PARTIAL DIFFERENTIAL EQUATIONS

TRANSFORMATIONS

Question 1

The smooth function $f = f(x, y)$ satisfies

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}.$$

Find the general solution of the above partial differential equation by using the transformation equations

$$x = u + v \quad \text{and} \quad y = u - v.$$

$$\boxed{f(x, y) = F(x + y)}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}, \quad f = f(x,y)$$

$$\left.\begin{array}{l} x = u + v \\ y = u - v \end{array}\right\} \Rightarrow \left.\begin{array}{l} 2u = x + y \\ 2v = x - y \end{array}\right\} \Rightarrow \boxed{\begin{array}{l} u = \frac{1}{2}x + \frac{1}{2}y \\ v = \frac{1}{2}x - \frac{1}{2}y \end{array}}$$

- $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = \frac{1}{2}\frac{\partial f}{\partial u} + \frac{1}{2}\frac{\partial f}{\partial v}$
- $\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial u} - \frac{1}{2}\frac{\partial f}{\partial v}$

SUB INTO THE P.D.E

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} \Rightarrow \cancel{\frac{1}{2}\frac{\partial f}{\partial u}} + \frac{1}{2}\frac{\partial f}{\partial v} = \cancel{\frac{1}{2}\frac{\partial f}{\partial u}} - \frac{1}{2}\frac{\partial f}{\partial v}$$

$$\frac{\partial f}{\partial v} = 0$$

$$f(u,v) = F(u)$$

$$f(x,y) = F(x+y)$$

Question 2

The smooth function $z = z(x, y)$ satisfies

$$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = xy .$$

Find the general solution of the above partial differential equation by using the transformation equations

$$u = x^2 + y^2 \quad \text{and} \quad v = x^2 - y^2 .$$

$$\boxed{z(x, y) = \tfrac{1}{4}\left(x^2 + y^2\right) + f\left(x^2 - y^2\right)}$$

$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = xy$

$u = x^2 + y^2$
$v = x^2 - y^2$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}(2x) + \frac{\partial z}{\partial v}(2x)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u}(2y) + \frac{\partial z}{\partial v}(-2y)$$

SUB INTO THE P.D.E

$$y\left[2x\frac{\partial z}{\partial u} + 2x\frac{\partial z}{\partial v}\right] + x\left[2y\frac{\partial z}{\partial u} - 2y\frac{\partial z}{\partial v}\right] = xy$$

$$2xy\left[\frac{\partial z}{\partial u} + \cancel{\frac{\partial z}{\partial v}} + \frac{\partial z}{\partial u} - \cancel{\frac{\partial z}{\partial v}}\right] = xy$$

$$4\frac{\partial z}{\partial u} = 1$$

$$\frac{\partial z}{\partial u} = \frac{1}{4}$$

$$z = \frac{1}{4}u + f(v)$$

$$z = \frac{1}{4}(x^2 + y^2) + f(x^2 - y^2)$$

Question 3

The smooth function $z = z(x, y)$ satisfies

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 6(x+y)^2 z^2 .$$

Find the general solution of the above partial differential equation by using the transformation equations

$$\xi = x + y \quad \text{and} \quad \eta = x - y .$$

$\boxed{}$, $\boxed{z(x,y) = -\dfrac{1}{(x+y)^3 - f(x-y)}}$

USING THE TRANSFORMATION EQUATIONS GIVEN

$\xi = x+y$ & $\eta = x-y$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial z}{\partial \xi}\times 1 + \frac{\partial z}{\partial \eta}\times 1 = \frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial z}{\partial \xi}\times 1 + \frac{\partial z}{\partial \eta}\times(-1) = \frac{\partial z}{\partial \xi} - \frac{\partial z}{\partial \eta}$$

SUBSTITUTE INTO THE O.D.E

$$\Rightarrow \frac{\partial z}{\partial x} + \frac{\partial z}{\partial x} = 6(x+y)^2 z^2$$

$$\Rightarrow \left(\frac{\partial z}{\partial \xi} + \cancel{\frac{\partial z}{\partial \eta}}\right) - \left(\frac{\partial z}{\partial \xi} - \cancel{\frac{\partial z}{\partial \eta}}\right) = 6\xi^2 z^2$$

$$\Rightarrow 2\frac{\partial z}{\partial \xi} = 6\xi^2 z^2$$

$$\Rightarrow \frac{\partial z}{\partial \xi} = 3\xi^2 z^2$$

SOLVE BY SEPARATION OF VARIABLES & DIRECT INTEGRATION

$$\Rightarrow \frac{1}{z^2}\,\partial z = 3\xi^2\,\partial \xi$$

$$\Rightarrow -\frac{1}{z} = \xi^3 + f(\eta)$$

$$\Rightarrow -z = \frac{1}{\xi^3 + f(\eta)}$$

$$\Rightarrow z = \frac{1}{-\xi^3 + f(\eta)}$$

$$\Rightarrow z(x,y) = \frac{1}{f(x-y) - (x+y)^3}$$

Question 4

The smooth function $z = z(x, y)$ satisfies

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1 .$$

Find the general solution of the above partial differential equation by using the transformation equations

$$x = u^2 + v^2 \quad \text{and} \quad y = u^2 - v^2 .$$

$\boxed{}$, $\boxed{z(x, y) = \frac{1}{2}(x + y) + f\left(\sqrt{x - y}\right)}$

REARRANGE THE TRANSFORMATION EQUATIONS, & OBTAIN PARTIAL DERIVATIVES

$$\left.\begin{array}{l} x = u^2 + v^2 \\ y = u^2 - v^2 \end{array}\right\} \text{ADDING \& SUBTRACTING} \quad \left.\begin{array}{l} x + y = 2u^2 \\ x - y = 2v^2 \end{array}\right\} \longrightarrow$$

$$\left.\begin{array}{l} u^2 = \frac{1}{2}x + \frac{1}{2}y \\ v^2 = \frac{1}{2}x - \frac{1}{2}y \end{array}\right\} \longrightarrow$$

- $2u \frac{\partial u}{\partial x} = \frac{1}{2}$ • $2v \frac{\partial v}{\partial x} = \frac{1}{2}$
- $2u \frac{\partial u}{\partial y} = \frac{1}{2}$ • $2v \frac{\partial v}{\partial y} = -\frac{1}{2}$

- $\frac{\partial u}{\partial x} = \frac{1}{4u}$ • $\frac{\partial v}{\partial x} = \frac{1}{4v}$
- $\frac{\partial u}{\partial y} = \frac{1}{4u}$ • $\frac{\partial v}{\partial y} = -\frac{1}{4v}$

BY THE CHAIN RULE WE HAVE

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}\frac{1}{4u} + \frac{\partial z}{\partial v}\frac{1}{4v}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u}\frac{1}{4u} + \frac{\partial z}{\partial v}\left(-\frac{1}{4v}\right)$$

SUBSTITUTE INTO THE P.D.E.

$$\Rightarrow \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1$$

$$\Rightarrow \left[\frac{1}{4u}\frac{\partial z}{\partial u} + \cancel{\frac{1}{4v}\frac{\partial z}{\partial v}}\right] + \left[\frac{1}{4u}\frac{\partial z}{\partial u} - \cancel{\frac{1}{4v}\frac{\partial z}{\partial v}}\right] = 1$$

$$\Rightarrow \frac{1}{2u}\frac{\partial z}{\partial u} = 1$$

$$\Rightarrow \frac{\partial z}{\partial u} = 2u$$

SOLVING BY DIRECT INTEGRATION

$$z(u, v) = u^2 + f(v)$$

$$z(x, y) = \left(\tfrac{1}{2}x + \tfrac{1}{2}y\right) + f\left(\sqrt{\tfrac{1}{2}x - \tfrac{1}{2}y}\right)$$

$$z(x, y) = \tfrac{1}{2}(x + y) + f\left(\sqrt{x - y}\right)$$

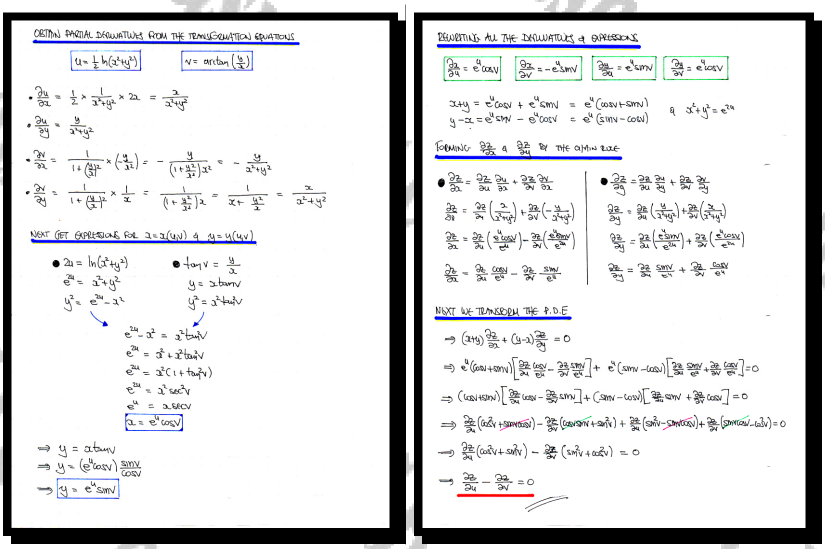
Question 5

$$(x+y)\frac{\partial z}{\partial x}+(y-x)\frac{\partial z}{\partial y}=0.$$

Transform the above partial differential equation using the equations

$$u=\frac{1}{2}\ln\left(x^2+y^2\right) \quad \text{and} \quad v=\arctan\left(\frac{y}{x}\right).$$

$$,\quad \boxed{\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=0}$$



OBTAIN PARTIAL DERIVATIVES FROM THE TRANSFORMATION EQUATIONS

$\boxed{u=\frac{1}{2}\ln(x^2+y^2)}$ $\boxed{v=\arctan\left(\frac{y}{x}\right)}$

- $\frac{\partial u}{\partial x}=\frac{1}{2}\times\frac{1}{x^2+y^2}\times 2x=\frac{x}{x^2+y^2}$
- $\frac{\partial u}{\partial y}=\frac{y}{x^2+y^2}$
- $\frac{\partial v}{\partial x}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\times\left(-\frac{y}{x^2}\right)=-\frac{y}{\left(1+\frac{y^2}{x^2}\right)x^2}=-\frac{y}{x^2+y^2}$
- $\frac{\partial v}{\partial y}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\times\frac{1}{x}=\frac{1}{\left(1+\frac{y^2}{x^2}\right)x}=\frac{1}{x+\frac{y^2}{x}}=\frac{x}{x^2+y^2}$

NEXT GET EXPRESSIONS FOR $x=x(u,v)$ & $y=y(u,v)$

- $2u=\ln(x^2+y^2)$, $e^{2u}=x^2+y^2$, $y^2=e^{2u}-x^2$
- $\tan v=\frac{y}{x}$, $y=x\tan v$, $y^2=x^2\tan^2 v$

$e^{2u}-x^2=x^2\tan^2 v$

$e^{2u}=x^2+x^2\tan^2 v$

$e^{2u}=x^2(1+\tan^2 v)$

$e^{2u}=x^2\sec^2 v$

$e^{u}=x\sec v$

$\boxed{x=e^{u}\cos v}$

$\Rightarrow y=x\tan v$

$\Rightarrow y=(e^{u}\cos v)\frac{\sin v}{\cos v}$

$\Rightarrow \boxed{y=e^{u}\sin v}$

REWRITING ALL THE DERIVATIVES & EXPRESSIONS

$\boxed{\frac{\partial x}{\partial u}=e^{u}\cos v}$ $\boxed{\frac{\partial x}{\partial v}=-e^{u}\sin v}$ $\boxed{\frac{\partial y}{\partial u}=e^{u}\sin v}$ $\boxed{\frac{\partial y}{\partial v}=e^{u}\cos v}$

$x+y=e^{u}\cos v+e^{u}\sin v=e^{u}(\cos v+\sin v)$

$y-x=e^{u}\sin v-e^{u}\cos v=e^{u}(\sin v-\cos v)$ & $x^2+y^2=e^{2u}$

FORMING $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ BY THE CHAIN RULE

- $\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$

$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\left(\frac{x}{x^2+y^2}\right)+\frac{\partial z}{\partial v}\left(-\frac{y}{x^2+y^2}\right)$

$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\left(\frac{e^{u}\cos v}{e^{2u}}\right)-\frac{\partial z}{\partial v}\left(\frac{e^{u}\sin v}{e^{2u}}\right)$

$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\cos v}{e^{u}}-\frac{\partial z}{\partial v}\frac{\sin v}{e^{u}}$

- $\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$

$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\left(\frac{y}{x^2+y^2}\right)+\frac{\partial z}{\partial v}\left(\frac{x}{x^2+y^2}\right)$

$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\left(\frac{e^{u}\sin v}{e^{2u}}\right)+\frac{\partial z}{\partial v}\left(\frac{e^{u}\cos v}{e^{2u}}\right)$

$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\sin v}{e^{u}}+\frac{\partial z}{\partial v}\frac{\cos v}{e^{u}}$

NEXT WE TRANSFORM THE P.D.E

$\Rightarrow (x+y)\frac{\partial z}{\partial x}+(y-x)\frac{\partial z}{\partial y}=0$

$\Rightarrow e^{u}(\cos v+\sin v)\left[\frac{\partial z}{\partial u}\frac{\cos v}{e^{u}}-\frac{\partial z}{\partial v}\frac{\sin v}{e^{u}}\right]+e^{u}(\sin v-\cos v)\left[\frac{\partial z}{\partial u}\frac{\sin v}{e^{u}}+\frac{\partial z}{\partial v}\frac{\cos v}{e^{u}}\right]=0$

$\Rightarrow (\cos v+\sin v)\left[\frac{\partial z}{\partial u}\cos v-\frac{\partial z}{\partial v}\sin v\right]+(\sin v-\cos v)\left[\frac{\partial z}{\partial u}\sin v+\frac{\partial z}{\partial v}\cos v\right]=0$

$\Rightarrow \frac{\partial z}{\partial u}(\cos^2 v+\cancel{\sin v\cos v})-\frac{\partial z}{\partial v}(\cancel{\cos v\sin v}+\sin^2 v)+\frac{\partial z}{\partial u}(\sin^2 v-\cancel{\sin v\cos v})+\frac{\partial z}{\partial v}(\cancel{\sin v\cos v}-\cos^2 v)=0$

$\Rightarrow \frac{\partial z}{\partial u}(\cos^2 v+\sin^2 v)-\frac{\partial z}{\partial v}(\sin^2 v+\cos^2 v)=0$

$\Rightarrow \frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=0$

Question 6

The function z depends on x and y so that

$$z = f(u,v), \quad u = x - 2\sqrt{y} \quad \text{and} \quad v = x + 2\sqrt{y}.$$

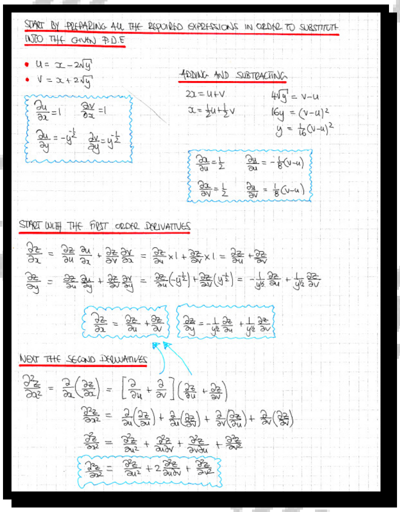
Show that the partial differential equation

$$2\frac{\partial^2 z}{\partial x^2} - 2y\frac{\partial^2 z}{\partial y^2} - \frac{\partial z}{\partial y} = 0,$$

can be simplified to

$$\frac{\partial^2 z}{\partial u \partial v} = 0.$$

$\boxed{}$, $\boxed{\text{proof}}$



START BY PREPARING ALL THE REQUIRED EXPRESSIONS IN ORDER TO SUBSTITUTE INTO THE GIVEN P.D.E

- $u = x - 2\sqrt{y}$
- $v = x + 2\sqrt{y}$

$$\boxed{\frac{\partial u}{\partial x} = 1 \quad \frac{\partial v}{\partial x} = 1 \quad \frac{\partial u}{\partial y} = -y^{-\frac{1}{2}} \quad \frac{\partial v}{\partial y} = y^{-\frac{1}{2}}}$$

ADDING AND SUBTRACTING

$2x = u + v \qquad 4\sqrt{y} = v - u$

$x = \frac{1}{2}u + \frac{1}{2}v \qquad 16y = (v-u)^2$

$y = \frac{1}{16}(v-u)^2$

$$\boxed{\frac{\partial x}{\partial u} = \frac{1}{2} \quad \frac{\partial y}{\partial u} = -\frac{1}{8}(v-u) \quad \frac{\partial x}{\partial v} = \frac{1}{2} \quad \frac{\partial y}{\partial v} = \frac{1}{8}(v-u)}$$

START WITH THE FIRST ORDER DERIVATIVES

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}\times 1 + \frac{\partial z}{\partial v}\times 1 = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u}\left(-y^{-\frac{1}{2}}\right) + \frac{\partial z}{\partial v}\left(y^{-\frac{1}{2}}\right) = -\frac{1}{y^{\frac{1}{2}}}\frac{\partial z}{\partial u} + \frac{1}{y^{\frac{1}{2}}}\frac{\partial z}{\partial v}$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}} \quad \boxed{\frac{\partial z}{\partial y} = -\frac{1}{y^{\frac{1}{2}}}\frac{\partial z}{\partial u} + \frac{1}{y^{\frac{1}{2}}}\frac{\partial z}{\partial v}}$$

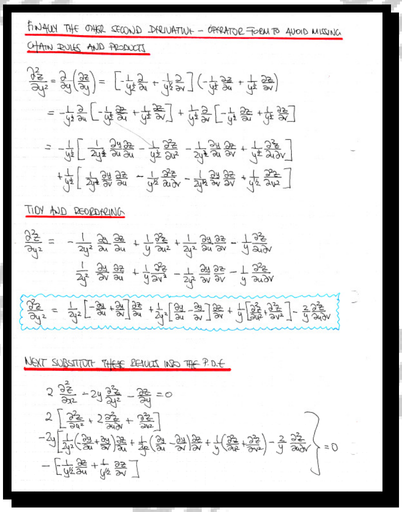
NEXT THE SECOND DERIVATIVES

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left[\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right]\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) + \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v \partial u} + \frac{\partial^2 z}{\partial v^2}$$

$$\boxed{\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}}$$



FINALLY THE OTHER SECOND DERIVATIVE – OPERATOR FORM TO AVOID MISSING CHAIN RULES AND PRODUCTS

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left[-\frac{1}{y^{\frac{1}{2}}}\frac{\partial}{\partial u} + \frac{1}{y^{\frac{1}{2}}}\frac{\partial}{\partial v}\right]\left(-\frac{1}{y^{\frac{1}{2}}}\frac{\partial z}{\partial u} + \frac{1}{y^{\frac{1}{2}}}\frac{\partial z}{\partial v}\right)$$

$$= -\frac{1}{y^{\frac{1}{2}}}\frac{\partial}{\partial u}\left[-\frac{1}{y^{\frac{1}{2}}}\frac{\partial z}{\partial u} + \frac{1}{y^{\frac{1}{2}}}\frac{\partial z}{\partial v}\right] + \frac{1}{y^{\frac{1}{2}}}\frac{\partial}{\partial v}\left[-\frac{1}{y^{\frac{1}{2}}}\frac{\partial z}{\partial u} + \frac{1}{y^{\frac{1}{2}}}\frac{\partial z}{\partial v}\right]$$

$$= -\frac{1}{y^{\frac{1}{2}}}\left[\frac{1}{2y^{\frac{3}{2}}}\frac{\partial y}{\partial u}\frac{\partial z}{\partial u} - \frac{1}{y^{\frac{1}{2}}}\frac{\partial^2 z}{\partial u^2} - \frac{1}{2y^{\frac{3}{2}}}\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} + \frac{1}{y^{\frac{1}{2}}}\frac{\partial^2 z}{\partial u \partial v}\right]$$

$$+\frac{1}{y^{\frac{1}{2}}}\left[\frac{1}{2y^{\frac{3}{2}}}\frac{\partial y}{\partial v}\frac{\partial z}{\partial u} - \frac{1}{y^{\frac{1}{2}}}\frac{\partial^2 z}{\partial u \partial v} - \frac{1}{2y^{\frac{3}{2}}}\frac{\partial y}{\partial v}\frac{\partial z}{\partial v} + \frac{1}{y^{\frac{1}{2}}}\frac{\partial^2 z}{\partial v^2}\right]$$

TIDY AND REORDERING

$$\frac{\partial^2 z}{\partial y^2} = -\frac{1}{2y^2}\frac{\partial y}{\partial u}\frac{\partial z}{\partial u} + \frac{1}{y}\frac{\partial^2 z}{\partial u^2} + \frac{1}{2y^2}\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{1}{y}\frac{\partial^2 z}{\partial u \partial v}$$

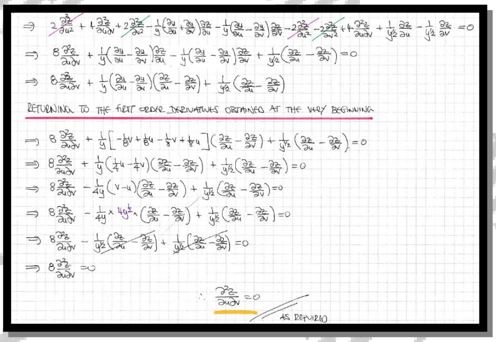
$$\frac{1}{2y^2}\frac{\partial y}{\partial v}\frac{\partial z}{\partial u} + \frac{1}{y}\frac{\partial^2 z}{\partial v^2} - \frac{1}{2y^2}\frac{\partial y}{\partial v}\frac{\partial z}{\partial v} - \frac{1}{y}\frac{\partial^2 z}{\partial u \partial v}$$

$$\boxed{\frac{\partial^2 z}{\partial y^2} = \frac{1}{2y^2}\left[-\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right]\frac{\partial z}{\partial u} + \frac{1}{2y^2}\left[\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v}\right]\frac{\partial z}{\partial v} + \frac{1}{y}\left[\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right] - \frac{2}{y}\frac{\partial^2 z}{\partial u \partial v}}$$

NEXT SUBSTITUTE THESE RESULTS INTO THE P.D.E

$$2\frac{\partial^2 z}{\partial x^2} - 2y\frac{\partial^2 z}{\partial y^2} - \frac{\partial z}{\partial y} = 0$$

$$\left.\begin{aligned}&2\left[\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right]\\ &-2y\left[\frac{1}{2y^2}\left(-\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right)\frac{\partial z}{\partial u} + \frac{1}{2y^2}\left(\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v}\right)\frac{\partial z}{\partial v} + \frac{1}{y}\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right) - \frac{2}{y}\frac{\partial^2 z}{\partial u \partial v}\right]\\ &-\left[-\frac{1}{y^{1/2}}\frac{\partial z}{\partial u} + \frac{1}{y^{1/2}}\frac{\partial z}{\partial v}\right]\end{aligned}\right\} = 0$$



$$\Rightarrow \cancel{2\frac{\partial^2 z}{\partial u^2}} + 4\frac{\partial^2 z}{\partial u \partial v} + \cancel{2\frac{\partial^2 z}{\partial v^2}} - \frac{1}{y}\left(-\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right)\frac{\partial z}{\partial u} - \frac{1}{y}\left(\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v}\right)\frac{\partial z}{\partial v} - \cancel{2\frac{\partial^2 z}{\partial u^2}} - \cancel{2\frac{\partial^2 z}{\partial v^2}} + 4\frac{\partial^2 z}{\partial u \partial v} + \frac{1}{y^{1/2}}\frac{\partial z}{\partial u} - \frac{1}{y^{1/2}}\frac{\partial z}{\partial v} = 0$$

$$\Rightarrow 8\frac{\partial^2 z}{\partial u \partial v} + \frac{1}{y}\left(\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v}\right)\frac{\partial z}{\partial u} - \frac{1}{y}\left(\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v}\right)\frac{\partial z}{\partial v} + \frac{1}{y^{1/2}}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) = 0$$

$$\Rightarrow 8\frac{\partial^2 z}{\partial u \partial v} + \frac{1}{y}\left(\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v}\right)\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + \frac{1}{y^{1/2}}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

RETURNING TO THE FIRST ORDER DERIVATIVES OBTAINED AT THE VERY BEGINNING

$$\Rightarrow 8\frac{\partial^2 z}{\partial u \partial v} + \frac{1}{y}\left[-\frac{1}{8}v + \frac{1}{8}u - \frac{1}{8}v + \frac{1}{8}u\right]\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + \frac{1}{y^{1/2}}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) = 0$$

$$\Rightarrow 8\frac{\partial^2 z}{\partial u \partial v} + \frac{1}{y}\left(\frac{1}{4}u - \frac{1}{4}v\right)\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + \frac{1}{y^{1/2}}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) = 0$$

$$\Rightarrow 8\frac{\partial^2 z}{\partial u \partial v} - \frac{1}{4y}(v-u)\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + \frac{1}{y^{1/2}}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) = 0$$

$$\Rightarrow 8\frac{\partial^2 z}{\partial u \partial v} - \frac{1}{4y} \times 4y^{\frac{1}{2}} \times \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + \frac{1}{y^{1/2}}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) = 0$$

$$\Rightarrow 8\frac{\partial^2 z}{\partial u \partial v} - \cancel{\frac{1}{y^{1/2}}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)} + \cancel{\frac{1}{y^{1/2}}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)} = 0$$

$$\Rightarrow 8\frac{\partial^2 z}{\partial u \partial v} = 0$$

$$\therefore \frac{\partial^2 z}{\partial u \partial v} = 0$$

AS REQUIRED

Question 7

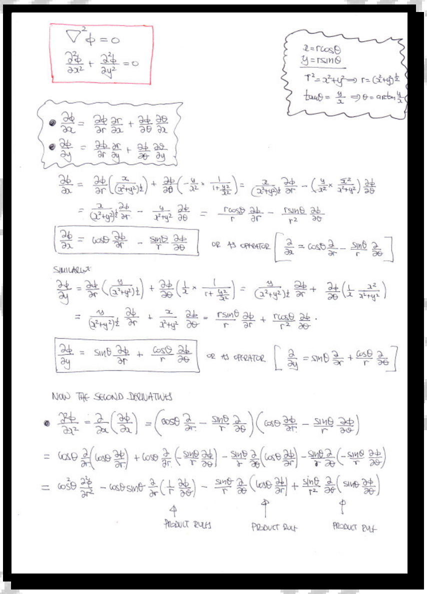
$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 .$$

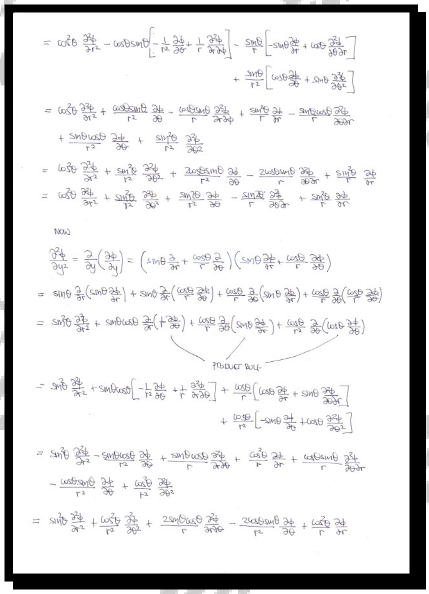
The above partial differential equation is Laplace's equation in a two dimensional Cartesian system of coordinates.

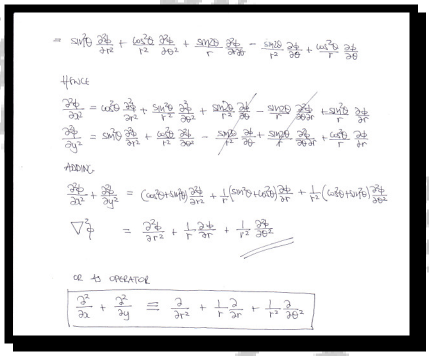
Show clearly that Laplace's equation in the standard two dimensional Polar system of coordinates is given by

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2} = 0 .$$

$\boxed{\text{proof}}$







FIRST ORDER P.D.E.s

$$\frac{\partial z}{\partial x}=F(x,y,z) \quad \text{or} \quad \frac{\partial z}{\partial y}=G(x,y,z) \quad \text{for} \quad z=z(x,y)$$

Question 1

It is given that $z = F(x, y)$ satisfies the partial differential equation

$$\frac{\partial z}{\partial y} + 2yz = xy^3 .$$

Determine a general solution of the above partial differential equation.

$\boxed{}$, $\boxed{z = \frac{1}{2}x\left(y^2 - 1\right) + e^{-y^2} f(x)}$

AS THIS IS A LINEAR FIRST ORDER P.D.E WITH ONLY ONE PARTIAL DERIVATIVE PRESENT WE CAN JUST SOLVE IT AS AN O.D.E WHERE THE OTHER INDEPENDENT VARIABLE IS TREATED AS A CONSTANT (x HERE)

$z = f(x,y)$

$\frac{\partial z}{\partial y} + 2yz = xy^3$

$$\text{I.F.} = e^{\int 2y\,dy} = e^{y^2}$$

$$\Longrightarrow \frac{\partial}{\partial y}\left(ze^{y^2}\right) = xy^3 e^{y^2}$$

$$\Longrightarrow ze^{y^2} = \int xy^3 e^{y^2}\,dy$$

$$\Longrightarrow ze^{y^2} = x\int y^2\left(ye^{y^2}\right)dy$$

BY PARTS (W.R.T y)

y^2	$2y$
$\frac{1}{2}e^{y^2}$	ye^{y^2}

$$\Longrightarrow ze^{y^2} = x\left[\frac{1}{2}y^2e^{y^2} - \int ye^{y^2}\,dy\right]$$

$$\Longrightarrow ze^{y^2} = x\left[\frac{1}{2}y^2e^{y^2} - \frac{1}{2}e^{y^2} + A(x)\right]$$

$$\Longrightarrow ze^{y^2} = \frac{1}{2}xy^2e^{y^2} - \frac{1}{2}xe^{y^2} + B(x)$$

$$\Longrightarrow z = \frac{1}{2}xy^2 - \frac{1}{2}x + B(x)e^{-y^2}$$

$$\Longrightarrow z(x,y) = \frac{1}{2}x(y^2-1) + B(x)e^{-y^2}$$

FIRST ORDER P.D.E.s

(by linear transformations)

$$A\frac{\partial z}{\partial x}+B\frac{\partial z}{\partial y}+Cz=G\left(x,y\right), \qquad z=z\left(x,y\right)$$

Question 1

It is given that $\psi = \psi(x, y)$ satisfies the partial differential equation

$$3\frac{\partial \psi}{\partial x} - 4\frac{\partial \psi}{\partial y} = x^2 .$$

Use the transformation equations

$$\xi = Ax + By \quad \text{and} \quad \eta = Cx + Dy , \quad AD - BC \neq 0$$

with suitable values of A, B, C and D, in order to determine a general solution of the above partial differential equation.

$\boxed{}$, $\boxed{\psi(x, y) = \frac{1}{9}x^3 + f(4x+3y)}$

$$3\frac{\partial \psi}{\partial x} - 4\frac{\partial \psi}{\partial y} = x^2$$

USING THE TRANSFORMATIONS GIVEN

$\xi = Ax + By$
$\eta = Cx + Dy$ $\quad AD - BC \neq 0$

$\frac{\partial \xi}{\partial x} = A \quad \frac{\partial \xi}{\partial y} = B$

$\frac{\partial \eta}{\partial x} = C \quad \frac{\partial \eta}{\partial y} = D$

BY THE CHAIN RULE

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial x} = A\frac{\partial \psi}{\partial \xi} + C\frac{\partial \psi}{\partial \eta}$$

$$\frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial y} = B\frac{\partial \psi}{\partial \xi} + D\frac{\partial \psi}{\partial \eta}$$

SUBSTITUTE INTO THE P.D.E

$$\Rightarrow 3\left[A\frac{\partial \psi}{\partial \xi} + C\frac{\partial \psi}{\partial \eta}\right] - 4\left[B\frac{\partial \psi}{\partial \xi} + D\frac{\partial \psi}{\partial \eta}\right] = x^2$$

$$\Rightarrow (3A - 4B)\frac{\partial \psi}{\partial \xi} + (3C - 4D)\frac{\partial \psi}{\partial \eta} = x^2$$

"KNOCK OFF" $\frac{\partial \psi}{\partial \eta}$, FURTHER SIMPLIFYING THE P.D.E.

$$\left.\begin{matrix} A=1 & B=0 \\ C=4 & D=3 \end{matrix}\right\} \Rightarrow \boxed{\begin{matrix} \xi = x \\ \eta = 4x+3y \end{matrix}}$$

THE PDE NOW TRANSFORMS

$$\Rightarrow 3\frac{\partial \psi}{\partial \xi} = \xi^2$$

$$\Rightarrow \frac{\partial \psi}{\partial \xi} = \frac{1}{3}\xi^3$$

$$\Rightarrow \psi(\xi, \eta) = \frac{1}{9}\xi^3 + f(\eta)$$

REVERSING THE TRANSFORMATIONS

$$\Rightarrow \psi(x, y) = \frac{1}{9}x^3 + f(4x+3y)$$

Question 2

It is given that $z = z(x, y)$ satisfies the partial differential equation

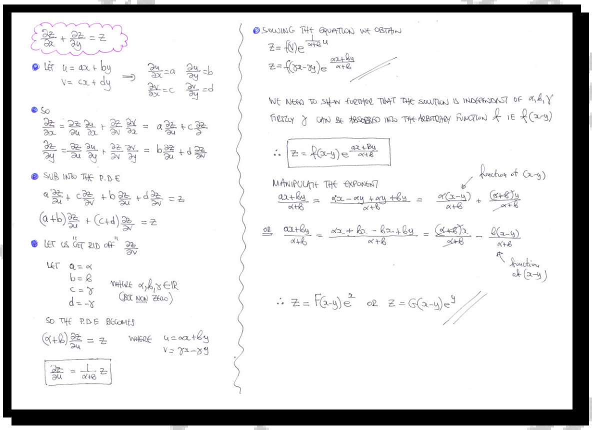
$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z \, .$$

Use the transformation equations

$$u = ax + by \quad \text{and} \quad v = cx + dy \, , \quad ad - bc \neq 0$$

in order to determine a general solution of the above partial differential equation, showing further that this general solution is independent of the choice of values of the constants of a, b, c and d.

$$\boxed{z = \mathrm{e}^{x} F(x-y) \quad \text{or} \quad z = \mathrm{e}^{y} G(x-y)}$$



$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z$

- LET $u = ax + by$, $v = cx + dy$ $\Longrightarrow$ $\frac{\partial u}{\partial x} = a$, $\frac{\partial u}{\partial y} = b$, $\frac{\partial v}{\partial x} = c$, $\frac{\partial v}{\partial y} = d$
- SO

$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = a\frac{\partial z}{\partial u} + c\frac{\partial z}{\partial v}$

$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = b\frac{\partial z}{\partial u} + d\frac{\partial z}{\partial v}$

- SUB INTO THE P.D.E

$a\frac{\partial z}{\partial u} + c\frac{\partial z}{\partial v} + b\frac{\partial z}{\partial u} + d\frac{\partial z}{\partial v} = z$

$(a+b)\frac{\partial z}{\partial u} + (c+d)\frac{\partial z}{\partial v} = z$

- LET US "GET RID OFF" $\frac{\partial z}{\partial v}$

LET $a = \alpha$, $b = \beta$, $c = \gamma$, $d = -\gamma$ WHERE $\alpha, \beta, \gamma \in \mathbb{R}$ (BUT NON ZERO)

SO THE P.D.E BECOMES

$(\alpha+\beta)\frac{\partial z}{\partial u} = z$ WHERE $u = \alpha x + \beta y$, $v = \gamma x - \gamma y$

$\boxed{\frac{\partial z}{\partial u} = \frac{1}{\alpha+\beta} z}$

- SOLVING THE EQUATION WE OBTAIN

$z = f(v)\mathrm{e}^{\frac{1}{\alpha+\beta}u}$

$z = f(\gamma x - \gamma y)\mathrm{e}^{\frac{\alpha x + \beta y}{\alpha+\beta}}$

WE NEED TO SHOW FURTHER THAT THE SOLUTION IS INDEPENDENT OF α, β, γ

FIRSTLY γ CAN BE ABSORBED INTO THE ARBITRARY FUNCTION f IE $f(x-y)$

$\therefore \boxed{z = f(x-y)\mathrm{e}^{\frac{\alpha x + \beta y}{\alpha+\beta}}}$

MANIPULATE THE EXPONENT

$\frac{\alpha x + \beta y}{\alpha+\beta} = \frac{\alpha x - \alpha y + \alpha y + \beta y}{\alpha+\beta} = \underbrace{\frac{\alpha(x-y)}{\alpha+\beta}}_{\text{function of } (x-y)} + \frac{\cancel{(\alpha+\beta)}y}{\cancel{\alpha+\beta}}$

OR $\frac{\alpha x + \beta y}{\alpha+\beta} = \frac{\alpha x + \beta x - \beta x + \beta y}{\alpha+\beta} = \frac{\cancel{(\alpha+\beta)}x}{\cancel{\alpha+\beta}} - \underbrace{\frac{\beta(x-y)}{\alpha+\beta}}_{\text{function of } (x-y)}$

$\therefore z = F(x-y)\mathrm{e}^{x}$ OR $z = G(x-y)\mathrm{e}^{y}$

Question 3

It is given that $z = z(x, y)$ satisfies the partial differential equation

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} + z = x .$$

Use the transformation equations

$$\xi = Ax + By \quad \text{and} \quad \eta = Cx + Dy , \qquad AD - BC \neq 0$$

with suitable values of A, B, C and D, in order to determine a general solution of the above partial differential equation.

$$\boxed{z = x - 1 + e^{-x} f(x - y)}$$

$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} + z = x$, FOR $z = z(x,y)$

PLAN IS TO USE A LINEAR TRANSFORMATION TO "KNOCK" ONE OF THE PARTIAL DERIVATIVES SO WE CAN USE AN INTEGRATING FACTOR

- LET $\xi = Ax + By$, $\eta = Cx + Dy$ $(AD - BC \neq 0)$ $\quad \frac{\partial \xi}{\partial x} = A \quad \frac{\partial \xi}{\partial y} = B \quad \frac{\partial \eta}{\partial x} = C \quad \frac{\partial \eta}{\partial y} = D$

- $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial x} = A\frac{\partial z}{\partial \xi} + C\frac{\partial z}{\partial \eta}$

  $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial y} = B\frac{\partial z}{\partial \xi} + D\frac{\partial z}{\partial \eta}$

- SUB INTO THE P.D.E.

  $A\frac{\partial z}{\partial \xi} + C\frac{\partial z}{\partial \eta} + B\frac{\partial z}{\partial \xi} + D\frac{\partial z}{\partial \eta} + z = x$

  WE MAY PICK THESE CONSTANTS SO WE CAN "KNOCK" ONE OF THE PARTIAL DERIVATIVES AND MAKE THE COEFFICIENT OF THE OTHER PARTIAL DERIVATIVE AS SIMPLE AS POSSIBLE

  E.G "KNOCK η OUT" $C = 1$, $D = -1$ & $A = 1$, $B = 0$

  SO THE TRANSFORMATION EQUATIONS BECOME

  $\left.\begin{array}{l}\xi = x \\ \eta = x - y\end{array}\right\} \Rightarrow \xi - \eta = y$ & $x = \xi$

HENCE THE P.D.E NOW BECOMES

$\frac{\partial z}{\partial \xi} + z = \xi$ WHICH CAN BE SOLVED BY INTEGRATING FACTOR

$\text{I.F} = e^{\int 1 \, d\xi} = e^{\xi}$

THUS $\frac{\partial}{\partial \xi}(ze^{\xi}) = \xi e^{\xi}$

$\Rightarrow ze^{\xi} = \int \xi e^{\xi} \, \partial\xi$ ← BY PARTS

ξ	1
e^{ξ}	e^{ξ}

$\Rightarrow ze^{\xi} = \xi e^{\xi} - \int e^{\xi} \, \partial \xi$

$\Rightarrow ze^{\xi} = \xi e^{\xi} - e^{\xi} + A(\eta)$

$\Rightarrow z = \xi - 1 + A(\eta)e^{-\xi}$

$\Rightarrow z = (x - 1) + A(x - y)e^{-x}$

$\Rightarrow z = (x - 1) + f(x - y)e^{-x}$

CHECK

IF $z = (x-1) + f(x-y)e^{-x}$

$\frac{\partial z}{\partial x} = 1 + f'(x-y)e^{-x} - f(x-y)e^{-x}$

$\frac{\partial z}{\partial y} = -f'(x-y)e^{-x}$

SUB INTO THE P.D.E

$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} + z = \left[\cancel{1} + \cancel{f'e^{-x}} - \cancel{fe^{-x}}\right] + \left[\cancel{-f'e^{-x}}\right] + \left[\cancel{(x-1)} + \cancel{fe^{-x}}\right] = x$

$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} + z = x$ $\quad$ ALTERNATIVE BY LAGRANGE'S METHOD

$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = x - z$

$1 \times \frac{\partial z}{\partial x} + 1 \times \frac{\partial z}{\partial y} = x - z$

$P(x,y,z) \quad Q(x,y,z) \quad R(x,y,z)$

- LAGRANGE'S ASSOCIATED O.D.Es ARE

  $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

  $dx = dy = \frac{dz}{x - z}$

- NOW $dx = dy$

  $y = x + C$

  OR $\boxed{y - x = C}$

  OR $x - y = C$

- $dx = \frac{dz}{x - z} \Rightarrow x - z = \frac{dz}{dx}$

  $\Rightarrow \frac{dz}{dx} + z = x$

  $\text{I.F}: e^{\int 1 \, dx} = e^{x}$

  $\Rightarrow \frac{d}{dx}(ze^{x}) = xe^{x}$

  $\Rightarrow ze^{x} = \int xe^{x} \, dx$

  $\Rightarrow ze^{x} = xe^{x} - e^{x} + C$

  $\Rightarrow z = x - 1 + Ce^{-x}$

  $\Rightarrow \boxed{(z - x + 1)e^{x} = C_2}$

- THUS $u(x,y,z) = y - x$

  $v(x,y,z) = e^{x}(z - x + 1)$

SO THE GENERAL SOLUTION OF THE P.D.E IS $F(u,v) = 0$

I.E $F\left(y - x, e^{x}(z - x + 1)\right) = 0$

OR

$y - x = f(z - x + 1)$ OR $e^{x}(z - x + 1) = g(y - x)$

$z - x + 1 = e^{-x}g(y - x)$

$z = x - 1 + e^{-x}g(y - x)$

Question 4

It is given that $\varphi = \varphi(x, y)$ satisfies the partial differential equation

$$\frac{\partial \varphi}{\partial x} - \frac{\partial \varphi}{\partial y} = \sin x + \cos y .$$

Use the transformation equations

$$u = ax + by \quad \text{and} \quad v = cx + dy , \quad ad - bc \neq 0$$

with suitable values of a, b, c and d, in order to determine a general solution of the above partial differential equation.

$$\boxed{\varphi(x, y) = F(x + y) - \cos x - \sin y}$$

$$\frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial y} = \sin x + \cos y$$

- $u = ax + by$, $v = cx + dy$, $ad - bc \neq 0$
- $\frac{\partial u}{\partial x} = a$, $\frac{\partial u}{\partial y} = b$, $\frac{\partial v}{\partial x} = c$, $\frac{\partial v}{\partial y} = d$
- $\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x} = a\frac{\partial \phi}{\partial u} + c\frac{\partial \phi}{\partial v}$

  $\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial y} = b\frac{\partial \phi}{\partial u} + d\frac{\partial \phi}{\partial v}$
- SUB INTO THE P.D.E

  $\left(a\frac{\partial \phi}{\partial u} + c\frac{\partial \phi}{\partial v}\right) - \left(b\frac{\partial \phi}{\partial u} + d\frac{\partial \phi}{\partial v}\right) = \sin x + \cos y$

  $(a-b)\frac{\partial \phi}{\partial u} + (c-d)\frac{\partial \phi}{\partial v} = \sin x + \cos y$
- "KNOCK OFF" $\frac{\partial \phi}{\partial v}$ BY $\left.\begin{array}{l} a=1 \\ b=0 \\ c=1 \\ d=1 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} u = x \\ v = x+y \end{array}\right\}$
- Thus

  $\frac{\partial \phi}{\partial u} = \sin u + \cos(v-u)$

  $\phi(u,v) = -\cos u - \sin(v-u) + f(v)$

  Thus

  $\phi(x,y) = -\cos x - \sin y + f(x+y)$

  $\phi(x,y) = f(x+y) - \cos x - \sin y$

Question 5

It is given that $z = z(x, y)$ satisfies the partial differential equation

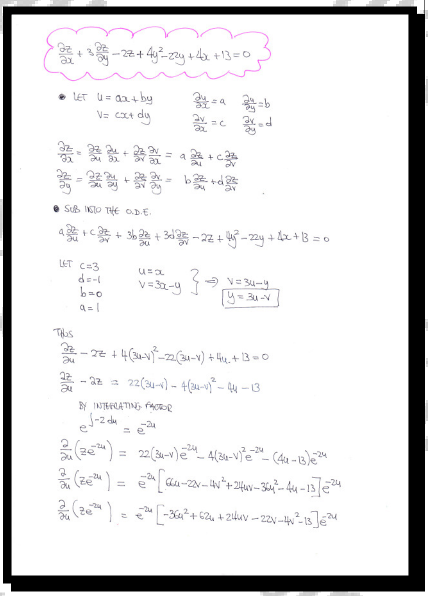
$$\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} - 2z + 4y^2 - 22y + 4x + 13 = 0 .$$

Use the transformation equations

$$u = ax + by \quad \text{and} \quad v = cx + dy , \quad ad - bc \neq 0$$

with suitable values of a, b, c and d, in order to determine a general solution of the above partial differential equation.

$$\boxed{z = 2y^2 + 2x - 5y + e^{2x} f(3x - y)}$$



$$\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} - 2z + 4y^2 - 22y + 4x + 13 = 0$$

- LET $u = ax + by$ $\quad \frac{\partial u}{\partial x} = a \quad \frac{\partial u}{\partial y} = b$
  $v = cx + dy$ $\quad \frac{\partial v}{\partial x} = c \quad \frac{\partial v}{\partial y} = d$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = a\frac{\partial z}{\partial u} + c\frac{\partial z}{\partial v}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = b\frac{\partial z}{\partial u} + d\frac{\partial z}{\partial v}$$

- SUB INTO THE O.D.E.

$$a\frac{\partial z}{\partial u} + c\frac{\partial z}{\partial v} + 3b\frac{\partial z}{\partial u} + 3d\frac{\partial z}{\partial v} - 2z + 4y^2 - 22y + 4x + 13 = 0$$

LET $c=3$, $d=-1$, $b=0$, $a=1$

$\left.\begin{matrix} u = x \\ v = 3x - y \end{matrix}\right\} \Rightarrow v = 3u - y$, $\boxed{y = 3u - v}$

THUS

$$\frac{\partial z}{\partial u} - 2z + 4(3u-v)^2 - 22(3u-v) + 4u + 13 = 0$$

$$\frac{\partial z}{\partial u} - 2z = 22(3u-v) - 4(3u-v)^2 - 4u - 13$$

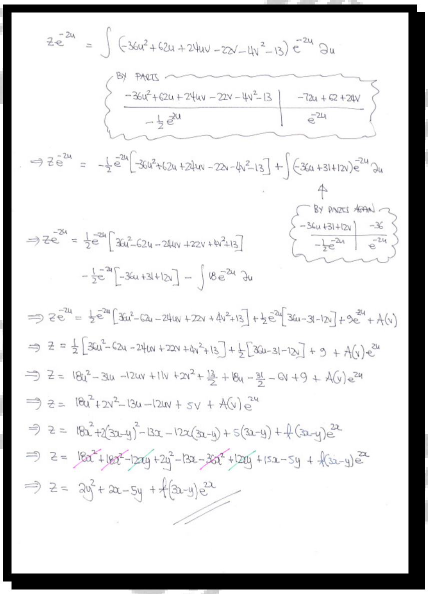
BY INTEGRATING FACTOR

$$e^{\int -2\,du} = e^{-2u}$$

$$\frac{\partial}{\partial u}\left(ze^{-2u}\right) = 22(3u-v)e^{-2u} - 4(3u-v)^2e^{-2u} - (4u-13)e^{-2u}$$

$$\frac{\partial}{\partial u}\left(ze^{-2u}\right) = e^{-2u}\left[66u - 22v - 4v^2 + 24uv - 36u^2 - 4u - 13\right]e^{-2u}$$

$$\frac{\partial}{\partial u}\left(ze^{-2u}\right) = e^{-2u}\left[-36u^2 + 62u + 24uv - 22v - 4v^2 - 13\right]e^{-2u}$$



$$ze^{-2u} = \int \left(-36u^2 + 62u + 24uv - 22v - 4v^2 - 13\right)e^{-2u}\,\partial u$$

BY PARTS

$-36u^2 + 62u + 24uv - 22v - 4v^2 - 13$	$-72u + 62 + 24v$
$-\frac{1}{2}e^{-2u}$	e^{-2u}

$$\Rightarrow ze^{-2u} = -\tfrac{1}{2}e^{-2u}\left[-36u^2 + 62u + 24uv - 22v - 4v^2 - 13\right] + \int \left(-36u + 31 + 12v\right)e^{-2u}\,\partial u$$

BY PARTS AGAIN

$-36u + 31 + 12v$	-36
$-\frac{1}{2}e^{-2u}$	e^{-2u}

$$\Rightarrow ze^{-2u} = \tfrac{1}{2}e^{-2u}\left[36u^2 - 62u - 24uv + 22v + 4v^2 + 13\right] - \tfrac{1}{2}e^{-2u}\left[-36u + 31 + 12v\right] - \int 18e^{-2u}\,\partial u$$

$$\Rightarrow ze^{-2u} = \tfrac{1}{2}e^{-2u}\left[36u^2 - 62u - 24uv + 22v + 4v^2 + 13\right] + \tfrac{1}{2}e^{-2u}\left[36u - 31 - 12v\right] + 9e^{-2u} + A(v)$$

$$\Rightarrow z = \tfrac{1}{2}\left[36u^2 - 62u - 24uv + 22v + 4v^2 + 13\right] + \tfrac{1}{2}\left[36u - 31 - 12v\right] + 9 + A(v)e^{2u}$$

$$\Rightarrow z = 18u^2 - 31u - 12uv + 11v + 2v^2 + \tfrac{13}{2} + 18u - \tfrac{31}{2} - 6v + 9 + A(v)e^{2u}$$

$$\Rightarrow z = 18u^2 + 2v^2 - 13u - 12uv + 5v + A(v)e^{2u}$$

$$\Rightarrow z = 18x^2 + 2(3x-y)^2 - 13x - 12x(3x-y) + 5(3x-y) + f(3x-y)e^{2x}$$

$$\Rightarrow z = \cancel{18x^2} + \cancel{18x^2} - \cancel{12xy} + 2y^2 - 13x - \cancel{36x^2} + \cancel{12xy} + 15x - 5y + f(3x-y)e^{2x}$$

$$\Rightarrow z = 2y^2 + 2x - 5y + f(3x-y)e^{2x}$$

Question 6

It is given that $\varphi = \varphi(x, y)$ satisfies the partial differential equation

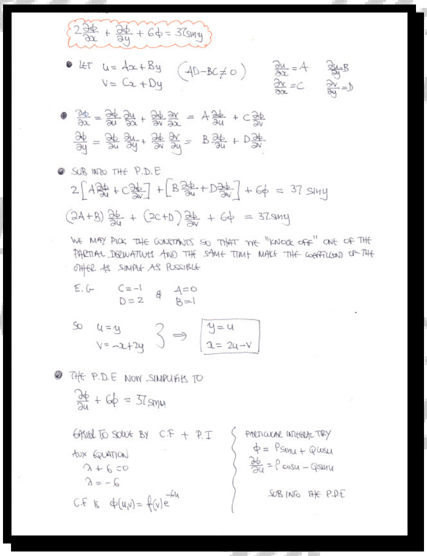
$$2\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y} + 6\varphi = 37\sin y .$$

Use the transformation equations

$$u = Ax + By \quad \text{and} \quad v = Cx + Dy , \quad AD - BC \neq 0$$

with suitable values of A, B, C and D, in order to determine a general solution of the above partial differential equation.

$$\boxed{\varphi(x, y) = 6\sin y - \cos y + e^{-3x} f(x - 2y) = 6\sin y - \cos y + e^{-6y} g(x - 2y)}$$



$$2\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} + 6\phi = 37\sin y$$

- LET $u = Ax + By$, $v = Cx + Dy$ $(AD - BC \neq 0)$ $\quad \frac{\partial u}{\partial x} = A \quad \frac{\partial u}{\partial y} = B \quad \frac{\partial v}{\partial x} = C \quad \frac{\partial v}{\partial y} = D$
- $\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x} = A\frac{\partial \phi}{\partial u} + C\frac{\partial \phi}{\partial v}$
  $\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial y} = B\frac{\partial \phi}{\partial u} + D\frac{\partial \phi}{\partial v}$
- SUB INTO THE P.D.E
  $2\left[A\frac{\partial \phi}{\partial u} + C\frac{\partial \phi}{\partial v}\right] + \left[B\frac{\partial \phi}{\partial u} + D\frac{\partial \phi}{\partial v}\right] + 6\phi = 37\sin y$
  $(2A + B)\frac{\partial \phi}{\partial u} + (2C + D)\frac{\partial \phi}{\partial v} + 6\phi = 37\sin y$
  WE MAY PICK THE CONSTANTS SO THAT WE "KNOCK OFF" ONE OF THE PARTIAL DERIVATIVES AND THE SAME TIME MAKE THE COEFFICIENT OF THE OTHER AS SIMPLE AS POSSIBLE
  E.G $\quad C = -1, D = 2 \quad \& \quad A = 0, B = 1$
  SO $\left.\begin{array}{l} u = y \\ v = -x + 2y \end{array}\right\} \Rightarrow \boxed{\begin{array}{l} y = u \\ x = 2u - v \end{array}}$
- THE P.D.E NOW SIMPLIFIES TO
  $\frac{\partial \phi}{\partial u} + 6\phi = 37\sin u$

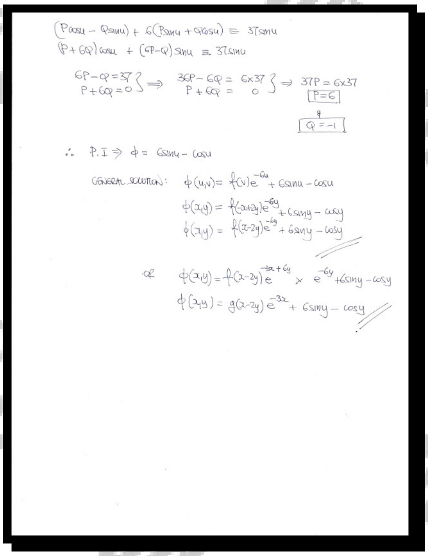
EASIER TO SOLVE BY C.F + P.I

AUX EQUATION
$\lambda + 6 = 0$
$\lambda = -6$

C.F IS $\phi(u, v) = f(v)e^{-6u}$

PARTICULAR INTEGRAL TRY
$\phi = P\sin u + Q\cos u$
$\frac{\partial \phi}{\partial u} = P\cos u - Q\sin u$

SUB INTO THE P.D.E



$(P\cos u - Q\sin u) + 6(P\sin u + Q\cos u) \equiv 37\sin u$
$(P + 6Q)\cos u + (6P - Q)\sin u \equiv 37\sin u$

$\left.\begin{array}{l} 6P - Q = 37 \\ P + 6Q = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 36P - 6Q = 6 \times 37 \\ P + 6Q = 0 \end{array}\right\} \Rightarrow 37P = 6 \times 37$

$\boxed{P = 6}$ & $\boxed{Q = -1}$

$\therefore$ P.I $\Rightarrow \phi = 6\sin u - \cos u$

GENERAL SOLUTION: $\phi(u, v) = f(v)e^{-6u} + 6\sin u - \cos u$
$\phi(x, y) = f(-x + 2y)e^{-6y} + 6\sin y - \cos y$
$\phi(x, y) = f(x - 2y)e^{-6y} + 6\sin y - \cos y$

OR $\quad \phi(x, y) = f(x - 2y)e^{-3x + 6y} \times e^{-6y} + 6\sin y - \cos y$
$\phi(x, y) = g(x - 2y)e^{-3x} + 6\sin y - \cos y$

Question 7

It is given that $\varphi = \varphi(x, y, z)$ satisfies the partial differential equation

$$\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y} + \frac{\partial \varphi}{\partial z} = \varphi .$$

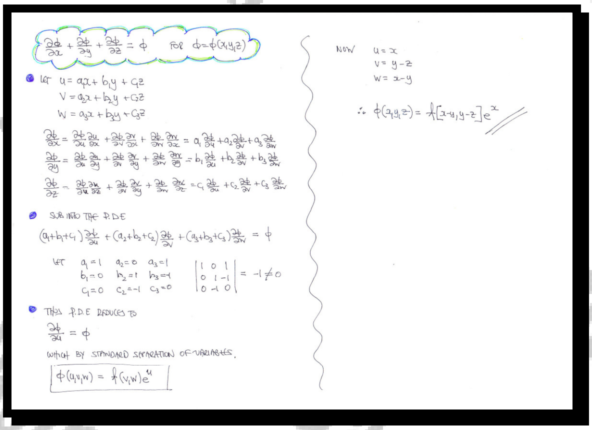
Use the transformation equations

$$u = a_1 x + b_1 y + c_1 z , \quad v = a_2 x + b_2 y + c_2 z \quad \text{and} \quad w = a_3 x + b_3 y + c_3 z ,$$

where $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$,

in order to determine a general solution of the above partial differential equation.

$$\boxed{\varphi(x, y, z) = f\left[x - y, y - z\right] e^{x}}$$



$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z} = \phi$ FOR $\phi = \phi(x,y,z)$

- LET $u = a_1 x + b_1 y + c_1 z$
  $v = a_2 x + b_2 y + c_2 z$
  $w = a_3 x + b_3 y + c_3 z$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial \phi}{\partial w}\frac{\partial w}{\partial x} = a_1 \frac{\partial \phi}{\partial u} + a_2 \frac{\partial \phi}{\partial v} + a_3 \frac{\partial \phi}{\partial w}$$
$$\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial \phi}{\partial w}\frac{\partial w}{\partial y} = b_1 \frac{\partial \phi}{\partial u} + b_2 \frac{\partial \phi}{\partial v} + b_3 \frac{\partial \phi}{\partial w}$$
$$\frac{\partial \phi}{\partial z} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial \phi}{\partial w}\frac{\partial w}{\partial z} = c_1 \frac{\partial \phi}{\partial u} + c_2 \frac{\partial \phi}{\partial v} + c_3 \frac{\partial \phi}{\partial w}$$

- SUB INTO THE P.D.E

$$(a_1+b_1+c_1)\frac{\partial \phi}{\partial u} + (a_2+b_2+c_2)\frac{\partial \phi}{\partial v} + (a_3+b_3+c_3)\frac{\partial \phi}{\partial w} = \phi$$

LET $a_1 = 1$, $a_2 = 0$, $a_3 = 1$; $b_1 = 0$, $b_2 = 1$, $b_3 = -1$; $c_1 = 0$, $c_2 = -1$, $c_3 = 0$

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 0 \end{vmatrix} = -1 \neq 0$$

- THUS P.D.E REDUCES TO

$$\frac{\partial \phi}{\partial u} = \phi$$

WHICH BY STANDARD SEPARATION OF VARIABLES,

$$\boxed{\phi(u,v,w) = f(v,w)e^{u}}$$

NOW $u = x$
$v = y - z$
$w = x - y$

$$\therefore \phi(x,y,z) = f[x-y, y-z]e^{x}$$

FIRST ORDER P.D.E.s

(by transformations)

$$A(x,y)\frac{\partial z}{\partial x}+B(x,y)\frac{\partial z}{\partial y}+C(x,y)z=G(x,y),$$

$$z=z(x,y)$$

Question 1

It is given that $\psi = f(x, y)$ satisfies the partial differential equation

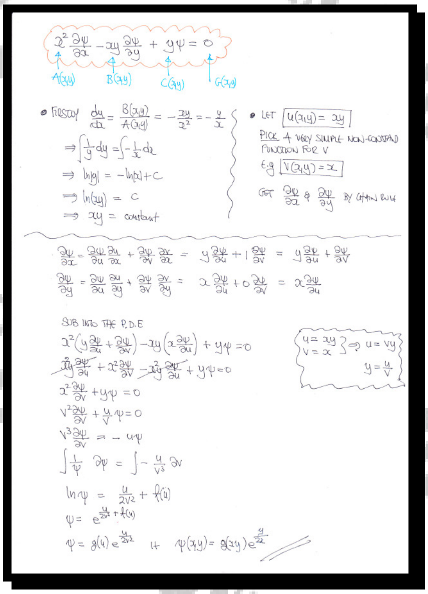
$$x^2 \frac{\partial \psi}{\partial x} - xy\frac{\partial \psi}{\partial y} + y\psi = 0\,.$$

Use the transformation equations

$$u = u(x, y) \quad \text{and} \quad v = v(x, y),$$

for suitable functions u and v, in order to determine a general solution of the above partial differential equation.

$$\boxed{\psi(x, y) = e^{\frac{y}{2x}}\, g(xy)}$$



$$x^2\frac{\partial \psi}{\partial x} - xy\frac{\partial \psi}{\partial y} + y\psi = 0$$

$A(x,y)$ $B(x,y)$ $C(x,y)$ $G(x,y)$

• FIRSTLY $\frac{dy}{dx} = \frac{B(x,y)}{A(x,y)} = -\frac{xy}{x^2} = -\frac{y}{x}$

$\Rightarrow \int \frac{1}{y}\,dy = \int -\frac{1}{x}\,dx$

$\Rightarrow \ln|y| = -\ln|x| + C$

$\Rightarrow \ln(xy) = C$

$\Rightarrow xy = \text{constant}$

• LET $\boxed{u(x,y) = xy}$

PICK A VERY SIMPLE NON-CONSTANT FUNCTION FOR V

E.g $\boxed{v(x,y) = x}$

GET $\frac{\partial \psi}{\partial x}$ & $\frac{\partial \psi}{\partial y}$ BY CHAIN RULE

$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \psi}{\partial v}\frac{\partial v}{\partial x} = y\frac{\partial \psi}{\partial u} + 1\frac{\partial \psi}{\partial v} = y\frac{\partial \psi}{\partial u} + \frac{\partial \psi}{\partial v}$

$\frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \psi}{\partial v}\frac{\partial v}{\partial y} = x\frac{\partial \psi}{\partial u} + 0\frac{\partial \psi}{\partial v} = x\frac{\partial \psi}{\partial u}$

SUB INTO THE P.D.E

$x^2\left(y\frac{\partial \psi}{\partial u} + \frac{\partial \psi}{\partial v}\right) - xy\left(x\frac{\partial \psi}{\partial u}\right) + y\psi = 0$

$\cancel{x^2y\frac{\partial \psi}{\partial u}} + x^2\frac{\partial \psi}{\partial v} - \cancel{x^2y\frac{\partial \psi}{\partial u}} + y\psi = 0$

$x^2\frac{\partial \psi}{\partial v} + y\psi = 0$

$v^2\frac{\partial \psi}{\partial v} + \frac{u}{v}\psi = 0$

$v^3\frac{\partial \psi}{\partial v} = -u\psi$

$\int \frac{1}{\psi}\,\partial\psi = \int -\frac{u}{v^3}\,\partial v$

$\ln\psi = \frac{u}{2v^2} + f(u)$

$\psi = e^{\frac{u}{2v^2} + f(u)}$

$\psi = g(u)\,e^{\frac{u}{2v^2}}$ ie $\psi(x,y) = g(xy)\,e^{\frac{y}{2x}}$

$\left.\begin{array}{l} u = xy \\ v = x \end{array}\right\} \Rightarrow u = vy$

$y = \frac{u}{v}$

Question 2

It is given that $z = z(x, y)$ satisfies the partial differential equation

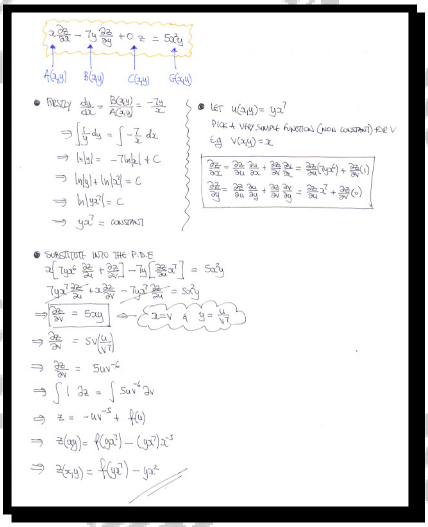
$$x\frac{\partial z}{\partial x} - 7y\frac{\partial z}{\partial y} = 5x^2 y.$$

Use the transformation equations

$$u = u(x, y) \quad \text{and} \quad v = v(x, y),$$

for suitable functions u and v, in order to determine a general solution of the above partial differential equation.

$$\boxed{z(x, y) = f\left(yx^7\right) - yx^2}$$



$$x\frac{\partial z}{\partial x} - 7y\frac{\partial z}{\partial y} + 0z = 5x^2y$$

$A(x,y)$ $B(x,y)$ $C(x,y)$ $G(x,y)$

- FIRSTLY $\frac{dy}{dx} = \frac{B(x,y)}{A(x,y)} = \frac{-7y}{x}$

$\Rightarrow \int \frac{1}{y}\,dy = \int -\frac{7}{x}\,dx$

$\Rightarrow \ln|y| = -7\ln|x| + C$

$\Rightarrow \ln|y| + \ln|x^7| = C$

$\Rightarrow \ln|yx^7| = C$

$\Rightarrow yx^7 = \text{CONSTANT}$

- LET $u(x,y) = yx^7$

PICK A VERY SIMPLE FUNCTION (NON CONSTANT) FOR V

E.g $V(x,y) = x$

$$\boxed{\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}(7yx^6) + \frac{\partial z}{\partial v}(1)\\ \frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u}x^7 + \frac{\partial z}{\partial v}(0)\end{aligned}}$$

- SUBSTITUTE INTO THE P.D.E

$x\left[7yx^6\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right] - 7y\left[\frac{\partial z}{\partial u}x^7\right] = 5x^2y$

$\cancel{7yx^7\frac{\partial z}{\partial u}} + x\frac{\partial z}{\partial v} - \cancel{7yx^7\frac{\partial z}{\partial u}} = 5x^2y$

$\Rightarrow \boxed{\frac{\partial z}{\partial v} = 5xy}$ $\Leftarrow$ $x = v$ & $y = \frac{u}{v^7}$

$\Rightarrow \frac{\partial z}{\partial v} = 5v\left(\frac{u}{v^7}\right)$

$\Rightarrow \frac{\partial z}{\partial v} = 5uv^{-6}$

$\Rightarrow \int 1\,\partial z = \int 5uv^{-6}\,\partial v$

$\Rightarrow z = -uv^{-5} + f(u)$

$\Rightarrow z(x,y) = f(yx^7) - (yx^7)x^{-5}$

$\Rightarrow z(x,y) = f(yx^7) - yx^2$

Question 3

It is given that $\varphi = \varphi(x, y)$ satisfies the partial differential equation

$$2\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y} + 6\varphi = 37\sin y .$$

Use the transformation equations

$$u = u(x, y) \quad \text{and} \quad v = v(x, y),$$

for suitable functions u and v, in order to determine a general solution of the above partial differential equation.

$$\boxed{\varphi(x, y) = 6\sin y - \cos y + e^{-3x} f(x - 2y) = 6\sin y - \cos y + e^{-6y} g(x - 2y)}$$

Question 4

It is given that $z = z(x, y)$ satisfies the partial differential equation

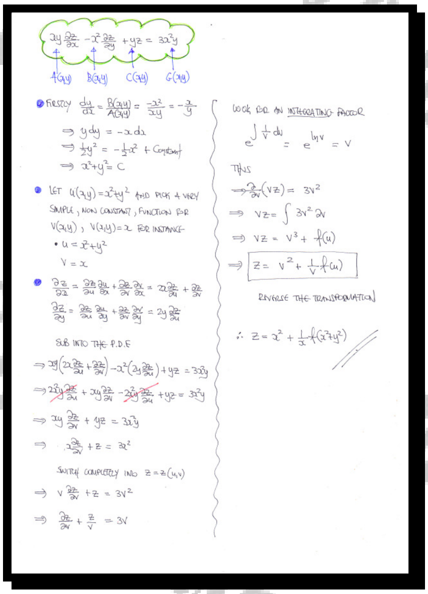
$$xy\frac{\partial z}{\partial x} - x^2\frac{\partial z}{\partial y} + yz = 3x^2 y\,.$$

Use the transformation equations

$$u = u(x, y) \quad \text{and} \quad v = v(x, y),$$

for suitable functions u and v, in order to determine a general solution of the above partial differential equation.

$$\boxed{z(x, y) = x^2 + \frac{1}{x} f\left(x^2 + y^2\right)}$$



$$\underset{A(x,y)}{xy}\frac{\partial z}{\partial x} \underset{B(x,y)}{- x^2}\frac{\partial z}{\partial y} + \underset{C(x,y)}{y}z = \underset{G(x,y)}{3x^2 y}$$

- FIRSTLY $\frac{dy}{dx} = \frac{B(x,y)}{A(x,y)} = \frac{-x^2}{xy} = -\frac{x}{y}$

$\Rightarrow y\,dy = -x\,dx$

$\Rightarrow \frac{1}{2}y^2 = -\frac{1}{2}x^2 + \text{Constant}$

$\Rightarrow x^2 + y^2 = C$

- LET $u(x,y) = x^2 + y^2$ AND PICK A VERY SIMPLE, NON CONSTANT, FUNCTION FOR $v(x,y)$, $v(x,y) = x$ FOR INSTANCE

• $u = x^2 + y^2$

$v = x$

- $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = 2x\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$

$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = 2y\frac{\partial z}{\partial u}$

SUB INTO THE P.D.E

$\Rightarrow xy\left(2x\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) - x^2\left(2y\frac{\partial z}{\partial u}\right) + yz = 3x^2y$

$\Rightarrow \cancel{2x^2y\frac{\partial z}{\partial u}} + xy\frac{\partial z}{\partial v} - \cancel{2x^2y\frac{\partial z}{\partial u}} + yz = 3x^2y$

$\Rightarrow xy\frac{\partial z}{\partial v} + yz = 3x^2y$

$\Rightarrow x\frac{\partial z}{\partial v} + z = 3x^2$

SWITCH COMPLETELY INTO $z = z(u,v)$

$\Rightarrow v\frac{\partial z}{\partial v} + z = 3v^2$

$\Rightarrow \frac{\partial z}{\partial v} + \frac{z}{v} = 3v$

LOOK FOR AN INTEGRATING FACTOR

$e^{\int \frac{1}{v}\,dv} = e^{\ln v} = v$

THUS

$\Rightarrow \frac{\partial}{\partial v}(vz) = 3v^2$

$\Rightarrow vz = \int 3v^2\,dv$

$\Rightarrow vz = v^3 + f(u)$

$\Rightarrow \boxed{z = v^2 + \frac{1}{v}f(u)}$

REVERSE THE TRANSFORMATION

$\therefore z = x^2 + \frac{1}{x}f(x^2+y^2)$

FIRST ORDER P.D.E.s

(by Lagrange's method)

$$P(x,y,z)\frac{\partial z}{\partial x}+Q(x,y,z)\frac{\partial z}{\partial y}=R(x,y,z), \quad z=z(x,y)$$

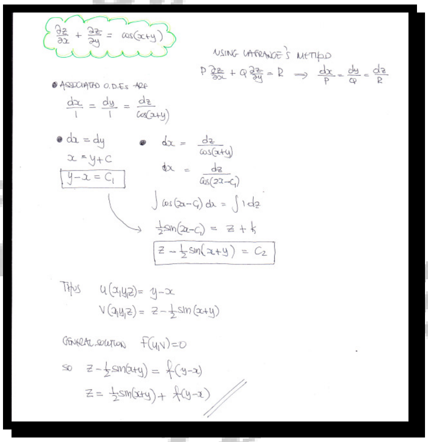
Question 1

It is given that $z = z(x, y)$ satisfies the partial differential equation

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \cos(x+y).$$

Use Lagrange's method, to determine the general solution of the above partial differential equation.

$$\boxed{z = \frac{1}{2}\sin(x+y) + f(y-x)}$$



$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \cos(x+y)$$

USING LAGRANGE'S METHOD

$$P\frac{\partial z}{\partial x} + Q\frac{\partial z}{\partial y} = R \implies \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

- ASSOCIATED O.D.Es ARE

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{\cos(x+y)}$$

- $dx = dy$

$x = y + C$

$\boxed{y - x = C_1}$

- $dx = \frac{dz}{\cos(x+y)}$

$dx = \frac{dz}{\cos(2x - C_1)}$

$\int \cos(2x - C_1)\,dx = \int 1\,dz$

$\frac{1}{2}\sin(2x - C_1) = z + k$

$\boxed{z - \frac{1}{2}\sin(x+y) = C_2}$

THUS $u(x,y,z) = y - x$

$v(x,y,z) = z - \frac{1}{2}\sin(x+y)$

GENERAL SOLUTION $f(u,v) = 0$

SO $z - \frac{1}{2}\sin(x+y) = f(y-x)$

$z = \frac{1}{2}\sin(x+y) + f(y-x)$

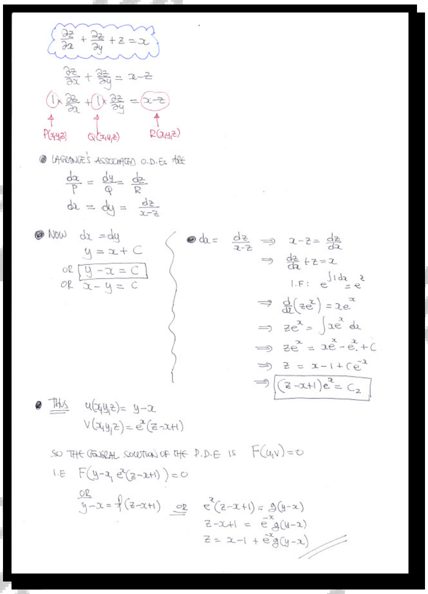
Question 2

It is given that $z = z(x, y)$ satisfies the partial differential equation

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} + z = x \, .$$

Use Lagrange's method to determine the general solution of the above partial differential equation.

$$\boxed{z = x - 1 + \mathrm{e}^{-x} f(x - y)}$$



$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} + z = x$$

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = x - z$$

$$\underbrace{1}_{P(x,y,z)} \times \frac{\partial z}{\partial x} + \underbrace{1}_{Q(x,y,z)} \times \frac{\partial z}{\partial y} = \underbrace{x - z}_{R(x,y,z)}$$

- LAGRANGE'S ASSOCIATED O.D.Es ARE

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$dx = dy = \frac{dz}{x - z}$$

- NOW $dx = dy$

$$y = x + C$$

OR $\boxed{y - x = C}$

OR $x - y = C$

- $dx = \frac{dz}{x - z} \Rightarrow x - z = \frac{dz}{dx}$

$$\Rightarrow \frac{dz}{dx} + z = x$$

$$\text{I.F}: \ e^{\int 1 \, dx} = e^{x}$$

$$\Rightarrow \frac{d}{dx}(ze^{x}) = xe^{x}$$

$$\Rightarrow ze^{x} = \int xe^{x} \, dx$$

$$\Rightarrow ze^{x} = xe^{x} - e^{x} + C$$

$$\Rightarrow z = x - 1 + Ce^{-x}$$

$$\Rightarrow \boxed{(z - x + 1)e^{x} = C_2}$$

- THUS $u(x,y,z) = y - x$

$$V(x,y,z) = e^{x}(z - x + 1)$$

SO THE GENERAL SOLUTION OF THE P.D.E IS $F(u,v) = 0$

I.E $F(y - x, e^{x}(z - x + 1)) = 0$

OR

$y - x = f(z - x + 1)$ OR $e^{x}(z - x + 1) = g(y - x)$

$$z - x + 1 = e^{-x} g(y - x)$$

$$z = x - 1 + e^{-x} g(y - x)$$

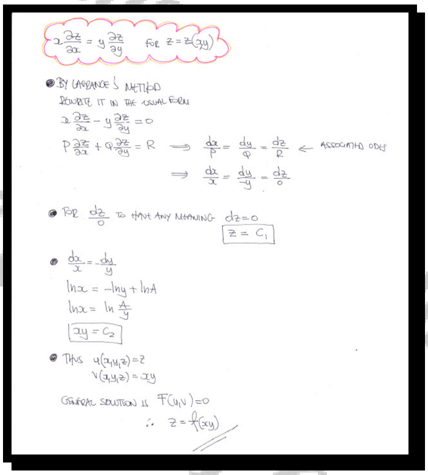
Question 3

It is given that $z = z(x, y)$ satisfies the partial differential equation

$$x\frac{\partial z}{\partial x} = y\frac{\partial z}{\partial y}.$$

Use Lagrange's method, to determine the general solution of the above partial differential equation.

$$\boxed{z = f(xy)}$$



$x\dfrac{\partial z}{\partial x} = y\dfrac{\partial z}{\partial y}$ FOR $z = z(x,y)$

- BY LAGRANGE'S METHOD

REWRITE IT IN THE USUAL FORM

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 0$$

$$P\frac{\partial z}{\partial x} + Q\frac{\partial z}{\partial y} = R \implies \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \leftarrow \text{ASSOCIATED ODEs}$$

$$\implies \frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{0}$$

- FOR $\dfrac{dz}{0}$ TO HAVE ANY MEANING $dz = 0$

$$\boxed{z = C_1}$$

- $\dfrac{dx}{x} = -\dfrac{dy}{y}$

$$\ln x = -\ln y + \ln A$$

$$\ln x = \ln \frac{A}{y}$$

$$\boxed{xy = C_2}$$

- THUS $u(x,y,z) = z$

$v(x,y,z) = xy$

GENERAL SOLUTION IS $F(u,v) = 0$

$$\therefore\ z = f(xy)$$

Question 4

It is given that $z = z(x, y)$ satisfies the partial differential equation

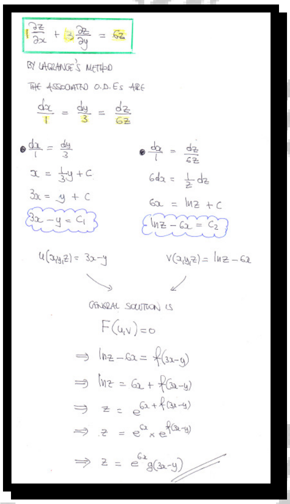
$$\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} = 6z .$$

Use Lagrange's method to show that the general solution of the above partial differential equation can be written as

$$z(x, y) = e^{6x}\, g(3x - y),$$

where g is an arbitrary function of $3x - y$.

$\boxed{\text{proof}}$



$$\boxed{1\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} = 6z}$$

BY LAGRANGE'S METHOD

THE ASSOCIATED O.D.Es ARE

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{6z}$$

- $\frac{dx}{1} = \frac{dy}{3}$

$x = \frac{1}{3}y + C$

$3x = y + C$

$3x - y = C_1$

- $\frac{dx}{1} = \frac{dz}{6z}$

$6dx = \frac{1}{z}dz$

$6x = \ln z + C$

$\ln z - 6x = C_2$

$u(x,y,z) = 3x - y$

$v(x,y,z) = \ln z - 6x$

GENERAL SOLUTION IS

$F(u,v) = 0$

$\Rightarrow \ln z - 6x = f(3x-y)$

$\Rightarrow \ln z = 6x + f(3x-y)$

$\Rightarrow z = e^{6x + f(3x-y)}$

$\Rightarrow z = e^{6x} \times e^{f(3x-y)}$

$\Rightarrow z = e^{6x} g(3x-y)$

Question 5

It is given that $z = z(x, y)$ satisfies the partial differential equation

$$x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = z^2 .$$

Use Lagrange's method to determine the general solution of the above partial differential equation.

$$\boxed{z = \frac{x}{1 + x f\left(\frac{1}{y} - \frac{1}{x}\right)}}$$

$$x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = z^2$$

$$\underbrace{x^2}_{P(x,y,z)} \frac{\partial z}{\partial x} + \underbrace{y^2}_{Q(x,y,z)} \frac{\partial z}{\partial y} = \underbrace{z^2}_{R(x,y,z)}$$

- LAGRANGE'S ASSOCIATED O.D.E ARE

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2}$$

- $\frac{dx}{x^2} = \frac{dy}{y^2}$

$$-\frac{1}{x} = -\frac{1}{y} + C$$

$$\boxed{\frac{1}{y} - \frac{1}{x} = C_1}$$

- $\frac{dx}{x^2} = \frac{dz}{z^2}$

$$-\frac{1}{x} = -\frac{1}{z} + C$$

$$\boxed{\frac{1}{z} - \frac{1}{x} = C_2}$$

- $u(x,y,z) = \frac{1}{y} - \frac{1}{x}$

$v(x,y,z) = \frac{1}{z} - \frac{1}{x}$

GENERAL SOLUTION IS $F(u,v) = 0$

$$F\left(\frac{1}{y} - \frac{1}{x}, \frac{1}{z} - \frac{1}{x}\right) = 0$$

$\frac{1}{y} - \frac{1}{x} = f\left(\frac{1}{z} - \frac{1}{x}\right)$ OR $\frac{1}{z} - \frac{1}{x} = g\left(\frac{1}{y} - \frac{1}{x}\right)$

$$\frac{1}{z} = \frac{1}{x} + g\left(\frac{1}{y} - \frac{1}{x}\right)$$

$$z = \frac{1}{\frac{1}{x} + g\left(\frac{1}{y} - \frac{1}{x}\right)}$$

$$z = \frac{x}{1 + xg\left(\frac{1}{y} - \frac{1}{x}\right)}$$

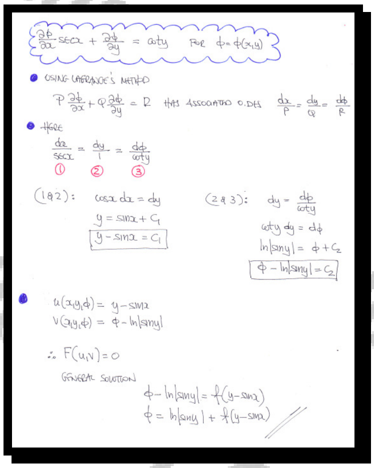
Question 6

It is given that $\varphi = \varphi(x, y)$ satisfies the partial differential equation

$$\frac{\partial \varphi}{\partial x}\sec x + \frac{\partial \varphi}{\partial y} = \cot y .$$

Use Lagrange's method, to determine the general solution of the above partial differential equation.

$$\boxed{\varphi(x, y) = \ln|\sin y| + f(y - \sin x)}$$



$\frac{\partial \phi}{\partial x}\sec x + \frac{\partial \phi}{\partial y} = \cot y$ FOR $\phi = \phi(x,y)$

- USING LAGRANGE'S METHOD

$P\frac{\partial \phi}{\partial x} + Q\frac{\partial \phi}{\partial y} = R$ HAS ASSOCIATED O.D.ES $\frac{dx}{P} = \frac{dy}{Q} = \frac{d\phi}{R}$

- HERE

$$\frac{dx}{\sec x} = \frac{dy}{1} = \frac{d\phi}{\cot y}$$
① ② ③

(1 & 2): $\cos x\,dx = dy$

$y = \sin x + C_1$

$\boxed{y - \sin x = C_1}$

(2 & 3): $dy = \frac{d\phi}{\cot y}$

$\cot y\,dy = d\phi$

$\ln|\sin y| = \phi + C_2$

$\boxed{\phi - \ln|\sin y| = C_2}$

- $u(x,y,\phi) = y - \sin x$

$v(x,y,\phi) = \phi - \ln|\sin y|$

$\therefore F(u,v) = 0$

GENERAL SOLUTION

$\phi - \ln|\sin y| = f(y - \sin x)$

$\phi = \ln|\sin y| + f(y - \sin x)$

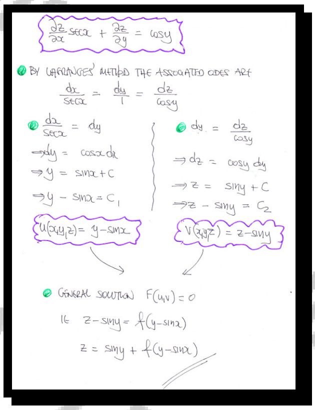
Question 7

It is given that $z = z(x, y)$ satisfies the partial differential equation

$$\frac{\partial z}{\partial x}\sec x + \frac{\partial z}{\partial y} = \cos y .$$

Use Lagrange's method, to determine the general solution of the above partial differential equation.

$$\boxed{z(x, y) = \sin y + f\left(y - \sin x\right)}$$



$\frac{\partial z}{\partial x}\sec x + \frac{\partial z}{\partial y} = \cos y$

BY LAGRANGE'S METHOD THE ASSOCIATED ODES ARE

$\frac{dx}{\sec x} = \frac{dy}{1} = \frac{dz}{\cos y}$

$\frac{dx}{\sec x} = dy$

$\Rightarrow dy = \cos x \, dx$

$\Rightarrow y = \sin x + C$

$\Rightarrow y - \sin x = C_1$

$u(x,y,z) = y - \sin x$

$dy = \frac{dz}{\cos y}$

$\Rightarrow dz = \cos y \, dy$

$\Rightarrow z = \sin y + C$

$\Rightarrow z - \sin y = C_2$

$v(x,y,z) = z - \sin y$

GENERAL SOLUTION $F(u,v) = 0$

IE $z - \sin y = f(y - \sin x)$

$z = \sin y + f(y - \sin x)$

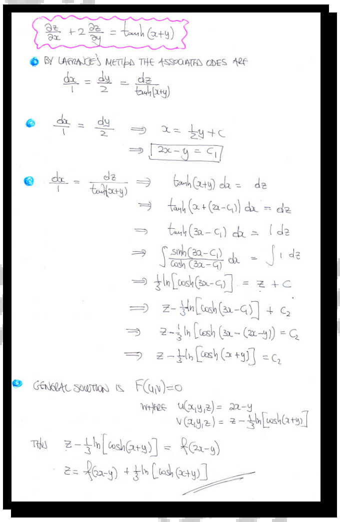
Question 8

It is given that $z = z(x, y)$ satisfies the partial differential equation

$$\frac{\partial z}{\partial x} + 2\frac{\partial z}{\partial y} = \tanh(x+y).$$

Use Lagrange's method to determine the general solution of the above partial differential equation.

$$\boxed{z(x,y) = f(2x-y) + \tfrac{1}{3}\ln\left[\cosh(x+y)\right]}$$



$$\frac{\partial z}{\partial x} + 2\frac{\partial z}{\partial y} = \tanh(x+y)$$

- BY LAGRANGE'S METHOD THE ASSOCIATED ODES ARE

$$\frac{dx}{1} = \frac{dy}{2} = \frac{dz}{\tanh(x+y)}$$

- $\frac{dx}{1} = \frac{dy}{2} \implies x = \frac{1}{2}y + C \implies \boxed{2x - y = C_1}$

- $\frac{dx}{1} = \frac{dz}{\tanh(x+y)} \implies \tanh(x+y)\,dx = dz$

$$\implies \tanh(x + (2x - C_1))\,dx = dz$$
$$\implies \tanh(3x - C_1)\,dx = 1\,dz$$
$$\implies \int \frac{\sinh(3x - C_1)}{\cosh(3x - C_1)}\,dx = \int 1\,dz$$
$$\implies \tfrac{1}{3}\ln\left[\cosh(3x - C_1)\right] = z + C$$
$$\implies z - \tfrac{1}{3}\ln\left[\cosh(3x - C_1)\right] + C_2$$
$$\implies z - \tfrac{1}{3}\ln\left[\cosh(3x - (2x - y))\right] = C_2$$
$$\implies z - \tfrac{1}{3}\ln\left[\cosh(x+y)\right] = C_2$$

- GENERAL SOLUTION IS $F(u,v) = 0$

WHERE $u(x,y,z) = 2x - y$

$v(x,y,z) = z - \frac{1}{3}\ln\left[\cosh(x+y)\right]$

THUS $z - \frac{1}{3}\ln\left[\cosh(x+y)\right] = f(2x - y)$

$z = f(2x-y) + \frac{1}{3}\ln\left[\cosh(x+y)\right]$

Question 9

It is given that $z = z(x, y)$ satisfies the partial differential equation

$$xz\frac{\partial z}{\partial x} + yz\frac{\partial z}{\partial y} + x^2 + y^2 = 0.$$

Use Lagrange's method to determine the general solution of the above partial differential equation.

$$\boxed{z^2 = f\left(\frac{y}{x}\right) - x^2 - y^2}$$

$$xz\frac{\partial z}{\partial x} + yz\frac{\partial z}{\partial y} + x^2+y^2 = 0$$

$$\underbrace{xz}_{P(x,y,z)}\frac{\partial z}{\partial x} + \underbrace{yz}_{Q(x,y,z)}\frac{\partial z}{\partial y} = \underbrace{-x^2-y^2}_{R(x,y,z)}$$

- LAGRANGES ASSOCIATED O.D.Es

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{-x^2-y^2}$$

- $\frac{dx}{xz} = \frac{dy}{yz}$

$$\frac{dx}{x} = \frac{dy}{y}$$

$$\ln x = \ln y + C$$

$$\ln\left(\frac{y}{x}\right) = C$$

$$\boxed{\frac{y}{x} = C_1}$$

- $\frac{dx}{xz} = -\frac{dz}{x^2+y^2} \Rightarrow \frac{dx}{xz} = -\frac{dz}{x^2+x^2C_1^2}$

$$\Rightarrow \frac{dx}{z} = -\frac{dz}{(1+C_1^2)x}$$

$$\Rightarrow (1+C_1^2)x\,dx = -z\,dz$$

$$\Rightarrow \tfrac{1}{2}(1+C_1^2)x^2 = -\tfrac{1}{2}z^2 + C$$

$$\Rightarrow (1+C_1^2)x^2 = -z^2 + C$$

$$\Rightarrow \left(1+\frac{y^2}{x^2}\right)x^2 = -z^2 + C$$

$$\Rightarrow \boxed{x^2+y^2+z^2 = C_2}$$

- THUS $u(x,y,z) = \frac{y}{x}$

$v(x,y,z) = x^2+y^2+z^2$

GENERAL SOLUTION IS $F(u,v) = 0$

$$F\left[x^2+y^2+z^2, \frac{y}{x}\right] = 0$$

$\frac{y}{x} = f(x^2+y^2+z^2)$ OR $x^2+y^2+z^2 = g\left(\frac{y}{x}\right)$

$$z^2 = -x^2-y^2+g\left(\frac{y}{x}\right)$$

Question 10

It is given that $z = z(x, y)$ satisfies the partial differential equation

$$z\frac{\partial z}{\partial x} + z\frac{\partial z}{\partial y} = y - x\,.$$

Use Lagrange's method, to determine the general solution of the above partial differential equation.

$$\boxed{z^2 = 2x(y-x) + f(x-y)}$$

$$z\frac{\partial z}{\partial x} + z\frac{\partial z}{\partial y} = y - x$$

- REWRITE THE P.D.E AS $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{y-x}{z}$
- USE LAGRANGE'S METHOD $P\frac{\partial z}{\partial x} + Q\frac{\partial z}{\partial x} = R$ WITH ASSOCIATED O.D.E.S $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

HERE $\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{\frac{y-x}{z}}$

1 2 3

1 & 2: $dx = dy$

$x = y + C$

$\boxed{x - y = C_1}$

1 & 3: $dx = \frac{z\,dz}{y-x}$

$dx = \frac{z\,dz}{-C_1}$

$-C_1\,dx = z\,dz$

$-C_1 x = \frac{1}{2}z^2 + C_2$

$\frac{1}{2}z^2 + C_1 x = C_2$

$\frac{1}{2}z^2 + (x-y)x = C_2$

$\boxed{z^2 + 2x(x-y) = C_2}$

$\therefore u(x,y,z) = x - y$

$v(x,y,z) = z^2 + 2x(x-y)$

$\therefore$ GENERAL SOLUTION $F(u,v) = 0$

$\therefore z^2 + 2x(x-y) = f(x-y)$

$z^2 = 2x(y-x) + f(x-y)$

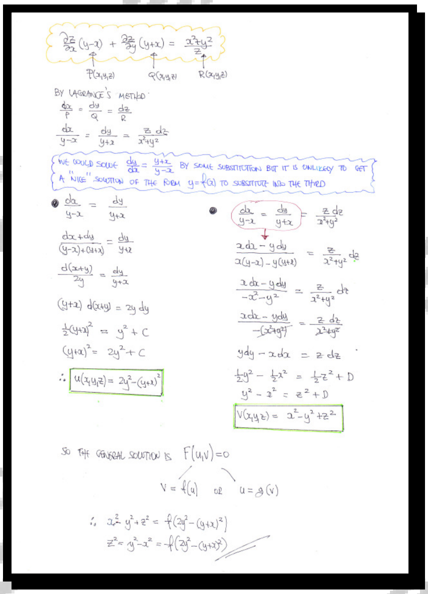
Question 11

It is given that $z = z(x, y)$ satisfies the partial differential equation

$$(y - x)\frac{\partial z}{\partial x} + (y + x)\frac{\partial z}{\partial y} = \frac{x^2 + y^2}{z}.$$

Use Lagrange's Multipliers method for derivatives, to find the general solution of the above partial differential equation.

$$\boxed{z^2 = y^2 - x^2 + f\left[2y^2 - (x + y)^2\right]}$$



$$\frac{\partial z}{\partial x}(y-x) + \frac{\partial z}{\partial y}(y+x) = \frac{x^2+y^2}{z}$$

$P(x,y,z)$ $Q(x,y,z)$ $R(x,y,z)$

BY LAGRANGE'S METHOD

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{y-x} = \frac{dy}{y+x} = \frac{z\,dz}{x^2+y^2}$$

WE COULD SOLVE $\frac{dy}{dx} = \frac{y+x}{y-x}$ BY SOME SUBSTITUTION BUT IT IS UNLIKELY TO GET A "NICE" SOLUTION OF THE FORM $y = f(x)$ TO SUBSTITUTE INTO THE THIRD

- $\frac{dx}{y-x} = \frac{dy}{y+x}$

$$\frac{dx+dy}{(y-x)+(y+x)} = \frac{dy}{y+x}$$

$$\frac{d(x+y)}{2y} = \frac{dy}{y+x}$$

$$(y+x)\,d(x+y) = 2y\,dy$$

$$\frac{1}{2}(y+x)^2 = y^2 + C$$

$$(y+x)^2 = 2y^2 + C$$

$$\therefore \boxed{u(x,y,z) = 2y^2 - (y+x)^2}$$

- $\frac{dx}{y-x} = \frac{dy}{y+x} = \frac{z\,dz}{x^2+y^2}$

$$\frac{x\,dx - y\,dy}{x(y-x) - y(y+x)} = \frac{z}{x^2+y^2}dz$$

$$\frac{x\,dx - y\,dy}{-x^2-y^2} = \frac{z}{x^2+y^2}dz$$

$$\frac{x\,dx - y\,dy}{-(x^2+y^2)} = \frac{z\,dz}{x^2+y^2}$$

$$y\,dy - x\,dx = z\,dz$$

$$\frac{1}{2}y^2 - \frac{1}{2}x^2 = \frac{1}{2}z^2 + D$$

$$y^2 - x^2 = z^2 + D$$

$$\boxed{V(x,y,z) = x^2 - y^2 + z^2}$$

SO THE GENERAL SOLUTION IS $F(u,v) = 0$

$v = f(u)$ OR $u = g(v)$

$$\therefore x^2 - y^2 + z^2 = f(2y^2 - (y+x)^2)$$

$$z^2 = y^2 - x^2 = f(2y^2 - (y+x)^2)$$

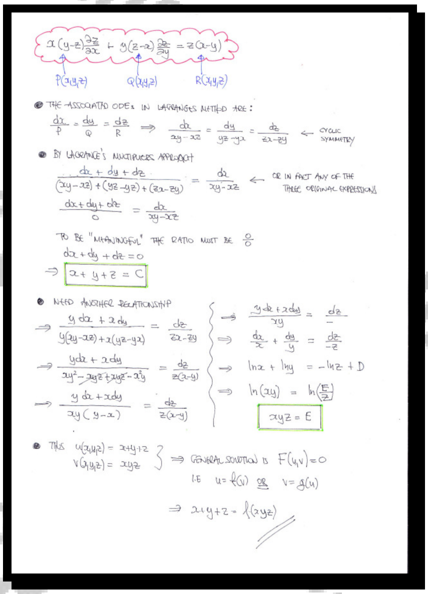
Question 12

It is given that $z = z(x, y)$ satisfies the partial differential equation

$$x(y-z)\frac{\partial z}{\partial x} + y(z-x)\frac{\partial z}{\partial y} = z(x-y).$$

Use Lagrange's Multipliers method for derivatives, to find the general solution of the above partial differential equation.

$$\boxed{x+y+z = f(xyz) \quad \text{or} \quad xyz = g(x+y+z)}$$



$$x(y-z)\frac{\partial z}{\partial x} + y(z-x)\frac{\partial z}{\partial y} = z(x-y)$$

$P(x,y,z)$ $Q(x,y,z)$ $R(x,y,z)$

- THE ASSOCIATED ODEs IN LAGRANGE'S METHOD ARE:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Longrightarrow \frac{dx}{xy-xz} = \frac{dy}{yz-yx} = \frac{dz}{zx-zy} \leftarrow \text{CYCLIC SYMMETRY}$$

- BY LAGRANGE'S MULTIPLIERS APPROACH

$$\frac{dx+dy+dz}{(xy-xz)+(yz-yz)+(zx-zy)} = \frac{dx}{xy-xz} \leftarrow \text{OR IN FACT ANY OF THE THREE ORIGINAL EXPRESSIONS}$$

$$\frac{dx+dy+dz}{0} = \frac{dx}{xy-xz}$$

TO BE "MEANINGFUL" THE RATIO MUST BE $\frac{0}{0}$

$$dx+dy+dz=0$$

$$\Rightarrow \boxed{x+y+z=C}$$

- NEED ANOTHER RELATIONSHIP

$$\Rightarrow \frac{y\,dx+x\,dy}{y(xy-xz)+x(yz-yx)} = \frac{dz}{zx-zy}$$

$$\Rightarrow \frac{y\,dx+x\,dy}{xy^2-\cancel{xyz}+\cancel{xyz}-x^2y} = \frac{dz}{z(x-y)}$$

$$\Rightarrow \frac{y\,dx+x\,dy}{xy(y-x)} = \frac{dz}{z(x-y)}$$

$$\Rightarrow \frac{y\,dx+x\,dy}{xy} = \frac{dz}{-}$$

$$\Rightarrow \frac{dx}{x}+\frac{dy}{y} = \frac{dz}{-z}$$

$$\Rightarrow \ln x + \ln y = -\ln z + D$$

$$\Rightarrow \ln(xy) = \ln\left(\frac{E}{z}\right)$$

$$\boxed{xyz=E}$$

- THUS $u(x,y,z) = x+y+z$, $v(x,y,z) = xyz$ $\Rightarrow$ GENERAL SOLUTION IS $F(u,v)=0$

I.E $u=f(v)$ OR $v=g(u)$

$$\Rightarrow x+y+z = f(xyz)$$

Question 13

It is given that $z = z(x, y)$ satisfies the partial differential equation

$$x\left(y^2 - z^2\right)\frac{\partial z}{\partial x} + y\left(z^2 - x^2\right)\frac{\partial z}{\partial y} = z\left(x^2 - y^2\right).$$

Use Lagrange's Multipliers method for derivatives, to find the general solution of the above partial differential equation.

$$\boxed{xyz = f\left(x^2 + y^2 + z^2\right) \quad \text{or} \quad x^2 + y^2 + z^2 = g\left(xyz\right)}$$

$$x(y^2-z^2)\frac{\partial z}{\partial x} + y(z^2-x^2)\frac{\partial z}{\partial y} = z(x^2-y^2)$$

- BY LAGRANGE'S METHOD, THE ASSOCIATED O.D.Es ARE

$$\Rightarrow \frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)}$$

$$\Rightarrow \boxed{\frac{dx}{xy^2-xz^2} = \frac{dy}{yz^2-yx^2} = \frac{dz}{zx^2-zy^2}}$$

$$\Rightarrow \frac{x\,dx}{x^2y^2-x^2z^2} = \frac{y\,dy}{y^2z^2-y^2x^2} = \frac{z\,dz}{z^2x^2-z^2y^2}$$

- THUS

$$\frac{x\,dx + y\,dy + z\,dz}{x^2y^2-x^2z^2+y^2z^2-y^2x^2+z^2x^2-z^2y^2} = \frac{z\,dz}{z^2x^2-z^2y^2}$$

$$\frac{x\,dx + y\,dy + z\,dz}{0} = \frac{z\,dz}{z^2x^2-z^2y^2}$$

- FOR THE RATIO TO BE MEANINGFUL THE NUMERATOR MUST ALSO BE ZERO

THUS

$$x\,dx + y\,dy + z\,dz = 0$$

$$x^2 + y^2 + z^2 = C_1$$

- NEXT STARTING AGAIN FROM THE EXPRESSION IN RED BOX

$$\frac{\frac{1}{x}dx}{y^2-z^2} = \frac{\frac{1}{y}dy}{z^2-x^2} = \frac{\frac{1}{z}dz}{x^2-y^2}$$

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y^2-z^2+z^2-x^2+x^2-y^2} = \frac{\frac{1}{z}dz}{x^2-y^2}$$

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0} = \frac{\frac{1}{z}dz}{x^2-y^2}$$

- AS BEFORE FOR A MEANINGFUL RATIO

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

$$\ln x + \ln y + \ln z = C$$

$$\ln(xyz) = C$$

$$xyz = C_2$$

- GENERAL SOLUTION

$F(u, v) = 0$ WHERE $u(x,y,z) = x^2+y^2+z^2$, $v(x,y,z) = xyz$

THUS $xyz = f(x^2+y^2+z^2)$

Question 14

The surface S has Cartesian equation

$$z = f(x, y).$$

The tangent plane at any point on S passes through the point $(0,0,-1)$.

a) Show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - z = 1.$$

b) Hence find the general expression for an equation for S.

$$\boxed{z = -1 + xG\left(\frac{x}{y}\right)}$$

a) $z = f(x,y)$

- $g(x,y,z) = f(x,y) - z$
- $\nabla g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1\right)$
- NORMAL AT A GENERAL POINT $(X,Y,Z) = \left[\left.\frac{\partial f}{\partial x}\right|_{(X,Y)}, \left.\frac{\partial f}{\partial y}\right|_{(X,Y)}, -1\right]$
- EQUATION OF THE TANGENT PLANE
$$z - Z = \left.\frac{\partial f}{\partial x}\right|_{(X,Y)}(x - X) + \left.\frac{\partial f}{\partial y}\right|_{(X,Y)}(y - Y)$$
- BUT THE TANGENT PLANE PASSES THROUGH $(x,y,z) = (0,0,-1)$
$$-1 - Z = \left.\frac{\partial f}{\partial x}\right|_{(X,Y)}(-X) + \left.\frac{\partial f}{\partial y}\right|_{(X,Y)}(-Y)$$
- DROPPING "SUBSCRIPTS" & NOTING $z = f(x,y)$
$$-1 - z = \frac{\partial z}{\partial x}(-x) + \frac{\partial z}{\partial y}(-y)$$
$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - z = 1$$
AS REQUIRED

b) TO SOLVE THE P.D.E USE LAGRANGE'S METHOD

REWRITE AS
$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 1 + z$$

ASSOCIATED O.D.Es ARE
$$\underset{①}{\frac{dx}{x}} = \underset{②}{\frac{dy}{y}} = \underset{③}{\frac{dz}{1+z}}$$

① & ② $\frac{dx}{x} = \frac{dy}{y} \Rightarrow \ln x = \ln y + \ln k$
$\Rightarrow x = ky$
$\Rightarrow \boxed{\frac{x}{y} = C_1}$

① & ③ $\frac{dx}{x} = \frac{dz}{1+z} \Rightarrow \ln x = \ln|1+z| + \ln C$
$\Rightarrow x = C(1+z)$
$\Rightarrow \boxed{\frac{z+1}{x} = C_2}$

THUS $\left.\begin{array}{l} u(x,y,z) = \left(\frac{x}{y}\right) \\ v(x,y,z) = \frac{z+1}{x} \end{array}\right\}$ GEN SOLUTION IS $F(u,v) = 0$

SO SOLUTION CAN BE WRITTEN AS
$$\frac{z+1}{x} = G\left(\frac{x}{y}\right)$$
$$z + 1 = xG\left(\frac{x}{y}\right)$$
$$z = -1 + xG\left(\frac{x}{y}\right)$$

FIRST ORDER P.D.E.s

(Boundary Value Problems)

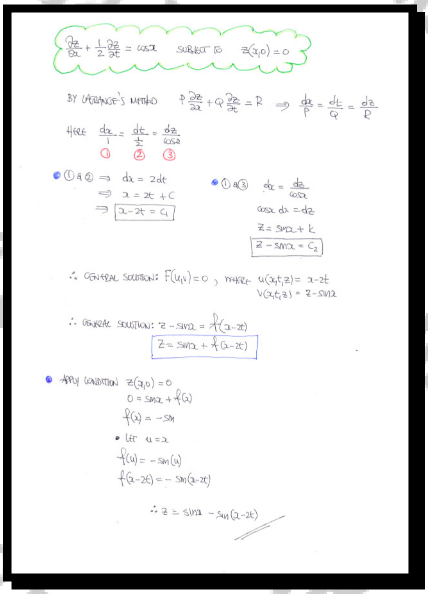
$$P\left(x, y, z\right)\frac{\partial z}{\partial x}+Q\left(x, y, z\right)\frac{\partial z}{\partial y}=R\left(x, y, z\right), \quad z=z\left(x, y\right)$$

Question 1

$$\frac{\partial z}{\partial x}+\frac{1}{2}\frac{\partial z}{\partial t}=\cos x \ .$$

Solve the above partial differential equation given that $z=z(x,t)$ and further satisfies the initial condition $z(x,0)=0$.

$$\boxed{z(x,y)=\sin x-\sin(x-2t)}$$



$\frac{\partial z}{\partial x}+\frac{1}{2}\frac{\partial z}{\partial t}=\cos x$ SUBJECT TO $z(x,0)=0$

BY LAGRANGE'S METHOD $P\frac{\partial z}{\partial x}+Q\frac{\partial z}{\partial t}=R \Rightarrow \frac{dx}{P}=\frac{dt}{Q}=\frac{dz}{R}$

HERE $\frac{dx}{1}=\frac{dt}{\frac{1}{2}}=\frac{dz}{\cos x}$

① ② ③

- ① & ② $\Rightarrow dx=2dt$
  $\Rightarrow x=2t+C$
  $\Rightarrow \boxed{x-2t=C_1}$

- ① & ③ $dx=\frac{dz}{\cos x}$
  $\cos x\,dx=dz$
  $z=\sin x+k$
  $\boxed{z-\sin x=C_2}$

$\therefore$ GENERAL SOLUTION: $F(u,v)=0$, WHERE $u(x,t,z)=x-2t$
$v(x,t,z)=z-\sin x$

$\therefore$ GENERAL SOLUTION: $z-\sin x=f(x-2t)$
$\boxed{z=\sin x+f(x-2t)}$

- APPLY CONDITION $z(x,0)=0$
  $0=\sin x+f(x)$
  $f(x)=-\sin$
  - LET $u=x$
  $f(u)=-\sin(u)$
  $f(x-2t)=-\sin(x-2t)$

$\therefore z=\sin x-\sin(x-2t)$

Question 2

It is given that $z = z(x, y)$ satisfies the partial differential equation

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 2z(x+y).$$

a) Use the transformation equations

$$u = x + y \quad \text{and} \quad v = x - y,$$

to find a general solution for the above partial differential equation.

b) Given further that when $z(x, y) = x^2$ at $x + y = 1$, find the value of $z(1,0)$.

$\boxed{}$, $\boxed{z(x,y) = g(x-y)e^{\frac{1}{2}(x+y)^2}}$, $\boxed{z(1,0) = 1}$

a) START BY PREPARING THE DERIVATIVES BY THE CHAIN RULE

$u = x+y \qquad v = x-y$

- $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}\cdot 1 + \frac{\partial z}{\partial v}\cdot 1 = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$
- $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u}\cdot 1 + \frac{\partial z}{\partial v}(-1) = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$

THE P.D.E NOW BECOMES

$\Rightarrow \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 2z(x+y)$

$\Rightarrow \left(\frac{\partial z}{\partial u} + \cancel{\frac{\partial z}{\partial v}}\right) + \left(\frac{\partial z}{\partial u} - \cancel{\frac{\partial z}{\partial v}}\right) = 2zu$

$\Rightarrow 2\frac{\partial z}{\partial u} = 2zu$

$\Rightarrow \frac{\partial z}{\partial u} = zu$

SOLVE BY SEPARATING VARIABLES – v IS TREATED AS A CONSTANT

$\Rightarrow \frac{1}{z}\,\partial z = u\,\partial u$

$\Rightarrow \ln|z| = \frac{1}{2}u^2 + A(v)$

$\Rightarrow z = e^{\frac{1}{2}u^2 + A(v)} = e^{\frac{1}{2}u^2} \times e^{A(v)} = B(v)e^{\frac{1}{2}u^2}$

$\Rightarrow z(x,y) = f(x-y)\,e^{\frac{1}{2}(x+y)^2}$

b) APPLYING THE BOUNDARY CONDITION

WHEN $x+y=1 \qquad z(x,y) = x^2$

$\Rightarrow z(x,y) = f(x-y)\,e^{\frac{1}{2}(x+y)^2}$

$\Rightarrow x^2 = f[x-(1-x)]\,e^{\frac{1}{2}(1)^2}$

$\Rightarrow x^2 = f(2x-1)\,e^{\frac{1}{2}}$

NOW LET $w = 2x-1 \iff x = \frac{1}{2}(w+1)$

$\Rightarrow \frac{1}{4}(w+1)^2 = f(w)\,e^{\frac{1}{2}}$

$\Rightarrow f(w) = \frac{1}{4}e^{-\frac{1}{2}}(w+1)^2$

$\Rightarrow f(x-y) = \frac{1}{4}e^{-\frac{1}{2}}(x-y+1)^2$

HENCE THE SPECIFIC SOLUTION IS

$$\boxed{z(x,y) = \frac{1}{4}(x-y+1)^2 e^{-\frac{1}{2}} \times e^{\frac{1}{2}(x+y)^2}}$$

$\therefore z(1,0) = \frac{1}{4}(1-0+1)^2 e^{-\frac{1}{2}} \times e^{\frac{1}{2}(1+0)^2}$

$z(1,0) = e^{-\frac{1}{2}}\,e^{\frac{1}{2}}$

$z(1,0) = 1$

Question 3

It is given that $z = z(x, y)$ satisfies the partial differential equation

$$2\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} = z .$$

Given further that $z = y$ at $x = 1$ for all y, find the solution of the above partial differential equation.

$\boxed{}$, $\boxed{z(x, y) = \frac{1}{2}(3 - 3x + 2y)\mathrm{e}^{\frac{1}{2}(x-1)}}$

SOLVE THE P.D.E. BY "LAGRANGE'S METHOD"

$$\underset{P}{2}\frac{\partial z}{\partial x} + \underset{Q}{3}\frac{\partial z}{\partial y} = \underset{R}{z}$$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \implies \underset{①}{\frac{dx}{2}} = \underset{②}{\frac{dy}{3}} = \underset{③}{\frac{dz}{z}}$$

SOLVING ① = ②

$\implies \frac{dx}{2} = \frac{dy}{3}$

$\implies 3dx = 2dy$

$\implies 3x = 2y + C$

$\implies 3x - 2y = C$

$u(x,y,z) = 3x - 2y$

SOLVING ① = ③

$\implies \frac{dx}{2} = \frac{dz}{z}$

$\implies \frac{1}{2}x = \ln z + D$

$\implies \frac{1}{2}x = \ln(Az)$

$\implies e^{\frac{1}{2}x} = Az$

$\implies z = Be^{\frac{1}{2}x}$

$\implies ze^{-\frac{1}{2}x} = B$

$\implies v(x,y,z) = ze^{-\frac{1}{2}x}$

THE GENERAL SOLUTION IS

$F(u,v) = 0$

$u = f(v)$ OR $v = g(u)$

$$\boxed{ze^{-\frac{1}{2}x} = g(3x - 2y)}$$

APPLY THE BOUNDARY CONDITION $z(1,y) = y$

$\implies ye^{-\frac{1}{2}} = g(3 - 2y)$

LET $w = 3 - 2y$
$2y = 3 - w$
$y = \frac{3-w}{2}$

$\implies \left(\frac{3-w}{2}\right)e^{-\frac{1}{2}} = g(w)$

$\implies g(3x - 2y) = \frac{3 - (3x - 2y)}{2}e^{-\frac{1}{2}}$

$\implies g(3x - 2y) = \frac{1}{2}e^{-\frac{1}{2}}(3 - 3x + 2y)$

FINALLY WE OBTAIN

$\implies ze^{-\frac{1}{2}x} = g(3x - 2y)$

$\implies ze^{-\frac{1}{2}x} = \frac{1}{2}e^{-\frac{1}{2}}(3 - 3x + 2y)$

$\implies z = \frac{1}{2}e^{\frac{1}{2}x}e^{-\frac{1}{2}}(3 - 3x + 2y)$

$\implies z(x,y) = \frac{1}{2}e^{\frac{1}{2}(x-1)}(3 - 3x + 2y)$

Question 4

It is given that $z = z(x, y)$ satisfies the partial differential equation

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z \, .$$

Given further that $z(x,0) = \cos x$, find the solution of the above partial differential equation.

$$\boxed{z(x,y) = \mathrm{e}^{y} \cos(x - y)}$$

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z \quad \text{SUBJECT TO} \quad z(x,0) = \cos x$$

BY LAGRANGES METHOD

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{z}$$

- $dx = dy$

$x = y + C$

$\boxed{x - y = C_1}$

$dx = \frac{dz}{z}$

$x = \ln z + C$

$\ln z = x + C$

$z = e^{x+C} = e^{x} \times e^{C}$

$z = C_2 e^{x}$

$\boxed{\frac{z}{e^{x}} = C_2}$

$\therefore \left. \begin{array}{l} u(x,y,z) = x - y \\ v(x,y,z) = \frac{z}{e^{x}} \end{array} \right\}$ GEN SOLUTION IS $F(u,v) = 0$

I.E $v = G(u)$

$\frac{z}{e^{x}} = G(x-y)$

$z = e^{x} G(x-y)$

NOW CONDITIONS

$z(x,0) = \cos x \implies \cos x = e^{x} G(x)$

$G(x) = e^{-x} \cos(x) \quad \forall x$

SAY $G(u) = e^{-u} \cos(u)$

LET $u = x - y$

$z(x,y) = e^{x} e^{-(x-y)} \cos(x-y)$

$z(x,y) = e^{y} \cos(x-y)$

Question 5

The surface S, with equation $z = z(x,y)$, satisfies the partial differential equation

$$x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} + z^2 = 0 .$$

The plane with equation $z = 1$ meets S on the curve with equation $xy = x + y$.

Find a Cartesian equation of S, in the form $z = f(x,y)$.

$\boxed{}$, $\boxed{z = \frac{1}{2}(3 - 3x + 2y)e^{\frac{1}{2}(x-1)}}$

$$x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} + z^2 = 0, \quad z = 1 \text{ AT } xy = x + y$$

BY LAGRANGE'S METHOD

$$\Rightarrow x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} = -z^2$$

$P(x,y,z)$ $Q(x,y,z)$ $R(x,y,z)$

$$\Rightarrow \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{-z^2}$$

SOLVING TWO SIMPLE O.D.Es

$$\Rightarrow \frac{dx}{x^2} = \frac{dy}{y^2}$$
$$\Rightarrow -\frac{1}{x} = -\frac{1}{y} + C$$
$$\Rightarrow -\frac{1}{x} + \frac{1}{y} = C$$
$$\Rightarrow \frac{1}{x} - \frac{1}{y} = C$$

$u(x,y,z) = C$

$$\Rightarrow \frac{dx}{x^2} = -\frac{dz}{z^2}$$
$$\Rightarrow -\frac{1}{x} = \frac{1}{z} + k$$
$$\Rightarrow \frac{1}{z} + \frac{1}{x} = k$$

$v(x,y,z) = k$

SETTING UP A GENERAL SOLUTION

$F(u,v) = 0$ OR $v = f(u)$ OR $u = g(v)$

$$\frac{1}{z} + \frac{1}{x} = f\left(\frac{1}{x} - \frac{1}{y}\right)$$

$$\frac{1}{z} = -\frac{1}{x} + f\left(\frac{1}{x} - \frac{1}{y}\right)$$

APPLY THE BOUNDARY CONDITION $z=1$ AT $xy = x+y$

$$\Rightarrow xy = x + y$$
$$\Rightarrow 1 = \frac{x+y}{xy}$$
$$\Rightarrow 1 = \frac{x}{xy} + \frac{y}{xy}$$
$$\Rightarrow 1 = \frac{1}{y} + \frac{1}{x}$$
$$\Rightarrow -\frac{1}{y} = \frac{1}{x} - 1$$

$$\Rightarrow \frac{1}{z} = -\frac{1}{x} + f\left(\frac{1}{x} - \frac{1}{y}\right)$$
$$\Rightarrow \frac{1}{1} = -\frac{1}{x} + f\left(\frac{1}{x} + \frac{1}{x} - 1\right)$$
$$\Rightarrow 1 = -\frac{1}{x} + f\left(\frac{2}{x} - 1\right)$$

LET $u = \frac{2}{x} - 1$

$u + 1 = \frac{2}{x}$

$x = \frac{2}{u+1}$

$\frac{1}{x} = \frac{u+1}{2}$

$-\frac{1}{x} = -\frac{u+1}{2}$

$$\Rightarrow 1 = -\frac{u+1}{2} + f(u)$$
$$\Rightarrow 1 + \frac{u+1}{2} = f(u)$$
$$\Rightarrow \frac{u+3}{2} = f(u)$$
$$\Rightarrow f(u) = \frac{1}{2}u + \frac{3}{2}$$
$$\Rightarrow f\left(\frac{1}{x} - \frac{1}{y}\right) = \frac{1}{2}\left(\frac{1}{x} - \frac{1}{y}\right) + \frac{3}{2}$$

FINALLY WE OBTAIN

$$\Rightarrow \frac{1}{z} = -\frac{1}{x} + \frac{1}{2}\left(\frac{1}{x} - \frac{1}{y}\right) + \frac{3}{2} = -\frac{1}{x} + \frac{1}{2x} - \frac{1}{2y} + \frac{3}{2}$$
$$\Rightarrow \frac{1}{z} = \frac{3}{2} - \frac{1}{2x} - \frac{1}{2y}$$
$$\Rightarrow \frac{1}{z} = \frac{3xy - y - x}{2xy}$$

$$\therefore z = \frac{2xy}{3xy - x - y}$$

Question 6

The surface S, with equation $z = z(x, y)$, satisfies the partial differential equation

$$xz\frac{\partial z}{\partial x} + yz\frac{\partial z}{\partial y} + xy = 0 .$$

S contains the curve with equation

$$xy = 1, \; z = x, \; \forall x .$$

Find a Cartesian equation of S, in the form $z = f(x, y)$.

$\boxed{}$, $\boxed{z(x, y) = \frac{x}{y} - xy + 1}$

REWRITE THE P.D.E & SOLVE BY "LAGRANGE'S METHOD"

$$xz\frac{\partial z}{\partial x} + yz\frac{\partial z}{\partial y} = -xy$$

$$\underset{P}{xz}, \quad \underset{Q}{yz}, \quad \underset{R}{-xy}$$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \implies \underset{①}{\frac{dx}{xz}} = \underset{②}{\frac{dy}{yz}} = \underset{③}{\frac{dz}{-xy}}$$

SOLVING ① = ②

$$\implies \frac{dx}{xz} = \frac{dy}{yz}$$
$$\implies \frac{dx}{x} = \frac{dy}{y}$$
$$\implies \ln x = \ln y + \ln A$$
$$\implies \ln x = \ln(Ay)$$
$$\implies x = Ay$$
$$\implies \frac{y}{x} = C_1$$

SOLVING ① = ③

$$\implies \frac{dx}{xz} = \frac{dz}{-xy}$$
$$\implies \frac{dx}{z} = -\frac{dz}{y}$$
$$\implies y\,dx = -z\,dz$$
$$\implies C_1 x\,dx = -z\,dz$$
$$\implies \frac{1}{2}C_1x^2 + C_2 = -\frac{1}{2}z^2$$
$$\implies \frac{1}{2}\left(\frac{y}{x}\right)x^2 + C_2 = -\frac{1}{2}z^2$$
$$\implies \frac{1}{2}xy + C_2 = -\frac{1}{2}z^2$$
$$\implies -xy + C_2 = z^2$$
$$\implies z^2 + xy = C_2$$

THE GENERAL SOLUTION IS GIVEN BY

$$F(u,v) = 0 \quad \text{WHERE} \quad u(x,y,z) = \frac{y}{x}, \quad v(x,y,z) = z^2 + xy$$

$$\therefore \; u = f(v) \quad \text{OR} \quad v = g(u)$$

$$\implies z^2 + xy = f\left(\frac{y}{x}\right)$$
$$\implies z^2 = f\left(\frac{y}{x}\right) - xy$$

APPLY BOUNDARY CONDITIONS NEXT, $xy = 1$ AT $z = x \; \forall x$,

$$\implies z^2 = f\left(\frac{y}{x}\right) - xy$$
$$\implies x^2 = f\left(\frac{1/x}{x}\right) - 1$$
$$\implies x^2 + 1 = f\left(\frac{1}{x^2}\right)$$

LET $u = \frac{1}{x^2}$, $x^2 = \frac{1}{u}$

$$\implies \frac{1}{u} + 1 = f(u)$$
$$\implies f(u) = \frac{1}{u} + 1$$
$$\implies f\left(\frac{y}{x}\right) = \frac{1}{y/x} + 1$$
$$\implies f\left(\frac{y}{x}\right) = \frac{x}{y} + 1$$

HENCE WE NOW HAVE

$$z^2 = \frac{x}{y} + 1 - xy$$

Question 7

The surface S, with equation $z = z(x,y)$, satisfies the partial differential equation

$$\left(x^2+1\right)\frac{\partial z}{\partial x} + 2xy\frac{\partial z}{\partial y} - xy = 0 .$$

S contains the curve with equation

$$z(x,1) = \left(x^2+1\right)^2, \quad \tfrac{1}{2} \le x \le \tfrac{2}{3} .$$

Find a Cartesian equation of S, in the form $z^2 = f(x,y)$.

$$\boxed{z^2 = \frac{y^5}{\left(x^2+1\right)^4}}$$

$$(x^2+1)\frac{\partial z}{\partial x} + 2xy\frac{\partial z}{\partial y} - xz = 0$$

$$(x^2+1)\frac{\partial z}{\partial x} + 2xy\frac{\partial z}{\partial y} = xz$$

- BY LAGRANGE'S METHOD, THE ASSOCIATED O.D.Es ARE

$$\frac{dx}{x^2+1} = \frac{dy}{2xy} = \frac{dz}{xz}$$

- $\dfrac{dy}{2xy} = \dfrac{dz}{xz}$

$\Rightarrow \dfrac{dy}{2y} = \dfrac{dz}{z}$

$\Rightarrow \ln y = \ln z + C$

$\Rightarrow \ln y = 2\ln z + C$

$\Rightarrow \ln y = \ln z^2 + \ln A$

$\Rightarrow \ln y = \ln(Az^2)$

$\Rightarrow y = Az^2$

$\Rightarrow \dfrac{y}{z^2} = C_1$

$\Rightarrow u(x,y,z) = \dfrac{y}{z^2}$

- $\dfrac{dx}{x^2+1} = \dfrac{dy}{xy}$

$\Rightarrow \dfrac{2x}{x^2+1}\,dx = \dfrac{1}{y}\,dy$

$\Rightarrow \ln[x^2+1] = \ln|y| + \ln A$

$\Rightarrow \ln(x^2+1) = \ln|Ay|$

$\Rightarrow x^2+1 = Ay$

$\Rightarrow \dfrac{y}{x^2+1} = C_2$

$\Rightarrow v(x,y,z) = \dfrac{y}{x^2+1}$

- GENERAL SOLUTION

$F(u,v) = 0$

$\dfrac{y}{z^2} = f\left(\dfrac{y}{x^2+1}\right)$ OR $\dfrac{y}{x^2+1} = g\left(\dfrac{y}{z^2}\right)$

- NOW USING $\dfrac{y}{z^2} = f\left(\dfrac{y}{x^2+1}\right)$

$z = (x^2+1)^2$ AT $y=1$ $\tfrac{1}{2} \le x \le \tfrac{2}{3}$

- $\dfrac{1}{(x^2+1)^4} = f\left(\dfrac{1}{x^2+1}\right)$ .... $\tfrac{1}{2} \le x \le \tfrac{2}{3}$

LET $u = \dfrac{1}{x^2+1} \Rightarrow x^2+1 = \dfrac{1}{u}$ ... $\tfrac{9}{13} \le u \le \tfrac{4}{5}$

$f(u) = \dfrac{1}{u^4}$ .... $\tfrac{9}{13} \le u \le \tfrac{4}{5}$

- LET $u = \dfrac{y}{x^2+1}$

$f\left(\dfrac{y}{x^2+1}\right) = \dfrac{1}{\frac{y^4}{(x^2+1)^4}} = \dfrac{(x^2+1)^4}{y^4}$ .... $\tfrac{1}{2} \le x \le \tfrac{2}{3}$

THUS $\dfrac{y}{z^2} = \dfrac{(x^2+1)^4}{y^4}$ .... $\tfrac{1}{2} \le x \le \tfrac{2}{3}$

$z^2(x^2+1)^4 = y^5$ .... $\tfrac{1}{2} \le x \le \tfrac{2}{3}$

$z^2 = \dfrac{y^5}{(x^2+1)^4}$ .... $\tfrac{1}{2} \le x \le \tfrac{2}{3}$

Question 8

It is given that $z = z(x, y)$ satisfies the partial differential equation

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 2z(x + y).$$

Given further that when $z(x, y) = x^2$ at $x + y = 1$, find the solution of the above partial differential equation.

$$\boxed{z(x, y) = \tfrac{1}{4}(x - y + 1)^2 \exp\left[\tfrac{1}{2}(x + y + 1)(x + y - 1)\right]}$$

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 2(x+y)z$$

BY LAGRANGE'S METHOD, THE ASSOCIATED O.D.Es ARE

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{2(x+y)z}$$

- $dx = dy$

$$\boxed{y = x + C_1}$$

- $dx = \dfrac{dz}{2(x+y)z}$

$$2(x+y)\,dx = \frac{1}{z}\,dz$$
$$2(x+x+C_1)\,dx = \frac{1}{z}\,dz$$
$$(4x + 2C_1)\,dx = \frac{1}{z}\,dz$$
$$2x^2 + 2C_1x = \ln z + C$$
$$\ln z = 2x^2 + 2C_1x + C$$
$$z = e^{2x^2 + 2C_1x + C}$$
$$z = C_2e^{2x^2 + 2x(y-x)} \quad (C_2 = e^C)$$
$$\boxed{z = C_2e^{2xy}}$$

GENERAL SOLUTION

- $F(u, v) = 0$ WHERE $u(x,y,z) = y - x$, $v(x,y,z) = \dfrac{z}{e^{2xy}}$

$$\therefore \frac{z}{e^{2xy}} = f(y-x)$$

NOW APPLY THE BOUNDARY CONDITION $z(x,y) = x^2$ AT $x + y = 1$

THUS $z(x,y) = e^{2xy}f(y-x)$

$$x^2 = e^{2x(1-x)}f(1-x-x)$$
$$x^2 = e^{2x-2x^2}f(1-2x)$$
$$f(1-2x) = \frac{x^2}{e^{2x-2x^2}}$$

LET $u = 1 - 2x \iff x = \dfrac{1-u}{2}$

$$f(u) = \frac{\left(\frac{1-u}{2}\right)^2}{e^{2\left(\frac{1-u}{2}\right) - 2\left(\frac{1-u}{2}\right)^2}}$$
$$f(u) = \frac{(1-u)^2}{4e^{1-u-\frac{1}{2}(1-2u+u^2)}}$$
$$f(u) = \frac{(1-u)^2}{4e^{\frac{1}{2}-\frac{1}{2}u^2}}$$
$$f(u) = \tfrac{1}{4}(1-u)^2e^{\frac{1}{2}(u^2-1)}$$

LET $u = y - x$

$$f(y-x) = \tfrac{1}{4}(1-y+x)^2e^{\frac{1}{2}(y^2-2xy+x^2-1)}$$
$$\therefore z(x,y) = \tfrac{1}{4}(1-y+x)^2e^{\frac{1}{2}(y^2-2xy+x^2-1)}e^{2xy}$$
$$z(x,y) = \tfrac{1}{4}(x-y+1)^2e^{\frac{1}{2}(y^2+2xy+x^2-1)}$$
$$z(x,y) = \tfrac{1}{4}(x-y+1)^2e^{\frac{1}{2}(x+y)^2-\frac{1}{2}}$$

Question 9

It is given that $z = z(x,t)$ satisfies the partial differential equation

$$e^{x}\frac{\partial z}{\partial x} + \frac{\partial z}{\partial t} = 0, \quad z(x,0) = \tanh x.$$

Find the solution of the above partial differential equation, in the form $z = f(x,t)$.

$\boxed{}$, $\boxed{z(x,y) = -\tanh\left[\ln\left(t + e^{-x}\right)\right]}$

SOLVE THE P.D.E BY "LAGRANGE'S METHOD"

$$\underbrace{e^{x}}_{P}\frac{\partial z}{\partial x} + \underbrace{1}_{Q}\frac{\partial z}{\partial t} = \underbrace{0}_{R}$$

$$\frac{dx}{P} = \frac{dt}{Q} = \frac{dz}{R} \implies \underbrace{\frac{dx}{e^{x}}}_{①} = \underbrace{\frac{dt}{1}}_{②} = \underbrace{\frac{dz}{0}}_{③}$$

EQUATING ① & ② YIELDS

$$\implies \frac{dx}{e^{x}} = \frac{dt}{1}$$

$$\implies \int e^{-x}\,dx = \int 1\,dt$$

$$\implies -e^{-x} = t + C_1$$

$$\implies t + e^{-x} = C \qquad \therefore\ u(x,t) = t + e^{-x}$$

EQUATION ③ INVOLVES DIVISION BY ZERO

$$\implies dz = 0 \qquad \text{(OTHERWISE THE RATIO IS MEANINGLESS)}$$

$$\implies z = C_2$$

$$\therefore\ v(z) = z$$

HENCE WE HAVE A GENERAL SOLUTION $F(u,v) = 0$

$$\therefore\ v = f(u) \qquad \text{or} \qquad u = g(v)$$

$$\therefore\ z = f(t + e^{-x})$$

APPLY THE INITIAL CONDITION, $t=0$ $z = \tanh x$

$$\implies \tanh x = f(e^{-x})$$

LET $w = e^{-x}$

$$\frac{1}{w} = e^{x}$$

$$x = \ln\frac{1}{w}$$

$$x = -\ln w$$

$$\implies \tanh(-\ln w) = f(w)$$

$$\implies f(w) = \tanh(-\ln w)$$

$$\implies f(w) = -\tanh(\ln w)$$

$$\implies f(t+e^{-x}) = -\tanh\left[\ln(t+e^{-x})\right]$$

FINALLY WE OBTAIN

$$z = -\tanh\left[\ln\left(t + e^{-x}\right)\right]$$

Question 10

It is given that $z = z(x, y)$ satisfies the partial differential equation

$$z\frac{\partial z}{\partial x} - z\frac{\partial z}{\partial t} = y - x, \qquad z(1, y) = y^2.$$

Find the solution of the above partial differential equation, in the form $z^2 = f(x, y)$.

$$\boxed{z^2 = 2xy + (x + y - 1)^4 - 2(x + y - 1)}$$

$$\boxed{\underset{P}{z}\frac{\partial z}{\partial x} \underset{Q}{- z}\frac{\partial z}{\partial y} = \underset{R}{y - x}}$$

BY LAGRANGES METHOD THE ASSOCIATED O.D.Es ARE

$\frac{dx}{P} = \frac{dy}{P} = \frac{dz}{Q}$ IE $\frac{dx}{z} = -\frac{dy}{z} = \frac{dz}{y-x}$

- $dx = -dy$
  $x = -y + C$
  $\boxed{x + y = C_1}$

- $\frac{dx}{z} = \frac{dz}{y-x}$
  $(y - x)\,dx = z\,dz$
  $(C_1 - 2x)\,dx = z\,dz$
  $C_1x - x^2 = \frac{1}{2}z^2 + C$
  $2C_1x - 2x^2 = z^2 + C$
  $z^2 - 2C_1x + 2x^2 = C_2$
  $z^2 - 2(x+y)x + 2x^2 = C_2$
  $\boxed{z^2 - 2xy = C_2}$

THE GENERAL SOLUTION IS

$F(u, v) = 0$ WHERE $u(x, y, z) = x + y$
$v(x, y, z) = z^2 - 2xy$

∴ THE GENERAL SOLUTION IS

$z^2 - 2xy = f(x+y)$

- NOW CONDITIONS

$\boxed{z(1, y) = y^2}$

$\Rightarrow z^2 - 2xy = f(x+y)$
$\Rightarrow y^4 - 2y = f(1+y)$

- LET $u = 1 + y \Rightarrow y = u - 1$
  $f(u) = (u-1)^4 - 2(u-1)$

- LET $u = x + y$
  $f(x+y) = (x+y-1)^4 - 2(x+y-1)$

∴ $z^2 = 2xy + (x+y-1)^4 - 2(x+y-1)$

Question 11

The surface S, with equation $z = z(x,y)$, satisfies the partial differential equation

$$x\frac{\partial z}{\partial x} - 3y\frac{\partial z}{\partial y} = x^2 y.$$

a) Use the transformation equations

$$\xi(x,y) = \ln x \quad \text{and} \quad \eta(x,y) = \ln y,$$

to transform the above partial differential equation into one with constant coefficients.

b) Given further that $z(1,y) = y$, find a Cartesian equation of S, giving the answer in the form $z = f(x,y)$.

$$\boxed{\frac{\partial z}{\partial x} - 3\frac{\partial z}{\partial y} = e^{2\xi+\eta}}, \quad \boxed{z(x,y) = 2x^3 y + x^2 y}$$

a) $x\frac{\partial z}{\partial x} - 3y\frac{\partial z}{\partial y} = x^2 y$ $\qquad$ $\xi = \ln x \Rightarrow x = e^{\xi}$, $\eta = \ln y \Rightarrow y = e^{\eta}$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial x} + \cancel{\frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial x}} = \frac{1}{x}\frac{\partial z}{\partial \xi}$$

$$\frac{\partial z}{\partial y} = \cancel{\frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial y}} + \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{1}{y}\frac{\partial z}{\partial \eta}$$

THE P.D.E NOW BECOMES

$$x\left(\frac{1}{x}\frac{\partial z}{\partial \xi}\right) - 3y\left(\frac{1}{y}\frac{\partial z}{\partial \eta}\right) = e^{2\xi}e^{\eta}$$

$$\frac{\partial z}{\partial \xi} - 3\frac{\partial z}{\partial \eta} = e^{2\xi+\eta}$$

b) BY LAGRANGE'S METHOD THE ASSOCIATED O.D.Es ARE

$$\frac{d\xi}{1} = \frac{d\eta}{-3} = \frac{dz}{e^{2\xi+\eta}}$$

- $3\,d\xi = -d\eta$

  $3\xi = -\eta + C$

  $\boxed{3\xi + \eta = C_1}$

  $\boxed{u(\xi,\eta,z) = 3\xi+\eta}$

- $d\xi = \frac{dz}{e^{2\xi+\eta}}$

  $d\xi\, e^{2\xi+\eta} = dz$

  $e^{2\xi+(C_1-3\xi)}\,d\xi = dz$

  $dz = e^{C_1-\xi}\,d\xi$

  $z = -e^{C_1-\xi} + C_2$

  $z + e^{3\xi+\eta-\xi} = C_2$

  $\boxed{z + e^{2\xi+\eta} = C_2}$

  $\boxed{v(\xi,\eta,z) = z + e^{2\xi+\eta}}$

GENERAL SOLUTION IS $F(u,v) = 0$ OR $v = G(u)$

IE $z + e^{2\xi+\eta} = G(3\xi+\eta)$

$z + e^{2\ln x + \ln y} = G(3\ln x + \ln y)$

$z + e^{\ln x^2} \times e^{\ln y} = G[\ln x^3 + \ln y]$

$z + x^2 y = G[\ln(x^3 y)]$

$z + x^2 y = f(x^3 y)$

$\boxed{z = f(x^3 y) - x^2 y}$

NOW APPLY CONDITION $z(1,y) = y$

$y = f(y) - y$

$f(y) = 2y \quad \forall y$

- LET $y = u$

  $f(u) = 2u$

- IN PARTICULAR LET $u = x^3 y \Rightarrow f(x^3 y) = 2x^3 y$

$\therefore z(x,y) = 2x^3 y - x^2 y$

Question 12

It is given that $u = u(x, y)$ satisfies the partial differential equation

$$3y^2 \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} - xy^2 u = 0 .$$

It is further given that when $x = y + y^3$, $u(x) = x\,e^{\frac{1}{6}x^2}$.

Find a simplified expression for $u = u(x, y)$, in the form $u(x, y) = f(x, y)e^{\frac{1}{6}x^2}$, where f is a function to be determined.

$$\boxed{u(x, y) = \left[\left(y^3 - x\right) + \left(y^3 - x\right)^3\right]e^{\frac{1}{6}x^2}}$$

$$\boxed{3y^2 \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} - xy^2 u = 0 \quad \text{SUBJECT TO WHEN } x = y + y^3,\ u(x) = xe^{\frac{1}{6}x^2}}$$

- REWRITE THE P.D.E AS FOLLOWS

$$3\frac{\partial u}{\partial x} + \frac{1}{y^2}\frac{\partial u}{\partial y} = xu$$

- USING LAGRANGE'S METHOD WE OBTAIN

$$\frac{dx}{3} = \frac{dy}{\frac{1}{y^2}} = \frac{du}{xu}$$

- SOLVING THE ABOVE ASSOCIATED O.D.E.S

① = ②

$\Rightarrow \frac{dx}{3} = y^2\,dy$

$\Rightarrow \int \frac{1}{3}\,dx = \int y^2\,dy$

$\Rightarrow \frac{1}{3}x = \frac{1}{3}y^3 + C$

$\Rightarrow x = y^3 + C_1$

$\Rightarrow x - y^3 = C_1$

② = ③

$\Rightarrow y^2\,dy = \frac{du}{xu}$

$\Rightarrow \int xy^2\,dy = \int \frac{1}{u}\,du$

$\Rightarrow \int (y^3 + C_1)y^2\,dy = \int \frac{1}{u}\,du$

$\Rightarrow \int \frac{1}{u}\,du = \int y^5 + C_1 y^2\,dy$

$\Rightarrow \ln u = \frac{1}{6}y^6 + \frac{1}{3}C_1 y^3 + C$

$\Rightarrow \ln u = \frac{1}{6}(x - C_1)^2 + \frac{1}{3}C_1(x - C_1) + C$

$\Rightarrow \ln u = \frac{1}{6}x^2 - \cancel{\frac{1}{3}C_1 x} + \frac{1}{6}C_1^2 + \cancel{\frac{1}{3}C_1 x} - \frac{1}{3}C_1^2 + C$

$\Rightarrow \ln u = \frac{1}{6}x^2 - \frac{1}{6}C_1^2 + C$

$\Rightarrow \ln u = \frac{1}{6}x^2 + C$

$\Rightarrow u = e^{\frac{1}{6}x^2 + C} = e^{\frac{1}{6}x^2} \times e^C$

$\Rightarrow u = C_2 e^{\frac{1}{6}x^2}$

$\Rightarrow \frac{u}{e^{\frac{1}{6}x^2}} = C_2$

- GENERAL SOLUTION IS

$$\frac{u}{e^{\frac{1}{6}x^2}} = f(x - y^3) \quad \text{OR} \quad x - y^3 = g\left(\frac{u}{e^{\frac{1}{6}x^2}}\right)$$

$$\boxed{u(x,y) = f(x - y^3)e^{\frac{1}{6}x^2}}$$

- APPLY CONDITION GIVEN, $u(x) = xe^{\frac{1}{6}x^2}$ WHEN $x = y + y^3$

$xe^{\frac{1}{6}x^2} = f(y + y^3 - y^3)e^{\frac{1}{6}x^2}$

$\Rightarrow f(-y) = x = y + y^3$

LET $v = -y \iff y = -v$

$\Rightarrow f(v) = (-v) + (-v)^3$

$\Rightarrow f(v) = -v - v^3$

HENCE $f(x - y^3) = -(x - y^3) - (x - y^3)^3$

$$\therefore \boxed{u(x,y) = \left[(y^3 - x) + (y^3 - x)^3\right]e^{\frac{1}{6}x^2}}$$

Question 13

The surface S, with equation $z = z(x, y)$, satisfies the partial differential equation

$$2\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} = z \ .$$

It is further given that

$$z(x,0) = \tan 3x \ , \quad 0 \le x \le 1 \ .$$

Find a Cartesian equation of S, in the form $z = f(x, y)$, further describing the relation of S to the x-y plane.

$$\boxed{z(x, y) = \frac{1}{6}\left(5\mathrm{e}^{x}+1\right)\tan(3x-y), \quad 3x-3 \le y \le 3x}$$

$$\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} = 6z + \tan(y-3x)$$

BY LAGRANGE'S METHOD, THE ASSOCIATED O.D.E.S ARE

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{6z+\tan(y-3x)}$$

- $dx = \frac{1}{3}dy$

$$x = \frac{1}{3}y + C$$
$$3x = y + C$$
$$\boxed{y - 3x = C_1}$$

- $dx = \dfrac{dz}{6z+\tan(y-3x)}$

$$dx = \frac{dz}{6z + \tan C_1}$$
$$dx = \frac{dz}{6z + A}$$
$$x = \frac{1}{6}\ln|6z+A| + C$$
$$6x = \ln|6z + \tan C_1| + C$$
$$\boxed{6x - \ln|6z + \tan(y-3x)| = C_2}$$

THE GENERAL SOLUTION IS GIVEN BY

$F(u,v) = 0$ WHERE $u(x,y,z) = y - 3x$
$v(x,y,z) = 6x - \ln|6z + \tan(y-3x)|$

THUS

$$6x - \ln|6z + \tan(y-3x)| = g(y-3x)$$
$$6x + g(y-3x) = \ln|6z + \tan(y-3x)|$$
$$6z + \tan(y-3x) = e^{6x + g(y-3x)}$$
$$6z + \tan(y-3x) = e^{6x}e^{g(y-3x)}$$
$$6z = e^{6x}f(y-3x) - \tan(y-3x)$$
$$\underline{\underline{z = e^{6x}f(y-3x) - \frac{1}{6}\tan(y-3x)}}$$

NOW THE BOUNDARY CONDITION $z(x,0) = \tan 3x$, $0 \le x \le 1$

$\tan 3x = e^{6x}f(-3x) - \frac{1}{6}\tan(-3x)$ .... $0 \le x \le 1$

$\tan 3x = e^{6x}f(-3x) + \frac{1}{6}\tan 3x$ .... $0 \le x \le 1$

$\frac{5}{6}\tan 3x = e^{6x}f(-3x)$ .... $0 \le x \le 1$

$f(-3x) = \left(\frac{5}{6}\tan 3x\right)e^{-6x}$ .... $0 \le x \le 1$

LET $u = -3x$
$x = -\frac{1}{3}u$ $\quad -3 \le u \le 0$

$f(u) = \frac{5}{6}e^{-6(-\frac{1}{3}u)}\tan(-u)$ .... $-3 \le u \le 0$

$\boxed{f(u) = -\frac{5}{6}e^{2u}\tan u}$ .... $-3 \le u \le 0$

$\therefore f(y-3x) = -\frac{5}{6}e^{2(y-3x)}\tan(y-3x)$ .... $-3 \le y-3x \le 0$

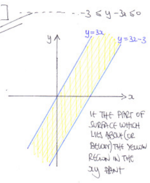
$\therefore z(x,y) = -\frac{5}{6}e^{6x}e^{2(y-3x)}\tan(y-3x) - \frac{1}{6}\tan(y-3x)$ ... $-3 \le y-3x \le 0$

$z(x,y) = -\frac{1}{6}\tan(y-3x)\left[5e^{2y}+1\right]$ .... $-3 \le y-3x \le 0$

(OR $z(x,y) = \frac{1}{6}(5e^{2y}+1)\tan(3x-y)$)

& $y - 3x \le 0$
$y \le 3x$

& $-3 \le y - 3x$
$-y \le -3x + 3$
$y \ge 3x - 3$



IE THE PART OF SURFACE WHICH LIES ABOVE (OR BELOW) THE YELLOW REGION IN THE xy PLANE

Question 14

The function f, with equation $z = f(x,y)$, satisfies the partial differential equation

$$(y-z)\frac{\partial z}{\partial x} + (z-x)\frac{\partial z}{\partial y} = x - y .$$

It is further given that

$$f(x,y) = 0 \text{, when } y = 2x .$$

Find a Cartesian equation of f, giving the answer in the form $z = f(x,y)$.

$$\boxed{z(x,y) = -x - y + \tfrac{3}{5}\sqrt{5x^2 + 5y^2 + 5z^2}}$$

$$(y-z)\frac{\partial z}{\partial x} + (z-x)\frac{\partial z}{\partial y} = x-y$$

- BY LAGRANGE'S METHOD, THE ASSOCIATED O.D.Es ARE

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y}$$

- ADDING THEM GIVES: $\dfrac{dx+dy+dz}{0} = \dots$ "ANY OF THE THREE RATIOS"

- FOR THE RATIO TO BE MEANINGFUL, THE NUMERATOR MUST BE ZERO

$dx+dy+dz=0$, AND BY INTEGRATING WE OBTAIN

$$\boxed{x+y+z = C_1}$$

- BY A SIMILAR APPROACH

$$\frac{x\,dx}{xy-xz} = \frac{y\,dy}{yz-xy} = \frac{z\,dz}{xz-yz}$$

- ADDING THEM: $\dfrac{x\,dx+y\,dy+z\,dz}{0} =$ "ANY OF THE 3 RATIOS"

- FOR THE RATIO TO BE MEANINGFUL, THE NUMERATOR MUST BE ZERO

$$\Rightarrow x\,dx + y\,dy + z\,dz = 0$$

$$\Rightarrow \tfrac{1}{2}x^2 + \tfrac{1}{2}y^2 + \tfrac{1}{2}z^2 = \text{constant}$$

$$\Rightarrow \boxed{x^2+y^2+z^2 = C_2}$$

THE GENERAL SOLUTION IS $F(u,v)=0$ WHERE $u(x,y,z) = x+y+z$

$v(x,y,z) = x^2+y^2+z^2$

$\therefore u = f(v)$ OR $v = g(u)$

I.E. $x+y+z = f(x^2+y^2+z^2)$

$x^2+y^2+z^2 = g(x+y+z)$

I.E. $z = f(x^2+y^2+z^2) - x - y$

OR $z^2 = g(x+y+z) - x^2 - y^2$

- APPLY BOUNDARY CONDITION

$$z(x,2x) = 0$$

$$\Rightarrow 0 = f(x^2+4x^2+0) - x - 2x$$

$$\Rightarrow f(5x^2) = 3x$$

LET $u = 5x^2$

$x^2 = \tfrac{1}{5}u$

$x = \sqrt{\tfrac{1}{5}u}$

$$\Rightarrow f(u) = 3\left(\tfrac{1}{5}u\right)^{\frac{1}{2}}$$

$$\Rightarrow f(u) = \frac{3}{\sqrt{5}}u^{\frac{1}{2}}$$

$$\Rightarrow f(u) = \tfrac{3}{5}\sqrt{5u}$$

$$\therefore f(x^2+y^2+z^2) = \tfrac{3}{5}\sqrt{5(x^2+y^2+z^2)}$$

$$f(x^2+y^2+z^2) = \tfrac{3}{5}\sqrt{5x^2+5y^2+5z^2}$$

HENCE

$$z = f(x^2+y^2+z^2) - x - y$$

$$z = \tfrac{3}{5}\sqrt{5x^2+5y^2+5z^2} - x - y$$

Question 15

The surface S, with equation $z = z(x, y)$, satisfies the partial differential equation

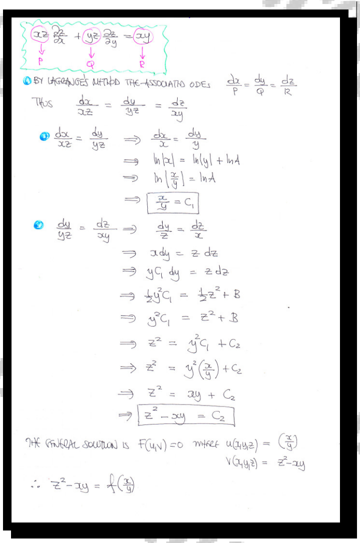
$$xz\frac{\partial z}{\partial x} + yz\frac{\partial z}{\partial y} = xy\,.$$

The plane with equation $z = 1$ intersects S along the curve with equation

$$y = 2x^2\,,\ -1 < x < 1\,.$$

Determine a Cartesian equation of S, giving the answer in the form $z^2 = f(x, y)$, sketching the projection of S on the x-y plane.

$$\boxed{z(x,y) = xy + 1 - \frac{y^3}{4x^3}}$$



$$xz\frac{\partial z}{\partial x} + yz\frac{\partial z}{\partial y} = xy$$

$P \quad Q \quad R$

BY LAGRANGE'S METHOD THE ASSOCIATED ODEs $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

THUS $\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy}$

$\frac{dx}{xz} = \frac{dy}{yz} \Rightarrow \frac{dx}{x} = \frac{dy}{y}$

$\Rightarrow \ln|x| = \ln|y| + \ln A$

$\Rightarrow \ln\left|\frac{x}{y}\right| = \ln A$

$\Rightarrow \boxed{\frac{x}{y} = C_1}$

$\frac{dy}{yz} = \frac{dz}{xy} \Rightarrow \frac{dy}{z} = \frac{dz}{x}$

$\Rightarrow x\,dy = z\,dz$

$\Rightarrow yC_1\,dy = z\,dz$

$\Rightarrow \frac{1}{2}y^2C_1 = \frac{1}{2}z^2 + B$

$\Rightarrow y^2C_1 = z^2 + B$

$\Rightarrow z^2 = y^2C_1 + C_2$

$\Rightarrow z^2 = y^2\left(\frac{x}{y}\right) + C_2$

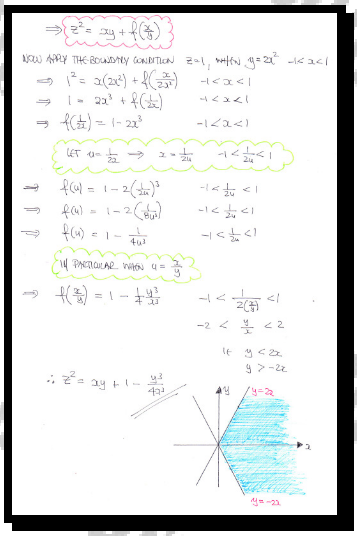
$\Rightarrow z^2 = xy + C_2$

$\Rightarrow \boxed{z^2 - xy = C_2}$

THE GENERAL SOLUTION IS $F(u,v) = 0$ WHERE $u(x,y,z) = \left(\frac{x}{y}\right)$

$v(x,y,z) = z^2 - xy$

$\therefore z^2 - xy = f\left(\frac{x}{y}\right)$



$\Rightarrow z^2 = xy + f\left(\frac{x}{y}\right)$

NOW APPLY THE BOUNDARY CONDITION $z = 1$, WHEN $y = 2x^2 \quad -1 < x < 1$

$\Rightarrow 1^2 = x(2x^2) + f\left(\frac{x}{2x^2}\right) \qquad -1 < x < 1$

$\Rightarrow 1 = 2x^3 + f\left(\frac{1}{2x}\right) \qquad -1 < x < 1$

$\Rightarrow f\left(\frac{1}{2x}\right) = 1 - 2x^3 \qquad -1 < x < 1$

LET $u = \frac{1}{2x} \Rightarrow x = \frac{1}{2u} \qquad -1 < \frac{1}{2u} < 1$

$\Rightarrow f(u) = 1 - 2\left(\frac{1}{2u}\right)^3 \qquad -1 < \frac{1}{2u} < 1$

$\Rightarrow f(u) = 1 - 2\left(\frac{1}{8u^3}\right) \qquad -1 < \frac{1}{2u} < 1$

$\Rightarrow f(u) = 1 - \frac{1}{4u^3} \qquad -1 < \frac{1}{2u} < 1$

IN PARTICULAR WHEN $u = \frac{x}{y}$

$\Rightarrow f\left(\frac{x}{y}\right) = 1 - \frac{1}{4}\frac{y^3}{x^3} \qquad -1 < \frac{1}{2\left(\frac{x}{y}\right)} < 1$

$-2 < \frac{y}{x} < 2$

IE $y < 2x$

$y > -2x$

$\therefore z^2 = xy + 1 - \frac{y^3}{4x^3}$

Question 16

The surface S, with equation $z = z(x, y)$, satisfies the partial differential equation

$$yz\frac{\partial z}{\partial x} - xz\frac{\partial z}{\partial y} = xy\,.$$

a) Find a general solution of the partial differential equation.

The plane with equation $y = 0$ intersects S along the curve with equation

$$z = \sin x\,,\ 1 < x < 2\,.$$

b) Find a Cartesian equation of S, giving the answer in the form $z^2 = f(x, y)$, sketching the projection of S on the x-y plane.

c) Show that the characteristic curves of the partial differential equation are the intersections of the families of two circular cylinders.

$$\boxed{z(x,y) = \frac{x}{y} - xy + 1}$$

a) $yz\frac{\partial z}{\partial x} - xz\frac{\partial z}{\partial y} = xy$

BY LAGRANGE'S METHOD THE ASSOCIATED O.D.Es ARE $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{yz} = \frac{dy}{-xz} = \frac{dz}{xy}$$

$\frac{dx}{yz} = \frac{dz}{xy}$

$x\,dx = z\,dz$

$\frac{1}{2}x^2 = \frac{1}{2}z^2 + C$

$\boxed{z^2 - x^2 = C_1}$

$u(x,y,z) = z^2 - x^2$

$\frac{dx}{yz} = -\frac{dy}{xz}$

$x\,dx = -y\,dy$

$\frac{1}{2}x^2 = -\frac{1}{2}y^2 = C$

$\boxed{x^2 + y^2 = C_2}$

$v(x,y,z) = x^2 + y^2$

GENERAL SOLUTION $F(u,v) = 0$

$z^2 - x^2 = f(x^2+y^2)$

$z^2 = x^2 + f(x^2+y^2)$

b) APPLY CONDITION:

$z(x,0) = \sin x\,,\ 1 \le x \le 2$

THUS

$\sin^2 x = x^2 + f(x^2) \qquad 1 < x < 2$

$f(x^2) = \sin^2 x - x^2 \qquad 1 < x < 2$

$\boxed{\text{LET } u = x^2 \implies x = \sqrt{u}}$

$f(u) = (\sin\sqrt{u})^2 - u \qquad 1 < u < 4$

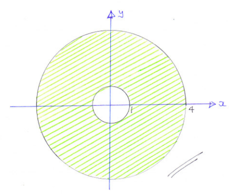
$\boxed{\text{IN PARTICULAR, LET } u = x^2+y^2}$

$f(x^2+y^2) = \left(\sin\sqrt{x^2+y^2}\right)^2 - (x^2+y^2) \qquad 1 < x^2+y^2 < 4$

THUS $z^2 = x^2 + f(x^2+y^2)$

$z^2 = x^2 + \left(\sin\sqrt{x^2+y^2}\right)^2 - (x^2+y^2) \qquad 1 < x^2+y^2 < 4$

$z^2 = \sin^2(x^2+y^2)^{\frac{1}{2}} - y^2 \qquad 1 < x^2+y^2 < 4$



c) THE CHARACTERISTICS

$\left.\begin{array}{l} y^2 + x^2 = C_1 \\ z^2 - x^2 = C_2 \end{array}\right\} \implies$

$y^2 + z^2 = C_1 + C_2$

$y^2 + z^2 = C_3$

& CHARACTERISTICS ARE

$y^2 + x^2 = C_1 \quad ①$

$y^2 + z^2 = C_3 \quad ②$

$z^2 - x^2 = C_2 \quad ③$

IF ① & ② ARE CYLINDERS

Question 17

It is given that $z = z(x,t)$ satisfies the partial differential equation

$$x\frac{\partial z}{\partial x} + (t-1)\frac{\partial z}{\partial t} = 0 .$$

It is further given that

$$z(x,0) = \begin{cases} 1-x^2 & |x|<1 \\ 0 & |x|\geq 1 \end{cases}.$$

Solve the above partial differential equation, and hence evaluate $z\left(\frac{1}{6},\frac{1}{3}\right)$ and $z\left(3,\frac{1}{3}\right)$.

$$\boxed{z\left(\frac{1}{6},\frac{1}{3}\right)=\frac{15}{16}}, \quad \boxed{z\left(3,\frac{1}{3}\right)=0}$$

$$\boxed{x\frac{\partial z}{\partial x} + (t-1)\frac{\partial z}{\partial t} = 0 \qquad z = z(x,t) \qquad z(x,0) = \begin{cases} 1-x^2 & |x|<1 \\ 0 & x\geq 1 \end{cases}}$$

BY LAGRANGE

$$\frac{dx}{x} = \frac{dt}{t-1} = \frac{dz}{0}$$

- ASSOCIATED O.D.E.S CAN ONLY HAVE A MEANING IF $dz=0$

THUS $\boxed{z = C_2}$

- $\frac{dx}{x} = \frac{dt}{t-1}$

$\ln x = \ln(t-1) + C$

$\ln\left(\frac{x}{t-1}\right) = \ln A$

$\boxed{\frac{x}{t-1} = C_2}$

$\therefore$ GEN. SOLUTION $F(u,v) = 0$ WHERE $u(x,t,z) = z$, $v(x,t,z) = \frac{x}{t-1}$

$$\boxed{\text{GEN. SOLUTION } \quad z = f\left(\frac{x}{t-1}\right)}$$

APPLY INITIAL CONDITION

$$z(x,0) = \begin{cases} 1-x^2 & |x|<1 \\ 0 & x\geq 1 \end{cases}$$

$$f(-x) = \begin{cases} 1-x^2 & |x|<1 \\ 0 & |x|\geq 1 \end{cases}$$

TWO CASES TO CONSIDER IF $|x|<1$ & $|x|\geq 1$

LET $u = -x \Rightarrow x = -u$

- IF $|x|<1$: $f(-x) = 1-x^2$, $f(u) = 1-(-u)^2$, $f(u) = 1-u^2$, $\quad |u|<1$

- IF $|x|\geq 1$: $f(-x) = 0$, $f(u) = 0$, $\quad |u|\geq 1$

THUS $f\left(\frac{x}{t-1}\right) = \begin{cases} 1-\left(\frac{x}{t-1}\right)^2 & \left|\frac{x}{t-1}\right|<1 \\ 0 & \left|\frac{x}{t-1}\right|\geq 1 \end{cases}$

$$\therefore z(x,t) = \begin{cases} 1-\left(\frac{x}{t-1}\right)^2 & \left|\frac{x}{t-1}\right|<1 \\ 0 & \left|\frac{x}{t-1}\right|\geq 1 \end{cases}$$

HENCE

- $z\left(\frac{1}{6},\frac{1}{3}\right)$
- FIRSTLY $\left|\frac{\frac{1}{6}}{\frac{1}{3}-1}\right| = \left|\frac{\frac{1}{6}}{-\frac{2}{3}}\right| = \frac{1}{4} < 1$
- $z\left(\frac{1}{6},\frac{1}{3}\right) = 1 - \left(\frac{\frac{1}{6}}{\frac{1}{3}-1}\right)^2 = \frac{15}{16}$

- $z\left(3,\frac{1}{3}\right)$
- FIRSTLY $\left|\frac{3}{\frac{1}{3}-1}\right| = \left|\frac{3}{-\frac{2}{3}}\right| = 3 > 1$
- $z\left(3,\frac{1}{3}\right) = 0$

Question 18

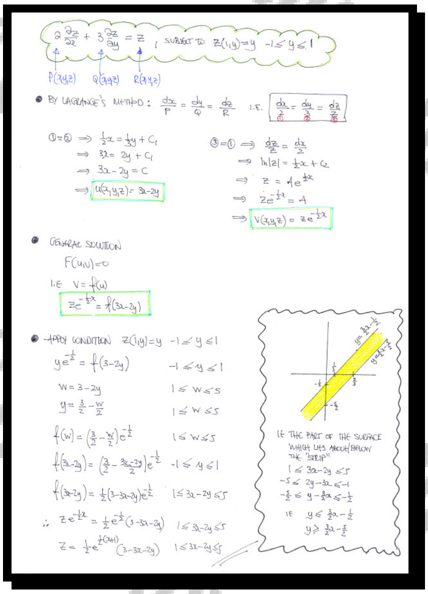
The surface S, with equation $z = z(x,y)$, satisfies the partial differential equation

$$2\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} = z \, .$$

It is further given that the plane with equation $x = 1$ meets S along the straight line with equation $z = y$, $-1 \le y \le 1$.

Find a Cartesian equation of S, in the form $z = f(x,y)$, further describing the relation of S to the x-y plane.

$$\boxed{z(x,y) = \frac{1}{2}(3 - 3x + 2y)\mathrm{e}^{\frac{1}{2}(x-1)}, \quad \frac{3}{2}x - \frac{5}{2} \le y \le \frac{3}{2}x - \frac{1}{2}}$$



Question 19

The surface S, with equation $z = z(x, y)$, is orthogonal to the sphere with Cartesian equation

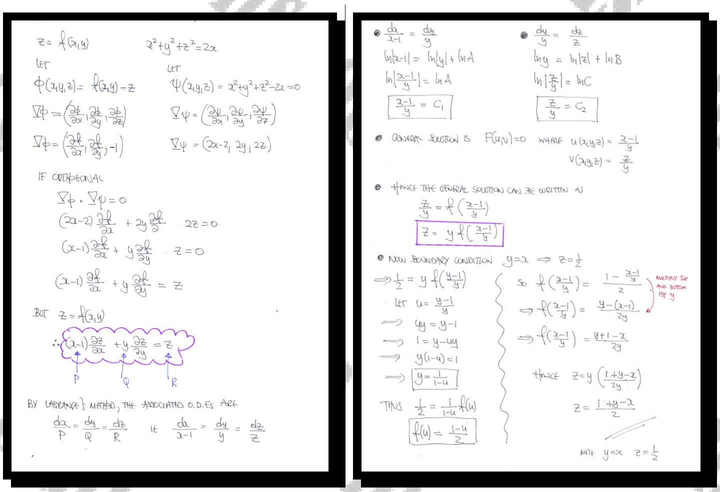
$$x^2 + y^2 + z^2 = 2x .$$

It is further given that S passes through the plane with equation $y = x$ at $z = \frac{1}{2}$.

$$z(x,0) = \tan 3x, \ 0 \le x \le 1 .$$

Find a Cartesian equation of S, in the form $z = f(x, y)$.

$$\boxed{z(x, y) = \tfrac{1}{2}(1 + y - x)}$$



$z = f(x,y)$ $\qquad$ $x^2+y^2+z^2=2x$

LET $\phi(x,y,z) = f(x,y) - z$ $\qquad$ LET $\psi(x,y,z) = x^2+y^2+z^2-2x = 0$

$\nabla\phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right)$ $\qquad$ $\nabla\psi = \left(\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial z}\right)$

$\nabla\phi = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1\right)$ $\qquad$ $\nabla\psi = (2x-2, 2y, 2z)$

IF ORTHOGONAL

$\nabla\phi \cdot \nabla\psi = 0$

$(2x-2)\frac{\partial f}{\partial x} + 2y\frac{\partial f}{\partial y} - 2z = 0$

$(x-1)\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} - z = 0$

$(x-1)\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = z$

BUT $z = f(x,y)$

$\therefore (x-1)\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$

P $\quad$ Q $\quad$ R

BY LAGRANGE'S METHOD, THE ASSOCIATED O.D.Es ARE

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ $\quad$ ie $\quad$ $\frac{dx}{x-1} = \frac{dy}{y} = \frac{dz}{z}$

- $\frac{dx}{x-1} = \frac{dy}{y}$

  $\ln|x-1| = \ln|y| + \ln A$

  $\ln\left|\frac{x-1}{y}\right| = \ln A$

  $\boxed{\frac{x-1}{y} = C_1}$

- $\frac{dy}{y} = \frac{dz}{z}$

  $\ln y = \ln|z| + \ln B$

  $\ln\left|\frac{z}{y}\right| = \ln C$

  $\boxed{\frac{z}{y} = C_2}$

- GENERAL SOLUTION IS $F(u,v) = 0$ WHERE $u(x,y,z) = \frac{x-1}{y}$, $v(x,y,z) = \frac{z}{y}$

- HENCE THE GENERAL SOLUTION CAN BE WRITTEN AS

  $\frac{z}{y} = f\left(\frac{x-1}{y}\right)$

  $\boxed{z = y f\left(\frac{x-1}{y}\right)}$

- NOW BOUNDARY CONDITION $y = x \Rightarrow z = \frac{1}{2}$

$\Rightarrow \frac{1}{2} = y f\left(\frac{y-1}{y}\right)$

LET $u = \frac{y-1}{y}$

$\Rightarrow uy = y-1$

$\Rightarrow 1 = y - uy$

$\Rightarrow y(1-u) = 1$

$\Rightarrow \boxed{y = \frac{1}{1-u}}$

THUS $\frac{1}{2} = \frac{1}{1-u} f(u)$

$\boxed{f(u) = \frac{1-u}{2}}$

SO $f\left(\frac{x-1}{y}\right) = \frac{1 - \frac{x-1}{y}}{2}$ $\quad$ MULTIPLY TOP AND BOTTOM BY y

$\Rightarrow f\left(\frac{x-1}{y}\right) = \frac{y-(x-1)}{2y}$

$\Rightarrow f\left(\frac{x-1}{y}\right) = \frac{y+1-x}{2y}$

HENCE $z = y\left(\frac{1+y-x}{2y}\right)$

$z = \frac{1+y-x}{2}$

NOTE $y = x \quad z = \frac{1}{2}$

SECOND ORDER P.D.E.s

$$P\frac{\partial^2 z}{\partial x^2}+Q\frac{\partial^2 z}{\partial x\,\partial y}+Q\frac{\partial^2 z}{\partial y^2}=0, \quad z=z(x,y)$$

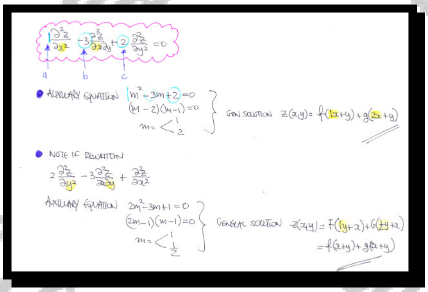
Question 1

It is given that $z = z(x, y)$ satisfies the partial differential equation

$$\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0 .$$

Find a general solution of the above partial differential equation.

$$\boxed{z = f(x+y) + g(2x+y)}$$



$$1\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0$$

a b c

- AUXILIARY EQUATION $1m^2 - 3m + 2 = 0$, $(m-2)(m-1) = 0$, $m = 1, 2$ GEN SOLUTION $z(x,y) = f(1x+y) + g(2x+y)$
- NOTE IF REWRITTEN

$$2\frac{\partial^2 z}{\partial y^2} - 3\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial x^2}$$

AUXILIARY EQUATION $2m^2 - 3m + 1 = 0$, $(2m-1)(m-1) = 0$, $m = 1, \frac{1}{2}$ GENERAL SOLUTION $z(x,y) = F(1y+x) + G(\frac{1}{2}y+x)$ $= f(x+y) + g(2x+y)$

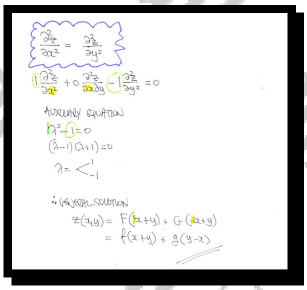
Question 2

It is given that $z = z(x, y)$ satisfies the partial differential equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2} .$$

Find a general solution of the above partial differential equation.

$$\boxed{z = f(y+x) + g(y-x)}$$



$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2}$$

$$1\frac{\partial^2 z}{\partial x^2} + 0\frac{\partial^2 z}{\partial x \partial y} - 1\frac{\partial^2 z}{\partial y^2} = 0$$

AUXILIARY EQUATION

$\lambda^2 - 1 = 0$

$(\lambda-1)(\lambda+1) = 0$

$\lambda = 1, -1$

$\therefore$ GENERAL SOLUTION

$z(x,y) = F(1x+y) + G(-1x+y)$

$= f(x+y) + g(y-x)$

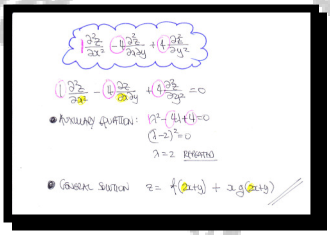
Question 3

It is given that $z = z(x, y)$ satisfies the partial differential equation

$$\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = 0.$$

Find a general solution of the above partial differential equation.

$$\boxed{z = f(2x + y) + x\,g(2x + y)}$$



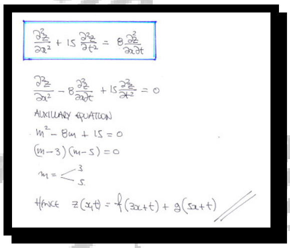
Question 4

It is given that $z = z(x, t)$ satisfies the partial differential equation

$$\frac{\partial^2 z}{\partial x^2} + 15\frac{\partial^2 z}{\partial t^2} = 8\frac{\partial^2 z}{\partial x \partial t}.$$

Find a general solution of the above partial differential equation.

$$\boxed{z(x,t) = f(3x + t) + g(5x + t)}$$



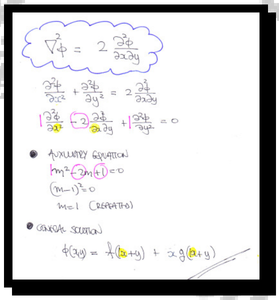
Question 5

It is given that $\varphi = \varphi(x, y)$ satisfies the partial differential equation

$$\nabla^2 \varphi = 2\frac{\partial^2 \varphi}{\partial x \partial y}.$$

Find a general solution of the above partial differential equation.

$$\boxed{z = f(x+y) + x\,g(x+y)}$$



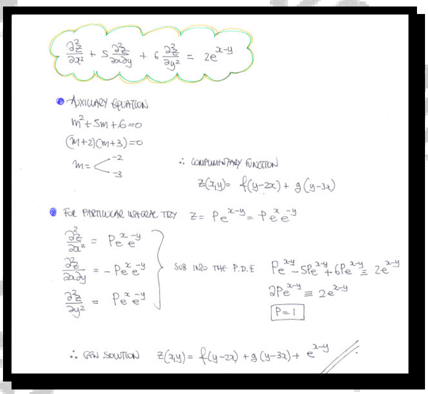
Question 6

It is given that $z = z(x, y)$ satisfies the partial differential equation

$$\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 6\frac{\partial^2 z}{\partial y^2} = 2\mathrm{e}^{x-y}.$$

Find a general solution of the above partial differential equation.

$$\boxed{z(x, y) = f(y-2x) + g(y-3x) + \mathrm{e}^{x-y}}$$



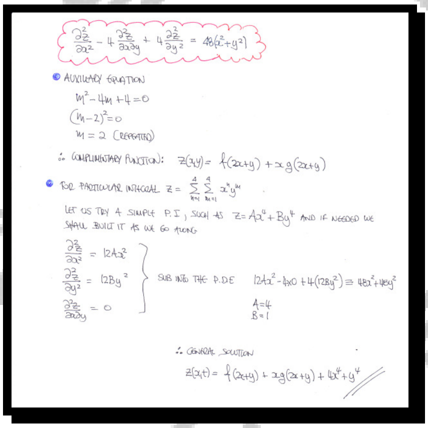
Question 7

It is given that $z = z(x,y)$ satisfies the partial differential equation

$$\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = 48\left(x^2 + y^2\right).$$

Find a general solution of the above partial differential equation.

$$\boxed{z(x,y) = f(2x+y) + x\,g(2x+y) + 4x^4 + y^4}$$



$$\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = 48(x^2+y^2)$$

- AUXILIARY EQUATION

$m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

$m = 2$ (REPEATED)

$\therefore$ COMPLIMENTARY FUNCTION: $z(x,y) = f(2x+y) + x\,g(2x+y)$

- FOR PARTICULAR INTEGRAL $z = \sum_{n=1}^{4}\sum_{m=1}^{4} x^n y^m$

LET US TRY A SIMPLE P.I, SUCH AS $z = Ax^4 + By^4$ AND IF NEEDED WE SHALL BUILT IT AS WE GO ALONG

$$\left.\begin{aligned} \frac{\partial^2 z}{\partial x^2} &= 12Ax^2 \\ \frac{\partial^2 z}{\partial y^2} &= 12By^2 \\ \frac{\partial^2 z}{\partial x \partial y} &= 0 \end{aligned}\right\} \text{SUB INTO THE P.D.E} \quad 12Ax^2 - 4\times 0 + 4(12By^2) \equiv 48x^2 + 48y^2$$

$A = 4$

$B = 1$

$\therefore$ GENERAL SOLUTION

$z(x,t) = f(2x+y) + x\,g(2x+y) + 4x^4 + y^4$

Question 8

It is given that $z = z(x, y)$ satisfies the partial differential equation

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 0 .$$

Find a general solution of the above partial differential equation.

$$\boxed{z(x, y) = f\left(\frac{y}{x}\right) + x f\left(\frac{y}{x}\right)}$$

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 0$$

THIS LOOK'S LIKE A "CAUCHY EULER TYPE" IN TWO VARIABLES

SO TRY $z = x^{\lambda} y^{\mu}$ AS THE SOLUTION

$\frac{\partial^2 z}{\partial x^2} = \lambda(\lambda-1)x^{\lambda-2}y^{\mu}$

$\frac{\partial^2 z}{\partial y^2} = \mu(\mu-1)x^{\lambda}y^{\mu-2}$

$\frac{\partial^2 z}{\partial x \partial y} = \lambda\mu\, x^{\lambda-1}y^{\mu-1}$

SUB INTO THE P.D.E

$\lambda(\lambda-1)x^{\lambda}y^{\mu} + 2\lambda\mu\, x^{\lambda}y^{\mu} + \mu(\mu-1)x^{\lambda}y^{\mu} = 0$

$(\lambda^2 - \lambda + 2\lambda\mu + \mu^2 - \mu) = 0$

$(\lambda^2 + 2\lambda\mu + \mu^2) - (\lambda + \mu) = 0$

$(\lambda + \mu)^2 - (\lambda + \mu) = 0$

$(\lambda + \mu)[\lambda + \mu - 1] = 0$

$\lambda = -\mu$ OR $\lambda = 1 - \mu$

$\therefore z_1 = x^{\lambda}y^{\mu} = x^{-\mu}y^{\mu} = \frac{y^{\mu}}{x^{\mu}} = \left(\frac{y}{x}\right)^{\mu}$

$z_2 = x^{\lambda}y^{\mu} = x^{1-\mu}y^{\mu} = \frac{y^{\mu}}{x^{\mu-1}} = \frac{y^{\mu}}{x^{\mu}x^{-1}} = x\left(\frac{y}{x}\right)^{\mu}$

$\therefore$ GENERAL SOLUTION

$z(x,y) = f\left(\frac{y}{x}\right) + x f\left(\frac{y}{x}\right)$