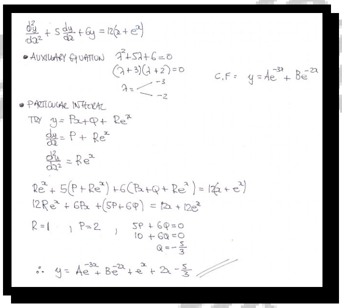
DIFFERENTIAL EQUATIONS

Question 1 (**)

Find a general solution of the differential equation

$$\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=12\left(x+e^x\right).$$

$$\boxed{y=Ae^{-3x}+Be^{-2x}+e^x+2x-\frac{5}{3}}$$



$$\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=12(x+e^x)$$

- AUXILIARY EQUATION $\lambda^2+5\lambda+6=0$
  $(\lambda+3)(\lambda+2)=0$
  $\lambda = -3, -2$
  C.F. $y=Ae^{-3x}+Be^{-2x}$
- PARTICULAR INTEGRAL
  TRY $y=Px+Q+Re^x$
  $\frac{dy}{dx}=P+Re^x$
  $\frac{d^2y}{dx^2}=Re^x$

$Re^x+5(P+Re^x)+6(Px+Q+Re^x)=12(x+e^x)$

$12Re^x+6Px+(5P+6Q)=12x+12e^x$

$R=1$, $P=2$, $5P+6Q=0$, $10+6Q=0$, $Q=-\frac{5}{3}$

$\therefore\ y=Ae^{-3x}+Be^{-2x}+e^x+2x-\frac{5}{3}$

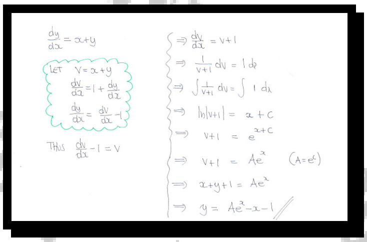
Question 2 (**)

By using a suitable substitution find a general solution of the differential equation

$$\frac{dy}{dx}=x+y,$$

giving the answer in the form $y=f(x)$.

$$\boxed{y=Ae^x-x-1}$$



$\frac{dy}{dx}=x+y$

LET $v=x+y$
$\frac{dv}{dx}=1+\frac{dy}{dx}$
$\frac{dy}{dx}=\frac{dv}{dx}-1$

THUS $\frac{dv}{dx}-1=v$

$\Rightarrow \frac{dv}{dx}=v+1$
$\Rightarrow \frac{1}{v+1}\,dv=1\,dx$
$\Rightarrow \int\frac{1}{v+1}\,dv=\int 1\,dx$
$\Rightarrow \ln|v+1|=x+C$
$\Rightarrow v+1=e^{x+C}$
$\Rightarrow v+1=Ae^x \quad (A=e^C)$
$\Rightarrow x+y+1=Ae^x$
$\Rightarrow y=Ae^x-x-1$

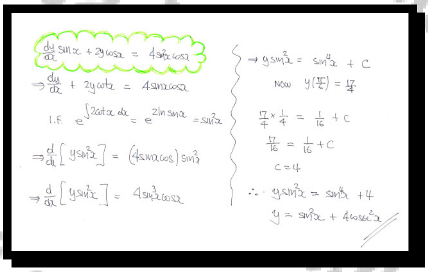
Question 3 (**)

Solve the differential equation

$$\frac{dy}{dx}\sin x + 2y\cos x = 4\sin^2 x\cos x, \quad y\left(\frac{1}{6}\pi\right) = \frac{17}{4}.$$

Give the answer in the form $y = f(x)$.

$$\boxed{y = \sin^2 x + 4\operatorname{cosec}^2 x}$$


$\frac{dy}{dx}\sin x + 2y\cos x = 4\sin^2 x\cos x$

$\Rightarrow \frac{dy}{dx} + 2y\cot x = 4\sin x\cos x$

I.F. $e^{\int 2\cot x\,dx} = e^{2\ln\sin x} = \sin^2 x$

$\Rightarrow \frac{d}{dx}\left[y\sin^2 x\right] = (4\sin x\cos x)\sin^2 x$

$\Rightarrow \frac{d}{dx}\left[y\sin^2 x\right] = 4\sin^3 x\cos x$

$\Rightarrow y\sin^2 x = \sin^4 x + C$

Now $y\left(\frac{\pi}{6}\right) = \frac{17}{4}$

$\frac{17}{4}\times\frac{1}{4} = \frac{1}{16} + C$

$\frac{17}{16} = \frac{1}{16} + C$

$C = 4$

$\therefore\ y\sin^2 x = \sin^4 x + 4$

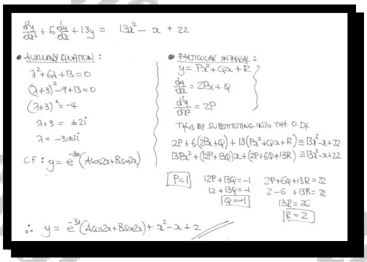
$y = \sin^2 x + 4\operatorname{cosec}^2 x$

Question 4 (**)

Find a general solution of the differential equation

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 13y = 13x^2 - x + 22.$$

$$\boxed{y = e^{-3x}\left(A\cos 2x + B\sin 2x\right) + x^2 - x + 2}$$


$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 13y = 13x^2 - x + 22$

• AUXILIARY EQUATION:

$\lambda^2 + 6\lambda + 13 = 0$

$(\lambda+3)^2 - 9 + 13 = 0$

$(\lambda+3)^2 = -4$

$\lambda + 3 = \pm 2i$

$\lambda = -3 \pm 2i$

C.F: $y = e^{-3x}(A\cos 2x + B\sin 2x)$

• PARTICULAR INTEGRAL:

$y = Px^2 + Qx + R$

$\frac{dy}{dx} = 2Px + Q$

$\frac{d^2y}{dx^2} = 2P$

THUS BY SUBSTITUTING INTO THE O.D.E

$2P + 6(2Px + Q) + 13(Px^2 + Qx + R) \equiv 13x^2 - x + 22$

$13Px^2 + (12P + 13Q)x + (2P + 6Q + 13R) \equiv 13x^2 - x + 22$

$\boxed{P = 1}$ $\quad 12P + 13Q = -1$, $12 + 13Q = -1$, $\boxed{Q = -1}$ $\quad 2P + 6Q + 13R = 22$, $2 - 6 + 13R = 22$, $13R = 26$, $\boxed{R = 2}$

$\therefore\ y = e^{-3x}(A\cos 2x + B\sin 2x) + x^2 - x + 2$

Question 5 (**)

$$\frac{dy}{dx}\sin x = \sin x \sin 2x + y\cos x\,.$$

Given that $y = \dfrac{3}{2}$ at $x = \dfrac{\pi}{6}$, find the exact value of y at $x = \dfrac{\pi}{4}$.

$\boxed{1+\sqrt{2}}$

$\frac{dy}{dx}\sin x = \sin x \sin 2x + y\cos x$

$\Rightarrow \frac{dy}{dx} = \sin 2x + y\cot x$

$\Rightarrow \frac{dy}{dx} - y\cot x = \sin 2x$

$\text{IF} = e^{\int -\cot x\,dx} = e^{-\ln\sin x} = \frac{1}{\sin x}$

$\Rightarrow \frac{d}{dx}\left(\frac{y}{\sin x}\right) = \frac{\sin 2x}{\sin x}$

$\Rightarrow \frac{y}{\sin x} = \int \frac{\sin 2x}{\sin x}\,dx$

$\Rightarrow \frac{y}{\sin x} = \int \frac{2\cancel{\sin x}\cos x}{\cancel{\sin x}}\,dx$

$\Rightarrow \frac{y}{\sin x} = \int 2\cos x\,dx$

$\Rightarrow \frac{y}{\sin x} = 2\sin x + C$

$\Rightarrow \boxed{y = 2\sin^2 x + C\sin x}$

when $x = \frac{\pi}{6}$ $y = \frac{3}{2}$

$\frac{3}{2} = 2 \times \frac{1}{4} + C \times \frac{1}{2}$

$3 = 1 + C$

$C = 2$

$\Rightarrow \boxed{y = 2\sin^2 x + 2\sin x}$

$\therefore$ when $x = \frac{\pi}{4}$

$y = 2 \times \frac{1}{2} + \sqrt{2}$

$y = 1 + \sqrt{2}$

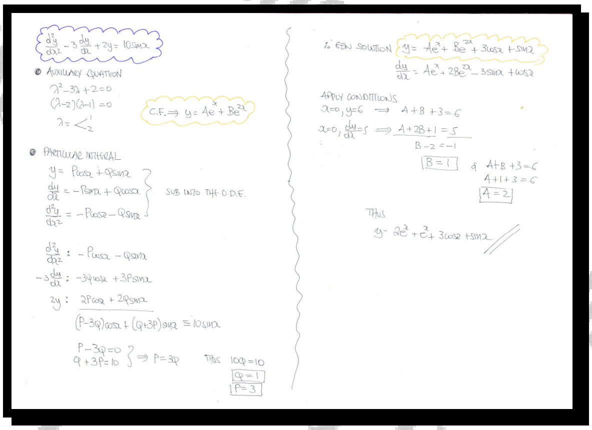
Question 6 (**)

Find a solution of the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 10\sin x ,$$

subject to the boundary conditions $y = 6$ and $\frac{dy}{dx} = 5$ at $x = 0$.

$$\boxed{y = 2e^{x} + e^{2x} + 3\cos x + \sin x}$$



$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 10\sin x$

• AUXILIARY EQUATION

$\lambda^2 - 3\lambda + 2 = 0$

$(\lambda - 2)(\lambda - 1) = 0$

$\lambda = \begin{cases} 1 \\ 2 \end{cases}$

C.F. $\Rightarrow y = Ae^{x} + Be^{2x}$

• PARTICULAR INTEGRAL

$y = P\cos x + Q\sin x$

$\frac{dy}{dx} = -P\sin x + Q\cos x$

$\frac{d^2y}{dx^2} = -P\cos x - Q\sin x$

SUB INTO THE O.D.E.

$\frac{d^2y}{dx^2}: \; -P\cos x - Q\sin x$

$-3\frac{dy}{dx}: \; -3Q\cos x + 3P\sin x$

$2y: \; 2P\cos x + 2Q\sin x$

$(P - 3Q)\cos x + (Q + 3P)\sin x \equiv 10\sin x$

$P - 3Q = 0$, $Q + 3P = 10$ $\Rightarrow P = 3Q$ THUS $10Q = 10$

$\boxed{Q = 1}$ $\boxed{P = 3}$

$\therefore$ GEN SOLUTION $y = Ae^{x} + Be^{2x} + 3\cos x + \sin x$

$\frac{dy}{dx} = Ae^{x} + 2Be^{2x} - 3\sin x + \cos x$

APPLY CONDITIONS

$x = 0, y = 6 \Rightarrow A + B + 3 = 6$

$x = 0, \frac{dy}{dx} = 5 \Rightarrow A + 2B + 1 = 5$

$B - 2 = -1$

$\boxed{B = 1}$ & $A + B + 3 = 6$, $A + 1 + 3 = 6$, $\boxed{A = 2}$

THUS

$y = 2e^{x} + e^{2x} + 3\cos x + \sin x$

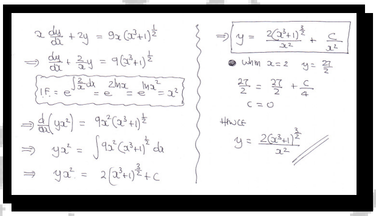
Question 7 (**)

$$x\frac{dy}{dx} + 2y = 9x\left(x^3 + 1\right)^{\frac{1}{2}} \text{ , with } y = \frac{27}{2} \text{ at } x = 2 .$$

Show that the solution of the above differential equation is

$$y = \frac{2}{x^2}\left(x^3 + 1\right)^{\frac{3}{2}} .$$

$\boxed{\text{proof}}$



$x\frac{dy}{dx} + 2y = 9x(x^3+1)^{\frac{1}{2}}$

$\Rightarrow \frac{dy}{dx} + \frac{2}{x}y = 9(x^3+1)^{\frac{1}{2}}$

$\text{I.F.} = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$

$\Rightarrow \frac{d}{dx}(yx^2) = 9x^2(x^3+1)^{\frac{1}{2}}$

$\Rightarrow yx^2 = \int 9x^2(x^3+1)^{\frac{1}{2}}\,dx$

$\Rightarrow yx^2 = 2(x^3+1)^{\frac{3}{2}} + C$

$\Rightarrow \boxed{y = \frac{2(x^3+1)^{\frac{3}{2}}}{x^2} + \frac{C}{x^2}}$

• when $x = 2$ $y = \frac{27}{2}$

$\frac{27}{2} = \frac{27}{2} + \frac{C}{4}$

$C = 0$

HENCE

$y = \frac{2(x^3+1)^{\frac{3}{2}}}{x^2}$

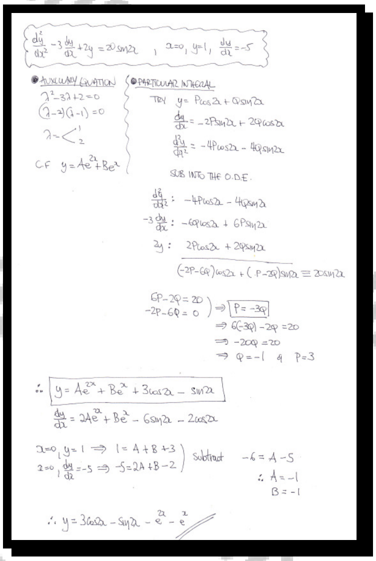
Question 8 (**)

Find a solution of the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 20\sin 2x ,$$

subject to the boundary conditions $y = 1$ and $\frac{dy}{dx} = -5$ at $x = 0$.

$$\boxed{y = 3\cos 2x - \sin 2x - e^{2x} - e^{x}}$$



$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 20\sin 2x$, $x=0$, $y=1$, $\frac{dy}{dx} = -5$

• AUXILIARY EQUATION

$\lambda^2 - 3\lambda + 2 = 0$

$(\lambda - 2)(\lambda - 1) = 0$

$\lambda = 1, 2$

C.F. $y = Ae^{2x} + Be^{x}$

• PARTICULAR INTEGRAL

TRY $y = P\cos 2x + Q\sin 2x$

$\frac{dy}{dx} = -2P\sin 2x + 2Q\cos 2x$

$\frac{d^2y}{dx^2} = -4P\cos 2x - 4Q\sin 2x$

SUB INTO THE O.D.E.

$\frac{d^2y}{dx^2}$: $-4P\cos 2x - 4Q\sin 2x$

$-3\frac{dy}{dx}$: $-6Q\cos 2x + 6P\sin 2x$

$2y$: $2P\cos 2x + 2Q\sin 2x$

$(-2P - 6Q)\cos 2x + (P - 2Q)\sin 2x \equiv 20\sin 2x$

$6P - 2Q = 20$, $-2P - 6Q = 0$ $\Rightarrow \boxed{P = -3Q}$

$\Rightarrow 6(-3Q) - 2Q = 20$

$\Rightarrow -20Q = 20$

$\Rightarrow Q = -1$ & $P = 3$

$\therefore \boxed{y = Ae^{2x} + Be^{x} + 3\cos 2x - \sin 2x}$

$\frac{dy}{dx} = 2Ae^{2x} + Be^{x} - 6\sin 2x - 2\cos 2x$

$x=0, y=1 \Rightarrow 1 = A + B + 3$

$x=0, \frac{dy}{dx} = -5 \Rightarrow -5 = 2A + B - 2$

subtract $-6 = A - 5$

$\therefore A = -1$

$B = -1$

$\therefore y = 3\cos 2x - \sin 2x - e^{2x} - e^{x}$

Question 9 (**)

$$\frac{dy}{dx} = x + 2y \text{, with } y = -\frac{1}{4} \text{ at } x = 0 .$$

By using a suitable substitution, show that the solution of the differential equation is

$$y = -\frac{1}{4}(2x+1) .$$

proof

$\frac{dy}{dx} = x + 2y$

$\Rightarrow \frac{1}{2}\left(\frac{dv}{dx} - 1\right) = v$

$\Rightarrow \frac{dv}{dx} - 1 = 2v$

$\Rightarrow \frac{dv}{dx} = 2v + 1$

$\Rightarrow \int \frac{1}{2v+1}\, dv = \int 1\, dx$

$\Rightarrow \frac{1}{2}\ln|2v+1| = x + C$

$\Rightarrow \ln|2v+1| = 2x + C$

$\Rightarrow 2v + 1 = Ae^{2x}$

$\Rightarrow v = \frac{1}{2}(-1 + Ae^{2x})$

$\Rightarrow \boxed{x + 2y = \frac{1}{2}(-1 + Ae^{2x})}$

$v = x + 2y$

$\frac{dv}{dx} = 1 + 2\frac{dy}{dx}$

$2\frac{dy}{dx} = \frac{dv}{dx} - 1$

APPLY CONDITION $(0, -\frac{1}{4})$

$-\frac{1}{2} = \frac{1}{2}(-1 + Ae^{0})$

$-\frac{1}{2} = -\frac{1}{2} + \frac{1}{2}A$

$\boxed{A = 0}$

$\therefore x + 2y = -\frac{1}{2}$

$\Rightarrow 2y = -\frac{1}{2} - x$

$\Rightarrow 2y = -\frac{1}{2}(1 + 2x)$

$\Rightarrow y = -\frac{1}{4}(2x+1)$

AS REQUIRED

ALTERNATIVE

$\frac{dy}{dx} - 2y = x$

$\text{IF} = e^{\int -2\, dx} = e^{-2x}$

$\Rightarrow \frac{d}{dx}\left(e^{-2x} y\right) = xe^{-2x}$

$\Rightarrow ye^{-2x} = \int xe^{-2x}\, dx$

BY PARTS

x	1
$-\frac{1}{2}e^{-2x}$	e^{-2x}

$ye^{-2x} = -\frac{1}{2}xe^{-2x} - \int -\frac{1}{2}e^{-2x}\, dx$

$ye^{-2x} = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + A$

$\boxed{y = -\frac{1}{2}x - \frac{1}{4} + Ae^{2x}}$

APPLY $(0, -\frac{1}{4}) \Rightarrow A = 0$

$\therefore y = -\frac{1}{2}x - \frac{1}{4}$

$y = -\frac{1}{4}(2x+1)$

AS BEFORE

Question 10 (**)

Find the general solution of the following differential equation.

$$4t^2\frac{d^2x}{dt^2} + 4t\frac{dx}{dt} - x = 0 .$$

$$\boxed{x = At^{\frac{1}{2}} + Bt^{-\frac{1}{2}}}$$

$4t^2\frac{d^2x}{dt^2} + 4t\frac{dx}{dt} - x = 0$

TRY A SOLUTION OF THE FORM $x = t^n$, WHERE n IS A CONSTANT TO BE FOUND

$\frac{dx}{dt} = nt^{n-1}$

$\frac{d^2x}{dt^2} = n(n-1)t^{n-2}$

- SUB INTO THE O.D.E

$\Rightarrow 4t^2\left[n(n-1)t^{n-2}\right] + 4t\left[nt^{n-1}\right] - t^n = 0$

$\Rightarrow 4n(n-1)t^n + 4nt^n - t^n = 0$

$\Rightarrow [4n(n-1) + 4n - 1]t^n = 0$

$\Rightarrow 4n^2 \cancel{-4n} \cancel{+4n} - 1 = 0$

$\Rightarrow (2n-1)(2n+1) = 0$

$n = \frac{1}{2}, -\frac{1}{2}$

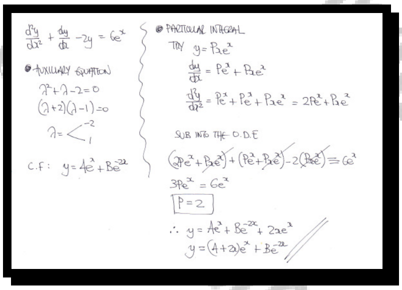
$\therefore x = At^{\frac{1}{2}} + Bt^{-\frac{1}{2}}$

Question 11 (**)

Find a general solution of the differential equation

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y=6e^x .$$

$$\boxed{y=(A+2x)e^x+Be^{-2x}}$$



Question 12 (**)

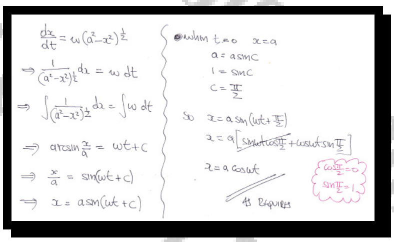
Show that if $y=a$ at $t=0$, the solution of the differential equation

$$\frac{dy}{dt}=\omega\left(a^2-y^2\right)^{\frac{1}{2}},$$

where a and ω are positive constants, can be written as

$$y=a\cos\omega t .$$

$\boxed{\text{proof}}$

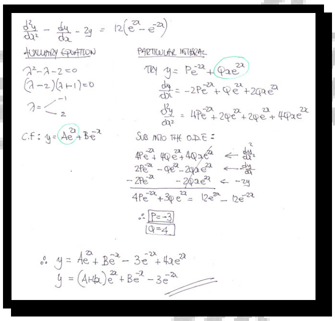


Question 13 (**)

Find a general solution of the differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 12\left(\mathrm{e}^{2x} - \mathrm{e}^{-2x}\right).$$

$$\boxed{y = (A+4x)\mathrm{e}^{2x} + B\mathrm{e}^{-x} - 3\mathrm{e}^{-2x}}$$



$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 12\left(e^{2x} - e^{-2x}\right)$

AUXILIARY EQUATION

$\lambda^2 - \lambda - 2 = 0$

$(\lambda - 2)(\lambda + 1) = 0$

$\lambda = \begin{cases} -1 \\ 2 \end{cases}$

C.F: $y = Ae^{2x} + Be^{-x}$

PARTICULAR INTEGRAL

TRY $y = Pe^{-2x} + Qxe^{2x}$

$\frac{dy}{dx} = -2Pe^{-2x} + Qe^{2x} + 2Qxe^{2x}$

$\frac{d^2y}{dx^2} = 4Pe^{-2x} + 2Qe^{2x} + 2Qe^{2x} + 4Qxe^{2x}$

SUB INTO THE O.D.E:

$4Pe^{-2x} + 4Qe^{2x} + \cancel{4Qxe^{2x}} \leftarrow \frac{d^2y}{dx^2}$

$2Pe^{-2x} - Qe^{2x} - \cancel{2Qxe^{2x}} \leftarrow -\frac{dy}{dx}$

$-2Pe^{-2x} \qquad - \cancel{2Qxe^{2x}} \leftarrow -2y$

$4Pe^{-2x} + 3Qe^{2x} = 12e^{2x} - 12e^{-2x}$

$\therefore \boxed{P = -3}$ $\boxed{Q = 4}$

$\therefore y = Ae^{2x} + Be^{-x} - 3e^{-2x} + 4xe^{2x}$

$y = (A + 4x)e^{2x} + Be^{-x} - 3e^{-2x}$

Question 14 (**)

20 grams of salt are dissolved into a beaker containing 1 litre of a certain chemical.

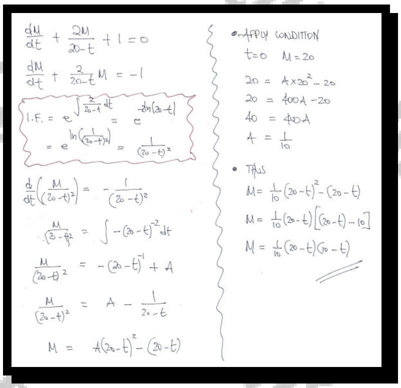
The mass of salt, M grams, which remains undissolved t seconds later, is modelled by the differential equation

$$\frac{dM}{dt} + \frac{2M}{20-t} + 1 = 0, \ t \geq 0.$$

Show clearly that

$$M = \frac{1}{10}(10-t)(20-t).$$

$\boxed{\text{proof}}$



$$\frac{dM}{dt} + \frac{2M}{20-t} + 1 = 0$$

$$\frac{dM}{dt} + \frac{2}{20-t}M = -1$$

$$\text{I.F.} = e^{\int \frac{2}{20-t}\,dt} = e^{-2\ln|20-t|} = e^{\ln\left(\frac{1}{(20-t)^2}\right)} = \frac{1}{(20-t)^2}$$

$$\frac{d}{dt}\left(\frac{M}{(20-t)^2}\right) = -\frac{1}{(20-t)^2}$$

$$\frac{M}{(20-t)^2} = \int -(20-t)^{-2}\,dt$$

$$\frac{M}{(20-t)^2} = -(20-t)^{-1} + A$$

$$\frac{M}{(20-t)^2} = A - \frac{1}{20-t}$$

$$M = A(20-t)^2 - (20-t)$$

- APPLY CONDITION

$t=0 \quad M=20$

$20 = A \times 20^2 - 20$

$20 = 400A - 20$

$40 = 400A$

$A = \frac{1}{10}$

- THUS

$M = \frac{1}{10}(20-t)^2 - (20-t)$

$M = \frac{1}{10}(20-t)\left[(20-t) - 10\right]$

$M = \frac{1}{10}(20-t)(10-t)$

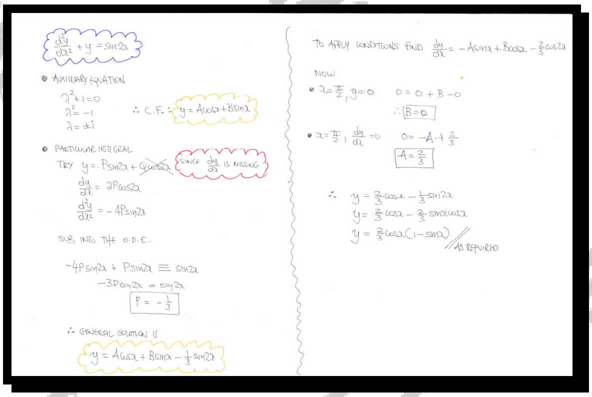
Question 15 (**)

$$\frac{d^2y}{dx^2} + y = \sin 2x, \text{ with } y = 0 \text{ , } \frac{dy}{dx} = 0 \text{ at } x = \frac{\pi}{2}.$$

Show that a solution of the above differential equation is

$$y = \frac{2}{3}\cos x(1 - \sin x).$$

proof



$$\frac{d^2y}{dx^2} + y = \sin 2x$$

• AUXILIARY EQUATION

$\lambda^2 + 1 = 0$
$\lambda^2 = -1$
$\lambda = \pm i$

$\therefore$ C.F. : $y = A\cos x + B\sin x$

• PARTICULAR INTEGRAL

TRY $y = P\sin 2x + \cancel{Q\cos 2x}$ SINCE $\frac{dy}{dx}$ IS MISSING

$\frac{dy}{dx} = 2P\cos 2x$

$\frac{d^2y}{dx^2} = -4P\sin 2x$

SUB INTO THE O.D.E.

$-4P\sin 2x + P\sin 2x \equiv \sin 2x$

$-3P\sin 2x = \sin 2x$

$\boxed{P = -\frac{1}{3}}$

$\therefore$ GENERAL SOLUTION IS

$y = A\cos x + B\sin x - \frac{1}{3}\sin 2x$

TO APPLY CONDITIONS FIND $\frac{dy}{dx} = -A\sin x + B\cos x - \frac{2}{3}\cos 2x$

NOW

• $x = \frac{\pi}{2}, y = 0$ $\quad 0 = 0 + B - 0$

$\therefore \boxed{B = 0}$

• $x = \frac{\pi}{2}, \frac{dy}{dx} = 0$ $\quad 0 = -A + \frac{2}{3}$

$\boxed{A = \frac{2}{3}}$

$\therefore$ $y = \frac{2}{3}\cos x - \frac{1}{3}\sin 2x$

$y = \frac{2}{3}\cos x - \frac{2}{3}\sin x\cos x$

$y = \frac{2}{3}\cos x(1 - \sin x)$ // AS REQUIRED

Question 16 (+)**

Show that a general solution of the differential equation

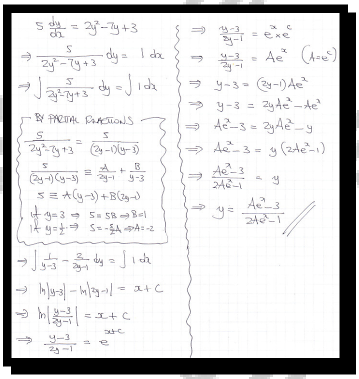
$$5\frac{dy}{dx} = 2y^2 - 7y + 3$$

is given by

$$y = \frac{A\mathrm{e}^x - 3}{2A\mathrm{e}^x - 1},$$

where A is an arbitrary constant.

proof



$$5\frac{dy}{dx} = 2y^2 - 7y + 3$$

$$\Rightarrow \frac{5}{2y^2 - 7y + 3}\,dy = 1\,dx$$

$$\Rightarrow \int \frac{5}{2y^2 - 7y + 3}\,dy = \int 1\,dx$$

BY PARTIAL FRACTIONS

$$\frac{5}{2y^2 - 7y + 3} = \frac{5}{(2y-1)(y-3)}$$

$$\frac{5}{(2y-1)(y-3)} \equiv \frac{A}{2y-1} + \frac{B}{y-3}$$

$$5 \equiv A(y-3) + B(2y-1)$$

If $y = 3 \Rightarrow 5 = 5B \Rightarrow B = 1$

If $y = \frac{1}{2} \Rightarrow 5 = -\frac{5}{2}A \Rightarrow A = -2$

$$\Rightarrow \int \frac{1}{y-3} - \frac{2}{2y-1}\,dy = \int 1\,dx$$

$$\Rightarrow \ln|y-3| - \ln|2y-1| = x + C$$

$$\Rightarrow \ln\left|\frac{y-3}{2y-1}\right| = x + C$$

$$\Rightarrow \frac{y-3}{2y-1} = e^{x+C}$$

$$\Rightarrow \frac{y-3}{2y-1} = e^x \times e^C$$

$$\Rightarrow \frac{y-3}{2y-1} = Ae^x \quad (A = e^C)$$

$$\Rightarrow y - 3 = (2y-1)Ae^x$$

$$\Rightarrow y - 3 = 2yAe^x - Ae^x$$

$$\Rightarrow Ae^x - 3 = 2yAe^x - y$$

$$\Rightarrow Ae^x - 3 = y(2Ae^x - 1)$$

$$\Rightarrow \frac{Ae^x - 3}{2Ae^x - 1} = y$$

$$\Rightarrow y = \frac{Ae^x - 3}{2Ae^x - 1}$$

Question 17 (**+)

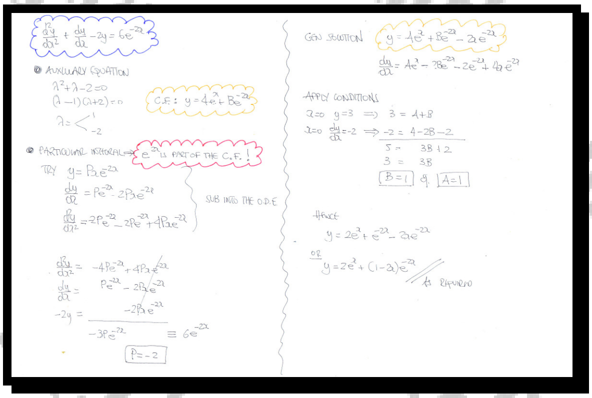
$$\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y=6e^{-2x},$$

with $y=3$ and $\frac{dy}{dx}=-2$ at $x=0$.

Show that the solution of the above differential equation is

$$y=2e^{x}+(1-2x)e^{-2x}.$$

proof

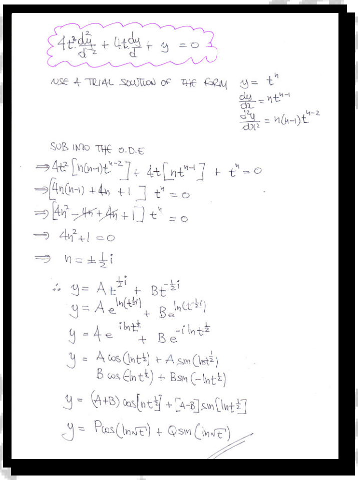


Question 18 **(**+)**

Find the general solution of the following differential equation.

$$4t^2\frac{d^2y}{dt^2}+4t\frac{dy}{dt}+y=0.$$

$$\boxed{y=P\cos\left[\ln\sqrt{t}\right]+P\sin\left[\ln\sqrt{t}\right]}$$



$4t^2\frac{d^2y}{dt^2}+4t\frac{dy}{dt}+y=0$

USE A TRIAL SOLUTION OF THE FORM $y=t^n$

$\frac{dy}{dx}=nt^{n-1}$

$\frac{d^2y}{dx^2}=n(n-1)t^{n-2}$

SUB INTO THE O.D.E

$\Rightarrow 4t^2\left[n(n-1)t^{n-2}\right]+4t\left[nt^{n-1}\right]+t^n=0$

$\Rightarrow \left[4n(n-1)+4n+1\right]t^n=0$

$\Rightarrow \left[4n^2-\cancel{4n}+\cancel{4n}+1\right]t^n=0$

$\Rightarrow 4n^2+1=0$

$\Rightarrow n=\pm\frac{1}{2}i$

$\therefore y=At^{\frac{1}{2}i}+Bt^{-\frac{1}{2}i}$

$y=Ae^{\ln(t^{\frac{1}{2}i})}+Be^{\ln(t^{-\frac{1}{2}i})}$

$y=Ae^{i\ln t^{\frac{1}{2}}}+Be^{-i\ln t^{\frac{1}{2}}}$

$y=A\cos(\ln t^{\frac{1}{2}})+A\sin(\ln t^{\frac{1}{2}})$
$B\cos(-\ln t^{\frac{1}{2}})+B\sin(-\ln t^{\frac{1}{2}})$

$y=(A+B)\cos[\ln t^{\frac{1}{2}}]+[A-B]\sin[\ln t^{\frac{1}{2}}]$

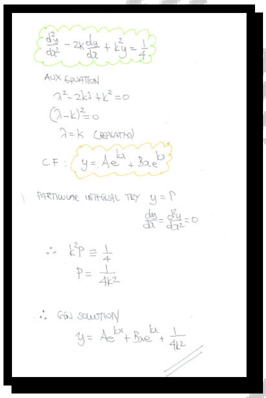
$y=P\cos(\ln\sqrt{t})+Q\sin(\ln\sqrt{t})$

Question 19 **(**+)**

Find a general solution of the differential equation

$$\frac{d^2y}{dx^2}-2k\frac{dy}{dx}+k^2y=\frac{1}{4},\ k>0.$$

$$\boxed{y=Ae^{kx}+Bxe^{kx}+\frac{1}{4k^2}}$$



$\frac{d^2y}{dx^2}-2k\frac{dy}{dx}+k^2y=\frac{1}{4}$

AUX EQUATION

$\lambda^2-2k\lambda+k^2=0$

$(\lambda-k)^2=0$

$\lambda=k$ (REPEATED)

C.F: $y=Ae^{kx}+Bxe^{kx}$

PARTICULAR INTEGRAL TRY $y=P$

$\frac{dy}{dx}=\frac{d^2y}{dx^2}=0$

$\therefore k^2P\equiv\frac{1}{4}$

$P=\frac{1}{4k^2}$

$\therefore$ GEN SOLUTION

$y=Ae^{kx}+Bxe^{kx}+\frac{1}{4k^2}$

Question 20 **(**+)**

Solve the differential equation

$$\frac{dy}{dx} = \frac{y}{x} - \left(\frac{y}{x}\right)^2, \quad x > 0,$$

subject to the condition $y = 1$ at $x = 1$.

$$\boxed{y = \frac{x}{1 + \ln x}}$$

$$\frac{dy}{dx} = \frac{y}{x} - \left(\frac{y}{x}\right)^2$$

$v = \frac{y}{x}$

$y = xv$

$\frac{dy}{dx} = 1 \times v + x \times \frac{dv}{dx}$

$\Rightarrow v + x\frac{dv}{dx} = v - v^2$

$\Rightarrow x\frac{dv}{dx} = -v^2$

$\Rightarrow -\frac{1}{v^2}\,dv = \frac{1}{x}\,dx$

$\Rightarrow \int -v^{-2}\,dv = \int \frac{1}{x}\,dx$

$\Rightarrow v^{-1} = \ln x + C$

$\Rightarrow \frac{x}{y} = \ln x + C$

$\Rightarrow \frac{x}{\ln x + C} = y$

$\Rightarrow$ APPLY CONDITION $(1,1)$

$\frac{1}{\ln 1 + C} = 1$

$C = 1$

$\therefore y = \frac{x}{1 + \ln x}$

Question 21 (**+)

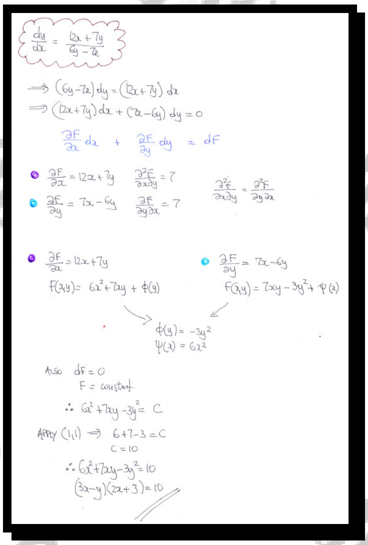
$$\frac{dy}{dx} = \frac{12x+7y}{6y-7x}, \quad y(1)=1.$$

Use a method involving partial differentiation to show that the solution of the above differential equation can be written as

$$(ax+by)(cx+dy)=10,$$

where a, b, c and d are integers to be found.

$$\boxed{(3x-y)(2x+3y)=k}$$



$\frac{dy}{dx} = \frac{12x+7y}{6y-7x}$

$\Rightarrow (6y-7x)\,dy = (12x+7y)\,dx$

$\Rightarrow (12x+7y)\,dx + (7x-6y)\,dy = 0$

$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = dF$

$\frac{\partial F}{\partial x} = 12x+7y \quad \frac{\partial^2 F}{\partial x \partial y} = 7$

$\frac{\partial F}{\partial y} = 7x-6y \quad \frac{\partial F}{\partial y \partial x} = 7$

$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}$

$\frac{\partial F}{\partial x} = 12x+7y$

$F(x,y) = 6x^2+7xy+\phi(y)$

$\frac{\partial F}{\partial y} = 7x-6y$

$F(x,y) = 7xy-3y^2+\psi(x)$

$\phi(y) = -3y^2$

$\psi(x) = 6x^2$

Also $dF=0$

$F = \text{constant}$

$\therefore 6x^2+7xy-3y^2 = C$

Apply $(1,1) \Rightarrow 6+7-3=C$

$C=10$

$\therefore 6x^2+7xy-3y^2=10$

$(3x-y)(2x+3)=10$

Question 22 (+)**

Find the solution of the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = 2x + 3,$$

subject to the conditions $y = 2$, $\frac{dy}{dx} = -5$ at $x = 0$.

$$\boxed{y = x^2 + x - 4 + 6e^{-x}}$$

$\frac{d^2y}{dx^2} + \frac{dy}{dx} = 2x+3$

- AUXILIARY EQUATION

$\lambda^2 + \lambda = 0$

$\lambda(\lambda+1) = 0$

$\lambda = 0, -1$

$\therefore$ C.F.: $y = A + Be^{-x}$

- PARTICULAR SOLUTION $\Rightarrow$ $y = A$ IS PART OF THE C.F.!

TRY $y = Px^2 + Qx$ (NORMALLY WE WOULD HAVE TRIED $y = Px + Q$)

$\frac{dy}{dx} = 2Px + Q$

$\frac{d^2y}{dx^2} = 2P$

SUB INTO THE O.D.E.

$2P + (2Px + Q) \equiv 2x + 3$

$2Px + (2P + Q) \equiv 2x + 3$

- $2P = 2$, $\boxed{P = 1}$

$2P + Q = 3$

$2 + Q = 3$

$\boxed{Q = 1}$

$\therefore$ GEN SOLUTION $y = A + Be^{-x} + x^2 + x$

APPLY CONDITIONS, NEED $\frac{dy}{dx} = -Be^{-x} + 2x + 1$

- $x = 0, y = 2 \Rightarrow 2 = A + B$
- $x = 0, \frac{dy}{dx} = -5 \Rightarrow -5 = -B + 1$

$\therefore \boxed{B = 6}$ & $A + B = 2$

$A + 6 = 2$

$\boxed{A = -4}$

$\therefore y = 6e^{-x} + x^2 + x - 4$

Question 23 (+)**

Find the general solution of the following differential equation.

$$\frac{d^4\psi}{dx^4} + 2\lambda\frac{d^2\psi}{dx^2} + \lambda^4\psi = 0.$$

$$\boxed{\psi = A\cos\lambda x + B\sin\lambda x}$$

$\frac{d^4\psi}{dx^4} + 2\lambda^2\frac{d^2\psi}{dx^2} + \lambda^4\psi = 0$

AUXILIARY EQUATION

$m^4 + 2\lambda^2 m^2 + \lambda^4 = 0$

$(m^2 + \lambda^2)^2 = 0$

$m^2 + \lambda^2 = 0$

$m^2 = -\lambda^2$

$m = \pm\lambda i$

$\therefore \psi = A\cos\lambda x + B\sin\lambda x$

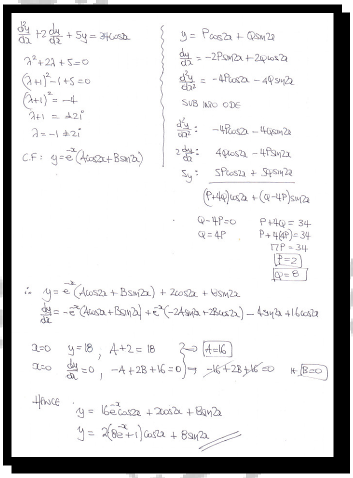
Question 24 (+)**

Find a solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 34\cos 2x,$$

subject to the boundary conditions $y = 18$ and $\frac{dy}{dx} = 0$ at $x = 0$.

$$\boxed{y = 2\left(8e^{-x} + 1\right)\cos 2x + 8\sin 2x}$$



$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 34\cos 2x$

$\lambda^2 + 2\lambda + 5 = 0$

$(\lambda+1)^2 - 1 + 5 = 0$

$(\lambda+1)^2 = -4$

$\lambda + 1 = \pm 2i$

$\lambda = -1 \pm 2i$

C.F: $y = e^{-x}(A\cos 2x + B\sin 2x)$

$y = P\cos 2x + Q\sin 2x$

$\frac{dy}{dx} = -2P\sin 2x + 2Q\cos 2x$

$\frac{d^2y}{dx^2} = -4P\cos 2x - 4Q\sin 2x$

SUB INTO ODE

$\frac{d^2y}{dx^2}$: $-4P\cos 2x - 4Q\sin 2x$

$2\frac{dy}{dx}$: $4Q\cos 2x - 4P\sin 2x$

$5y$: $5P\cos 2x + 5Q\sin 2x$

$(P+4Q)\cos 2x + (Q-4P)\sin 2x$

$Q - 4P = 0$, $Q = 4P$

$P + 4Q = 34$, $P + 4(4P) = 34$, $17P = 34$, $\boxed{P=2}$, $\boxed{Q=8}$

$\therefore y = e^{-x}(A\cos 2x + B\sin 2x) + 2\cos 2x + 8\sin 2x$

$\frac{dy}{dx} = -e^{-x}(A\cos 2x + B\sin 2x) + e^{-x}(-2A\sin 2x + 2B\cos 2x) - 4\sin 2x + 16\cos 2x$

$x=0$ $y=18$, $A + 2 = 18$ $\Rightarrow \boxed{A=16}$

$x=0$ $\frac{dy}{dx}=0$, $-A + 2B + 16 = 0$ $\Rightarrow \cancel{-16} + 2B \cancel{+16} = 0$ $\boxed{B=0}$

HENCE

$y = 16e^{-x}\cos 2x + 2\cos 2x + 8\sin 2x$

$y = 2(8e^{-x}+1)\cos 2x + 8\sin 2x$

Question 25 (**+)

$$\frac{dy}{dx} = \frac{(4x+y)(x+y)}{x^2}, \quad x > 0.$$

a) Use a suitable substitution to show that the above differential equation can be transformed to

$$x\frac{dv}{dx} = (v+2)^2.$$

b) Hence find the general solution of the original differential equation, giving the answer in the form $y = f(x)$.

c) Use the boundary condition $y = -1$ at $x = 1$, to show that a specific solution of the original differential equation is

$$y = \frac{x}{1 - \ln x} - 2x.$$

$$\boxed{y = \frac{x}{A - \ln x} - 2x}$$

$y = vx$

$\frac{dy}{dx} = \frac{dv}{dx}x + v \times 1$

$\boxed{\frac{dy}{dx} = v + x\frac{dv}{dx}}$

$v = \frac{y}{x}$

(a) $\frac{dy}{dx} = \frac{(4x+y)(x+y)}{x^2}$

$\Rightarrow \frac{dy}{dx} = \frac{4x^2+5xy+y^2}{x^2}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{4x^2+5x(vx)+(vx)^2}{x^2}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{4x^2+5vx^2+v^2x^2}{x^2}$

$\Rightarrow v + x\frac{dv}{dx} = 4 + 5v + v^2$

$\Rightarrow x\frac{dv}{dx} = v^2 + 4v + 4$

$\Rightarrow x\frac{dv}{dx} = (v+2)^2$ // AS REQUIRED

b) $\frac{1}{(v+2)^2}\,dv = \frac{1}{x}\,dx$

$\Rightarrow \int \frac{1}{(v+2)^2}\,dv = \int \frac{1}{x}\,dx$

$\Rightarrow -\frac{1}{v+2} = \ln x + C$

$\Rightarrow \frac{1}{v+2} = A - \ln x$

$\Rightarrow v + 2 = \frac{1}{A - \ln x}$

$\Rightarrow v = \frac{1}{A - \ln x} - 2$

$\Rightarrow \frac{y}{x} = \frac{1}{A - \ln x} - 2$

$\Rightarrow y = \frac{x}{A - \ln x} - 2x$ //

(c) $x = 1 \quad y = -1$

$-1 = \frac{1}{A - \cancel{\ln 1}} - 2$

$1 = \frac{1}{A}$

$\boxed{A = 1}$

$\therefore y = \frac{x}{1 - \ln x} - 2x$ //

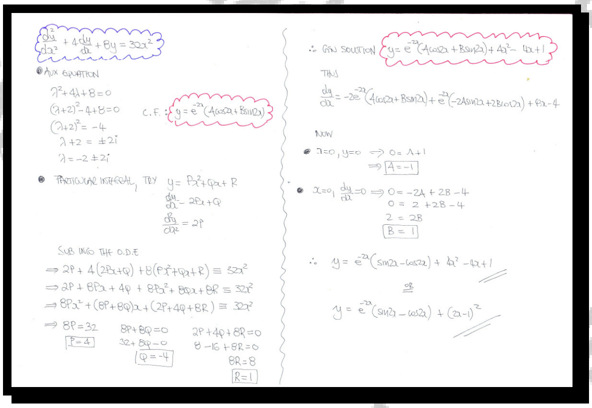
Question 26 (**+)

The curve C has a local minimum at the origin and satisfies the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 8y = 32x^2 .$$

Find an equation for C.

$$\boxed{y = e^{x}\left(\sin 2x + \cos 2x\right) + \left(2x-1\right)^2}$$



$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 8y = 32x^2$

- AUX EQUATION

$\lambda^2 + 4\lambda + 8 = 0$

$(\lambda+2)^2 - 4 + 8 = 0$

$(\lambda+2)^2 = -4$

$\lambda + 2 = \pm 2i$

$\lambda = -2 \pm 2i$

C.F.: $y = e^{-2x}(A\cos 2x + B\sin 2x)$

- PARTICULAR INTEGRAL, TRY $y = Px^2 + Qx + R$

$\frac{dy}{dx} = 2Px + Q$

$\frac{d^2y}{dx^2} = 2P$

SUB INTO THE O.D.E

$\Rightarrow 2P + 4(2Px+Q) + 8(Px^2+Qx+R) \equiv 32x^2$

$\Rightarrow 2P + 8Px + 4Q + 8Px^2 + 8Qx + 8R \equiv 32x^2$

$\Rightarrow 8Px^2 + (8P+8Q)x + (2P+4Q+8R) \equiv 32x^2$

$\Rightarrow 8P = 32$, $\boxed{P=4}$; $8P + 8Q = 0$, $32 + 8Q = 0$, $\boxed{Q=-4}$; $2P + 4Q + 8R = 0$, $8 - 16 + 8R = 0$, $8R = 8$, $\boxed{R=1}$

$\therefore$ GEN SOLUTION $y = e^{-2x}(A\cos 2x + B\sin 2x) + 4x^2 - 4x + 1$

THUS

$\frac{dy}{dx} = -2e^{-2x}(A\cos 2x + B\sin 2x) + e^{-2x}(-2A\sin 2x + 2B\cos 2x) + 8x - 4$

NOW

- $x=0, y=0 \Rightarrow 0 = A + 1$
$\Rightarrow \boxed{A=-1}$

- $x=0, \frac{dy}{dx}=0 \Rightarrow 0 = -2A + 2B - 4$
$0 = 2 + 2B - 4$
$2 = 2B$
$\boxed{B=1}$

$\therefore \quad y = e^{-2x}(\sin 2x - \cos 2x) + 4x^2 - 4x + 1$

OR

$y = e^{-2x}(\sin 2x - \cos 2x) + (2x-1)^2$

Question 27 (+)**

Show that a general solution of the differential equation

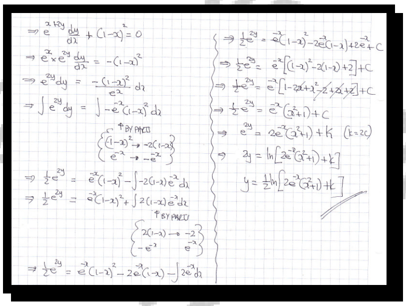
$$e^{x+2y}\frac{dy}{dx}+(1-x)^2=0$$

is given by

$$y=\frac{1}{2}\ln\left[2e^{-x}\left(x^2+1\right)+K\right],$$

where K is an arbitrary constant.

$\boxed{\text{proof}}$



$$\Rightarrow e^{x+2y}\frac{dy}{dx}+(1-x)^2=0$$

$$\Rightarrow e^{x}\times e^{2y}\frac{dy}{dx}=-(1-x)^2$$

$$\Rightarrow e^{2y}\,dy=\frac{-(1-x)^2}{e^x}\,dx$$

$$\Rightarrow \int e^{2y}\,dy=\int -e^{-x}(1-x)^2\,dx$$

BY PARTS

$$\left\{\begin{array}{l}(1-x)^2 \to -2(1-x)\\ e^{-x} \to -e^{-x}\end{array}\right\}$$

$$\Rightarrow \frac{1}{2}e^{2y}=e^{-x}(1-x)^2-\int -2(1-x)e^{-x}\,dx$$

$$\Rightarrow \frac{1}{2}e^{2y}=e^{-x}(1-x)^2+\int 2(1-x)e^{-x}\,dx$$

BY PARTS

$$\left\{\begin{array}{ll}2(1-x) \to -2\\ -e^{-x} & e^{-x}\end{array}\right\}$$

$$\Rightarrow \frac{1}{2}e^{2y}=e^{-x}(1-x)^2-2e^{-x}(1-x)-\int 2e^{-x}\,dx$$

$$\Rightarrow \frac{1}{2}e^{2y}=e^{-x}(1-x)^2-2e^{-x}(1-x)+2e^{-x}+C$$

$$\Rightarrow \frac{1}{2}e^{2y}=e^{-x}\left[(1-x)^2-2(1-x)+2\right]+C$$

$$\Rightarrow \frac{1}{2}e^{2y}=e^{-x}\left[1-\cancel{2x}+x^2-\cancel{2}+\cancel{2x}+\cancel{2}\right]+C$$

$$\Rightarrow \frac{1}{2}e^{2y}=e^{-x}(x^2+1)+C$$

$$\Rightarrow e^{2y}=2e^{-x}(x^2+1)+K \quad (K=2C)$$

$$\Rightarrow 2y=\ln\left[2e^{-x}(x^2+1)+K\right]$$

$$y=\frac{1}{2}\ln\left[2e^{-x}(x^2+1)+K\right]$$

Question 28 (**+)

$$\frac{d^2x}{dt^2}+9x+12\sin 3t=0\,,\ t\geq 0\,,$$

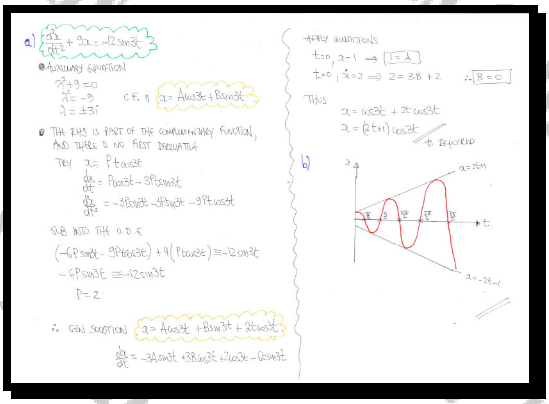
with $x=1$, $\dfrac{dx}{dt}=2$ at $t=0$.

a) Show that a solution of the differential equation is

$$x=(2t+1)\cos 3t\,.$$

b) Sketch the graph of x.

proof


a) $\frac{d^2x}{dt^2}+9x=-12\sin 3t$

• AUXILIARY EQUATION

$\lambda^2+9=0$

$\lambda^2=-9$

$\lambda=\pm 3i$

C.F. ∴ $x=A\cos 3t+B\sin 3t$

• THE RHS IS PART OF THE COMPLIMENTARY FUNCTION, AND THERE IS NO FIRST DERIVATIVE

TRY $x=Pt\cos 3t$

$\frac{dx}{dt}=P\cos 3t-3Pt\sin 3t$

$\frac{d^2x}{dt^2}=-3P\sin 3t-3P\sin 3t-9Pt\cos 3t$

SUB INTO THE O.D.E

$(-6P\sin 3t-9Pt\cos 3t)+9(Pt\cos 3t)\equiv -12\sin 3t$

$-6P\sin 3t\equiv -12\sin 3t$

$P=2$

∴ GEN SOLUTION $x=A\cos 3t+B\sin 3t+2t\cos 3t$

$\frac{dx}{dt}=-3A\sin 3t+3B\cos 3t+2\cos 3t-6t\sin 3t$

APPLY CONDITIONS

$t=0,\ x=1 \Rightarrow \boxed{1=A}$

$t=0,\ \dot{x}=2 \Rightarrow 2=3B+2$ ∴ $\boxed{B=0}$

THUS

$x=\cos 3t+2t\cos 3t$

$x=(2t+1)\cos 3t$ AS REQUIRED

b)

Created by T. Madas

Question 29 (+)**

By using a suitable substitution, solve the differential equation

$$xy\frac{dy}{dx} = x^2 + y^2 ,\ \ x > 0 ,$$

subject to the boundary condition $y = 1$ at $x = 1$.

$$\boxed{y = x^2\left(1 + 2\ln x\right)}$$

$$xy\frac{dy}{dx} = x^2+y^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2+y^2}{xy}$$

$$y = zx$$

$$\frac{dy}{dx} = \frac{dz}{dx}x + z$$

$$\Rightarrow x\frac{dz}{dx} + z = \frac{x^2 + z^2x^2}{x(zx)}$$

$$\Rightarrow x\frac{dz}{dx} = \frac{1+z^2}{z} - z$$

$$\Rightarrow x\frac{dz}{dx} = \frac{1}{z} + \cancel{z} - \cancel{z}$$

$$\Rightarrow z\,dz = \frac{1}{x}\,dx$$

$$\Rightarrow \int z\,dz = \int \frac{1}{x}\,dx$$

$$\Rightarrow \frac{1}{2}z^2 = \ln|x| + A$$

$$\Rightarrow z^2 = 2\ln x + A$$

$$\Rightarrow \frac{y^2}{x^2} = A + 2\ln x$$

$$\Rightarrow \boxed{y^2 = Ax^2 + 2x\ln x}$$

- APPLY CONDITION

$$1 = A$$

$$\therefore y^2 = x^2 + 2x\ln x$$

Question 30 (+)**

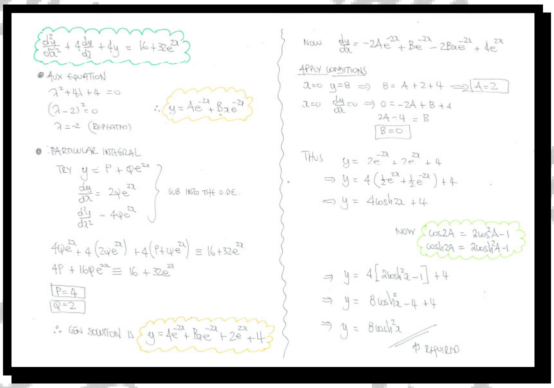
$$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=16+32e^{2x},$$

with $y=8$ and $\frac{dy}{dx}=0$ at $x=0$.

Show that the solution of the above differential equation is

$$y=8\cosh^2 x.$$

proof



$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=16+32e^{2x}$

- AUX EQUATION

$\lambda^2+4\lambda+4=0$

$(\lambda-2)^2=0$

$\lambda=-2$ (REPEATED)

$\therefore\ y=Ae^{-2x}+Bxe^{-2x}$

- PARTICULAR INTEGRAL

TRY $y=P+Qe^{2x}$

$\frac{dy}{dx}=2Qe^{2x}$

$\frac{d^2y}{dx^2}=4Qe^{2x}$

SUB INTO THE O.D.E.

$4Qe^{2x}+4(2Qe^{2x})+4(P+Qe^{2x})\equiv16+32e^{2x}$

$4P+16Qe^{2x}\equiv16+32e^{2x}$

$\boxed{P=4}$

$\boxed{Q=2}$

$\therefore$ GEN SOLUTION IS $y=Ae^{-2x}+Bxe^{-2x}+2e^{2x}+4$

NOW $\frac{dy}{dx}=-2Ae^{-2x}+Be^{-2x}-2Bxe^{-2x}+4e^{2x}$

APPLY CONDITIONS

$x=0\ \ y=8 \Rightarrow 8=A+2+4 \Rightarrow \boxed{A=2}$

$x=0\ \ \frac{dy}{dx}=0 \Rightarrow 0=-2A+B+4$

$2A-4=B$

$\boxed{B=0}$

THUS $y=2e^{-2x}+2e^{2x}+4$

$\Rightarrow y=4\left(\frac{1}{2}e^{2x}+\frac{1}{2}e^{-2x}\right)+4$

$\Rightarrow y=4\cosh2x+4$

NOW $\cos2A=2\cos^2A-1$

$\cosh2A=2\cosh^2A-1$

$\Rightarrow y=4\left[2\cosh^2x-1\right]+4$

$\Rightarrow y=8\cosh^2x-4+4$

$\Rightarrow y=8\cosh^2x$

AS REQUIRED

Question 31 (**+)

$$x\frac{dy}{dx} = \sqrt{y^2+1}\ ,\ x>0\text{, with } y=0 \text{ at } x=2\,.$$

Show that the solution of the above differential equation is

$$y = \frac{x}{4} - \frac{1}{x}.$$

proof

$$x\frac{dy}{dx} = \sqrt{y^2+1}$$

$$\Rightarrow \int \frac{1}{\sqrt{y^2+1}}\,dy = \int \frac{1}{x}\,dx$$

$$\Rightarrow \operatorname{arsinh} y = \ln x + C$$

$$\Rightarrow \ln\left(y+\sqrt{y^2+1}\right) = \ln x + \ln A$$

$$\Rightarrow \ln\left(y+\sqrt{y^2+1}\right) = \ln Ax$$

$$\Rightarrow \boxed{y+\sqrt{y^2+1} = Ax}$$

when $x=2$ $y=0$

$1 = 2A$

$\boxed{A=\frac{1}{2}}$

$$\Rightarrow y+\sqrt{y^2+1} = \tfrac{1}{2}x$$

$$\Rightarrow \sqrt{y^2+1} = \tfrac{1}{2}x - y$$

$$\Rightarrow \cancel{y^2}+1 = \tfrac{1}{4}x^2 - xy + \cancel{y^2}$$

$$\Rightarrow xy = \tfrac{1}{4}x^2 - 1$$

$$\Rightarrow y = \tfrac{1}{4}x - \tfrac{1}{x}$$

$$\Rightarrow y = \frac{x}{4} - \frac{1}{x}$$

AS REQUIRED

Question 32 (**+)

$$\frac{d^2y}{dx^2} - 2k\frac{dy}{dx} + k^2 y = 12x\mathrm{e}^{kx}, \; k > 0$$

a) Find a general solution of the differential equation given that $y = Px^3\mathrm{e}^{kx}$, where P is a constant, is part of the solution.

b) Given further that $y = 1$, $\frac{dy}{dx} = 0$ at $x = 0$ show that

$$y = \mathrm{e}^{kx}\left(2x^3 - kx + 1\right).$$

$$\boxed{y = \mathrm{e}^{kx}\left(2x^3 + Ax + B\right)}$$

(a) $\frac{d^2y}{dx^2} - 2k\frac{dy}{dx} + k^2 y = 0$

AUXILIARY EQUATION

$\lambda^2 - 2k\lambda + k^2 = 0$

$(\lambda - k)^2 = 0$

$\lambda = k$ (REPEATED)

COMPLEMENTARY FUNCTION

$y = Ae^{kx} + Bxe^{kx}$

FOR PARTICULAR INTEGRAL TRY

$y = Px^3e^{kx}$

$\frac{dy}{dx} = 3Px^2e^{kx} + Pkx^3e^{kx}$

$\frac{d^2y}{dx^2} = 6Pxe^{kx} + 3Pkx^2e^{kx} + 3Pkx^2e^{kx} + Pk^2x^3e^{kx}$

SUB INTO THE O.D.E

$\frac{d^2y}{dx^2} = \cancel{Pk^2x^3e^{kx}} + \cancel{6Pkx^2e^{kx}} + 6Pxe^{kx}$

$-2k\frac{dy}{dx} = \cancel{-2Pk^2x^3e^{kx}} - \cancel{6Pkx^2e^{kx}}$

$k^2y = \cancel{Pk^2x^3e^{kx}}$

ADD $\quad 6Pxe^{kx} \equiv 12xe^{kx}$

$\therefore P = 2$

$\therefore$ GEN SOLUTION $y = Ae^{kx} + Bxe^{kx} + 2x^3e^{kx}$

$y = e^{kx}(A + Bx + 2x^3)$

(b) $\frac{dy}{dx} = ke^{kx}(A + Bx + 2x^3) + e^{kx}(B + 6x)$

$x=0 \; y=1 \Rightarrow \boxed{1 = A}$

$x=0 \; \frac{dy}{dx}=0 \Rightarrow 0 = k \times 1 + B \quad \therefore \boxed{B = -k}$

$\therefore y = e^{kx}(1 - kx + 2x^3)$

Question 33 (+)**

By using a suitable substitution, or otherwise, solve the differential equation

$$\frac{dy}{dx} = x^2 + 2xy + y^2,$$

subject to the condition $y(0) = 0$.

$\boxed{y = -x + \tan x}$

$\frac{dy}{dx} = x^2 + 2xy + y^2$

$\frac{dy}{dx} = (x+y)^2$

$u = x+y$ or $y = u - x$

$\frac{du}{dx} = 1 + \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{du}{dx} - 1$

$\Rightarrow \frac{du}{dx} - 1 = u^2$

$\Rightarrow \frac{du}{dx} = u^2 + 1$

$\Rightarrow \frac{1}{u^2+1}\,du = 1\,dx$

$\Rightarrow \int \frac{1}{u^2+1}\,du = \int 1\,dx$

$\Rightarrow \arctan u = x + C$

$\Rightarrow \boxed{\arctan(x+y) = x + C}$

using $(0,0)$

$\arctan 0 = 0 + C$

$\boxed{C = 0}$

$\Rightarrow \arctan(x+y) = x$

$\Rightarrow x + y = \tan x$

$\Rightarrow y = -x + \tan x$

Question 34 (**+)

Show that the solution of the differential equation

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 24e^{4x},$$

subject to the boundary conditions $y = -1$, $\frac{dy}{dx} = -4$ at $x = 0$, can be written as

$$y = \left(12x^2 - 1\right)e^{4x}.$$

proof

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 24e^{4x}$$

- AUXILIARY EQUATION

$\lambda^2 - 8\lambda + 16 = 0$

$(\lambda - 4)^2 = 0$

$\lambda = 4$ REPEATED

C.F. IS $y = Ae^{4x} + Bxe^{4x}$

- AS e^{4x} IS PART OF THE C.F & xe^{4x} IS ALSO PART OF C.F THEN WE TRY

- $y = Px^2e^{4x}$
- $\frac{dy}{dx} = 2Pxe^{4x} + 4Px^2e^{4x} = 2P(x + 2x^2)e^{4x}$
- $\frac{d^2y}{dx^2} = 2P(1+4x)e^{4x} + 8P(x+2x^2)e^{4x}$

$= 2Pe^{4x}\left[1 + 4x + 4(x + 2x^2)\right]$

$= 2Pe^{4x}\left[8x^2 + 8x + 1\right]$

SUB INTO THE O.D.E

$$\underbrace{2Pe^{4x}\left[8x^2+8x+1\right]}_{\frac{d^2y}{dx^2}} - 8\times\underbrace{2P\left[x+2x^2\right]e^{4x}}_{\frac{dy}{dx}} + 16\underbrace{Px^2e^{4x}}_{y} \equiv 24e^{4x}$$

$$\Rightarrow Pe^{4x}\left[\cancel{16x^2} + \cancel{16x} + 2 - \cancel{16x} - \cancel{32x^2} + \cancel{16x^2}\right] \equiv 24e^{4x}$$

$\therefore 2Pe^{4x} \equiv 24e^{4x}$

$\boxed{P = 12}$

$\therefore$ GEN SOLUTION $y = Ae^{4x} + Bxe^{4x} + 12x^2e^{4x}$

$y = \left(A + Bx + 12x^2\right)e^{4x}$

THUS

$\frac{dy}{dx} = (B + 24x)e^{4x} + 4(A + Bx + 12x^2)e^{4x}$

APPLY CONDITIONS

$x = 0,\ y = -1 \Rightarrow \boxed{-1 = A}$

$x = 0,\ \frac{dy}{dx} = -4 \Rightarrow -4 = B + 4A$

$\Rightarrow -4 = B - 4$

$\Rightarrow \boxed{B = 0}$

$\therefore y = (12x^2 - 1)e^{4x}$

Question 35 (+)**

$$\frac{dy}{dx} + ky = \cos 3x, \quad k \text{ is a non zero constant.}$$

By finding a complimentary function and a particular integral, or otherwise, find the general of the above differential equation.

$$\boxed{y = A\mathrm{e}^{-x} + \frac{k}{9+k^2}\cos 3x + \frac{3}{9+k^2}\sin 3x}$$

$\frac{dy}{dx} + ky = \cos 3x$

- AUXILIARY EQUATION

$\lambda + k = 0$

$\lambda = -k$

COMPLIMENTARY FUNCTION

$\therefore\ y = Ae^{-k}$

- PARTICULAR INTEGRAL

TRY

$y = P\cos 3x + Q\sin 3x$

$y' = -3P\sin 3x + 3Q\cos 3x$

SUB INTO THE O.D.E

$(3Q + kP)\cos 3x + (kQ - 3P)\sin 3x \equiv \cos 3x$

$\left.\begin{array}{l} 3Q + kP = 1 \\ kQ - 3P = 0 \end{array}\right\} \Rightarrow \boxed{P = \frac{1}{3}kQ}$

$\Rightarrow 3Q + k\left(\frac{1}{3}kQ\right) = 1$

$\Rightarrow 3Q + \frac{1}{3}k^2 Q = 1$

$\Rightarrow Q\left(3 + \frac{1}{3}k^2\right) = 1$

$\Rightarrow Q = \frac{1}{3 + \frac{1}{3}k^2}$

$\Rightarrow \boxed{Q = \frac{3}{9+k^2}}$ & $\boxed{P = \frac{k}{9+k^2}}$

$\therefore$ GENERAL SOLUTION

$y = Ae^{-k} + \frac{k}{9+k^2}\cos 3x + \frac{3}{9+k^2}\sin 3x$

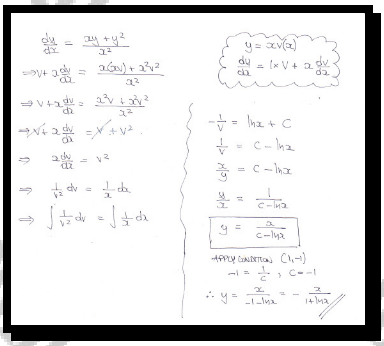
Question 36 (+)**

By using a suitable substitution, solve the differential equation

$$\frac{dy}{dx} = \frac{xy + y^2}{x^2}, \ x > 0,$$

subject to the condition $y = -1$ at $x = 1$.

$$\boxed{y = -\frac{x}{1 + \ln x}}$$



$$\frac{dy}{dx} = \frac{xy + y^2}{x^2}$$

$$y = xv(x)$$
$$\frac{dy}{dx} = 1 \times v + x\frac{dv}{dx}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{x(xv) + x^2v^2}{x^2}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{x^2v + x^2v^2}{x^2}$$

$$\Rightarrow \cancel{v} + x\frac{dv}{dx} = \cancel{v} + v^2.$$

$$\Rightarrow x\frac{dv}{dx} = v^2$$

$$\Rightarrow \frac{1}{v^2}\,dv = \frac{1}{x}\,dx$$

$$\Rightarrow \int \frac{1}{v^2}\,dv = \int \frac{1}{x}\,dx$$

$$-\frac{1}{v} = \ln x + C$$

$$\frac{1}{v} = C - \ln x$$

$$\frac{x}{y} = C - \ln x$$

$$\frac{y}{x} = \frac{1}{C - \ln x}$$

$$\boxed{y = \frac{x}{C - \ln x}}$$

APPLY CONDITION $(1, -1)$

$$-1 = \frac{1}{C}, \quad C = -1$$

$$\therefore y = \frac{x}{-1 - \ln x} = -\frac{x}{1 + \ln x}$$

Question 37 (**+)

Given that $z = f(x)$ and $y = g(x)$ satisfy the following differential equations

$$\frac{dz}{dx} + 2z = e^{-2x} \quad \text{and} \quad \frac{dy}{dx} + 2y = z,$$

a) Find z in the form $z = f(x)$

b) Express y in the form $y = g(x)$, given further that at $x = 0$, $y = 1$, $\frac{dy}{dx} = 0$

$\boxed{z = (x + C)e^{-2x}}$, $\boxed{y = \left(\frac{1}{2}x^2 + 2x + 1\right)e^{-2x}}$

(a) $\frac{dz}{dx} + 2z = e^{-2x}$

- I.F. $e^{\int 2\,dx} = e^{2x}$

$\Rightarrow \frac{d}{dx}\left(ze^{2x}\right) = e^{-2x}e^{2x}$

$\Rightarrow \frac{d}{dx}\left(ze^{2x}\right) = 1$

$\Rightarrow ze^{2x} = \int 1\,dx$

$\Rightarrow ze^{2x} = x + C$

$\Rightarrow z = xe^{-2x} + Ce^{-2x}$

(b) $\frac{dy}{dx} + 2y = z$

$\Rightarrow \frac{dy}{dx} + 2y = xe^{-2x} + Ce^{-2x}$

- IF $= e^{2x}$ AS BEFORE

$\Rightarrow \frac{d}{dx}\left(ye^{2x}\right) = \left(xe^{-2x} + Ce^{-2x}\right)e^{2x}$

$\Rightarrow \frac{d}{dx}\left(ye^{2x}\right) = x + C$

$\Rightarrow ye^{2x} = \int x + C\,dx$

$\Rightarrow ye^{2x} = \frac{1}{2}x^2 + Cx + D$

$\Rightarrow \boxed{y = \left(\frac{1}{2}x^2 + Cx + D\right)e^{-2x}}$

- $x = 0$ $y = 1$

$\boxed{1 = D}$

$\Rightarrow y = \left(\frac{1}{2}x^2 + Cx + 1\right)e^{-2x}$

- WHEN $x = 0$, $y = 1$, $\frac{dy}{dx} = 0$

FROM THE 2ND O.D.E

$0 + 2 = z$

$\therefore \boxed{z = 2}$

HENCE FROM 1ST ODE:

$\boxed{2 = C}$

$\therefore y = \left(\frac{1}{2}x^2 + 2x + 1\right)e^{-2x}$

Question 38 (**+)

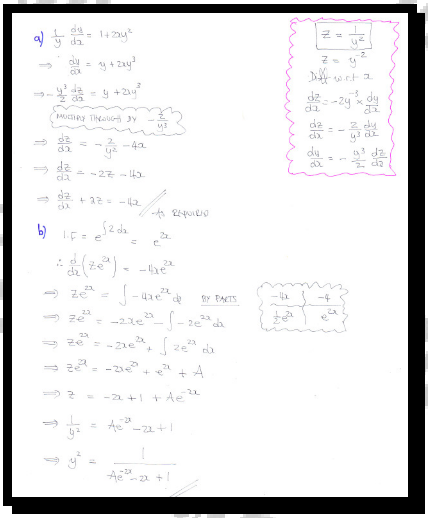
$$\frac{1}{y}\frac{dy}{dx} = 1 + 2xy^2, \quad y > 0.$$

a) Show that the substitution $z = \dfrac{1}{y^2}$ transforms the above differential equation into the new differential equation

$$\frac{dz}{dx} + 2z = -4x.$$

b) Hence find the general solution of the original differential equation, giving the answer in the form $y^2 = f(x)$.

$$\boxed{y^2 = \frac{1}{Ae^{x} - 2x + 1}}$$



a) $\frac{1}{y}\frac{dy}{dx} = 1 + 2xy^2$

$\Rightarrow \frac{dy}{dx} = y + 2xy^3$

$\Rightarrow -\frac{y^3}{2}\frac{dz}{dx} = y + 2xy^3$

MULTIPLY THROUGH BY $-\frac{2}{y^3}$

$\Rightarrow \frac{dz}{dx} = -\frac{2}{y^2} - 4x$

$\Rightarrow \frac{dz}{dx} = -2z - 4x$

$\Rightarrow \frac{dz}{dx} + 2z = -4x$ // AS REQUIRED

$\boxed{z = \frac{1}{y^2}}$

$z = y^{-2}$

Diff w.r.t x

$\frac{dz}{dx} = -2y^{-3} \times \frac{dy}{dx}$

$\frac{dz}{dx} = -\frac{2}{y^3}\frac{dy}{dx}$

$\frac{dy}{dx} = -\frac{y^3}{2}\frac{dz}{dx}$

b) $\text{I.F} = e^{\int 2\,dx} = e^{2x}$

$\therefore \frac{d}{dx}\left(ze^{2x}\right) = -4xe^{2x}$

$\Rightarrow ze^{2x} = \int -4xe^{2x}\,dx$ BY PARTS

$-4x$	-4
$\frac{1}{2}e^{2x}$	e^{2x}

$\Rightarrow ze^{2x} = -2xe^{2x} - \int -2e^{2x}\,dx$

$\Rightarrow ze^{2x} = -2xe^{2x} + \int 2e^{2x}\,dx$

$\Rightarrow ze^{2x} = -2xe^{2x} + e^{2x} + A$

$\Rightarrow z = -2x + 1 + Ae^{-2x}$

$\Rightarrow \frac{1}{y^2} = Ae^{-2x} - 2x + 1$

$\Rightarrow y^2 = \frac{1}{Ae^{-2x} - 2x + 1}$ //

Question 39 (***)

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 4e^{3x}.$$

a) Find a solution of the differential equation given that $y = 1$, $\frac{dy}{dx} = 0$ at $x = 0$.

b) Sketch the graph of y.

The sketch must include …

- the coordinates of any points where the graph meets the coordinate axes.
- the coordinates of any stationary points of the curve.
- clear indications of how the graph looks for large positive or negative values of x.

$$\boxed{y = e^{3x}\left(2x^2 - 3x + 1\right)}$$

(a) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 4e^{3x}$

AUXILIARY EQUATION

$\lambda^2 - 6\lambda + 9 = 0$

$(\lambda - 3)^2 = 0$

$\lambda = 3$ REPEATED

COMPLEMENTARY FUNCTION

$y = Ae^{3x} + Bxe^{3x}$

FOR PARTICULAR INTEGRAL TRY $y = Px^2e^{3x}$

$\frac{dy}{dx} = 2Pxe^{3x} + 3Px^2e^{3x}$

$\frac{d^2y}{dx^2} = 2Pe^{3x} + 6Pxe^{3x} + 6Pxe^{3x} + 9Px^2e^{3x}$

SUB INTO THE O.D.E

$\frac{d^2y}{dx^2} = 2Pe^{3x} + \cancel{12Pxe^{3x}} + \cancel{9Px^2e^{3x}}$

$-6\frac{dy}{dx} = \cancel{-12Pxe^{3x}} \cancel{-18Px^2e^{3x}}$

$+9y = \cancel{9Px^2e^{3x}}$

ADD TO GET $2Pe^{3x} \equiv 4e^{3x}$

$P = 2$

$\therefore y = Ae^{3x} + Bxe^{3x} + 2x^2e^{3x}$

$y = e^{3x}[A + Bx + 2x^2]$

$\frac{dy}{dx} = 3e^{3x}[A + Bx + 2x^2] + e^{3x}[B + 4x]$

- $x=0 \quad y=1 \implies \boxed{1 = A}$
- $x=0 \quad \frac{dy}{dx}=0 \implies 0 = 3A + B \implies \boxed{B = -3}$

$y = e^{3x}[1 - 3x + 2x^2]$

- $y = e^{3x}(2x^2 - 3x + 1)$

  $y = e^{3x}(2x - 1)(x - 1)$

AXIS INTERCEPTS $(0,1)$ $(\frac{1}{2},0)$ $(1,0)$

- $\frac{dy}{dx} = 3e^{3x}(2x^2 - 3x + 1) + e^{3x}(4x - 3)$

  $= e^{3x}(6x^2 - 9x \cancel{+3} + 4x \cancel{-3})$

  $= e^{3x}(6x^2 - 5x)$

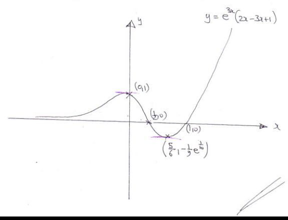
  $= xe^{3x}(6x - 5)$

$\therefore$ T.P $\quad x = 0 \quad y = 1$

$x = \frac{5}{6} \quad y = -\frac{1}{9}e^{\frac{5}{2}}$

- As $x \to +\infty \quad y \to +\infty$

  As $x \to -\infty \quad y \to 0$



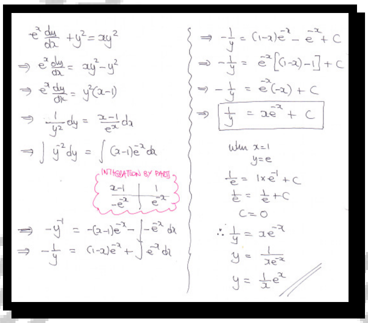
Question 40 (***)

$$e^{x}\frac{dy}{dx}+y^{2}=xy^{2}\,,\ \ x>0\,,\ \ y>0$$

Show that the solution of the above differential equation subject to $y=e$ at $x=1$, is

$$y=\frac{1}{x}e^{x}.$$

proof



Question 41 (***)

$$2y\frac{d^2y}{dx^2}-8y\frac{dy}{dx}+16y^2=\left(\frac{dy}{dx}\right)^2,\ \ y\neq 0,$$

Find the general solution of the above differential equation by using the transformation equation $t=\sqrt{y}$.

Give the answer in the form $y=f(x)$.

$$\boxed{y=\left(A\mathrm{e}^{2x}+Bx\mathrm{e}^{2x}\right)^2}$$

$2y\frac{d^2y}{dx^2}-8y\frac{dy}{dx}+16y^2=\left(\frac{dy}{dx}\right)^2$

$\Rightarrow 2t^2\left[2t\frac{d^2t}{dx^2}+2\left(\frac{dt}{dx}\right)^2\right]-8t^2\left(2t\frac{dt}{dx}\right)+16t^4=\left(2t\frac{dt}{dx}\right)^2$

$\Rightarrow 4t^3\frac{d^2t}{dx^2}+\cancel{4t^2\left(\frac{dt}{dx}\right)^2}-16t^3\frac{dt}{dx}+16t^4=\cancel{4t^2\left(\frac{dt}{dx}\right)^2}$

$\Rightarrow \frac{d^2t}{dx^2}-4\frac{dt}{dx}+4t=0$

- AUXILIARY EQUATION

$\lambda^2-4\lambda+4=0$

$(\lambda-2)^2=0$

$\lambda=2$ (REPEATED)

$\therefore t=Ae^{2x}+Bxe^{2x}$

$\sqrt{y}=Ae^{2x}+Bxe^{2x}$

$y=\left(Ae^{2x}+Bxe^{2x}\right)^2$

$t=y^{\frac{1}{2}}$

$t^2=y$

$\boxed{2t\frac{dt}{dx}=\frac{dy}{dx}}$

$\frac{d^2y}{dx^2}=2\frac{dt}{dx}\frac{dt}{dx}+2t\frac{d^2t}{dx^2}$

$\boxed{\frac{d^2y}{dx^2}=2t\frac{d^2t}{dx^2}+2\left(\frac{dt}{dx}\right)^2}$

Question 42 **(***)**

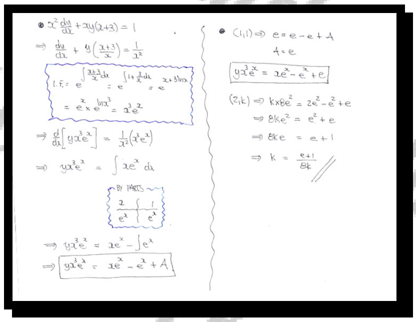
A curve C, with equation $y = f(x)$, passes through the points with coordinates $(1,1)$ and $(2,k)$, where k is a constant.

Given further that the equation of C satisfies the differential equation

$$x^2 \frac{dy}{dx} + xy(x+3) = 1,$$

determine the exact value of k.

$$\boxed{k = \frac{e+1}{8e}}$$



Question 43 (***)

A curve C, with equation $y = f(x)$, meets the y axis the point $(0,1)$.

It is further given that the equation of C satisfies the differential equation

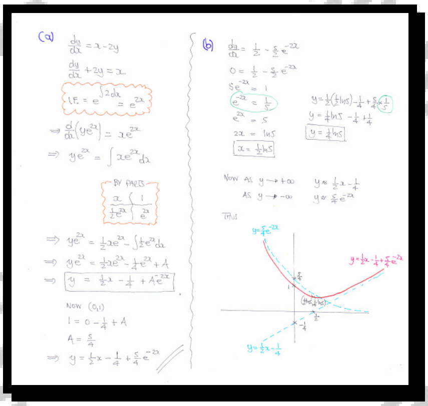
$$\frac{dy}{dx} = x - 2y.$$

a) Determine an equation of C.

b) Sketch the graph of C.

The graph must include in exact simplified form the coordinates of the stationary point of the curve and the equation of its asymptote.

$$\boxed{y = \frac{1}{2}x - \frac{1}{4} + \frac{5}{4}e^{-2x}}$$



Question 44 **(***)**

A curve $y = f(x)$ satisfies the differential equation

$$y = 1 - \frac{dy}{dx}\frac{x+1}{(x-1)(x+2)}, \quad y > 1, x > -1$$

a) Solve the differential equation to show that

$$\ln(y-5) + \frac{1}{2}x^2 + 4x - 2\ln(x+1) = C .$$

When $x = 0$, $y = 2$.

b) Show further that

$$y = 1 + (x+1)^2 e^{-\frac{1}{2}x^2} .$$

$\boxed{\text{proof}}$

(a) $y = 1 - \frac{dy}{dx}\frac{x+1}{(x-1)(x+2)}$

$\Rightarrow \frac{dy}{dx}\frac{x+1}{(x-1)(x+2)} = 1 - y$

$\Rightarrow \frac{1}{1-y}\,dy = \frac{(x-1)(x+2)}{x+1}\,dx$

$\Rightarrow \int \frac{1}{1-y}\,dy = \int \frac{x^2+x-2}{x+1}\,dx$

BY SUBSTITUTION $u = x+1$ OR BY ALGEBRAIC MANIPULATION

$\Rightarrow \int \frac{1}{1-y}\,dy = \int \frac{x(x+1)-2}{x+1}\,dx$

$\Rightarrow \int \frac{1}{1-y}\,dy = \int x - \frac{2}{x+1}\,dx$

$\Rightarrow -\ln|1-y| = \frac{1}{2}x^2 - 2\ln|x+1| + C$

BUT $y>1$ & $x>-1$

$\Rightarrow -\ln(y-1) = \frac{1}{2}x^2 - 2\ln(x+1) + C$

$\Rightarrow \ln(y-1) = -\frac{1}{2}x^2 + 2\ln(x+1) + C$

$\Rightarrow \ln(y-1) + \frac{1}{2}x^2 - 2\ln(x+1) = C$

AS REQUIRED

(b) when $x=0$ $y=2$

$\ln 1 + 0 - 2\ln 1 = C$

$C = 0$

$\therefore \ln(y-1) + \frac{1}{2}x^2 - 2\ln(x+1) = 0$

$\Rightarrow \ln(y-1) = 2\ln(x+1) - \frac{1}{2}x^2$

$\Rightarrow \ln(y-1) = \ln(x+1)^2 - \frac{1}{2}x^2$

$\Rightarrow y - 1 = e^{\ln(x+1)^2 - \frac{1}{2}x^2}$

$\Rightarrow y - 1 = e^{\ln(x+1)^2} \times e^{-\frac{1}{2}x^2}$

$\Rightarrow y - 1 = (x+1)^2 \times e^{-\frac{1}{2}x^2}$

$\Rightarrow y = 1 + (x+1)^2 e^{-\frac{1}{2}x^2}$

AS REQUIRED

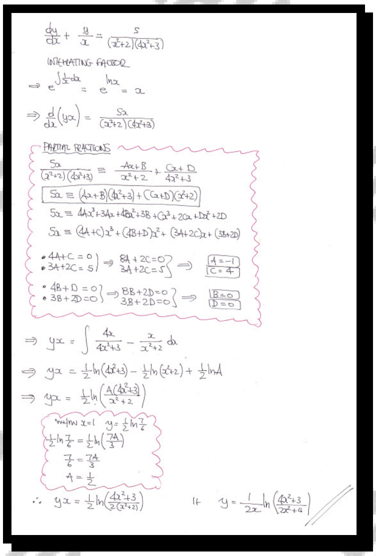
Question 45 (***)

$$\frac{dy}{dx}+\frac{y}{x}=\frac{5}{\left(x^2+2\right)\left(4x^2+3\right)},\ x>0.$$

Given that $y=\frac{1}{2}\ln\frac{7}{6}$ at $x=1$, show that the solution of the above differential equation can be written as

$$y=\frac{1}{2x}\ln\left(\frac{4x^2+3}{2x^2+4}\right).$$

proof



$$\frac{dy}{dx}+\frac{y}{x}=\frac{5}{(x^2+2)(4x^2+3)}$$

INTEGRATING FACTOR

$\Rightarrow e^{\int \frac{1}{x}dx}=e^{\ln x}=x$

$\Rightarrow \frac{d}{dx}(yx)=\frac{5x}{(x^2+2)(4x^2+3)}$

PARTIAL FRACTIONS

$\frac{5x}{(x^2+2)(4x^2+3)}\equiv\frac{Ax+B}{x^2+2}+\frac{Cx+D}{4x^2+3}$

$\boxed{5x\equiv(Ax+B)(4x^2+3)+(Cx+D)(x^2+2)}$

$5x\equiv 4Ax^3+3Ax+4Bx^2+3B+Cx^3+2Cx+Dx^2+2D$

$5x\equiv(4A+C)x^3+(4B+D)x^2+(3A+2C)x+(3B+2D)$

- $4A+C=0$, $3A+2C=5$ $\Rightarrow$ $8A+2C=0$, $3A+2C=5$ $\Rightarrow$ $\boxed{A=-1}$, $\boxed{C=4}$
- $4B+D=0$, $3B+2D=0$ $\Rightarrow$ $8B+2D=0$, $3B+2D=0$ $\Rightarrow$ $\boxed{B=0}$, $\boxed{D=0}$

$\Rightarrow yx=\int\frac{4x}{4x^2+3}-\frac{x}{x^2+2}\,dx$

$\Rightarrow yx=\frac{1}{2}\ln(4x^2+3)-\frac{1}{2}\ln(x^2+2)+\frac{1}{2}\ln A$

$\Rightarrow yx=\frac{1}{2}\ln\left(\frac{A(4x^2+3)}{x^2+2}\right)$

when $x=1$ $y=\frac{1}{2}\ln\frac{7}{6}$

$\frac{1}{2}\ln\frac{7}{6}=\frac{1}{2}\ln\left(\frac{7A}{3}\right)$

$\frac{7}{6}=\frac{7A}{3}$

$A=\frac{1}{2}$

$\therefore\ yx=\frac{1}{2}\ln\left(\frac{4x^2+3}{2(x^2+2)}\right)$ // $y=\frac{1}{2x}\ln\left(\frac{4x^2+3}{2x^2+4}\right)$

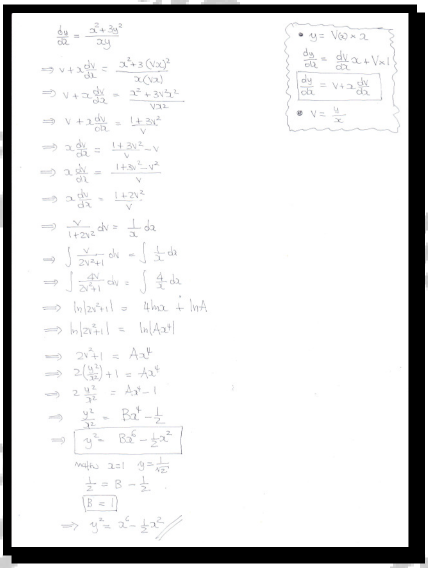
Question 46 (***)

$$\frac{dy}{dx} = \frac{x^2 + 3y^2}{xy}, \quad x > 0, \quad y > 0.$$

Given the boundary condition $y(1) = \frac{1}{\sqrt{2}}$, show that

$$y^2 = x^6 - \frac{1}{2}x^2.$$

proof



$$\frac{dy}{dx} = \frac{x^2+3y^2}{xy}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{x^2+3(vx)^2}{x(vx)}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{x^2+3v^2x^2}{vx^2}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{1+3v^2}{v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+3v^2}{v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+3v^2-v^2}{v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+2v^2}{v}$$

$$\Rightarrow \frac{v}{1+2v^2}\,dv = \frac{1}{x}\,dx$$

$$\Rightarrow \int \frac{v}{2v^2+1}\,dv = \int \frac{1}{x}\,dx$$

$$\Rightarrow \int \frac{4v}{2v^2+1}\,dv = \int \frac{4}{x}\,dx$$

$$\Rightarrow \ln|2v^2+1| = 4\ln x + \ln A$$

$$\Rightarrow \ln|2v^2+1| = \ln|Ax^4|$$

$$\Rightarrow 2v^2+1 = Ax^4$$

$$\Rightarrow 2\left(\frac{y^2}{x^2}\right)+1 = Ax^4$$

$$\Rightarrow 2\frac{y^2}{x^2} = Ax^4 - 1$$

$$\Rightarrow \frac{y^2}{x^2} = Bx^4 - \frac{1}{2}$$

$$\Rightarrow \boxed{y^2 = Bx^6 - \frac{1}{2}x^2}$$

• $y = V(x) \times x$

$\frac{dy}{dx} = \frac{dV}{dx}x + V \times 1$

$\boxed{\frac{dy}{dx} = V + x\frac{dV}{dx}}$

• $V = \frac{y}{x}$

with $x=1$ $y=\frac{1}{\sqrt{2}}$

$\frac{1}{2} = B - \frac{1}{2}$

$\boxed{B=1}$

$$\Rightarrow y^2 = x^6 - \frac{1}{2}x^2$$

Question 47 (***)

The differential equation

$$x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=3x\,,\ \ x\neq 0\,,$$

is to be solved subject to the boundary conditions $y=\frac{3}{2}$, $\frac{dy}{dx}=\frac{1}{2}$ at $x=1$.

a) Show that the substitution $v=\frac{dy}{dx}$, transforms the above differential equation into

$$\frac{dv}{dx}+\frac{2v}{x}=3\,.$$

b) Hence find the solution of the original differential equation, giving the answer in the form $y=f(x)$.

$$\boxed{y=\frac{1}{2}\left(x^2+\frac{1}{x}+1\right)}$$

(a) $x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=3x$

$x\frac{dv}{dx}+2v=3x$

$\frac{dv}{dx}+\frac{2v}{x}=3$

$v=\frac{dy}{dx}$

$\frac{dv}{dx}=\frac{d^2y}{dx^2}$

(b) • $\frac{dv}{dx}+\frac{2v}{x}=0$

$\frac{dv}{dx}=-\frac{2v}{x}$

$\int\frac{1}{v}\,dv=\int-\frac{2}{x}\,dx$

$\ln|v|=-2\ln|x|+C$

$\ln|v|=\ln\left|\frac{A}{x^2}\right|$

$v=\frac{A}{x^2}$

• TRY $v=Px$

$\frac{dv}{dx}=P$

$P+\frac{2}{x}(Px)\equiv 3$

$P+2P=3$

$\boxed{P=1}$

$\therefore\ v=\frac{A}{x^2}+x$ (OR DO IT BY INTEGRATING FACTOR)

$\Rightarrow \frac{dy}{dx}=\frac{A}{x^2}+x$

$\Rightarrow y=-\frac{A}{x}+\frac{1}{2}x^2+B$

• APPLY CONDITION $x=1$, $\frac{dy}{dx}=\frac{1}{2}$ $\Rightarrow \frac{1}{2}=\frac{A}{1}+1$

$\Rightarrow \boxed{A=-\frac{1}{2}}$

• APPLY CONDITION $x=1$, $y=\frac{3}{2}$ $\Rightarrow \frac{3}{2}=\frac{1}{2}+\frac{1}{2}+B$

$\Rightarrow \boxed{B=\frac{1}{2}}$

$\therefore\ y=\frac{1}{2}+\frac{1}{2}x^2+\frac{1}{2x}$

$y=\frac{1}{2x}\left(1+x+x^3\right)$

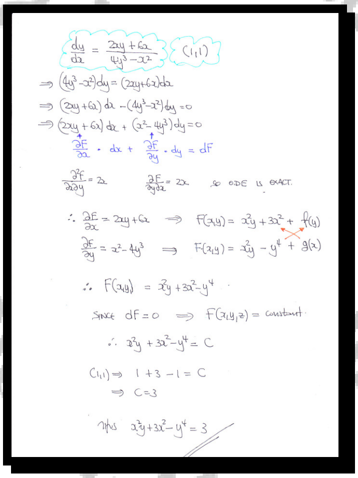
Question 48 (***)

Solve the differential equation

$$\frac{dy}{dx} = \frac{2xy+6x}{4y^3-x^2},$$

subject to the boundary condition $y=1$ at $x=1$.

$$\boxed{x^2y+3x^2-y^4=3}$$



$\frac{dy}{dx} = \frac{2xy+6x}{4y^3-x^2}$ (1,1)

$\Rightarrow (4y^3-x^2)dy = (2xy+6x)dx$

$\Rightarrow (2xy+6x)\,dx - (4y^3-x^2)\,dy = 0$

$\Rightarrow (2xy+6x)\,dx + (x^2-4y^3)\,dy = 0$

$\frac{\partial F}{\partial x} \cdot dx + \frac{\partial F}{\partial y} \cdot dy = dF$

$\frac{\partial^2 F}{\partial x \partial y} = 2x \qquad \frac{\partial^2 F}{\partial y \partial x} = 2x$ so O.D.E is EXACT.

$\therefore \frac{\partial F}{\partial x} = 2xy+6x \Rightarrow F(x,y) = x^2y + 3x^2 + f(y)$

$\frac{\partial F}{\partial y} = x^2 - 4y^3 \Rightarrow F(x,y) = x^2y - y^4 + g(x)$

$\therefore F(x,y) = x^2y + 3x^2 - y^4$

Since $dF = 0 \Rightarrow F(x,y,z) = \text{constant}$

$\therefore x^2y + 3x^2 - y^4 = C$

$(1,1) \Rightarrow 1 + 3 - 1 = C$

$\Rightarrow C = 3$

Thus $x^2y + 3x^2 - y^4 = 3$

Question 49 (***)

By using a suitable substitution, solve the differential equation

$$\frac{dy}{dx} = \frac{x^3 + y^3}{xy^2},$$

subject to the condition $y = 1$ at $x = 1$.

$$\boxed{y^3 = x^3\left(3\ln x + 1\right)}$$

$y = xv$

$\frac{dy}{dx} = 1 \times v + x\frac{dv}{dx}$

$v = \frac{y}{x}$

$$\frac{dy}{dx} = \frac{x^3+y^3}{xy^2}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{x^3 + x^3v^3}{x(x^2v^2)}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{1+v^3}{v^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1}{v^2} + \cancel{v - v}$$

$$\Rightarrow v^2\,dv = \frac{1}{x}\,dx$$

$$\Rightarrow \int v^2\,dv = \int \frac{1}{x}\,dx$$

$$\Rightarrow \frac{1}{3}v^3 = \ln|x| + A$$

$$\Rightarrow v^3 = 3\ln|x| + B$$

$$\Rightarrow \frac{y^3}{x^3} = 3\ln|x| + B$$

$$\Rightarrow y^3 = 3x^3\ln|x| + Bx^3$$

$1 = B$ SINCE $x=1$, $y=1$

$$\Rightarrow y^3 = 3x^3\ln x + x^3$$

Question 50 (***)

$$\left(1-x^2\right)\frac{dy}{dx}+y=\left(1-x^2\right)(1-x)^{\frac{1}{2}}\ ,\ -1<x<1.$$

Given that $y=\frac{\sqrt{2}}{2}$ at $x=\frac{1}{2}$, show that the solution of the above differential equation can be written as

$$y=\frac{2}{3}\sqrt{\left(1-x^2\right)(1+x)}.$$

$\boxed{}$, $\boxed{\text{proof}}$

$$\boxed{(1-x^2)\frac{dy}{dx}+y=(1-x^2)(1-x)^{\frac{1}{2}}}$$

REWRITE THE O.D.E IN "STANDARD" FORM AND LOOK FOR AN INTEGRATING FACTOR

$$\Rightarrow \frac{dy}{dx}+\frac{1}{1-x^2}\frac{dy}{dx}=(1-x)^{\frac{1}{2}}$$

- $\text{I.F.}=e^{\int\frac{1}{1-x^2}dx}=e^{\int\frac{1}{(1+x)(1-x)}dx}=\dots$ PARTIAL FRACTIONS BY INSPECTION (COVER UP)

$$=e^{\int\frac{\frac{1}{2}}{1+x}+\frac{\frac{1}{2}}{1-x}dx}=e^{\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|}=e^{\ln\sqrt{\frac{1+x}{1-x}}}=\frac{\sqrt{1+x}}{\sqrt{1-x}}$$

$$\Rightarrow \frac{d}{dx}\left[y\left(\frac{\sqrt{1+x}}{\sqrt{1-x}}\right)\right]=\cancel{(1-x)^{\frac{1}{2}}}\left(\frac{\sqrt{1+x}}{\cancel{\sqrt{1-x}}}\right)$$

$$\Rightarrow \frac{y(1+x)^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}}=\int(1+x)^{\frac{1}{2}}\,dx$$

$$\Rightarrow \frac{y(1+x)^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}}=\frac{2}{3}(1+x)^{\frac{3}{2}}+A$$

$$\Rightarrow \boxed{y=\frac{2}{3}(1+x)^{1}(1-x)^{\frac{1}{2}}+A\frac{(1-x)^{\frac{1}{2}}}{(1+x)^{\frac{1}{2}}}}$$

APPLY $x=\frac{1}{2}$, $y=\frac{\sqrt{2}}{2}$

$$\Rightarrow \frac{\sqrt{2}}{2}=\frac{2}{3}\times\frac{3}{2}\times\frac{\sqrt{2}}{2}+A\frac{\sqrt{2}/2}{3/2}$$

$$\Rightarrow \frac{\sqrt{2}}{2}=\frac{\sqrt{2}}{2}+A\frac{\sqrt{2}}{3}$$

$$\Rightarrow A=0$$

$$\Rightarrow y=\frac{2}{3}(1+x)(1-x)^{\frac{1}{2}}$$

$$\Rightarrow y=\frac{2}{3}(1+x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}}(1-x)^{\frac{1}{2}}$$

$$\Rightarrow y=\frac{2}{3}(1+x)^{\frac{1}{2}}\sqrt{(1+x)(1-x)}$$

$$\Rightarrow y=\frac{2}{3}(1+x)^{\frac{1}{2}}\sqrt{1-x^2}$$

$$\Rightarrow y=\frac{2}{3}\sqrt{(1+x)(1-x^2)}$$

// AS REQUIRED

Question 51 (***)

By using a suitable substitution, solve the differential equation

$$2x^2 \frac{dy}{dx} = x^2 + y^2, \ x > 0,$$

subject to the condition $y(1) = 0$.

$$\boxed{y = x - \frac{2x}{2 + \ln x}}$$

$$2x^2 \frac{dy}{dx} = x^2 + y^2$$

$$\boxed{\begin{gathered} y = xz(x) \\ \frac{dy}{dx} = 1 \times z + x\frac{dz}{dx} \\ z = \frac{y}{x} \end{gathered}}$$

$$\Rightarrow 2\cancel{x^2}\left[z + x\frac{dz}{dx}\right] = \cancel{x^2} + \cancel{x^2}z^2$$

$$\Rightarrow 2z + 2x\frac{dz}{dx} = 1 + z^2$$

$$\Rightarrow 2x\frac{dz}{dx} = 1 - 2z + z^2$$

$$\Rightarrow 2x\frac{dz}{dx} = (z-1)^2$$

$$\Rightarrow \frac{1}{(z-1)^2}\,dz = \frac{1}{2x}\,dx$$

$$\Rightarrow \int \frac{1}{(z-1)^2}\,dz = \int \frac{1}{2x}\,dx$$

$$\Rightarrow -\frac{1}{z-1} = \frac{1}{2}\ln x + C$$

$$\Rightarrow \frac{1}{z-1} = C - \frac{1}{2}\ln x$$

$$\Rightarrow z - 1 = \frac{1}{C - \frac{1}{2}\ln x}$$

$$\Rightarrow z = \frac{1}{C - \frac{1}{2}\ln x} + 1$$

$$\Rightarrow \frac{y}{x} = \frac{2}{A - \ln x} + 1$$

$$\Rightarrow \boxed{y = \frac{2x}{A - \ln x} + x}$$

Apply $(1,0) \Rightarrow 0 = \frac{2}{A} + 1$

$\Rightarrow A = -$

$$\therefore y = x - \frac{2x}{2 + \ln x}$$

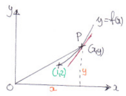
Question 52 (***)

The general point P lies on the curve with equation $y = f(x)$.

The gradient of the curve at P is 2 more than the gradient of the straight line segment OP.

Given further that the curve passes through $Q(1,2)$, express y in terms of x.

$$\boxed{y = 2x(1+\ln x)}$$



GRAD AT P ON THE CURVE

GRAD OF OP

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} + 2$$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = 2$$

$$\text{I.F. is } e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{y}{x}\right) = \frac{2}{x}$$

$$\Rightarrow \frac{y}{x} = \int \frac{2}{x}\,dx$$

$$\Rightarrow \frac{y}{x} = 2\ln x + A$$

$$\Rightarrow \boxed{y = 2x\ln x + Ax}$$

APPLY $(1,2) \Rightarrow 2 = \cancel{2\ln 1} + A$

$A = 2$

$$\therefore y = 2x\ln x + 2x$$

$$y = 2x(1+\ln x)$$

Question 53 (***)

By using a suitable substitution, solve the differential equation

$$x\frac{dy}{dx} - y = x\cos\left(\frac{y}{x}\right),\ x \neq 0,$$

subject to the condition $y(4) = \pi$.

The final answer may not involve natural logarithms.

$$\boxed{\sec\left(\frac{y}{x}\right) + \tan\left(\frac{y}{x}\right) = \frac{1}{4}x\left(1+\sqrt{2}\right)}$$

$$x\frac{dy}{dx} - y = x\cos\left(\frac{y}{x}\right)$$

$$y = xv$$
$$\frac{dy}{dx} = 1 \times v + x\frac{dv}{dx}$$
$$v = \frac{y}{x}$$

$$\Rightarrow x\left(v + x\frac{dv}{dx}\right) - xv = x\cos\left(\frac{xv}{x}\right)$$

$$\Rightarrow \cancel{xv} + x^2\frac{dv}{dx} - \cancel{xv} = x\cos v$$

$$\Rightarrow x\frac{dv}{dx} = \cos v$$

$$\Rightarrow \frac{1}{\cos v}\,dv = \frac{1}{x}\,dx$$

$$\Rightarrow \int \sec v\,dv = \int \frac{1}{x}\,dx$$

$$\Rightarrow \ln|\sec v + \tan v| = \ln|x| + \ln A$$

$$\Rightarrow \ln|\sec v + \tan v| = \ln|Ax|$$

$$\Rightarrow \sec v + \tan v = Ax$$

$$\Rightarrow \boxed{\sec\left(\frac{y}{x}\right) + \tan\left(\frac{y}{x}\right) = Ax}$$

APPLY CONDITION $x=4$ $y=\pi$

$$\sec\frac{\pi}{4} + \tan\frac{\pi}{4} = 4A$$
$$\sqrt{2} + 1 = 4A$$
$$A = \frac{1}{4}(1+\sqrt{2})$$

$$\therefore \sec\left(\frac{y}{x}\right) + \tan\left(\frac{y}{x}\right) = \frac{1}{4}x\left(1+\sqrt{2}\right)$$

Question 54 (***)

The curve C has equation $y = f(x)$ and satisfies the differential equation

$$x^2 \frac{d^2 y}{dx^2} - 2x\frac{dy}{dx} - 2y\left(2x^2 - 1\right) = 3x^3 \mathrm{e}^x, \quad x \neq 0$$

is to be solved subject to the boundary conditions $y = \frac{3}{2}$, $\frac{dy}{dx} = \frac{1}{2}$ at $x = 1$.

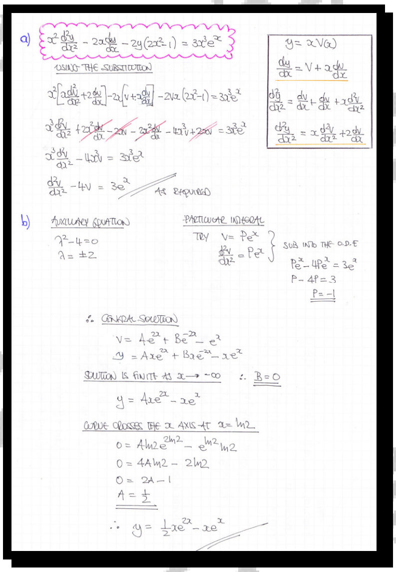
a) Show that the substitution $y = xv$, where v is a function of x transforms the above differential equation into

$$\frac{d^2 v}{dx^2} - 4v = 3\mathrm{e}^x.$$

It is further given that C meets the x axis at $x = \ln 2$ and has a finite value for y as x gets infinitely negatively large.

b) Express the equation of C in the form $y = f(x)$.

$$\boxed{y = \frac{1}{2}x\mathrm{e}^{2x} - x\mathrm{e}^x}$$



a) $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 2y(2x^2-1) = 3x^3e^x$

$y = xv(x)$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\frac{d^2y}{dx^2} = \frac{dv}{dx} + \frac{dv}{dx} + x\frac{d^2v}{dx^2}$

$\frac{d^2y}{dx^2} = x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}$

USING THE SUBSTITUTION

$x^2\left[x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}\right] - 2x\left[v + x\frac{dv}{dx}\right] - 2vx(2x^2-1) = 3x^3e^x$

$x^3\frac{d^2v}{dx^2} + \cancel{2x^2\frac{dv}{dx}} - \cancel{2xv} - \cancel{2x^2\frac{dv}{dx}} - 4x^3v + \cancel{2xv} = 3x^3e^x$

$x^3\frac{d^2v}{dx^2} - 4x^3v = 3x^3e^x$

$\frac{d^2v}{dx^2} - 4v = 3e^x$ AS REQUIRED

b) AUXILIARY EQUATION

$\lambda^2 - 4 = 0$

$\lambda = \pm 2$

PARTICULAR INTEGRAL

TRY $v = Pe^x$, $\frac{d^2v}{dx^2} = Pe^x$ SUB INTO THE O.D.E

$Pe^x - 4Pe^x = 3e^x$

$P - 4P = 3$

$P = -1$

$\therefore$ GENERAL SOLUTION

$v = Ae^{2x} + Be^{-2x} - e^x$

$y = Axe^{2x} + Bxe^{-2x} - xe^x$

SOLUTION IS FINITE AS $x \to -\infty$ $\therefore$ $B = 0$

$y = Axe^{2x} - xe^x$

CURVE CROSSES THE x AXIS AT $x = \ln 2$

$0 = A\ln 2\, e^{2\ln 2} - e^{\ln 2}\ln 2$

$0 = 4A\ln 2 - 2\ln 2$

$0 = 2A - 1$

$A = \frac{1}{2}$

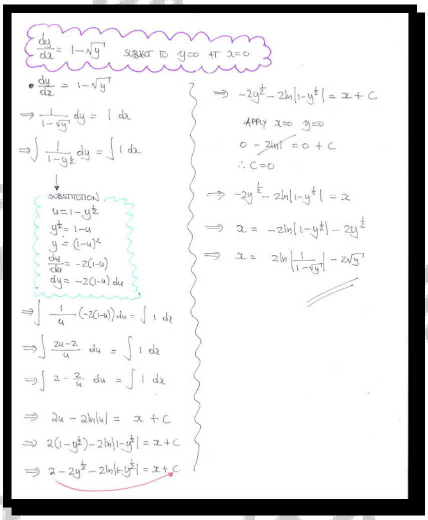
$\therefore$ $y = \frac{1}{2}xe^{2x} - xe^x$

Question 55 (***)

$$\frac{dy}{dx} = 1 - \sqrt{y}\,, \quad y \ge 0\,, \quad y \ne 1\,.$$

Find the solution of the above differential equation subject to the condition $y = 0$ at $x = 0$, giving the answer in the form $x = f(y)$.

$$\boxed{x = 2\ln\left|\frac{1}{1-\sqrt{y}}\right| - 2\sqrt{y}}$$



$\frac{dy}{dx} = 1-\sqrt{y}$ SUBJECT TO $y=0$ AT $x=0$

- $\frac{dy}{dx} = 1-\sqrt{y}$

$\Rightarrow \frac{1}{1-\sqrt{y}}\,dy = 1\,dx$

$\Rightarrow \int \frac{1}{1-y^{\frac{1}{2}}}\,dy = \int 1\,dx$

SUBSTITUTION
$u = 1-y^{\frac{1}{2}}$
$y^{\frac{1}{2}} = 1-u$
$y = (1-u)^2$
$\frac{dy}{du} = -2(1-u)$
$dy = -2(1-u)\,du$

$\Rightarrow \int \frac{1}{u}(-2(1-u))\,du = \int 1\,dx$

$\Rightarrow \int \frac{2u-2}{u}\,du = \int 1\,dx$

$\Rightarrow \int 2 - \frac{2}{u}\,du = \int 1\,dx$

$\Rightarrow 2u - 2\ln|u| = x + C$

$\Rightarrow 2(1-y^{\frac{1}{2}}) - 2\ln|1-y^{\frac{1}{2}}| = x + C$

$\Rightarrow 2 - 2y^{\frac{1}{2}} - 2\ln|1-y^{\frac{1}{2}}| = x + C$

$\Rightarrow -2y^{\frac{1}{2}} - 2\ln|1-y^{\frac{1}{2}}| = x + C$

APPLY $x=0$ $y=0$

$0 - \cancel{2\ln 1} = 0 + C$

$\therefore C = 0$

$\Rightarrow -2y^{\frac{1}{2}} - 2\ln|1-y^{\frac{1}{2}}| = x$

$\Rightarrow x = -2\ln|1-y^{\frac{1}{2}}| - 2y^{\frac{1}{2}}$

$\Rightarrow x = 2\ln\left|\frac{1}{1-\sqrt{y}}\right| - 2\sqrt{y}$

Question 56 (***)

Solve the differential equation

$$x^2\frac{dy}{dx} + xy = y^2, \quad y\left(\frac{1}{2}\right) = 2.$$

$$\boxed{y = \frac{2x}{1-2x^2}}$$

$$x^2\frac{dy}{dx} + xy = y^2$$

$$\Rightarrow x^2\left(-y^2\frac{dz}{dx}\right) + xy = y^2$$

$$\Rightarrow -x^2y^2\frac{dz}{dx} + xy = y^2$$

$$\Rightarrow \frac{dz}{dx} - \frac{1}{yx} = -\frac{1}{x^2}$$

$$\Rightarrow \frac{dz}{dx} - \frac{z}{x} = -\frac{1}{x^2}$$

$$\Rightarrow \frac{d}{dx}\left(z \times \frac{1}{x}\right) = -\frac{1}{x^3}$$

$$\Rightarrow \frac{z}{x} = \int -\frac{1}{x^3}\,dx$$

$$\Rightarrow \frac{z}{x} = \frac{1}{2x^2} + A$$

$$\Rightarrow z = \frac{1}{2x} + Ax$$

$$\Rightarrow \frac{1}{y} = \frac{1 + Bx^2}{2x}$$

$$\Rightarrow \boxed{y = \frac{2x}{1+Bx^2}}$$

• $z = \frac{1}{y}$

$$\frac{dz}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$$

$$\frac{dy}{dx} = -y^2\frac{dz}{dx}$$

• $\text{I.F.} = e^{\int -\frac{1}{x}\,dx} = e^{-\ln x} = \frac{1}{x}$

when $x = \frac{1}{2}$ $y = 2$

$$2 = \frac{1}{1+\frac{1}{4}B}$$

$$2 + \frac{1}{2}B = 1$$

$$\frac{1}{2}B = -1$$

$$\boxed{B = -2}$$

$$\therefore y = \frac{2x}{1-2x^2}$$

Question 57 (***)

Find a general solution of the following differential equation

$$\frac{dy}{dx}=\frac{y\left(y^2-3x^2+1\right)}{x\left(x^2-3y^2-1\right)}.$$

$$\boxed{xy\left(x^2-y^2-1\right)=\text{constant}}$$

$$\frac{dy}{dx}=\frac{y(y^2-3x^2+1)}{x(x^2-3y^2-1)}=\frac{y^3-3x^2y+y}{x^3-3xy^2-x}$$

$\frac{dy}{dx}=-\frac{\partial\phi/\partial x}{\partial\phi/\partial y}$ WHERE $\phi(x,y)=0$

DIFFERENTIATE AGAIN TO CHECK

$$\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right)=3y^2-3x^2+1$$

$$\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right)=-(3x^2-3y^2-1)=3y^2-3x^2+1$$

SO "EXACT"

$\frac{\partial\phi}{\partial x}=y^3-3x^2y+y$

$\phi(x,y)=y^3x-x^3y+xy+F(y)$

$\frac{\partial\phi}{\partial y}=-(x^3-3xy^2-x)$

$\frac{\partial\phi}{\partial y}=-x^3+3xy^2+x$

$\phi(x,y)=-x^3y+y^3x+xy+G(x)$

COMPARING

$\phi(x,y)=xy+y^3x-x^3y+C$

OR

$xy\left[y^2-x^2+1\right]=k$

Question 58 (***)

$$x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} - 4y = 9x^8.$$

Determine the solution of the above differential equation subject to the boundary conditions

$$y = \frac{3}{2}, \ \frac{dy}{dx} = 2 \ \text{at} \ x = 1.$$

$$\boxed{}, \ \boxed{y = \frac{1}{4}x^4\left(x^4+1\right) + \frac{1}{x}}$$

$$\boxed{x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = 9x^8 \quad \text{WITH} \quad x=1, \ y=\tfrac{3}{2}, \ \tfrac{dy}{dx}=2}$$

ASSUME A SOLUTION OF THE FORM $y = x^{\lambda}$

$y' = \lambda x^{\lambda-1}$

$y'' = \lambda(\lambda-1)x^{\lambda-2}$

SUBSTITUTE INTO THE L.H.S OF THE O.D.E (IGNORE R.H.S)

$\Rightarrow x^2\left[\lambda(\lambda-1)x^{\lambda-2}\right] - 2x\left[\lambda x^{\lambda-1}\right] - 4\left[x^{\lambda}\right] = 0$

$\Rightarrow \lambda(\lambda-1)x^{\lambda} - 2\lambda x^{\lambda} - 4x^{\lambda} = 0$

$\Rightarrow \left[\lambda(\lambda-1) - 2\lambda - 4\right]x^{\lambda} = 0$

$\Rightarrow \lambda^2 - 3\lambda - 4 = 0$

$\Rightarrow (\lambda-4)(\lambda+1) = 0$

$\Rightarrow \lambda = \begin{cases} -1 \\ 4 \end{cases}$

$\therefore$ COMPLEMENTARY FUNCTION $\quad y = Ax^{-1} + Bx^4$

PARTICULAR INTEGRAL BY INSPECTION

$$\left.\begin{aligned} y &= Px^8 \\ y' &= 8Px^7 \\ y'' &= 56Px^6 \end{aligned}\right\} \Rightarrow x^2\left[56Px^6\right] - 2x\left[8Px^7\right] - 4Px^8 \equiv 9x^8$$

$\Rightarrow 56Px^8 - 16Px^8 - 4Px^8 \equiv 9x^8$

$\Rightarrow 36P = 9$

$\Rightarrow P = \frac{1}{4}$

$\therefore$ GENERAL SOLUTION IS

$$y = \frac{A}{x} + Bx^4 + \frac{1}{4}x^8$$

APPLYING CONDITIONS $x=1, \ y=\frac{3}{2}, \ \frac{dy}{dx}=2$

$$\left.\begin{aligned} y &= \frac{A}{x} + Bx^4 + \frac{1}{4}x^8 \\ \frac{dy}{dx} &= -\frac{A}{x^2} + 4Bx^3 + 2x^7 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \tfrac{3}{2} &= A + B + \tfrac{1}{4} \\ 2 &= -A + 4B + 2 \end{aligned}\right\} \text{ADDING}$$

$\Rightarrow \frac{7}{2} = 5B + \frac{9}{4}$

$\Rightarrow 14 = 20B + 9$

$\Rightarrow 5 = 20B$

$\Rightarrow B = \frac{1}{4}$

& $A = 4B \Rightarrow A = 1$

FINALLY WE HAVE A SOLUTION

$$y = \frac{1}{x} + \frac{1}{4}x^4 + \frac{1}{4}x^8$$

$$y = \frac{1}{x} + \frac{1}{4}x^4(1+x^4)$$

Question 59 (***)

$$xy\frac{dy}{dx} = (x-y)^2 + xy, \quad y(1) = 0.$$

Show that the solution of the above differential equation is

$$(x-y)e^{\frac{y}{x}} = 1.$$

proof

$$xy\frac{dy}{dx} = (x-y)^2 + xy$$
$$xy\frac{dy}{dx} = x^2 - 2xy + y^2 + xy$$
$$xy\frac{dy}{dx} = x^2 - xy + y^2$$
$$\frac{dy}{dx} = \frac{x^2 - xy + y^2}{xy}$$

I.E A STANDARD HOMOGENEOUS R.H.S.

LET $y = xV(x)$

$$\frac{dy}{dx} = V + x\frac{dV}{dx}$$

$$\Rightarrow V + x\frac{dV}{dx} = \frac{x^2 - x(xV) + (xV)^2}{x(xV)}$$
$$\Rightarrow V + x\frac{dV}{dx} = \frac{x^2 - x^2V + x^2V^2}{x^2V}$$
$$\Rightarrow V + x\frac{dV}{dx} = \frac{1 - V + V^2}{V}$$
$$\Rightarrow x\frac{dV}{dx} = \frac{1 - V + V^2}{V} - V$$
$$\Rightarrow x\frac{dV}{dx} = \frac{1 - V + V^2 - V^2}{V}$$
$$\Rightarrow x\frac{dV}{dx} = \frac{1 - V}{V}$$
$$\Rightarrow \frac{V}{1-V}\,dV = \frac{1}{x}\,dx$$

$$\Rightarrow \int \frac{V}{1-V}\,dV = \int \frac{1}{x}\,dx$$
$$\Rightarrow \int \frac{1-(1-V)}{1-V}\,dV = \int \frac{1}{x}\,dx$$
$$\Rightarrow \int \frac{1}{1-V} - 1\,dV = \int \frac{1}{x}\,dx$$
$$-\ln|1-V| - V = \ln|x| + \ln A$$
$$-\left[\ln|1-V| + \ln e^V\right] = \ln|Ax|$$
$$-\ln|(1-V)e^V| = \ln|Ax|$$
$$\ln|(1-V)e^V| = -\ln|Ax|$$
$$\ln|(1-V)e^V| = \ln\left|\frac{1}{Ax}\right|$$
$$\boxed{(1-V)e^V = \frac{C}{x}}$$
$$\left(1 - \frac{y}{x}\right)e^{\frac{y}{x}} = \frac{C}{x}$$
$$(x-y)e^{\frac{y}{x}} = C$$

NOW $y(1) = 0$

$$(1-0)e^0 = C$$
$$1 = C$$

$$(x-y)e^{\frac{y}{x}} = 1$$

Question 60 (***)

Solve the differential equation

$$\frac{dy}{dx} = (9x + 4y + 1)^2, \quad y(0) = -\frac{1}{4}.$$

Give the answer in the form $y = f(x)$.

$$\boxed{y = -\frac{1}{4} - \frac{9}{4}x + \frac{3}{8}\tan 6x}$$

$$\frac{dy}{dx} = (9x+4y+1)^2, \quad y(0) = -\frac{1}{4}$$

- LET $u = 9x + 4y + 1$
- $\frac{du}{dx} = 9 + 4\frac{dy}{dx}$

$\Rightarrow 4\frac{dy}{dx} = 4(9x+4y+1)^2$

$\Rightarrow 4\frac{dy}{dx} + 9 = 4(9x+4y+1)^2 + 9$

$\Rightarrow \frac{du}{dx} = 4u^2 + 9$

$\Rightarrow \frac{1}{4u^2+9}\,du = 1\,dx$

$\Rightarrow \frac{4}{4u^2+9}\,du = 4\,dx$

$\Rightarrow \frac{1}{u^2+\frac{9}{4}}\,du = 4\,dx$

$\Rightarrow \int \frac{1}{u^2+\left(\frac{3}{2}\right)^2}\,du = \int 4\,dx$

$\Rightarrow \frac{1}{\frac{3}{2}}\arctan\left(\frac{u}{\frac{3}{2}}\right) = 4x + A$

$\Rightarrow \frac{2}{3}\arctan\left(\frac{2}{3}u\right) = 4x + A$

$\Rightarrow \frac{2}{3}\arctan\left[\frac{2}{3}(9x+4y+1)\right] = 4x + A$

$\Rightarrow \frac{2}{3}\arctan\left[6x + \frac{8}{3}y + \frac{2}{3}\right] = 4x + A$

$\Rightarrow \arctan\left(6x + \frac{8}{3}y + \frac{2}{3}\right) = 6x + A$

$\Rightarrow 6x + \frac{8}{3}y + \frac{2}{3} = \tan(6x + A)$

$\Rightarrow \boxed{18x + 8y + 2 = 3\tan(6x + A)}$

when $x = 0$ $y = -\frac{1}{4}$

$0 - 2 + 2 = 3\tan A$

$\tan A = 0$

$A = 0$

$18x + 8y + 2 = 3\tan 6x$

$8y = -2 - 18x + 3\tan 6x$

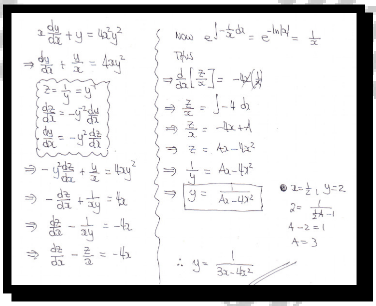
$y = -\frac{1}{4} - \frac{9}{4}x + \frac{3}{8}\tan 6x$

Question 61 (***)

Solve the differential equation

$$x\frac{dy}{dx} + y = 4x^2y^2, \quad y\left(\frac{1}{2}\right) = 2.$$

$$\boxed{y = \frac{1}{3x - 4x^2}}$$



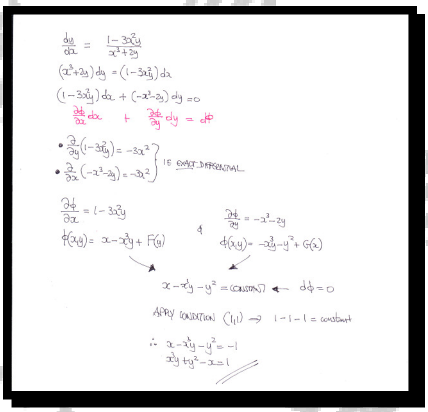
Question 62 (***)

Find the solution of the following differential equation

$$\frac{dy}{dx} = \frac{1 - 3x^2y}{x^3 + 2y},$$

subject to the boundary condition $y = 1$ at $x = 1$.

$$\boxed{x^3y + y^2 - x = 1}$$



Question 63 (***)

Solve the differential equation

$$\frac{dy}{dx} + y = 4xy^3, \quad y(0) = \frac{1}{\sqrt{2}}.$$

Give the answer in the form $y^2 = f(x)$.

$$\boxed{y^2 = \frac{1}{4x+2}}$$

$$\frac{dy}{dx} + y = 4xy^3$$

$$\bullet\; t = \frac{1}{y^2} = y^{-2}$$

$$\frac{dt}{dx} = -2y^{-3}\frac{dy}{dx}$$

$$\frac{dt}{dx} = -\frac{2}{y^3}\frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{y^3}{2}\frac{dt}{dx}$$

$$\Rightarrow -\frac{y^3}{2}\frac{dt}{dx} + y = 4xy^3$$

$$\Rightarrow -\frac{1}{2}\frac{dt}{dx} + \frac{1}{y^2} = 4x$$

$$\Rightarrow \frac{dt}{dx} - \frac{2}{y^2} = -8x$$

$$\Rightarrow \frac{dt}{dx} - 2t = -8x$$

$$\text{IF} = e^{\int -2\,dx} = e^{-2x}$$

HENCE

$$\Rightarrow \frac{d}{dx}\left(te^{-2x}\right) = -8xe^{-2x}$$

$$\Rightarrow te^{-2x} = \int -8xe^{-2x}\,dx$$

BY PARTS

$$-8x \rightarrow -8$$

$$-\frac{1}{2}e^{-2x} \rightarrow e^{-2x}$$

$$\Rightarrow te^{-2x} = 4xe^{-2x} - \int 4e^{-2x}\,dx$$

$$\Rightarrow te^{-2x} = 4xe^{-2x} + 2e^{-2x} + A$$

$$\Rightarrow t = 4x + 2 + Ae^{2x}$$

$$\Rightarrow \frac{1}{y^2} = 4x + 2 + Ae^{2x}$$

$$\Rightarrow \boxed{y^2 = \frac{1}{4x+2+Ae^{2x}}}$$

APPLY CONDITION $x=0$ $y = \frac{1}{\sqrt{2}}$

$$\frac{1}{2} = \frac{1}{0+2+A}$$

$$A = 0$$

HENCE $y^2 = \dfrac{1}{4x+2}$

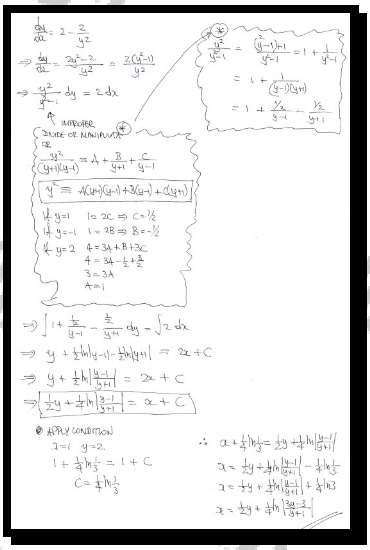
Question 64 (***)

Solve the differential equation

$$\frac{dy}{dx} = 2 - \frac{2}{y^2},$$

subject to the condition $y = 2$ at $x = 1$, giving the answer in the form $x = f(y)$.

$$\boxed{x = \frac{1}{2}y + \frac{1}{4}\ln\left|\frac{3y-3}{y+1}\right|}$$



$$\frac{dy}{dx} = 2 - \frac{2}{y^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2y^2-2}{y^2} = \frac{2(y^2-1)}{y^2}$$

$$\Rightarrow \frac{y^2}{y^2-1}\,dy = 2\,dx$$

IMPROPER
DIVIDE OR MANIPULATE
OR

$$\frac{y^2}{(y+1)(y-1)} \equiv A + \frac{B}{y+1} + \frac{C}{y-1}$$

$$\boxed{y^2 \equiv A(y+1)(y-1) + B(y-1) + C(y+1)}$$

If $y=1$: $1 = 2C \Rightarrow C = \frac{1}{2}$

If $y=-1$: $1 = 2B \Rightarrow B = -\frac{1}{2}$

If $y=2$: $4 = 3A + B + 3C$

$4 = 3A - \frac{1}{2} + \frac{3}{2}$

$3 = 3A$

$A = 1$

$$\frac{y^2}{y^2-1} = \frac{(y^2-1)+1}{y^2-1} = 1 + \frac{1}{y^2-1} = 1 + \frac{1}{(y-1)(y+1)} = 1 + \frac{\frac{1}{2}}{y-1} - \frac{\frac{1}{2}}{y+1}$$

$$\Rightarrow \int 1 + \frac{\frac{1}{2}}{y-1} - \frac{\frac{1}{2}}{y+1}\,dy = \int 2\,dx$$

$$\Rightarrow y + \frac{1}{2}\ln|y-1| - \frac{1}{2}\ln|y+1| = 2x + C$$

$$\Rightarrow y + \frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| = 2x + C$$

$$\Rightarrow \boxed{\frac{1}{2}y + \frac{1}{4}\ln\left|\frac{y-1}{y+1}\right| = x + C}$$

- APPLY CONDITION

$x=1 \quad y=2$

$1 + \frac{1}{4}\ln\frac{1}{3} = 1 + C$

$C = \frac{1}{4}\ln\frac{1}{3}$

$$\therefore x + \frac{1}{4}\ln\frac{1}{3} = \frac{1}{2}y + \frac{1}{4}\ln\left|\frac{y-1}{y+1}\right|$$

$$x = \frac{1}{2}y + \frac{1}{4}\ln\left|\frac{y-1}{y+1}\right| - \frac{1}{4}\ln\frac{1}{3}$$

$$x = \frac{1}{2}y + \frac{1}{4}\ln\left|\frac{y-1}{y+1}\right| + \frac{1}{4}\ln 3$$

$$x = \frac{1}{2}y + \frac{1}{4}\ln\left|\frac{3y-3}{y+1}\right|$$

Question 65 (***)

By using a suitable substitution, solve the differential equation

$$xy\frac{dy}{dx} + 2y^2 = x, \quad y(1) = 0.$$

Give the answer in the form $y^2 = f(x)$.

$$\boxed{y^2 = \frac{2}{5}\left(x - \frac{1}{x^4}\right)}$$

$$z = y^2$$
$$\frac{dz}{dx} = 2y\frac{dy}{dx}$$
$$\frac{1}{2}\frac{dz}{dx} = y\frac{dy}{dx}$$

$$xy\frac{dy}{dx} + 2y^2 = x$$
$$x\left(\frac{1}{2}\frac{dz}{dx}\right) + 2y^2 = x$$
$$\frac{dz}{dx} + \frac{4y^2}{x} = 2$$
$$\frac{dz}{dx} + \frac{4z}{x} = 2$$
$$\text{IF} = e^{\int \frac{4}{x}\,dx} = e^{4\ln x} = x^4$$
$$\Rightarrow \frac{d}{dx}\left(zx^4\right) = 2x^4$$
$$\Rightarrow zx^4 = \int 2x^4\,dx$$
$$\Rightarrow zx^4 = \frac{2}{5}x^5 + A$$
$$\Rightarrow z = \frac{2}{5}x + \frac{A}{x^4}$$
$$\Rightarrow \boxed{y^2 = \frac{2}{5}x + \frac{A}{x^4}}$$

$x = 1,\ y = 0$

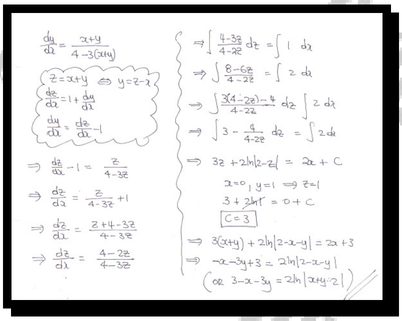
$$0 = \frac{2}{5} + A$$
$$A = -\frac{2}{5}$$
$$\therefore y^2 = \frac{2}{5}x - \frac{2/5}{x^4}$$
$$y^2 = \frac{2}{5}\left(x - \frac{1}{x^4}\right)$$

Question 67 (***)

Use a suitable substitution to solve the differential equation

$$\frac{dy}{dx} = \frac{x+y}{4-3(x+y)}, \quad y(0)=1.$$

$$\boxed{2\ln|x+y-2| = 3-x-3y}$$



$\frac{dy}{dx} = \frac{x+y}{4-3(x+y)}$

$z = x+y \Leftrightarrow y = z-x$

$\frac{dz}{dx} = 1 + \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{dz}{dx} - 1$

$\Rightarrow \frac{dz}{dx} - 1 = \frac{z}{4-3z}$

$\Rightarrow \frac{dz}{dx} = \frac{z}{4-3z} + 1$

$\Rightarrow \frac{dz}{dx} = \frac{z+4-3z}{4-3z}$

$\Rightarrow \frac{dz}{dx} = \frac{4-2z}{4-3z}$

$\Rightarrow \int \frac{4-3z}{4-2z}\,dz = \int 1\,dx$

$\Rightarrow \int \frac{8-6z}{4-2z} = \int 2\,dx$

$\Rightarrow \int \frac{3(4-2z)-4}{4-2z}\,dz \int 2\,dx$

$\Rightarrow \int 3 - \frac{4}{4-2z}\,dz = \int 2\,dx$

$\Rightarrow 3z + 2\ln|2-z| = 2x + C$

$x=0,\ y=1 \Rightarrow z=1$

$3 + 2\ln 1 = 0 + C$

$\boxed{C=3}$

$\Rightarrow 3(x+y) + 2\ln|2-x-y| = 2x+3$

$\Rightarrow -x-3y+3 = 2\ln|2-x-y|$

$\left(\text{OR } 3-x-3y = 2\ln|x+y-2|\right)$

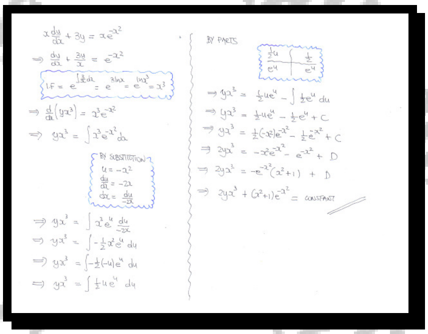
Question 68 (***)

$$x\frac{dy}{dx} + 3y = x\mathrm{e}^{-x^2}, \quad x>0.$$

Show clearly that the general solution of the above differential equation can be written in the form

$$2yx^3 + \left(x^2+1\right)\mathrm{e}^{-x^2} = \text{constant}.$$

$\boxed{\text{proof}}$



$x\frac{dy}{dx} + 3y = xe^{-x^2}$

$\Rightarrow \frac{dy}{dx} + \frac{3y}{x} = e^{-x^2}$

$\text{I.F.} = e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3$

$\Rightarrow \frac{d}{dx}(yx^3) = x^3e^{-x^2}$

$\Rightarrow yx^3 = \int x^3e^{-x^2}\,dx$

BY SUBSTITUTION

$u = -x^2$

$\frac{du}{dx} = -2x$

$dx = \frac{du}{-2x}$

$\Rightarrow yx^3 = \int x^3e^u \frac{du}{-2x}$

$\Rightarrow yx^3 = \int -\frac{1}{2}x^2e^u\,du$

$\Rightarrow yx^3 = \int -\frac{1}{2}(-u)e^u\,du$

$\Rightarrow yx^3 = \int \frac{1}{2}ue^u\,du$

BY PARTS

$\frac{1}{2}u$	$\frac{1}{2}$
e^u	e^u

$\Rightarrow yx^3 = \frac{1}{2}ue^u - \int \frac{1}{2}e^u\,du$

$\Rightarrow yx^3 = \frac{1}{2}ue^u - \frac{1}{2}e^u + C$

$\Rightarrow yx^3 = \frac{1}{2}(-x^2)e^{-x^2} - \frac{1}{2}e^{-x^2} + C$

$\Rightarrow 2yx^3 = -x^2e^{-x^2} - e^{-x^2} + D$

$\Rightarrow 2yx^3 = -e^{-x^2}(x^2+1) + D$

$\Rightarrow 2yx^3 + (x^2+1)e^{-x^2} = \text{CONSTANT}$

Question 69 (***)

Solve the following differential equation

$$\frac{dy}{dx} = \frac{3x+2y}{3y-2x}, \quad y(1)=3.$$

Give the final answer in the form $F(x, y) = 12$

$\boxed{}$, $\boxed{3y^2 - 4xy - 3x^2 = 12}$

USING THE SUBSTITUTION GIVEN

$\Rightarrow y = xV(x)$

$\Rightarrow \frac{dy}{dx} = 1 \times V(x) + x\frac{dv}{dx}$

$\Rightarrow \frac{dy}{dx} = V + x\frac{dv}{dx}$

SUBSTITUTING INTO THE O.D.E.

$\Rightarrow \frac{dy}{dx} = \frac{3x+2y}{3y-2x}$

$\Rightarrow V + x\frac{dv}{dx} = \frac{3x+2xV}{3xV-2x}$

$\Rightarrow V + x\frac{dv}{dx} = \frac{3+2V}{3V-2}$

$\Rightarrow x\frac{dv}{dx} = \frac{3+2V}{3V-2} - V$

$\Rightarrow x\frac{dv}{dx} = \frac{3+2V-3V^2+2V}{3V-2}$

$\Rightarrow x\frac{dv}{dx} = \frac{3+4V-3V^2}{3V-2}$

SEPARATING VARIABLES

$\Rightarrow \int \frac{3V-2}{3+4V-3V^2}\,dv = \int \frac{1}{x}\,dx$

$\Rightarrow \int \frac{3V-2}{3V^2-4V-3}\,dv = \int -\frac{1}{x}\,dx$

$\Rightarrow \int \frac{6V-4}{3V^2-4V-3}\,dv = \int -\frac{2}{x}\,dx$

$\Rightarrow \ln|3V^2-4V-3| = -2\ln|x| + \ln A$

$\Rightarrow \ln|3V^2-4V-3| = \ln\left(\frac{1}{x^2}\right) + \ln A$

$\Rightarrow \ln|3V^2-4V-3| = \ln\left|\frac{A}{x^2}\right|$

$\Rightarrow 3V^2-4V-3 = \frac{A}{x^2}$

REVERSING THE TRANSFORMATION

$\Rightarrow 3\left(\frac{y}{x}\right)^2 - 4\left(\frac{y}{x}\right) - 3 = \frac{A}{x^2}$

$\Rightarrow \frac{3y^2}{x^2} - \frac{4y}{x} - 3 = \frac{A}{x^2}$

$\Rightarrow 3y^2 - 4xy - 3x^2 = A$

APPLY CONDITION $(1,3)$

$3\times3^2 - 4\times1\times3 - 3\times1^2 = A$

$27 - 12 - 3 = A$

$A = 12$

$\therefore 3y^2 - 4xy - 3x^2 = 12$

ALTERNATIVE SUBSTITUTION

LET $V(x) = 3y - 2x$

$\frac{dv}{dx} = 3\frac{dy}{dx} - 2$

$3\frac{dy}{dx} = \frac{dv}{dx} + 2$

ALSO WE HAVE

$3y = V + 2x$

$6y = 2V + 4x$

$\Rightarrow \frac{dy}{dx} = \frac{3x+2y}{3y-2x}$

$\Rightarrow 3\frac{dy}{dx} = \frac{9x+6y}{3y-2x}$

$\Rightarrow \frac{dv}{dx} + 2 = \frac{9x+(2V+4x)}{V}$

$\Rightarrow \frac{dv}{dx} + 2 = \frac{2V+13x}{V}$

$\Rightarrow \frac{dv}{dx} + \cancel{2} = \cancel{2} + \frac{13x}{V}$

$\Rightarrow \int V\,dv = \int 13x\,dx$

$\Rightarrow \frac{1}{2}V^2 = \frac{13}{2}x^2 + C$

$\Rightarrow V^2 = 13x^2 + C$

$\Rightarrow (3y-2x)^2 = 13x^2 + C$

APPLY $(1,3)$ $\Rightarrow (9-2)^2 = 13\times1^2 + C$

$\Rightarrow 49 = 13 + C$

$\Rightarrow C = 36$

$\Rightarrow (3y-2x)^2 = 13x^2 + 36$

$\Rightarrow 9y^2 - 12xy + 4x^2 = 13x^2 + 36$

$\Rightarrow 9y^2 - 12xy - 9x^2 = 36$

$\Rightarrow 3y^2 - 4xy - 3x^2 = 12$

AS BEFORE

ALTERNATIVE BY MULTIVARIABLE CALCULUS

$\Rightarrow \frac{dy}{dx} = \frac{3x+2y}{3y-2x}$

$\Rightarrow (3y-2x)\,dy = (3x+2y)\,dx$

$\Rightarrow (3x+2y)\,dx + (2x-3y)\,dy = 0$

$\frac{\partial G}{\partial x}\,dx + \frac{\partial G}{\partial y}\,dy = dG$

EVIDENTLY THIS IS EXACT AS $\frac{\partial^2 G}{\partial x \partial y} = \frac{\partial^2 G}{\partial y \partial x} = 2$

HENCE WE HAVE BY DIRECT INTEGRATION

- $dG = 0 \Rightarrow G(x,y) = \text{constant}$
- $\frac{\partial G}{\partial x} = 3x+2y \Rightarrow G(x,y) = \frac{3}{2}x^2 + 2xy + f(y)$
- $\frac{\partial G}{\partial y} = 2x-3y \Rightarrow G(x,y) = 2xy - \frac{3}{2}y^2 + g(x)$

$\therefore f(y) = -\frac{3}{2}y^2$ & $g(x) = \frac{3}{2}x^2$

THUS WE OBTAIN

$G(x,y) = \text{constant}$

$\frac{3}{2}x^2 + 2xy - \frac{3}{2}y^2 = \text{constant}$

$3x^2 + 4xy - 3y^2 = \text{constant}$

$(1,3)$ NOW YIELDS

$3 + 12 - 27 = \text{constant}$

$\text{constant} = -12$

$3x^2 + 4xy - 3y^2 = -12$

$3y^2 - 4xy - 3x^2 = 12$

AS BEFORE

Question 70 **(***)**

Find the general solution of the following differential equation

$$x^2 \frac{d^2 y}{dx^2} + 2x\frac{dy}{dx} - n(n+1)y = 0.$$

$$\boxed{y = Ax^n + \frac{B}{x^{n+1}}}$$

$$x^2 \frac{d^2 y}{dx^2} + 2x\frac{dy}{dx} - n(n+1)y = 0$$

THIS IS A STANDARD "EULER TYPE" EQUATION

LET $y = x^\lambda$

$\frac{dy}{dx} = \lambda x^{\lambda-1}$

$\frac{d^2y}{dx^2} = \lambda(\lambda-1)x^{\lambda-2}$

SUB INTO THE O.D.E

$\Rightarrow x^2 \lambda(\lambda-1)x^{\lambda-2} + 2x\left[\lambda x^{\lambda-1}\right] - n(n+1)x^\lambda = 0$

$\Rightarrow \lambda(\lambda-1)x^\lambda + 2\lambda x^\lambda - n(n+1)x^\lambda = 0$

$\Rightarrow \left[\lambda(\lambda-1) + 2\lambda - n(n+1)\right]x^\lambda = 0$

$\Rightarrow \lambda^2 + \lambda - n^2 - n = 0$

$\Rightarrow \lambda^2 + \lambda = n^2 + n$

$\Rightarrow \left(\lambda + \frac{1}{2}\right)^2 - \frac{1}{4} = n^2 + n$

$\Rightarrow \left(\lambda + \frac{1}{2}\right)^2 = n^2 + n + \frac{1}{4}$

$\Rightarrow \left(\lambda + \frac{1}{2}\right)^2 = \left(n + \frac{1}{2}\right)^2$

$\Rightarrow \lambda + \frac{1}{2} = \pm\left(n + \frac{1}{2}\right)$

$\Rightarrow \lambda + \frac{1}{2} = \begin{cases} n + \frac{1}{2} \\ -n - \frac{1}{2} \end{cases}$

$\Rightarrow \lambda = \begin{cases} n \\ -n-1 \end{cases}$

$\therefore$ GEN SOLUTION $y = Ax^n + Bx^{-n-1}$

$y = Ax^n + \frac{B}{x^{n+1}}$

Question 71 **(***)**

Use the substitution $y = e^z$ to solve the differential equation

$$x\frac{dy}{dx} + y\ln y = 2xy, \quad y(1) = e^2.$$

$$\boxed{y = e^{x + \frac{1}{x}}}$$

$x\frac{dy}{dx} + y\ln y = 2xy$

$y = e^z$

$\frac{dy}{dx} = e^z \frac{dz}{dx}$

$\Rightarrow x\left(e^z \frac{dz}{dx}\right) + e^z\left(\ln e^z\right) = 2xe^z$

$\Rightarrow x\frac{dz}{dx} + z = 2x$

BY INTEGRATING FACTOR OR NOTICING LHS IS EXACT

$\Rightarrow \frac{d}{dx}(xz) = 2x$

$\Rightarrow xz = \int 2x \, dx$

$\Rightarrow xz = x^2 + A$

$\Rightarrow \boxed{x\ln y = x^2 + A}$

$(1, e^2) \Rightarrow 2 = 1 + A$

$\Rightarrow \boxed{A = 1}$

$\Rightarrow x\ln y = x^2 + 1$

$\Rightarrow \ln y = x + \frac{1}{x}$

$\Rightarrow y = e^{x + \frac{1}{x}}$

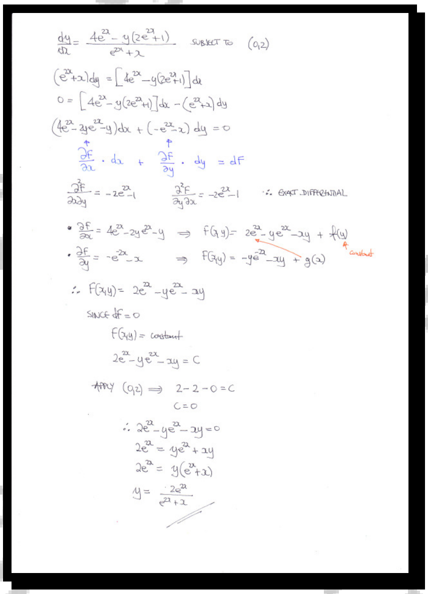
Question 72 (***)

Solve the differential equation

$$\frac{dy}{dx}=\frac{4e^{2x}-y\left(2e^{2x}+1\right)}{e^{2x}+x},$$

subject to the boundary condition $y=2$ at $x=0$.

$$\boxed{y=\frac{2e^{2x}}{e^{2x}+x}}$$



$\frac{dy}{dx}=\frac{4e^{2x}-y(2e^{2x}+1)}{e^{2x}+x}$ SUBJECT TO $(0,2)$

$(e^{2x}+x)dy=\left[4e^{2x}-y(2e^{2x}+1)\right]dx$

$0=\left[4e^{2x}-y(2e^{2x}+1)\right]dx-(e^{2x}+x)dy$

$(4e^{2x}-2ye^{2x}-y)dx+(-e^{2x}-x)\,dy=0$

$\frac{\partial F}{\partial x}\cdot dx+\frac{\partial F}{\partial y}\cdot dy=dF$

$\frac{\partial^2 F}{\partial x\partial y}=-2e^{2x}-1$ $\quad\frac{\partial^2 F}{\partial y\partial x}=-2e^{2x}-1$ $\quad\therefore$ EXACT DIFFERENTIAL

- $\frac{\partial F}{\partial x}=4e^{2x}-2ye^{2x}-y \Rightarrow F(x,y)=2e^{2x}-ye^{2x}-xy+f(y)$
- $\frac{\partial F}{\partial y}=-e^{-2x}-x \Rightarrow F(x,y)=-ye^{-2x}-xy+g(x)$

constant

$\therefore F(x,y)=2e^{2x}-ye^{2x}-xy$

SINCE $dF=0$

$F(x,y)=$ constant

$2e^{2x}-ye^{2x}-xy=C$

APPLY $(0,2)\Rightarrow 2-2-0=C$

$C=0$

$\therefore 2e^{2x}-ye^{2x}-xy=0$

$2e^{2x}=ye^{2x}+xy$

$2e^{2x}=y(e^{2x}+x)$

$y=\frac{2e^{2x}}{e^{2x}+x}$

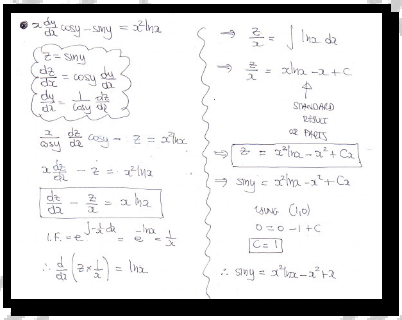
Question 73 (***)

Use the substitution $z = \sin y$ to solve the differential equation

$$x\frac{dy}{dx}\cos y - \sin y = x^2 \ln x\,, \quad y(1) = 0$$

subject to the condition $y = 0$ at $x = 1$.

$$\boxed{\sin y = x^2 \ln x - x^2 + x}$$



$\bullet\; x\frac{dy}{dx}\cos y - \sin y = x^2 \ln x$

$z = \sin y$

$\frac{dz}{dx} = \cos y \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{\cos y}\frac{dz}{dx}$

$\frac{x}{\cos y}\frac{dz}{dx}\cos y - z = x^2 \ln x$

$x\frac{dz}{dx} - z = x^2 \ln x$

$\boxed{\frac{dz}{dx} - \frac{z}{x} = x \ln x}$

$\text{I.F.} = e^{\int -\frac{1}{x}\,dx} = e^{-\ln x} = \frac{1}{x}$

$\therefore \frac{d}{dx}\left(z \times \frac{1}{x}\right) = \ln x$

$\Rightarrow \frac{z}{x} = \int \ln x \, dx$

$\Rightarrow \frac{z}{x} = x\ln x - x + C$

STANDARD RESULT OR PARTS

$\Rightarrow \boxed{z = x^2 \ln x - x^2 + Cx}$

$\Rightarrow \sin y = x^2 \ln x - x^2 + Cx$

USING $(1,0)$

$0 = 0 - 1 + C$

$\boxed{C = 1}$

$\therefore \sin y = x^2 \ln x - x^2 + x$

Question 74 (***+)

The differential equation

$$\left(x^3+1\right)\frac{d^2y}{dx^2}-3x^2\frac{dy}{dx}=2-4x^3,$$

is to be solved subject to the boundary conditions $y=0$, $\frac{dy}{dx}=4$ at $x=0$.

Use the substitution $u=\frac{dy}{dx}-2x$, where u is a function of x, to show that the solution of the above differential equation is

$$y=x^4+x^2+4x.$$

proof

USING THE SUBSTITUTION GIVEN

$$\Rightarrow u=\frac{dy}{dx}-2x$$
$$\Rightarrow \frac{dy}{dx}=u+2x$$
$$\Rightarrow \frac{d^2y}{dx^2}=\frac{du}{dx}+2$$

SUBSTITUTE INTO THE O.D.E.

$$\Rightarrow (x^3+1)\frac{d^2y}{dx^2}-3x^2\frac{dy}{dx}=2-4x^3$$
$$\Rightarrow (x^3+1)\left(\frac{du}{dx}+2\right)-3x^2(u+2x)=2-4x^3$$
$$\Rightarrow (x^3+1)\frac{du}{dx}+2(x^3+1)-3x^2u-6x^3=2-4x^3$$
$$\Rightarrow (x^3+1)\frac{du}{dx}+2x^3\cancel{+2}-3ux^2-6x^3=\cancel{2}-4x^3$$
$$\Rightarrow (x^3+1)\frac{du}{dx}-3ux^2\cancel{-4x^3}=\cancel{-4x^3}$$
$$\Rightarrow (x^3+1)\frac{du}{dx}=3ux^2$$

SEPARATE VARIABLES

$$\Rightarrow \frac{1}{u}\,du=\frac{3x^2}{x^3+1}\,dx$$
$$\Rightarrow \int\frac{1}{u}\,du=\int\frac{3x^2}{x^3+1}\,dx$$
$$\Rightarrow \ln|u|=\ln|x^3+1|+\ln A$$
$$\Rightarrow \ln|u|=\ln|A(x^3+1)|$$
$$\Rightarrow \boxed{u=A(x^3+1)}$$

REVERSING THE TRANSFORMATION

$$\Rightarrow \frac{dy}{dx}-2x=A(x^3+1)$$
$$\Rightarrow \frac{dy}{dx}=A(x^3+1)+2x$$

INTEGRATING W.R.T x

$$\Rightarrow y=A\left(\tfrac{1}{4}x^4+x\right)+x^2+B$$

USING THE CONDITION GIVEN

$$x=0,\ y=0 \Rightarrow 0=B$$
$$x=0,\ \frac{dy}{dx}=4 \Rightarrow 4=A$$

$$\therefore\ y=4\left(\tfrac{1}{4}x^4+x\right)+x^2$$
$$y=x^4+4x+x^2$$
$$y=x^4+x^2+4x$$

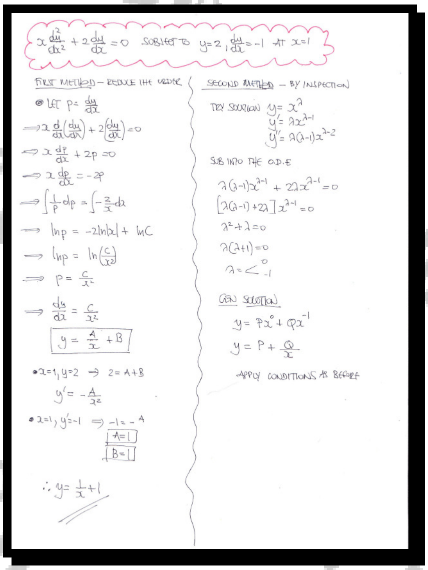
Question 75 (***+)

Solve the differential equation

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0,$$

subject to the boundary conditions $y = 2$, $\frac{dy}{dx} = -1$ at $x = 1$.

$$\boxed{y = \frac{2e^{2x}}{e^{2x} + x}}$$



$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$ SUBJECT TO $y = 2$, $\frac{dy}{dx} = -1$ AT $x = 1$

FIRST METHOD – REDUCE THE ORDER

- LET $p = \frac{dy}{dx}$

$\Rightarrow x\frac{d}{dx}\left(\frac{dy}{dx}\right) + 2\left(\frac{dy}{dx}\right) = 0$

$\Rightarrow x\frac{dp}{dx} + 2p = 0$

$\Rightarrow x\frac{dp}{dx} = -2p$

$\Rightarrow \int \frac{1}{p}\,dp = \int -\frac{2}{x}\,dx$

$\Rightarrow \ln p = -2\ln|x| + \ln C$

$\Rightarrow \ln p = \ln\left(\frac{C}{x^2}\right)$

$\Rightarrow p = \frac{C}{x^2}$

$\Rightarrow \frac{dy}{dx} = \frac{C}{x^2}$

$\boxed{y = \frac{A}{x} + B}$

- $x = 1, y = 2 \Rightarrow 2 = A + B$

$y' = -\frac{A}{x^2}$

- $x = 1, y' = -1 \Rightarrow -1 = -A$

$\boxed{A = 1}$

$\boxed{B = 1}$

$\therefore y = \frac{1}{x} + 1$

SECOND METHOD – BY INSPECTION

TRY SOLUTION $y = x^\lambda$

$y' = \lambda x^{\lambda-1}$

$y'' = \lambda(\lambda-1)x^{\lambda-2}$

SUB INTO THE O.D.E

$\lambda(\lambda-1)x^{\lambda-1} + 2\lambda x^{\lambda-1} = 0$

$[\lambda(\lambda-1) + 2\lambda]x^{\lambda-1} = 0$

$\lambda^2 + \lambda = 0$

$\lambda(\lambda+1) = 0$

$\lambda = 0, -1$

GEN SOLUTION

$y = Px^0 + Qx^{-1}$

$y = P + \frac{Q}{x}$

APPLY CONDITIONS AS BEFORE

Question 76 (***+)

$$\frac{dy}{dx} + \frac{2y}{x} = y^4, \quad x > 0, \quad y > 0.$$

Given that $y(1) = 1$, show that

$$y^3 = \frac{5}{3x + 2x^6}.$$

$\boxed{\text{proof}}$

$$\boxed{\frac{dy}{dx} + \frac{2y}{x} = y^4}$$

$u = \frac{1}{y^3}$ OR $u = y^{-3}$

$\frac{du}{dx} = -3y^{-4}\frac{dy}{dx}$

$\frac{du}{dx} = -\frac{3}{y^4}\frac{dy}{dx}$

$\boxed{-\frac{y^4}{3}\frac{du}{dx} = \frac{dy}{dx}}$

THUS TRANSFORMING THE O.D.E

$\Rightarrow -\frac{y^4}{3}\frac{du}{dx} + \frac{2y}{x} = y^4$ (MULTIPLY BY $\frac{-3}{y^4}$)

$\Rightarrow \frac{du}{dx} - \frac{6}{xy^3} = -3$

$\Rightarrow \frac{du}{dx} - \frac{6}{x}\left(\frac{1}{y^3}\right) = -3$

$\Rightarrow \boxed{\frac{du}{dx} - \frac{6u}{x} = -3}$

$\text{I.F.} = e^{\int -\frac{6}{x}\,dx} = e^{-6\ln x} = e^{\ln x^{-6}} = \frac{1}{x^6}$

THUS

$\frac{d}{dx}\left(u \times \frac{1}{x^6}\right) = -3 \times \frac{1}{x^6}$

$\Rightarrow \frac{u}{x^6} = \int -\frac{3}{x^6}\,dx$

$\Rightarrow \frac{u}{x^6} = \int -3x^{-6}\,dx$

$\Rightarrow \frac{u}{x^6} = \frac{3}{5}x^{-5} + A$

$\Rightarrow u = \frac{3}{5}x + Ax^6$

$\Rightarrow \frac{1}{y^3} = \frac{3}{5}x + Ax^6$

$\Rightarrow y^3 = \frac{1}{\frac{3}{5}x + Ax^6}$

$\Rightarrow y^3 = \frac{5}{3x + Bx^6}$

NOW $x=1$ $y=1$ $\Rightarrow 1 = \frac{5}{3+B}$

$\Rightarrow B + 3 = 5$

$\Rightarrow \boxed{B = 2}$

$\therefore y^3 = \frac{5}{3x + 2x^6}$

AS REQUIRED

Question 77 (*+)**

A curve with equation $y = f(x)$ passes through the origin and satisfies the differential equation

$$2y\left(1+x^2\right)\frac{dy}{dx} + xy^2 = \left(1+x^2\right)^{\frac{3}{2}}.$$

By finding a suitable integrating factor, or otherwise, show that

$$y^2 = \frac{x^3+3x}{3\sqrt{x^2+1}}.$$

$\boxed{\text{proof}}$

$$2y(1+x^2)\frac{dy}{dx} + xy^2 = (1+x^2)^{\frac{3}{2}}$$

$$\Rightarrow 2y\frac{dy}{dx} + \frac{x}{1+x^2}y^2 = (1+x^2)^{\frac{1}{2}}$$

$$\Rightarrow \frac{d}{dx}(y^2) + \frac{x}{1+x^2}y^2 = (1+x^2)^{\frac{1}{2}}$$

$$\text{I.F.} = e^{\int \frac{x}{1+x^2}\,dx} = e^{\frac{1}{2}\ln(1+x^2)} = (1+x^2)^{\frac{1}{2}}$$

$$\Rightarrow \frac{d}{dx}\left(y^2(1+x^2)^{\frac{1}{2}}\right) = 1+x^2$$

$$\Rightarrow y^2(1+x^2)^{\frac{1}{2}} = \int 1+x^2\,dx$$

$$\Rightarrow y^2(1+x^2)^{\frac{1}{2}} = x + \frac{1}{3}x^3 + C$$

$$\Rightarrow y^2 = \frac{x+\frac{1}{3}x^3+C}{(1+x^2)^{\frac{1}{2}}}$$

$$\Rightarrow y^2 = \frac{3x+x^3+A}{3(1+x^2)^{\frac{1}{2}}}$$

Now $(0,0) \Rightarrow A=0$

$$\Rightarrow y^2 = \frac{x^3+3x}{3\sqrt{x^2+1}}$$

Question 78 (***+)

Given that if $x = e^t$ and $y = f(x)$, show clearly that …

a) … $x\dfrac{dy}{dx} = \dfrac{dy}{dt}$.

b) … $x^2\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{dt^2} - \dfrac{dy}{dt}$.

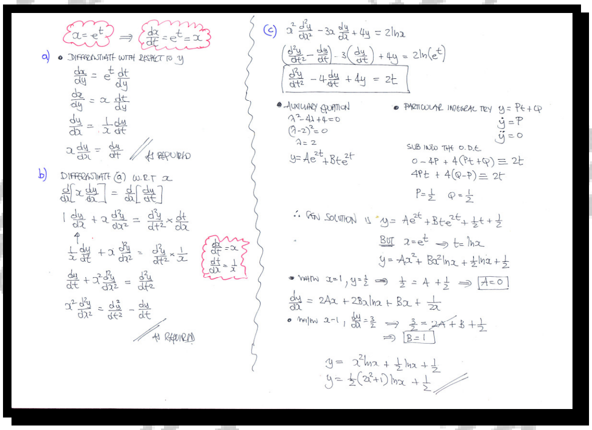
The following differential equation is to be solved

$$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2\ln x$$

subject to the boundary conditions $y = \dfrac{1}{2}, \dfrac{dy}{dx} = \dfrac{3}{2}$ at $x = 1$.

c) Use the substitution $x = e^t$ to solve the above differential equation.

$$\boxed{y = \frac{1}{2} + \frac{1}{2}\left(2x^2 + 1\right)\ln x}$$



$x = e^t \Rightarrow \dfrac{dx}{dt} = e^t = x$

a) • DIFFERENTIATE WITH RESPECT TO y

$\dfrac{dx}{dy} = e^t\dfrac{dt}{dy}$

$\dfrac{dx}{dy} = x\dfrac{dt}{dy}$

$\dfrac{dy}{dx} = \dfrac{1}{x}\dfrac{dy}{dt}$

$x\dfrac{dy}{dx} = \dfrac{dy}{dt}$ // AS REQUIRED

b) DIFFERENTIATE (a) W.R.T x

$\dfrac{d}{dx}\left[x\dfrac{dy}{dx}\right] = \dfrac{d}{dx}\left[\dfrac{dy}{dt}\right]$

$1\dfrac{dy}{dx} + x\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{dt^2}\times\dfrac{dt}{dx}$

$\dfrac{1}{x}\dfrac{dy}{dt} + x\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{dt^2}\times\dfrac{1}{x}$

$\dfrac{dx}{dt} = x$, $\dfrac{dt}{dx} = \dfrac{1}{x}$

$\dfrac{dy}{dt} + x^2\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{dt^2}$

$x^2\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{dt^2} - \dfrac{dy}{dt}$ // AS REQUIRED

(c) $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 4y = 2\ln x$

$\left(\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt}\right) - 3\left(\dfrac{dy}{dt}\right) + 4y = 2\ln(e^t)$

$\boxed{\dfrac{d^2y}{dt^2} - 4\dfrac{dy}{dt} + 4y = 2t}$

• AUXILIARY EQUATION

$\lambda^2 - 4\lambda + 4 = 0$

$(\lambda - 2)^2 = 0$

$\lambda = 2$

$y = Ae^{2t} + Bte^{2t}$

• PARTICULAR INTEGRAL TRY $y = Pt + Q$, $\dot{y} = P$, $\ddot{y} = 0$

SUB INTO THE O.D.E.

$0 - 4P + 4(Pt + Q) \equiv 2t$

$4Pt + 4(Q - P) \equiv 2t$

$P = \frac{1}{2}$ $Q = \frac{1}{2}$

∴ GEN SOLUTION IS $y = Ae^{2t} + Bte^{2t} + \frac{1}{2}t + \frac{1}{2}$

BUT $x = e^t \Rightarrow t = \ln x$

$y = Ax^2 + Bx^2\ln x + \frac{1}{2}\ln x + \frac{1}{2}$

• WHEN $x = 1, y = \frac{1}{2} \Rightarrow \frac{1}{2} = A + \frac{1}{2} \Rightarrow \boxed{A = 0}$

$\dfrac{dy}{dx} = 2Ax + 2Bx\ln x + Bx + \dfrac{1}{2x}$

• WHEN $x = 1$, $\dfrac{dy}{dx} = \dfrac{3}{2} \Rightarrow \dfrac{3}{2} = \cancel{2A} + B + \dfrac{1}{2}$

$\Rightarrow \boxed{B = 1}$

$y = x^2\ln x + \frac{1}{2}\ln x + \frac{1}{2}$

$y = \frac{1}{2}(2x^2 + 1)\ln x + \frac{1}{2}$ //

Question 79 (*+)**

Solve the differential equation

$$\frac{dy}{dx} + \frac{xy}{1+x^2} = y^3, \quad y(0) = 1.$$

Give the answer in the form $y^2 = f(x)$.

$$\boxed{y^2 = \frac{1}{\left(1+x^2\right)\left(1-2\arctan x\right)}}$$

$$\frac{dy}{dx} + \frac{xy}{1+x^2} = y^3$$

$$t = \frac{1}{y^2}$$

$$\frac{dt}{dx} = -\frac{2}{y^3}\frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{1}{2}y^3\frac{dt}{dx}$$

$$\Rightarrow -\frac{1}{2}y^3\frac{dt}{dx} + \frac{xy}{1+x^2} = y^3$$

$$\Rightarrow \frac{dt}{dx} - \frac{2x}{y^2(1+x^2)} = -2$$

$$\Rightarrow \frac{dt}{dx} - \frac{2xt}{1+x^2} = -2$$

$$\text{I.F} = e^{\int \frac{-2x}{1+x^2}\,dx} = e^{-\ln(1+x^2)} = \frac{1}{1+x^2}$$

$$\Rightarrow \frac{d}{dx}\left(t \times \frac{1}{1+x^2}\right) = \frac{-2}{1+x^2}$$

$$\Rightarrow \frac{t}{1+x^2} = \int \frac{-2}{1+x^2}\,dx$$

$$\Rightarrow \frac{t}{1+x^2} = A - 2\arctan x$$

$$\Rightarrow t = A(1+x^2) - 2(1+x^2)\arctan x$$

$$\Rightarrow \frac{1}{y^2} = (1+x^2)(A - 2\arctan x)$$

$$\Rightarrow \boxed{y^2 = \frac{1}{(1+x^2)(A-2\arctan x)}}$$

when $x=0$ $y=1$ $\quad 1 = \frac{1}{1\times(A-0)}$

$A = 1$

$$\therefore y^2 = \frac{1}{(1+x^2)(1-2\arctan x)}$$

Question 80 (***+)

The function $y = f(x)$ satisfies the differential equation

$$\frac{dy}{dx}\sin^2\left(x+\frac{\pi}{6}\right) = 2xy(y+1),$$

subject to the condition $y=1$ at $x=0$.

Find the exact value of y when $x=\frac{\pi}{12}$.

$$\boxed{y = \frac{1}{e^{\frac{\pi}{6}}-1}}$$

$$\frac{dy}{dx}\sin^2\left(x+\frac{\pi}{6}\right) = 2xy(y+1)$$

$$\Rightarrow \frac{1}{y(y+1)}\,dy = \frac{2x}{\sin^2\left(x+\frac{\pi}{6}\right)}\,dx$$

$$\Rightarrow \int \frac{1}{y(y+1)}\,dy = \int 2x\,\text{cosec}^2\left(x+\frac{\pi}{6}\right)dx$$

PARTIAL FRACTIONS

$$\frac{1}{y(y+1)} \equiv \frac{A}{y} + \frac{B}{y+1}$$

$$1 \equiv A(y+1) + By$$

If $y=0$ $\quad 1=A$

If $y=-1$ $\quad -B=1$

BY PARTS

$$2x \longrightarrow 2$$

$$-\cot\left(x+\frac{\pi}{6}\right) \longrightarrow \text{cosec}^2\left(x+\frac{\pi}{6}\right)$$

$$\Rightarrow \int \frac{1}{y} - \frac{1}{y+1}\,dy = -2x\cot\left(x+\frac{\pi}{6}\right) + \int +2\cot\left(x+\frac{\pi}{6}\right)dx$$

$$\Rightarrow \boxed{\ln|y| - \ln|y+1| = -2x\cot\left(x+\frac{\pi}{6}\right) + 2\ln\left|\sin\left(x+\frac{\pi}{6}\right)\right| + C}$$

APPLY $x=0$ $y=1$

$$\cancel{\ln 1} - \ln 2 = 2\ln\frac{1}{2} + C$$

$$-\ln 2 = -2\ln 2 + C$$

$$C = \ln 2$$

$$\boxed{\therefore \ln\left|\frac{y}{y+1}\right| = -2x\cot\left(x+\frac{\pi}{6}\right) + 2\ln\left|\sin\left(x+\frac{\pi}{6}\right)\right| + \ln 2}$$

When $x=\frac{\pi}{12}$

$$\ln\left|\frac{y}{y+1}\right| = -2\left(\frac{\pi}{12}\right)\times 1 + 2\times\ln\left(\frac{1}{\sqrt{2}}\right) + \ln 2$$

$$\ln\left|\frac{y}{y+1}\right| = -\frac{\pi}{6} + 2\ln 2^{-\frac{1}{2}} + \ln 2$$

$$\ln\left|\frac{y}{y+1}\right| = -\frac{\pi}{6} - \cancel{\ln 2} + \cancel{\ln 2}$$

$$\frac{y}{y+1} = e^{-\frac{\pi}{6}}$$

$$\frac{y+1}{y} = e^{\frac{\pi}{6}}$$

$$1 + \frac{1}{y} = e^{\frac{\pi}{6}}$$

$$\frac{1}{y} = e^{\frac{\pi}{6}} - 1$$

$$y = \frac{1}{e^{\frac{\pi}{6}}-1}$$

Question 81 (***+)

Solve the differential equation

$$\frac{dy}{dx} = y\left(1 + xy^4\right), \quad y(0) = 1.$$

$$\boxed{\frac{1}{y^4} = \frac{1}{4}\left(1 + 3e^{-4x}\right) - x}$$

$$\frac{dy}{dx} = y(1+xy^4)$$

$$\Rightarrow \frac{dy}{dx} = y + xy^5$$

$$\Rightarrow \frac{dy}{dx} - y = xy^5$$

$$\Rightarrow -\frac{1}{4}y^5\frac{du}{dx} - y = xy^5$$

$$\Rightarrow \frac{du}{dx} + \frac{4}{y^4} = -4x$$

$$\Rightarrow \frac{du}{dx} + 4u = -4x$$

$u = \frac{1}{y^4}$

$\frac{du}{dx} = -\frac{4}{y^5}\frac{dy}{dx}$

$\frac{dy}{dx} = -\frac{1}{4}y^5\frac{du}{dx}$

- $\text{I.F.} = e^{\int 4\,dx} = e^{4x}$

$$\Rightarrow \frac{d}{dx}\left(ue^{4x}\right) = -4xe^{4x}$$

$$\Rightarrow ue^{4x} = \int -4xe^{4x} \leftarrow \text{BY PARTS}$$

$-4x$	-4
$\frac{1}{4}e^{4x}$	e^{4x}

$$\Rightarrow ue^{4x} = -xe^{4x} + \int e^{4x}\,dx$$

$$\Rightarrow ue^{4x} = -xe^{4x} + \frac{1}{4}e^{4x} + A$$

$$\Rightarrow u = -x + \frac{1}{4} + Ae^{-4x}$$

$$\Rightarrow \frac{1}{y^4} = \left(\frac{1}{4} - x\right) + Ae^{-4x}$$

APPLY CONDITION:

$1 = \left(\frac{1}{4} - 0\right) + Ae^0$

$1 = \frac{1}{4} + A$

$\boxed{A = \frac{3}{4}}$

$$\Rightarrow \frac{1}{y^4} = \left(\frac{1}{4} - x\right) + \frac{3}{4}e^{-4x}$$

$$\Rightarrow \frac{1}{y^4} = \frac{1}{4}\left(1 + e^{-4x}\right) - x$$

OR $y^4 = \frac{4}{1 - 4x + e^{-4x}}$

Question 82 (***+)

$$x\frac{d^2y}{dx^2}+(6x+2)\frac{dy}{dx}+9xy=27x-6y.$$

Use the substitution $u=xy$, where u is a function of x, to find a general solution of the above differential equation.

$$\boxed{y=\frac{A}{x}\mathrm{e}^{-3x}+B\mathrm{e}^{-3x}+3-\frac{2}{x}}$$

$$x\frac{d^2y}{dx^2}+(6x+2)\frac{dy}{dx}+9xy=27x-6y$$

$$u=xy \Rightarrow \frac{du}{dx}=y+x\frac{dy}{dx} \qquad \Rightarrow \frac{d^2u}{dx^2}=\frac{dy}{dx}+\frac{dy}{dx}+x\frac{d^2y}{dx^2}$$

$$\Rightarrow \boxed{x\frac{dy}{dx}=\frac{du}{dx}-y} \qquad \Rightarrow \boxed{x\frac{d^2y}{dx^2}=\frac{d^2u}{dx^2}-2\frac{dy}{dx}}$$

THUS

$$\Rightarrow \left(\frac{d^2u}{dx^2}-2\frac{dy}{dx}\right)+(6x+2)\frac{dy}{dx}+9u=27x-6y$$

$$\Rightarrow \frac{d^2u}{dx^2}\cancel{-2\frac{dy}{dx}}+6x\frac{dy}{dx}\cancel{+2\frac{dy}{dx}}+9u=27x-6y$$

$$\Rightarrow \frac{d^2u}{dx^2}+6\left[\frac{du}{dx}-y\right]+9u=27x-6y$$

$$\Rightarrow \frac{d^2u}{dx^2}+6\frac{du}{dx}\cancel{-6y}+9u=27x\cancel{-6y}$$

$$\Rightarrow \boxed{\frac{d^2u}{dx^2}+6\frac{du}{dx}+9u=27x}$$

- AUXILIARY EQUATION

$$\lambda^2+6\lambda+9=0$$
$$(\lambda+3)^2=0$$
$$\lambda=-3$$

- PARTICULAR INTEGRAL $u=Px+Q$

$$\frac{du}{dx}=P$$
$$\frac{d^2u}{dx^2}=0$$

$$\therefore\ 6P+9(Px+Q)=27x$$
$$6P+9Px+9Q=27x$$

$$9P=27 \qquad 6P+9Q=0$$
$$\boxed{P=3} \qquad 18+9Q=0$$
$$\boxed{Q=-2}$$

$\therefore$ GEN SOLUTION $u=Ae^{-3x}+Bxe^{-3x}+3x-2$

$$xy=Ae^{-3x}+Bxe^{-3x}+3x-2$$

$$y=\frac{A}{x}e^{-3x}+Be^{-3x}+3-\frac{2}{x}$$

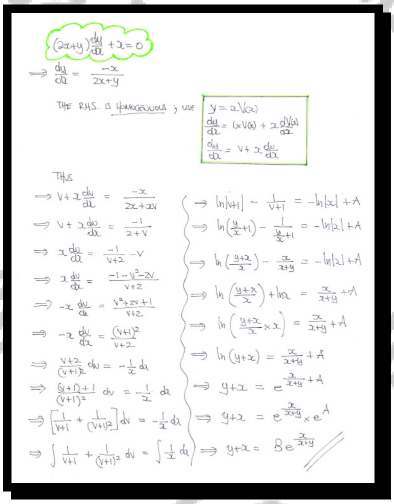
Question 83 (***+)

Find a general solution for the following differential equation

$$\left(2x+y\right)\frac{dy}{dx}+x=0\,.$$

The final answer must not contain natural logarithms.

$$\boxed{y+x=A\,\mathrm{e}^{\frac{x}{x+y}}}$$



$(2x+y)\frac{dy}{dx}+x=0$

$\Rightarrow \frac{dy}{dx}=\frac{-x}{2x+y}$

THE R.H.S. IS HOMOGENEOUS ; USE

$$\boxed{\begin{aligned} y&=xV(x)\\ \frac{dy}{dx}&=1\times V(x)+x\frac{dV(x)}{dx}\\ \frac{dy}{dx}&=V+x\frac{dv}{dx}\end{aligned}}$$

THUS

$\Rightarrow V+x\frac{dv}{dx}=\frac{-x}{2x+xV}$

$\Rightarrow V+x\frac{dv}{dx}=\frac{-1}{2+V}$

$\Rightarrow x\frac{dv}{dx}=\frac{-1}{V+2}-V$

$\Rightarrow x\frac{dv}{dx}=\frac{-1-V^2-2V}{V+2}$

$\Rightarrow -x\frac{dv}{dx}=\frac{V^2+2V+1}{V+2}$

$\Rightarrow -x\frac{dv}{dx}=\frac{(V+1)^2}{V+2}$

$\Rightarrow \frac{V+2}{(V+1)^2}\,dv=-\frac{1}{x}\,dx$

$\Rightarrow \frac{(V+1)+1}{(V+1)^2}\,dv=-\frac{1}{x}\,dx$

$\Rightarrow \left[\frac{1}{V+1}+\frac{1}{(V+1)^2}\right]dv=-\frac{1}{x}\,dx$

$\Rightarrow \int \frac{1}{V+1}+\frac{1}{(V+1)^2}\,dv=\int -\frac{1}{x}\,dx$

$\Rightarrow \ln|V+1|-\frac{1}{V+1}=-\ln|x|+A$

$\Rightarrow \ln\left(\frac{y}{x}+1\right)-\frac{1}{\frac{y}{x}+1}=-\ln|x|+A$

$\Rightarrow \ln\left(\frac{y+x}{x}\right)-\frac{x}{x+y}=-\ln|x|+A$

$\Rightarrow \ln\left(\frac{y+x}{x}\right)+\ln x=\frac{x}{x+y}+A$

$\Rightarrow \ln\left(\frac{y+x}{x}\times x\right)=\frac{x}{x+y}+A$

$\Rightarrow \ln(y+x)=\frac{x}{x+y}+A$

$\Rightarrow y+x=e^{\frac{x}{x+y}+A}$

$\Rightarrow y+x=e^{\frac{x}{x+y}}\times e^{A}$

$\Rightarrow y+x=Be^{\frac{x}{x+y}}$

Question 84 (***+)

a) By using the substitution $z = x^2 + y^2$, solve the following differential equation

$$2xy\frac{dy}{dx} + y^2 = 2x - 3x^2,$$

subject to the condition $y = 1$ at $x = 1$.

b) Verify the answer to part **(a)** by using the substitution $z = y^2$ to solve the same differential equation and subject to the same condition.

$$\boxed{y^2 = x - x^2 + \frac{1}{x}}$$

a) USING THE SUBSTITUTION GIVEN

$$\Rightarrow z = x^2 + y^2$$
$$\Rightarrow \frac{dz}{dx} = 2x + 2y\frac{dy}{dx}$$
$$\Rightarrow 2y\frac{dy}{dx} = \frac{dz}{dx} - 2x$$
$$\Rightarrow 2xy\frac{dy}{dx} = x\frac{dz}{dx} - 2x^2$$

SUBSTITUTE INTO THE O.D.E

$$\Rightarrow 2xy\frac{dy}{dx} + y^2 = 2x - 3x^2 \qquad [x=1,\ y=1]$$
$$\Rightarrow \left[x\frac{dz}{dx} - 2x^2\right] + y^2 = 2x - 3x^2 \qquad [x=1,\ z=2]$$
$$\Rightarrow x\frac{dz}{dx} - 2x^2 + (z - x^2) = 2x - 3x^2$$
$$\Rightarrow x\frac{dz}{dx} \cancel{-3x^2} + z = 2x \cancel{-3x^2}$$
$$\Rightarrow \frac{dz}{dx} + \frac{z}{x} = 2$$

INTEGRATING FACTOR NEXT (IN FACT THE ODE WAS EXACT)

$$e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x$$

THUS WE FINALLY HAVE

$$\Rightarrow \frac{d}{dx}(zx) = 2x$$
$$\Rightarrow \Big[zx\Big]_{(1,2)}^{(x,z)} = \Big[x^2\Big]_1^x$$
$$\Rightarrow zx - 2 = x^2 - 1$$
$$\Rightarrow z = x^2 + 1$$
$$\Rightarrow z = \frac{x^2+1}{x}$$
$$\Rightarrow x^2 + y^2 = x + \frac{1}{x}$$
$$\Rightarrow y^2 = x + \frac{1}{x} - x^2$$

b) REWRITE THE O.D.E AS

$$\Rightarrow 2xy\frac{dy}{dx} + y^2 = 2x - 3x^2$$
$$\Rightarrow \frac{dy}{dx} + \frac{y}{2x} = \frac{2x - 3x^2}{2xy}$$
$$\Rightarrow \frac{dy}{dx} + \frac{y}{2x} = \left(1 - \frac{3}{2}x\right)y^{-1} \leftarrow n = -1$$

THIS IS A BERNOULLI TYPE, SO WE USE THE SUBSTITUTION

$$z = \frac{1}{y^{n-1}} \qquad \text{HERE} \quad z = \frac{1}{y^{-2}}$$

- $z = y^2$

$$\frac{dz}{dx} = 2y\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{1}{2y}\frac{dz}{dx}$$

RETURNING TO THE O.D.E

$$\Rightarrow \frac{dy}{dx} + \frac{y}{2x} = \left(1 - \frac{3}{2}x\right)y^{-1}$$
$$\Rightarrow \frac{1}{2y}\frac{dz}{dx} + \frac{y}{2x} = \left(1 - \frac{3}{2}x\right)y^{-1}$$
$$\Rightarrow \frac{dz}{dx} + \frac{y^2}{x} = 2\left(1 - \frac{3}{2}x\right)$$
$$\Rightarrow \frac{dz}{dx} + \frac{z}{x} = 2 - 3x$$

MULTIPLY THROUGH BY x — OR INTEGRATING FACTOR

$$\Rightarrow x\frac{dz}{dx} + z = 2x - 3x^2 \qquad [x=1,\ y=1,\ z=1]$$
$$\Rightarrow \frac{d}{dx}(xz) = 2x - 3x^2$$
$$\Rightarrow \Big[xz\Big]_{(1,1)}^{(x,z)} = \int_1^x 2x - 3x^2\,dx$$
$$\Rightarrow xz - 1 = \Big[x^2 - x^3\Big]_1^x$$
$$\Rightarrow xz - 1 = (x^2 - x^3) - \cancel{(1-1)}$$
$$\Rightarrow xz = x^2 + 1 - x^3$$
$$\Rightarrow z = x + \frac{1}{x} - x^2$$
$$\Rightarrow y^2 = x + \frac{1}{x} - x^2$$

AS BEFORE

Question 85 (***+)

A curve with equation $y = f(x)$ passes through the point with coordinates $(0,1)$ and satisfies the differential equation

$$y^2 \frac{dy}{dx} + y^3 = 4e^x .$$

By finding a suitable integrating factor, or otherwise, show that

$$y^3 = 3e^x - 2e^{-3x} .$$

proof

$$y^2 \frac{dy}{dx} + y^3 = 4e^x$$
$$\Rightarrow 3y^2 \frac{dy}{dx} + 3y^3 = 12e^x$$
$$\Rightarrow \frac{d}{dx}(y^3) + 3y^3 = 12e^x$$
$$\text{IF} = e^{\int 3\,dx} = e^{3x}$$
$$\Rightarrow \frac{d}{dx}(y^3e^{3x}) = 12e^xe^{3x}$$
$$\Rightarrow y^3e^{3x} = \int 12e^xe^{3x}\,dx$$
$$\Rightarrow y^3e^{3x} = \int 12e^{4x}\,dx$$
$$\Rightarrow y^3e^{3x} = 3e^{4x} + A$$
$$\Rightarrow \boxed{y^3 = 3e^x + Ae^{-3x}}$$
$$(0,1) \Rightarrow 1 = 3 + A$$
$$\Rightarrow A = -2$$
$$\therefore y^3 = 3e^x - 2e^{-3x}$$

Question 86 (***+)

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx}\tan x - y\sec^4 x = 0.$$

The above differential equation is to be solved by a substitution.

a) If $t = \tan x$ show that …

i. … $\dfrac{dy}{dx} = \dfrac{dy}{dt}\sec^2 x$

ii. … $\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{dt^2}\sec^4 x + 2\dfrac{dy}{dt}\sec^2 x \tan x$

b) Use the results obtained in part **(a)** to find a general solution of the differential equation in the form $y = f(x)$.

$$\boxed{y = A\mathrm{e}^{\tan x} + B\mathrm{e}^{-\tan x}}$$

(a) (I) $t = \tan x$

$\frac{dt}{dy} = \sec^2 x \frac{dx}{dy}$

$\frac{dx}{dy} = \frac{dt}{dy} \times \frac{1}{\sec^2 x}$

$\frac{dy}{dx} = \frac{dy}{dt}\sec^2 x$

(II) $\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2}\frac{dt}{dx}\sec^2 x + \frac{dy}{dt}(2\sec^2 x \tan x)$

BUT $t = \tan x$

$\frac{dt}{dx} = \sec^2 x$

$\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2}\sec^4 x + 2\frac{dy}{dt}\sec^2 x \tan x$

(b) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx}\tan x - y\sec^4 x = 0$

$\frac{d^2y}{dt^2}\sec^4 x + \cancel{2\frac{dy}{dt}\sec^2 x \tan x} - \cancel{2\left(\frac{dy}{dt}\sec^2 x\right)\tan x} - y\sec^4 x = 0$

$\frac{d^2y}{dt^2}\sec^4 x - y\sec^4 x = 0$

$\frac{d^2y}{dt^2} - y = 0$

NOW

AUXILIARY EQUATION IS $\lambda^2 - 1 = 0$

$\lambda = \pm 1$

$y = Ae^{t} + Be^{-t}$

$y = Ae^{\tan x} + Be^{-\tan x}$

Question 87 (***+)

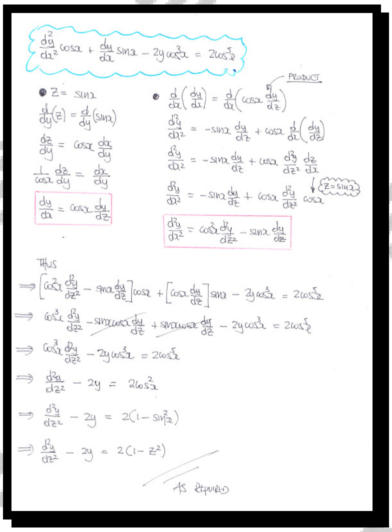
Show clearly that the substitution $z = \sin x$, transforms the differential equation

$$\frac{d^2y}{dx^2}\cos x + \frac{dy}{dx}\sin x - 2y\cos^3 x = 2\cos^5 x,$$

into the differential equation

$$\frac{d^2y}{dz^2} - 2y = 2\left(1 - z^2\right)$$

proof



$$\frac{d^2y}{dx^2}\cos x + \frac{dy}{dx}\sin x - 2y\cos^3 x = 2\cos^5 x$$

- $z = \sin x$

$\frac{d}{dy}(z) = \frac{d}{dy}(\sin x)$

$\frac{dz}{dy} = \cos x \frac{dx}{dy}$

$\frac{1}{\cos x}\frac{dz}{dy} = \frac{dx}{dy}$

$\boxed{\frac{dy}{dx} = \cos x \frac{dy}{dz}}$

- $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\cos x \frac{dy}{dz}\right)$ PRODUCT

$\frac{d^2y}{dx^2} = -\sin x \frac{dy}{dz} + \cos x \frac{d}{dx}\left(\frac{dy}{dz}\right)$

$\frac{d^2y}{dx^2} = -\sin x \frac{dy}{dz} + \cos x \frac{d^2y}{dz^2}\frac{dz}{dx}$

$\frac{d^2y}{dx^2} = -\sin x \frac{dy}{dz} + \cos x \frac{d^2y}{dz^2}\cos x$ $z = \sin x$

$\boxed{\frac{d^2y}{dx^2} = \cos^2 x \frac{d^2y}{dz^2} - \sin x \frac{dy}{dz}}$

Thus

$\Rightarrow \left[\cos^2 x \frac{d^2y}{dz^2} - \sin x \frac{dy}{dz}\right]\cos x + \left[\cos x \frac{dy}{dz}\right]\sin x - 2y\cos^3 x = 2\cos^5 x$

$\Rightarrow \cos^3 x \frac{d^2y}{dz^2} \cancel{- \sin x\cos x \frac{dy}{dz}} \cancel{+ \sin x\cos x \frac{dy}{dz}} - 2y\cos^3 x = 2\cos^5 x$

$\Rightarrow \cos^3 x \frac{d^2y}{dz^2} - 2y\cos^3 x = 2\cos^5 x$

$\Rightarrow \frac{d^2y}{dz^2} - 2y = 2\cos^2 x$

$\Rightarrow \frac{d^2y}{dz^2} - 2y = 2(1 - \sin^2 x)$

$\Rightarrow \frac{d^2y}{dz^2} - 2y = 2(1 - z^2)$

AS REQUIRED

Question 88 (***+)

$$x^3 \frac{d^2 y}{dx^2} - 2x^2 \frac{dy}{dx} - 4xy = 5.$$

Find the solution of the above differential equation subject to the boundary conditions $y = 4$, $\frac{dy}{dx} = 20$ at $x = 0$.

$$\boxed{y = 5x^4 - \frac{1}{x}(1 + \ln x)}$$

$$\boxed{x^3 \frac{d^2y}{dx^2} - 2x^2\frac{dy}{dx} - 4xy = 5, \quad x=0,\ y=4,\ \frac{dy}{dx}=20}$$

ASSUME A SOLUTION OF THE FORM $y = x^n$, $y' = nx^{n-1}$, $y'' = n(n-1)x^{n-2}$

DIVIDE x THROUGH, THEN SUB INTO THE HOMOGENEOUS O.D.E

$$\boxed{x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y} = \frac{5}{x}$$

$\Rightarrow x^2\left[n(n-1)x^{n-2}\right] - 2x\left[nx^{n-1}\right] - 4x^n = 0$

$\Rightarrow \left[n(n-1) - 2n - 4\right]x^n = 0$

$\Rightarrow n^2 - 3n - 4 = 0$

$\Rightarrow (n-4)(n+1) = 0$

$n = 4, -1 \qquad \therefore$ C.F.: $y = Ax^4 + Bx^{-1}$

FOR PARTICULAR INTEGRAL TRY $y = \frac{P}{x}\ln x$ (SINCE $\frac{1}{x} = x^{-1}$ IS PART OF C.F.)

$y' = -\frac{P}{x^2}\ln x + \frac{P}{x^2} = \frac{P}{x^2}\left[1 - \ln x\right]$

$y'' = -\frac{2P}{x^3}\left[1 - \ln x\right] - \frac{P}{x^3} = -\frac{P}{x^3}\left[2 - 2\ln x + 1\right] = \frac{P}{x^3}\left[2\ln x - 3\right]$

SUB INTO THE O.D.E $\quad x^2\left[\frac{P}{x^3}(2\ln x - 3)\right] - 2x\left[\frac{P}{x^2}(1 - \ln x)\right] - 4\left[\frac{P}{x}\ln x\right] \equiv \frac{5}{x}$

$\frac{P}{x}\left[\cancel{2\ln x} - 3 - 2 + \cancel{2\ln x} - \cancel{4\ln x}\right] \equiv \frac{5}{x}$

$-5P = 5$

$P = -1$

$\therefore y = Ax^4 + \frac{B}{x} - \frac{1}{x}\ln x$, $\frac{dy}{dx} = 4Ax^3 - \frac{B}{x^2} + \frac{1}{x^2}\ln x - \frac{1}{x^2}$ $\Rightarrow$ $4 = A + B$, $20 = 4A - B - 1$ $\Rightarrow$ $5A = 25$, $\boxed{A = 5}$, $\boxed{B = -1}$

$\therefore y = 5x^4 - \frac{1}{x} - \frac{1}{x}\ln x$

$y = 5x^4 - \frac{1}{x}(1 + \ln x)$

Question 89 (***+)

Find a general solution of the following differential equation

$$\frac{dy}{dx} = \frac{\cos x \cos y + \sin^2 x}{\sin x \sin y + \cos^2 y}.$$

$$\boxed{\sin x \cos y - \frac{1}{4}(\sin 2x + \sin 2y) + \frac{1}{2}(x - y) = \text{constant}}$$

$$\frac{dy}{dx} = \frac{\cos x \cos y + \sin^2 x}{\sin x \sin y + \cos^2 y}$$

$$\Rightarrow (\sin x \sin y + \cos^2 y)\,dy = (\cos x \cos y + \sin^2 x)\,dx$$

$$\Rightarrow \underbrace{(\cos x \cos y + \sin^2 x)}_{M(x,y)}\,dx \underbrace{-(\sin x \sin y + \cos^2 y)}_{N(x,y)}\,dy = 0$$

- $\frac{\partial M}{\partial y} = -\cos x \sin y$
- $\frac{\partial N}{\partial x} = -\cos x \sin y$

i.e. $\frac{\partial M}{\partial y} = \frac{\partial M}{\partial x}$

$$\Longrightarrow dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = 0$$

$$\Longrightarrow dF = (\cos x \cos y + \sin^2 x)\,dx + (-\sin x \sin y - \cos^2 y)\,dy = 0$$

Thus

$$\frac{\partial}{\partial x}[F(x,y)] = \cos x \cos y + \sin^2 x$$

$$\frac{\partial}{\partial x}[F(x,y)] = \cos x \cos y + \frac{1}{2} - \frac{1}{2}\cos 2x$$

$$F(x,y) = \sin x \cos y + \frac{1}{2}x - \frac{1}{4}\sin 2x + f(y)$$

$$\frac{\partial}{\partial y}[F(x,y)] = -\sin x \sin y - \cos^2 y$$

$$\frac{\partial}{\partial y}[F(x,y)] = -\sin x \sin y - \frac{1}{2} - \frac{1}{2}\cos 2y$$

$$F(x,y) = \sin x \cos y - \frac{1}{2}y - \frac{1}{4}\sin 2y + g(x)$$

COMPARING AND NOTING THAT $dF = 0$ IMPLIES $F(x,y) = \text{constant}$

$$F(x,y) = \sin x \cos y - \frac{1}{4}\sin 2x - \frac{1}{4}\sin 2y + \frac{1}{2}x - \frac{1}{2}y + \text{constant}$$

$$\therefore \sin x \cos y - \frac{1}{4}(\sin 2x + \sin 2y) + \frac{1}{2}(x - y) = \text{constant}$$

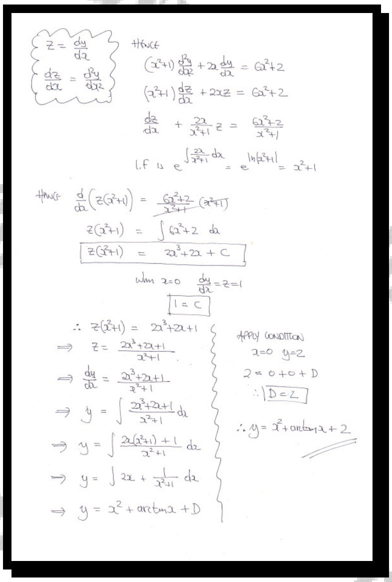
Question 90 **(***+)**

By using the substitution $z = \frac{dy}{dx}$, or otherwise, solve the differential equation

$$\left(x^2+1\right)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 6x^2 + 2,$$

subject to the conditions $x = 0$, $y = 2$, $\frac{dy}{dx} = 1$

$$\boxed{y = x^2 + 2 + \arctan x}$$



$z = \frac{dy}{dx}$

$\frac{dz}{dx} = \frac{d^2y}{dx^2}$

Hence

$(x^2+1)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 6x^2+2$

$(x^2+1)\frac{dz}{dx} + 2xz = 6x^2+2$

$\frac{dz}{dx} + \frac{2x}{x^2+1}z = \frac{6x^2+2}{x^2+1}$

I.F is $e^{\int \frac{2x}{x^2+1}\,dx} = e^{\ln|x^2+1|} = x^2+1$

Hence $\frac{d}{dx}\left(z(x^2+1)\right) = \frac{6x^2+2}{\cancel{x^2+1}}\cancel{(x^2+1)}$

$z(x^2+1) = \int 6x^2+2 \, dx$

$\boxed{z(x^2+1) = 2x^3+2x+C}$

when $x=0$ $\frac{dy}{dx} = z = 1$

$\boxed{1 = C}$

$\therefore z(x^2+1) = 2x^3+2x+1$

$\Rightarrow z = \frac{2x^3+2x+1}{x^2+1}$

$\Rightarrow \frac{dy}{dx} = \frac{2x^3+2x+1}{x^2+1}$

$\Rightarrow y = \int \frac{2x^3+2x+1}{x^2+1}\,dx$

$\Rightarrow y = \int \frac{2x(x^2+1)+1}{x^2+1}\,dx$

$\Rightarrow y = \int 2x + \frac{1}{x^2+1}\,dx$

$\Rightarrow y = x^2 + \arctan x + D$

APPLY CONDITION

$x=0$ $y=2$

$2 = 0+0+D$

$\therefore \boxed{D=2}$

$\therefore y = x^2 + \arctan x + 2$

Question 91 (****)

A curve C passes through the point $(1,1)$ and satisfies the differential equation

$$\frac{dy}{dx} - \frac{y}{x} = \frac{x^3}{4y^3}, \quad x > 0, \quad y > 0,$$

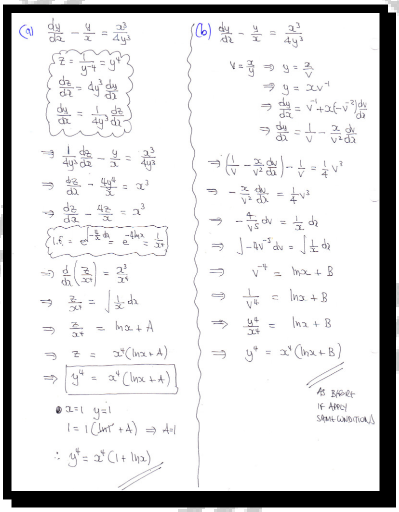
subject to the condition $y = 1$ at $x = 1$.

a) Find an equation of C by using the substitution $z = y^4$.

b) Find an equation of C by using the substitution $v = \dfrac{x}{y}$.

Give the answer in the form $y^4 = f(x)$.

$$\boxed{y^4 = x^4(1 + \ln x)}$$



(a) $\frac{dy}{dx} - \frac{y}{x} = \frac{x^3}{4y^3}$

$z = \frac{1}{y^{-4}} = y^4$

$\frac{dz}{dx} = 4y^3 \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{4y^3}\frac{dz}{dx}$

$\Rightarrow \frac{1}{4y^3}\frac{dz}{dx} - \frac{y}{x} = \frac{x^3}{4y^3}$

$\Rightarrow \frac{dz}{dx} - \frac{4y^4}{x} = x^3$

$\Rightarrow \frac{dz}{dx} - \frac{4z}{x} = x^3$

$\text{I.F.} = e^{\int -\frac{4}{x}\,dx} = e^{-4\ln x} = \frac{1}{x^4}$

$\Rightarrow \frac{d}{dx}\left(\frac{z}{x^4}\right) = \frac{x^3}{x^4}$

$\Rightarrow \frac{z}{x^4} = \int \frac{1}{x}\,dx$

$\Rightarrow \frac{z}{x^4} = \ln x + A$

$\Rightarrow z = x^4(\ln x + A)$

$\Rightarrow \boxed{y^4 = x^4(\ln x + A)}$

• $x = 1 \quad y = 1$

$1 = 1(\ln 1 + A) \Rightarrow A = 1$

$\therefore y^4 = x^4(1 + \ln x)$

(b) $\frac{dy}{dx} - \frac{y}{x} = \frac{x^3}{4y^3}$

$v = \frac{x}{y} \Rightarrow y = \frac{x}{v}$

$\Rightarrow y = xv^{-1}$

$\Rightarrow \frac{dy}{dx} = v^{-1} + x(-v^{-2})\frac{dv}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{v} - \frac{x}{v^2}\frac{dv}{dx}$

$\Rightarrow \left(\frac{1}{v} - \frac{x}{v^2}\frac{dv}{dx}\right) - \frac{1}{v} = \frac{1}{4}v^3$

$\Rightarrow -\frac{x}{v^2}\frac{dv}{dx} = \frac{1}{4}v^3$

$\Rightarrow -\frac{4}{v^5}\,dv = \frac{1}{x}\,dx$

$\Rightarrow \int -4v^{-5}\,dv = \int \frac{1}{x}\,dx$

$\Rightarrow v^{-4} = \ln x + B$

$\Rightarrow \frac{1}{v^4} = \ln x + B$

$\Rightarrow \frac{y^4}{x^4} = \ln x + B$

$\Rightarrow y^4 = x^4(\ln x + B)$

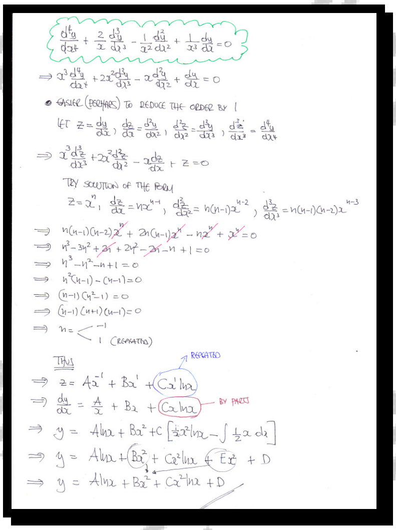
AS BEFORE IF APPLY SAME CONDITIONS

Question 92 (****)

Find the general solution of the following differential equation

$$\frac{d^4y}{dx^4}+\frac{2}{x}\frac{d^3y}{dx^3}-\frac{1}{x^2}\frac{d^2y}{dx^2}+\frac{1}{x^3}\frac{dy}{dx}=0.$$

$$\boxed{y = A\ln x + Bx^2 + Cx^2\ln x + D}$$



$$\frac{d^4y}{dx^4}+\frac{2}{x}\frac{d^3y}{dx^3}-\frac{1}{x^2}\frac{d^2y}{dx^2}+\frac{1}{x^3}\frac{dy}{dx}=0$$

$$\Rightarrow x^3\frac{d^4y}{dx^4}+2x^2\frac{d^3y}{dx^3}-x\frac{d^2y}{dx^2}+\frac{dy}{dx}=0$$

- EASIER (PERHAPS) TO REDUCE THE ORDER BY 1

LET $z=\frac{dy}{dx}$, $\frac{dz}{dx}=\frac{d^2y}{dx^2}$, $\frac{d^2z}{dx^2}=\frac{d^3y}{dx^3}$, $\frac{d^3z}{dx^3}=\frac{d^4y}{dx^4}$

$$\Rightarrow x^3\frac{d^3z}{dx^3}+2x^2\frac{d^2z}{dx^2}-x\frac{dz}{dx}+z=0$$

TRY SOLUTION OF THE FORM

$z=x^n$, $\frac{dz}{dx}=nx^{n-1}$, $\frac{d^2z}{dx^2}=n(n-1)x^{n-2}$, $\frac{d^3z}{dx^3}=n(n-1)(n-2)x^{n-3}$

$$\Rightarrow n(n-1)(n-2)\cancel{x^n}+2n(n-1)\cancel{x^n}-n\cancel{x^n}+\cancel{x^n}=0$$

$$\Rightarrow n^3-3n^2+\cancel{2n}+2n^2-\cancel{2n}-n+1=0$$

$$\Rightarrow n^3-n^2-n+1=0$$

$$\Rightarrow n^2(n-1)-(n-1)=0$$

$$\Rightarrow (n-1)(n^2-1)=0$$

$$\Rightarrow (n-1)(n+1)(n-1)=0$$

$$\Rightarrow n = -1, \; 1 \text{ (REPEATED)}$$

THUS

REPEATED

$$\Rightarrow z = Ax^{-1}+Bx^{1}+Cx^{1}\ln x$$

$$\Rightarrow \frac{dy}{dx}=\frac{A}{x}+Bx+Cx\ln x \quad \text{BY PARTS}$$

$$\Rightarrow y = A\ln x + Bx^2 + C\left[\tfrac{1}{2}x^2\ln x-\int \tfrac{1}{2}x\,dx\right]$$

$$\Rightarrow y = A\ln x + Bx^2 + Cx^2\ln x + Ex^2 + D$$

$$\Rightarrow y = A\ln x + Bx^2 + Cx^2\ln x + D$$

Question 93 (****)

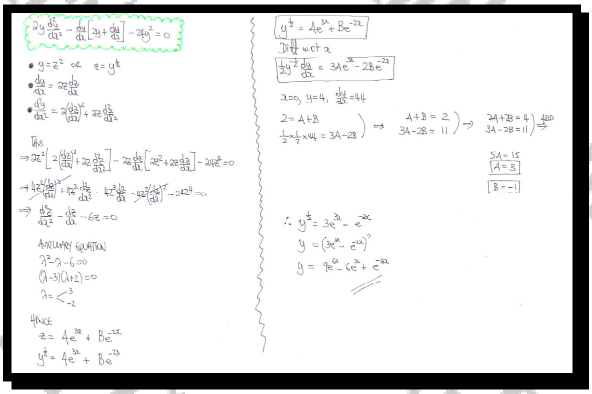
Use the substitution $z=\sqrt{y}$, where $y=f(x)$, to solve the differential equation

$$\frac{d^2y}{dx^2}+\frac{1}{y}\left(\frac{dy}{dx}\right)^2-5\frac{dy}{dx}+2y=0,$$

subject to the boundary conditions $y=4$, $\frac{dy}{dx}=44$ at $x=0$.

Give the answer in the form $y=f(x)$.

$$\boxed{y=9e^{6x}-6e^{x}+e^{-4x}}$$



Question 94 (****)

Solve the differential equation

$$\frac{dy}{dx} = \frac{x-y}{x+y}, \quad y(1) = 1 .$$

$\boxed{}$, $\boxed{y^2 + 2xy - x^2 = 2}$

AS THIS IS A STANDARD HOMOGENEOUS O.D.E WE USE THE SUBSTITUTION $y = vx$, WHERE $v = v(x)$

$$\Rightarrow y = vx$$
$$\Rightarrow \frac{dy}{dx} = \frac{dv}{dx} \times x + v \times 1$$
$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

HENCE WE CAN TRANSFORM THE O.D.E

$$\Rightarrow \frac{dy}{dx} = \frac{x-y}{x+y}$$
$$\Rightarrow \frac{dy}{dx} = \frac{x-vx}{x+vx}$$
$$\Rightarrow v + x\frac{dv}{dx} = \frac{1-v}{1+v}$$
$$\Rightarrow x\frac{dv}{dx} = \frac{1-v}{1+v} - v$$
$$\Rightarrow x\frac{dv}{dx} = \frac{1-v-v(1+v)}{1+v}$$
$$\Rightarrow x\frac{dv}{dx} = \frac{1-2v-v^2}{1+v}$$
$$\Rightarrow \frac{v+1}{-v^2-2v+1}\,dv = \frac{1}{x}\,dx$$

$\times(-2)$

$$\Rightarrow \int \frac{-2v-2}{-v^2-2v+1}\,dv = \int \frac{-2}{x}\,dx$$
$$\Rightarrow \ln|-v^2-2v+1| = -2\ln|x| + \ln A$$
$$\Rightarrow \ln|1-2v-v^2| = \ln\left|\frac{A}{x^2}\right|$$
$$\Rightarrow 1-2v-v^2 = \frac{A}{x^2}$$

REVERSING THE TRANSFORMATIONS WE OBTAIN

$$\Rightarrow 1 - 2\left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)^2 = \frac{A}{x^2}$$
$$\Rightarrow 1 - \frac{2y}{x} - \frac{y^2}{x^2} = \frac{A}{x^2}$$
$$\Rightarrow x^2 - 2xy - y^2 = A$$

APPLYING THE CONDITION $(1,1)$ YIELDS $A = -2$

$$\Rightarrow x^2 - 2xy - y^2 = -2$$
$$\Rightarrow y^2 + 2xy - x^2 = 2$$

ALTERNATIVE USING PARTIAL DIFFERENTIATION

$$\Rightarrow \frac{dy}{dx} = \frac{x-y}{x+y}$$
$$\Rightarrow (x-y)\,dx = (x+y)\,dy$$
$$\Rightarrow (x-y)\,dx + (-x-y)\,dy = 0$$
$$\frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = dF$$

CHECK FOR "EXACTNESS"

- $\frac{\partial F}{\partial x} = x - y \Rightarrow \frac{\partial^2 F}{\partial y \partial x} = -1$
- $\frac{\partial F}{\partial y} = -x - y \Rightarrow \frac{\partial^2 F}{\partial x \partial y} = -1$

$\therefore$ EXACT

- $\frac{\partial F}{\partial x} = x - y$

  $F(x,y) = \frac{1}{2}x^2 - xy + f(y)$
- $\frac{\partial F}{\partial y} = -x - y$

  $F(x,y) = -xy - \frac{1}{2}y^2 + g(x)$

COMPARING EXPRESSIONS FOR $F(x,y)$ GIVES

$$f(y) = -\tfrac{1}{2}y^2 \quad \& \quad g(x) = \tfrac{1}{2}x^2$$

FINALLY WE HAVE

$$F(x,y) = \tfrac{1}{2}x^2 - xy - \tfrac{1}{2}y^2$$

& SINCE $dF = 0$

$$F(x,y) = \text{CONSTANT}$$

$$\Rightarrow \tfrac{1}{2}x^2 - xy - \tfrac{1}{2}y^2 = \text{CONSTANT}$$
$$\Rightarrow y^2 + 2xy - x^2 = \text{CONSTANT}$$

& USING $(1,1)$ FINDS THE CONSTANT AS 2

$$\therefore\ y^2 + 2yx - x^2 = 2$$

AS BEFORE

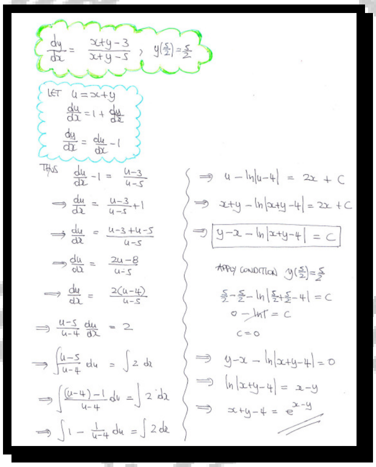
Question 95 (****)

Solve the differential equation

$$\frac{dy}{dx} = \frac{x+y-3}{x+y-5},$$

subject to the condition $y = \frac{5}{2}$ at $x = \frac{5}{2}$.

$$\boxed{x+y-4 = e^{x-y}}$$



$\frac{dy}{dx} = \frac{x+y-3}{x+y-5}, \quad y\left(\frac{5}{2}\right) = \frac{5}{2}$

LET $u = x+y$

$\frac{du}{dx} = 1 + \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{du}{dx} - 1$

THUS $\frac{du}{dx} - 1 = \frac{u-3}{u-5}$

$\Rightarrow \frac{du}{dx} = \frac{u-3}{u-5} + 1$

$\Rightarrow \frac{du}{dx} = \frac{u-3+u-5}{u-5}$

$\Rightarrow \frac{du}{dx} = \frac{2u-8}{u-5}$

$\Rightarrow \frac{du}{dx} = \frac{2(u-4)}{u-5}$

$\Rightarrow \frac{u-5}{u-4}\frac{du}{dx} = 2$

$\Rightarrow \int \frac{u-5}{u-4}\,du = \int 2\,dx$

$\Rightarrow \int \frac{(u-4)-1}{u-4}\,du = \int 2\,dx$

$\Rightarrow \int 1 - \frac{1}{u-4}\,du = \int 2\,dx$

$\Rightarrow u - \ln|u-4| = 2x + C$

$\Rightarrow x+y - \ln|x+y-4| = 2x + C$

$\Rightarrow \boxed{y - x - \ln|x+y-4| = C}$

APPLY CONDITION $y\left(\frac{5}{2}\right) = \frac{5}{2}$

$\frac{5}{2} - \frac{5}{2} - \ln\left|\frac{5}{2}+\frac{5}{2}-4\right| = C$

$0 - \ln 1 = C$

$C = 0$

$\Rightarrow y - x - \ln|x+y-4| = 0$

$\Rightarrow \ln|x+y-4| = x-y$

$\Rightarrow x+y-4 = e^{x-y}$

Question 96 (****)

$$\frac{dy}{dx} = \frac{x+y+3}{x+y-1}, \quad y(0)=0.$$

Show that the solution of the above differential equation is

$$y - x - 2\ln(x+y+1) = 0.$$

proof

$$\frac{dy}{dx} = \frac{x+y+3}{x+y-1}$$

$z = x+y \Rightarrow y = z-x$

$\frac{dz}{dx} = 1 + \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{dz}{dx} - 1$

$\Rightarrow \frac{dz}{dx} - 1 = \frac{z+3}{z-1}$

$\Rightarrow \frac{dz}{dx} = \frac{z+3}{z-1} + 1$

$\Rightarrow \frac{dz}{dx} = \frac{z+3+z-1}{z-1}$

$\Rightarrow \frac{dz}{dx} = \frac{2z+2}{z-1}$

$\Rightarrow \frac{z-1}{2(z+1)}\,dz = 1\,dx$

$\Rightarrow \int \frac{z-1}{z+1}\,dz = \int 2\,dx$

$\Rightarrow \int \frac{(z+1)-2}{(z+1)}\,dz = \int 2\,dx$

$\Rightarrow \int 1 - \frac{2}{z+1}\,dx = \int 2\,dx$

$\Rightarrow z - 2\ln|z+1| = 2x + C$

$\Rightarrow \boxed{x+y-2\ln|x+y+1| = 2x + C}$

USING $(0,0) \Rightarrow 0+0-2\ln 1 = 0 + C$

$\Rightarrow \boxed{C=0}$

$\Rightarrow x+y-2\ln|x+y+1| = 2x$

$\Rightarrow y-x-2\ln|x+y+1| = 0$

AS REQUIRED

Question 97 **(****)**

$$2x\frac{d^2y}{dx^2}+\left(1-3x^{\frac{1}{2}}\right)\frac{dy}{dx}+y=0.$$

The above differential equation is to be solved by a substitution.

a) Given that $y=f(x)$ and $t=x^{\frac{1}{2}}$, show clearly that …

i. … $\dfrac{dy}{dx}=\dfrac{1}{2t}\dfrac{dy}{dt}$.

ii. … $\dfrac{d^2y}{dx^2}=\dfrac{1}{4t^2}\dfrac{d^2y}{dt^2}-\dfrac{1}{4t^3}\dfrac{dy}{dt}$.

b) Hence show further that the differential equation

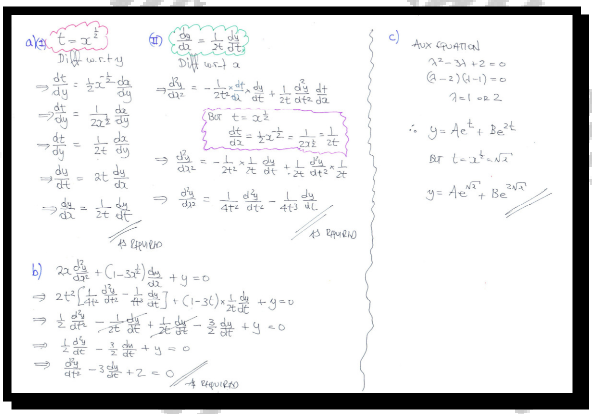
$$2x\frac{d^2y}{dx^2}+\left(1-3x^{\frac{1}{2}}\right)\frac{dy}{dx}+y=0,$$

can be transformed to the differential equation

$$\frac{d^2y}{dt^2}-3\frac{dy}{dt}+2y=0.$$

c) Find a general solution of the **original** differential equation, giving the answer in the form $y=f(x)$.

$$\boxed{y=Ae^{\sqrt{x}}+Be^{2\sqrt{x}}}$$



a)(I) $t=x^{\frac{1}{2}}$

Diff w.r.t y

$\Rightarrow \frac{dt}{dy}=\frac{1}{2}x^{-\frac{1}{2}}\frac{dx}{dy}$

$\Rightarrow \frac{dt}{dy}=\frac{1}{2x^{\frac{1}{2}}}\frac{dx}{dy}$

$\Rightarrow \frac{dt}{dy}=\frac{1}{2t}\frac{dx}{dy}$

$\Rightarrow \frac{dy}{dt}=2t\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}=\frac{1}{2t}\frac{dy}{dt}$

AS REQUIRED

(II) $\frac{dy}{dx}=\frac{1}{2t}\frac{dy}{dt}$

Diff w.r.t x

$\Rightarrow \frac{d^2y}{dx^2}=-\frac{1}{2t^2}\times\frac{dt}{dx}\times\frac{dy}{dt}+\frac{1}{2t}\frac{d^2y}{dt^2}\frac{dt}{dx}$

BUT $t=x^{\frac{1}{2}}$

$\frac{dt}{dx}=\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2x^{\frac{1}{2}}}=\frac{1}{2t}$

$\Rightarrow \frac{d^2y}{dx^2}=-\frac{1}{2t^2}\times\frac{1}{2t}\frac{dy}{dt}+\frac{1}{2t}\frac{d^2y}{dt^2}\times\frac{1}{2t}$

$\Rightarrow \frac{d^2y}{dx^2}=\frac{1}{4t^2}\frac{d^2y}{dt^2}-\frac{1}{4t^3}\frac{dy}{dt}$

AS REQUIRED

b) $2x\frac{d^2y}{dx^2}+(1-3x^{\frac{1}{2}})\frac{dy}{dx}+y=0$

$\Rightarrow 2t^2\left[\frac{1}{4t^2}\frac{d^2y}{dt^2}-\frac{1}{4t^3}\frac{dy}{dt}\right]+(1-3t)\times\frac{1}{2t}\frac{dy}{dt}+y=0$

$\Rightarrow \frac{1}{2}\frac{d^2y}{dt^2}-\cancel{\frac{1}{2t}\frac{dy}{dt}}+\cancel{\frac{1}{2t}\frac{dy}{dt}}-\frac{3}{2}\frac{dy}{dt}+y=0$

$\Rightarrow \frac{1}{2}\frac{d^2y}{dt}-\frac{3}{2}\frac{dy}{dt}+y=0$

$\Rightarrow \frac{d^2y}{dt^2}-3\frac{dy}{dt}+2=0$

AS REQUIRED

c) AUX EQUATION

$\lambda^2-3\lambda+2=0$

$(\lambda-2)(\lambda-1)=0$

$\lambda=1$ OR 2

$\therefore\ y=Ae^{t}+Be^{2t}$

BUT $t=x^{\frac{1}{2}}=\sqrt{x}$

$y=Ae^{\sqrt{x}}+Be^{2\sqrt{x}}$

Question 98 (****)

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-\left(x^2+n^2\right)y=0.$$

The above differential equation is known as modified Bessel's Equation.

Use the Frobenius method to show that the general solution of this differential equation, for $n=\frac{1}{2}$, is

$$y=x^{-\frac{1}{2}}\left[A\cosh x+B\sinh x\right].$$

proof

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-(x^2+n^2)y=0$$
$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-x^2y-\frac{1}{4}y=0$$

- ASSUME A SOLUTION OF THE FORM $y=\sum_{r=0}^{\infty}a_rx^{r+p}$, $a_0\neq 0$

$$\frac{dy}{dx}=\sum_{r=0}^{\infty}a_r(r+p)x^{r+p-1}$$
$$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty}a_r(r+p)(r+p-1)x^{r+p-2}$$

- SUBSTITUTE INTO THE O.D.E

$$\sum_{r=0}^{\infty}a_r(r+p)(r+p-1)x^{r+p}+\sum_{r=0}^{\infty}a_r(r+p)x^{r+p}-\sum_{r=0}^{\infty}a_rx^{r+p+2}-\frac{1}{4}\sum_{r=0}^{\infty}a_rx^{r+p}=0$$

- WHEN $r=0$ THE LOWEST POWER OF x IS x^p, AND THE HIGHEST x^{p+2}
PULL x^p AND x^{p+1} OUT OF THE SUMMATIONS

$$\underbrace{\left[a_0p(p-1)+a_0p-\frac{1}{4}a_0\right]}_{\text{INDICIAL EQUATION}}x^p+\left[a_1(p+1)p+a_1(p+1)-\frac{1}{4}a_1\right]x^{p+1}$$
$$+\sum_{r=2}^{\infty}a_r(r+p)(r+p-1)x^{r+p}+\sum_{r=2}^{\infty}a_r(r+p)x^{r+p}-\sum_{r=0}^{\infty}a_rx^{r+p+2}-\frac{1}{4}\sum_{r=2}^{\infty}a_rx^{r+p}=0$$

- INDICIAL EQUATION, $a_0\neq 0$

$$p(p-1)+p-\frac{1}{4}=0$$
$$p^2-\frac{1}{4}=0$$
$p=\pm\frac{1}{2}$ TWO DISTINCT SOLUTIONS DIFFERING BY AN INTEGER

- CHECK THE NEXT POWER UP FOR UNDETERMINED COEFFICIENTS

$$\left[(p+1)p+(p+1)-\frac{1}{4}\right]a_1=0$$
$$\left[p^2+p+p+1-\frac{1}{4}\right]a_1=0$$
$$\left[p^2+2p+\frac{3}{4}\right]a_1=0$$

IF $p=-\frac{1}{2}$

$$a_1\left[p^2+2p+\frac{3}{4}\right]=0$$
$$a_1\left[\frac{1}{4}-1+\frac{3}{4}\right]=0$$
$$a_1\times 0=0$$

$\therefore$ a_1 IS UNDETERMINED

THUS THE <u>ENTIRE</u> SOLUTION WILL BE OBTAINED FROM $p=-\frac{1}{2}$
($p=\frac{1}{2}$ MERELY PRODUCES PART OF THE SOLUTION)

- ADJUST THE SUMMATIONS SO THEY ALL START FROM $r=0$

$$\sum_{r=0}^{\infty}a_{r+2}(r+p+2)(r+p+1)x^{r+p+2}+\sum_{r=0}^{\infty}a_{r+2}(r+p+2)x^{r+p+2}-\sum_{r=0}^{\infty}a_rx^{r+p+2}-\frac{1}{4}\sum_{r=0}^{\infty}a_{r+2}x^{r+p+2}=0$$

$$a_{r+2}\left[(r+p+2)(r+p+1)+(r+p+2)-\frac{1}{4}\right]=a_r$$
$$a_{r+2}\left[4(r+p+2)(r+p+1)+4(r+p+2)-1\right]=4a_r$$
$$a_{r+2}=\frac{4a_r}{4(r+p+2)(r+p+1)+4(r+p+2)-1}$$

TIDY $4(w+2)(w+1)+4(w+2)-1$ $\quad(w=r+p)$
$$=4w^2+12w+8+4w+8-1$$
$$=4w^2+16w+15$$
$$=(2w+3)(2w+5)$$
$$=[2(r+p)+3][2(r+p)+5]$$
$$=[2r+2p+3][2r+2p+5]$$
BUT $p=-\frac{1}{2}$
$$=[2r-1+3][2r-1+5]$$
$$=(2r+2)(2r+4)$$
$$=4(r+1)(r+2)$$

$$\therefore\ a_{r+2}=\frac{4a_r}{4(r+1)(r+2)}$$

$$a_{r+2}=\frac{a_r}{(r+1)(r+2)}$$

$r=0$: $a_2=\frac{a_0}{1\times 2}$

$r=1$: $a_3=\frac{a_1}{2\times 3}$

$r=2$: $a_4=\frac{a_2}{3\times 4}=\frac{a_0}{1\times 2\times 3\times 4}$

$r=3$: $a_5=\frac{a_3}{4\times 5}=\frac{a_1}{1\times 2\times 3\times 4\times 5}$

$r=4$: $a_6=\frac{a_4}{5\times 6}=\frac{a_0}{1\times 2\times 3\times 4\times 5\times 6}$

$r=5$: $a_7=\frac{a_5}{6\times 7}=\frac{a_1}{1\times 2\times 3\times 4\times 5\times 6\times 7}$

$r=6$: $a_8=\frac{a_6}{7\times 8}=\frac{a_0}{1\times 2\times 3\times 4\times 5\times 6\times 7\times 8}$ ETC

THUS

$$y=x^{-\frac{1}{2}}\left[a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6+a_7x^7+a_8x^8+\dots\right]$$
$$y=x^{-\frac{1}{2}}\left[a_0+a_1x+\frac{a_0}{2!}x^2+\frac{a_1}{3!}x^3+\frac{a_0}{4!}x^4+\frac{a_1}{5!}x^5+\frac{a_0}{6!}x^6+\dots\right]$$
$$y=a_0x^{-\frac{1}{2}}\left[1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\dots\right]+a_1x^{-\frac{1}{2}}\left[x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\dots\right]$$
$$y=\frac{A}{\sqrt{x}}\sum_{r=0}^{\infty}\frac{x^{2r}}{(2r)!}+\frac{B}{\sqrt{x}}\sum_{r=0}^{\infty}\frac{x^{2r+1}}{(2r+1)!}$$
$$y=\frac{A}{\sqrt{x}}\cosh x+\frac{B}{\sqrt{x}}\sinh x$$

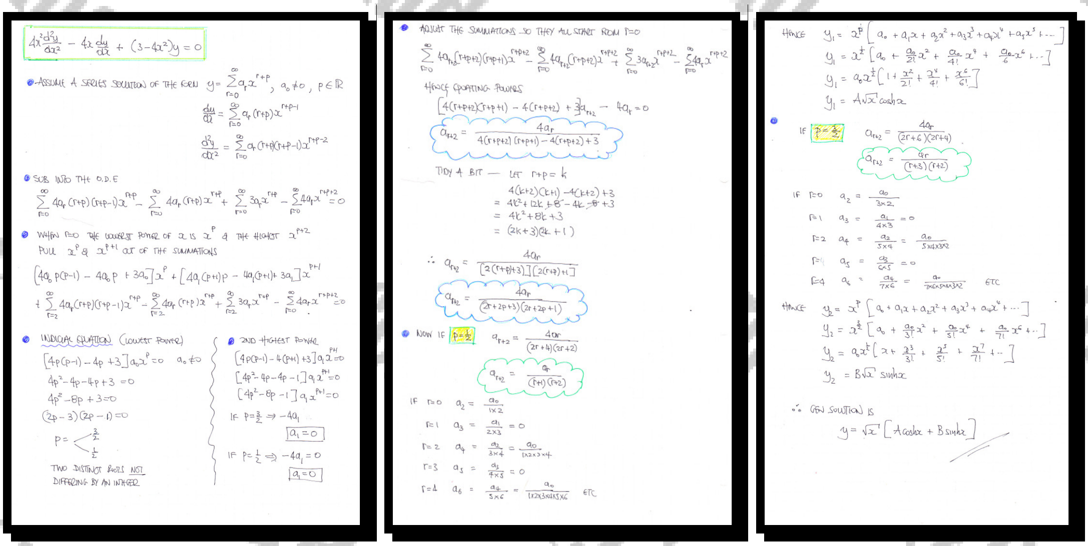
Question 99 (****)

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$4x^2\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+\left(3-4x^2\right)y=0.$$

Give the final answer in terms of elementary functions.

$$\boxed{y=\sqrt{x}\left(A\cosh x+B\sinh x\right)}$$



Question 100 (****)

Show clearly that the substitution $z = y^2$, where $y = f(x)$, transforms the differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - 5\frac{dy}{dx} + 2y = 0,$$

into the differential equation

$$\frac{d^2z}{dx^2} - 5\frac{dz}{dx} + 4z = 0$$

$\boxed{\text{proof}}$

- $z = y^2$

Diff w.r.t x

$$\frac{dz}{dx} = 2y\frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{2y}\frac{dz}{dx}}$$

- $\frac{dy}{dx} = \frac{1}{2y}\frac{dz}{dx}$

Diff w.r.t x

$$\frac{d^2y}{dx^2} = -\frac{1}{2y^2}\frac{dy}{dx}\frac{dz}{dx} + \frac{1}{2y}\frac{d^2z}{dx^2}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{1}{2y}\frac{d^2z}{dx^2} - \frac{1}{2y^2}\frac{dy}{dx}\frac{dz}{dx}}$$

- $\frac{d^2y}{dx^2} + \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - 5\frac{dy}{dx} + 2y = 0$

$$\Rightarrow \frac{1}{2y}\frac{d^2z}{dx^2} - \frac{1}{2y^2}\frac{dy}{dx}\frac{dz}{dx} + \frac{1}{y}\left(\frac{1}{2y}\frac{dz}{dx}\right)^2 - 5\left(\frac{1}{2y}\frac{dz}{dx}\right) + 2y = 0$$

$$\Rightarrow \frac{1}{2y}\frac{d^2z}{dx^2} - \frac{1}{2y^2}\left(\frac{1}{2y}\frac{dz}{dx}\right)\frac{dz}{dx} + \frac{1}{y}\times\frac{1}{4y^2}\left(\frac{dz}{dx}\right)^2 - \frac{5}{2y}\frac{dz}{dx} + 2y = 0$$

$$\Rightarrow \frac{1}{2y}\frac{d^2z}{dx^2} \cancel{- \frac{1}{4y^3}\left(\frac{dz}{dx}\right)^2} + \cancel{\frac{1}{4y^3}\left(\frac{dz}{dx}\right)^2} - \frac{5}{2y}\frac{dz}{dx} + 2y = 0$$

$$\Rightarrow \frac{1}{2y}\frac{d^2z}{dx^2} - \frac{5}{2y}\frac{dz}{dx} + 2y = 0$$

$$\Rightarrow \frac{d^2z}{dx^2} - 5\frac{dz}{dx} + 4y^2 = 0$$

$$\Rightarrow \frac{d^2z}{dx^2} - 5\frac{dz}{dx} + 4z = 0$$

// AS REQUIRED

Question 101 (****)

The curve with equation $y = f(x)$ has the line $y = 1$ as an asymptote and satisfies the differential equation

$$x^3 \frac{dy}{dx} - x = xy + 1, \quad x \neq 0.$$

Solve the above differential equation, giving the solution in the form $y = f(x)$.

$$\boxed{y = \mathrm{e}^{-\frac{1}{x}} - \frac{1}{x}}$$

$$x^3 \frac{dy}{dx} - x = xy + 1$$

$$\frac{dy}{dx} - \frac{1}{x^2} = \frac{y}{x^2} + \frac{1}{x^3}$$

$$\frac{dy}{dx} - \frac{y}{x^2} = \frac{1}{x^2} + \frac{1}{x^3}$$

$$\text{IF} = e^{\int -\frac{1}{x^2}\,dx} = e^{\frac{1}{x}}$$

$$\frac{d}{dx}\left(ye^{\frac{1}{x}}\right) = \left(\frac{1}{x^2} + \frac{1}{x^3}\right)e^{\frac{1}{x}}$$

$$ye^{\frac{1}{x}} = \int \left(\frac{1}{x^2} + \frac{1}{x^3}\right)e^{\frac{1}{x}}\,dx$$

BY SUBSTITUTION

$$\boxed{u = \frac{1}{x}} \Rightarrow \boxed{x = \frac{1}{u}}$$

$$\frac{du}{dx} = -\frac{1}{x^2}$$

$$dx = -x^2\,du = -\frac{1}{u^2}\,du$$

$$ye^{\frac{1}{x}} = \int (u^2 + u^3)e^u\left(-\frac{1}{u^2}\,du\right)$$

$$ye^{\frac{1}{x}} = -\int (1+u)e^u\,du$$

BY PARTS

$1+u$	1
e^u	e^u

$$ye^{\frac{1}{x}} = -\left[(1+u)e^u - \int e^u\,du\right]$$

$$ye^{\frac{1}{x}} = -\left[(1+u)e^u - e^u\right] + C$$

$$\Rightarrow ye^{\frac{1}{x}} = -\left[\cancel{e^u} + ue^u - \cancel{e^u}\right] + C$$

$$\Rightarrow ye^{\frac{1}{x}} = -ue^u + C$$

$$\Rightarrow ye^{\frac{1}{x}} = -\frac{1}{x}e^{\frac{1}{x}} + C$$

$$\Rightarrow \boxed{y = -\frac{1}{x} + Ce^{-\frac{1}{x}}}$$

IF $y=1$ IS AN ASYMPTOTE THEN

AS $x \mapsto \infty$

$y \mapsto 1$

$\therefore C = 1$

$$y = -\frac{1}{x} + e^{-\frac{1}{x}}$$

$$y = e^{-\frac{1}{x}} - \frac{1}{x}$$

Question 102 (****)

Given that if $x = t^{\frac{1}{2}}$, where $y = f(x)$, show clearly that

a) $\dfrac{dy}{dx} = 2t^{\frac{1}{2}}\dfrac{dy}{dt}$.

b) $\dfrac{d^2y}{dx^2} = 4t\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt}$.

The following differential equation is to be solved

$$x\frac{d^2y}{dx^2} - \left(8x^2+1\right)\frac{dy}{dx} + 12x^3y = 12x^5,$$

subject to the boundary conditions $y = \dfrac{10}{3}$, $\dfrac{d^2y}{dx^2} = 10$ at $x = 0$.

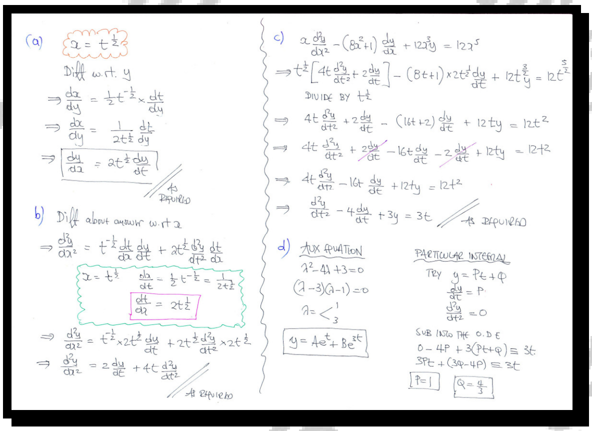
c) Show further that the substitution $x = t^{\frac{1}{2}}$, where $y = f(x)$, transforms the above differential equation into the differential equation

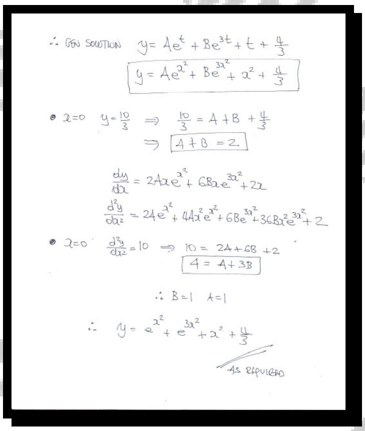
$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y = 3t.$$

d) Show that a solution of the **original** differential equation is

$$y = e^{3x^2} + e^{x^2} + x^2 + \frac{4}{3}.$$

$\boxed{\text{proof}}$





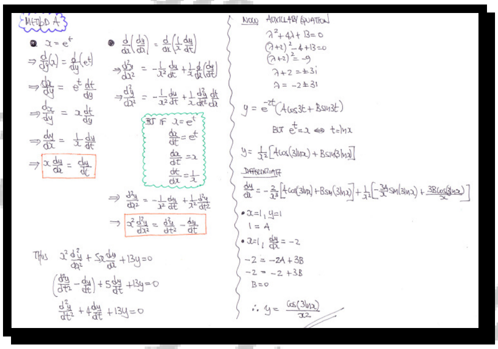
Question 103 (****)

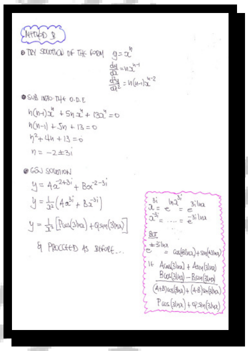
The curve with equation $y = f(x)$ satisfies

$$x^2 \frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 13y = 0, \quad x > 0.$$

By using the substitution $x = e^t$, or otherwise, determine an equation for $y = f(x)$, given further that $y = 1$ and $\frac{dy}{dx} = -2$ at $x = 1$.

$$\boxed{y = \frac{\cos(3\ln x)}{x^2}}$$





Question 104 (****)

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx}\cot x + 2y\operatorname{cosec}^2 x = 2\cos x - 2\cos^3 x.$$

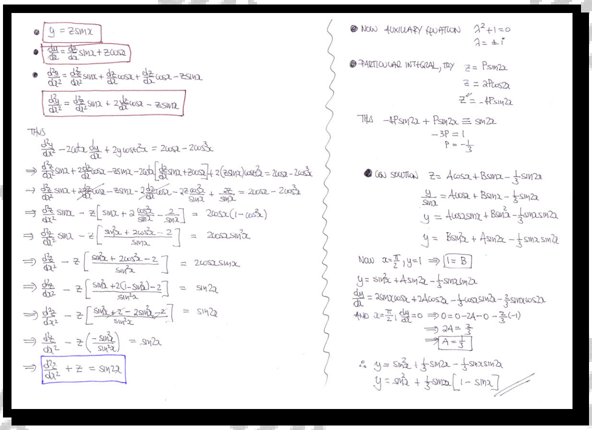
Use the substitution $y = z\sin x$, where z is a function of x, to solve the above differential equation subject to the boundary conditions $y = 1$, $\frac{dy}{dx} = 0$ at $x = \frac{\pi}{2}$.

Give the answer in the form

$$y = a\sin^2 x + b(1 - \sin x)\sin 2x,$$

where a and b are constants to be found.

$\boxed{a = 1}$, $\boxed{b = \frac{1}{3}}$

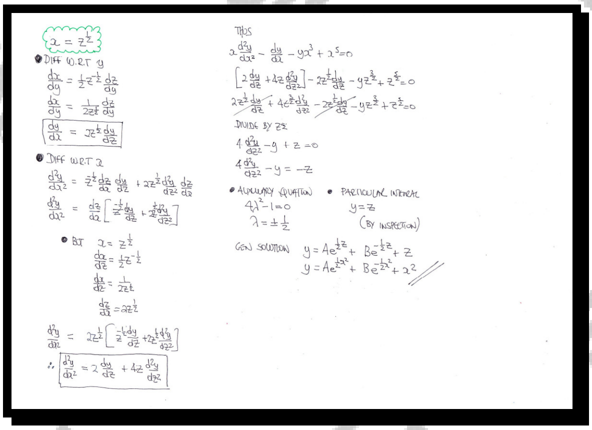


Question 105 (****)

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} - x^3y + x^5 = 0.$$

Use the substitution $x = z^{\frac{1}{2}}$, where $y = f(x)$, to find a general solution of the above differential equation.

$$\boxed{y = A\mathrm{e}^{\frac{1}{2}x^2} + B\mathrm{e}^{-\frac{1}{2}x^2} + x^2}$$



$x = z^{\frac{1}{2}}$

- DIFF W.R.T y

$\frac{dx}{dy} = \frac{1}{2}z^{-\frac{1}{2}}\frac{dz}{dy}$

$\frac{dx}{dy} = \frac{1}{2z^{\frac{1}{2}}}\frac{dz}{dy}$

$\boxed{\frac{dy}{dx} = 2z^{\frac{1}{2}}\frac{dy}{dz}}$

- DIFF W.R.T x

$\frac{d^2y}{dx^2} = z^{-\frac{1}{2}}\frac{dz}{dx}\frac{dy}{dz} + 2z^{\frac{1}{2}}\frac{d^2y}{dz^2}\frac{dz}{dx}$

$\frac{d^2y}{dx^2} = \frac{dz}{dx}\left[z^{-\frac{1}{2}}\frac{dy}{dz} + 2z^{\frac{1}{2}}\frac{d^2y}{dz^2}\right]$

- BUT $x = z^{\frac{1}{2}}$

$\frac{dx}{dz} = \frac{1}{2}z^{-\frac{1}{2}}$

$\frac{dx}{dz} = \frac{1}{2z^{\frac{1}{2}}}$

$\frac{dz}{dx} = 2z^{\frac{1}{2}}$

$\frac{d^2y}{dx^2} = 2z^{\frac{1}{2}}\left[z^{-\frac{1}{2}}\frac{dy}{dz} + 2z^{\frac{1}{2}}\frac{d^2y}{dz^2}\right]$

$\therefore \boxed{\frac{d^2y}{dx^2} = 2\frac{dy}{dz} + 4z\frac{d^2y}{dz^2}}$

THUS

$x\frac{d^2y}{dx^2} - \frac{dy}{dx} - yx^3 + x^5 = 0$

$\left[2\frac{dy}{dz} + 4z\frac{d^2y}{dz^2}\right] - 2z^{\frac{1}{2}}\frac{dy}{dz} - yz^{\frac{3}{2}} + z^{\frac{5}{2}} = 0$

$\cancel{2z^{\frac{1}{2}}\frac{dy}{dz}} + 4z^{\frac{3}{2}}\frac{d^2y}{dz^2} - \cancel{2z^{\frac{1}{2}}\frac{dy}{dz}} - yz^{\frac{3}{2}} + z^{\frac{5}{2}} = 0$

DIVIDE BY $z^{\frac{3}{2}}$

$4\frac{d^2y}{dz^2} - y + z = 0$

$4\frac{d^2y}{dz^2} - y = -z$

- AUXILIARY EQUATION

$4\lambda^2 - 1 = 0$

$\lambda = \pm\frac{1}{2}$

- PARTICULAR INTEGRAL

$y = z$

(BY INSPECTION)

GEN SOLUTION $y = Ae^{\frac{1}{2}z} + Be^{-\frac{1}{2}z} + z$

$y = Ae^{\frac{1}{2}x^2} + Be^{-\frac{1}{2}x^2} + x^2$

Question 106 (****)

Use variation of parameters to determine the specific solution of the following differential equation

$$x^2\frac{d^2y}{dx^2}-7x\frac{dy}{dx}+16y=16\ln x\,,$$

given further that $y=\frac{1}{2}$, $\frac{dy}{dx}=2$ at $x=1$.

$$\boxed{y=\frac{1}{2}+\left(1+x^4\right)\ln x}$$

$x^2\frac{d^2y}{dx^2}-7x\frac{dy}{dx}+16y=16\ln x$ SUBJECT TO $x=1$, $y=1/2$, $y'=2$

- ASSUME SOLUTION OF THE FORM

$y=x^\lambda \Rightarrow y'=\lambda x^{\lambda-1}$ & $y''=\lambda(\lambda-1)x^{\lambda-2}$

SUB INTO THE O.D.E WITH R.H.S ZERO

$\lambda(\lambda-1)x^\lambda-7\lambda x^\lambda+16x^\lambda=0$

$(\lambda^2-\lambda-7\lambda+16)x^\lambda=0$

$\lambda^2-8\lambda+16=0$

$(\lambda-4)^2=0$

$\lambda=4$ REPEATED $\quad\therefore$ COMPLEMENTARY FUNCTION

$y=Ax^4+Bx^4\ln x$

- PARTICULAR INTEGRAL BY VARIATION OF PARAMETERS

$x^2\frac{d^2y}{dx^2}-7x\frac{dy}{dx}+16y=16\ln x$

$a(x)=x^2$

$\boxed{\begin{aligned}e_1&=x^4\\ e_2&=x^4\ln x\end{aligned}}$

WRONSKIAN $=\begin{vmatrix}e_1 & e_2\\ e_1' & e_2'\end{vmatrix}$

$=\begin{vmatrix}x^4 & x^4\ln x\\ 4x^3 & 4x^3\ln x+x^3\end{vmatrix}$

$=4x^7\ln x+x^7-4x^7\ln x$

$=x^7$

- THUS THE PARTICULAR INTEGRAL IS GIVEN BY

$y_p=-e_1\int\frac{e_2 f}{aW}\,dx+e_2\int\frac{e_1 f}{aW}\,dx$

$y_p=-x^4\int\frac{x^4\ln x\times16\ln x}{x^2\times x^7}\,dx+x^4\ln x\int\frac{x^4\times16\ln x}{x^2\times x^7}\,dx$

$y_p=-x^4\int\frac{16(\ln x)^2}{x^5}\,dx+x^4\ln x\int\frac{16\ln x}{x^5}\,dx$

- EACH BY PARTS

- $\int16x^{-5}(\ln x)^2\,dx$

$(\ln x)^2$	$(2\ln x)\frac{1}{x}$
$-4x^{-4}$	$16x^{-5}$

$=-\frac{4}{x^4}(\ln x)^2+\int8x^{-5}(\ln x)\,dx$

BY PARTS AGAIN

$\ln x$	$\frac{1}{x}$
$-2x^{-4}$	$8x^{-5}$

$=-\frac{4}{x^4}(\ln x)^2-\frac{2}{x^4}\ln x+\int2x^{-5}\,dx$

$=-\frac{4}{x^4}(\ln x)^2-\frac{2}{x^4}\ln x-\frac{1}{2}x^{-4}$

- $\int16x^{-5}(\ln x)\,dx$

$\ln x$	$\frac{1}{x}$
$-4x^{-4}$	$16x^{-5}$

$=-\frac{4}{x^4}\ln x+\int4x^{-5}\,dx$

$=-\frac{4}{x^4}\ln x-x^{-4}$

$\therefore y_p=-x^4\left[-\frac{4}{x^4}(\ln x)^2-\frac{2}{x^4}\ln x-\frac{1}{2}x^{-4}\right]+x^4\ln x\left[-\frac{4}{x^4}\ln x-x^{-4}\right]$

$y_p=\cancel{4(\ln x)^2}+2\ln x+\frac{1}{2}-\cancel{4(\ln x)^2}-\ln x$

SO $y_p=\ln x+\frac{1}{2}$

- GEN SOLUTION

$y=Ax^4+Bx^4\ln x+\ln x+\frac{1}{2}$

- APPLY CONDITION $x=1$ $y=\frac{1}{2}$

$\frac{1}{2}=A+\frac{1}{2}\Rightarrow\boxed{A=0}$

$\therefore y=Bx^4\ln x+\ln x+\frac{1}{2}$

$\frac{dy}{dx}=4Bx^3\ln x+Bx^3+\frac{1}{x}$

- APPLY CONDITION $x=1$ $\frac{dy}{dx}=2$

$2=B+1$

$\boxed{B=1}$

$\therefore y=x^4\ln x+\ln x+\frac{1}{2}$

$y=\frac{1}{2}+(x^4+1)\ln x$

Question 107 (****+)

Solve the differential equation

$$\frac{dy}{dx} = \frac{1 - xy + x^2 y^2}{x^2 - yx^3}, \quad x > 0,$$

subject to the condition $y(1) = 0$.

$$\boxed{2xy - x^2 y^2 = 2\ln x}$$

$$\frac{dy}{dx} = \frac{1 - xy + x^2y^2}{x^2 - yx^3}$$

$$v = xy \quad \text{or} \quad y = \frac{v}{x}$$

$$\frac{dv}{dx} = 1 \times y + x\frac{dy}{dx}$$

$$x\frac{dy}{dx} = \frac{dv}{dx} - y \quad \text{or} \quad x\frac{dy}{dx} = \frac{dv}{dx} - \frac{v}{x}$$

MULTIPLY O.D.E BY x

$$\Rightarrow x\frac{dy}{dx} = \frac{x - x^2y + x^3y^2}{x^2 - yx^3}$$

$$\Rightarrow \frac{dv}{dx} - \frac{v}{x} = \frac{x - xv + xv^2}{x^2 - x^2v}$$

$$\Rightarrow \frac{dv}{dx} - \frac{v}{x} = \frac{x(1 - v + v^2)}{x^2(1 - v)}$$

$$\Rightarrow \frac{dv}{dx} - \frac{v}{x} = \frac{1 - v + v^2}{x(1 - v)}$$

$$\Rightarrow x\frac{dv}{dx} - v = \frac{1 - v + v^2}{1 - v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1 - v + v^2}{1 - v} + v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1 \cancel{- v} \cancel{+ v^2} \cancel{+ v} \cancel{- v^2}}{1 - v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1}{1 - v}$$

$$\Rightarrow (1 - v)\,dv = \frac{1}{x}\,dx$$

$$\Rightarrow \int 1 - v \; dv = \int \frac{1}{x}\,dx$$

$$\Rightarrow v - \frac{1}{2}v^2 = \ln x + C$$

$$\Rightarrow xy - \frac{1}{2}x^2y^2 = \ln x + C$$

WHEN $x = 1$ $y = 0$

$$0 = \ln 1 + C$$

$$C = 0$$

$$\Rightarrow xy - \frac{1}{2}x^2y^2 = \ln x$$

$$\Rightarrow 2xy - x^2y^2 = 2\ln x$$

AS REQUIRED

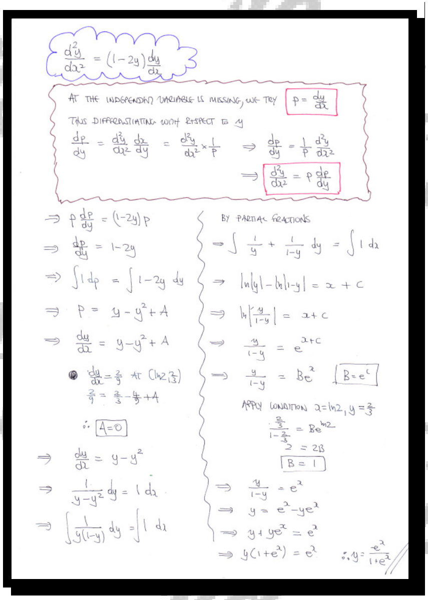
Question 108 (****+)

The curve C, has gradient $\frac{2}{9}$ at the point with coordinates $\left(\ln 2, \frac{2}{3}\right)$, and satisfies the differential relationship

$$\frac{d^2y}{dx^2} = (1-2y)\frac{dy}{dx}, \quad y < \frac{1}{2}.$$

Find an equation for C, giving the answer in the form $y = f(x)$.

$$\boxed{y = \frac{e^x}{1+e^x} = \frac{1}{e^x + e^{-x}} = \frac{1}{2}\operatorname{sech} x}$$



$\frac{d^2y}{dx^2} = (1-2y)\frac{dy}{dx}$

AS THE INDEPENDENT VARIABLE IS MISSING, WE TRY $\boxed{p = \frac{dy}{dx}}$

THUS DIFFERENTIATING WITH RESPECT TO y

$\frac{dp}{dy} = \frac{d^2y}{dx^2}\frac{dx}{dy} = \frac{d^2y}{dx^2} \times \frac{1}{p} \Rightarrow \frac{dp}{dy} = \frac{1}{p}\frac{d^2y}{dx^2}$

$\Rightarrow \boxed{\frac{d^2y}{dx^2} = p\frac{dp}{dy}}$

$\Rightarrow p\frac{dp}{dy} = (1-2y)p$

$\Rightarrow \frac{dp}{dy} = 1-2y$

$\Rightarrow \int 1\,dp = \int 1-2y\,dy$

$\Rightarrow p = y - y^2 + A$

$\Rightarrow \frac{dy}{dx} = y - y^2 + A$

$\frac{dy}{dx} = \frac{2}{9}$ AT $\left(\ln 2, \frac{2}{3}\right)$

$\frac{2}{9} = \frac{2}{3} - \frac{4}{9} + A$

$\therefore \boxed{A = 0}$

$\Rightarrow \frac{dy}{dx} = y - y^2$

$\Rightarrow \frac{1}{y-y^2}\,dy = 1\,dx$

$\Rightarrow \int \frac{1}{y(1-y)}\,dy = \int 1\,dx$

BY PARTIAL FRACTIONS

$\Rightarrow \int \frac{1}{y} + \frac{1}{1-y}\,dy = \int 1\,dx$

$\Rightarrow \ln|y| - \ln|1-y| = x + C$

$\Rightarrow \ln\left|\frac{y}{1-y}\right| = x + C$

$\Rightarrow \frac{y}{1-y} = e^{x+C}$

$\Rightarrow \frac{y}{1-y} = Be^x \quad \boxed{B = e^C}$

APPLY CONDITION $x = \ln 2$, $y = \frac{2}{3}$

$\frac{\frac{2}{3}}{1-\frac{2}{3}} = Be^{\ln 2}$

$2 = 2B$

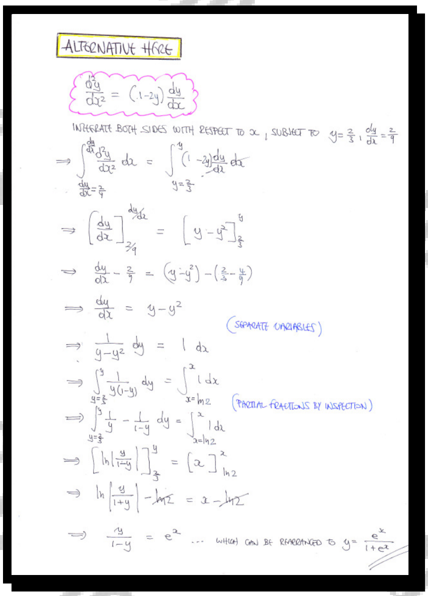
$\boxed{B = 1}$

$\Rightarrow \frac{y}{1-y} = e^x$

$\Rightarrow y = e^x - ye^x$

$\Rightarrow y + ye^x = e^x$

$\Rightarrow y(1+e^x) = e^x \quad \therefore y = \frac{e^x}{1+e^x}$



ALTERNATIVE HERE

$\frac{d^2y}{dx^2} = (1-2y)\frac{dy}{dx}$

INTEGRATE BOTH SIDES WITH RESPECT TO x, SUBJECT TO $y = \frac{2}{3}$, $\frac{dy}{dx} = \frac{2}{9}$

$\Rightarrow \int_{\frac{dy}{dx}=\frac{2}{9}}^{\frac{dy}{dx}} \frac{d^2y}{dx^2}\,dx = \int_{y=\frac{2}{3}}^{y} (1-2y)\frac{dy}{\cancel{dx}}\,\cancel{dx}$

$\Rightarrow \left[\frac{dy}{dx}\right]_{2/9}^{dy/dx} = \left[y - y^2\right]_{\frac{2}{3}}^{y}$

$\Rightarrow \frac{dy}{dx} - \frac{2}{9} = (y-y^2) - \left(\frac{2}{3} - \frac{4}{9}\right)$

$\Rightarrow \frac{dy}{dx} = y - y^2$

(SEPARATE VARIABLES)

$\Rightarrow \frac{1}{y-y^2}\,dy = 1\,dx$

$\Rightarrow \int_{y=\frac{2}{3}}^{y} \frac{1}{y(1-y)}\,dy = \int_{x=\ln 2}^{x} 1\,dx$

(PARTIAL FRACTIONS BY INSPECTION)

$\Rightarrow \int_{y=\frac{2}{3}}^{y} \frac{1}{y} - \frac{1}{1-y}\,dy = \int_{x=\ln 2}^{x} 1\,dx$

$\Rightarrow \left[\ln\left|\frac{y}{1-y}\right|\right]_{\frac{2}{3}}^{y} = \left[x\right]_{\ln 2}^{x}$

$\Rightarrow \ln\left|\frac{y}{1+y}\right| - \cancel{\ln 2} = x - \cancel{\ln 2}$

$\Rightarrow \frac{y}{1-y} = e^x$ ... WHICH CAN BE REARRANGED TO $y = \frac{e^x}{1+e^x}$

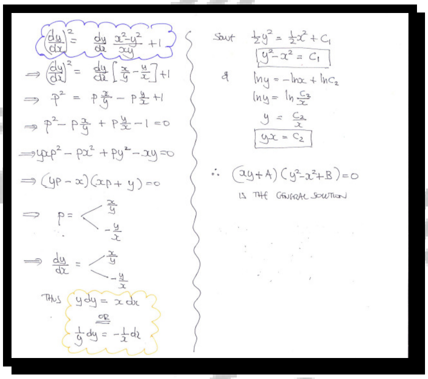
Question 109 (****+)

By writing $\frac{dy}{dx} = p$ and seeking a suitable factorization find a general solution for the non linear differential equation

$$\left(\frac{dy}{dx}\right)^2 = \frac{dy}{dx}\left(\frac{x^2 - y^2}{xy}\right) + 1.$$

Give the solution in the form $F(x, y)G(x, y) = 0$.

$$\boxed{(xy + A)(x^2 - y^2 + B) = 0}$$



$\left(\frac{dy}{dx}\right)^2 = \frac{dy}{dx}\,\frac{x^2-y^2}{xy} + 1$

$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{dy}{dx}\left[\frac{x}{y} - \frac{y}{x}\right] + 1$

$\Rightarrow p^2 = p\frac{x}{y} - p\frac{y}{x} + 1$

$\Rightarrow p^2 - p\frac{x}{y} + p\frac{y}{x} - 1 = 0$

$\Rightarrow yxp^2 - px^2 + py^2 - xy = 0$

$\Rightarrow (yp - x)(xp + y) = 0$

$\Rightarrow p = \frac{x}{y}$ or $-\frac{y}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{x}{y}$ or $-\frac{y}{x}$

THUS $y\,dy = x\,dx$ OR $\frac{1}{y}dy = -\frac{1}{x}dx$

SOLN $\frac{1}{2}y^2 = \frac{1}{2}x^2 + C_1$

$\boxed{y^2 - x^2 = C_1}$

& $\ln y = -\ln x + \ln C_2$

$\ln y = \ln \frac{C_2}{x}$

$y = \frac{C_2}{x}$

$\boxed{yx = C_2}$

$\therefore (xy + A)(y^2 - x^2 + B) = 0$

IS THE GENERAL SOLUTION

Question 110 (****+)

Find a general solution for the differential equation

$$\frac{dy}{dx} = \frac{y - xy^2}{x + yx^2}, \ x \neq 0.$$

$$\boxed{y e^{xy} = Cx}$$

$$\frac{dy}{dx} = \frac{y - xy^2}{x + x^2 y}, \ x \neq 0$$

$v = xy$ OR $y = \frac{v}{x}$

$$\frac{dv}{dx} = 1xy + x\frac{dy}{dx}$$

$$\frac{dv}{dx} = \frac{v}{x} + x\frac{dy}{dx}$$

$$x\frac{dy}{dx} = \frac{dv}{dx} - \frac{v}{x}$$

MULTIPLY O.D.E BY x

$$\Rightarrow x\frac{dy}{dx} = \frac{xy - x^2y^2}{x + x^2y}$$

$$\Rightarrow \frac{dv}{dx} - \frac{v}{x} = \frac{v - v^2}{x + xv}$$

$$\Rightarrow \frac{dv}{dx} - \frac{v}{x} = \frac{v - v^2}{x(1+v)}$$

$$\Rightarrow x\frac{dv}{dx} - v = \frac{v - v^2}{1+v}$$

$$\Rightarrow x\frac{dv}{dx} = v + \frac{v - v^2}{1+v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v + \cancel{v^2} + v - \cancel{v^2}}{1+v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v}{1+v}$$

$$\Rightarrow \frac{v+1}{2v}\,dv = \frac{1}{x}\,dx$$

$$\Rightarrow \int \frac{v+1}{v}\,dv = \int \frac{2}{x}\,dx$$

$$\Rightarrow \int 1 + \frac{1}{v}\,dv = \int \frac{2}{x}\,dx$$

$$\Rightarrow v + \ln|v| = 2\ln|x| + C$$

$$\Rightarrow \ln(e^v) + \ln|v| = \ln(x^2) + \ln A$$

$$\Rightarrow \ln(ve^v) = \ln(Ax^2)$$

$$\Rightarrow ve^v = Ax^2$$

$$\Rightarrow xye^{xy} = Ax^2$$

$$\Rightarrow ye^{xy} = Ax$$

Question 111 (****+)

The curve C, has a stationary point at $(0,2)$ and satisfies the differential relationship

$$\frac{d^2y}{dx^2}=\frac{4}{y^3}, \quad y \neq 0 .$$

a) Given further that $\frac{dy}{dx} \geq 0$ along C, determine a simplified expression for the Cartesian equation of C.

b) Verify by differentiation the answer to part (**a**).

$$\boxed{y^2-x^2=4}$$

a) $\frac{d^2y}{dx^2}=\frac{4}{y^3}$ STATIONARY POINT AT $(0,2)$, $\frac{dy}{dx} \geq 0$

LET $p=\frac{dy}{dx}$ (SINCE THE INDEPENDENT TERM IS MISSING)

$\frac{dp}{dy}=\frac{d}{dy}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}\times\frac{dx}{dy}=\frac{d^2y}{dx^2}\times\frac{1}{p}$

$\boxed{\frac{d^2y}{dx^2}=p\frac{dp}{dy}}$

$\Rightarrow p\frac{dp}{dy}=\frac{4}{y^3}$

$\Rightarrow \int p\,dp=\int \frac{4}{y^3}\,dy$

$\Rightarrow \frac{1}{2}p^2=-\frac{2}{y^2}+C$

$\Rightarrow p^2=C-\frac{4}{y^2}$

$\Rightarrow \left(\frac{dy}{dx}\right)^2=C-\frac{4}{y^2}$

APPLY CONDITION, $y=2$, $\frac{dy}{dx}=0$

$0=C-\frac{4}{2^2}$

$C=1$

$\Rightarrow \left(\frac{dy}{dx}\right)^2=1-\frac{4}{y^2}$

$\Rightarrow \left(\frac{dy}{dx}\right)^2=\frac{y^2-4}{y^2}$

$\Rightarrow \frac{dy}{dx}=+\frac{\sqrt{y^2-4}}{y}$ $\left(\frac{dy}{dx}\geq 0\right)$

$\Rightarrow \frac{y}{\sqrt{y^2-4}}\,dy=1\,dx$

$\Rightarrow \int y(y^2-4)^{-\frac{1}{2}}\,dy=\int 1\,dx$

$\Rightarrow (y^2-4)^{\frac{1}{2}}=x+B$

APPLY CONDITION

$x=0$ $y=2$

$0=0+B$

$B=0$

$\Rightarrow x=(y^2-4)^{\frac{1}{2}}$

$\Rightarrow x^2=y^2-4$

$\Rightarrow y^2-x^2=4$

b) $y^2-x^2=4$

$y^2=x^2+4$ Differentiate w.r.t x

$\Rightarrow 2y\frac{dy}{dx}=2x$

$\Rightarrow y\frac{dy}{dx}=x$

DIFFERENTIATE W.R.T x AGAIN

$\Rightarrow \frac{dy}{dx}\times\frac{dy}{dx}+y\frac{d^2y}{dx^2}=1$

$\Rightarrow \left(\frac{dy}{dx}\right)^2+y\frac{d^2y}{dx^2}=1$

BUT

$\frac{dy}{dx}=\frac{x}{y}$

$\Rightarrow \frac{x^2}{y^2}+y\frac{d^2y}{dx^2}=1$

$\Rightarrow \frac{y^2-4}{y^2}+y\frac{d^2y}{dx^2}=1$

$\Rightarrow \cancel{1}-\frac{4}{y^2}+y\frac{d^2y}{dx^2}=\cancel{1}$

$\Rightarrow y\frac{d^2y}{dx^2}=\frac{4}{y^2}$

$\Rightarrow \frac{d^2y}{dx^2}=\frac{4}{y^3}$

AS REQUIRED

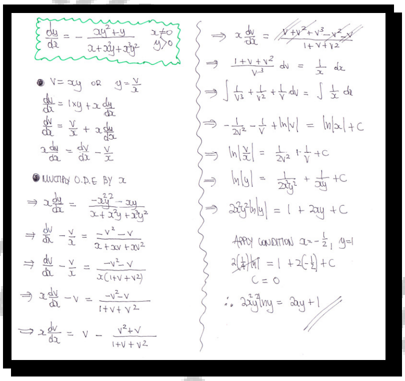
Question 112 (****+)

Solve the differential equation

$$\frac{dy}{dx} = -\frac{xy^2 + y}{x + yx^2 + x^3y^2}, \quad x \neq 0, \quad y > 0,$$

subject to the condition $y\left(\frac{1}{2}\right) = 1$.

$$\boxed{2x^2y^2 \ln y = 2xy + 1}$$



$$\boxed{\frac{dy}{dx} = -\frac{xy^2+y}{x+x^2y+x^3y^2} \quad x \neq 0,\ y>0}$$

- $v = xy$ OR $y = \frac{v}{x}$

$\frac{dv}{dx} = 1 \times y + x\frac{dy}{dx}$

$\frac{dv}{dx} = \frac{v}{x} + x\frac{dy}{dx}$

$x\frac{dy}{dx} = \frac{dv}{dx} - \frac{v}{x}$

- MULTIPLY O.D.E BY x

$\Rightarrow x\frac{dy}{dx} = \frac{-x^2y^2 - xy}{x + x^2y + x^3y^2}$

$\Rightarrow \frac{dv}{dx} - \frac{v}{x} = \frac{-v^2 - v}{x + xv + xv^2}$

$\Rightarrow \frac{dv}{dx} - \frac{v}{x} = \frac{-v^2 - v}{x(1+v+v^2)}$

$\Rightarrow x\frac{dv}{dx} - v = \frac{-v^2 - v}{1+v+v^2}$

$\Rightarrow x\frac{dv}{dx} = v - \frac{v^2+v}{1+v+v^2}$

$\Rightarrow x\frac{dv}{dx} = \frac{\cancel{v} + \cancel{v^2} + v^3 - \cancel{v^2} - \cancel{v}}{1+v+v^2}$

$\Rightarrow \frac{1+v+v^2}{v^3}\,dv = \frac{1}{x}\,dx$

$\Rightarrow \int \frac{1}{v^3} + \frac{1}{v^2} + \frac{1}{v}\,dv = \int \frac{1}{x}\,dx$

$\Rightarrow -\frac{1}{2v^2} - \frac{1}{v} + \ln|v| = \ln|x| + C$

$\Rightarrow \ln\left|\frac{v}{x}\right| = \frac{1}{2v^2} + \frac{1}{v} + C$

$\Rightarrow \ln|y| = \frac{1}{2x^2y^2} + \frac{1}{xy} + C$

$\Rightarrow 2x^2y^2\ln|y| = 1 + 2xy + C$

APPLY CONDITION $x = -\frac{1}{2}$, $y = 1$

$\cancel{2\left(\frac{1}{4}\right)\ln 1} = 1 + 2\left(-\frac{1}{2}\right) + C$

$C = 0$

$\therefore 2x^2y^2\ln y = 2xy + 1$

Question 113 (****+)

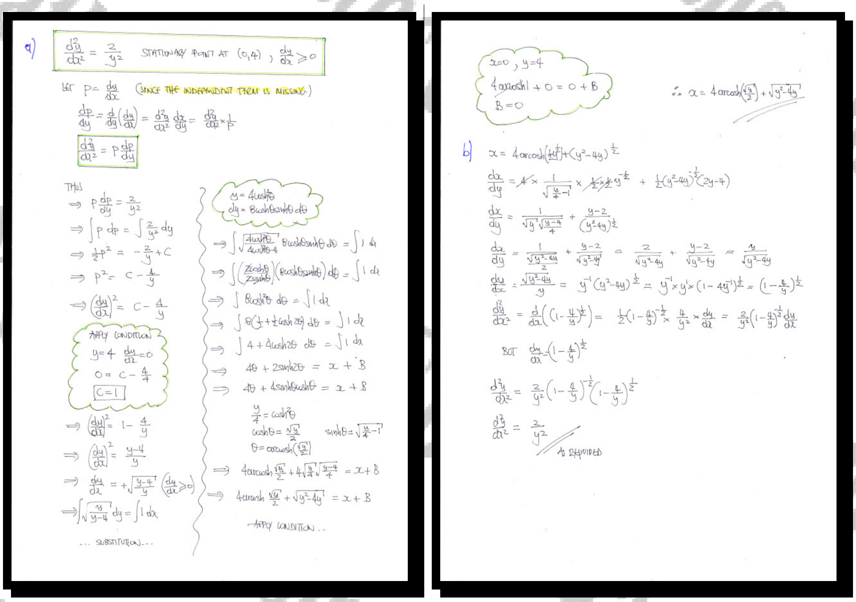
The curve C, has a stationary point at $(0,4)$ and satisfies the differential equation

$$\frac{d^2y}{dx^2} = \frac{2}{y^2}, \quad y \neq 0 .$$

a) Given further that $\frac{dy}{dx} \geq 0$ along C, determine a simplified expression for the Cartesian equation of C, giving the answer in the form $x = f(y)$.

b) Verify by differentiation the answer to part **(a)**.

$$\boxed{x = 4\operatorname{arcosh}\left(\tfrac{1}{2}\sqrt{y}\right) + \sqrt{y^2 - 4y}}$$



Question 114 (****+)

The curve C with Cartesian equation $f(x, y) = 0$, satisfies the differential equation

$$(1-y)\,y'' = (2-y)(y')^2 .$$

It is further given that $y(0) = 0$ and $y'(0) = 1$

a) Determine a simplified expression for the Cartesian equation of C.

b) Verify by differentiation the answer to part (**a**).

$$\boxed{x = y\mathrm{e}^{-y}}$$

a) $(1-y)\dfrac{d^2y}{dx^2} = (2-y)\left(\dfrac{dy}{dx}\right)^2, \quad x=0,\ y=0,\ \dfrac{dy}{dx}=1$

- SINCE THE DEPENDENT VARIABLE x IS MISSING WE USE THE STANDARD SUBSTITUTION

$\dfrac{dy}{dx} = p$

DIFF W.R.T y

$\Rightarrow \dfrac{d}{dy}\left(\dfrac{dy}{dx}\right) = \dfrac{dp}{dy}$

$\Rightarrow \dfrac{d^2y}{dx^2}\dfrac{dx}{dy} = \dfrac{dp}{dy}$

$\Rightarrow \dfrac{d^2y}{dx^2} \times \dfrac{1}{p} = \dfrac{dp}{dy}$

$\Rightarrow \dfrac{d^2y}{dx^2} = p\dfrac{dp}{dy}$

- $(1-y)\,p\dfrac{dp}{dy} = (2-y)\,p^2$

$\Rightarrow (1-y)\dfrac{dp}{dy} = (2-y)\,p$

$\Rightarrow \dfrac{1}{p}\,dp = \dfrac{2-y}{1-y}\,dy$

$\Rightarrow \dfrac{1}{p}\,dp = \dfrac{1+(1-y)}{1-y}\,dy$

$\Rightarrow \displaystyle\int \frac{1}{p}\,dp = \int \frac{1}{1-y} + 1\,dy$

$\Rightarrow \ln|p| = -\ln|1-y| + y + C$

$\Rightarrow p = e^{y - \ln|1-y| + C}$

$\Rightarrow p = \dfrac{Ae^{y}}{1-y} \quad (A = e^{c})$

... APPLY CONDITION $y=0$, $p = \dfrac{dy}{dx} = 1$

$1 = \dfrac{A}{1}$ so $\boxed{A=1}$

THUS $p = \dfrac{e^{y}}{1-y}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{e^{y}}{1-y}$

$\Rightarrow \displaystyle\int (1-y)e^{-y}\,dy = \int 1\,dx$

BY PARTS

$1-y$	-1
$-e^{-y}$	e^{-y}

$\Rightarrow -(1-y)e^{-y} - \displaystyle\int e^{-y}\,dy = x + D$

$\Rightarrow \cancel{-e^{-y}} + ye^{-y} \cancel{+e^{-y}} = x + D$

$\Rightarrow ye^{-y} = x + D$

APPLY CONDITION

$x=0,\ y=0 \Rightarrow D=0$

$\therefore\ x = ye^{-y}$ OR $xe^{y} = y$

b) DIFF $x = ye^{-y}$ W.R.T y

$\dfrac{dx}{dy} = 1 \times e^{-y} + y(-e^{-y})$

$\dfrac{dx}{dy} = e^{-y} - ye^{-y}$

$\dfrac{dx}{dy} = e^{-y}(1-y)$

$\dfrac{dy}{dx} = \dfrac{e^{y}}{1-y}$

$(1-y)\dfrac{dy}{dx} = e^{y}$

DIFF W.R.T x

$-\dfrac{dy}{dx} \times \dfrac{dy}{dx} + (1-y)\dfrac{d^2y}{dx^2} = e^{y}\dfrac{dy}{dx}$

$-\left(\dfrac{dy}{dx}\right)^2 + (1-y)\dfrac{d^2y}{dx^2} = \left[(1-y)\dfrac{dy}{dx}\right] \times \dfrac{dy}{dx}$

$(1-y)\dfrac{d^2y}{dx^2} = (1-y)\left(\dfrac{dy}{dx}\right)^2 + \left(\dfrac{dy}{dx}\right)^2$

$(1-y)\dfrac{d^2y}{dx^2} = (2-y)\left(\dfrac{dy}{dx}\right)^2$

AS REQUIRED

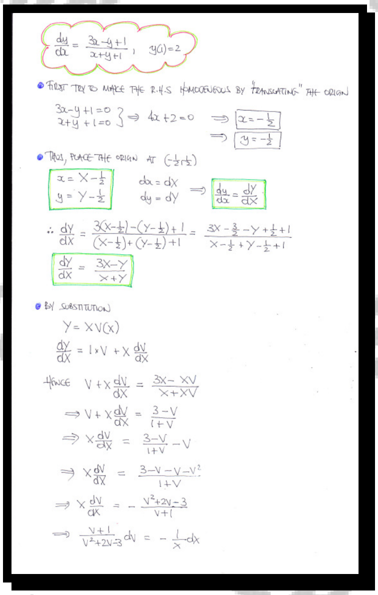
Question 115 (****+)

$$\frac{dy}{dx} = \frac{3x - y + 1}{x + y + 1}, \quad y(1) = 2.$$

Solve the differential equation to show that

$$(y - x)(y + 3x + 2) = 7.$$

proof



$$\frac{dy}{dx} = \frac{3x-y+1}{x+y+1}, \quad y(1)=2$$

- FIRST TRY TO MAKE THE R.H.S HOMOGENEOUS BY "TRANSLATING" THE ORIGIN

$$\left.\begin{array}{l} 3x-y+1=0 \\ x+y+1=0 \end{array}\right\} \Rightarrow 4x+2=0 \Rightarrow \boxed{x=-\tfrac{1}{2}} \Rightarrow \boxed{y=-\tfrac{1}{2}}$$

- THUS, PLACE THE ORIGIN AT $\left(-\frac{1}{2},-\frac{1}{2}\right)$

$$\boxed{\begin{array}{l} x = X-\frac{1}{2} \\ y = Y-\frac{1}{2} \end{array}} \quad \begin{array}{l} dx = dX \\ dy = dY \end{array} \Rightarrow \boxed{\frac{dy}{dx} = \frac{dY}{dX}}$$

$$\therefore \frac{dY}{dX} = \frac{3(X-\frac{1}{2})-(Y-\frac{1}{2})+1}{(X-\frac{1}{2})+(Y-\frac{1}{2})+1} = \frac{3X-\frac{3}{2}-Y+\frac{1}{2}+1}{X-\frac{1}{2}+Y-\frac{1}{2}+1}$$

$$\boxed{\frac{dY}{dX} = \frac{3X-Y}{X+Y}}$$

- BY SUBSTITUTION

$$Y = XV(X)$$

$$\frac{dY}{dX} = 1 \times V + X\frac{dV}{dX}$$

HENCE $V + X\frac{dV}{dX} = \frac{3X-XV}{X+XV}$

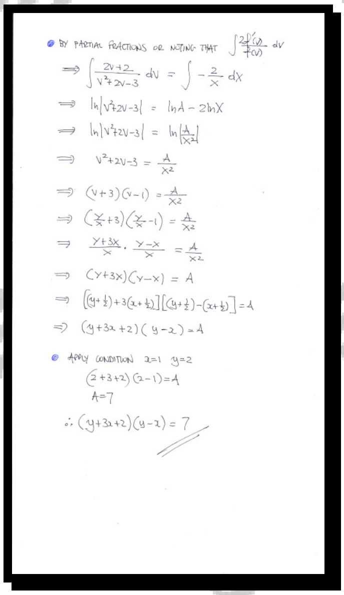
$$\Rightarrow V + X\frac{dV}{dX} = \frac{3-V}{1+V}$$

$$\Rightarrow X\frac{dV}{dX} = \frac{3-V}{1+V} - V$$

$$\Rightarrow X\frac{dV}{dX} = \frac{3-V-V-V^2}{1+V}$$

$$\Rightarrow X\frac{dV}{dX} = -\frac{V^2+2V-3}{V+1}$$

$$\Rightarrow \frac{V+1}{V^2+2V-3}\,dV = -\frac{1}{X}\,dX$$



- BY PARTIAL FRACTIONS OR NOTING THAT $\int \frac{2f'(V)}{f(V)}\,dV$

$$\Rightarrow \int \frac{2V+2}{V^2+2V-3}\,dV = \int -\frac{2}{X}\,dX$$

$$\Rightarrow \ln|V^2+2V-3| = \ln A - 2\ln X$$

$$\Rightarrow \ln|V^2+2V-3| = \ln\left|\frac{A}{X^2}\right|$$

$$\Rightarrow V^2+2V-3 = \frac{A}{X^2}$$

$$\Rightarrow (V+3)(V-1) = \frac{A}{X^2}$$

$$\Rightarrow \left(\frac{Y}{X}+3\right)\left(\frac{Y}{X}-1\right) = \frac{A}{X^2}$$

$$\Rightarrow \frac{Y+3X}{X}\cdot\frac{Y-X}{X} = \frac{A}{X^2}$$

$$\Rightarrow (Y+3X)(Y-X) = A$$

$$\Rightarrow \left[(y+\tfrac{1}{2})+3(x+\tfrac{1}{2})\right]\left[(y+\tfrac{1}{2})-(x+\tfrac{1}{2})\right] = A$$

$$\Rightarrow (y+3x+2)(y-x) = A$$

- APPLY CONDITION $x=1$ $y=2$

$$(2+3+2)(2-1) = A$$

$$A = 7$$

$$\therefore (y+3x+2)(y-x) = 7$$

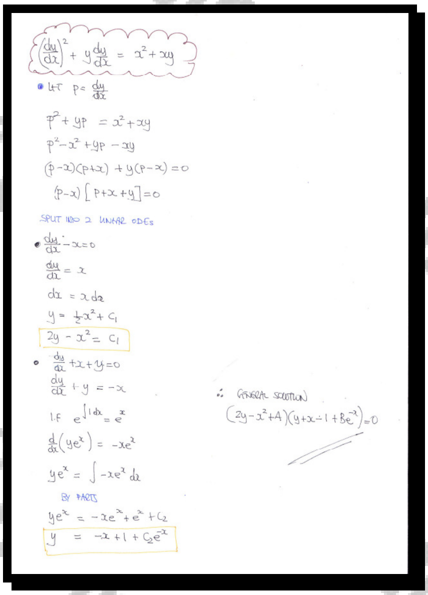
Question 116 (****+)

By writing $\frac{dy}{dx} = p$ and seeking a suitable factorization find a general solution for the non linear differential equation

$$\left(\frac{dy}{dx}\right)^2 + y\frac{dy}{dx} = x^2 + xy .$$

Give the solution in the form $F(x,y)G(x,y) = 0$.

$$\boxed{\left(2y - x^2 + A\right)\left(x + y - 1 + B\,\mathrm{e}^{-x}\right) = 0}$$



$$\left(\frac{dy}{dx}\right)^2 + y\frac{dy}{dx} = x^2 + xy$$

• LET $p = \frac{dy}{dx}$

$p^2 + yp = x^2 + xy$

$p^2 - x^2 + yp - xy$

$(p-x)(p+x) + y(p-x) = 0$

$(p-x)[p+x+y] = 0$

SPLIT INTO 2 LINEAR ODEs

• $\frac{dy}{dx} - x = 0$

$\frac{dy}{dx} = x$

$dx = x\,dx$

$y = \frac{1}{2}x^2 + C_1$

$\boxed{2y - x^2 = C_1}$

• $\frac{dy}{dx} + x + y = 0$

$\frac{dy}{dx} + y = -x$

I.F. $e^{\int 1\,dx} = e^x$

$\frac{d}{dx}(ye^x) = -xe^x$

$ye^x = \int -xe^x\,dx$

BY PARTS

$ye^x = -xe^x + e^x + C_2$

$\boxed{y = -x + 1 + C_2e^{-x}}$

∴ GENERAL SOLUTION

$(2y - x^2 + A)(y + x - 1 + Be^{-x}) = 0$

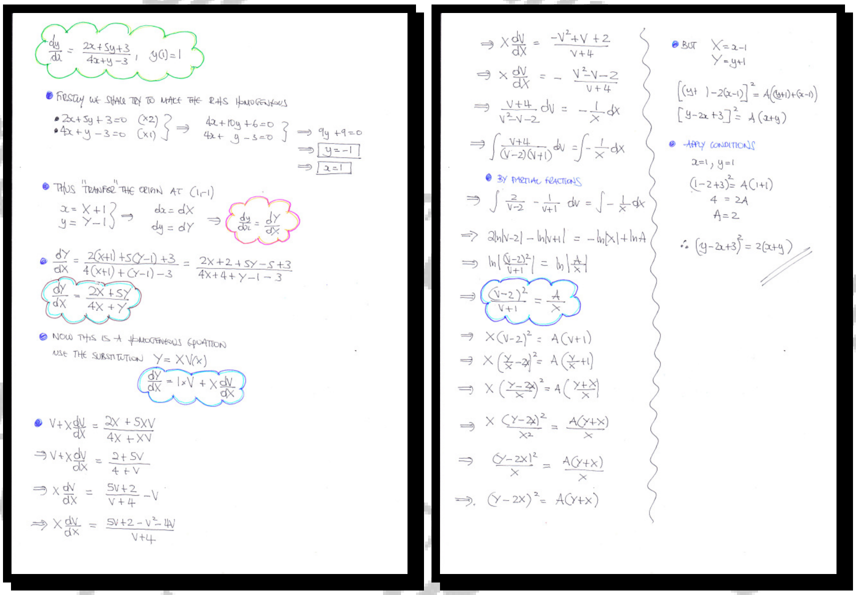
Question 117 (****+)

$$\frac{dy}{dx} = \frac{2x+5y+3}{4x+y-3}, \quad y(1)=1.$$

Solve the differential equation to show that

$$(y-2x+3)^2 = 2(x+y).$$

$\boxed{\text{proof}}$



$\frac{dy}{dx} = \frac{2x+5y+3}{4x+y-3}, \quad y(1)=1$

- FIRSTLY WE SHALL TRY TO MAKE THE RHS HOMOGENEOUS

$\left.\begin{array}{l} \bullet\ 2x+5y+3=0 \quad (\times 2) \\ \bullet\ 4x+y-3=0 \quad (\times 1) \end{array}\right\} \Rightarrow \left.\begin{array}{l} 4x+10y+6=0 \\ 4x+y-3=0 \end{array}\right\} \Rightarrow 9y+9=0$

$\Rightarrow \boxed{y=-1}$

$\Rightarrow \boxed{x=1}$

- THUS "TRANFER" THE ORIGIN AT $(1,-1)$

$\left.\begin{array}{l} x = X+1 \\ y = Y-1 \end{array}\right\} \Rightarrow \begin{array}{l} dx = dX \\ dy = dY \end{array} \Rightarrow \frac{dy}{dx} = \frac{dY}{dX}$

- $\frac{dY}{dX} = \frac{2(X+1)+5(Y-1)+3}{4(X+1)+(Y-1)-3} = \frac{2X+2+5Y-5+3}{4X+4+Y-1-3}$

$\frac{dY}{dX} = \frac{2X+5Y}{4X+Y}$

- NOW THIS IS A HOMOGENEOUS EQUATION
USE THE SUBSTITUTION $Y = XV(X)$

$\frac{dY}{dX} = 1 \times V + X\frac{dV}{dX}$

- $V + X\frac{dV}{dX} = \frac{2X+5XV}{4X+XV}$

$\Rightarrow V + X\frac{dV}{dX} = \frac{2+5V}{4+V}$

$\Rightarrow X\frac{dV}{dX} = \frac{5V+2}{V+4} - V$

$\Rightarrow X\frac{dV}{dX} = \frac{5V+2-V^2-4V}{V+4}$

$\Rightarrow X\frac{dV}{dX} = \frac{-V^2+V+2}{V+4}$

$\Rightarrow X\frac{dV}{dX} = -\frac{V^2-V-2}{V+4}$

$\Rightarrow \frac{V+4}{V^2-V-2}\,dV = -\frac{1}{X}\,dX$

$\Rightarrow \int \frac{V+4}{(V-2)(V+1)}\,dV = \int -\frac{1}{X}\,dX$

- BY PARTIAL FRACTIONS

$\Rightarrow \int \frac{2}{V-2} - \frac{1}{V+1}\,dV = \int -\frac{1}{X}\,dX$

$\Rightarrow 2\ln|V-2| - \ln|V+1| = -\ln|X| + \ln A$

$\Rightarrow \ln\left|\frac{(V-2)^2}{V+1}\right| = \ln\left|\frac{A}{X}\right|$

$\Rightarrow \frac{(V-2)^2}{V+1} = \frac{A}{X}$

$\Rightarrow X(V-2)^2 = A(V+1)$

$\Rightarrow X\left(\frac{Y}{X}-2\right)^2 = A\left(\frac{Y}{X}+1\right)$

$\Rightarrow X\left(\frac{Y-2X}{X}\right)^2 = A\left(\frac{Y+X}{X}\right)$

$\Rightarrow X\frac{(Y-2X)^2}{X^2} = \frac{A(Y+X)}{X}$

$\Rightarrow \frac{(Y-2X)^2}{X} = \frac{A(Y+X)}{X}$

$\Rightarrow (Y-2X)^2 = A(Y+X)$

- BUT $X = x-1$
$Y = y+1$

$[(y+1)-2(x-1)]^2 = A((y+1)+(x-1))$

$[y-2x+3]^2 = A(x+y)$

- APPLY CONDITIONS

$x=1,\ y=1$

$(1-2+3)^2 = A(1+1)$

$4 = 2A$

$A = 2$

$\therefore (y-2x+3)^2 = 2(x+y)$

Question 118 (*****)

The curve with equation $y = f(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} = 6y^2 + 4y, \quad \frac{dy}{dx} \geq 0.$$

If $y = 3$, $\frac{dy}{dx} = 12$ at $x = -\frac{1}{2}\ln 3$, solve the differential equation to show that

$$y = \operatorname{cosech}^2 x.$$

proof

$$\frac{d^2y}{dx^2} = 6y^2 + 4y \quad \text{SUBJECT TO} \quad x = -\tfrac{1}{2}\ln 3,\ y = 3,\ \frac{dy}{dx} = 12$$

- SINCE THE INDEPENDENT VARIABLE IS MISSING WE USE $p = \frac{dy}{dx}$

SO $\frac{dp}{dy} = \frac{d}{dy}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}\frac{dx}{dy} = \frac{d^2y}{dx^2}\frac{1}{p}$

THUS $\boxed{\frac{d^2y}{dx^2} = p\frac{dp}{dy}}$

- THE O.D.E TRANSFORMS TO

$\Rightarrow p\frac{dp}{dy} = 6y^2 + 4y$

$\Rightarrow p\,dp = (6y^2+4y)\,dy$

$\Rightarrow \int p\,dp = \int 6y^2+4y\,dy$

$\Rightarrow \frac{1}{2}p^2 = 2y^3 + 2y^2 + C$

$\Rightarrow p^2 = 4y^3 + 4y^2 + C$

- APPLY CONDITION $y = 3$ $p = \frac{dy}{dx} = 12$

$144 = 4\times27 + 4\times9 + C$

$144 = 108 + 36 + C$

$C = 0$

$\Rightarrow p^2 = 4y^3 + 4y^2$

$\Rightarrow p^2 = 4y^2(y+1)$

$\Rightarrow \left(\frac{dy}{dx}\right)^2 = 4y^2(y+1)$

$\Rightarrow \frac{dy}{dx} = 2y(y+1)^{\frac{1}{2}}$ $\quad \frac{dy}{dx} > 0$

- SEPARATE VARIABLES

$\Rightarrow \int \frac{1}{y(y+1)^{\frac{1}{2}}}\,dy = \int 2\,dx$

$\Rightarrow \int \frac{1}{(u^2-1)\cancel{u}}(2\cancel{u}\,du) = \int 2\,dx$

$\Rightarrow \int \frac{2}{u^2-1}\,du = \int 2\,dx$

$\Rightarrow \int \frac{2}{(u-1)(u+1)}\,du = \int 2\,dx$

BY SUBSTITUTION

$u = (y+1)^{\frac{1}{2}}$

$u^2 = y+1$

$2u\,du = dy$

$y = u^2 - 1$

- BY PARTIAL FRACTIONS

$\Rightarrow \int \frac{1}{u-1} - \frac{1}{u+1}\,du = \int 2\,dx$

$\Rightarrow \ln|u-1| - \ln|u+1| = 2x + k$

$\Rightarrow \ln\left|\frac{u-1}{u+1}\right| = 2x + k$

$\Rightarrow \frac{u-1}{u+1} = e^{2x+k}$

$\Rightarrow \frac{u-1}{u+1} = Ae^{2x}$ $\quad (A = e^k)$

$\Rightarrow u - 1 = Aue^{2x} + Ae^{2x}$

$\Rightarrow u - Aue^{2x} = 1 + Ae^{2x}$

$\Rightarrow u(1 - Ae^{2x}) = 1 + Ae^{2x}$

$\Rightarrow u = \frac{1+Ae^{2x}}{1-Ae^{2x}}$

$\Rightarrow (y+1)^{\frac{1}{2}} = \frac{1+Ae^{2x}}{1-Ae^{2x}}$

- APPLY THE LAST CONDITION

$y = 3$, $x = -\frac{1}{2}\ln 3$ (I.E $e^{2x} = \frac{1}{3}$)

$\sqrt{3+1} = \frac{1+\frac{1}{3}A}{1-\frac{1}{3}A}$

$2 = \frac{3+A}{3-A}$

$6 - 2A = 3 + A$

$3 = 3A$

$\boxed{A = 1}$

HENCE

$\Rightarrow \sqrt{y+1} = \frac{1+e^{2x}}{1-e^{2x}}$

$\Rightarrow \sqrt{y+1} = \frac{e^{-x}+e^{x}}{e^{-x}-e^{x}}$

$\Rightarrow \sqrt{y+1} = -\frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}$

$\Rightarrow \sqrt{y+1} = -\coth x$

$\Rightarrow y+1 = \coth^2 x$

$\Rightarrow y = \coth^2 x - 1$

$\Rightarrow y = \operatorname{cosech}^2 x$

WHICH CAN EASILY BE CHECKED AGAINST THE ORIGINAL O.D.E

Question 119 (*****)

The curve with equation $y = f(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 8y .$$

Given further that the curve has a stationary point at $\left(\frac{1}{2}, \frac{1}{4}\right)$, solve the differential equation to show that

$$y = x^2 + x + \frac{1}{2} .$$

proof

$\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 8y$ SUBJECT TO $y = \frac{1}{4}$, $\frac{dy}{dx} = 0$, AT $x = \frac{1}{2}$

- SINCE THE INDEPENDENT TERM IS MISSING (x), LET $p = \frac{dy}{dx}$

THUS $\frac{dp}{dy} = \frac{d}{dy}(p) = \frac{d}{dy}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} \times \frac{dx}{dy} = \frac{d^2y}{dx^2} \times \frac{1}{p}$

$\therefore$ $p = \frac{dy}{dx}$ & $\frac{d^2y}{dx^2} = p\frac{dp}{dy}$

- HENCE

$\Rightarrow p\frac{dp}{dy} + 2p^2 = 8y$

$\Rightarrow \frac{dp}{dy} + 2p = \frac{8y}{p}$

$\Rightarrow \boxed{\frac{dp}{dy} + 2p = 8yp^{-1}}$ ← STANDARD BERNOULLI TYPE

SUBSTITUTION $z = \frac{1}{p^{n-1}}$

HERE $z = \frac{1}{p^{-1-1}} = p^2$

IE $\boxed{z = p^2}$

$\boxed{\frac{dz}{dy} = 2p\frac{dp}{dy}}$

MULTIPLY THROUGH BY $2p$

$\Rightarrow 2p\frac{dp}{dy} + 4p^2 = 16y$

$\Rightarrow \boxed{\frac{dz}{dy} + 4z = 16y}$

- NOW

INTEGRATING FACTOR IS

$e^{\int 4\,dy} = e^{4y}$

$\Rightarrow \frac{d}{dy}\left(ze^{4y}\right) = 16ye^{4y}$

$\Rightarrow ze^{4y} = \int 16ye^{4y}\,dy$ (BY PARTS)

$\Rightarrow ze^{4y} = \int 16ye^{4y}\,dy$

$16y$	16
$\frac{1}{4}e^{4y}$	e^{4y}

$\Rightarrow ze^{4y} = 4ye^{4y} - \int 4e^{4y}\,dy$

$\Rightarrow ze^{4y} = 4ye^{4y} - e^{4y} + A$

$\Rightarrow z = (4y-1) + Ae^{4y}$

$\Rightarrow p^2 = 4y - 1 + Ae^{4y}$

$\Rightarrow \boxed{\left(\frac{dy}{dx}\right)^2 = 4y - 1 + Ae^{4y}}$

NOW STATIONARY AT $\left(\frac{1}{2}, \frac{1}{4}\right) \Rightarrow$
$0 = 4 \times \frac{1}{4} - 1 + Ae^{4 \times \frac{1}{4}}$
$0 = 1 - 1 + Ae$
$Ae = 0$
$\boxed{A = 0}$

$\Rightarrow \left(\frac{dy}{dx}\right)^2 = 4y - 1$

$\Rightarrow \frac{dy}{dx} = \pm(4y-1)^{\frac{1}{2}}$

$\Rightarrow \frac{1}{\pm(4y-1)^{\frac{1}{2}}}\,dy = 1\,dx$

$\Rightarrow \int \pm(4y-1)^{-\frac{1}{2}}\,dy = \int 1\,dx$

$\Rightarrow \pm\frac{1}{2}(4y-1)^{\frac{1}{2}} = x + C$

NOW $\left(\frac{1}{2}, \frac{1}{4}\right) \Rightarrow \pm\frac{1}{2} \times 0 = \frac{1}{2} + C$
$\Rightarrow 0 = \frac{1}{2} + C$
$\Rightarrow \boxed{C = \frac{1}{2}}$

$\Rightarrow \pm\frac{1}{2}(4y-1)^{\frac{1}{2}} = x + \frac{1}{2}$

$\Rightarrow \pm(4y-1)^{\frac{1}{2}} = 2x + 1$ (SQUARE)

$\Rightarrow 4y - 1 = 4x^2 + 4x + 1$

$\therefore 4y = 4x^2 + 4x + 2$

$y = x^2 + x + \frac{1}{2}$

Question 120 (*****)

The curve C, has gradient 1 at the origin and satisfies the differential relationship

$$\frac{d^2y}{dx^2}\sqrt{1-2y} = \frac{dy}{dx}(3y-2), \quad y < \frac{1}{2}.$$

Find an equation for C, giving the answer in the form $y = f(x)$.

$$\boxed{y = \frac{\sin x}{1+\sin x} = (\sec x - \tan x)\tan x}$$

$$\frac{d^2y}{dx^2} = \frac{dy}{dx}\left(\frac{3y-2}{\sqrt{1-2y}}\right) \qquad x=0,\ y=0,\ \frac{dy}{dx}=1$$

LET $p = \frac{dy}{dx}$ DIFF W.R.T y

$$\frac{dp}{dy} = \frac{d^2y}{dx^2}\frac{dx}{dy} = \frac{d^2y}{dx^2} \times \frac{1}{\frac{dy}{dx}}$$

$$\frac{dp}{dy} = \frac{d^2y}{dx^2} \times \frac{1}{p}$$

$$\frac{d^2y}{dx^2} = p\frac{dp}{dy}$$

THUS

$$\cancel{p}\frac{dp}{dy} = \cancel{p}\frac{3y-2}{\sqrt{1-2y}}$$

$$\Rightarrow \int 1\, dp = \int \frac{3y-2}{\sqrt{1-2y}}\, dy$$

$$\Rightarrow p = \int \frac{3y-2}{\sqrt{1-2y}}\, dy$$

BY SUBSTITUTION

$$u = \sqrt{1-2y}$$
$$u^2 = 1-2y$$
$$2u\,du = -2\,dy$$
$$dy = -u\,du$$

$$\Rightarrow \frac{dy}{dx} = \int \frac{3y-2}{u}(-2u\,du)$$

$$\Rightarrow \frac{dy}{dx} = \int 2-3y\ du$$

$$\Rightarrow \frac{dy}{dx} = \int 2-\tfrac{3}{2}(2y)\ du$$

$$\Rightarrow \frac{dy}{dx} = \int 2-\tfrac{3}{2}(1-u^2)\ du$$

$$\Rightarrow \frac{dy}{dx} = \int 2-\tfrac{3}{2}+\tfrac{3}{2}u^2\ du$$

$$\Rightarrow \frac{dy}{dx} = \int \tfrac{1}{2}+\tfrac{3}{2}u^2\ du$$

$$\Rightarrow \frac{dy}{dx} = \tfrac{1}{2}u + \tfrac{1}{2}u^3 + C$$

$$\Rightarrow \frac{dy}{dx} = \tfrac{1}{2}u(1+u^2) + C$$

$$\Rightarrow \frac{dy}{dx} = \tfrac{1}{2}\sqrt{1-2y}\,(1+1-2y) + C$$

$$\Rightarrow \frac{dy}{dx} = \tfrac{1}{2}\sqrt{1-2y}\,(2-2y) + C$$

$$\Rightarrow \frac{dy}{dx} = (1-y)\sqrt{1-2y} + C$$

APPLY CONDITIONS

$y=0 \quad \frac{dy}{dx}=1$

$1 = 1 + C$

$\boxed{C=0}$

$$\boxed{\frac{dy}{dx} = (1-y)(1-2y)^{\frac{1}{2}}}$$

SEPARATE VARIABLES

$$\int \frac{1}{(1-y)(1-2y)^{\frac{1}{2}}}\,dy = \int 1\, dx$$

USE A SUBSTITUTION

$$2y = \sin^2\theta$$
$$2\,dy = 2\sin\theta\cos\theta\, d\theta$$
$$dy = \sin\theta\cos\theta\, d\theta$$

$$\Rightarrow x = \int \frac{\sin\theta\,\cancel{\cos\theta}\, d\theta}{\left(1-\frac{1}{2}\sin^2\theta\right)\cancel{(1-\sin^2\theta)^{\frac{1}{2}}}}$$

$$\Rightarrow x = \int \frac{2\sin\theta}{2-\sin^2\theta}\, d\theta$$

$$\Rightarrow x = \int \frac{\frac{2\sin\theta}{\cos^2\theta}}{\frac{2}{\cos^2\theta}-\frac{\sin^2\theta}{\cos^2\theta}}\, d\theta$$

$$\Rightarrow x = \int \frac{2\sec\theta\tan\theta}{2\sec^2\theta-\tan^2\theta}\, d\theta$$

$$\Rightarrow x = \int \frac{2\sec\theta\tan\theta}{2\sec^2\theta-(\sec^2\theta-1)}\, d\theta$$

$$\Rightarrow x = \int \frac{2\sec\theta\tan\theta}{\sec^2\theta+1}\, d\theta$$

ANOTHER SUBSTITUTION

$u = \sec\theta$

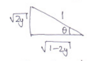
OR

RECOGNITION

$$\boxed{x = 2\arctan(\sec\theta) + C}$$

NOW

$\sin^2\theta = 2y$



$$\therefore \sec\theta = \frac{1}{\sqrt{1-2y}}$$

$$\Rightarrow x = 2\arctan\left(\frac{1}{\sqrt{1-2y}}\right) + k$$

$$\Rightarrow x = -2\arctan\left(\sqrt{1-2y}\right) + k$$

APPLY CONDITION

$x=0,\ y=0 \Rightarrow 0 = -2\times\frac{\pi}{4} + k$

$k = \frac{\pi}{2}$

$$\boxed{x = -2\arctan\sqrt{1-2y} + \frac{\pi}{2}}$$

TIDY FURTHER

$$\Rightarrow 2\arctan\sqrt{1-2y} = \frac{\pi}{2} - x$$

$$\Rightarrow \arctan\sqrt{1-2y} = \frac{\pi}{4} - \frac{x}{2}$$

$$\Rightarrow \sqrt{1-2y} = \tan\left(\frac{\pi}{4}-\frac{x}{2}\right)$$

TIDY FURTHER

$$\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}} = \frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}$$

(MULTIPLY TOP & BOTTOM BY $\cos\frac{x}{2}$)

$$= \frac{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}$$

$$= \frac{\cos^2\frac{x}{2}-2\cos\frac{x}{2}\sin\frac{x}{2}+\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}$$

$$= \frac{1-\sin x}{\cos x}$$

$$\Rightarrow \sqrt{1-2y} = \frac{1-\sin x}{\cos x}$$

$$\Rightarrow \sqrt{1-2y} = \sec x - \tan x$$

$$\Rightarrow 1-2y = \sec^2 x + \tan^2 x - 2\sec x\tan x$$

$$\Rightarrow 1-\sec^2 x - \tan^2 x + 2\sec x\tan x = 2y$$

$$\Rightarrow -2\tan^2 x + 2\sec x\tan x = 2y$$

$$\Rightarrow y = \sec x\tan x - \tan^2 x$$

$$\Rightarrow y = \tan x(\sec x - \tan x)$$

ALTERNATIVE ANSWER

$$\Rightarrow \sqrt{1-2y} = \frac{1-\sin x}{\cos x}$$

$$\Rightarrow 1-2y = \frac{(1-\sin x)^2}{\cos^2 x}$$

$$\Rightarrow 1-2y = \frac{(1-\sin x)^2}{1-\sin^2 x}$$

$$\Rightarrow 1-2y = \frac{(1-\sin x)^2}{(1-\sin x)(1+\sin x)}$$

$$\Rightarrow 1-2y = \frac{1-\sin x}{1+\sin x}$$

$$\Rightarrow 1-\frac{1-\sin x}{1+\sin x} = 2y$$

$$\Rightarrow 2y = \frac{1+\sin x-1+\sin x}{1+\sin x}$$

$$\Rightarrow 2y = \frac{2\sin x}{1+\sin x}$$

$$\Rightarrow y = \frac{\sin x}{1+\sin x}$$

Question 121 (*****)

The curve C, has gradient $\frac{1}{8}$ at the point with coordinates $\left(1,\frac{1}{2}\right)$ and further satisfies the differential relationship

$$2y^2\frac{d^2y}{dx^2}+(2y+1)(y-1)^2\frac{dy}{dx}=0, \qquad y \neq 0.$$

Find an equation for C, giving the answer in the form $y=f(x)$.

$$\boxed{y=\frac{\sqrt{x}}{1+\sqrt{x}}}$$

$$\boxed{2y^2\frac{d^2y}{dx^2}+(2y+1)(y-1)^2\frac{dy}{dx}=0 \qquad x=1,\ y=\frac{1}{2},\ \frac{dy}{dx}=\frac{1}{8}}$$

$\Rightarrow 2y^2\frac{d^2y}{dx^2}=-(2y+1)(y-1)^2\frac{dy}{dx}$

$\Rightarrow \frac{d^2y}{dx^2}=-\frac{(2y+1)(y^2-2y+1)}{2y^2}\frac{dy}{dx}$

$\Rightarrow \frac{d^2y}{dx^2}=-\frac{\begin{matrix}2y^3-4y^2+2y\\ y^2-2y+1\end{matrix}}{2y^2}\frac{dy}{dx}$

$\Rightarrow \frac{d^2y}{dx^2}=-\frac{2y^3-3y^2+1}{2y^2}\frac{dy}{dx}$

$\Rightarrow \frac{d^2y}{dx^2}=\left(-y+\frac{3}{2}-\frac{1}{2}y^{-2}\right)\frac{dy}{dx}$

- INTEGRATE BOTH SIDES WITH RESPECT TO x SUBJECT TO $y=\frac{1}{2}$ $dy/dx=\frac{1}{8}$

$\Rightarrow \int_{\frac{1}{8}}^{y'}\frac{d^2y}{dx^2}\,dx=\int_{\frac{1}{2}}^{y}\left(-y+\frac{3}{2}-\frac{1}{2}y^{-2}\right)\cancel{\frac{dy}{dx}}\,\cancel{dx}$

$\Rightarrow \left[\frac{dy}{dx}\right]_{\frac{1}{8}}^{y'}=\left[-\frac{1}{2}y^2+\frac{3}{2}y+\frac{1}{2}y^{-1}\right]_{\frac{1}{2}}^{y}$

$\Rightarrow \frac{dy}{dx}-\frac{1}{8}=\left(-\frac{1}{2}y^2+\frac{3}{2}y+\frac{1}{2y}\right)-\left(-\frac{1}{8}+\frac{3}{4}+1\right)$

$\Rightarrow \frac{dy}{dx}=-\frac{1}{2}y^2+\frac{3}{2}y-\frac{3}{2}+\frac{1}{2y}$

TIDY UP FIRST

$\Rightarrow \frac{dy}{dx}=-\frac{1}{2y}\left[y^3-3y^2+3y-1\right]$

$\Rightarrow \frac{dy}{dx}=-\frac{(y-1)^3}{2y}$

$\Rightarrow \frac{dy}{dx}=\frac{(1-y)^3}{2y}$

- SEPARATE VARIABLES

$\Rightarrow \frac{2y}{(1-y)^3}\,dy=1\,dx$

- INTEGRATE SUBJECT TO THE CONDITIONS $x=1, y=\frac{1}{2}$

$\Rightarrow \int_{\frac{1}{2}}^{y}\frac{2y}{(1-y)^3}\,dy=\int_{1}^{x}1\,dx$

- SUBSTITUTION (OR PARTIAL FRACTIONS)

$u=1-y$
$y=1-u$
$dy=-du$

$y=\frac{1}{2} \mapsto u=\frac{1}{2}$
$y=y \mapsto u=1-y$

$\Rightarrow \int_{\frac{1}{2}}^{1-y}\frac{2-2u}{u^3}(-du)=\int_{1}^{x}1\,dx$

$\Rightarrow 2\int_{\frac{1}{2}}^{1-y}-u^{-3}+u^{-2}\,du=\int_{1}^{x}1\,dx$

$\Rightarrow 2\left[\frac{1}{2}u^{-2}-u^{-1}\right]_{\frac{1}{2}}^{1-y}=\left[x\right]_{1}^{x}$

$\Rightarrow \left[\frac{1}{u^2}-\frac{2}{u}\right]_{\frac{1}{2}}^{1-y}=x-1$

$\Rightarrow \left[\frac{1-2u}{u^2}\right]_{\frac{1}{2}}^{1-y}=x-1$

$\Rightarrow \frac{1-2(1-y)}{(1-y)^2}-0=x-1$

$\Rightarrow \frac{2y-1}{(y-1)^2}=x-1$

$\Rightarrow 2y-1=(x-1)(y^2-2y+1)$

$\Rightarrow 2y-1=(x-1)y^2-2(x-1)y+(x-1)$

$\Rightarrow \cancel{2y}\cancel{-1}=(x-1)y^2-2xy+\cancel{2y}+x\cancel{-1}$

$\Rightarrow (x-1)y^2-2xy+x=0$ $\quad \boxed{\left[y(\sqrt{x}-1)-\sqrt{x}\right]\left[y[\sqrt{x}+1]-\sqrt{x}\right]=0}$

FACTORIZE OR COMPLETE THE SQUARE

$\Rightarrow y^2-\frac{2x}{x-1}y+\frac{x}{x-1}=0$

$\Rightarrow \left[y-\frac{x}{x-1}\right]^2-\frac{x^2}{(x-1)^2}+\frac{x}{x-1}=0$

$\Rightarrow \left[y-\frac{x}{x-1}\right]^2+\frac{-x^2+x(x-1)}{(x-1)^2}=0$

$\Rightarrow \left[y-\frac{x}{x-1}\right]^2+\frac{-x}{(x-1)^2}=0$

$\Rightarrow \left[y-\frac{x}{x-1}\right]^2=\frac{x}{x-1}$

$\Rightarrow y-\frac{x}{x-1}=\frac{\pm\sqrt{x}}{x-1}$

$\Rightarrow y=\frac{x\pm\sqrt{x}}{x-1}$

$\Rightarrow y=\frac{\sqrt{x}(\sqrt{x}\pm1)}{(\sqrt{x})^2-1}$

$\Rightarrow y=\frac{\sqrt{x}[\sqrt{x}\pm1]}{(\sqrt{x}-1)(\sqrt{x}+1)}$

$\Rightarrow y=\begin{cases}\cancel{\frac{\sqrt{x}}{\sqrt{x}-1}} & \text{NOT DEFINED AT } x=1\\ \frac{\sqrt{x}}{\sqrt{x}+1}\end{cases}$

$\Rightarrow y=\frac{\sqrt{x}}{\sqrt{x}+1}$

ALTERNATIVE REARRANGEMENT.

$\Rightarrow \frac{2y-1}{(y-1)^2}=x-1$

$\Rightarrow x=\frac{2y-1}{(y-1)^2}+1$

$\Rightarrow x=\frac{(2y-1)+(y-1)^2}{(y-1)^2}$

$\Rightarrow x=\frac{2y-1+y^2-2y+1}{(y-1)^2}$

$\Rightarrow x=\frac{y^2}{(y-1)^2}$

$\Rightarrow \frac{y}{y-1}=\begin{cases}\sqrt{x}\\ -\sqrt{x}\end{cases}$

ORIGINAL IS SATISFIED BY $x=1, y=\frac{1}{2}$

HENCE

$\frac{y}{y-1}=-\sqrt{x}$

$y=-y\sqrt{x}+\sqrt{x}$

$y+y\sqrt{x}=\sqrt{x}$

$y(1+\sqrt{x})=\sqrt{x}$

$y=\frac{\sqrt{x}}{1+\sqrt{x}}$

AS BEFORE

Question 122 (*****)

Find a general solution of the following differential equation.

$$y = x\frac{dy}{dx} + e^{\frac{dy}{dx}} .$$

$\boxed{}$, $\boxed{(y + Ax + B)(y - x\ln x + Cx) = 0}$

$$y = x\frac{dy}{dx} + e^{\frac{dy}{dx}}$$

START BY DIFFERENTIATING THE O.D.E. WITH RESPECT TO x

$$\Rightarrow \frac{dy}{dx} = \left[1 \times \frac{dy}{dx} + x\frac{d^2y}{dx^2}\right] + e^{\frac{dy}{dx}} \times \frac{d^2y}{dx^2}$$

$$\Rightarrow \cancel{\frac{dy}{dx}} = \cancel{\frac{dy}{dx}} + x\frac{d^2y}{dx^2} + e^{\frac{dy}{dx}} \times \frac{d^2y}{dx^2}$$

$$\Rightarrow 0 = \frac{d^2y}{dx^2}\left[x + e^{\frac{dy}{dx}}\right]$$

NOW REARRANGING THE ORIGINAL O.D.E

$$\Rightarrow 0 = \frac{d^2y}{dx^2}\left[x + \left(y - x\frac{dy}{dx}\right)\right]$$

THUS WE HAVE TWO SEPARATE O.D.E TO SOLVE

- $\frac{d^2y}{dx^2} = 0 \Rightarrow y = Ax + B$
- $x + y - x\frac{dy}{dx} = 0$

$$\Rightarrow x\frac{dy}{dx} - y = x$$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = 1$$

INTEGRATING FACTOR $= e^{\int -\frac{1}{x}\,dx} = e^{-\ln x} = \frac{1}{x}$

$$\Rightarrow \frac{d}{dx}\left(\frac{y}{x}\right) = \frac{1}{x}$$

$$\Rightarrow \frac{y}{x} = \int \frac{1}{x}\,dx$$

$$\Rightarrow \frac{y}{x} = \ln|x| + C$$

$$\Rightarrow y = x\ln x + Cx$$

COMBINING THE SOLUTIONS WE HAVE

$$y = \begin{cases} Ax + B \\ x\ln x + C \end{cases}$$

THIS CAN BE WRITTEN AS

$$\Rightarrow (y - Ax - B)(y - x\ln x - C) = 0$$

$$\Rightarrow (y + Px + Q)(y - x\ln x + R) = 0$$