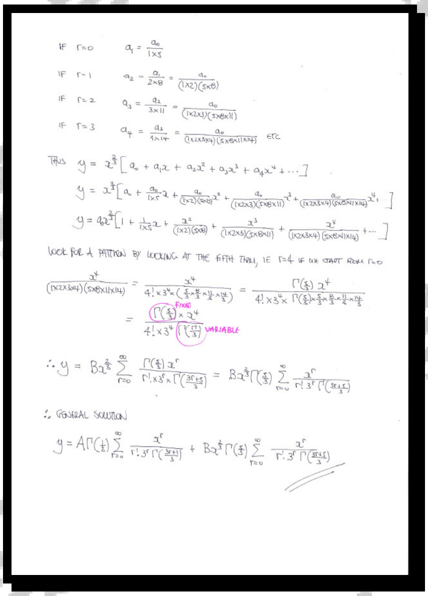
$\frac{x^4}{(1\times2\times3\times4)(1\times4\times7\times10)}=\frac{x^4}{4!\times3^4\times\frac{1}{3}\times\frac{4}{3}\times\frac{7}{3}\times\frac{10}{3}}=\frac{x^4\,\Gamma(\frac{1}{3})}{4!\times3^4\times\Gamma(\frac{1}{3})\times\frac{1}{3}\times\frac{4}{3}\times\frac{7}{3}\times\frac{10}{3}}$

$=\frac{x^4\,\Gamma(\frac{1}{3})}{4!\times3^4\times\Gamma(\frac{13}{3})}$ FIXED VARIABLE

$\therefore y=A\sum_{r=0}^{\infty}\frac{x^r\,\Gamma(\frac{1}{3})}{r!\,3^r\,\Gamma(\frac{3r+1}{3})}=A\Gamma(\tfrac{1}{3})\sum_{r=0}^{\infty}\frac{x^r}{r!\,3^r\,\Gamma(\frac{3r+1}{3})}$

● IF $k=\frac{2}{3}$ $\quad a_{r+1}=\frac{a_r}{(r+\frac{5}{3})(3r+3)}$

$a_{r+1}=\frac{a_r}{(r+1)(3r+5)}$



IF $r=0$ $\quad a_1=\frac{a_0}{1\times5}$

IF $r=1$ $\quad a_2=\frac{a_1}{2\times8}=\frac{a_0}{(1\times2)(5\times8)}$

IF $r=2$ $\quad a_3=\frac{a_2}{3\times11}=\frac{a_0}{(1\times2\times3)(5\times8\times11)}$

IF $r=3$ $\quad a_4=\frac{a_3}{4\times14}=\frac{a_0}{(1\times2\times3\times4)(5\times8\times11\times14)}$ ETC

THUS $y=x^{\frac{2}{3}}\left[a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\dots\right]$

$y=x^{\frac{2}{3}}\left[a_0+\frac{a_0}{1\times5}x+\frac{a_0}{(1\times2)(5\times8)}x^2+\frac{a_0}{(1\times2\times3)(5\times8\times11)}x^3+\frac{a_0}{(1\times2\times3\times4)(5\times8\times11\times14)}x^4+\dots\right]$

$y=a_0x^{\frac{2}{3}}\left[1+\frac{1}{1\times5}x+\frac{x^2}{(1\times2)(5\times8)}+\frac{x^3}{(1\times2\times3)(5\times8\times11)}+\frac{x^4}{(1\times2\times3\times4)(5\times8\times11\times14)}+\dots\right]$

LOOK FOR A PATTERN BY LOOKING AT THE FIFTH TERM, IE $r=4$ IF WE START FROM $r=0$

$\frac{x^4}{(1\times2\times3\times4)(5\times8\times11\times14)}=\frac{x^4}{4!\times3^4\times(\frac{5}{3}\times\frac{8}{3}\times\frac{11}{3}\times\frac{14}{3})}=\frac{\Gamma(\frac{5}{3})\,x^4}{4!\times3^4\times\Gamma(\frac{5}{3})\times\frac{5}{3}\times\frac{8}{3}\times\frac{11}{3}\times\frac{14}{3}}$

$=\frac{\Gamma(\frac{5}{3})\times x^4}{4!\times3^4\,\Gamma(\frac{17}{3})}$ FIXED VARIABLE

$\therefore y=Bx^{\frac{2}{3}}\sum_{r=0}^{\infty}\frac{\Gamma(\frac{5}{3})\,x^r}{r!\times3^r\times\Gamma(\frac{3r+5}{3})}=Bx^{\frac{2}{3}}\Gamma(\tfrac{5}{3})\sum_{r=0}^{\infty}\frac{x^r}{r!\,3^r\,\Gamma(\frac{3r+5}{3})}$

$\therefore$ GENERAL SOLUTION

$y=A\Gamma(\tfrac{1}{3})\sum_{r=0}^{\infty}\frac{x^r}{r!\,3^r\,\Gamma(\frac{3r+1}{3})}+Bx^{\frac{2}{3}}\Gamma(\tfrac{5}{3})\sum_{r=0}^{\infty}\frac{x^r}{r!\,3^r\,\Gamma(\frac{3r+5}{3})}$

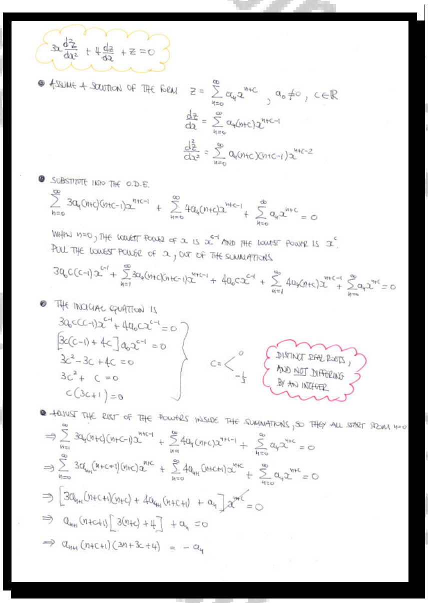
Question 4

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$3x\frac{d^2z}{dx^2}+4\frac{dz}{dx}+z=0\,.$$

Give the final answer in simplified Sigma notation.

$$\boxed{z=A\,\Gamma\left(\tfrac{4}{3}\right)\sum_{n=0}^{\infty}\left[\frac{(-x)^n}{n!\times 3^n\times\Gamma\left(\frac{3n+4}{3}\right)}\right]+Bx^{-\frac{1}{3}}\,\Gamma\left(\tfrac{2}{3}\right)\sum_{n=0}^{\infty}\left[\frac{x^n}{n!\times 3^n\times\Gamma\left(\frac{3n+2}{3}\right)}\right]}$$



$$3x\frac{d^2z}{dx^2}+4\frac{dz}{dx}+z=0$$

- ASSUME A SOLUTION OF THE FORM $z=\sum_{n=0}^{\infty}a_nx^{n+c}\,,\ a_0\neq 0\,,\ c\in\mathbb{R}$

$$\frac{dz}{dx}=\sum_{n=0}^{\infty}a_n(n+c)x^{n+c-1}$$

$$\frac{d^2z}{dx^2}=\sum_{n=0}^{\infty}a_n(n+c)(n+c-1)x^{n+c-2}$$

- SUBSTITUTE INTO THE O.D.E.

$$\sum_{n=0}^{\infty}3a_n(n+c)(n+c-1)x^{n+c-1}+\sum_{n=0}^{\infty}4a_n(n+c)x^{n+c-1}+\sum_{n=0}^{\infty}a_nx^{n+c}=0$$

WHEN $n=0$, THE LOWEST POWER OF x IS x^{c-1} AND THE LOWEST POWER IS x^c.
PULL THE LOWEST POWER OF x, OUT OF THE SUMMATIONS

$$3a_0c(c-1)x^{c-1}+\sum_{n=1}^{\infty}3a_n(n+c)(n+c-1)x^{n+c-1}+4a_0cx^{c-1}+\sum_{n=1}^{\infty}4a_n(n+c)x^{n+c-1}+\sum_{n=0}^{\infty}a_nx^{n+c}=0$$

- THE INDICIAL EQUATION IS

$$\left.\begin{aligned}&3a_0c(c-1)x^{c-1}+4a_0cx^{c-1}=0\\&\left[3c(c-1)+4c\right]a_0x^{c-1}=0\\&3c^2-3c+4c=0\\&3c^2+c=0\\&c(3c+1)=0\end{aligned}\right\}\quad c=\begin{cases}0\\-\frac{1}{3}\end{cases}$$

DISTINCT REAL ROOTS, AND NOT DIFFERING BY AN INTEGER

- ADJUST THE REST OF THE POWERS INSIDE THE SUMMATIONS, SO THEY ALL START FROM $n=0$

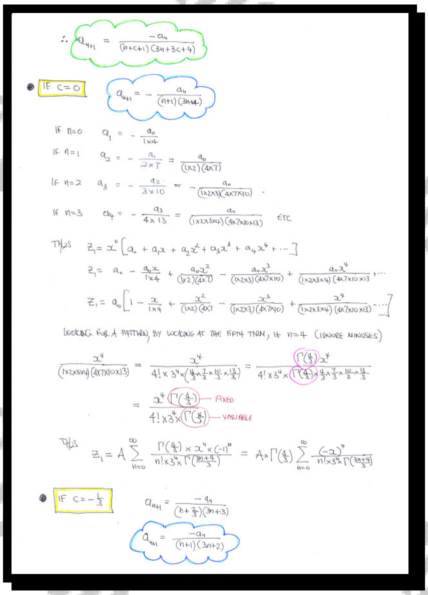
$$\Rightarrow\sum_{n=1}^{\infty}3a_n(n+c)(n+c-1)x^{n+c-1}+\sum_{n=1}^{\infty}4a_n(n+c)x^{n+c-1}+\sum_{n=0}^{\infty}a_nx^{n+c}=0$$

$$\Rightarrow\sum_{n=0}^{\infty}3a_{n+1}(n+c+1)(n+c)x^{n+c}+\sum_{n=0}^{\infty}4a_{n+1}(n+c+1)x^{n+c}+\sum_{n=0}^{\infty}a_nx^{n+c}=0$$

$$\Rightarrow\left[3a_{n+1}(n+c+1)(n+c)+4a_{n+1}(n+c+1)+a_n\right]\cancel{x^{n+c}}=0$$

$$\Rightarrow a_{n+1}(n+c+1)\left[3(n+c)+4\right]+a_n=0$$

$$\Rightarrow a_{n+1}(n+c+1)(3n+3c+4)=-a_n$$



$$\therefore\ a_{n+1}=\frac{-a_n}{(n+c+1)(3n+3c+4)}$$

- IF $c=0$

$$a_{n+1}=-\frac{a_n}{(n+1)(3n+4)}$$

IF $n=0$ $\quad a_1=-\frac{a_0}{1\times 4}$

IF $n=1$ $\quad a_2=-\frac{a_1}{2\times 7}=\frac{a_0}{(1\times 2)(4\times 7)}$

IF $n=2$ $\quad a_3=-\frac{a_2}{3\times 10}=-\frac{a_0}{(1\times 2\times 3)(4\times 7\times 10)}$.

IF $n=3$ $\quad a_4=-\frac{a_3}{4\times 13}=\frac{a_0}{(1\times 2\times 3\times 4)(4\times 7\times 10\times 13)}$ ETC

THUS $\quad z_1=x^0\left[a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\dots\right]$

$$z_1=a_0-\frac{a_0x}{1\times 4}+\frac{a_0x^2}{(1\times 2)(4\times 7)}-\frac{a_0x^3}{(1\times 2\times 3)(4\times 7\times 10)}+\frac{a_0x^4}{(1\times 2\times 3\times 4)(4\times 7\times 10\times 13}+\dots$$

$$z_1=a_0\left[1-\frac{x}{1\times 4}+\frac{x^2}{(1\times 2)(4\times 7}-\frac{x^3}{(1\times 2\times 3)(4\times 7\times 10)}+\frac{x^4}{(1\times 2\times 3\times 4)(4\times 7\times 10\times 13)}\dots\right]$$

LOOKING FOR A PATTERN, BY LOOKING AT THE FIFTH TERM, IE $n=4$ (IGNORE MINUSES)

$$\frac{x^4}{(1\times 2\times 3\times 4)(4\times 7\times 10\times 13)}=\frac{x^4}{4!\times 3^4\times\left(\frac{4}{3}\times\frac{7}{3}\times\frac{10}{3}\times\frac{13}{3}\right)}=\frac{\Gamma\left(\frac{4}{3}\right)x^4}{4!\times 3^4\times\Gamma\left(\frac{4}{3}\right)\times\frac{4}{3}\times\frac{7}{3}\times\frac{10}{3}\times\frac{13}{3}}$$

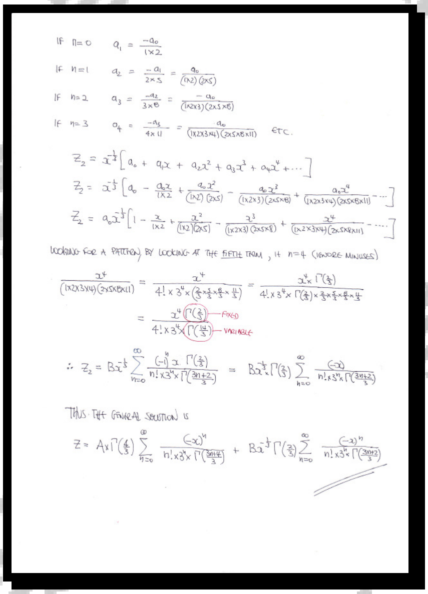
$$=\frac{x^4\,\Gamma\left(\frac{4}{3}\right)}{4!\times 3^4\times\Gamma\left(\frac{16}{3}\right)}\begin{matrix}\text{— FIXED}\\ \text{— VARIABLE}\end{matrix}$$

THUS $\quad z_1=A\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{4}{3}\right)\times x^n\times(-1)^n}{n!\times 3^n\times\Gamma\left(\frac{3n+4}{3}\right)}=A\times\Gamma\left(\frac{4}{3}\right)\sum_{n=0}^{\infty}\frac{(-x)^n}{n!\times 3^n\,\Gamma\left(\frac{3n+4}{3}\right)}$

- IF $c=-\frac{1}{3}$

$$a_{n+1}=\frac{-a_n}{\left(n+\frac{2}{3}\right)(3n+3)}$$

$$a_{n+1}=\frac{-a_n}{(n+1)(3n+2)}$$



IF $n=0$ $\quad a_1=\frac{-a_0}{1\times 2}$

IF $n=1$ $\quad a_2=\frac{-a_1}{2\times 5}=\frac{a_0}{(1\times 2)(2\times 5)}$

IF $n=2$ $\quad a_3=\frac{-a_2}{3\times 8}=\frac{-a_0}{(1\times 2\times 3)(2\times 5\times 8)}$

IF $n=3$ $\quad a_4=\frac{-a_3}{4\times 11}=\frac{a_0}{(1\times 2\times 3\times 4)(2\times 5\times 8\times 11)}$ ETC.

$$z_2=x^{-\frac{1}{3}}\left[a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\dots\right]$$

$$z_2=x^{-\frac{1}{3}}\left[a_0-\frac{a_0x}{1\times 2}+\frac{a_0x^2}{(1\times 2)(2\times 5)}-\frac{a_0x^3}{(1\times 2\times 3)(2\times 5\times 8)}+\frac{a_0x^4}{(1\times 2\times 3\times 4)(2\times 5\times 8\times 11)}-\dots\right]$$

$$z_2=a_0x^{-\frac{1}{3}}\left[1-\frac{x}{1\times 2}+\frac{x^2}{(1\times 2)(2\times 5)}-\frac{x^3}{(1\times 2\times 3)(2\times 5\times 8)}+\frac{x^4}{(1\times 2\times 3\times 4)(2\times 5\times 8\times 11)}-\dots\right]$$

LOOKING FOR A PATTERN, BY LOOKING AT THE FIFTH TERM, IE $n=4$ (IGNORE MINUSES)

$$\frac{x^4}{(1\times 2\times 3\times 4)(2\times 5\times 8\times 11)}=\frac{x^4}{4!\times 3^4\times\left(\frac{2}{3}\times\frac{5}{3}\times\frac{8}{3}\times\frac{11}{3}\right)}=\frac{x^4\times\Gamma\left(\frac{2}{3}\right)}{4!\times 3^4\times\Gamma\left(\frac{2}{3}\right)\times\frac{2}{3}\times\frac{5}{3}\times\frac{8}{3}\times\frac{4}{3}}$$

$$=\frac{x^4\,\Gamma\left(\frac{2}{3}\right)}{4!\times 3^4\times\Gamma\left(\frac{14}{3}\right)}\begin{matrix}\text{— FIXED}\\ \text{— VARIABLE}\end{matrix}$$

$$\therefore\ z_2=Bx^{-\frac{1}{3}}\sum_{n=0}^{\infty}\frac{(-1)^n x\,.\,\Gamma\left(\frac{2}{3}\right)}{n!\times 3^n\times\Gamma\left(\frac{3n+2}{3}\right)}=Bx^{-\frac{1}{3}}\times\Gamma\left(\frac{2}{3}\right)\sum_{n=0}^{\infty}\frac{(-x)}{n!\times 3^n\times\Gamma\left(\frac{3n+2}{3}\right)}$$

THUS THE GENERAL SOLUTION IS

$$z=A\times\Gamma\left(\frac{4}{3}\right)\sum_{n=0}^{\infty}\frac{(-x)^n}{n!\times 3^n\times\Gamma\left(\frac{3n+4}{3}\right)}+Bx^{-\frac{1}{3}}\,\Gamma\left(\frac{2}{3}\right)\sum_{n=0}^{\infty}\frac{(-x)^n}{n!\times 3^n\times\Gamma\left(\frac{3n+2}{3}\right)}$$

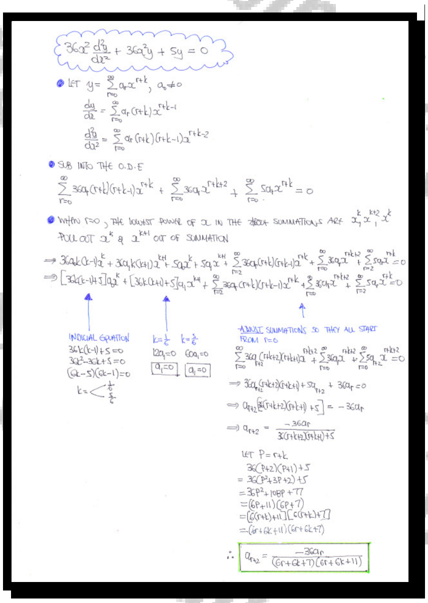
Question 5

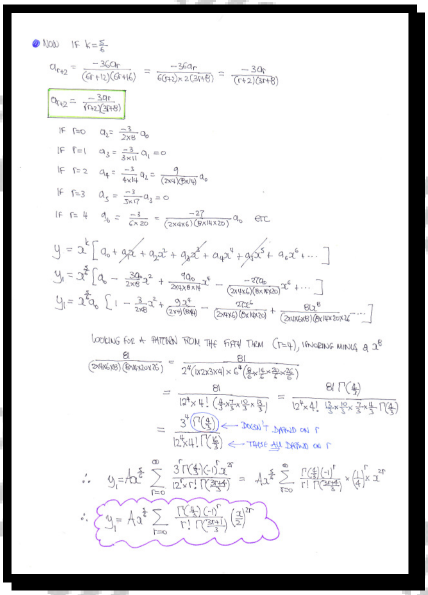
Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

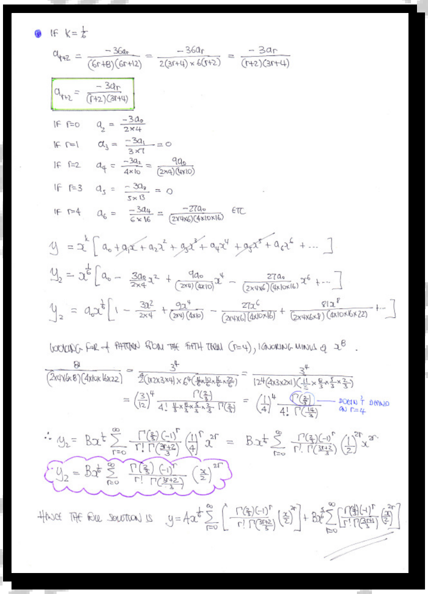
$$36x^2\frac{d^2y}{dx^2}+36x^2y+5y=0\,.$$

Give the final answer in simplified Sigma notation.

$$\boxed{z=Ax^{\frac{1}{6}}\Gamma\left(\tfrac{2}{3}\right)\sum_{r=0}^{\infty}\left[\frac{(-1)^r}{r!\times\Gamma\left(\frac{3r+2}{3}\right)}\left(\frac{x}{2}\right)^{2r}\right]+Bx^{\frac{5}{6}}\Gamma\left(\tfrac{4}{3}\right)\sum_{r=0}^{\infty}\left[\frac{(-1)^r}{r!\times\Gamma\left(\frac{3r+1}{3}\right)}\left(\frac{x}{2}\right)^{2r}\right]}$$







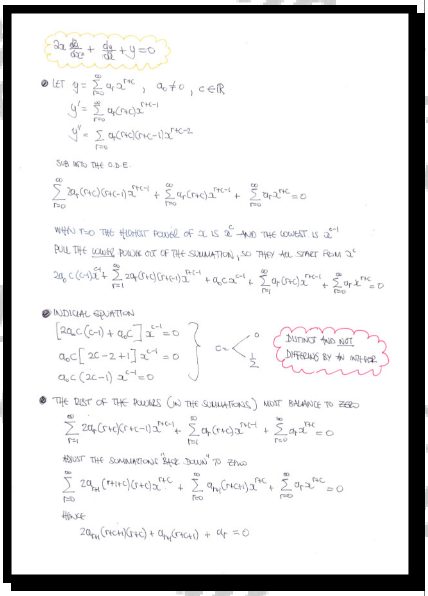
Question 6

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$2x\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=0.$$

Give the final answer in simplified Sigma notation.

$$\boxed{y=\sum_{r=0}^{\infty}\left[\frac{(-2x)^r}{(2r)!}\left(A+\frac{B\sqrt{x}}{2r+1}\right)\right]}$$



$$2x\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=0$$

- LET $y=\sum_{r=0}^{\infty}a_r x^{r+c}$, $a_0\neq 0$, $c\in\mathbb{R}$

$$y'=\sum_{r=0}^{\infty}a_r(r+c)x^{r+c-1}$$

$$y''=\sum_{r=0}a_r(r+c)(r+c-1)x^{r+c-2}$$

SUB INTO THE O.D.E.

$$\sum_{r=0}^{\infty}2a_r(r+c)(r+c-1)x^{r+c-1}+\sum_{r=0}^{\infty}a_r(r+c)x^{r+c-1}+\sum_{r=0}^{\infty}a_r x^{r+c}=0$$

WHEN $r=0$ THE HIGHEST POWER OF x IS x^c AND THE LOWEST IS x^{c-1}

PULL THE LOWER POWER OUT OF THE SUMMATION, SO THEY ALL START FROM x^c

$$2a_0c(c-1)x^{c-1}+\sum_{r=1}^{\infty}2a_r(r+c)(r+c-1)x^{r+c-1}+a_0cx^{c-1}+\sum_{r=1}^{\infty}a_r(r+c)x^{r+c-1}+\sum_{r=0}^{\infty}a_r x^{r+c}=0$$

- INDICIAL EQUATION

$$\left.\begin{array}{l}\left[2a_0c(c-1)+a_0c\right]x^{c-1}=0\\ a_0c\left[2c-2+1\right]x^{c-1}=0\\ a_0c(2c-1)x^{c-1}=0\end{array}\right\}\quad c=\begin{cases}0\\ \frac{1}{2}\end{cases}$$

DISTINCT AND NOT DIFFERING BY AN INTEGER

- THE REST OF THE POWERS (IN THE SUMMATIONS) MUST BALANCE TO ZERO

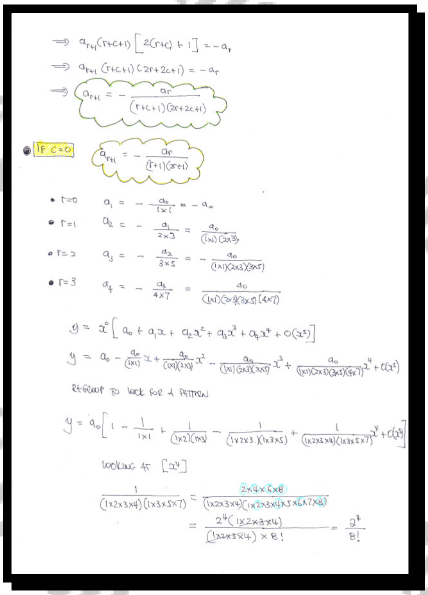
$$\sum_{r=1}^{\infty}2a_r(r+c)(r+c-1)x^{r+c-1}+\sum_{r=1}^{\infty}a_r(r+c)x^{r+c-1}+\sum_{r=0}^{\infty}a_r x^{r+c}=0$$

ADJUST THE SUMMATIONS "BACK DOWN" TO ZERO

$$\sum_{r=0}^{\infty}2a_{r+1}(r+1+c)(r+c)x^{r+c}+\sum_{r=0}^{\infty}a_{r+1}(r+c+1)x^{r+c}+\sum_{r=0}^{\infty}a_r x^{r+c}=0$$

HENCE

$$2a_{r+1}(r+c+1)(r+c)+a_{r+1}(r+c+1)+a_r=0$$



$$\Rightarrow a_{r+1}(r+c+1)\left[2(r+c)+1\right]=-a_r$$

$$\Rightarrow a_{r+1}(r+c+1)(2r+2c+1)=-a_r$$

$$\Rightarrow a_{r+1}=-\frac{a_r}{(r+c+1)(2r+2c+1)}$$

- IF $c=0$ $\qquad a_{r+1}=-\frac{a_r}{(r+1)(2r+1)}$

- $r=0$ $\quad a_1=-\frac{a_0}{1\times 1}=-a_0$
- $r=1$ $\quad a_2=-\frac{a_1}{2\times 3}=\frac{a_0}{(1\times1)(2\times3)}$
- $r=2$ $\quad a_3=-\frac{a_2}{3\times 5}=-\frac{a_0}{(1\times1)(2\times3)(3\times5)}$
- $r=3$ $\quad a_4=-\frac{a_3}{4\times 7}=\frac{a_0}{(1\times1)(2\times3)(3\times5)(4\times7)}$

$$y=x^0\left[a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+O(x^5)\right]$$

$$y=a_0-\frac{a_0}{(1\times1)}x+\frac{a_0}{(1\times1)(2\times3)}x^2-\frac{a_0}{(1\times1)(2\times3)(3\times5)}x^3+\frac{a_0}{(1\times1)(2\times3)(3\times5)(4\times7)}x^4+O(x^5)$$

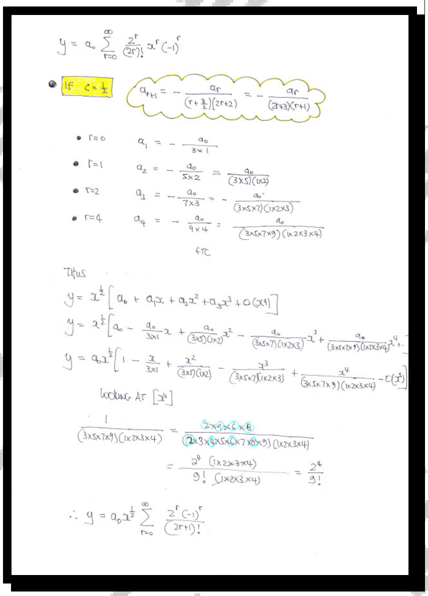
REGROUP TO LOOK FOR A PATTERN

$$y=a_0\left[1-\frac{1}{1\times1}+\frac{1}{(1\times2)(1\times3)}-\frac{1}{(1\times2\times3)(1\times3\times5)}+\frac{1}{(1\times2\times3\times4)(1\times3\times5\times7)}x^4+O(x^5)\right]$$

LOOKING AT $[x^4]$

$$\frac{1}{(1\times2\times3\times4)(1\times3\times5\times7)}=\frac{2\times4\times6\times8}{(1\times2\times3\times4)(1\times2\times3\times4\times5\times6\times7\times8)}$$

$$=\frac{2^4(\cancel{1\times2\times3\times4})}{(\cancel{1\times2\times3\times4})\times 8!}=\frac{2^4}{8!}$$



$$y=a_0\sum_{r=0}^{\infty}\frac{2^r}{(2r)!}x^r(-1)^r$$

- IF $c=\frac{1}{2}$ $\qquad a_{r+1}=-\frac{a_r}{(r+\frac{3}{2})(2r+2)}=-\frac{a_r}{(2r+3)(r+1)}$

- $r=0$ $\quad a_1=-\frac{a_0}{3\times1}$
- $r=1$ $\quad a_2=-\frac{a_0}{5\times2}=\frac{a_0}{(3\times5)(1\times2)}$
- $r=2$ $\quad a_3=-\frac{a_0}{7\times3}=-\frac{a_0}{(3\times5\times7)(1\times2\times3)}$
- $r=4$ $\quad a_4=-\frac{a_0}{9\times4}=\frac{a_0}{(3\times5\times7\times9)(1\times2\times3\times4)}$

ETC

THUS

$$y=x^{\frac{1}{2}}\left[a_0+a_1x+a_2x^2+a_3x^3+O(x^4)\right]$$

$$y=x^{\frac{1}{2}}\left[a_0-\frac{a_0}{3\times1}x+\frac{a_0}{(3\times5)(1\times2)}x^2-\frac{a_0}{(3\times5\times7)(1\times2\times3)}x^3+\frac{a_0}{(3\times5\times7\times9)(1\times2\times3\times4)}x^4\dots\right.$$

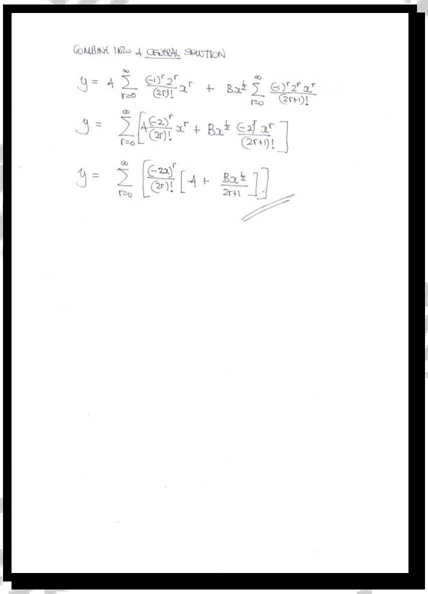
$$y=a_0x^{\frac{1}{2}}\left[1-\frac{x}{3\times1}+\frac{x^2}{(3\times5)(1\times2)}-\frac{x^3}{(3\times5\times7)(1\times2\times3)}+\frac{x^4}{(3\times5\times7\times9)(1\times2\times3\times4)}-O(x^5)\right]$$

LOOKING AT $[x^4]$

$$\frac{1}{(3\times5\times7\times9)(1\times2\times3\times4)}=\frac{2\times4\times6\times8}{(2\times3\times4\times5\times6\times7\times8\times9)(1\times2\times3\times4)}$$

$$=\frac{2^4(\cancel{1\times2\times3\times4})}{9!(\cancel{1\times2\times3\times4})}=\frac{2^4}{9!}$$

$$\therefore y=a_0x^{\frac{1}{2}}\sum_{r=0}^{\infty}\frac{2^r(-1)^r}{(2r+1)!}$$



COMBINE INTO A GENERAL SOLUTION

$$y=A\sum_{r=0}^{\infty}\frac{(-1)^r2^r}{(2r)!}x^r+Bx^{\frac{1}{2}}\sum_{r=0}^{\infty}\frac{(-1)^r2^rx^r}{(2r+1)!}$$

$$y=\sum_{r=0}^{\infty}\left[A\frac{(-2)^r}{(2r)!}x^r+Bx^{\frac{1}{2}}\frac{(-2)^rx^r}{(2r+1)!}\right]$$

$$y=\sum_{r=0}^{\infty}\left[\frac{(-2x)^r}{(2r)!}\left[A+\frac{Bx^{\frac{1}{2}}}{2r+1}\right]\right].$$

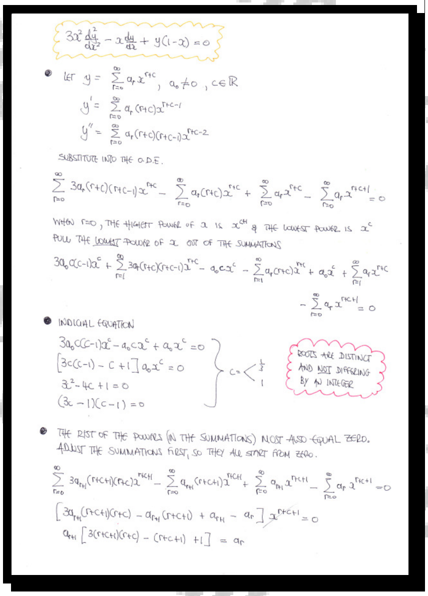
Question 7

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$3x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y - xy = 0 .$$

Give the final answer in simplified Sigma notation.

$$\boxed{y = Ax\Gamma\left(\frac{5}{3}\right)\sum_{r=0}^{\infty}\left[\frac{1}{r!\,\Gamma\left(\frac{3r+5}{3}\right)}\left(\frac{1}{3}x\right)^r\right] + Bx^{\frac{1}{3}}\Gamma\left(\frac{1}{3}\right)\sum_{r=0}^{\infty}\left[\frac{1}{r!\,\Gamma\left(\frac{3r+1}{3}\right)}\left(\frac{1}{3}x\right)^r\right]}$$



$$3x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y(1-x) = 0$$

- LET $y = \sum_{r=0}^{\infty} a_r x^{r+c}$, $a_0 \neq 0$, $c \in \mathbb{R}$

$$y' = \sum_{r=0}^{\infty} a_r(r+c)x^{r+c-1}$$

$$y'' = \sum_{r=0}^{\infty} a_r(r+c)(r+c-1)x^{r+c-2}$$

SUBSTITUTE INTO THE O.D.E.

$$\sum_{r=0}^{\infty} 3a_r(r+c)(r+c-1)x^{r+c} - \sum_{r=0}^{\infty} a_r(r+c)x^{r+c} + \sum_{r=0}^{\infty} a_r x^{r+c} - \sum_{r=0}^{\infty} a_r x^{r+c+1} = 0$$

WHEN $r=0$, THE HIGHEST POWER OF x IS x^{c+1} & THE LOWEST POWER IS x^c

PULL THE LOWEST POWER OF x OUT OF THE SUMMATIONS

$$3a_0c(c-1)x^c + \sum_{r=1}^{\infty} 3a_r(r+c)(r+c-1)x^{r+c} - a_0cx^c - \sum_{r=1}^{\infty} a_r(r+c)x^{r+c} + a_0x^c + \sum_{r=1}^{\infty} a_r x^{r+c}$$

$$- \sum_{r=0}^{\infty} a_r x^{r+c+1} = 0$$

- INDICIAL EQUATION

$$\left.\begin{aligned} &3a_0c(c-1)x^c - a_0cx^c + a_0x^c = 0 \\ &[3c(c-1) - c + 1]a_0x^c = 0 \\ &3c^2 - 4c + 1 = 0 \\ &(3c-1)(c-1) = 0 \end{aligned}\right\} c = \left\langle \begin{matrix} \frac{1}{3} \\ 1 \end{matrix} \right.$$

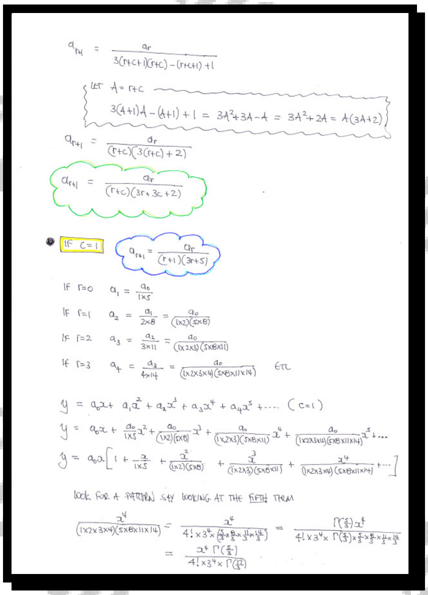
ROOTS ARE DISTINCT AND NOT DIFFERING BY AN INTEGER

- THE REST OF THE POWERS (IN THE SUMMATIONS) MUST ALSO EQUAL ZERO. ADJUST THE SUMMATIONS FIRST, SO THEY ALL START FROM ZERO.

$$\sum_{r=0}^{\infty} 3a_{r+1}(r+c+1)(r+c)x^{r+c+1} - \sum_{r=0}^{\infty} a_{r+1}(r+c+1)x^{r+c+1} + \sum_{r=0}^{\infty} a_{r+1}x^{r+c+1} - \sum_{r=0}^{\infty} a_r x^{r+c+1} = 0$$

$$[3a_{r+1}(r+c+1)(r+c) - a_{r+1}(r+c+1) + a_{r+1} - a_r]\cancel{x^{r+c+1}} = 0$$

$$a_{r+1}[3(r+c+1)(r+c) - (r+c+1) + 1] = a_r$$



$$a_{r+1} = \frac{a_r}{3(r+c+1)(r+c) - (r+c+1) + 1}$$

LET $A = r+c$

$$3(A+1)A - (A+1) + 1 = 3A^2 + 3A - A = 3A^2 + 2A = A(3A+2)$$

$$a_{r+1} = \frac{a_r}{(r+c)(3(r+c)+2)}$$

$$a_{r+1} = \frac{a_r}{(r+c)(3r+3c+2)}$$

- IF $c=1$ $\quad a_{r+1} = \frac{a_r}{(r+1)(3r+5)}$

IF $r=0$ $\quad a_1 = \frac{a_0}{1\times5}$

IF $r=1$ $\quad a_2 = \frac{a_1}{2\times8} = \frac{a_0}{(1\times2)(5\times8)}$

IF $r=2$ $\quad a_3 = \frac{a_2}{3\times11} = \frac{a_0}{(1\times2\times3)(5\times8\times11)}$

IF $r=3$ $\quad a_4 = \frac{a_3}{4\times14} = \frac{a_0}{(1\times2\times3\times4)(5\times8\times11\times14)}$ ETC

$$y = a_0x + a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5 + \dots \quad (c=1)$$

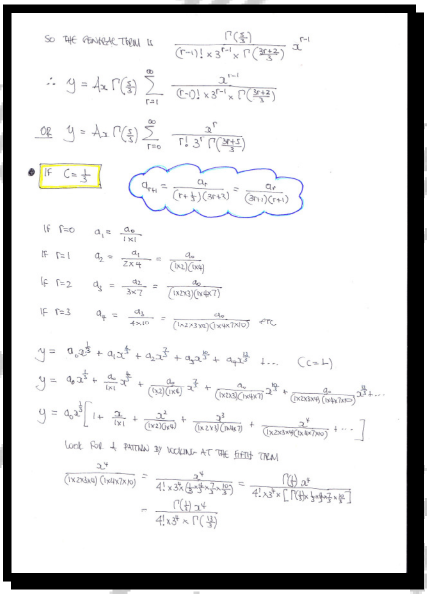
$$y = a_0x + \frac{a_0}{1\times5}x^2 + \frac{a_0}{(1\times2)(5\times8)}x^3 + \frac{a_0}{(1\times2\times3)(5\times8\times11)}x^4 + \frac{a_0}{(1\times2\times3\times4)(5\times8\times11\times14)}x^5 + \dots$$

$$y = a_0x\left[1 + \frac{x}{1\times5} + \frac{x^2}{(1\times2)(5\times8)} + \frac{x^3}{(1\times2\times3)(5\times8\times11)} + \frac{x^4}{(1\times2\times3\times4)(5\times8\times11\times14)} + \dots\right]$$

LOOK FOR A PATTERN SAY LOOKING AT THE FIFTH TERM

$$\frac{x^4}{(1\times2\times3\times4)(5\times8\times11\times14)} = \frac{x^4}{4!\times3^4\times\left(\frac{5}{3}\times\frac{8}{3}\times\frac{11}{3}\times\frac{14}{3}\right)} = \frac{\Gamma\left(\frac{5}{3}\right)x^4}{4!\times3^4\times\Gamma\left(\frac{5}{3}\right)\times\frac{5}{3}\times\frac{8}{3}\times\frac{11}{3}\times\frac{14}{3}}$$

$$= \frac{x^4\,\Gamma\left(\frac{5}{3}\right)}{4!\times3^4\times\Gamma\left(\frac{17}{3}\right)}$$



SO THE GENERAL TERM IS $\frac{\Gamma\left(\frac{5}{3}\right)}{(r-1)!\times3^{r-1}\times\Gamma\left(\frac{3r+2}{3}\right)}x^{r-1}$

$$\therefore\ y = Ax\Gamma\left(\frac{5}{3}\right)\sum_{r=1}^{\infty}\frac{x^{r-1}}{(r-1)!\times3^{r-1}\times\Gamma\left(\frac{3r+2}{3}\right)}$$

$$\text{OR}\quad y = Ax\Gamma\left(\frac{5}{3}\right)\sum_{r=0}^{\infty}\frac{x^r}{r!\,3^r\,\Gamma\left(\frac{3r+5}{3}\right)}$$

- IF $c=\frac{1}{3}$ $\quad a_{r+1} = \frac{a_r}{\left(r+\frac{1}{3}\right)(3r+3)} = \frac{a_r}{(3r+1)(r+1)}$

IF $r=0$ $\quad a_1 = \frac{a_0}{1\times1}$

IF $r=1$ $\quad a_2 = \frac{a_1}{2\times4} = \frac{a_0}{(1\times2)(1\times4)}$

IF $r=2$ $\quad a_3 = \frac{a_2}{3\times7} = \frac{a_0}{(1\times2\times3)(1\times4\times7)}$

IF $r=3$ $\quad a_4 = \frac{a_3}{4\times10} = \frac{a_0}{(1\times2\times3\times4)(1\times4\times7\times10)}$ ETC

$$y = a_0x^{\frac{1}{3}} + a_1x^{\frac{4}{3}} + a_2x^{\frac{7}{3}} + a_3x^{\frac{10}{3}} + a_4x^{\frac{13}{3}} + \dots \quad (c=\tfrac{1}{3})$$

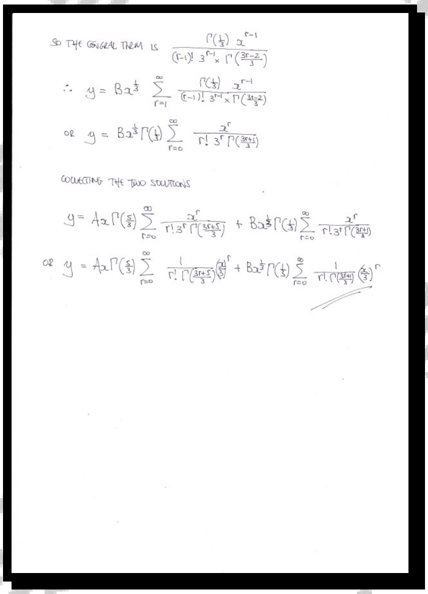
$$y = a_0x^{\frac{1}{3}} + \frac{a_0}{1\times1}x^{\frac{4}{3}} + \frac{a_0}{(1\times2)(1\times4)}x^{\frac{7}{3}} + \frac{a_0}{(1\times2\times3)(1\times4\times7)}x^{\frac{10}{3}} + \frac{a_0}{(1\times2\times3\times4)(1\times4\times7\times10)}x^{\frac{13}{3}} + \dots$$

$$y = a_0x^{\frac{1}{3}}\left[1 + \frac{x}{1\times1} + \frac{x^2}{(1\times2)(1\times4)} + \frac{x^3}{(1\times2\times3)(1\times4\times7)} + \frac{x^4}{(1\times2\times3\times4)(1\times4\times7\times10)} + \dots\right]$$

LOOK FOR A PATTERN BY LOOKING AT THE FIFTH TERM

$$\frac{x^4}{(1\times2\times3\times4)(1\times4\times7\times10)} = \frac{x^4}{4!\times3^4\times\left(\frac{1}{3}\times\frac{4}{3}\times\frac{7}{3}\times\frac{10}{3}\right)} = \frac{\Gamma\left(\frac{1}{3}\right)x^4}{4!\times3^4\times\left[\Gamma\left(\frac{1}{3}\right)\times\frac{1}{3}\times\frac{4}{3}\times\frac{7}{3}\times\frac{10}{3}\right]}$$

$$= \frac{\Gamma\left(\frac{1}{3}\right)x^4}{4!\times3^4\times\Gamma\left(\frac{13}{3}\right)}$$



SO THE GENERAL TERM IS $\frac{\Gamma\left(\frac{1}{3}\right)x^{r-1}}{(r-1)!\,3^{r-1}\times\Gamma\left(\frac{3r-2}{3}\right)}$

$$\therefore\ y = Bx^{\frac{1}{3}}\sum_{r=1}^{\infty}\frac{\Gamma\left(\frac{1}{3}\right)x^{r-1}}{(r-1)!\,3^{r-1}\times\Gamma\left(\frac{3r-2}{3}\right)}$$

$$\text{OR}\quad y = Bx^{\frac{1}{3}}\Gamma\left(\frac{1}{3}\right)\sum_{r=0}^{\infty}\frac{x^r}{r!\,3^r\,\Gamma\left(\frac{3r+1}{3}\right)}$$

COLLECTING THE TWO SOLUTIONS

$$y = Ax\Gamma\left(\frac{5}{3}\right)\sum_{r=0}^{\infty}\frac{x^r}{r!\,3^r\,\Gamma\left(\frac{3r+5}{3}\right)} + Bx^{\frac{1}{3}}\Gamma\left(\frac{1}{3}\right)\sum_{r=0}^{\infty}\frac{x^r}{r!\,3^r\,\Gamma\left(\frac{3r+1}{3}\right)}$$

$$\text{OR}\quad y = Ax\Gamma\left(\frac{5}{3}\right)\sum_{r=0}^{\infty}\frac{1}{r!\,\Gamma\left(\frac{3r+5}{3}\right)}\left(\frac{x}{3}\right)^r + Bx^{\frac{1}{3}}\Gamma\left(\frac{1}{3}\right)\sum_{r=0}^{\infty}\frac{1}{r!\,\Gamma\left(\frac{3r+1}{3}\right)}\left(\frac{x}{3}\right)^r$$

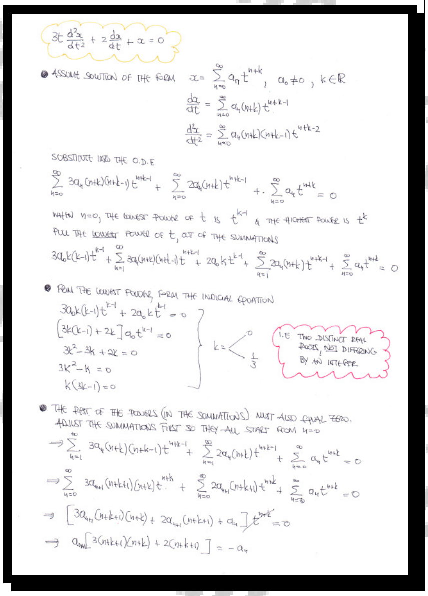
Question 8

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$3t\frac{d^2x}{dt^2}+2\frac{dx}{dt}+x=0.$$

Give the final answer in simplified Sigma notation.

$$\boxed{y=A\times\Gamma\left(\frac{2}{3}\right)\sum_{n=0}^{\infty}\left[\frac{(-t)^n}{n!\times3^n\times\Gamma\left(\frac{3n+2}{3}\right)}\right]+B\times t^{\frac{1}{3}}\times\Gamma\left(\frac{4}{3}\right)\sum_{n=0}^{\infty}\left[\frac{(-t)^n}{n!\times3^n\times\Gamma\left(\frac{3n+4}{3}\right)}\right]}$$



$$3t\frac{d^2x}{dt^2}+2\frac{dx}{dt}+x=0$$

- ASSUME SOLUTION OF THE FORM $x=\sum_{n=0}^{\infty}a_nt^{n+k}$, $a_0\neq0$, $k\in\mathbb{R}$

$$\frac{dx}{dt}=\sum_{n=0}^{\infty}a_n(n+k)t^{n+k-1}$$

$$\frac{d^2x}{dt^2}=\sum_{n=0}^{\infty}a_n(n+k)(n+k-1)t^{n+k-2}$$

SUBSTITUTE INTO THE O.D.E

$$\sum_{n=0}^{\infty}3a_n(n+k)(n+k-1)t^{n+k-1}+\sum_{n=0}^{\infty}2a_n(n+k)t^{n+k-1}+\sum_{n=0}^{\infty}a_nt^{n+k}=0$$

WHEN $n=0$, THE LOWEST POWER OF t IS t^{k-1} & THE HIGHEST POWER IS t^k

PULL THE LOWEST POWER OF t, OUT OF THE SUMMATIONS

$$3a_0k(k-1)t^{k-1}+\sum_{n=1}^{\infty}3a_n(n+k)(n+k-1)t^{n+k-1}+2a_0kt^{k-1}+\sum_{n=1}^{\infty}2a_n(n+k)t^{n+k-1}+\sum_{n=0}^{\infty}a_nt^{n+k}=0$$

- FROM THE LOWEST POWER, FORM THE INDICIAL EQUATION

$$\left.\begin{aligned}&3a_0k(k-1)t^{k-1}+2a_0kt^{k-1}=0\\&[3k(k-1)+2k]a_0t^{k-1}=0\\&3k^2-3k+2k=0\\&3k^2-k=0\\&k(3k-1)=0\end{aligned}\right\}\quad k=\begin{cases}0\\ \frac{1}{3}\end{cases}$$

I.E TWO DISTINCT REAL ROOTS, NOT DIFFERING BY AN INTEGER

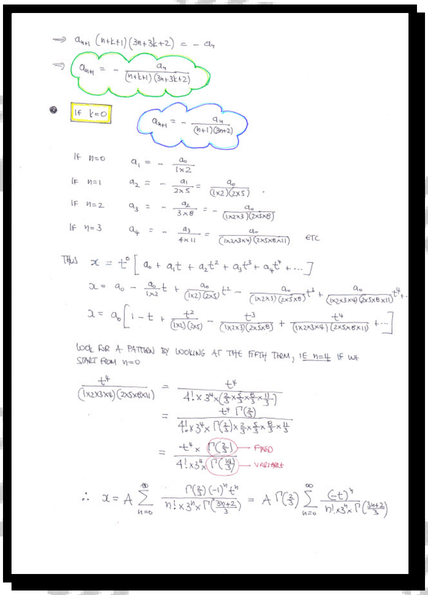
- THE REST OF THE POWERS (IN THE SUMMATIONS) MUST ALSO EQUAL ZERO. ADJUST THE SUMMATIONS FIRST SO THEY ALL START FROM $n=0$

$$\Rightarrow\sum_{n=1}^{\infty}3a_n(n+k)(n+k-1)t^{n+k-1}+\sum_{n=1}^{\infty}2a_n(n+k)t^{n+k-1}+\sum_{n=0}^{\infty}a_nt^{n+k}=0$$

$$\Rightarrow\sum_{n=0}^{\infty}3a_{n+1}(n+k+1)(n+k)t^{n+k}+\sum_{n=0}^{\infty}2a_{n+1}(n+k+1)t^{n+k}+\sum_{n=0}^{\infty}a_nt^{n+k}=0$$

$$\Rightarrow\left[3a_{n+1}(n+k+1)(n+k)+2a_{n+1}(n+k+1)+a_n\right]\cancel{t^{n+k}}=0$$

$$\Rightarrow a_{n+1}\left[3(n+k+1)(n+k)+2(n+k+1)\right]=-a_n$$



$$\Rightarrow a_{n+1}(n+k+1)(3n+3k+2)=-a_n$$

$$\Rightarrow a_{n+1}=-\frac{a_n}{(n+k+1)(3n+3k+2)}$$

- IF $k=0$

$$a_{n+1}=-\frac{a_n}{(n+1)(3n+2)}$$

IF $n=0$ $\quad a_1=-\frac{a_0}{1\times2}$

IF $n=1$ $\quad a_2=-\frac{a_1}{2\times5}=\frac{a_0}{(1\times2)(2\times5)}$

IF $n=2$ $\quad a_3=-\frac{a_2}{3\times8}=-\frac{a_0}{(1\times2\times3)(2\times5\times8)}$

IF $n=3$ $\quad a_4=-\frac{a_3}{4\times11}=\frac{a_0}{(1\times2\times3\times4)(2\times5\times8\times11)}$ ETC

THUS $x=t^0\left[a_0+a_1t+a_2t^2+a_3t^3+a_4t^4+\dots\right]$

$$x=a_0-\frac{a_0}{1\times2}t+\frac{a_0}{(1\times2)(2\times5)}t^2-\frac{a_0}{(1\times2\times3)(2\times5\times8)}t^3+\frac{a_0}{(1\times2\times3\times4)(2\times5\times8\times11)}t^4+\dots$$

$$x=a_0\left[1-t+\frac{t^2}{(1\times2)(2\times5)}-\frac{t^3}{(1\times2\times3)(2\times5\times8)}+\frac{t^4}{(1\times2\times3\times4)(2\times5\times8\times11)}+\dots\right]$$

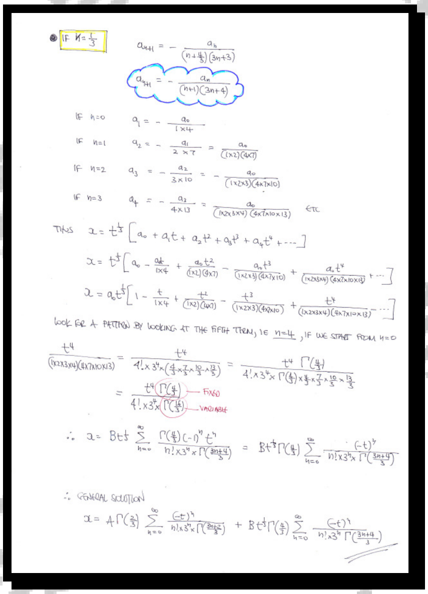
LOOK FOR A PATTERN BY LOOKING AT THE FIFTH TERM, IE $n=4$ IF WE START FROM $n=0$

$$\frac{t^4}{(1\times2\times3\times4)(2\times5\times8\times11)}=\frac{t^4}{4!\times3^4\times\left(\frac{2}{3}\times\frac{5}{3}\times\frac{8}{3}\times\frac{11}{3}\right)}$$

$$=\frac{t^4\,\Gamma\left(\frac{2}{3}\right)}{4!\times3^4\times\Gamma\left(\frac{2}{3}\right)\times\frac{2}{3}\times\frac{5}{3}\times\frac{8}{3}\times\frac{11}{3}}$$

$$=\frac{t^4\times\Gamma\left(\frac{2}{3}\right)}{4!\times3^4\times\Gamma\left(\frac{14}{3}\right)}\begin{matrix}\leftarrow\text{FIXED}\\ \leftarrow\text{VARIABLE}\end{matrix}$$

$$\therefore\ x=A\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{2}{3}\right)(-1)^nt^n}{n!\times3^n\times\Gamma\left(\frac{3n+2}{3}\right)}=A\,\Gamma\left(\frac{2}{3}\right)\sum_{n=0}^{\infty}\frac{(-t)^n}{n!\times3^n\times\Gamma\left(\frac{3n+2}{3}\right)}$$



- IF $k=\frac{1}{3}$

$$a_{n+1}=-\frac{a_n}{\left(n+\frac{4}{3}\right)(3n+3)}$$

$$a_{n+1}=-\frac{a_n}{(n+1)(3n+4)}$$

IF $n=0$ $\quad a_1=-\frac{a_0}{1\times4}$

IF $n=1$ $\quad a_2=-\frac{a_1}{2\times7}=\frac{a_0}{(1\times2)(4\times7)}$

IF $n=2$ $\quad a_3=-\frac{a_2}{3\times10}=-\frac{a_0}{(1\times2\times3)(4\times7\times10)}$

IF $n=3$ $\quad a_4=-\frac{a_3}{4\times13}=\frac{a_0}{(1\times2\times3\times4)(4\times7\times10\times13)}$ ETC

THUS $x=t^{\frac{1}{3}}\left[a_0+a_1t+a_2t^2+a_3t^3+a_4t^4+\dots\right]$

$$x=t^{\frac{1}{3}}\left[a_0-\frac{a_0t}{1\times4}+\frac{a_0t^2}{(1\times2)(4\times7)}-\frac{a_0t^3}{(1\times2\times3)(4\times7\times10)}+\frac{a_0t^4}{(1\times2\times3\times4)(4\times7\times10\times13)}+\dots\right]$$

$$x=a_0t^{\frac{1}{3}}\left[1-\frac{t}{1\times4}+\frac{t^2}{(1\times2)(4\times7)}-\frac{t^3}{(1\times2\times3)(4\times7\times10)}+\frac{t^4}{(1\times2\times3\times4)(4\times7\times10\times13)}-\dots\right]$$

LOOK FOR A PATTERN BY LOOKING AT THE FIFTH TERM, IE $n=4$, IF WE START FROM $n=0$

$$\frac{t^4}{(1\times2\times3\times4)(4\times7\times10\times13)}=\frac{t^4}{4!\times3^4\times\left(\frac{4}{3}\times\frac{7}{3}\times\frac{10}{3}\times\frac{13}{3}\right)}=\frac{t^4\,\Gamma\left(\frac{4}{3}\right)}{4!\times3^4\times\Gamma\left(\frac{4}{3}\right)\times\frac{4}{3}\times\frac{7}{3}\times\frac{10}{3}\times\frac{13}{3}}$$

$$=\frac{t^4\,\Gamma\left(\frac{4}{3}\right)}{4!\times3^4\times\Gamma\left(\frac{16}{3}\right)}\begin{matrix}\leftarrow\text{FIXED}\\ \leftarrow\text{VARIABLE}\end{matrix}$$

$$\therefore\ x=Bt^{\frac{1}{3}}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{4}{3}\right)(-1)^nt^n}{n!\times3^n\times\Gamma\left(\frac{3n+4}{3}\right)}=Bt^{\frac{1}{3}}\Gamma\left(\frac{4}{3}\right)\sum_{n=0}^{\infty}\frac{(-t)^n}{n!\times3^n\times\Gamma\left(\frac{3n+4}{3}\right)}$$

$\therefore$ GENERAL SOLUTION

$$x=A\Gamma\left(\frac{2}{3}\right)\sum_{n=0}^{\infty}\frac{(-t)^n}{n!\times3^n\times\Gamma\left(\frac{3n+2}{3}\right)}+Bt^{\frac{1}{3}}\Gamma\left(\frac{4}{3}\right)\sum_{n=0}^{\infty}\frac{(-t)^n}{n!\times3^n\,\Gamma\left(\frac{3n+4}{3}\right)}$$

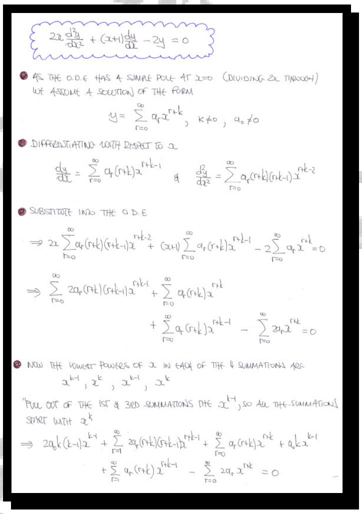
Question 9

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$2x\frac{d^2y}{dx^2}+(x+1)\frac{dy}{dx}-2y=0.$$

Give the final answer in simplified Sigma notation.

$$\boxed{y=A\left(1+2x+\frac{1}{3}x^2\right)+Bx^{\frac{1}{2}}\left[1+\frac{1}{2}x+\frac{1}{2}x^2-3\sum_{n=0}^{\infty}\left[\frac{(-1)^n\ (2n+1)!\ x^{n+3}}{2^n\ n!(2n+7)!}\right]\right]}$$



$$2x\frac{d^2y}{dx^2}+(x+1)\frac{dy}{dx}-2y=0$$

- AS THE O.D.E HAS A SIMPLE POLE AT $x=0$ (DIVIDING $2x$ THROUGH) WE ASSUME A SOLUTION OF THE FORM

$$y=\sum_{r=0}^{\infty}a_rx^{r+k},\quad k\neq 0,\quad a_0\neq 0$$

- DIFFERENTIATING WITH RESPECT TO x

$$\frac{dy}{dx}=\sum_{r=0}^{\infty}a_r(r+k)x^{r+k-1}\quad \& \quad \frac{d^2y}{dx^2}=\sum_{r=0}^{\infty}a_r(r+k)(r+k-1)x^{r+k-2}$$

- SUBSTITUTE INTO THE O.D.E

$$\Rightarrow 2x\sum_{r=0}^{\infty}a_r(r+k)(r+k-1)x^{r+k-2}+(x+1)\sum_{r=0}^{\infty}a_r(r+k)x^{r+k-1}-2\sum_{r=0}^{\infty}a_rx^{r+k}=0$$

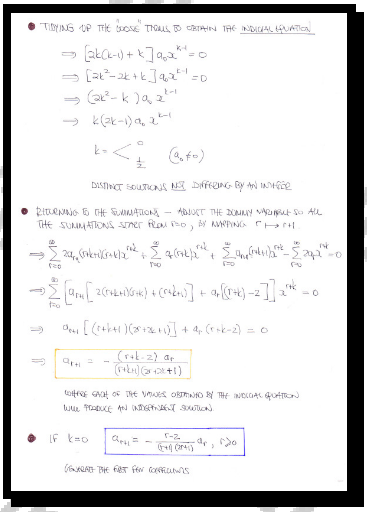
$$\Rightarrow \sum_{r=0}^{\infty}2a_r(r+k)(r+k-1)x^{r+k-1}+\sum_{r=0}^{\infty}a_r(r+k)x^{r+k}$$
$$+\sum_{r=0}^{\infty}a_r(r+k)x^{r+k-1}-\sum_{r=0}^{\infty}2a_rx^{r+k}=0$$

- NOW THE LOWEST POWERS OF x IN EACH OF THE 4 SUMMATIONS ARE

$$x^{k-1},\ x^k,\ x^{k-1},\ x^k$$

"PULL OUT OF THE 1ST & 3RD SUMMATIONS THE x^{k-1}, SO ALL THE SUMMATIONS START WITH x^k

$$\Rightarrow 2a_0k(k-1)x^{k-1}+\sum_{r=1}^{\infty}2a_r(r+k)(r+k-1)x^{r+k-1}+\sum_{r=0}^{\infty}a_r(r+k)x^{r+k}+a_0kx^{k-1}$$
$$+\sum_{r=1}^{\infty}a_r(r+k)x^{r+k-1}-\sum_{r=0}^{\infty}2a_rx^{r+k}=0$$



- TIDYING UP THE "LOOSE" TERMS TO OBTAIN THE INDICIAL EQUATION

$$\Rightarrow [2k(k-1)+k]a_0x^{k-1}=0$$
$$\Rightarrow [2k^2-2k+k]a_0x^{k-1}=0$$
$$\Rightarrow (2k^2-k)a_0x^{k-1}$$
$$\Rightarrow k(2k-1)a_0x^{k-1}$$

$$k=<\begin{matrix}0\\\frac{1}{2}\end{matrix}\quad (a_0\neq 0)$$

DISTINCT SOLUTIONS NOT DIFFERING BY AN INTEGER

- RETURNING TO THE SUMMATIONS – ADJUST THE DUMMY VARIABLE SO ALL THE SUMMATIONS START FROM $r=0$, BY MAPPING $r\mapsto r+1$.

$$\Rightarrow \sum_{r=0}^{\infty}2a_{r+1}(r+k+1)(r+k)x^{r+k}+\sum_{r=0}^{\infty}a_r(r+k)x^{r+k}+\sum_{r=0}^{\infty}a_{r+1}(r+k+1)x^{r+k}-\sum_{r=0}^{\infty}2a_rx^{r+k}=0$$

$$\Rightarrow \sum_{r=0}^{\infty}\Big[a_{r+1}\big[2(r+k+1)(r+k)+(r+k+1)\big]+a_r\big[(r+k)-2\big]\Big]x^{r+k}=0$$

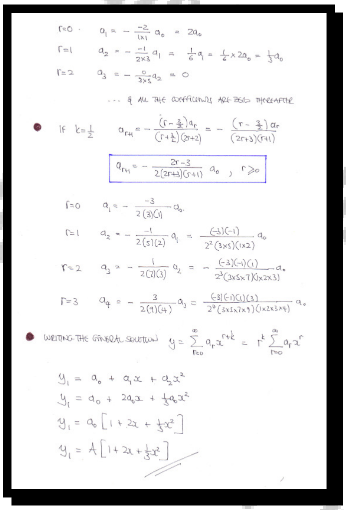
$$\Rightarrow a_{r+1}\big[(r+k+1)(2r+2k+1)\big]+a_r(r+k-2)=0$$

$$\Rightarrow \boxed{a_{r+1}=-\frac{(r+k-2)\ a_r}{(r+k+1)(2r+2k+1)}}$$

WHERE EACH OF THE VALUES OBTAINED BY THE INDICIAL EQUATION WILL PRODUCE AN INDEPENDENT SOLUTION.

- IF $k=0$ $\boxed{a_{r+1}=-\frac{r-2}{(r+1)(2r+1)}a_r,\ r\geqslant 0}$

GENERATE THE FIRST FEW COEFFICIENTS



$r=0$ $\quad a_1=-\frac{-2}{1\times 1}a_0=2a_0$

$r=1$ $\quad a_2=-\frac{-1}{2\times 3}a_1=\frac{1}{6}a_1=\frac{1}{6}\times 2a_0=\frac{1}{3}a_0$

$r=2$ $\quad a_3=-\frac{0}{3\times 5}a_2=0$

... & ALL THE COEFFICIENTS ARE ZERO THEREAFTER

- IF $k=\frac{1}{2}$ $\quad a_{r+1}=-\frac{(r-\frac{3}{2})a_r}{(r+\frac{3}{2})(2r+2)}=-\frac{(r-\frac{3}{2})a_r}{(2r+3)(r+1)}$

$$\boxed{a_{r+1}=-\frac{2r-3}{2(2r+3)(r+1)}a_0,\quad r\geqslant 0}$$

$r=0$ $\quad a_1=-\frac{-3}{2(3)(1)}a_0.$

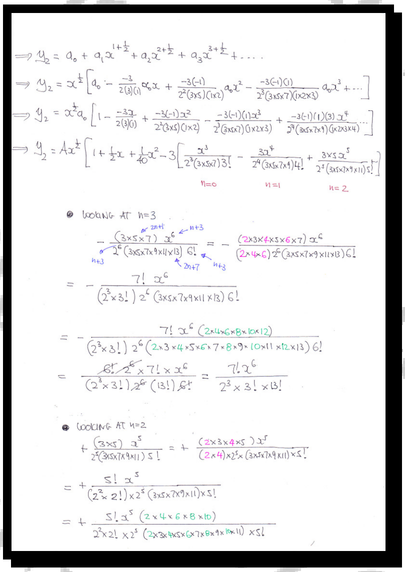
$r=1$ $\quad a_2=-\frac{-1}{2(5)(2)}a_1=\frac{(-3)(-1)}{2^2(3\times 5)(1\times 2)}a_0$

$r=2$ $\quad a_3=-\frac{1}{2(7)(3)}a_2=-\frac{(-3)(-1)(1)}{2^3(3\times 5\times 7)(1\times 2\times 3)}a_0$

$r=3$ $\quad a_4=-\frac{3}{2(9)(4)}a_3=\frac{(-3)(-1)(1)(3)}{2^4(3\times 5\times 7\times 9)(1\times 2\times 3\times 4)}a_0$

- WRITING THE GENERAL SOLUTION $y=\sum_{r=0}^{\infty}a_rx^{r+k}=r^k\sum_{r=0}^{\infty}a_rx^r$

$$y_1=a_0+a_1x+a_2x^2$$
$$y_1=a_0+2a_0x+\frac{1}{3}a_0x^2$$
$$y_1=a_0\left[1+2x+\frac{1}{3}x^2\right]$$
$$y_1=A\left[1+2x+\frac{1}{3}x^2\right]$$



$$\Rightarrow y_2=a_0+a_1x^{1+\frac{1}{2}}+a_2x^{2+\frac{1}{2}}+a_3x^{3+\frac{1}{2}}+\dots$$

$$\Rightarrow y_2=x^{\frac{1}{2}}\left[a_0-\frac{-3}{2(3)(1)}a_0x+\frac{-3(-1)}{2^2(3\times5)(1\times2)}a_0x^2-\frac{-3(-1)(1)}{2^3(3\times5\times7)(1\times2\times3)}a_0x^3+\dots\right]$$

$$\Rightarrow y_2=x^{\frac{1}{2}}a_0\left[1-\frac{-3x}{2(3)(1)}+\frac{-3(-1)x^2}{2^2(3\times5)(1\times2)}-\frac{-3(-1)(1)x^3}{2^3(3\times5\times7)(1\times2\times3)}+\frac{-3(-1)(1)(3)x^4}{2^4(3\times5\times7\times9)(1\times2\times3\times4)}\dots\right]$$

$$\Rightarrow y_2=Ax^{\frac{1}{2}}\left[1+\frac{1}{2}x+\frac{1}{40}x^2-3\left[\underset{n=0}{\frac{x^3}{2^3(3\times5\times7)3!}}-\underset{n=1}{\frac{3x^4}{2^4(3\times5\times7\times9)4!}}+\underset{n=2}{\frac{3\times5x^5}{2^5(3\times5\times7\times9\times11)5!}}\dots\right]\right]$$

- LOOKING AT $n=3$

$$-\frac{\overset{2n+1}{(3\times5\times7)}\ \overset{n+3}{x^6}}{\underset{n+3}{2^6}\underset{2n+7}{(3\times5\times7\times9\times11\times13)}\ \underset{n+3}{6!}}=-\frac{(2\times3\times4\times5\times6\times7)x^6}{(2\times4\times6)2^6(3\times5\times7\times9\times11\times13)6!}$$

$$=-\frac{7!\ x^6}{(2^3\times3!)2^6(3\times5\times7\times9\times11\times13)6!}$$

$$=-\frac{7!\ x^6(2\times4\times6\times8\times10\times12)}{(2^3\times3!)2^6(2\times3\times4\times5\times6\times7\times8\times9\times10\times11\times12\times13)6!}$$

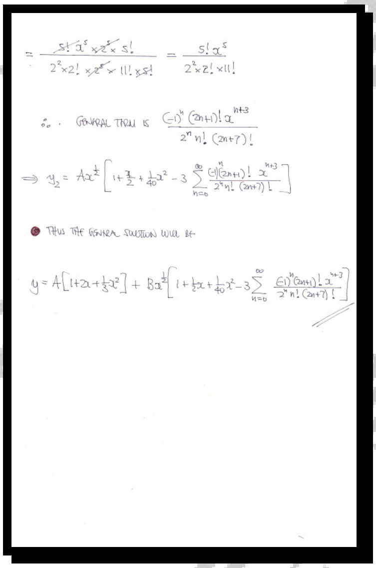
$$=\frac{\cancel{6!}\ \cancel{2^6}\times7!\times x^6}{(2^3\times3!)\cancel{2^6}(13!)\cancel{6!}}=\frac{7!x^6}{2^3\times3!\times13!}$$

- LOOKING AT $n=2$

$$+\frac{(3\times5)\ x^5}{2^5(3\times5\times7\times9\times11)5!}=+\frac{(2\times3\times4\times5)x^5}{(2\times4)\times2^5\times(3\times5\times7\times9\times11)\times5!}$$

$$=+\frac{5!\ x^5}{(2^2\times2!)\times2^5(3\times5\times7\times9\times11)\times5!}$$

$$=+\frac{5!x^5(2\times4\times6\times8\times10)}{2^2\times2!\times2^5(2\times3\times4\times5\times6\times7\times8\times9\times10\times11)\times5!}$$



$$=\frac{\cancel{5!}\ x^5\times\cancel{2^5}\times5!}{2^2\times2!\times\cancel{2^5}\times11!\times\cancel{5!}}=\frac{5!x^5}{2^2\times2!\times11!}$$

$\therefore$ GENERAL TERM IS $\frac{(-1)^n(2n+1)!x^{n+3}}{2^nn!(2n+7)!}$

$$\Rightarrow y_2=Ax^{\frac{1}{2}}\left[1+\frac{x}{2}+\frac{1}{40}x^2-3\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)!\ x^{n+3}}{2^nn!(2n+7)!}\right]$$

- THUS THE GENERAL SOLUTION WILL BE

$$y=A\left[1+2x+\frac{1}{3}x^2\right]+Bx^{\frac{1}{2}}\left[1+\frac{1}{2}x+\frac{1}{40}x^2-3\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)!\ x^{n+3}}{2^nn!(2n+7)!}\right]$$

FROBENIUS METHOD

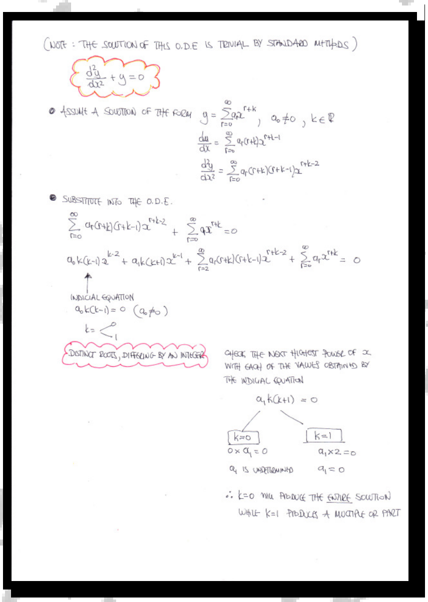
[2^{nd} order O.D.E.s, where the roots of the indicial equation differ by an integer but one of the coefficients is undetermined]

Question 1

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$\frac{d^2y}{dx^2}+y=0.$$

$$\boxed{y=A\cos x+B\sin x}$$



(NOTE : THE SOLUTION OF THIS O.D.E IS TRIVIAL BY STANDARD METHODS)

$\frac{d^2y}{dx^2}+y=0$

- ASSUME A SOLUTION OF THE FORM $y=\sum_{r=0}^{\infty}a_rx^{r+k}$, $a_0\neq0$, $k\in\mathbb{R}$

$$\frac{du}{dx}=\sum_{r=0}^{\infty}a_r(r+k)x^{r+k-1}$$

$$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty}a_r(r+k)(r+k-1)x^{r+k-2}$$

- SUBSTITUTE INTO THE O.D.E.

$$\sum_{r=0}^{\infty}a_r(r+k)(r+k-1)x^{r+k-2}+\sum_{r=0}^{\infty}a_rx^{r+k}=0$$

$$a_0k(k-1)x^{k-2}+a_1k(k+1)x^{k-1}+\sum_{r=2}^{\infty}a_r(r+k)(r+k-1)x^{r+k-2}+\sum_{r=0}^{\infty}a_rx^{r+k}=0$$

INDICIAL EQUATION

$a_0k(k-1)=0\quad(a_0\neq0)$

$k=\begin{cases}0\\1\end{cases}$

DISTINCT ROOTS, DIFFERING BY AN INTEGER

CHECK THE NEXT HIGHEST POWER OF x WITH EACH OF THE VALUES OBTAINED BY THE INDICIAL EQUATION

$a_1k(k+1)=0$

$\boxed{k=0}$

$0\times a_1=0$

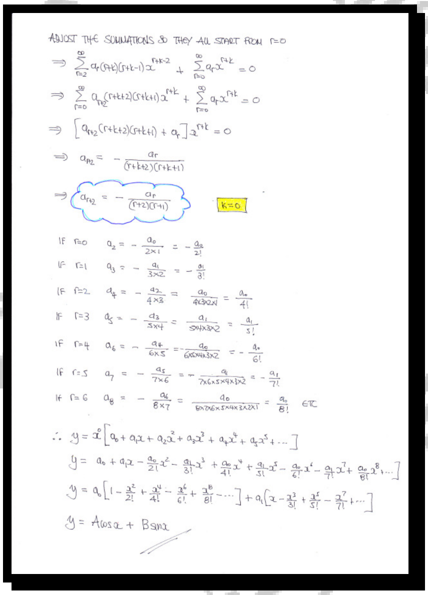
a_1 IS UNDETERMINED

$\boxed{k=1}$

$a_1\times2=0$

$a_1=0$

$\therefore$ $k=0$ WILL PRODUCE THE ENTIRE SOLUTION WHILE $k=1$ PRODUCES A MULTIPLE OR PART



ADJUST THE SUMMATIONS SO THEY ALL START FROM $r=0$

$$\Rightarrow\sum_{r=2}^{\infty}a_r(r+k)(r+k-1)x^{r+k-2}+\sum_{r=0}^{\infty}a_rx^{r+k}=0$$

$$\Rightarrow\sum_{r=0}^{\infty}a_{r+2}(r+k+2)(r+k+1)x^{r+k}+\sum_{r=0}^{\infty}a_rx^{r+k}=0$$

$$\Rightarrow\left[a_{r+2}(r+k+2)(r+k+1)+a_r\right]x^{r+k}=0$$

$$\Rightarrow a_{r+2}=-\frac{a_r}{(r+k+2)(r+k+1)}$$

$$\Rightarrow a_{r+2}=-\frac{a_r}{(r+2)(r+1)}\qquad k=0$$

IF $r=0$ $\quad a_2=-\frac{a_0}{2\times1}=-\frac{a_0}{2!}$

IF $r=1$ $\quad a_3=-\frac{a_1}{3\times2}=-\frac{a_1}{3!}$

IF $r=2$ $\quad a_4=-\frac{a_2}{4\times3}=\frac{a_0}{4\times3\times2\times1}=\frac{a_0}{4!}$

IF $r=3$ $\quad a_5=-\frac{a_3}{5\times4}=\frac{a_1}{5\times4\times3\times2}=\frac{a_1}{5!}$

IF $r=4$ $\quad a_6=-\frac{a_4}{6\times5}=-\frac{a_0}{6\times5\times4\times3\times2}=-\frac{a_0}{6!}$

IF $r=5$ $\quad a_7=-\frac{a_5}{7\times6}=-\frac{a_1}{7\times6\times5\times4\times3\times2}=-\frac{a_1}{7!}$

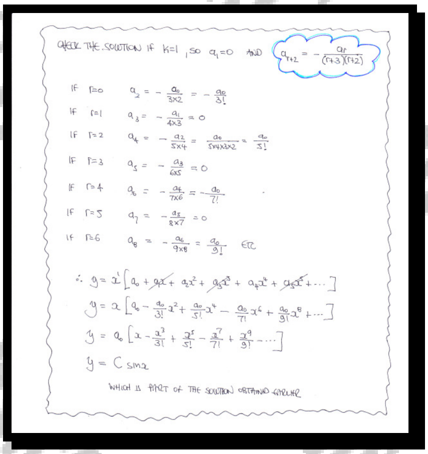
IF $r=6$ $\quad a_8=-\frac{a_6}{8\times7}=\frac{a_0}{8\times7\times6\times5\times4\times3\times2\times1}=\frac{a_0}{8!}$ ETC

$$\therefore\ y=x^0\left[a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+\dots\right]$$

$$y=a_0+a_1x-\frac{a_0}{2!}x^2-\frac{a_1}{3!}x^3+\frac{a_0}{4!}x^4+\frac{a_1}{5!}x^5-\frac{a_0}{6!}x^6-\frac{a_1}{7!}x^7+\frac{a_0}{8!}x^8+\dots]$$

$$y=a_0\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\dots\right]+a_1\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots\right]$$

$$y=A\cos x+B\sin x$$



CHECK THE SOLUTION IF $k=1$, SO $a_1=0$ AND $a_{r+2}=-\frac{a_r}{(r+3)(r+2)}$

IF $r=0$ $\quad a_2=-\frac{a_0}{3\times2}=-\frac{a_0}{3!}$

IF $r=1$ $\quad a_3=-\frac{a_1}{4\times3}=0$

IF $r=2$ $\quad a_4=-\frac{a_2}{5\times4}=\frac{a_0}{5\times4\times3\times2}=\frac{a_0}{5!}$

IF $r=3$ $\quad a_5=-\frac{a_3}{6\times5}=0$

IF $r=4$ $\quad a_6=-\frac{a_4}{7\times6}=-\frac{a_0}{7!}$

IF $r=5$ $\quad a_7=-\frac{a_5}{8\times7}=0$

IF $r=6$ $\quad a_8=-\frac{a_6}{9\times8}=\frac{a_0}{9!}$ ETC

$$\therefore\ y=x^1\left[a_0+\cancel{a_1x}+a_2x^2+\cancel{a_3x^3}+a_4x^4+\cancel{a_5x^5}+\dots\right]$$

$$y=x\left[a_0-\frac{a_0}{3!}x^2+\frac{a_0}{5!}x^4-\frac{a_0}{7!}x^6+\frac{a_0}{9!}x^8+\dots\right]$$

$$y=a_0\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\dots\right]$$

$$y=C\sin x$$

WHICH IS PART OF THE SOLUTION OBTAINED EARLIER

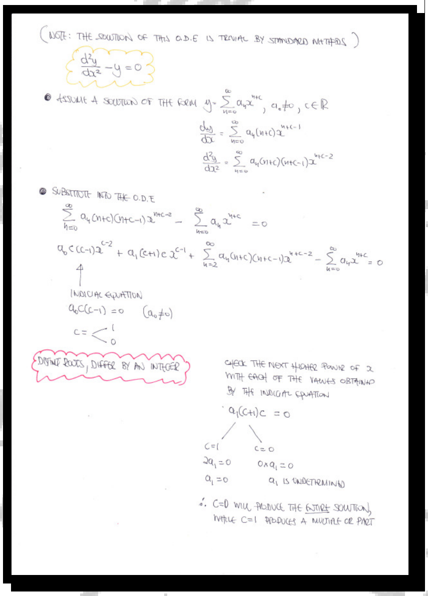
Question 2

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$\frac{d^2y}{dy^2} - y = 0 .$$

Give the final answer in a simplified form.

$$\boxed{y = A\sinh x + B\cosh x}$$



(NOTE: THE SOLUTION OF THIS O.D.E IS TRIVIAL BY STANDARD METHODS)

$$\frac{d^2y}{dx^2} - y = 0$$

- ASSUME A SOLUTION OF THE FORM $y = \sum_{n=0}^{\infty} a_n x^{n+c}$, $a_0 \neq 0$, $c \in \mathbb{R}$

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} a_n (n+c) x^{n+c-1}$$

$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} a_n (n+c)(n+c-1) x^{n+c-2}$$

- SUBSTITUTE INTO THE O.D.E

$$\sum_{n=0}^{\infty} a_n (n+c)(n+c-1) x^{n+c-2} - \sum_{n=0}^{\infty} a_n x^{n+c} = 0$$

$$a_0 c(c-1) x^{c-2} + a_1 (c+1) c x^{c-1} + \sum_{n=2}^{\infty} a_n (n+c)(n+c-1) x^{n+c-2} - \sum_{n=0}^{\infty} a_n x^{n+c} = 0$$

INDICIAL EQUATION

$a_0 c(c-1) = 0 \quad (a_0 \neq 0)$

$c = \begin{cases} 1 \\ 0 \end{cases}$

DISTINCT ROOTS, DIFFER BY AN INTEGER

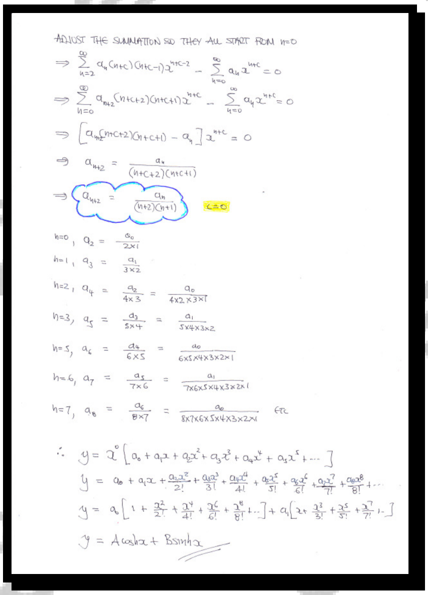
CHECK THE NEXT HIGHER POWER OF x WITH EACH OF THE VALUES OBTAINED BY THE INDICIAL EQUATION

$a_1 (c+1) c = 0$

$c = 1$: $2a_1 = 0$, $a_1 = 0$

$c = 0$: $0 \times a_1 = 0$, a_1 IS UNDETERMINED

$\therefore$ $c=0$ WILL PRODUCE THE ENTIRE SOLUTION, WHILE $c=1$ PRODUCES A MULTIPLE OR PART



ADJUST THE SUMMATION SO THEY ALL START FROM $n=0$

$$\Rightarrow \sum_{n=2}^{\infty} a_n (n+c)(n+c-1) x^{n+c-2} - \sum_{n=0}^{\infty} a_n x^{n+c} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} a_{n+2} (n+c+2)(n+c+1) x^{n+c} - \sum_{n=0}^{\infty} a_n x^{n+c} = 0$$

$$\Rightarrow \left[a_{n+2}(n+c+2)(n+c+1) - a_n \right] x^{n+c} = 0$$

$$\Rightarrow a_{n+2} = \frac{a_n}{(n+c+2)(n+c+1)}$$

$$\Rightarrow a_{n+2} = \frac{a_n}{(n+2)(n+1)} \quad c=0$$

$n=0$, $a_2 = \frac{a_0}{2 \times 1}$

$n=1$, $a_3 = \frac{a_1}{3 \times 2}$

$n=2$, $a_4 = \frac{a_2}{4 \times 3} = \frac{a_0}{4 \times 2 \times 3 \times 1}$

$n=3$, $a_5 = \frac{a_3}{5 \times 4} = \frac{a_1}{5 \times 4 \times 3 \times 2}$

$n=5$, $a_6 = \frac{a_4}{6 \times 5} = \frac{a_0}{6 \times 5 \times 4 \times 3 \times 2 \times 1}$

$n=6$, $a_7 = \frac{a_5}{7 \times 6} = \frac{a_1}{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$

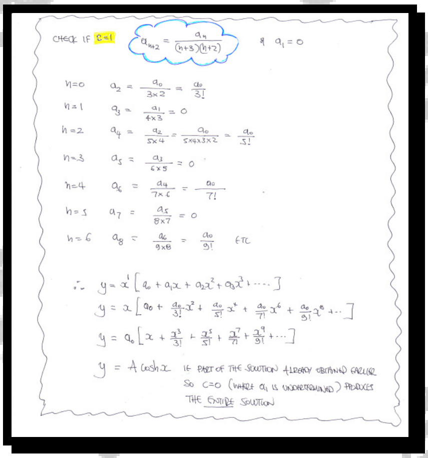
$n=7$, $a_8 = \frac{a_6}{8 \times 7} = \frac{a_0}{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$ ETC

$$\therefore \quad y = x^0 \left[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots \right]$$

$$y = a_0 + a_1 x + \frac{a_2 x^2}{2!} + \frac{a_3 x^3}{3!} + \frac{a_4 x^4}{4!} + \frac{a_5 x^5}{5!} + \frac{a_6 x^6}{6!} + \frac{a_7 x^7}{7!} + \frac{a_8 x^8}{8!} + \dots$$

$$y = a_0 \left[1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \dots \right] + a_1 \left[x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots \right]$$

$$y = A\cosh x + B\sinh x$$



CHECK IF $c=1$ $\quad a_{n+2} = \frac{a_n}{(n+3)(n+2)}$ & $a_1 = 0$

$n=0$ $\quad a_2 = \frac{a_0}{3 \times 2} = \frac{a_0}{3!}$

$n=1$ $\quad a_3 = \frac{a_1}{4 \times 3} = 0$

$n=2$ $\quad a_4 = \frac{a_2}{5 \times 4} = \frac{a_0}{5 \times 4 \times 3 \times 2} = \frac{a_0}{5!}$

$n=3$ $\quad a_5 = \frac{a_3}{6 \times 5} = 0$

$n=4$ $\quad a_6 = \frac{a_4}{7 \times 6} = \frac{a_0}{7!}$

$n=5$ $\quad a_7 = \frac{a_5}{8 \times 7} = 0$

$n=6$ $\quad a_8 = \frac{a_6}{9 \times 8} = \frac{a_0}{9!}$ ETC

$$\therefore \quad y = x^1 \left[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \right]$$

$$y = x \left[a_0 + \frac{a_0}{3!} x^2 + \frac{a_0}{5!} x^4 + \frac{a_0}{7!} x^6 + \frac{a_0}{9!} x^8 + \dots \right]$$

$$y = a_0 \left[x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \dots \right]$$

$y = A\cosh x$ IS PART OF THE SOLUTION ALREADY OBTAINED EARLIER SO $c=0$ (WHERE a_1 IS UNDETERMINED) PRODUCES THE ENTIRE SOLUTION

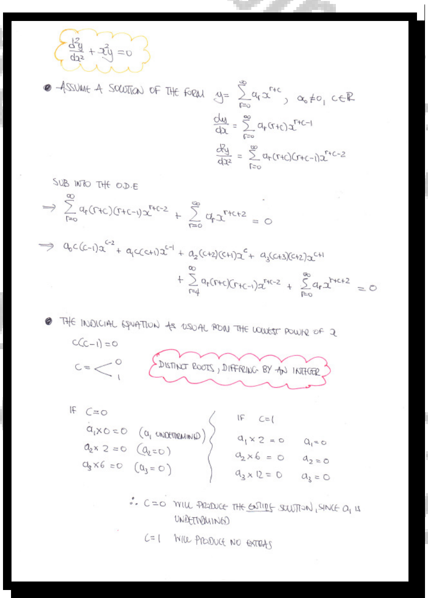
Question 3

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$\frac{d^2y}{dx^2} - yx^2 = 0 .$$

Give the final answer in simplified Sigma notation.

$$\boxed{y = A\Gamma\left(\tfrac{3}{4}\right)\sum_{r=0}^{\infty}\left[\frac{(-1)^r x^{4r}}{2^{4r}\times r!\times\Gamma\left(\frac{4r+3}{4}\right)}\right] + B\Gamma\left(\tfrac{5}{4}\right)\sum_{r=0}^{\infty}\left[\frac{(-1)^r x^{4r+1}}{2^{4r}\times r!\times\Gamma\left(\frac{4r+5}{4}\right)}\right]}$$



$$\frac{d^2y}{dx^2} + x^2y = 0$$

- ASSUME A SOLUTION OF THE FORM $y = \sum_{r=0}^{\infty} a_r x^{r+c}$, $a_0 \neq 0$, $c \in \mathbb{R}$

$$\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r(r+c)x^{r+c-1}$$

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r(r+c)(r+c-1)x^{r+c-2}$$

SUB INTO THE O.D.E

$$\Rightarrow \sum_{r=0}^{\infty} a_r(r+c)(r+c-1)x^{r+c-2} + \sum_{r=0}^{\infty} a_r x^{r+c+2} = 0$$

$$\Rightarrow a_0c(c-1)x^{c-2} + a_1c(c+1)x^{c-1} + a_2(c+2)(c+1)x^{c} + a_3(c+3)(c+2)x^{c+1} + \sum_{r=4}^{\infty} a_r(r+c)(r+c-1)x^{r+c-2} + \sum_{r=0}^{\infty} a_r x^{r+c+2} = 0$$

- THE INDICIAL EQUATION AS USUAL FROM THE LOWEST POWER OF x

$c(c-1) = 0$

$c = \begin{cases} 0 \\ 1 \end{cases}$ DISTINCT ROOTS, DIFFERING BY AN INTEGER

IF $c=0$

$a_1 \times 0 = 0$ (a_1 UNDETERMINED)

$a_2 \times 2 = 0$ ($a_2 = 0$)

$a_3 \times 6 = 0$ ($a_3 = 0$)

IF $c=1$

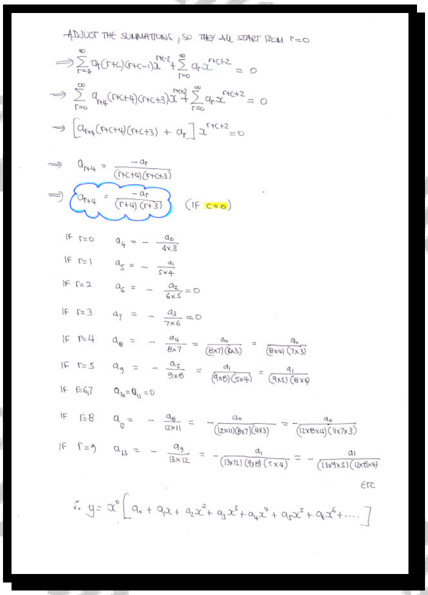
$a_1 \times 2 = 0$ $\quad a_1 = 0$

$a_2 \times 6 = 0$ $\quad a_2 = 0$

$a_3 \times 12 = 0$ $\quad a_3 = 0$

$\therefore$ $c=0$ WILL PRODUCE THE ENTIRE SOLUTION, SINCE a_1 IS UNDETERMINED

$c=1$ WILL PRODUCE NO EXTRAS



ADJUST THE SUMMATIONS, SO THEY ALL START FROM $r=0$

$$\Rightarrow \sum_{r=4}^{\infty} a_r(r+c)(r+c-1)x^{r+c-2} + \sum_{r=0}^{\infty} a_r x^{r+c+2} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_{r+4}(r+c+4)(r+c+3)x^{r+c+2} + \sum_{r=0}^{\infty} a_r x^{r+c+2} = 0$$

$$\Rightarrow \left[a_{r+4}(r+c+4)(r+c+3) + a_r\right]x^{r+c+2} = 0$$

$$\Rightarrow a_{r+4} = \frac{-a_r}{(r+c+4)(r+c+3)}$$

$$\Rightarrow a_{r+4} = \frac{-a_r}{(r+4)(r+3)} \quad \text{(IF } c=0\text{)}$$

IF $r=0$ $\quad a_4 = -\frac{a_0}{4\times3}$

IF $r=1$ $\quad a_5 = -\frac{a_1}{5\times4}$

IF $r=2$ $\quad a_6 = -\frac{a_2}{6\times5} = 0$

IF $r=3$ $\quad a_7 = -\frac{a_3}{7\times6} = 0$

IF $r=4$ $\quad a_8 = -\frac{a_4}{8\times7} = \frac{a_0}{(8\times7)(4\times3)} = \frac{a_0}{(8\times4)(7\times3)}$

IF $r=5$ $\quad a_9 = -\frac{a_5}{9\times8} = \frac{a_1}{(9\times8)(5\times4)} = \frac{a_1}{(9\times5)(8\times4)}$

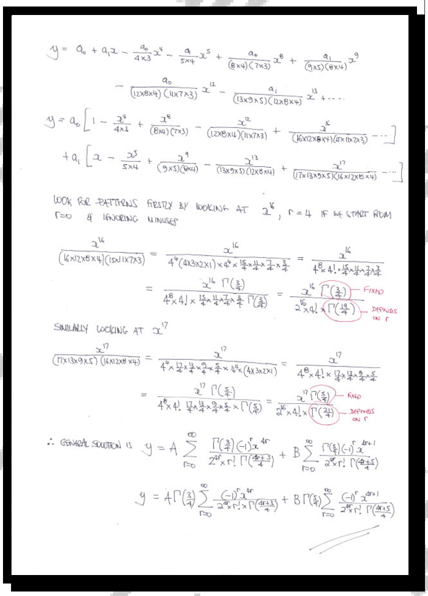
IF $r=6,7$ $\quad a_{10} = a_{11} = 0$

IF $r=8$ $\quad a_{12} = -\frac{a_8}{12\times11} = -\frac{a_0}{(12\times11)(8\times7)(4\times3)} = -\frac{a_0}{(12\times8\times4)(11\times7\times3)}$

IF $r=9$ $\quad a_{13} = -\frac{a_9}{13\times12} = -\frac{a_1}{(13\times12)(9\times8)(5\times4)} = -\frac{a_1}{(13\times9\times5)(12\times8\times4)}$

ETC

$$\therefore y = x^0\left[a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \dots\right]$$



$$y = a_0 + a_1x - \frac{a_0}{4\times3}x^4 - \frac{a_1}{5\times4}x^5 + \frac{a_0}{(8\times4)(7\times3)}x^8 + \frac{a_1}{(9\times5)(8\times4)}x^9 - \frac{a_0}{(12\times8\times4)(11\times7\times3)}x^{12} - \frac{a_1}{(13\times9\times5)(12\times8\times4)}x^{13} + \dots$$

$$y = a_0\left[1 - \frac{x^4}{4\times3} + \frac{x^8}{(8\times4)(7\times3)} - \frac{x^{12}}{(12\times8\times4)(11\times7\times3)} + \frac{x^{16}}{(16\times12\times8\times4)(15\times11\times7\times3)} - \dots\right] + a_1\left[x - \frac{x^5}{5\times4} + \frac{x^9}{(9\times5)(8\times4)} - \frac{x^{13}}{(13\times9\times5)(12\times8\times4)} + \frac{x^{17}}{(17\times13\times9\times5)(16\times12\times8\times4)} - \dots\right]$$

LOOK FOR PATTERNS FIRSTLY BY LOOKING AT x^{16}, $r=4$ IF WE START FROM $r=0$ & IGNORING MINUSES

$$\frac{x^{16}}{(16\times12\times8\times4)(15\times11\times7\times3)} = \frac{x^{16}}{4^4(4\times3\times2\times1)\times4^4\times\frac{15}{4}\times\frac{11}{4}\times\frac{7}{4}\times\frac{3}{4}} = \frac{x^{16}}{4^8\times4!\times\frac{15}{4}\times\frac{11}{4}\times\frac{7}{4}\times\frac{3}{4}}$$

$$= \frac{x^{16}\Gamma\left(\frac{3}{4}\right)}{4^8\times4!\times\frac{15}{4}\times\frac{11}{4}\times\frac{7}{4}\times\frac{3}{4}\Gamma\left(\frac{3}{4}\right)} = \frac{x^{16}\Gamma\left(\frac{3}{4}\right)}{2^{16}\times4!\times\Gamma\left(\frac{19}{4}\right)}$$

FIXED

DEPENDS ON r

SIMILARLY LOOKING AT x^{17}

$$\frac{x^{17}}{(17\times13\times9\times5)(16\times12\times8\times4)} = \frac{x^{17}}{4^4\times\frac{17}{4}\times\frac{13}{4}\times\frac{9}{4}\times\frac{5}{4}\times4^4\times(4\times3\times2\times1)} = \frac{x^{17}}{4^8\times4!\times\frac{17}{4}\times\frac{13}{4}\times\frac{9}{4}\times\frac{5}{4}}$$

$$= \frac{x^{17}\Gamma\left(\frac{5}{4}\right)}{4^8\times4!\,\frac{17}{4}\times\frac{13}{4}\times\frac{9}{4}\times\frac{5}{4}\times\Gamma\left(\frac{5}{4}\right)} = \frac{x^{17}\Gamma\left(\frac{5}{4}\right)}{2^{16}\times4!\times\Gamma\left(\frac{21}{4}\right)}$$

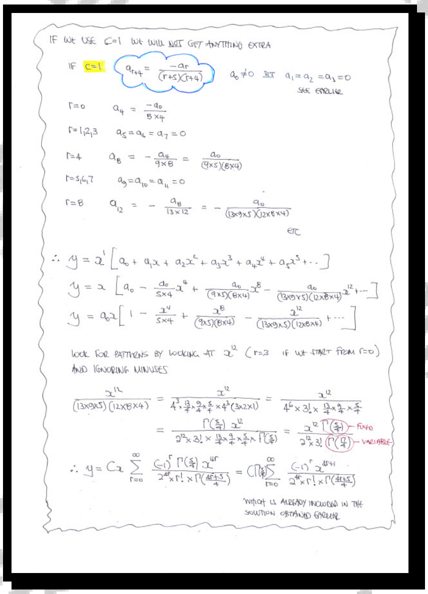
FIXED

DEPENDS ON r

$\therefore$ GENERAL SOLUTION IS

$$y = A\sum_{r=0}^{\infty}\frac{\Gamma\left(\frac{3}{4}\right)(-1)^r x^{4r}}{2^{4r}\times r!\,\Gamma\left(\frac{4r+3}{4}\right)} + B\sum_{r=0}^{\infty}\frac{\Gamma\left(\frac{5}{4}\right)(-1)^r x^{4r+1}}{2^{4r}\times r!\,\Gamma\left(\frac{4r+5}{4}\right)}$$

$$y = A\Gamma\left(\tfrac{3}{4}\right)\sum_{r=0}^{\infty}\frac{(-1)^r x^{4r}}{2^{4r}\times r!\times\Gamma\left(\frac{4r+3}{4}\right)} + B\Gamma\left(\tfrac{5}{4}\right)\sum_{r=0}^{\infty}\frac{(-1)^r x^{4r+1}}{2^{4r}\times r!\,\Gamma\left(\frac{4r+5}{4}\right)}$$



IF WE USE $c=1$ WE WILL NOT GET ANYTHING EXTRA

IF $c=1$ $\quad a_{r+4} = \frac{-a_r}{(r+5)(r+4)}$ $\quad a_0 \neq 0$ BUT $a_1 = a_2 = a_3 = 0$ SEE EARLIER

$r=0$ $\quad a_4 = \frac{-a_0}{5\times4}$

$r=1,2,3$ $\quad a_5 = a_6 = a_7 = 0$

$r=4$ $\quad a_8 = -\frac{a_4}{9\times8} = \frac{a_0}{(9\times5)(8\times4)}$

$r=5,6,7$ $\quad a_9 = a_{10} = a_{11} = 0$

$r=8$ $\quad a_{12} = -\frac{a_8}{13\times12} = -\frac{a_0}{(13\times9\times5)(12\times8\times4)}$

ETC

$$\therefore y = x^1\left[a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \dots\right]$$

$$y = x\left[a_0 - \frac{a_0}{5\times4}x^4 + \frac{a_0}{(9\times5)(8\times4)}x^8 - \frac{a_0}{(13\times9\times5)(12\times8\times4)}x^{12} + \dots\right]$$

$$y = a_0x\left[1 - \frac{x^4}{5\times4} + \frac{x^8}{(9\times5)(8\times4)} - \frac{x^{12}}{(13\times9\times5)(12\times8\times4)} + \dots\right]$$

LOOK FOR PATTERNS BY LOOKING AT x^{12} ($r=3$ IF WE START FROM $r=0$) AND IGNORING MINUSES

$$\frac{x^{12}}{(13\times9\times5)(12\times8\times4)} = \frac{x^{12}}{4^3\times\frac{13}{4}\times\frac{9}{4}\times\frac{5}{4}\times4^3(3\times2\times1)} = \frac{x^{12}}{4^6\times3!\times\frac{13}{4}\times\frac{9}{4}\times\frac{5}{4}}$$

$$= \frac{\Gamma\left(\frac{5}{4}\right)x^{12}}{2^{12}\times3!\times\frac{13}{4}\times\frac{9}{4}\times\frac{5}{4}\times\Gamma\left(\frac{5}{4}\right)} = \frac{x^{12}\Gamma\left(\frac{5}{4}\right)}{2^{12}\times3!\,\Gamma\left(\frac{17}{4}\right)}$$

FIXED

VARIABLE

$$\therefore y = Cx\sum_{r=0}^{\infty}\frac{(-1)^r\Gamma\left(\frac{5}{4}\right)x^{4r}}{2^{4r}\times r!\times\Gamma\left(\frac{4r+5}{4}\right)} = C\Gamma\left(\tfrac{5}{4}\right)\sum_{r=0}^{\infty}\frac{(-1)^r x^{4r+1}}{2^{4r}\times r!\times\Gamma\left(\frac{4r+5}{4}\right)}$$

WHICH IS ALREADY INCLUDED IN THE SOLUTION OBTAINED EARLIER

Question 4

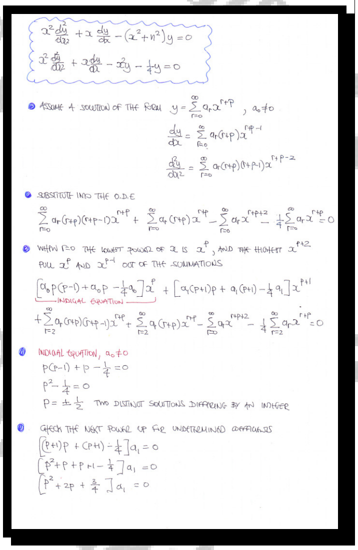
$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-\left(x^2+n^2\right)y=0.$$

The above differential equation is known as modified Bessel's Equation.

Use the Frobenius method to show that the general solution of this differential equation, for $n=\frac{1}{2}$, is

$$y=x^{-\frac{1}{2}}\left[A\cosh x+B\sinh x\right].$$

$\boxed{\text{proof}}$



$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-(x^2+n^2)y=0$$

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-x^2y-\frac{1}{4}y=0$$

- ASSUME A SOLUTION OF THE FORM $y=\sum_{r=0}^{\infty}a_rx^{r+p}$, $a_0\neq0$

$$\frac{dy}{dx}=\sum_{r=0}^{\infty}a_r(r+p)x^{r+p-1}$$

$$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty}a_r(r+p)(r+p-1)x^{r+p-2}$$

- SUBSTITUTE INTO THE O.D.E

$$\sum_{r=0}^{\infty}a_r(r+p)(r+p-1)x^{r+p}+\sum_{r=0}^{\infty}a_r(r+p)x^{r+p}-\sum_{r=0}^{\infty}a_rx^{r+p+2}-\frac{1}{4}\sum_{r=0}^{\infty}a_rx^{r+p}=0$$

- WHEN $r=0$ THE LOWEST POWER OF x IS x^p, AND THE HIGHEST x^{p+2}

PULL x^p AND x^{p-1} OUT OF THE SUMMATIONS

$$\underbrace{\left[a_0p(p-1)+a_0p-\frac{1}{4}a_0\right]x^p}_{\text{INDICIAL EQUATION}}+\left[a_1(p+1)p+a_1(p+1)-\frac{1}{4}a_1\right]x^{p+1}$$

$$+\sum_{r=2}^{\infty}a_r(r+p)(r+p-1)x^{r+p}+\sum_{r=2}^{\infty}a_r(r+p)x^{r+p}-\sum_{r=0}^{\infty}a_rx^{r+p+2}-\frac{1}{4}\sum_{r=2}^{\infty}a_rx^{r+p}=0$$

- INDICIAL EQUATION, $a_0\neq0$

$$p(p-1)+p-\frac{1}{4}=0$$

$$p^2-\frac{1}{4}=0$$

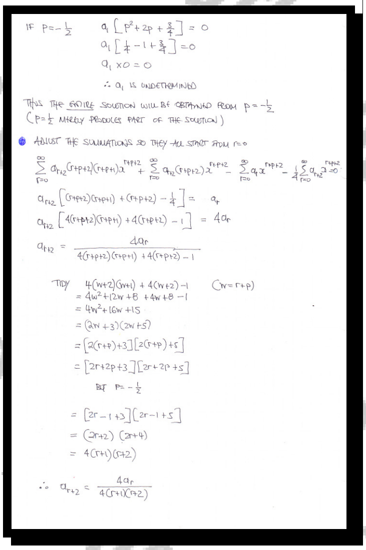
$p=\pm\frac{1}{2}$ TWO DISTINCT SOLUTIONS DIFFERING BY AN INTEGER

- CHECK THE NEXT POWER UP FOR UNDETERMINED COEFFICIENTS

$$\left[(p+1)p+(p+1)-\frac{1}{4}\right]a_1=0$$

$$\left[p^2+p+p+1-\frac{1}{4}\right]a_1=0$$

$$\left[p^2+2p+\frac{3}{4}\right]a_1=0$$



IF $p=-\frac{1}{2}$

$$a_1\left[p^2+2p+\frac{3}{4}\right]=0$$

$$a_1\left[\frac{1}{4}-1+\frac{3}{4}\right]=0$$

$$a_1\times0=0$$

$\therefore a_1$ IS UNDETERMINED

THUS THE ENTIRE SOLUTION WILL BE OBTAINED FROM $p=-\frac{1}{2}$

($p=\frac{1}{2}$ MERELY PRODUCES PART OF THE SOLUTION)

- ADJUST THE SUMMATIONS SO THEY ALL START FROM $r=0$

$$\sum_{r=0}^{\infty}a_{r+2}(r+p+2)(r+p+1)x^{r+p+2}+\sum_{r=0}^{\infty}a_{r+2}(r+p+2)x^{r+p+2}-\sum_{r=0}^{\infty}a_rx^{r+p+2}-\frac{1}{4}\sum_{r=0}^{\infty}a_{r+2}x^{r+p+2}=0$$

$$a_{r+2}\left[(r+p+2)(r+p+1)+(r+p+2)-\frac{1}{4}\right]=a_r$$

$$a_{r+2}\left[4(r+p+2)(r+p+1)+4(r+p+2)-1\right]=4a_r$$

$$a_{r+2}=\frac{4a_r}{4(r+p+2)(r+p+1)+4(r+p+2)-1}$$

TIDY $4(w+2)(w+1)+4(w+2)-1$ $\quad(w=r+p)$

$$=4w^2+12w+8+4w+8-1$$

$$=4w^2+16w+15$$

$$=(2w+3)(2w+5)$$

$$=\left[2(r+p)+3\right]\left[2(r+p)+5\right]$$

$$=\left[2r+2p+3\right]\left[2r+2p+5\right]$$

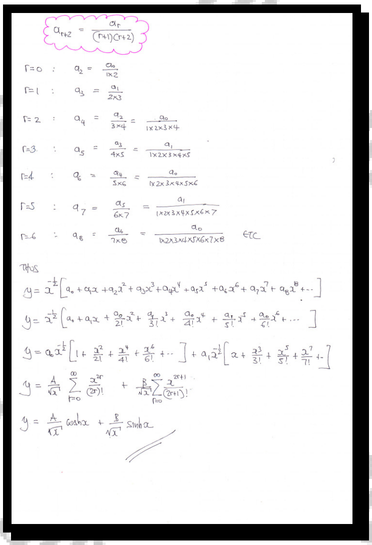
BUT $p=-\frac{1}{2}$

$$=\left[2r-1+3\right]\left[2r-1+5\right]$$

$$=(2r+2)(2r+4)$$

$$=4(r+1)(r+2)$$

$$\therefore a_{r+2}=\frac{4a_r}{4(r+1)(r+2)}$$



$$a_{r+2}=\frac{a_r}{(r+1)(r+2)}$$

$r=0$: $a_2=\frac{a_0}{1\times2}$

$r=1$: $a_3=\frac{a_1}{2\times3}$

$r=2$: $a_4=\frac{a_2}{3\times4}=\frac{a_0}{1\times2\times3\times4}$

$r=3$: $a_5=\frac{a_3}{4\times5}=\frac{a_1}{1\times2\times3\times4\times5}$

$r=4$: $a_6=\frac{a_4}{5\times6}=\frac{a_0}{1\times2\times3\times4\times5\times6}$

$r=5$: $a_7=\frac{a_5}{6\times7}=\frac{a_1}{1\times2\times3\times4\times5\times6\times7}$

$r=6$: $a_8=\frac{a_6}{7\times8}=\frac{a_0}{1\times2\times3\times4\times5\times6\times7\times8}$ ETC

THUS

$$y=x^{-\frac{1}{2}}\left[a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6+a_7x^7+a_8x^8+\dots\right]$$

$$y=x^{-\frac{1}{2}}\left[a_0+a_1x+\frac{a_0}{2!}x^2+\frac{a_1}{3!}x^3+\frac{a_0}{4!}x^4+\frac{a_1}{5!}x^5+\frac{a_0}{6!}x^6+\dots\right]$$

$$y=a_0x^{-\frac{1}{2}}\left[1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\dots\right]+a_1x^{-\frac{1}{2}}\left[x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\dots\right]$$

$$y=\frac{A}{\sqrt{x}}\sum_{r=0}^{\infty}\frac{x^{2r}}{(2r)!}+\frac{B}{\sqrt{x}}\sum_{r=0}^{\infty}\frac{x^{2r+1}}{(2r+1)!}$$

$$y=\frac{A}{\sqrt{x}}\cosh x+\frac{B}{\sqrt{x}}\sinh x$$

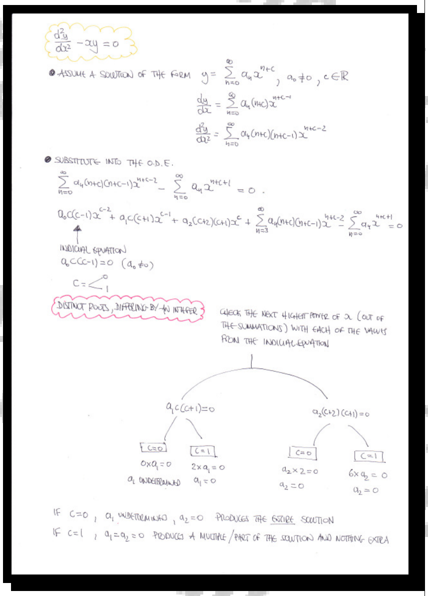
Question 5

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$\frac{d^2y}{dx^2} - xy = 0 .$$

Give the final answer in simplified Sigma notation.

$$\boxed{y = \sum_{r=0}^{\infty}\left\{\frac{x^{3r}}{9^r \times r!}\left[\frac{A\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{3r+2}{3}\right)} + \frac{Bx\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{3r+4}{3}\right)}\right]\right\}}$$



$$\frac{d^2y}{dx^2} - xy = 0$$

- ASSUME A SOLUTION OF THE FORM $y = \sum_{n=0}^{\infty} a_n x^{n+c}$, $a_0 \neq 0$, $c \in \mathbb{R}$

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} a_n (n+c) x^{n+c-1}$$

$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} a_n (n+c)(n+c-1) x^{n+c-2}$$

- SUBSTITUTE INTO THE O.D.E.

$$\sum_{n=0}^{\infty} a_n(n+c)(n+c-1)x^{n+c-2} - \sum_{n=0}^{\infty} a_n x^{n+c+1} = 0 .$$

$$a_0c(c-1)x^{c-2} + a_1c(c+1)x^{c-1} + a_2(c+2)(c+1)x^c + \sum_{n=3}^{\infty} a_n(n+c)(n+c-1)x^{n+c-2} - \sum_{n=0}^{\infty} a_n x^{n+c+1} = 0$$

INDICIAL EQUATION

$a_0c(c-1) = 0 \quad (a_0 \neq 0)$

$c = \begin{cases} 0 \\ 1 \end{cases}$

DISTINCT ROOTS, DIFFERING BY AN INTEGER

CHECK THE NEXT HIGHEST POWER OF x (OUT OF THE SUMMATIONS) WITH EACH OF THE VALUES FROM THE INDICIAL EQUATION

$a_1c(c+1) = 0$

$\boxed{c=0}$ $0 \times a_1 = 0$, a_1 UNDETERMINED

$\boxed{c=1}$ $2 \times a_1 = 0$, $a_1 = 0$

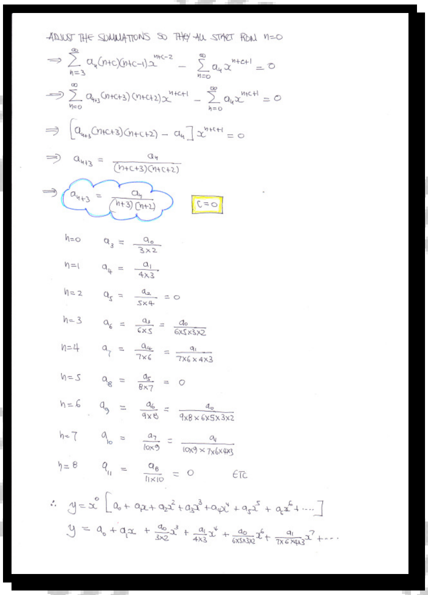
$a_2(c+2)(c+1) = 0$

$\boxed{c=0}$ $a_2 \times 2 = 0$, $a_2 = 0$

$\boxed{c=1}$ $6 \times a_2 = 0$, $a_2 = 0$

IF $c=0$, a_1 UNDETERMINED, $a_2 = 0$ PRODUCES THE ENTIRE SOLUTION

IF $c=1$, $a_1 = a_2 = 0$ PRODUCES A MULTIPLE / PART OF THE SOLUTION AND NOTHING EXTRA



ADJUST THE SUMMATIONS SO THEY ALL START FROM $n=0$

$$\Rightarrow \sum_{n=3}^{\infty} a_n(n+c)(n+c-1)x^{n+c-2} - \sum_{n=0}^{\infty} a_n x^{n+c+1} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} a_{n+3}(n+c+3)(n+c+2)x^{n+c+1} - \sum_{n=0}^{\infty} a_n x^{n+c+1} = 0$$

$$\Rightarrow \left[a_{n+3}(n+c+3)(n+c+2) - a_n\right]x^{n+c+1} = 0$$

$$\Rightarrow a_{n+3} = \frac{a_n}{(n+c+3)(n+c+2)}$$

$$\Rightarrow a_{n+3} = \frac{a_n}{(n+3)(n+2)} \qquad \boxed{c=0}$$

$n=0 \quad a_3 = \frac{a_0}{3\times2}$

$n=1 \quad a_4 = \frac{a_1}{4\times3}$

$n=2 \quad a_5 = \frac{a_2}{5\times4} = 0$

$n=3 \quad a_6 = \frac{a_3}{6\times5} = \frac{a_0}{6\times5\times3\times2}$

$n=4 \quad a_7 = \frac{a_4}{7\times6} = \frac{a_1}{7\times6\times4\times3}$

$n=5 \quad a_8 = \frac{a_5}{8\times7} = 0$

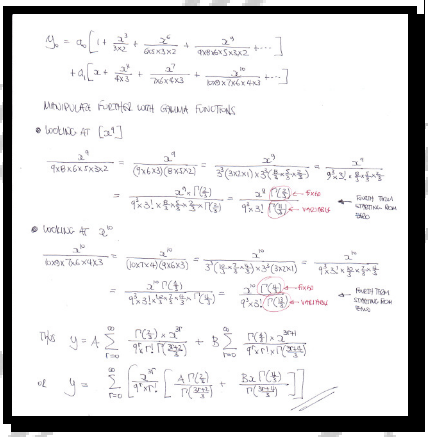
$n=6 \quad a_9 = \frac{a_6}{9\times8} = \frac{a_0}{9\times8\times6\times5\times3\times2}$

$n=7 \quad a_{10} = \frac{a_7}{10\times9} = \frac{a_1}{10\times9\times7\times6\times4\times3}$

$n=8 \quad a_{11} = \frac{a_8}{11\times10} = 0$ ETC

$$\therefore y = x^0\left[a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \dots\right]$$

$$y = a_0 + a_1x + \frac{a_0}{3\times2}x^3 + \frac{a_1}{4\times3}x^4 + \frac{a_0}{6\times5\times3\times2}x^6 + \frac{a_1}{7\times6\times4\times3}x^7 + \dots$$



$$y_0 = a_0\left[1 + \frac{x^3}{3\times2} + \frac{x^6}{6\times5\times3\times2} + \frac{x^9}{9\times8\times6\times5\times3\times2} + \dots\right]$$

$$+ a_1\left[x + \frac{x^4}{4\times3} + \frac{x^7}{7\times6\times4\times3} + \frac{x^{10}}{10\times9\times7\times6\times4\times3} + \dots\right]$$

MANIPULATE FURTHER WITH GAMMA FUNCTIONS

- LOOKING AT $[x^9]$

$$\frac{x^9}{9\times8\times6\times5\times3\times2} = \frac{x^9}{(9\times6\times3)(8\times5\times2)} = \frac{x^9}{3^3(3\times2\times1)\times3^3\left(\frac{8}{3}\times\frac{5}{3}\times\frac{2}{3}\right)} = \frac{x^9}{9^3\times3!\times\frac{8}{3}\times\frac{5}{3}\times\frac{2}{3}}$$

$$= \frac{x^9\times\Gamma\left(\frac{2}{3}\right)}{9^3\times3!\times\frac{8}{3}\times\frac{5}{3}\times\frac{2}{3}\times\Gamma\left(\frac{2}{3}\right)} = \frac{x^9\,\Gamma\left(\frac{2}{3}\right)\leftarrow \text{FIXED}}{9^3\times3!\,\Gamma\left(\frac{11}{3}\right)\leftarrow \text{VARIABLE}}$$

← FOURTH TERM STARTING FROM ZERO

- LOOKING AT x^{10}

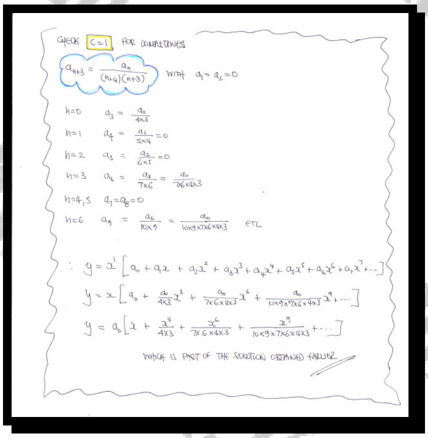
$$\frac{x^{10}}{10\times9\times7\times6\times4\times3} = \frac{x^{10}}{(10\times7\times4)(9\times6\times3)} = \frac{x^{10}}{3^3\left(\frac{10}{3}\times\frac{7}{3}\times\frac{4}{3}\right)\times3^3(3\times2\times1)} = \frac{x^{10}}{9^3\times3!\times\frac{10}{3}\times\frac{7}{3}\times\frac{4}{3}}$$

$$= \frac{x^{10}\,\Gamma\left(\frac{4}{3}\right)}{9^3\times3!\times\frac{10}{3}\times\frac{7}{3}\times\frac{4}{3}\times\Gamma\left(\frac{4}{3}\right)} = \frac{x^{10}\,\Gamma\left(\frac{4}{3}\right)\leftarrow \text{FIXED}}{9^3\times3!\,\Gamma\left(\frac{13}{3}\right)\leftarrow \text{VARIABLE}}$$

← FOURTH TERM STARTING FROM ZERO

THUS $y = A\sum_{r=0}^{\infty}\frac{\Gamma\left(\frac{2}{3}\right)\times x^{3r}}{9^r\times r!\,\Gamma\left(\frac{3r+2}{3}\right)} + B\sum_{r=0}^{\infty}\frac{\Gamma\left(\frac{4}{3}\right)\times x^{3r+1}}{9^r\times r!\times\Gamma\left(\frac{3r+4}{3}\right)}$

OR $y = \sum_{r=0}^{\infty}\left[\frac{x^{3r}}{9^r\times r!}\left[\frac{A\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{3r+2}{3}\right)} + \frac{Bx\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{3r+4}{3}\right)}\right]\right]$



CHECK $\boxed{c=1}$ FOR COMPLETENESS

$$a_{n+3} = \frac{a_n}{(n+4)(n+3)} \quad \text{WITH } a_1 = a_2 = 0$$

$n=0 \quad a_3 = \frac{a_0}{4\times3}$

$n=1 \quad a_4 = \frac{a_1}{5\times4} = 0$

$n=2 \quad a_5 = \frac{a_2}{6\times5} = 0$

$n=3 \quad a_6 = \frac{a_3}{7\times6} = \frac{a_0}{7\times6\times4\times3}$

$n=4,5 \quad a_7 = a_8 = 0$

$n=6 \quad a_9 = \frac{a_6}{10\times9} = \frac{a_0}{10\times9\times7\times6\times4\times3}$ ETC

$$\therefore y = x^1\left[a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + \dots\right]$$

$$y = x\left[a_0 + \frac{a_0}{4\times3}x^3 + \frac{a_0}{7\times6\times4\times3}x^6 + \frac{a_0}{10\times9\times7\times6\times4\times3}x^9 + \dots\right]$$

$$y = a_0\left[x + \frac{x^4}{4\times3} + \frac{x^6}{7\times6\times4\times3} + \frac{x^9}{10\times9\times7\times6\times4\times3} + \dots\right]$$

WHICH IS PART OF THE SOLUTION OBTAINED EARLIER

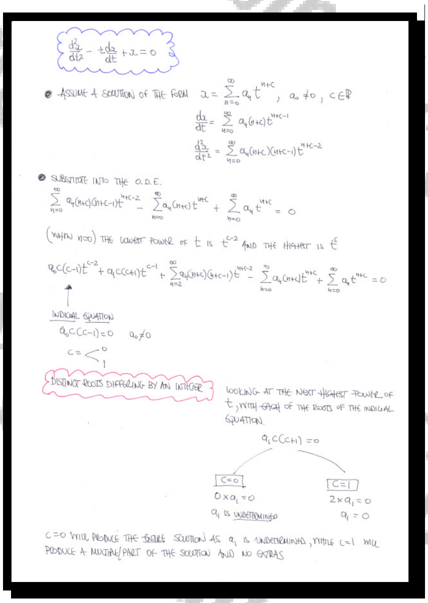
Question 6

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$\frac{d^2x}{dt^2} - t\frac{dx}{dt} + x = 0 .$$

Give the final answer in simplified Sigma notation.

$$\boxed{x = At + \frac{B}{\sqrt{\pi}}\sum_{n=0}^{\infty}\left[\frac{2^{n-1}\Gamma\left(n-\frac{1}{2}\right)t^{2n}}{(2n)!}\right]}$$



$$\frac{d^2x}{dt^2} - t\frac{dx}{dt} + x = 0$$

- ASSUME A SOLUTION OF THE FORM $x = \sum_{n=0}^{\infty} a_n t^{n+c}$, $a_0 \neq 0$, $c \in \mathbb{R}$

$$\frac{dx}{dt} = \sum_{n=0}^{\infty} a_n (n+c) t^{n+c-1}$$

$$\frac{d^2x}{dt^2} = \sum_{n=0}^{\infty} a_n (n+c)(n+c-1) t^{n+c-2}$$

- SUBSTITUTE INTO THE O.D.E.

$$\sum_{n=0}^{\infty} a_n(n+c)(n+c-1)t^{n+c-2} - \sum_{n=0}^{\infty} a_n(n+c)t^{n+c} + \sum_{n=0}^{\infty} a_n t^{n+c} = 0$$

(WHEN $n=0$) THE LOWEST POWER OF t IS t^{c-2} AND THE HIGHEST IS t^c

$$\underset{\uparrow}{a_0c(c-1)t^{c-2}} + a_1c(c+1)t^{c-1} + \sum_{n=2}^{\infty} a_n(n+c)(n+c-1)t^{n+c-2} - \sum_{n=0}^{\infty} a_n(n+c)t^{n+c} + \sum_{n=0}^{\infty} a_n t^{n+c} = 0$$

INDICIAL EQUATION

$a_0c(c-1) = 0 \quad a_0 \neq 0$

$c = \begin{cases} 0 \\ 1 \end{cases}$

DISTINCT ROOTS DIFFERING BY AN INTEGER

LOOKING AT THE NEXT HIGHEST POWER OF t, WITH ~~EACH~~ OF THE ROOTS OF THE INDICIAL EQUATION.

$a_1c(c+1) = 0$

$\boxed{c=0}$

$0 \times a_1 = 0$

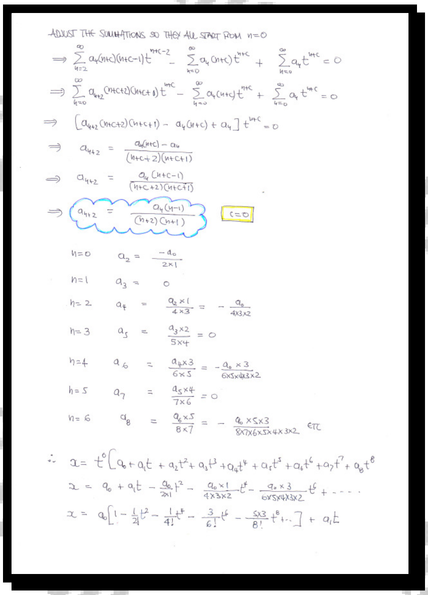
a_1 IS UNDETERMINED

$\boxed{c=1}$

$2 \times a_1 = 0$

$a_1 = 0$

$c=0$ WILL PRODUCE THE ENTIRE SOLUTION AS a_1 IS UNDETERMINED, WHILE $c=1$ WILL PRODUCE A MULTIPLE/PART OF THE SOLUTION AND NO EXTRAS



ADJUST THE SUMMATIONS SO THEY ALL START FROM $n=0$

$$\Rightarrow \sum_{n=2}^{\infty} a_n(n+c)(n+c-1)t^{n+c-2} - \sum_{n=0}^{\infty} a_n(n+c)t^{n+c} + \sum_{n=0}^{\infty} a_n t^{n+c} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} a_{n+2}(n+c+2)(n+c+1)t^{n+c} - \sum_{n=0}^{\infty} a_n(n+c)t^{n+c} + \sum_{n=0}^{\infty} a_n t^{n+c} = 0$$

$$\Rightarrow \left[a_{n+2}(n+c+2)(n+c+1) - a_n(n+c) + a_n\right]t^{n+c} = 0$$

$$\Rightarrow a_{n+2} = \frac{a_n(n+c) - a_n}{(n+c+2)(n+c+1)}$$

$$\Rightarrow a_{n+2} = \frac{a_n(n+c-1)}{(n+c+2)(n+c+1)}$$

$$\Rightarrow a_{n+2} = \frac{a_n(n-1)}{(n+2)(n+1)} \qquad \boxed{c=0}$$

$n=0 \quad a_2 = \frac{-a_0}{2\times1}$

$n=1 \quad a_3 = 0$

$n=2 \quad a_4 = \frac{a_2\times1}{4\times3} = -\frac{a_0}{4\times3\times2}$

$n=3 \quad a_5 = \frac{a_3\times2}{5\times4} = 0$

$n=4 \quad a_6 = \frac{a_4\times3}{6\times5} = -\frac{a_0\times3}{6\times5\times4\times3\times2}$

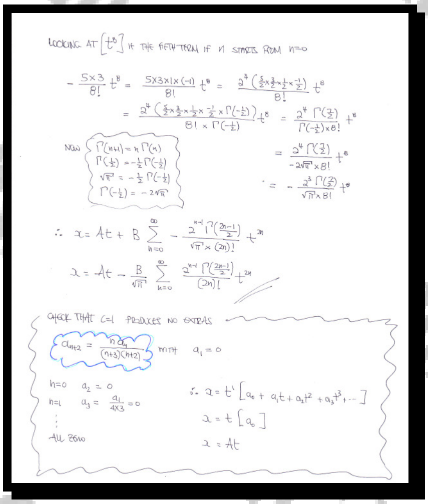
$n=5 \quad a_7 = \frac{a_5\times4}{7\times6} = 0$

$n=6 \quad a_8 = \frac{a_6\times5}{8\times7} = -\frac{a_0\times5\times3}{8\times7\times6\times5\times4\times3\times2}$ ETC

$$\therefore x = t^0\left[a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5 + a_6t^6 + a_7t^7 + a_8t^8\right.$$

$$x = a_0 + a_1t - \frac{a_0}{2\times1}t^2 - \frac{a_0\times1}{4\times3\times2}t^4 - \frac{a_0\times3}{6\times5\times4\times3\times2}t^6 + \dots$$

$$x = a_0\left[1 - \frac{1}{2!}t^2 - \frac{1}{4!}t^4 - \frac{3}{6!}t^6 - \frac{5\times3}{8!}t^8 + \dots\right] + a_1t$$



LOOKING AT $\left[t^8\right]$ IT THE FIFTH TERM IF n STARTS FROM $n=0$

$$-\frac{5\times3}{8!}t^8 = \frac{5\times3\times1\times(-1)}{8!}t^8 = \frac{2^4\left(\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\frac{-1}{2}\right)}{8!}t^8$$

$$= \frac{2^4\left(\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\frac{-1}{2}\times\Gamma\left(-\frac{1}{2}\right)\right)}{8!\times\Gamma\left(-\frac{1}{2}\right)}t^8 = \frac{2^4\,\Gamma\left(\frac{7}{2}\right)}{\Gamma\left(-\frac{1}{2}\right)\times8!}t^8$$

NOW

$\Gamma(n+1) = n\Gamma(n)$

$\Gamma\left(\frac{1}{2}\right) = -\frac{1}{2}\Gamma\left(-\frac{1}{2}\right)$

$\sqrt{\pi} = -\frac{1}{2}\Gamma\left(-\frac{1}{2}\right)$

$\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$

$$= \frac{2^4\,\Gamma\left(\frac{7}{2}\right)}{-2\sqrt{\pi}\times8!}t^8$$

$$= -\frac{2^3\,\Gamma\left(\frac{7}{2}\right)}{\sqrt{\pi}\times8!}t^8$$

$$\therefore x = At + B\sum_{n=0}^{\infty} -\frac{2^{n-1}\Gamma\left(\frac{2n-1}{2}\right)}{\sqrt{\pi}\times(2n)!}t^{2n}$$

$$x = At - \frac{B}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{2^{n-1}\Gamma\left(\frac{2n-1}{2}\right)}{(2n)!}t^{2n}$$

CHECK THAT $C=1$ PRODUCES NO EXTRAS

$$a_{n+2} = \frac{na_n}{(n+3)(n+2)} \quad \text{WITH} \quad a_1 = 0$$

$n=0 \quad a_2 = 0$

$n=1 \quad a_3 = \frac{a_1}{4\times3} = 0$

$\vdots$

ALL ZERO

$\therefore x = t^1\left[a_0 + a_1t + a_2t^2 + a_3t^3 + \dots\right]$

$x = t\left[a_0\right]$

$x = At$

FROBENIUS METHOD

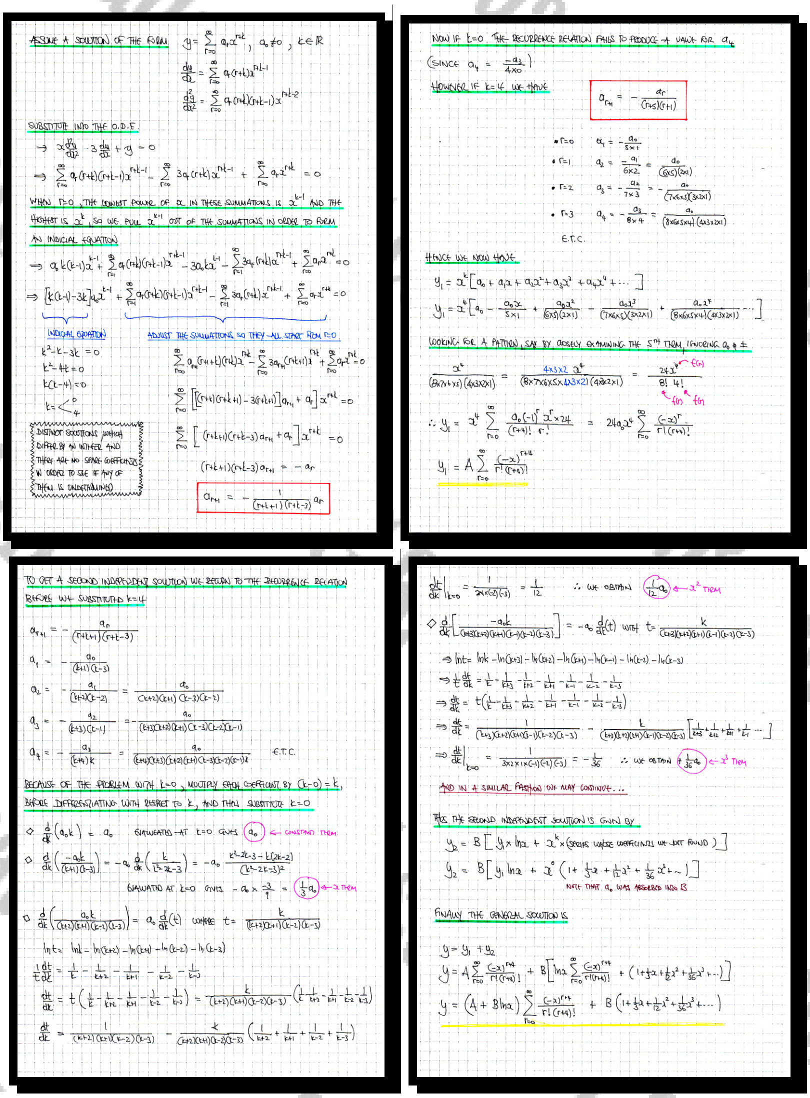
[2nd order O.D.E.s, where the roots of the indicial equation differ by an integer but no coefficient is undetermined]

Question 1

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$x\frac{d^2y}{dx^2}-3\frac{dy}{dx}+y=0.$$

$$\boxed{}\ ,\quad \boxed{y=\left(A+B\ln x\right)\sum_{r=0}^{\infty}\left[\frac{(-x)^{r+4}}{r!(r+4)!}\right]+B\left[1+\frac{1}{3}x+\frac{1}{12}x^2+\frac{1}{36}x^3+O\left(x^4\right)\right]}$$



ASSUME A SOLUTION OF THE FORM $y=\sum_{r=0}^{\infty}a_r x^{r+k}$, $a_0\neq 0$, $k\in\mathbb{R}$

$$\frac{dy}{dx}=\sum_{r=0}^{\infty}a_r(r+k)x^{r+k-1}$$

$$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty}a_r(r+k)(r+k-1)x^{r+k-2}$$

SUBSTITUTE INTO THE O.D.E.

$$\rightarrow x\frac{d^2y}{dx^2}-3\frac{dy}{dx}+y=0$$

$$\Rightarrow \sum_{r=0}^{\infty}a_r(r+k)(r+k-1)x^{r+k-1}-\sum_{r=0}^{\infty}3a_r(r+k)x^{r+k-1}+\sum_{r=0}^{\infty}a_r x^{r+k}=0$$

WHEN $r=0$, THE LOWEST POWER OF x IN THESE SUMMATIONS IS x^{k-1} AND THE HIGHEST IS x^k, SO WE PULL x^{k-1} OUT OF THE SUMMATIONS IN ORDER TO FORM AN INDICIAL EQUATION

$$\Rightarrow a_0k(k-1)x^{k-1}+\sum_{r=1}^{\infty}a_r(r+k)(r+k-1)x^{r+k-1}-3a_0kx^{k-1}-\sum_{r=1}^{\infty}3a_r(r+k)x^{r+k-1}+\sum_{r=0}^{\infty}a_rx^{r+k}=0$$

$$\Rightarrow \underbrace{[k(k-1)-3k]a_0x^{k-1}}_{\text{INDICIAL EQUATION}}+\underbrace{\sum_{r=1}^{\infty}a_r(r+k)(r+k-1)x^{r+k-1}-\sum_{r=1}^{\infty}3a_r(r+k)x^{r+k-1}+\sum_{r=0}^{\infty}a_rx^{r+k}}_{\text{ADJUST THE SUMMATIONS SO THEY ALL START FROM } r=0}=0$$

$$k^2-k-3k=0$$
$$k^2-4k=0$$
$$k(k-4)=0$$
$$k=\begin{cases}0\\4\end{cases}$$

DISTINCT SOLUTIONS WHICH DIFFER BY AN INTEGER AND THERE ARE NO SPARE COEFFICIENTS IN ORDER TO SEE IF ANY OF THEM IS UNDETERMINED

$$\sum_{r=0}^{\infty}a_{r+1}(r+1+k)(r+k)x^{r+k}-\sum_{r=0}^{\infty}3a_{r+1}(r+k+1)x^{r+k}+\sum_{r=0}^{\infty}a_rx^{r+k}=0$$

$$\sum_{r=0}^{\infty}\Big[\big[(r+k)(r+k+1)-3(r+k+1)\big]a_{r+1}+a_r\Big]x^{r+k}=0$$

$$\sum_{r=0}^{\infty}\Big[(r+k+1)(r+k-3)a_{r+1}+a_r\Big]x^{r+k}=0$$

$$(r+k+1)(r+k-3)a_{r+1}=-a_r$$

$$\boxed{a_{r+1}=-\frac{1}{(r+k+1)(r+k-3)}a_r}$$

NOW IF $k=0$ THE RECURRENCE RELATION FAILS TO PRODUCE A VALUE FOR a_4

(SINCE $a_4=\frac{-a_3}{4\times 0}$)

HOWEVER IF $k=4$ WE HAVE

$$\boxed{a_{r+1}=-\frac{a_r}{(r+5)(r+1)}}$$

- $r=0$ $\quad a_1=-\frac{a_0}{5\times 1}$
- $r=1$ $\quad a_2=-\frac{a_1}{6\times 2}=\frac{a_0}{(6\times5)(2\times1)}$
- $r=2$ $\quad a_3=-\frac{a_2}{7\times 3}=-\frac{a_0}{(7\times6\times5)(3\times2\times1)}$
- $r=3$ $\quad a_4=-\frac{a_3}{8\times 4}=\frac{a_0}{(8\times6\times5\times4)(4\times3\times2\times1)}$

E.T.C.

HENCE WE NOW HAVE

$$y_1=x^k\left[a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\dots\right]$$

$$y_1=x^4\left[a_0-\frac{a_0x}{5\times1}+\frac{a_0x^2}{(6\times5)(2\times1)}-\frac{a_0x^3}{(7\times6\times5)(3\times2\times1)}+\frac{a_0x^4}{(8\times6\times5\times4)(4\times3\times2\times1)}\dots\right]$$

LOOKING FOR A PATTERN, SAY BY CLOSELY EXAMINING THE 5TH TERM, IGNORING a_0 & $\pm$

$$\frac{x^4}{(8\times7\times6\times5)(4\times3\times2\times1)}=\frac{4\times3\times2\ x^4}{(8\times7\times6\times5\times4\times3\times2)(4\times3\times2\times1)}=\frac{24x^{4\ \leftarrow f(r)}}{\underset{\uparrow f(r)}{8!}\ \underset{\uparrow f(r)}{4!}}$$

$$\therefore y_1=x^4\sum_{r=0}^{\infty}\frac{a_0(-1)^rx^r\times24}{(r+4)!\ r!}=24a_0x^4\sum_{r=0}^{\infty}\frac{(-x)^r}{r!(r+4)!}$$

$$y_1=A\sum_{r=0}^{\infty}\frac{(-x)^{r+4}}{r!(r+4)!}$$

TO GET A SECOND INDEPENDENT SOLUTION WE RETURN TO THE RECURRENCE RELATION BEFORE WE SUBSTITUTED $k=4$

$$a_{r+1}=-\frac{a_r}{(r+k+1)(r+k-3)}$$

$$a_1=-\frac{a_0}{(k+1)(k-3)}$$

$$a_2=-\frac{a_1}{(k+2)(k-2)}=\frac{a_0}{(k+2)(k+1)(k-3)(k-2)}$$

$$a_3=-\frac{a_2}{(k+3)(k-1)}=-\frac{a_0}{(k+3)(k+2)(k+1)(k-3)(k-2)(k-1)}$$

$$a_4=-\frac{a_3}{(k+4)k}=\frac{a_0}{(k+4)(k+3)(k+2)(k+1)(k-3)(k-2)(k-1)k}\qquad \text{E.T.C.}$$

BECAUSE OF THE PROBLEM WITH $k=0$, MULTIPLY EACH COEFFICIENT BY $(k-0)=k$, BEFORE DIFFERENTIATING WITH RESPECT TO k, AND THEN SUBSTITUTE $k=0$

◇ $\frac{d}{dk}(a_0k)=a_0$ EVALUATED AT $k=0$ GIVES $\textcircled{a_0}$ ← CONSTANT TERM

◇ $\frac{d}{dk}\left(\frac{-a_0k}{(k+1)(k-3)}\right)=-a_0\frac{d}{dk}\left(\frac{k}{k^2-2k-3}\right)=-a_0\frac{k^2-2k-3-k(2k-2)}{(k^2-2k-3)^2}$

EVALUATED AT $k=0$ GIVES $-a_0\times\frac{-3}{9}=\textcircled{\frac{1}{3}a_0}$ ← x TERM

◇ $\frac{d}{dk}\left(\frac{a_0k}{(k+2)(k+1)(k-2)(k-3)}\right)=a_0\frac{d}{dk}(t)$ WHERE $t=\frac{k}{(k+2)(k+1)(k-2)(k-3)}$

$$\ln t=\ln k-\ln(k+2)-\ln(k+1)-\ln(k-2)-\ln(k-3)$$

$$\frac{1}{t}\frac{dt}{dk}=\frac{1}{k}-\frac{1}{k+2}-\frac{1}{k+1}-\frac{1}{k-2}-\frac{1}{k-3}$$

$$\frac{dt}{dk}=t\left(\frac{1}{k}-\frac{1}{k+2}-\frac{1}{k+1}-\frac{1}{k-2}-\frac{1}{k-3}\right)=\frac{k}{(k+2)(k+1)(k-2)(k-3)}\left(\frac{1}{k}-\frac{1}{k+2}-\frac{1}{k+1}-\frac{1}{k-2}-\frac{1}{k-3}\right)$$

$$\frac{dt}{dk}=\frac{1}{(k+2)(k+1)(k-2)(k-3)}-\frac{k}{(k+2)(k+1)(k-2)(k-3)}\left(\frac{1}{k+2}+\frac{1}{k+1}+\frac{1}{k-2}+\frac{1}{k-3}\right)$$

$$\left.\frac{dt}{dk}\right|_{k=0}=\frac{1}{2\times1\times(-2)(-3)}=\frac{1}{12}$$ $\therefore$ WE OBTAIN $\textcircled{\frac{1}{12}a_0}$ ← x^2 TERM

◇ $\frac{d}{dk}\left[\frac{-a_0k}{(k+3)(k+2)(k+1)(k-1)(k-2)(k-3)}\right]=-a_0\frac{d}{dk}(t)$ WITH $t=\frac{k}{(k+3)(k+2)(k+1)(k-1)(k-2)(k-3)}$

$$\Rightarrow \ln t=\ln k-\ln(k+3)-\ln(k+2)-\ln(k+1)-\ln(k-1)-\ln(k-2)-\ln(k-3)$$

$$\Rightarrow \frac{1}{t}\frac{dt}{dk}=\frac{1}{k}-\frac{1}{k+3}-\frac{1}{k+2}-\frac{1}{k+1}-\frac{1}{k-1}-\frac{1}{k-2}-\frac{1}{k-3}$$

$$\Rightarrow \frac{dt}{dk}=t\left(\frac{1}{k}-\frac{1}{k+3}-\frac{1}{k+2}-\frac{1}{k+1}-\frac{1}{k-1}-\frac{1}{k-2}-\frac{1}{k-3}\right)$$

$$\Rightarrow \frac{dt}{dk}=\frac{1}{(k+3)(k+2)(k+1)(k-1)(k-2)(k-3)}-\frac{k}{(k+3)(k+2)(k+1)(k-1)(k-2)(k-3)}\left[\frac{1}{k+3}+\frac{1}{k+2}+\frac{1}{k+1}+\frac{1}{k-1}\dots\right]$$

$$\Rightarrow \left.\frac{dt}{dk}\right|_{k=0}=\frac{1}{3\times2\times1\times(-1)(-2)(-3)}=-\frac{1}{36}$$ $\therefore$ WE OBTAIN $\textcircled{+\frac{1}{36}a_0}$ ← x^3 TERM

AND IN A SIMILAR FASHION WE MAY CONTINUE...

THUS THE SECOND INDEPENDENT SOLUTION IS GIVEN BY

$$y_2=B\left[y_1\times\ln x+x^k\times(\text{SERIES WHOSE COEFFICIENTS WE JUST FOUND})\right]$$

$$y_2=B\left[y_1\ln x+x^0\left(1+\frac{1}{3}x+\frac{1}{12}x^2+\frac{1}{36}x^3+\dots\right)\right]$$

NOTE THAT a_0 WAS ABSORBED INTO B

FINALLY THE GENERAL SOLUTION IS

$$y=y_1+y_2$$

$$y=A\sum_{r=0}^{\infty}\frac{(-x)^{r+4}}{r!(r+4)!}+B\left[\ln x\sum_{r=0}^{\infty}\frac{(-x)^{r+4}}{r!(r+4)!}+\left(1+\frac{1}{3}x+\frac{1}{12}x^2+\frac{1}{36}x^3+\dots\right)\right]$$

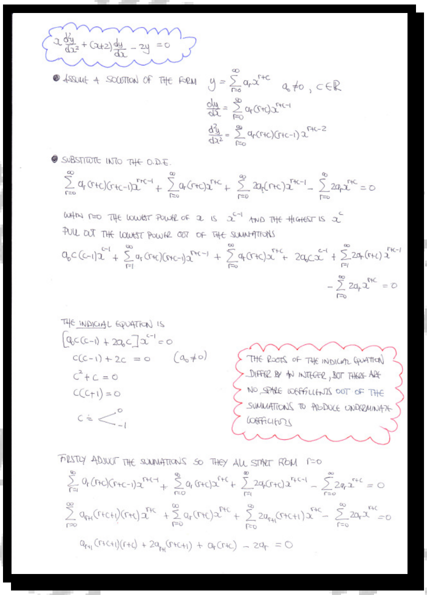
$$y=(A+B\ln x)\sum_{r=0}^{\infty}\frac{(-x)^{r+4}}{r!(r+4)!}+B\left(1+\frac{1}{3}x+\frac{1}{12}x^2+\frac{1}{36}x^3+\dots\right)$$

Question 2

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$x\frac{d^2y}{dx^2}+(x+2)\frac{dy}{dx}-2y=0\,.$$

$$\boxed{y=A\left[1+x+\frac{1}{6}x^2\right]+B\left[\left(1+x+\frac{1}{6}x^2\right)\ln x+\frac{1}{x}\left[1-4x-10x^2-\frac{31}{12}x^3+O\left(x^4\right)\right]\right]}$$



$$x\frac{d^2y}{dx^2}+(x+2)\frac{dy}{dx}-2y=0$$

- ASSUME A SOLUTION OF THE FORM $y=\sum_{r=0}^{\infty}a_rx^{r+c}$ $a_0\neq0$, $c\in\mathbb{R}$

$$\frac{dy}{dx}=\sum_{r=0}^{\infty}a_r(r+c)x^{r+c-1}$$

$$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty}a_r(r+c)(r+c-1)x^{r+c-2}$$

- SUBSTITUTE INTO THE O.D.E.

$$\sum_{r=0}^{\infty}a_r(r+c)(r+c-1)x^{r+c-1}+\sum_{r=0}^{\infty}a_r(r+c)x^{r+c}+\sum_{r=0}^{\infty}2a_r(r+c)x^{r+c-1}-\sum_{r=0}^{\infty}2a_rx^{r+c}=0$$

WHEN $r=0$ THE LOWEST POWER OF x IS x^{c-1} AND THE HIGHEST IS x^c

PULL OUT THE LOWEST POWER OUT OF THE SUMMATIONS

$$a_0c(c-1)x^{c-1}+\sum_{r=1}^{\infty}a_r(r+c)(r+c-1)x^{r+c-1}+\sum_{r=0}^{\infty}a_r(r+c)x^{r+c}+2a_0cx^{c-1}+\sum_{r=1}^{\infty}2a_r(r+c)x^{r+c-1}$$

$$-\sum_{r=0}^{\infty}2a_rx^{r+c}=0$$

THE INDICIAL EQUATION IS

$$\left[a_0c(c-1)+2a_0c\right]x^{c-1}=0$$

$$c(c-1)+2c=0 \qquad (a_0\neq0)$$

$$c^2+c=0$$

$$c(c+1)=0$$

$$c=\begin{cases}0\\-1\end{cases}$$

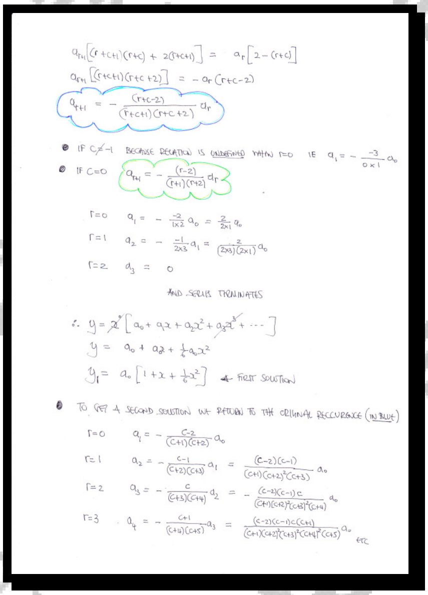
THE ROOTS OF THE INDICIAL EQUATION DIFFER BY AN INTEGER, BUT THERE ARE NO SPARE COEFFICIENTS OUT OF THE SUMMATIONS TO PRODUCE UNDERMINATE COEFFICIENTS

FIRSTLY ADJUST THE SUMMATIONS SO THEY ALL START FROM $r=0$

$$\sum_{r=1}^{\infty}a_r(r+c)(r+c-1)x^{r+c-1}+\sum_{r=0}^{\infty}a_r(r+c)x^{r+c}+\sum_{r=1}^{\infty}2a_r(r+c)x^{r+c-1}-\sum_{r=0}^{\infty}2a_rx^{r+c}=0$$

$$\sum_{r=0}^{\infty}a_{r+1}(r+c+1)(r+c)x^{r+c}+\sum_{r=0}^{\infty}a_r(r+c)x^{r+c}+\sum_{r=0}^{\infty}2a_{r+1}(r+c+1)x^{r+c}-\sum_{r=0}^{\infty}2a_rx^{r+c}=0$$

$$a_{r+1}(r+c+1)(r+c)+2a_{r+1}(r+c+1)+a_r(r+c)-2a_r=0$$



$$a_{r+1}\left[(r+c+1)(r+c)+2(r+c+1)\right]=a_r\left[2-(r+c)\right]$$

$$a_{r+1}\left[(r+c+1)(r+c+2)\right]=-a_r(r+c-2)$$

$$a_{r+1}=-\frac{(r+c-2)}{(r+c+1)(r+c+2)}a_r$$

- IF $c\neq-1$ BECAUSE RELATION IS UNDEFINED WHEN $r=0$ IE $a_1=-\frac{-3}{0\times1}a_0$
- IF $c=0$ $a_{r+1}=-\frac{(r-2)}{(r+1)(r+2)}a_r$

$r=0$ $a_1=-\frac{-2}{1\times2}a_0=\frac{2}{2\times1}a_0$

$r=1$ $a_2=-\frac{-1}{2\times3}a_1=\frac{2}{(2\times3)(2\times1)}a_0$

$r=2$ $a_3=0$

AND SERIES TERMINATES

$$\therefore\ y=\cancel{x^0}\left[a_0+a_1x+a_2x^2+\cancel{a_3x^3}+\cdots\right]$$

$$y=a_0+a_0x+\tfrac{1}{6}a_0x^2$$

$$y_1=a_0\left[1+x+\tfrac{1}{6}x^2\right]$$ ← FIRST SOLUTION

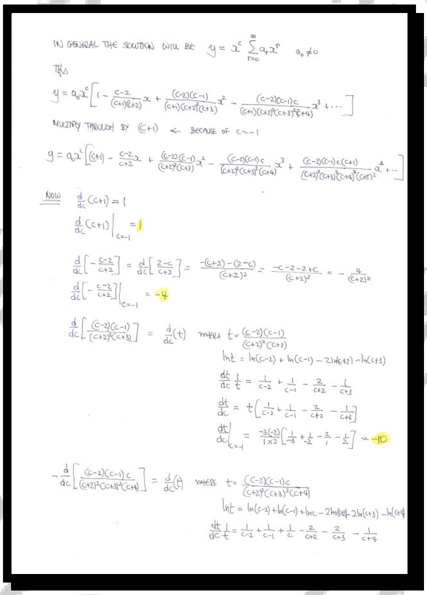
- TO GET A SECOND SOLUTION WE RETURN TO THE ORIGINAL RECCURENCE (IN BLUE)

$r=0$ $a_1=-\frac{c-2}{(c+1)(c+2)}a_0$

$r=1$ $a_2=-\frac{c-1}{(c+2)(c+3)}a_1=\frac{(c-2)(c-1)}{(c+1)(c+2)^2(c+3)}a_0$

$r=2$ $a_3=-\frac{c}{(c+3)(c+4)}a_2=-\frac{(c-2)(c-1)c}{(c+1)(c+2)^2(c+3)^2(c+4)}a_0$

$r=3$ $a_4=-\frac{c+1}{(c+4)(c+5)}a_3=\frac{(c-2)(c-1)c(c+1)}{(c+1)(c+2)^2(c+3)^2(c+4)^2(c+5)}a_0$ ETC



IN GENERAL THE SOLUTION WILL BE $y=x^c\sum_{r=0}^{\infty}a_rx^r$ $a_0\neq0$

THUS

$$y=a_0x^c\left[1-\frac{c-2}{(c+1)(c+2)}x+\frac{(c-2)(c-1)}{(c+1)(c+2)^2(c+3)}x^2-\frac{(c-2)(c-1)c}{(c+1)(c+2)^2(c+3)^2(c+4)}x^3+\cdots\right]$$

MULTIPLY THROUGH BY $(c+1)$ ← BECAUSE OF $c=-1$

$$y=a_0x^c\left[(c+1)-\frac{c-2}{c+2}x+\frac{(c-2)(c-1)}{(c+2)^2(c+3)}x^2-\frac{(c-2)(c-1)c}{(c+2)^2(c+3)^2(c+4)}x^3+\frac{(c-2)(c-1)c(c+1)}{(c+2)^2(c+3)^2(c+4)^2(c+5)^2}x^4+\cdots\right]$$

NOW $\frac{d}{dc}(c+1)=1$

$$\left.\frac{d}{dc}(c+1)\right|_{c=-1}=1$$

$$\frac{d}{dc}\left[-\frac{c-2}{c+2}\right]=\frac{d}{dc}\left[\frac{2-c}{c+2}\right]=\frac{-(c+2)-(2-c)}{(c+2)^2}=\frac{-c-2-2+c}{(c+2)^2}=-\frac{4}{(c+2)^2}$$

$$\left.\frac{d}{dc}\left[-\frac{c-2}{c+2}\right]\right|_{c=-1}=-4$$

$$\frac{d}{dc}\left[\frac{(c-2)(c-1)}{(c+2)^2(c+3)}\right]=\frac{d}{dc}(t) \quad \text{WHERE } t=\frac{(c-2)(c-1)}{(c+2)^2(c+3)}$$

$$\ln t=\ln(c-2)+\ln(c-1)-2\ln(c+2)-\ln(c+3)$$

$$\frac{dt}{dc}\frac{1}{t}=\frac{1}{c-2}+\frac{1}{c-1}-\frac{2}{c+2}-\frac{1}{c+3}$$

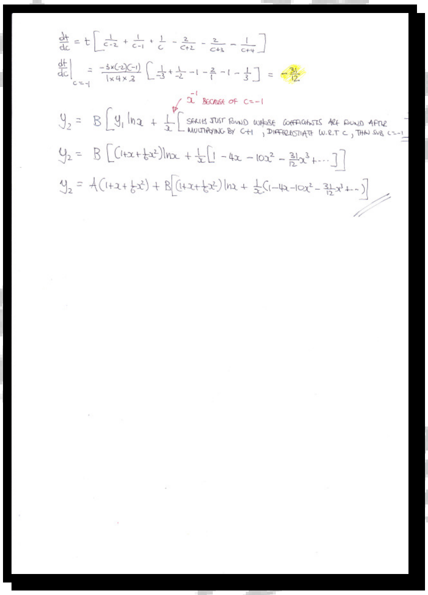
$$\frac{dt}{dc}=t\left[\frac{1}{c-2}+\frac{1}{c-1}-\frac{2}{c+2}-\frac{1}{c+3}\right]$$

$$\left.\frac{dt}{dc}\right|_{c=-1}=\frac{-3(-2)}{1\times2}\left[\frac{1}{-3}+\frac{1}{-2}-\frac{2}{1}-\frac{1}{2}\right]=-10$$

$$-\frac{d}{dc}\left[\frac{(c-2)(c-1)c}{(c+2)^2(c+3)^2(c+4)}\right]=\frac{d}{dc}(t) \quad \text{WHERE } t=\frac{(c-2)(c-1)c}{(c+2)^2(c+3)^2(c+4)}$$

$$\ln t=\ln(c-2)+\ln(c-1)+\ln c-2\ln(c+2)-2\ln(c+3)-\ln(c+4)$$

$$\frac{dt}{dc}\frac{1}{t}=\frac{1}{c-2}+\frac{1}{c-1}+\frac{1}{c}-\frac{2}{c+2}-\frac{2}{c+3}-\frac{1}{c+4}$$



$$\frac{dt}{dc}=t\left[\frac{1}{c-2}+\frac{1}{c-1}+\frac{1}{c}-\frac{2}{c+2}-\frac{2}{c+3}-\frac{1}{c+4}\right]$$

$$\left.\frac{dt}{dc}\right|_{c=-1}=\frac{-3\times(-2)(-1)}{1\times4\times3}\left[\frac{1}{-3}+\frac{1}{-2}-1-\frac{2}{1}-1-\frac{1}{3}\right]=-\frac{31}{12}$$

x^{-1} BECAUSE OF $c=-1$

$$y_2=B\left[y_1\ln x+\frac{1}{x}\left[\text{SERIES JUST FOUND WHOSE COEFFICIENTS ARE FOUND AFTER MULTIPLYING BY } c+1\text{, DIFFERENTIATE W.R.T } c\text{, THEN SUB } c=-1\right]\right]$$

$$y_2=B\left[\left(1+x+\tfrac{1}{6}x^2\right)\ln x+\frac{1}{x}\left[1-4x-10x^2-\frac{31}{12}x^3+\cdots\right]\right]$$

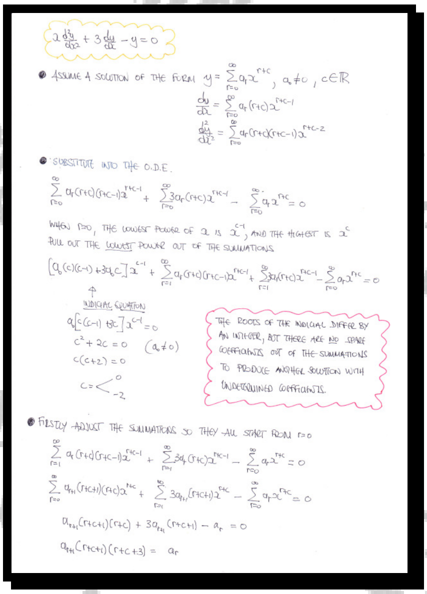
$$y_2=A\left(1+x+\tfrac{1}{6}x^2\right)+B\left[\left(1+x+\tfrac{1}{6}x^2\right)\ln x+\frac{1}{x}\left(1-4x-10x^2-\frac{31}{12}x^3+\cdots\right)\right]$$

Question 3

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$x\frac{d^2y}{dx^2}+3\frac{dy}{dx}-y=0\,.$$

$$\boxed{y=A\left[\sum_{r=0}^{\infty}\frac{x^r}{r!(r+2)!}\right]+B\left[\ln x\sum_{r=0}^{\infty}\frac{x^r}{r!(r+2)!}+\frac{1}{x^2}\left[1-x+\frac{1}{4}x^2+\frac{11}{36}x^3+O\left(x^4\right)\right]\right]}$$



$x\frac{d^2y}{dx^2}+3\frac{dy}{dx}-y=0$

- ASSUME A SOLUTION OF THE FORM $y=\sum_{r=0}^{\infty}a_rx^{r+c}$, $a_0\neq0$, $c\in\mathbb{R}$

$\frac{dy}{dx}=\sum_{r=0}^{\infty}a_r(r+c)x^{r+c-1}$

$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty}a_r(r+c)(r+c-1)x^{r+c-2}$

- SUBSTITUTE INTO THE O.D.E.

$$\sum_{r=0}^{\infty}a_r(r+c)(r+c-1)x^{r+c-1}+\sum_{r=0}^{\infty}3a_r(r+c)x^{r+c-1}-\sum_{r=0}^{\infty}a_rx^{r+c}=0$$

WHEN r=0, THE LOWEST POWER OF x IS x^{c-1}, AND THE HIGHEST IS x^c
PULL OUT THE LOWEST POWER OUT OF THE SUMMATIONS

$$[a_0(c)(c-1)+3a_0c]x^{c-1}+\sum_{r=1}^{\infty}a_r(r+c)(r+c-1)x^{r+c-1}+\sum_{r=1}^{\infty}3a_r(r+c)x^{r+c-1}-\sum_{r=0}^{\infty}a_rx^{r+c}=0$$

↑
INDICIAL EQUATION

$a_0[c(c-1)+3c]x^{c-1}=0$

$c^2+2c=0$ $(a_0\neq0)$

$c(c+2)=0$

$c=\begin{cases}0\\-2\end{cases}$

THE ROOTS OF THE INDICIAL DIFFER BY AN INTEGER, BUT THERE ARE NO SPARE COEFFICIENTS OUT OF THE SUMMATIONS TO PRODUCE ANOTHER SOLUTION WITH UNDETERMINED COEFFICIENTS.

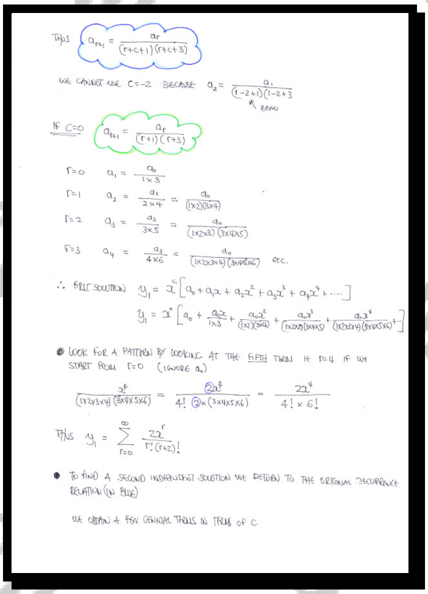
- FIRSTLY ADJUST THE SUMMATIONS SO THEY ALL START FROM r=0

$$\sum_{r=1}^{\infty}a_r(r+c)(r+c-1)x^{r+c-1}+\sum_{r=1}^{\infty}3a_r(r+c)x^{r+c-1}-\sum_{r=0}^{\infty}a_rx^{r+c}=0$$

$$\sum_{r=0}^{\infty}a_{r+1}(r+c+1)(r+c)x^{r+c}+\sum_{r=1}^{\infty}3a_{r+1}(r+c+1)x^{r+c}-\sum_{r=0}^{\infty}a_rx^{r+c}=0$$

$a_{r+1}(r+c+1)(r+c)+3a_{r+1}(r+c+1)-a_r=0$

$a_{r+1}(r+c+1)(r+c+3)=a_r$



THUS $\boxed{a_{r+1}=\frac{a_r}{(r+c+1)(r+c+3)}}$

WE CANNOT USE $c=-2$ BECAUSE $a_2=\frac{a_1}{(1-2+1)(1-2+3)}$
ZERO

IF $c=0$ $\boxed{a_{r+1}=\frac{a_r}{(r+1)(r+3)}}$

$r=0$ $a_1=\frac{a_0}{1\times3}$

$r=1$ $a_2=\frac{a_1}{2\times4}=\frac{a_0}{(1\times2)(3\times4)}$

$r=2$ $a_3=\frac{a_2}{3\times5}=\frac{a_0}{(1\times2\times3)(3\times4\times5)}$

$r=3$ $a_4=\frac{a_3}{4\times6}=\frac{a_0}{(1\times2\times3\times4)(3\times4\times5\times6)}$ ETC.

∴ FIRST SOLUTION $y_1=x^c\left[a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\dots\right]$

$y_1=x^0\left[a_0+\frac{a_0x}{1\times3}+\frac{a_0x^2}{(1\times2)(3\times4)}+\frac{a_0x^3}{(1\times2\times3)(3\times4\times5)}+\frac{a_0x^4}{(1\times2\times3\times4)(3\times4\times5\times6)}+\dots\right]$

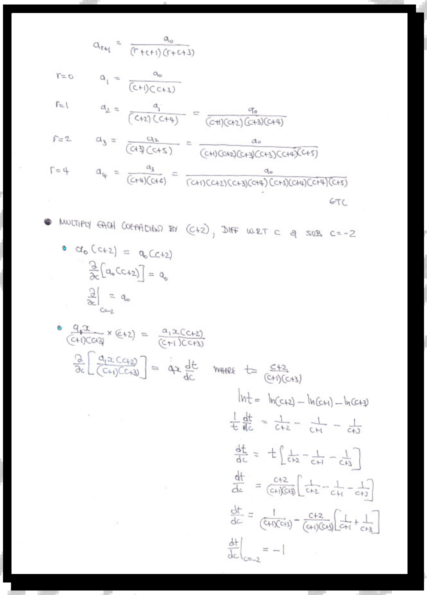
- LOOK FOR A PATTERN BY LOOKING AT THE FIFTH TERM IT r=4 IF WE START FROM r=0 (IGNORE a_0)

$$\frac{x^4}{(1\times2\times3\times4)(3\times4\times5\times6)}=\frac{\textcircled{2}x^4}{4!\,\textcircled{2}\times(3\times4\times5\times6)}=\frac{2x^4}{4!\times6!}$$

THUS $y_1=\sum_{r=0}^{\infty}\frac{2x^r}{r!(r+2)!}$

- TO FIND A SECOND INDEPENDENT SOLUTION WE RETURN TO THE ORIGINAL RECURRENCE RELATION (IN BLUE)

WE OBTAIN A FEW GENERAL TERMS IN TERMS OF c



$a_{r+1}=\frac{a_0}{(r+c+1)(r+c+3)}$

$r=0$ $a_1=\frac{a_0}{(c+1)(c+3)}$

$r=1$ $a_2=\frac{a_1}{(c+2)(c+4)}=\frac{a_0}{(c+1)(c+2)(c+3)(c+4)}$

$r=2$ $a_3=\frac{a_2}{(c+3)(c+5)}=\frac{a_0}{(c+1)(c+2)(c+3)(c+3)(c+4)(c+5)}$

$r=4$ $a_4=\frac{a_3}{(c+4)(c+6)}=\frac{a_0}{(c+1)(c+2)(c+3)(c+4)(c+3)(c+4)(c+4)(c+5)}$

ETC

- MULTIPLY EACH COEFFICIENT BY $(c+2)$, DIFF W.R.T c & SUB $c=-2$
  - $a_0(c+2)=a_0(c+2)$

    $\frac{\partial}{\partial c}[a_0(c+2)]=a_0$

    $\left.\frac{\partial}{\partial c}\right|_{c=-2}=a_0$
  - $\frac{a_0x}{(c+1)(c+3)}\times(c+2)=\frac{a_1x(c+2)}{(c+1)(c+3)}$

    $\frac{\partial}{\partial c}\left[\frac{a_1x(c+2)}{(c+1)(c+3)}\right]=a_1x\frac{dt}{dc}$ WHERE $t=\frac{c+2}{(c+1)(c+3)}$

$\ln t=\ln(c+2)-\ln(c+1)-\ln(c+3)$

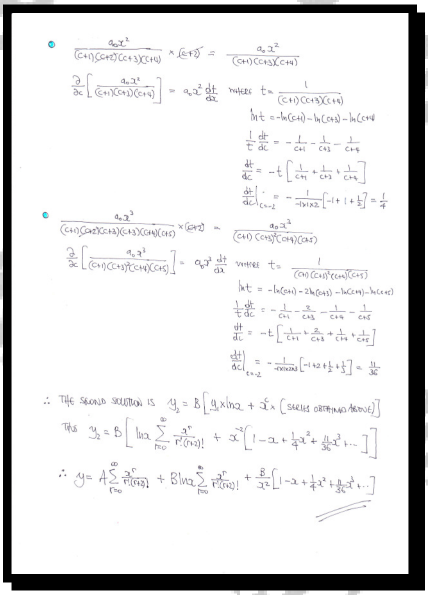
$\frac{1}{t}\frac{dt}{dc}=\frac{1}{c+2}-\frac{1}{c+1}-\frac{1}{c+3}$

$\frac{dt}{dc}=t\left[\frac{1}{c+2}-\frac{1}{c+1}-\frac{1}{c+3}\right]$

$\frac{dt}{dc}=\frac{c+2}{(c+1)(c+3)}\left[\frac{1}{c+2}-\frac{1}{c+1}-\frac{1}{c+3}\right]$

$\frac{dt}{dc}=\frac{1}{(c+1)(c+3)}-\frac{c+2}{(c+1)(c+3)}\left[\frac{1}{c+1}+\frac{1}{c+3}\right]$

$\left.\frac{dt}{dc}\right|_{c=-2}=-1$



- $\frac{a_0x^2}{(c+1)\cancel{(c+2)}(c+3)(c+4)}\times\cancel{(c+2)}=\frac{a_0x^2}{(c+1)(c+3)(c+4)}$

$\frac{\partial}{\partial c}\left[\frac{a_0x^2}{(c+1)(c+3)(c+4)}\right]=a_0x^2\frac{dt}{dx}$ WHERE $t=\frac{1}{(c+1)(c+3)(c+4)}$

$\ln t=-\ln(c+1)-\ln(c+3)-\ln(c+4)$

$\frac{1}{t}\frac{dt}{dc}=-\frac{1}{c+1}-\frac{1}{c+3}-\frac{1}{c+4}$

$\frac{dt}{dc}=-t\left[\frac{1}{c+1}+\frac{1}{c+3}+\frac{1}{c+4}\right]$

$\left.\frac{dt}{dc}\right|_{c=-2}=-\frac{1}{-1\times1\times2}\left[-1+1+\frac{1}{2}\right]=\frac{1}{4}$

- $\frac{a_0x^3}{(c+1)\cancel{(c+2)}(c+3)(c+3)(c+4)(c+5)}\times\cancel{(c+2)}=\frac{a_0x^3}{(c+1)(c+3)^2(c+4)(c+5)}$

$\frac{\partial}{\partial c}\left[\frac{a_0x^3}{(c+1)(c+3)^2(c+4)(c+5)}\right]=a_0x^3\frac{dt}{dx}$ WHERE $t=\frac{1}{(c+1)(c+3)^2(c+4)(c+5)}$

$\ln t=-\ln(c+1)-2\ln(c+3)-\ln(c+4)-\ln(c+5)$

$\frac{1}{t}\frac{dt}{dc}=-\frac{1}{c+1}-\frac{2}{c+3}-\frac{1}{c+4}-\frac{1}{c+5}$

$\frac{dt}{dc}=-t\left[\frac{1}{c+1}+\frac{2}{c+3}+\frac{1}{c+4}+\frac{1}{c+5}\right]$

$\left.\frac{dt}{dc}\right|_{c=-2}=-\frac{1}{-1\times1\times2\times3}\left[-1+2+\frac{1}{2}+\frac{1}{3}\right]=\frac{11}{36}$

∴ THE SECOND SOLUTION IS $y_2=B\left[y_1\times\ln x+x^c\times(\text{SERIES OBTAINED ABOVE})\right]$

THUS $y_2=B\left[\ln x\sum_{r=0}^{\infty}\frac{x^r}{r!(r+2)!}+x^{-2}\left[1-x+\frac{1}{4}x^2+\frac{11}{36}x^3+\dots\right]\right]$

∴ $y=A\sum_{r=0}^{\infty}\frac{x^r}{r!(r+2)!}+B\ln x\sum_{r=0}^{\infty}\frac{x^r}{r!(r+2)!}+\frac{B}{x^2}\left[1-x+\frac{1}{4}x^2+\frac{11}{36}x^3+\dots\right]$

FROBENIUS METHOD

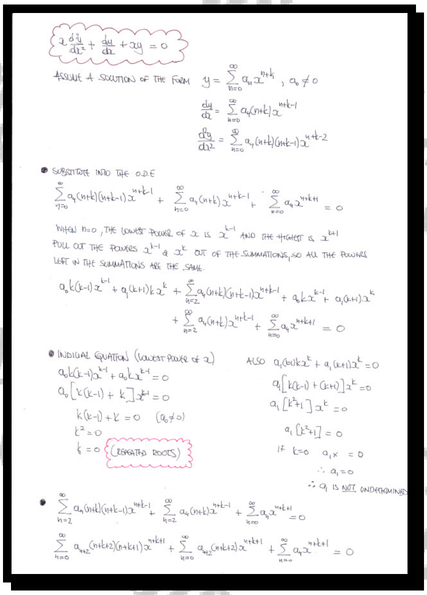
[2nd order O.D.E.s, where the indicial equation has repeated roots]

Question 1

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+xy=0\,.$$

$$\boxed{y=A\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{(n!)^2}\left(\frac{1}{2}x\right)^{2n}\right]+B\left[\ln x\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{(n!)^2}\left(\frac{1}{2}x\right)^{2n}\right]+\sum_{n=1}^{\infty}\sum_{m=1}^{n}\left[\frac{(-1)^n}{m(n!)^2}\left(\frac{1}{2}x\right)^{2n}\right]\right]}$$



$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+xy=0$$

ASSUME A SOLUTION OF THE FORM $y=\sum_{n=0}^{\infty}a_nx^{n+k}\,,\ a_0\neq 0$

$$\frac{dy}{dx}=\sum_{n=0}^{\infty}a_n(n+k)x^{n+k-1}$$

$$\frac{d^2y}{dx^2}=\sum_{n=0}^{\infty}a_n(n+k)(n+k-1)x^{n+k-2}$$

- SUBSTITUTE INTO THE O.D.E

$$\sum_{n=0}^{\infty}a_n(n+k)(n+k-1)x^{n+k-1}+\sum_{n=0}^{\infty}a_n(n+k)x^{n+k-1}+\sum_{n=0}^{\infty}a_nx^{n+k+1}=0$$

WHEN $n=0$, THE LOWEST POWER OF x IS x^{k-1} AND THE HIGHEST IS x^{k+1}
PULL OUT THE POWERS x^{k-1} & x^k OUT OF THE SUMMATIONS, SO ALL THE POWERS LEFT IN THE SUMMATIONS ARE THE SAME.

$$a_0k(k-1)x^{k-1}+a_1(k+1)kx^k+\sum_{n=2}^{\infty}a_n(n+k)(n+k-1)x^{n+k-1}+a_0kx^{k-1}+a_1(k+1)x^k$$
$$+\sum_{n=2}^{\infty}a_n(n+k)x^{n+k-1}+\sum_{n=0}^{\infty}a_nx^{n+k+1}=0$$

- INDICIAL EQUATION (LOWEST POWER OF x)

$$a_0k(k-1)x^{k-1}+a_0kx^{k-1}=0$$
$$a_0\left[k(k-1)+k\right]x^{k-1}=0$$
$$k(k-1)+k=0\quad(a_0\neq0)$$
$$k^2=0$$
$$k=0\ \text{(REPEATED ROOTS)}$$

ALSO $a_1(k+1)kx^k+a_1(k+1)x^k=0$

$$a_1\left[k(k-1)+(k+1)\right]x^k=0$$
$$a_1\left[k^2+1\right]x^k=0$$
$$a_1\left[k^2+1\right]=0$$

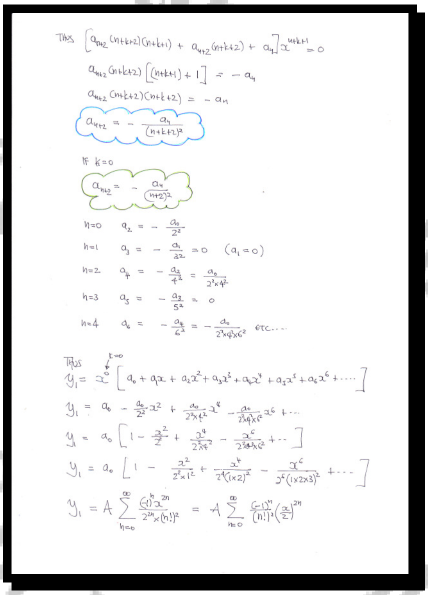
IF $k=0$ $\quad a_1x=0$

$$\therefore a_1=0$$

$\therefore a_1$ IS NOT UNDETERMINED

- $$\sum_{n=2}^{\infty}a_n(n+k)(n+k-1)x^{n+k-1}+\sum_{n=2}^{\infty}a_n(n+k)x^{n+k-1}+\sum_{n=0}^{\infty}a_nx^{n+k+1}=0$$

$$\sum_{n=0}^{\infty}a_{n+2}(n+k+2)(n+k+1)x^{n+k+1}+\sum_{n=0}^{\infty}a_{n+2}(n+k+2)x^{n+k+1}+\sum_{n=0}^{\infty}a_nx^{n+k+1}=0$$



THUS $\left[a_{n+2}(n+k+2)(n+k+1)+a_{n+2}(n+k+2)+a_n\right]x^{n+k+1}=0$

$$a_{n+2}(n+k+2)\left[(n+k+1)+1\right]=-a_n$$
$$a_{n+2}(n+k+2)(n+k+2)=-a_n$$
$$a_{n+2}=-\frac{a_n}{(n+k+2)^2}$$

IF $k=0$

$$a_{n+2}=-\frac{a_n}{(n+2)^2}$$

$n=0$ $\quad a_2=-\frac{a_0}{2^2}$

$n=1$ $\quad a_3=-\frac{a_1}{3^2}=0\quad(a_1=0)$

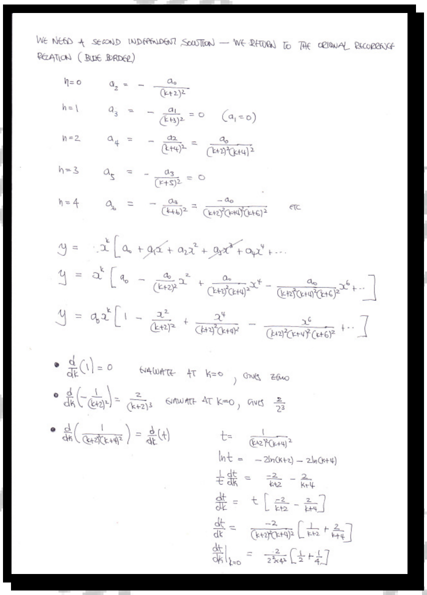
$n=2$ $\quad a_4=-\frac{a_2}{4^2}=\frac{a_0}{2^2\times4^2}$

$n=3$ $\quad a_5=-\frac{a_3}{5^2}=0$

$n=4$ $\quad a_6=-\frac{a_4}{6^2}=-\frac{a_0}{2^2\times4^2\times6^2}$ ETC....

THUS

$$y_1=\overset{k=0}{x^0}\left[a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6+\dots\right]$$
$$y_1=a_0-\frac{a_0}{2^2}x^2+\frac{a_0}{2^2\times4^2}x^4-\frac{a_0}{2^2\times4^2\times6^2}x^6+\dots$$
$$y_1=a_0\left[1-\frac{x^2}{2^2}+\frac{x^4}{2^2\times4^2}-\frac{x^6}{2^2\times4^2\times6^2}+\dots\right]$$
$$y_1=a_0\left[1-\frac{x^2}{2^2\times1^2}+\frac{x^4}{2^4(1\times2)^2}-\frac{x^6}{2^6(1\times2\times3)^2}+\dots\right]$$
$$y_1=A\sum_{n=0}^{\infty}\frac{(-1)^nx^{2n}}{2^{2n}\times(n!)^2}=A\sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{x}{2}\right)^{2n}$$



WE NEED A SECOND INDEPENDENT SOLUTION — WE RETURN TO THE ORIGINAL RECURRENCE RELATION (BLUE BORDER)

$n=0$ $\quad a_2=-\frac{a_0}{(k+2)^2}$

$n=1$ $\quad a_3=-\frac{a_1}{(k+3)^2}=0\quad(a_1=0)$

$n=2$ $\quad a_4=-\frac{a_2}{(k+4)^2}=\frac{a_0}{(k+2)^2(k+4)^2}$

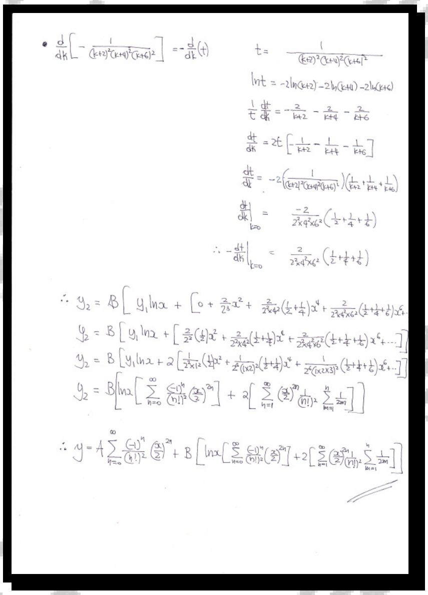
$n=3$ $\quad a_5=-\frac{a_3}{(k+5)^2}=0$

$n=4$ $\quad a_6=-\frac{a_4}{(k+6)^2}=\frac{-a_0}{(k+2)^2(k+4)^2(k+6)^2}$ ETC

$$y=x^k\left[a_0+\cancel{a_1x}+a_2x^2+\cancel{a_3x^3}+a_4x^4+\dots\right.$$
$$y=x^k\left[a_0-\frac{a_0}{(k+2)^2}x^2+\frac{a_0}{(k+2)^2(k+4)^2}x^4-\frac{a_0}{(k+2)^2(k+4)^2(k+6)^2}x^6+\dots\right]$$
$$y=a_0x^k\left[1-\frac{x^2}{(k+2)^2}+\frac{x^4}{(k+2)^2(k+4)^2}-\frac{x^6}{(k+2)^2(k+4)^2(k+6)^2}+\dots\right]$$

- $\frac{d}{dk}(1)=0$ EVALUATE AT $k=0$, GIVES ZERO
- $\frac{d}{dk}\left(-\frac{1}{(k+2)^2}\right)=\frac{2}{(k+2)^3}$ EVALUATE AT $k=0$, GIVES $\frac{2}{2^3}$
- $\frac{d}{dk}\left(\frac{1}{(k+2)^2(k+4)^2}\right)=\frac{d}{dk}(t)$

$$t=\frac{1}{(k+2)^2(k+4)^2}$$
$$\ln t=-2\ln(k+2)-2\ln(k+4)$$
$$\frac{1}{t}\frac{dt}{dk}=\frac{-2}{k+2}-\frac{2}{k+4}$$
$$\frac{dt}{dk}=t\left[\frac{-2}{k+2}-\frac{2}{k+4}\right]$$
$$\frac{dt}{dk}=\frac{-2}{(k+2)^2(k+4)^2}\left[\frac{1}{k+2}+\frac{2}{k+4}\right]$$
$$\left.\frac{dt}{dk}\right|_{k=0}=\frac{2}{2^2\times4^2}\left[\frac{1}{2}+\frac{1}{4}\right]$$



- $\frac{d}{dk}\left[-\frac{1}{(k+2)^2(k+4)^2(k+6)^2}\right]=-\frac{d}{dk}(t)$

$$t=\frac{1}{(k+2)^2(k+4)^2(k+6)^2}$$
$$\ln t=-2\ln(k+2)-2\ln(k+4)-2\ln(k+6)$$
$$\frac{1}{t}\frac{dt}{dk}=-\frac{2}{k+2}-\frac{2}{k+4}-\frac{2}{k+6}$$
$$\frac{dt}{dk}=2t\left[-\frac{1}{k+2}-\frac{1}{k+4}-\frac{1}{k+6}\right]$$
$$\frac{dt}{dk}=-2\left(\frac{1}{(k+2)^2(k+4)^2(k+6)^2}\right)\left(\frac{1}{k+2}+\frac{1}{k+4}+\frac{1}{k+6}\right)$$
$$\left.\frac{dt}{dk}\right|_{k=0}=\frac{-2}{2^2\times4^2\times6^2}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}\right)$$
$$\therefore\left.-\frac{dt}{dk}\right|_{k=0}=\frac{2}{2^2\times4^2\times6^2}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}\right)$$

$$\therefore y_2=B\left[y_1\ln x+\left[0+\frac{2}{2^3}x^2+\frac{2}{2^2\times4^2}\left(\frac{1}{2}+\frac{1}{4}\right)x^4+\frac{2}{2^2\times4^2\times6^2}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}\right)x^6+\dots\right.\right.$$
$$y_2=B\left[y_1\ln x+\left[\frac{2}{2^2}\left(\frac{1}{2}\right)x^2+\frac{2}{2^2\times4^2}\left(\frac{1}{2}+\frac{1}{4}\right)x^4+\frac{2}{2^2\times4^2\times6^2}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}\right)x^6+\dots\right]\right]$$
$$y_2=B\left[y_1\ln x+2\left[\frac{1}{2^2\times1^2}\left(\frac{1}{2}\right)x^2+\frac{1}{2^4(1\times2)^2}\left(\frac{1}{2}+\frac{1}{4}\right)x^4+\frac{1}{2^6(1\times2\times3)^2}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}\right)x^6+\dots\right]\right]$$
$$y_2=B\left[\ln x\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{x}{2}\right)^{2n}\right]+2\left[\sum_{n=1}^{\infty}\left(\frac{x}{2}\right)^{2n}\frac{1}{(n!)^2}\sum_{m=1}^{n}\frac{1}{2m}\right]\right]$$

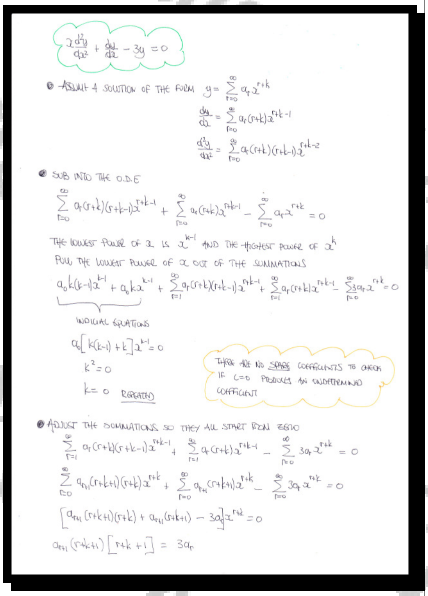
$$\therefore y=A\sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{x}{2}\right)^{2n}+B\left[\ln x\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{x}{2}\right)^{2n}\right]+2\left[\sum_{n=1}^{\infty}\left(\frac{x}{2}\right)^{2n}\frac{1}{(n!)^2}\sum_{m=1}^{n}\frac{1}{2m}\right]\right]$$

Question 2

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}-3y=0.$$

$$\boxed{y=A\sum_{r=0}^{\infty}\left[\frac{(3x)^r}{(r!)^2}\right]+B\left[\ln x\sum_{r=0}^{\infty}\left[\frac{(3x)^r}{(r!)^2}\right]+\sum_{n=1}^{\infty}\sum_{m=1}^{n}\left[\frac{(3x)^r}{m(n!)^2}\right]\right]}$$



$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}-3y=0$$

- ASSUME A SOLUTION OF THE FORM $y=\sum_{r=0}^{\infty}a_r x^{r+k}$

$$\frac{dy}{dx}=\sum_{r=0}^{\infty}a_r(r+k)x^{r+k-1}$$

$$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty}a_r(r+k)(r+k-1)x^{r+k-2}$$

- SUB INTO THE O.D.E

$$\sum_{r=0}^{\infty}a_r(r+k)(r+k-1)x^{r+k-1}+\sum_{r=0}^{\infty}a_r(r+k)x^{r+k-1}-\sum_{r=0}^{\infty}a_r x^{r+k}=0$$

THE LOWEST POWER OF x IS x^{k-1} AND THE HIGHEST POWER OF x^k

PULL THE LOWEST POWER OF x OUT OF THE SUMMATIONS

$$\underbrace{a_0k(k-1)x^{k-1}+a_0kx^{k-1}}_{\text{INDICIAL EQUATIONS}}+\sum_{r=1}^{\infty}a_r(r+k)(r+k-1)x^{r+k-1}+\sum_{r=1}^{\infty}a_r(r+k)x^{r+k-1}-\sum_{r=0}^{\infty}3a_rx^{r+k}=0$$

$$a_0\left[k(k-1)+k\right]x^{k-1}=0$$

$$k^2=0$$

$k=0$ REPEATED

THERE ARE NO SPARE COEFFICIENTS TO CHECK IF $k=0$ PRODUCES AN UNDETERMINED COEFFICIENT

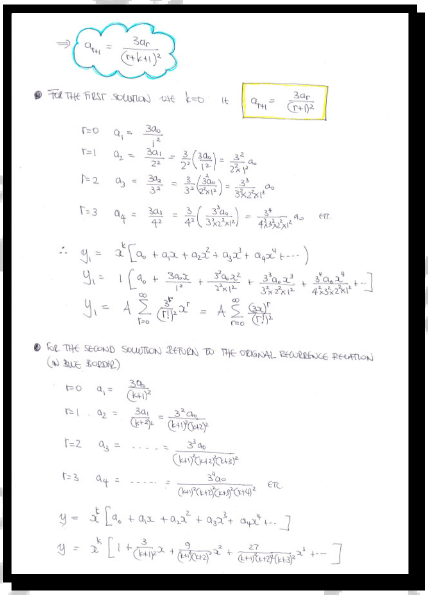
- ADJUST THE SUMMATIONS SO THEY ALL START FROM ZERO

$$\sum_{r=1}^{\infty}a_r(r+k)(r+k-1)x^{r+k-1}+\sum_{r=1}^{\infty}a_r(r+k)x^{r+k-1}-\sum_{r=0}^{\infty}3a_rx^{r+k}=0$$

$$\sum_{r=0}^{\infty}a_{r+1}(r+k+1)(r+k)x^{r+k}+\sum_{r=0}^{\infty}a_{r+1}(r+k+1)x^{r+k}-\sum_{r=0}^{\infty}3a_rx^{r+k}=0$$

$$\left[a_{r+1}(r+k+1)(r+k)+a_{r+1}(r+k+1)-3a_r\right]x^{r+k}=0$$

$$a_{r+1}(r+k+1)\left[r+k+1\right]=3a_r$$



$$\Rightarrow a_{r+1}=\frac{3a_r}{(r+k+1)^2}$$

- FOR THE FIRST SOLUTION USE $k=0$ IE $\boxed{a_{r+1}=\frac{3a_r}{(r+1)^2}}$

$r=0$ $a_1=\frac{3a_0}{1^2}$

$r=1$ $a_2=\frac{3a_1}{2^2}=\frac{3}{2^2}\left(\frac{3a_0}{1^2}\right)=\frac{3^2}{2^2\times1^2}a_0$

$r=2$ $a_3=\frac{3a_2}{3^2}=\frac{3}{3^2}\left(\frac{3^2a_0}{2^2\times1^2}\right)=\frac{3^3}{3^2\times2^2\times1^2}a_0$

$r=3$ $a_4=\frac{3a_3}{4^2}=\frac{3}{4^2}\left(\frac{3^3a_0}{3^2\times2^2\times1^2}\right)=\frac{3^4}{4^2\times3^2\times2^2\times1^2}a_0$ ETC.

$$\therefore y_1=x^k\left[a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\dots\right)$$

$$y_1=1\left[a_0+\frac{3a_0x}{1^2}+\frac{3^2a_0x^2}{2^2\times1^2}+\frac{3^3a_0x^3}{3^2\times2^2\times1^2}+\frac{3^4a_0x^4}{4^2\times3^2\times2^2\times1^2}+\dots\right]$$

$$y_1=A\sum_{r=0}^{\infty}\frac{3^r}{(r!)^2}x^r=A\sum_{r=0}^{\infty}\frac{(3x)^r}{(r!)^2}$$

- FOR THE SECOND SOLUTION RETURN TO THE ORIGINAL RECURRENCE RELATION (IN BLUE BORDER)

$r=0$ $a_1=\frac{3a_0}{(k+1)^2}$

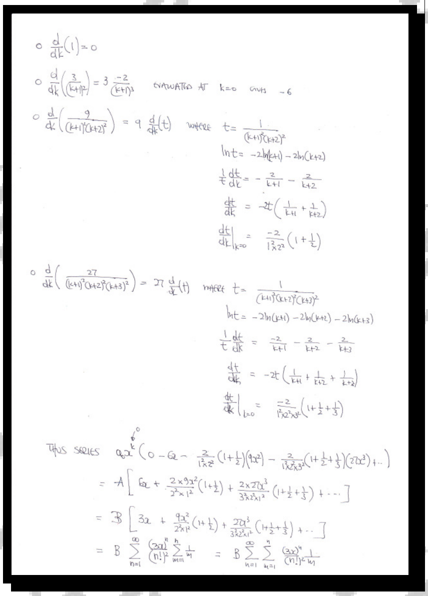
$r=1$ $a_2=\frac{3a_1}{(k+2)^2}=\frac{3^2a_0}{(k+1)^2(k+2)^2}$

$r=2$ $a_3=\dots\dots=\frac{3^3a_0}{(k+1)^2(k+2)^2(k+3)^2}$

$r=3$ $a_4=\dots\dots=\frac{3^4a_0}{(k+1)^2(k+2)^2(k+3)^2(k+4)^2}$ ETC.

$$y=x^k\left[a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\dots\right]$$

$$y=x^k\left[1+\frac{3}{(k+1)^2}x+\frac{9}{(k+1)^2(k+2)^2}x^2+\frac{27}{(k+1)^2(k+2)^2(k+3)^2}x^3+\dots\right]$$



- $\frac{d}{dk}(1)=0$
- $\frac{d}{dk}\left(\frac{3}{(k+1)^2}\right)=3\frac{-2}{(k+1)^3}$ EVALUATED AT $k=0$ GIVES -6
- $\frac{d}{dk}\left(\frac{9}{(k+1)^2(k+2)^2}\right)=9\frac{d}{dk}(t)$ WHERE $t=\frac{1}{(k+1)^2(k+2)^2}$

$$\ln t=-2\ln(k+1)-2\ln(k+2)$$

$$\frac{1}{t}\frac{dt}{dk}=-\frac{2}{k+1}-\frac{2}{k+2}$$

$$\frac{dt}{dk}=-2t\left(\frac{1}{k+1}+\frac{1}{k+2}\right)$$

$$\left.\frac{dt}{dk}\right|_{k=0}=\frac{-2}{1^2\times2^2}\left(1+\frac{1}{2}\right)$$

- $\frac{d}{dk}\left(\frac{27}{(k+1)^2(k+2)^2(k+3)^2}\right)=27\frac{d}{dk}(t)$ WHERE $t=\frac{1}{(k+1)^2(k+2)^2(k+3)^2}$

$$\ln t=-2\ln(k+1)-2\ln(k+2)-2\ln(k+3)$$

$$\frac{1}{t}\frac{dt}{dk}=\frac{-2}{k+1}-\frac{2}{k+2}-\frac{2}{k+3}$$

$$\frac{dt}{dk}=-2t\left(\frac{1}{k+1}+\frac{1}{k+2}+\frac{1}{k+3}\right)$$

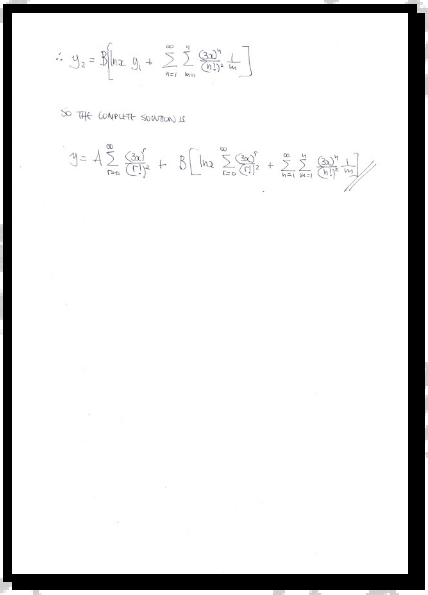
$$\left.\frac{dt}{dk}\right|_{k=0}=\frac{-2}{1^2\times2^2\times3^2}\left(1+\frac{1}{2}+\frac{1}{3}\right)$$

THUS SERIES $a_0\overset{0}{\cancel{x^k}}\left(0-6x-\frac{2}{1^2\times2^2}\left(1+\frac{1}{2}\right)(9x^2)-\frac{2}{1^2\times2^2\times3^2}\left(1+\frac{1}{2}+\frac{1}{3}\right)(27x^3)+\dots\right)$

$$=-A\left[6x+\frac{2\times9x^2}{2^2\times1^2}\left(1+\frac{1}{2}\right)+\frac{2\times27x^3}{3^2\times2^2\times1^2}\left(1+\frac{1}{2}+\frac{1}{3}\right)+\dots\right]$$

$$=B\left[3x+\frac{9x^2}{2^2\times1^2}\left(1+\frac{1}{2}\right)+\frac{27x^3}{3^2\times2^2\times1^2}\left(1+\frac{1}{2}+\frac{1}{3}\right)+\dots\right]$$

$$=B\sum_{n=1}^{\infty}\frac{(3x)^n}{(n!)^2}\sum_{m=1}^{n}\frac{1}{m}=B\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{(3x)^n}{(n!)^2}\frac{1}{m}$$



$$\therefore y_2=B\left[\ln x\,y_1+\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{(3x)^n}{(n!)^2}\frac{1}{m}\right]$$

SO THE COMPLETE SOLUTION IS

$$y=A\sum_{r=0}^{\infty}\frac{(3x)^r}{(r!)^2}+B\left[\ln x\sum_{r=0}^{\infty}\frac{(3x)^r}{(r!)^2}+\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{(3x)^n}{(n!)^2}\frac{1}{m}\right]$$