PARTIAL DIFFERENTIAL EQUATIONS

(by integral transformations)

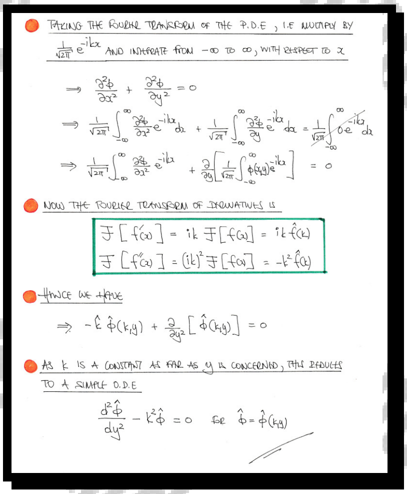
Question 1

The function $\varphi = \varphi(x, y)$ satisfies Laplace's equation in Cartesian coordinates

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 .$$

Use Fourier transforms to convert the above partial differential equation into an ordinary differential equation for $\hat{\varphi}(k, y)$, where $\hat{\varphi}(k, y)$ is the Fourier transform of $\varphi(x, y)$ with respect to x.

$\boxed{}$, $\boxed{\frac{d^2\hat{\varphi}}{dx^2} - k^2\hat{\varphi} = 0}$



- TAKING THE FOURIER TRANSFORM OF THE P.D.E, I.E MULTIPLY BY $\frac{1}{\sqrt{2\pi}}e^{-ikx}$ AND INTEGRATE FROM $-\infty$ TO ∞, WITH RESPECT TO x

$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\partial^2 \phi}{\partial x^2} e^{-ikx}\,dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\partial^2 \phi}{\partial y} e^{-ikx}\,dx = \cancel{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 0 e^{-ikx}\,dx}$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty} \frac{\partial^2 \phi}{\partial x^2} e^{-ikx} + \frac{\partial}{\partial y}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \phi(x,y) e^{-ikx}\right] = 0$$

- NOW THE FOURIER TRANSFORM OF DERIVATIVES IS

$$\boxed{\begin{aligned} \mathcal{F}\left[f'(x)\right] &= ik\,\mathcal{F}\left[f(x)\right] = ik\hat{f}(k) \\ \mathcal{F}\left[f''(x)\right] &= (ik)^2\mathcal{F}\left[f(x)\right] = -k^2\hat{f}(k) \end{aligned}}$$

- HENCE WE HAVE

$$\Rightarrow -\hat{k}\,\hat{\phi}(k,y) + \frac{\partial}{\partial y^2}\left[\hat{\phi}(k,y)\right] = 0$$

- AS k IS A CONSTANT AS FAR AS y IS CONCERNED, THIS REDUCES TO A SIMPLE O.D.E

$$\frac{d^2\hat{\phi}}{dy^2} - k^2\hat{\phi} = 0 \quad \text{FOR} \quad \hat{\phi} = \hat{\phi}(k,y)$$

Question 2

The function $\varphi = \varphi(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0,$$

in the part of the x-y plane for which $y \geq 0$.

It is further given that

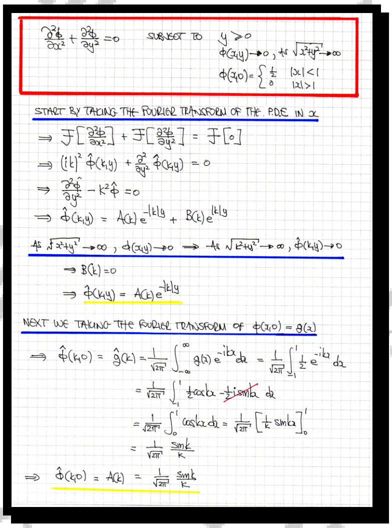
- $\varphi(x, y) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$
- $\varphi(x, 0) = \begin{cases} \frac{1}{2} & |x| < 1 \\ 0 & |x| > 1 \end{cases}$

Use Fourier transforms to show that

$$\varphi(x, y) = \frac{1}{\pi}\int_0^{\infty} \frac{1}{k} \mathrm{e}^{-ky} \sin k \cos kx \ \ dk,$$

and hence deduce the value of $\varphi(\pm 1, 0)$.

$\boxed{}$, $\boxed{\varphi(\pm 1, 0) = \frac{1}{4}}$



$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$ SUBJECT TO $y > 0$

$\phi(x,y) \to 0$, AS $\sqrt{x^2+y^2} \to \infty$

$\phi(x,0) = \begin{cases} \frac{1}{2} & |x|<1 \\ 0 & |x|>1 \end{cases}$

START BY TAKING THE FOURIER TRANSFORM OF THE P.D.E IN x

$\Rightarrow \mathcal{F}\left[\frac{\partial^2\phi}{\partial x^2}\right] + \mathcal{F}\left[\frac{\partial^2\phi}{\partial y^2}\right] = \mathcal{F}[0]$

$\Rightarrow (ik)^2 \hat{\phi}(k,y) + \frac{\partial^2}{\partial y^2}\hat{\phi}(k,y) = 0$

$\Rightarrow \frac{\partial^2\hat{\phi}}{\partial y^2} - k^2\hat{\phi} = 0$

$\Rightarrow \hat{\phi}(k,y) = A(k)e^{-|k|y} + B(k)e^{|k|y}$

AS $\sqrt{x^2+y^2} \to \infty$, $\phi(x,y) \to 0 \Rightarrow$ AS $\sqrt{k^2+y^2} \to \infty$, $\hat{\phi}(k,y) \to 0$

$\Rightarrow B(k) = 0$

$\Rightarrow \hat{\phi}(k,y) = A(k)e^{-|k|y}$

NEXT WE TAKING THE FOURIER TRANSFORM OF $\phi(x,0) = g(x)$

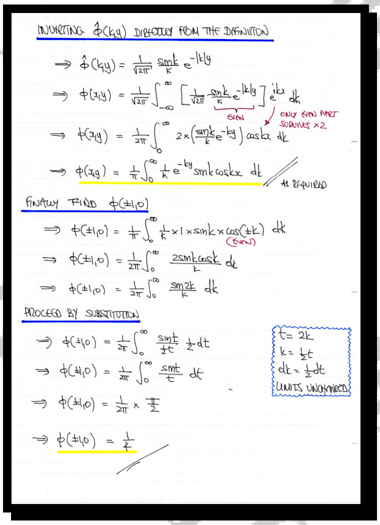
$\Rightarrow \hat{\phi}(k,0) = \hat{g}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)e^{-ikx}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}\frac{1}{2}e^{-ikx}\,dx$

$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}\frac{1}{2}\cos kx - \cancel{\frac{1}{2}i\sin kx}\,dx$

$= \frac{1}{\sqrt{2\pi}}\int_0^1 \cos kx\,dx = \frac{1}{\sqrt{2\pi}}\left[\frac{1}{k}\sin kx\right]_0^1$

$= \frac{1}{\sqrt{2\pi}}\frac{\sin k}{k}$

$\Rightarrow \hat{\phi}(k,0) = A(k) = \frac{1}{\sqrt{2\pi}}\frac{\sin k}{k}$



INVERTING $\hat{\phi}(k,y)$ DIRECTLY FROM THE DEFINITION

$\Rightarrow \hat{\phi}(k,y) = \frac{1}{\sqrt{2\pi}}\frac{\sin k}{k}e^{-|k|y}$

$\Rightarrow \phi(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\underbrace{\frac{\sin k}{k}e^{-|k|y}}_{\text{EVEN}}\right]e^{ikx}\,dk$

ONLY EVEN PART SURVIVES ×2

$\Rightarrow \phi(x,y) = \frac{1}{2\pi}\int_0^{\infty} 2\times\left(\frac{\sin k}{k}e^{-ky}\right)\cos kx\,dk$

$\Rightarrow \phi(x,y) = \frac{1}{\pi}\int_0^{\infty}\frac{1}{k}e^{-ky}\sin k\cos kx\,dk$ // AS REQUIRED

FINALLY FIND $\phi(\pm1,0)$

$\Rightarrow \phi(\pm1,0) = \frac{1}{\pi}\int_0^{\infty}\frac{1}{k}\times1\times\sin k\times\underset{\text{(EVEN)}}{\cos(\pm k)}\,dk$

$\Rightarrow \phi(\pm1,0) = \frac{1}{2\pi}\int_0^{\infty}\frac{2\sin k\cos k}{k}\,dk$

$\Rightarrow \phi(\pm1,0) = \frac{1}{2\pi}\int_0^{\infty}\frac{\sin 2k}{k}\,dk$

PROCEED BY SUBSTITUTION

$\Rightarrow \phi(\pm1,0) = \frac{1}{2\pi}\int_0^{\infty}\frac{\sin t}{\frac{1}{2}t}\,\frac{1}{2}dt$

$\Rightarrow \phi(\pm1,0) = \frac{1}{2\pi}\int_0^{\infty}\frac{\sin t}{t}\,dt$

$\Rightarrow \phi(\pm1,0) = \frac{1}{2\pi}\times\frac{\pi}{2}$

$\Rightarrow \phi(\pm1,0) = \frac{1}{4}$

$t = 2k$
$k = \frac{1}{2}t$
$dk = \frac{1}{2}dt$
LIMITS UNCHANGED

Question 3

The function $\psi = \psi(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 ,$$

in the part of the x-y plane for which $y \geq 0$.

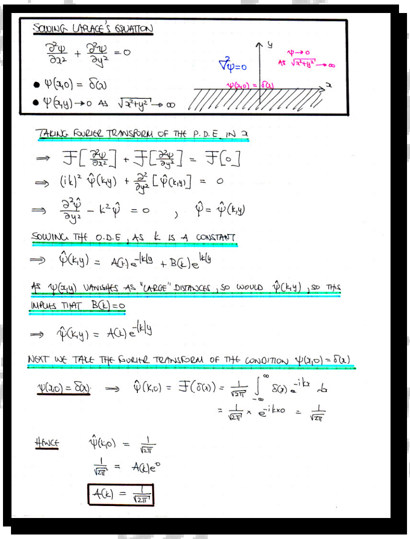
It is further given that

- $\psi(x,0) = \delta(x)$
- $\psi(x,y) \to 0$ as $\sqrt{x^2+y^2} \to \infty$

Use Fourier transforms to convert the above partial differential equation into an ordinary differential equation and hence show that

$$\psi(x,y) = \frac{1}{\pi}\left(\frac{y}{x^2+y^2}\right).$$

[illegible], proof



SOLVING LAPLACE'S EQUATION

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

- $\psi(x,0) = \delta(x)$
- $\psi(x,y) \to 0$ AS $\sqrt{x^2+y^2} \to \infty$

TAKING FOURIER TRANSFORM OF THE P.D.E IN x

$$\Rightarrow \mathcal{F}\left[\frac{\partial^2 \psi}{\partial x^2}\right] + \mathcal{F}\left[\frac{\partial^2 \psi}{\partial y^2}\right] = \mathcal{F}[0]$$

$$\Rightarrow (ik)^2 \hat{\psi}(k,y) + \frac{\partial^2}{\partial y^2}[\hat{\psi}(k,y)] = 0$$

$$\Rightarrow \frac{\partial^2 \hat{\psi}}{\partial y^2} - k^2\hat{\psi} = 0 \quad , \quad \hat{\psi} = \hat{\psi}(k,y)$$

SOLVING THE O.D.E, AS k IS A CONSTANT

$$\Rightarrow \hat{\psi}(k,y) = A(k)e^{-|k|y} + B(k)e^{|k|y}$$

AS $\psi(x,y)$ VANISHES AS "LARGE" DISTANCES, SO WOULD $\hat{\psi}(k,y)$, SO THIS IMPLIES THAT $B(k)=0$

$$\Rightarrow \hat{\psi}(k,y) = A(k)e^{-|k|y}$$

NEXT WE TAKE THE FOURIER TRANSFORM OF THE CONDITION $\psi(x,0) = \delta(x)$

$$\underline{\psi(x,0) = \delta(x)} \Rightarrow \hat{\psi}(k,0) = \mathcal{F}(\delta(x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \delta(x)e^{-ikx}\,dx$$

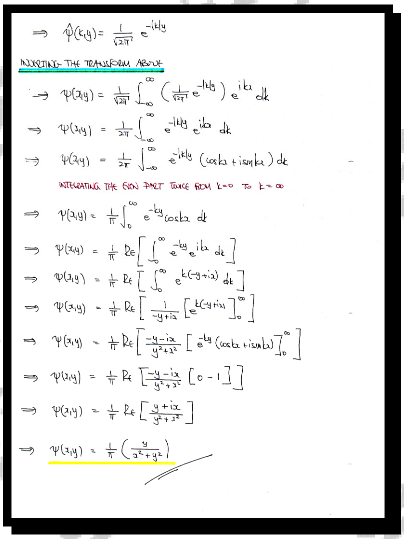
$$= \frac{1}{\sqrt{2\pi}} \times e^{-ik\times 0} = \frac{1}{\sqrt{2\pi}}$$

HENCE

$$\hat{\psi}(k,0) = \frac{1}{\sqrt{2\pi}}$$

$$\frac{1}{\sqrt{2\pi}} = A(k)e^0$$

$$\boxed{A(k) = \frac{1}{\sqrt{2\pi}}}$$



$$\Rightarrow \hat{\psi}(k,y) = \frac{1}{\sqrt{2\pi}}e^{-|k|y}$$

INVERTING THE TRANSFORM ABOUT

$$\Rightarrow \psi(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}e^{-|k|y}\right)e^{ikx}\,dk$$

$$\Rightarrow \psi(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|k|y}e^{ikx}\,dk$$

$$\Rightarrow \psi(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|k|y}(\cos kx + i\sin kx)\,dk$$

INTEGRATING THE EVEN PART TWICE FROM $k=0$ TO $k=\infty$

$$\Rightarrow \psi(x,y) = \frac{1}{\pi}\int_0^{\infty} e^{-ky}\cos kx\,dk$$

$$\Rightarrow \psi(x,y) = \frac{1}{\pi}\text{Re}\left[\int_0^{\infty} e^{-ky}e^{ikx}\,dk\right]$$

$$\Rightarrow \psi(x,y) = \frac{1}{\pi}\text{Re}\left[\int_0^{\infty} e^{k(-y+ix)}\,dk\right]$$

$$\Rightarrow \psi(x,y) = \frac{1}{\pi}\text{Re}\left[\frac{1}{-y+ix}\left[e^{k(-y+ix)}\right]_0^{\infty}\right]$$

$$\Rightarrow \psi(x,y) = \frac{1}{\pi}\text{Re}\left[\frac{-y-ix}{y^2+x^2}\left[e^{-ky}(\cos kx + i\sin kx)\right]_0^{\infty}\right]$$

$$\Rightarrow \psi(x,y) = \frac{1}{\pi}\text{Re}\left[\frac{-y-ix}{y^2+x^2}[0-1]\right]$$

$$\Rightarrow \psi(x,y) = \frac{1}{\pi}\text{Re}\left[\frac{y+ix}{y^2+x^2}\right]$$

$$\Rightarrow \psi(x,y) = \frac{1}{\pi}\left(\frac{y}{x^2+y^2}\right)$$

Question 4

The function $u = u(x,t)$ satisfies the partial differential equation

$$\frac{\partial u}{\partial t} + \frac{1}{3}\frac{\partial^3 u}{\partial x^3} = 0 .$$

It is further given that

- $u(x,0) = \delta(x)$
- $u(x,t) \to 0$ as $|x| \to \infty$

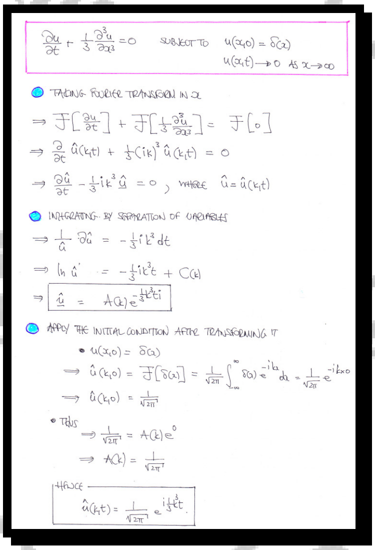
Use Fourier transforms to convert the above partial differential equation into an ordinary differential equation and hence show that

$$u(x,t) = \frac{1}{t^{\frac{1}{3}}}\mathrm{Ai}\left(\frac{x}{t^{\frac{1}{3}}}\right),$$

where the $\mathrm{Ai}(x)$ is the Airy function, defined as

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^{\infty} \cos\left[\tfrac{1}{3}k^3 + kx\right]\, dk .$$

proof



$$\boxed{\frac{\partial u}{\partial t} + \frac{1}{3}\frac{\partial^3 u}{\partial x^3} = 0 \quad \text{SUBJECT TO} \quad \begin{array}{l} u(x,0) = \delta(x) \\ u(x,t) \to 0 \text{ AS } x \to \infty \end{array}}$$

● TAKING FOURIER TRANSFORM IN x

$$\Rightarrow \mathcal{F}\left[\frac{\partial u}{\partial t}\right] + \mathcal{F}\left[\frac{1}{3}\frac{\partial^3 u}{\partial x^3}\right] = \mathcal{F}[0]$$

$$\Rightarrow \frac{\partial}{\partial t}\hat{u}(k,t) + \frac{1}{3}(ik)^3\hat{u}(k,t) = 0$$

$$\Rightarrow \frac{\partial \hat{u}}{\partial t} - \frac{1}{3}ik^3\hat{u} = 0, \quad \text{WHERE } \hat{u} = \hat{u}(k,t)$$

● INTEGRATING BY SEPARATION OF VARIABLES

$$\Rightarrow \frac{1}{\hat{u}}\,\partial\hat{u} = -\frac{1}{3}ik^3\,dt$$

$$\Rightarrow \ln \hat{u} = -\frac{1}{3}ik^3t + C(k)$$

$$\Rightarrow \boxed{\hat{u} = A(k)e^{-\frac{1}{3}k^3ti}}$$

● APPLY THE INITIAL CONDITION AFTER TRANSFORMING IT

- $u(x,0) = \delta(x)$

$$\Rightarrow \hat{u}(k,0) = \mathcal{F}[\delta(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x)e^{-ikx}\,dx = \frac{1}{\sqrt{2\pi}}e^{-ik\times 0}$$

$$\Rightarrow \hat{u}(k,0) = \frac{1}{\sqrt{2\pi}}$$

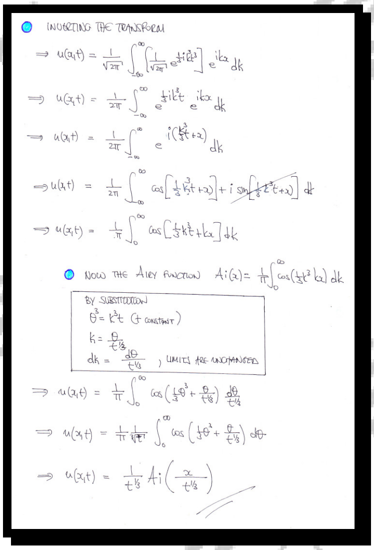
- THUS

$$\Rightarrow \frac{1}{\sqrt{2\pi}} = A(k)e^{0}$$

$$\Rightarrow A(k) = \frac{1}{\sqrt{2\pi}}$$

HENCE

$$\boxed{\hat{u}(k,t) = \frac{1}{\sqrt{2\pi}}e^{i\frac{1}{3}k^3t}.}$$



● INVERTING THE TRANSFORM

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}e^{\frac{1}{3}ik^3t^3}\right]e^{ikx}\,dk$$

$$\Rightarrow u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{\frac{1}{3}ik^3t}e^{ikx}\,dk$$

$$\Rightarrow u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i(\frac{k^3}{3}t+x)}\,dk$$

$$\Rightarrow u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\cos\left[\tfrac{1}{3}k^3t+x)\right] + \cancel{i\sin\left[\tfrac{1}{3}k^3t+x)\right]}\,dk$$

$$\Rightarrow u(x,t) = \frac{1}{\pi}\int_0^{\infty}\cos\left[\tfrac{1}{3}k^3t+kx\right]dk$$

● NOW THE AIRY FUNCTION $\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^{\infty}\cos(\frac{1}{3}k^3\,kx)\,dk$

BY SUBSTITUTION
$\theta^3 = k^3t$ (t CONSTANT)
$k = \frac{\theta}{t^{1/3}}$
$dk = \frac{d\theta}{t^{1/3}}$, LIMITS ARE UNCHANGED

$$\Rightarrow u(x,t) = \frac{1}{\pi}\int_0^{\infty}\cos\left(\tfrac{1}{3}\theta^3 + \frac{\theta}{t^{1/3}}\right)\frac{d\theta}{t^{1/3}}$$

$$\Rightarrow u(x,t) = \frac{1}{\pi}\frac{1}{\sqrt[3]{t}}\int_0^{\infty}\cos\left(\tfrac{1}{3}\theta^3 + \frac{\theta}{t^{1/3}}\right)d\theta$$

$$\Rightarrow u(x,t) = \frac{1}{t^{1/3}}\mathrm{Ai}\left(\frac{x}{t^{1/3}}\right)$$

Question 5

The function $\Phi = \Phi(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0,$$

in the part of the x-y plane for which $y \geq 0$.

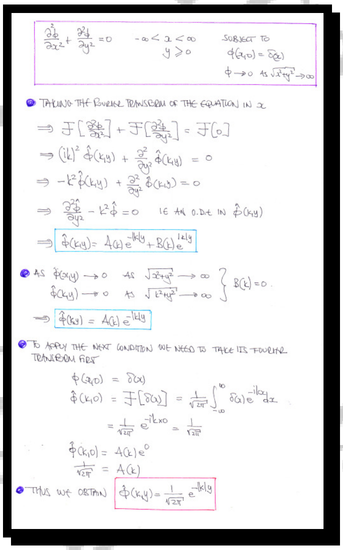
It is further given that

- $\Phi(x,0) = \delta(x)$
- $\Phi(x, y) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$

Use Fourier transforms to find the solution of the above partial differential equation and hence show that

$$\delta(x) = \lim_{\alpha \to 0}\left[\frac{1}{\pi\alpha}\left(1 + \frac{y^2}{\alpha^2}\right)^{-1}\right].$$

proof



$$\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \begin{matrix} -\infty < x < \infty \\ y \geq 0 \end{matrix} \qquad \begin{matrix} \text{SUBJECT TO} \\ \phi(x,0) = \delta(x) \\ \phi \to 0 \text{ AS } \sqrt{x^2+y^2} \to \infty \end{matrix}}$$

- TAKING THE FOURIER TRANSFORM OF THE EQUATION IN x

$$\Rightarrow \mathcal{F}\left[\frac{\partial^2 \phi}{\partial x^2}\right] + \mathcal{F}\left[\frac{\partial^2 \phi}{\partial y^2}\right] = \mathcal{F}[0]$$

$$\Rightarrow (ik)^2 \hat{\phi}(k,y) + \frac{\partial^2}{\partial y^2}\hat{\phi}(k,y) = 0$$

$$\Rightarrow -k^2 \hat{\phi}(k,y) + \frac{\partial^2}{\partial y^2}\hat{\phi}(k,y) = 0$$

$$\Rightarrow \frac{\partial^2 \hat{\phi}}{\partial y^2} - k^2 \hat{\phi} = 0 \quad \text{IE AN O.D.E IN } \hat{\phi}(k,y)$$

$$\Rightarrow \boxed{\hat{\phi}(k,y) = A(k)e^{-|k|y} + B(k)e^{|k|y}}$$

- AS $\left.\begin{matrix} \phi(x,y) \to 0 \text{ AS } \sqrt{x^2+y^2} \to \infty \\ \hat{\phi}(k,y) \to 0 \text{ AS } \sqrt{k^2+y^2} \to \infty \end{matrix}\right\} B(k) = 0.$

$$\Rightarrow \boxed{\hat{\phi}(k,y) = A(k)e^{-|k|y}}$$

- TO APPLY THE NEXT CONDITION WE NEED TO TAKE ITS FOURIER TRANSFORM FIRST

$$\phi(x,0) = \delta(x)$$

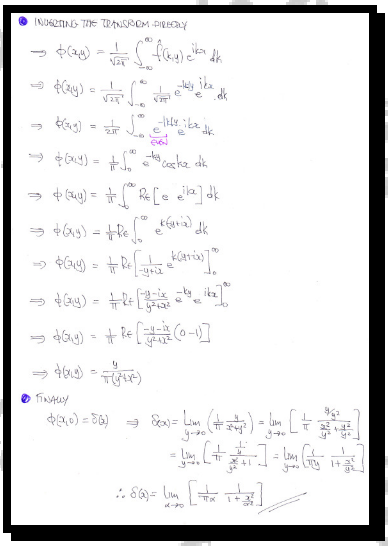
$$\hat{\phi}(k,0) = \mathcal{F}[\delta(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \delta(x)e^{-ikx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}e^{-ik\times 0} = \frac{1}{\sqrt{2\pi}}$$

$$\hat{\phi}(k,0) = A(k)e^0$$

$$\frac{1}{\sqrt{2\pi}} = A(k)$$

- THUS WE OBTAIN $\boxed{\hat{\phi}(k,y) = \frac{1}{\sqrt{2\pi}}e^{-|k|y}}$



- INVERTING THE TRANSFORM DIRECTLY

$$\Rightarrow \phi(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(k,y)e^{ikx}\,dk$$

$$\Rightarrow \phi(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-|k|y}e^{ikx}.dk$$

$$\Rightarrow \phi(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{e^{-|k|y}}_{\text{EVEN}}e^{ikx}\,dk$$

$$\Rightarrow \phi(x,y) = \frac{1}{\pi}\int_0^{\infty} e^{-ky}\cos kx\,dk$$

$$\Rightarrow \phi(x,y) = \frac{1}{\pi}\int_0^{\infty} \text{Re}\left[e\ e^{ikx}\right]dk$$

$$\Rightarrow \phi(x,y) = \frac{1}{\pi}\text{Re}\int_0^{\infty} e^{k(-y+ix)}\,dk$$

$$\Rightarrow \phi(x,y) = \frac{1}{\pi}\text{Re}\left[\frac{1}{-y+ix}e^{k(-y+ix)}\right]_0^{\infty}$$

$$\Rightarrow \phi(x,y) = \frac{1}{\pi}\text{Re}\left[\frac{-y-ix}{y^2+x^2}e^{-ky}e^{ikx}\right]_0^{\infty}$$

$$\Rightarrow \phi(x,y) = \frac{1}{\pi}\text{Re}\left[\frac{-y-ix}{y^2+x^2}(0-1)\right]$$

$$\Rightarrow \phi(x,y) = \frac{y}{\pi(y^2+x^2)}$$

- FINALLY

$$\phi(x,0) = \delta(x) \Rightarrow \delta(x) = \lim_{y\to 0}\left(\frac{1}{\pi}\frac{y}{x^2+y^2}\right) = \lim_{y\to 0}\left[\frac{1}{\pi}\frac{y/y^2}{\frac{x^2}{y^2}+\frac{y^2}{y^2}}\right]$$

$$= \lim_{y\to 0}\left[\frac{1}{\pi}\frac{\frac{1}{y}}{\frac{x^2}{y^2}+1}\right] = \lim_{y\to 0}\left[\frac{1}{\pi y}\frac{1}{1+\frac{x^2}{y^2}}\right]$$

$$\therefore \delta(x) = \lim_{\alpha\to 0}\left[\frac{1}{\pi\alpha}\frac{1}{1+\frac{x^2}{\alpha^2}}\right]$$

Question 6

The function $u = u(t, y)$ satisfies the partial differential equation

$$\frac{\partial u}{\partial t} + y\frac{\partial u}{\partial y} = y\,,\ t \ge 0\,,\ y > 0\,,$$

subject to the following conditions

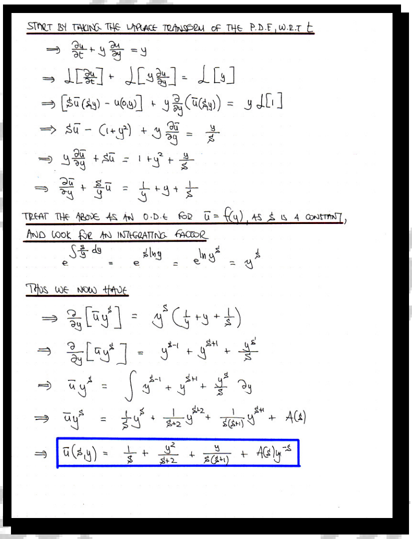
i. $u(0, y) = 1 + y^2\,,\ y > 0$

ii. $u(t, 0) = 1\,,\ t \ge 0$

Use Laplace transforms in t to show that

$$u(t, y) = 1 + y - y\mathrm{e}^{-t} + y^2\mathrm{e}^{-2t}\,.$$

$\boxed{}$, $\boxed{\text{proof}}$



START BY TAKING THE LAPLACE TRANSFORM OF THE P.D.E, W.R.T t

$$\Rightarrow \frac{\partial u}{\partial t} + y\frac{\partial u}{\partial y} = y$$

$$\Rightarrow \mathcal{L}\left[\frac{\partial u}{\partial t}\right] + \mathcal{L}\left[y\frac{\partial u}{\partial y}\right] = \mathcal{L}[y]$$

$$\Rightarrow \left[s\bar{u}(s,y) - u(0,y)\right] + y\frac{\partial}{\partial y}\left(\bar{u}(s,y)\right) = y\,\mathcal{L}[1]$$

$$\Rightarrow s\bar{u} - (1+y^2) + y\frac{\partial \bar{u}}{\partial y} = \frac{y}{s}$$

$$\Rightarrow y\frac{\partial \bar{u}}{\partial y} + s\bar{u} = 1 + y^2 + \frac{y}{s}$$

$$\Rightarrow \frac{\partial \bar{u}}{\partial y} + \frac{s}{y}\bar{u} = \frac{1}{y} + y + \frac{1}{s}$$

TREAT THE ABOVE AS AN O.D.E FOR $\bar{u} = f(y)$, AS s IS A CONSTANT, AND LOOK FOR AN INTEGRATING FACTOR

$$e^{\int \frac{s}{y}\,dy} = e^{s\ln y} = e^{\ln y^s} = y^s$$

THUS WE NOW HAVE

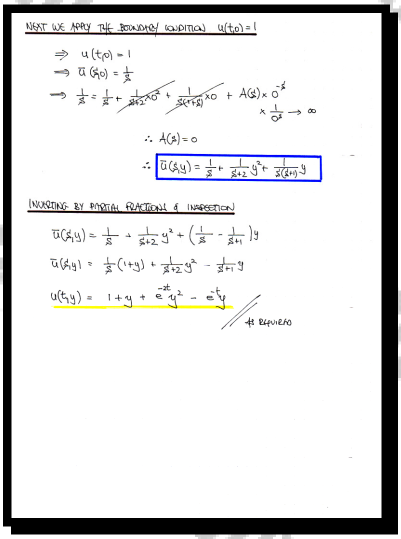
$$\Rightarrow \frac{\partial}{\partial y}\left[\bar{u}y^s\right] = y^s\left(\frac{1}{y} + y + \frac{1}{s}\right)$$

$$\Rightarrow \frac{\partial}{\partial y}\left[\bar{u}y^s\right] = y^{s-1} + y^{s+1} + \frac{y^s}{s}$$

$$\Rightarrow \bar{u}y^s = \int y^{s-1} + y^{s+1} + \frac{y^s}{s}\,\partial y$$

$$\Rightarrow \bar{u}y^s = \frac{1}{s}y^s + \frac{1}{s+2}y^{s+2} + \frac{1}{s(s+1)}y^{s+1} + A(s)$$

$$\Rightarrow \boxed{\bar{u}(s,y) = \frac{1}{s} + \frac{y^2}{s+2} + \frac{y}{s(s+1)} + A(s)y^{-s}}$$



NEXT WE APPLY THE BOUNDARY CONDITION $u(t,0) = 1$

$$\Rightarrow u(t,0) = 1$$

$$\Rightarrow \bar{u}(s,0) = \frac{1}{s}$$

$$\Rightarrow \frac{1}{s} = \frac{1}{s} + \cancel{\frac{1}{s+2}\times 0^2} + \cancel{\frac{1}{s(1+s)}\times 0} + A(s)\times 0^{-s}$$

$$\times\frac{1}{0^s} \rightarrow \infty$$

$$\therefore A(s) = 0$$

$$\therefore \boxed{\bar{u}(s,y) = \frac{1}{s} + \frac{1}{s+2}y^2 + \frac{1}{s(s+1)}y}$$

INVERTING BY PARTIAL FRACTIONS & INSPECTION

$$\bar{u}(s,y) = \frac{1}{s} + \frac{1}{s+2}y^2 + \left(\frac{1}{s} - \frac{1}{s+1}\right)y$$

$$\bar{u}(s,y) = \frac{1}{s}(1+y) + \frac{1}{s+2}y^2 - \frac{1}{s+1}y$$

$$u(t,y) = 1 + y + e^{-2t}y^2 - e^{-t}y$$

// AS REQUIRED

Question 7

The function $\varphi = \varphi(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0,$$

in the part of the x-y plane for which $x \geq 0$ and $y \geq 0$.

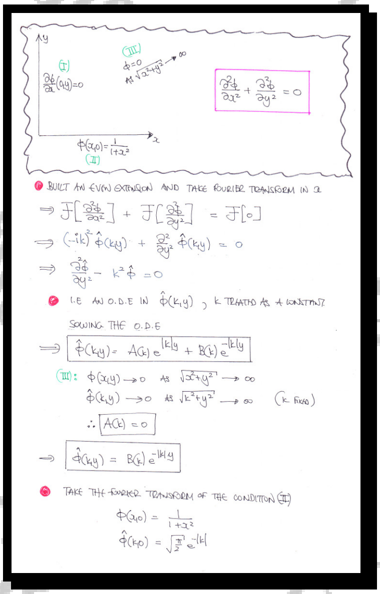
It is further given that

- $\varphi(x, 0) = \dfrac{1}{1 + x^2}$
- $\varphi(x, y) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$
- $\dfrac{\partial}{\partial x}\left[\varphi(x, 0)\right] = 0$

Use Fourier transforms to convert the above partial differential equation into an ordinary differential equation and hence show that

$$\varphi(x, y) = \frac{y + 1}{x^2 + (y + 1)^2}.$$

proof



$$\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0}$$

- BUILT AN EVEN EXTENSION AND TAKE FOURIER TRANSFORM IN x

$$\Rightarrow \mathcal{F}\left[\frac{\partial^2 \phi}{\partial x^2}\right] + \mathcal{F}\left[\frac{\partial^2 \phi}{\partial y^2}\right] = \mathcal{F}[0]$$

$$\Rightarrow (-ik)^2 \hat{\phi}(k,y) + \frac{\partial^2}{\partial y^2}\hat{\phi}(k,y) = 0$$

$$\Rightarrow \frac{\partial^2 \hat{\phi}}{\partial y^2} - k^2 \hat{\phi} = 0$$

- I.E AN O.D.E IN $\hat{\phi}(k,y)$, k TREATED AS A CONSTANT.

SOLVING THE O.D.E

$$\Rightarrow \boxed{\hat{\phi}(k,y) = A(k)e^{|k|y} + B(k)e^{-|k|y}}$$

(III): $\phi(x,y) \to 0$ AS $\sqrt{x^2+y^2} \to \infty$

$\hat{\phi}(k,y) \to 0$ AS $\sqrt{k^2+y^2} \to \infty$ (k FIXED)

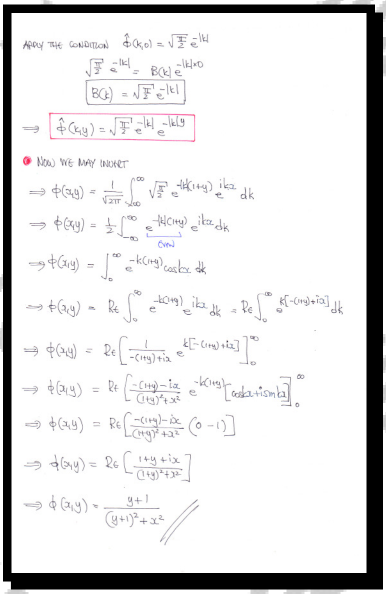
$\therefore \boxed{A(k) = 0}$

$$\Rightarrow \boxed{\hat{\phi}(k,y) = B(k)e^{-|k|y}}$$

- TAKE THE FOURIER TRANSFORM OF THE CONDITION (II)

$$\phi(x,0) = \frac{1}{1+x^2}$$

$$\hat{\phi}(k,0) = \sqrt{\frac{\pi}{2}}e^{-|k|}$$



APPLY THE CONDITION $\hat{\phi}(k,0) = \sqrt{\frac{\pi}{2}}e^{-|k|}$

$$\sqrt{\frac{\pi}{2}}e^{-|k|} = B(k)e^{-|k|\times 0}$$

$$\boxed{B(k) = \sqrt{\frac{\pi}{2}}e^{-|k|}}$$

$$\Rightarrow \boxed{\hat{\phi}(k,y) = \sqrt{\frac{\pi}{2}}e^{-|k|}e^{-|k|y}}$$

- NOW WE MAY INVERT

$$\Rightarrow \phi(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{\pi}{2}}e^{-|k|(1+y)}e^{ikx}\,dk$$

$$\Rightarrow \phi(x,y) = \frac{1}{2}\int_{-\infty}^{\infty}\underbrace{e^{-|k|(1+y)}}_{\text{EVEN}}e^{ikx}\,dk$$

$$\Rightarrow \phi(x,y) = \int_0^{\infty}e^{-k(1+y)}\cos kx\,dk$$

$$\Rightarrow \phi(x,y) = \mathrm{Re}\int_0^{\infty}e^{-k(1+y)}e^{ikx}\,dk = \mathrm{Re}\int_0^{\infty}e^{k[-(1+y)+ix]}\,dk$$

$$\Rightarrow \phi(x,y) = \mathrm{Re}\left[\frac{1}{-(1+y)+ix}e^{k[-(1+y)+ix]}\right]_0^{\infty}$$

$$\Rightarrow \phi(x,y) = \mathrm{Re}\left[\frac{-(1+y)-ix}{(1+y)^2+x^2}e^{-k(1+y)}\left[\cos kx + i\sin kx\right]\right]_0^{\infty}$$

$$\Rightarrow \phi(x,y) = \mathrm{Re}\left[\frac{-(1+y)-ix}{(1+y)^2+x^2}(0-1)\right]$$

$$\Rightarrow \phi(x,y) = \mathrm{Re}\left[\frac{1+y+ix}{(1+y)^2+x^2}\right]$$

$$\Rightarrow \phi(x,y) = \frac{y+1}{(y+1)^2+x^2} \quad //$$

Question 8

The function $z = z(x,t)$ satisfies the partial differential equation

$$\frac{\partial u}{\partial x} = 2\frac{\partial z}{\partial t} + z, \quad x \ge 0, \quad t \ge 0,$$

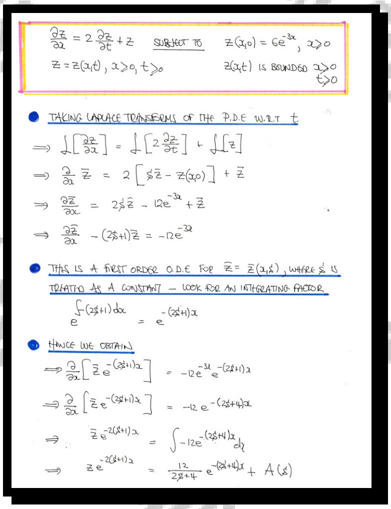
subject to the following conditions

i. $z(x,0) = 6\mathrm{e}^{-3x}$, $x > 0$.

ii. $z(x,t)$, is bounded for all $x \ge 0$ and $t \ge 0$.

Find the solution of partial differential equation by using Laplace transforms.

$\boxed{}$, $\boxed{z(x,t) = 6\mathrm{e}^{-(3x+2t)}}$



$\frac{\partial z}{\partial x} = 2\frac{\partial z}{\partial t} + z$ SUBJECT TO $z(x,0) = 6e^{-3x}, x \geqslant 0$

$z = z(x,t), x \geqslant 0, t \geqslant 0$ $z(x,t)$ IS BOUNDED $x \geqslant 0$ $t \geqslant 0$

- TAKING LAPLACE TRANSFORMS OF THE P.D.E W.R.T t

$$\Rightarrow \mathcal{L}\left[\frac{\partial z}{\partial x}\right] = \mathcal{L}\left[2\frac{\partial z}{\partial t}\right] + \mathcal{L}[z]$$

$$\Rightarrow \frac{\partial}{\partial x}\bar{z} = 2\left[s\bar{z} - z(x,0)\right] + \bar{z}$$

$$\Rightarrow \frac{\partial \bar{z}}{\partial x} = 2s\bar{z} - 12e^{-3x} + \bar{z}$$

$$\Rightarrow \frac{\partial \bar{z}}{\partial x} - (2s+1)\bar{z} = -12e^{-3x}$$

- THIS IS A FIRST ORDER O.D.E FOR $\bar{z} = \bar{z}(x,s)$, WHERE s IS TREATED AS A CONSTANT – LOOK FOR AN INTEGRATING FACTOR

$$e^{\int (2s+1)\,dx} = e^{-(2s+1)x}$$

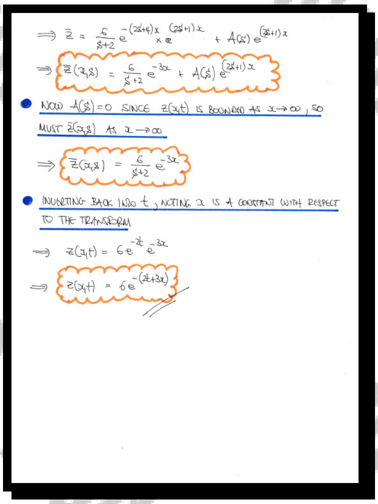
- HENCE WE OBTAIN

$$\Rightarrow \frac{\partial}{\partial x}\left[\bar{z}e^{-(2s+1)x}\right] = -12e^{-3x}e^{-(2s+1)x}$$

$$\Rightarrow \frac{\partial}{\partial x}\left[\bar{z}e^{-(2s+1)x}\right] = -12e^{-(2s+4)x}$$

$$\Rightarrow \bar{z}e^{-2(s+1)x} = \int -12e^{-(2s+4)x}\,dx$$

$$\Rightarrow \bar{z}e^{-2(s+1)x} = \frac{12}{2s+4}e^{-(2s+4)x} + A(s)$$



$$\Rightarrow \bar{z} = \frac{6}{s+2}e^{-(2s+4)x} \times e^{(2s+1)x} + A(s)e^{(2s+1)x}$$

$$\Rightarrow \boxed{\bar{z}(x,s) = \frac{6}{s+2}e^{-3x} + A(s)e^{(2s+1)x}}$$

- NOW $A(s) = 0$ SINCE $z(x,t)$ IS BOUNDED AS $x \to \infty$, SO MUST $\bar{z}(x,s)$ AS $x \to \infty$

$$\Rightarrow \boxed{\bar{z}(x,s) = \frac{6}{s+2}e^{-3x}}$$

- INVERTING BACK INTO t, NOTING x IS A CONSTANT WITH RESPECT TO THE TRANSFORM

$$\Rightarrow z(x,t) = 6e^{-2t}e^{-3x}$$

$$\Rightarrow \boxed{z(x,t) = 6e^{-(2t+3x)}}$$

Question 9

$$\theta(x) = 8\sin(2\pi x),\ 0 \le x \le 1$$

The above equation represents the temperature distribution θ °C, maintained along the 1 m length of a thin rod.

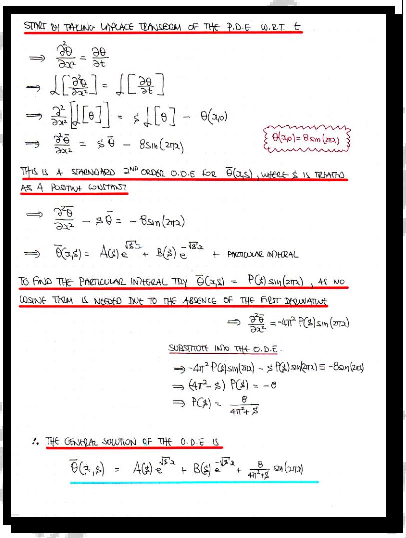
At time $t = 0$, the temperature θ is suddenly dropped to $\theta = 0$ °C at both the ends of the rod at $x = 0$, and at $x = 1$, and the source which was previously maintaining the temperature distribution is removed.

The new temperature distribution along the rod $\theta(x,t)$, satisfies the heat equation

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial \theta}{\partial t},\quad 0 \le x \le 1,\quad t \ge 0.$$

Use Laplace transforms to determine an expression for $\theta(x,t)$.

$$\boxed{},\ \boxed{\theta(x,t) = 8e^{-4\pi^2 t}\sin(2\pi x)}$$



START BY TAKING LAPLACE TRANSFORM OF THE P.D.E W.R.T. t

$$\Longrightarrow \frac{\partial^2\theta}{\partial x^2} = \frac{\partial \theta}{\partial t}$$

$$\Longrightarrow \mathcal{L}\left[\frac{\partial^2\theta}{\partial x^2}\right] = \mathcal{L}\left[\frac{\partial\theta}{\partial t}\right]$$

$$\Longrightarrow \frac{\partial^2}{\partial x^2}\Big[\mathcal{L}[\theta]\Big] = s\mathcal{L}[\theta] - \theta(x,0)$$

$$\Longrightarrow \frac{\partial^2\bar{\theta}}{\partial x^2} = s\bar{\theta} - 8\sin(2\pi x)$$

$\theta(x,0) = 8\sin(2\pi x)$

THIS IS A STANDARD 2ND ORDER O.D.E FOR $\bar{\theta}(x,s)$, WHERE s IS TREATED AS A POSITIVE CONSTANT

$$\Longrightarrow \frac{\partial^2\bar{\theta}}{\partial x^2} - s\bar{\theta} = -8\sin(2\pi x)$$

$$\Longrightarrow \bar{\theta}(x,s) = A(s)e^{\sqrt{s}x} + B(s)e^{-\sqrt{s}x} + \text{PARTICULAR INTEGRAL}$$

TO FIND THE PARTICULAR INTEGRAL TRY $\bar{\theta}(x,s) = P(s)\sin(2\pi x)$, AS NO COSINE TERM IS NEEDED DUE TO THE ABSENCE OF THE FIRST DERIVATIVE

$$\Longrightarrow \frac{\partial^2\bar{\theta}}{\partial x^2} = -4\pi^2 P(s)\sin(2\pi x)$$

SUBSTITUTE INTO THE O.D.E.

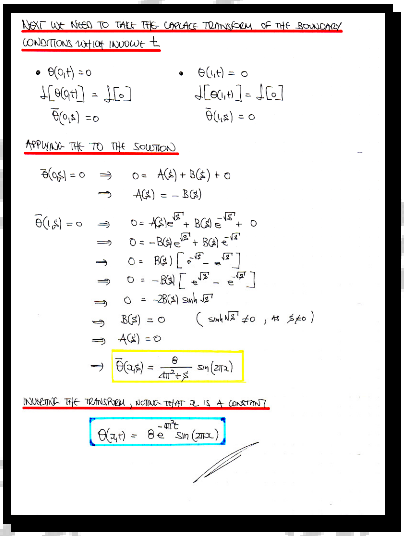
$$\Longrightarrow -4\pi^2 P(s)\sin(2\pi x) - sP(s)\sin(2\pi x) \equiv -8\sin(2\pi x)$$

$$\Longrightarrow (4\pi^2 - s)P(s) = -8$$

$$\Longrightarrow P(s) = \frac{8}{4\pi^2 + s}$$

∴ THE GENERAL SOLUTION OF THE O.D.E IS

$$\bar{\theta}(x,s) = A(s)e^{\sqrt{s}x} + B(s)e^{-\sqrt{s}x} + \frac{8}{4\pi^2+s}\sin(2\pi x)$$



NEXT WE NEED TO TAKE THE LAPLACE TRANSFORM OF THE BOUNDARY CONDITIONS WHICH INVOLVE t

- $\theta(0,t) = 0$
  $\mathcal{L}[\theta(0,t)] = \mathcal{L}[0]$
  $\bar{\theta}(0,s) = 0$
- $\theta(1,t) = 0$
  $\mathcal{L}[\theta(1,t)] = \mathcal{L}[0]$
  $\bar{\theta}(1,s) = 0$

APPLYING THE TO THE SOLUTION

$$\bar{\theta}(0,s) = 0 \Longrightarrow 0 = A(s) + B(s) + 0$$

$$\Longrightarrow A(s) = -B(s)$$

$$\bar{\theta}(1,s) = 0 \Longrightarrow 0 = A(s)e^{\sqrt{s}} + B(s)e^{-\sqrt{s}} + 0$$

$$\Longrightarrow 0 = -B(s)e^{\sqrt{s}} + B(s)e^{-\sqrt{s}}$$

$$\Longrightarrow 0 = B(s)\left[e^{-\sqrt{s}} - e^{\sqrt{s}}\right]$$

$$\Longrightarrow 0 = -B(s)\left[e^{\sqrt{s}} - e^{-\sqrt{s}}\right]$$

$$\Longrightarrow 0 = -2B(s)\sinh\sqrt{s}$$

$$\Longrightarrow B(s) = 0 \qquad (\sinh\sqrt{s} \ne 0,\ \text{as } s \ne 0)$$

$$\Longrightarrow A(s) = 0$$

$$\Longrightarrow \boxed{\bar{\theta}(x,s) = \frac{8}{4\pi^2+s}\sin(2\pi x)}$$

INVERTING THE TRANSFORM, NOTING THAT x IS A CONSTANT

$$\boxed{\theta(x,t) = 8e^{-4\pi^2 t}\sin(2\pi x)}$$

Question 10

The function $\varphi = \varphi(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0,$$

in the semi-infinite region of the x-y plane for which $y \geq 0$.

It is further given that

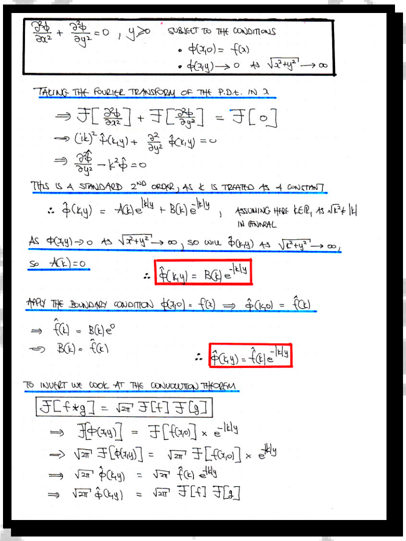
$$\varphi(x, 0) = f(x)$$

$$\varphi(x, y) \to 0 \ \text{ as } \sqrt{x^2 + y^2} \to \infty$$

Use Fourier transforms to convert the above partial differential equation into an ordinary differential equation and hence show that

$$\varphi(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(x-u)}{u^2 + y^2} \, du.$$

proof



$$\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \ , \ y \geqslant 0 \quad \text{SUBJECT TO THE CONDITIONS} \quad \bullet\ \phi(x,0) = f(x) \quad \bullet\ \phi(x,y) \to 0 \text{ AS } \sqrt{x^2+y^2} \to \infty}$$

TAKING THE FOURIER TRANSFORM OF THE P.D.E. IN x

$$\Rightarrow \mathcal{F}\left[\frac{\partial^2 \phi}{\partial x^2}\right] + \mathcal{F}\left[\frac{\partial^2 \phi}{\partial y^2}\right] = \mathcal{F}[0]$$

$$\Rightarrow (ik)^2 \hat{\phi}(k,y) + \frac{\partial^2}{\partial y^2} \hat{\phi}(k,y) = 0$$

$$\Rightarrow \frac{\partial^2 \hat{\phi}}{\partial y^2} - k^2 \hat{\phi} = 0$$

THIS IS A STANDARD 2ND ORDER, AS k IS TREATED AS A CONSTANT

$$\therefore \hat{\phi}(k,y) = A(k) e^{|k|y} + B(k) e^{-|k|y}, \quad \text{ASSUMING HERE } k \in \mathbb{R}, \text{ AS } \sqrt{k^2} \neq |k| \text{ IN GENERAL}$$

AS $\phi(x,y) \to 0$ AS $\sqrt{x^2+y^2} \to \infty$, SO WILL $\hat{\phi}(k,y)$ AS $\sqrt{k^2+y^2} \to \infty$,

SO $A(k) = 0$

$$\therefore \boxed{\hat{\phi}(k,y) = B(k) e^{-|k|y}}$$

APPLY THE BOUNDARY CONDITION $\phi(x,0) = f(x) \Rightarrow \hat{\phi}(k,0) = \hat{f}(k)$

$$\Rightarrow \hat{f}(k) = B(k) e^{0}$$

$$\Rightarrow B(k) = \hat{f}(k)$$

$$\therefore \boxed{\hat{\phi}(k,y) = \hat{f}(k) e^{-|k|y}}$$

TO INVERT WE LOOK AT THE CONVOLUTION THEOREM

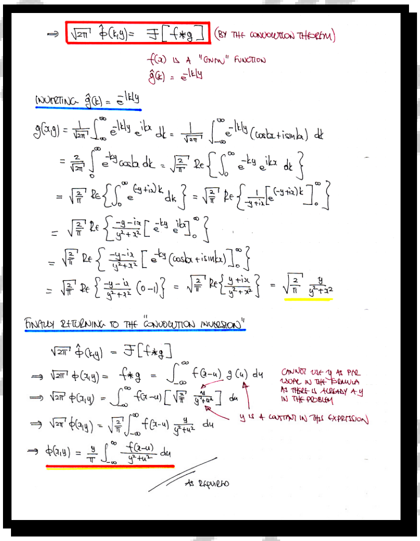
$$\boxed{\mathcal{F}[f*g] = \sqrt{2\pi}\, \mathcal{F}[f]\, \mathcal{F}[g]}$$

$$\Rightarrow \mathcal{F}[\phi(x,y)] = \mathcal{F}[f(x,0)] \times e^{-|k|y}$$

$$\Rightarrow \sqrt{2\pi}\, \mathcal{F}[\phi(x,y)] = \sqrt{2\pi}\, \mathcal{F}[f(x,0)] \times e^{-|k|y}$$

$$\Rightarrow \sqrt{2\pi}\, \hat{\phi}(k,y) = \sqrt{2\pi}\, \hat{f}(k)\, e^{-|k|y}$$

$$\Rightarrow \sqrt{2\pi}\, \hat{\phi}(k,y) = \sqrt{2\pi}\, \mathcal{F}[f]\, \mathcal{F}[g]$$



$$\Rightarrow \boxed{\sqrt{2\pi}\, \hat{\phi}(k,y) = \mathcal{F}[f*g]} \quad \text{(BY THE CONVOLUTION THEOREM)}$$

$f(x)$ IS A "GIVEN" FUNCTION

$\hat{g}(k) = e^{-|k|y}$

INVERTING $\hat{g}(k) = e^{-|k|y}$

$$g(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|k|y} e^{ikx} \, dk = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|k|y} (\cos kx + i \sin kx) \, dk$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-ky} \cos kx \, dk = \sqrt{\frac{2}{\pi}}\, \text{Re}\left\{ \int_0^{\infty} e^{-ky} e^{ikx} \, dk \right\}$$

$$= \sqrt{\frac{2}{\pi}}\, \text{Re}\left\{ \int_0^{\infty} e^{(-y+ix)k} \, dk \right\} = \sqrt{\frac{2}{\pi}}\, \text{Re}\left\{ \frac{1}{-y+ix} \left[e^{(-y+ix)k} \right]_0^{\infty} \right\}$$

$$= \sqrt{\frac{2}{\pi}}\, \text{Re}\left\{ \frac{-y-ix}{y^2+x^2} \left[e^{-ky} e^{ikx} \right]_0^{\infty} \right\}$$

$$= \sqrt{\frac{2}{\pi}}\, \text{Re}\left\{ \frac{-y-ix}{y^2+x^2} \left[e^{-ky} (\cos kx + i \sin kx) \right]_0^{\infty} \right\}$$

$$= \sqrt{\frac{2}{\pi}}\, \text{Re}\left\{ \frac{-y-ix}{y^2+x^2} (0-1) \right\} = \sqrt{\frac{2}{\pi}}\, \text{Re}\left\{ \frac{y+ix}{y^2+x^2} \right\} = \sqrt{\frac{2}{\pi}}\, \frac{y}{y^2+x^2}$$

FINALLY RETURNING TO THE "CONVOLUTION INVERSION"

$$\sqrt{2\pi}\, \hat{\phi}(k,y) = \mathcal{F}[f*g]$$

$$\Rightarrow \sqrt{2\pi}\, \phi(x,y) = f*g = \int_{-\infty}^{\infty} f(x-u)\, g(u) \, du$$

CANNOT USE y AS PER USUAL IN THE FORMULA AS THERE IS ALREADY A y IN THE PROBLEM

$$\Rightarrow \sqrt{2\pi}\, \phi(x,y) = \int_{-\infty}^{\infty} f(x-u) \left[\sqrt{\frac{2}{\pi}}\, \frac{y}{y^2+u^2} \right] du$$

y IS A CONSTANT IN THIS EXPRESSION

$$\Rightarrow \sqrt{2\pi}\, \phi(x,y) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} f(x-u)\, \frac{y}{y^2+u^2} \, du$$

$$\Rightarrow \phi(x,y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(x-u)}{y^2+u^2} \, du$$

AS REQUIRED

Question 11

The function $\varphi = \varphi(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0,$$

in the semi-infinite region of the x-y plane for which $y \geq 0$.

It is further given that for a given function $f = f(x)$

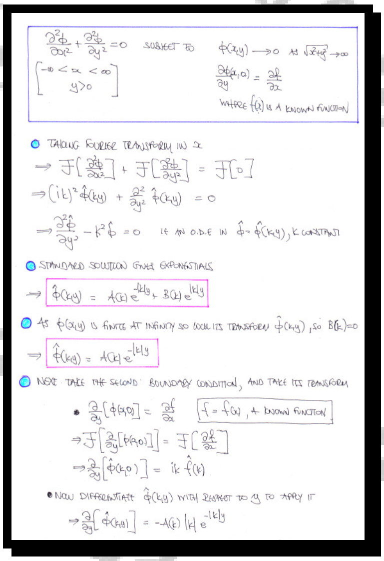
- $\frac{\partial}{\partial y}\left[\varphi(x,0)\right] = \frac{\partial}{\partial x}\left[f(x)\right]$
- $\varphi(x, y) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$

Use Fourier transforms to convert the above partial differential equation into an ordinary differential equation and hence show that

$$\varphi(x,0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(u)}{x-u}\, du .$$

$\boxed{\text{proof}}$

[**solution overleaf**]



$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$ SUBJECT TO $\phi(x,y)\longrightarrow 0$ AS $\sqrt{x^2+y^2}\to\infty$

$$\left[\begin{matrix}-\infty<x<\infty\\ y>0\end{matrix}\right]$$ $$\frac{\partial\phi}{\partial y}(x,0)=\frac{\partial f}{\partial x}$$

WHERE $f(x)$ IS A KNOWN FUNCTION

- TAKING FOURIER TRANSFORM IN x

$$\Longrightarrow \mathcal{F}\left[\frac{\partial^2\phi}{\partial x^2}\right]+\mathcal{F}\left[\frac{\partial^2\phi}{\partial y^2}\right]=\mathcal{F}[0]$$

$$\Rightarrow (ik)^2\hat{\phi}(k,y)+\frac{\partial^2}{\partial y^2}\hat{\phi}(k,y)=0$$

$$\Longrightarrow \frac{\partial^2\hat{\phi}}{\partial y^2}-k^2\hat{\phi}=0$$ IE AN O.D.E IN $\hat{\phi}=\hat{\phi}(k,y)$, k CONSTANT

- STANDARD SOLUTION GIVES EXPONENTIALS

$$\Rightarrow \boxed{\hat{\phi}(k,y)=A(k)e^{-|k|y}+B(k)e^{|k|y}}$$

- AS $\phi(x,y)$ IS FINITE AT INFINITY SO WILL ITS TRANSFORM $\hat{\phi}(k,y)$, SO $B(k)=0$

$$\Rightarrow \boxed{\hat{\phi}(k,y)=A(k)e^{-|k|y}}$$

- NEXT TAKE THE SECOND BOUNDARY CONDITION, AND TAKE ITS TRANSFORM

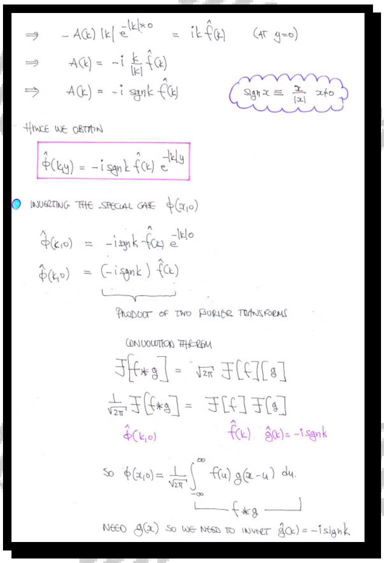
• $$\frac{\partial}{\partial y}[\phi(x,0)]=\frac{\partial f}{\partial x}$$ $\boxed{f=f(x)\text{, A KNOWN FUNCTION}}$

$$\Rightarrow \mathcal{F}\left[\frac{\partial}{\partial y}[\phi(x,0)]\right]=\mathcal{F}\left[\frac{\partial f}{\partial x}\right]$$

$$\Rightarrow \frac{\partial}{\partial y}\left[\hat{\phi}(k,0)\right]=ik\hat{f}(k)$$

• NOW DIFFERENTIATE $\hat{\phi}(k,y)$ WITH RESPECT TO y TO APPLY IT

$$\Rightarrow \frac{\partial}{\partial y}\left[\hat{\phi}(k,y)\right]=-A(k)|k|e^{-|k|y}$$



$$\Rightarrow \quad -A(k)|k|e^{-|k|\times 0}=ik\hat{f}(k) \quad (\text{AT } y=0)$$

$$\Rightarrow \quad A(k)=-i\frac{k}{|k|}\hat{f}(k)$$

$$\Rightarrow \quad A(k)=-i\,\mathrm{sgn}\,k\,\hat{f}(k)$$

$$\mathrm{sgn}\,x\equiv\frac{x}{|x|} \quad x\neq 0$$

HENCE WE OBTAIN

$$\boxed{\hat{\phi}(k,y)=-i\,\mathrm{sgn}\,k\,\hat{f}(k)\,e^{-|k|y}}$$

- INVERTING THE SPECIAL CASE $\phi(x,0)$

$$\hat{\phi}(k,0)=-i\,\mathrm{sgn}\,k\,\hat{f}(k)\,e^{-|k|0}$$

$$\hat{\phi}(k,0)=\underbrace{(-i\,\mathrm{sgn}\,k)\,\hat{f}(k)}_{\text{PRODUCT OF TWO FOURIER TRANSFORMS}}$$

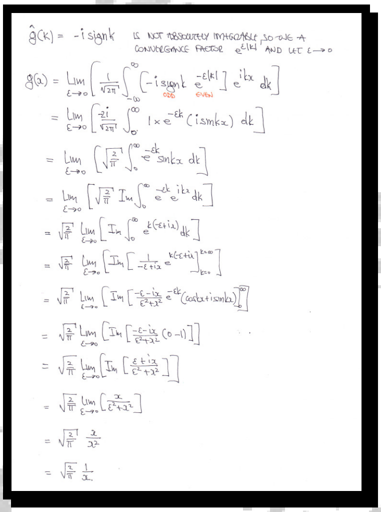
CONVOLUTION THEOREM

$$\mathcal{F}[f*g]=\sqrt{2\pi}\,\mathcal{F}[f][g]$$

$$\underbrace{\frac{1}{\sqrt{2\pi}}\mathcal{F}[f*g]}_{\hat{\phi}(k,0)}=\underbrace{\mathcal{F}[f]}_{\hat{f}(k)}\underbrace{\mathcal{F}[g]}_{\hat{g}(k)=-i\,\mathrm{sgn}\,k}$$

$$\text{SO } \phi(x,0)=\underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(u)\,g(x-u)\,du}_{f*g}.$$

NEED $g(x)$ SO WE NEED TO INVERT $\hat{g}(k)=-i\,\mathrm{sign}\,k$



$\hat{g}(k)=-i\,\mathrm{sign}\,k$ IS NOT ABSOLUTELY INTEGRABLE, SO USE A CONVERGENCE FACTOR $e^{\varepsilon|k|}$ AND LET $\varepsilon\to 0$

$$g(x)=\lim_{\varepsilon\to 0}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Big[-i\,\underset{\text{ODD}}{\mathrm{sign}\,k}\ \underset{\text{EVEN}}{e^{-\varepsilon|k|}}\Big]e^{ikx}\,dk\right]$$

$$=\lim_{\varepsilon\to 0}\left[\frac{-2i}{\sqrt{2\pi}}\int_{0}^{\infty}1\times e^{-\varepsilon k}(i\sin kx)\,dk\right]$$

$$=\lim_{\varepsilon\to 0}\left[\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}e^{-\varepsilon k}\sin kx\,dk\right]$$

$$=\lim_{\varepsilon\to 0}\left[\sqrt{\frac{2}{\pi}}\,\mathrm{Im}\int_{0}^{\infty}e^{-\varepsilon k}e^{ikx}\,dk\right]$$

$$=\sqrt{\frac{2}{\pi}}\lim_{\varepsilon\to 0}\left[\mathrm{Im}\int_{0}^{\infty}e^{k(-\varepsilon+ix)}\,dk\right]$$

$$=\sqrt{\frac{2}{\pi}}\lim_{\varepsilon\to 0}\left[\mathrm{Im}\left[\frac{1}{-\varepsilon+ix}e^{k(-\varepsilon+ix)}\right]_{k=0}^{k=\infty}\right]$$

$$=\sqrt{\frac{2}{\pi}}\lim_{\varepsilon\to 0}\left[\mathrm{Im}\left[\frac{-\varepsilon-ix}{\varepsilon^2+x^2}e^{-\varepsilon k}(\cos kx+i\sin kx)\right]_{0}^{\infty}\right]$$

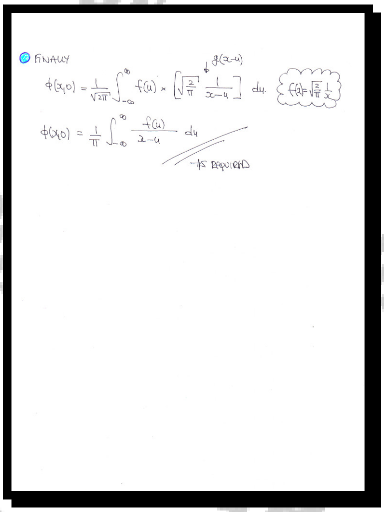
$$=\sqrt{\frac{2}{\pi}}\lim_{\varepsilon\to 0}\left[\mathrm{Im}\left[\frac{-\varepsilon-ix}{\varepsilon^2+x^2}(0-1)\right]\right]$$

$$=\sqrt{\frac{2}{\pi}}\lim_{\varepsilon\to 0}\left[\mathrm{Im}\left[\frac{\varepsilon+ix}{\varepsilon^2+x^2}\right]\right]$$

$$=\sqrt{\frac{2}{\pi}}\lim_{\varepsilon\to 0}\left[\frac{x}{\varepsilon^2+x^2}\right]$$

$$=\sqrt{\frac{2}{\pi}}\,\frac{x}{x^2}$$

$$=\sqrt{\frac{2}{\pi}}\,\frac{1}{x}.$$



- FINALLY

$$\phi(x,0)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(u)\times\overset{g(x-u)}{\left[\sqrt{\frac{2}{\pi}}\frac{1}{x-u}\right]}\,du. \qquad \left\{f(x)=\sqrt{\frac{2}{\pi}}\frac{1}{x}\right\}$$

$$\phi(x,0)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(u)}{x-u}\,du$$

AS REQUIRED

Question 12

The function $\varphi = \varphi(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0, \quad -\infty < x < \infty, \; y \geq 0.$$

It is further given that

- $\varphi(x, y) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$
- $\varphi(x, 0) = \mathrm{H}(x)$, the Heaviside function.

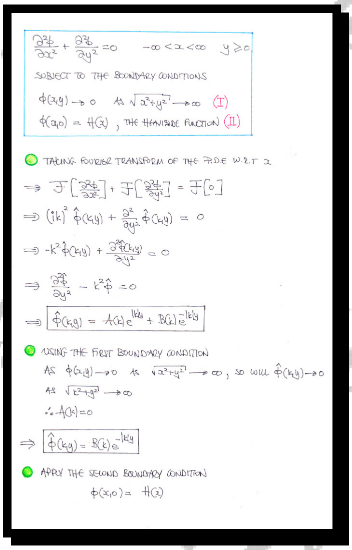
Use Fourier transforms to show that

$$\varphi(x, y) = \frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{x}{y}\right).$$

You may assume that

$$\mathcal{F}\left[\mathrm{H}(x)\right] = \frac{1}{\sqrt{2\pi}}\left[\pi \delta(k) + \frac{1}{\mathrm{i}k}\right].$$

proof



$$\boxed{\begin{array}{l}\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad -\infty < x < \infty \quad y \geq 0 \\ \text{SUBJECT TO THE BOUNDARY CONDITIONS} \\ \phi(x,y) \to 0 \quad \text{AS } \sqrt{x^2+y^2} \to \infty \quad (\mathrm{I}) \\ \phi(x,0) = H(x) \text{, THE HEAVISIDE FUNCTION } (\mathrm{II})\end{array}}$$

TAKING FOURIER TRANSFORM OF THE P.D.E W.R.T x

$$\Rightarrow \mathcal{F}\left[\frac{\partial^2 \phi}{\partial x^2}\right] + \mathcal{F}\left[\frac{\partial^2 \phi}{\partial y^2}\right] = \mathcal{F}[0]$$

$$\Rightarrow (ik)^2 \hat{\phi}(k,y) + \frac{\partial^2}{\partial y^2}\hat{\phi}(k,y) = 0$$

$$\Rightarrow -k^2 \hat{\phi}(k,y) + \frac{\partial^2 \hat{\phi}(k,y)}{\partial y^2} = 0$$

$$\Rightarrow \frac{\partial^2 \hat{\phi}}{\partial y^2} - k^2 \hat{\phi} = 0$$

$$\Rightarrow \boxed{\hat{\phi}(k,y) = A(k) e^{|k|y} + B(k) e^{-|k|y}}$$

USING THE FIRST BOUNDARY CONDITION

AS $\phi(x,y) \to 0$ AS $\sqrt{x^2+y^2} \to \infty$, SO WILL $\hat{\phi}(k,y) \to 0$

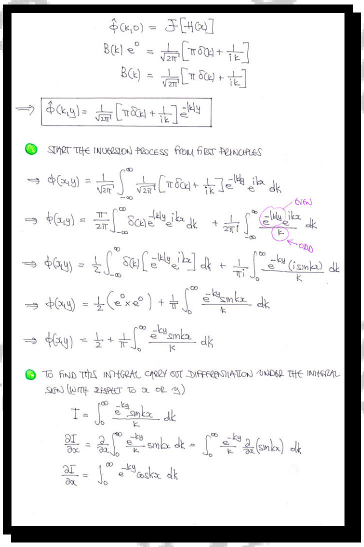
AS $\sqrt{k^2+y^2} \to \infty$

$\therefore A(k) = 0$

$$\Rightarrow \boxed{\hat{\phi}(k,y) = B(k) e^{-|k|y}}$$

APPLY THE SECOND BOUNDARY CONDITION

$$\phi(x,0) = H(x)$$



$$\hat{\phi}(k,0) = \mathcal{F}[H(x)]$$

$$B(k)e^{0} = \frac{1}{\sqrt{2\pi}}\left[\pi\delta(k) + \frac{1}{ik}\right]$$

$$B(k) = \frac{1}{\sqrt{2\pi}}\left[\pi\delta(k) + \frac{1}{ik}\right]$$

$$\Rightarrow \boxed{\hat{\phi}(k,y) = \frac{1}{\sqrt{2\pi}}\left[\pi\delta(k) + \frac{1}{ik}\right] e^{-|k|y}}$$

START THE INVERSION PROCESS FROM FIRST PRINCIPLES

$$\Rightarrow \phi(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\left[\pi\delta(k) + \frac{1}{ik}\right] e^{-|k|y} e^{ikx} \, dk$$

$$\Rightarrow \phi(x,y) = \frac{\pi}{2\pi}\int_{-\infty}^{\infty} \delta(k) e^{-|k|y} e^{ikx} \, dk + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\overset{\text{EVEN}}{e^{-|k|y}} e^{ikx}}{\underset{\text{ODD}}{k}} \, dk$$

$$\Rightarrow \phi(x,y) = \frac{1}{2}\int_{-\infty}^{\infty} \delta(k)\left[e^{-|k|y} e^{ikx}\right] dk + \frac{1}{\pi i}\int_{0}^{\infty} \frac{e^{-ky}(i\sin kx)}{k} \, dk$$

$$\Rightarrow \phi(x,y) = \frac{1}{2}\left(e^{0} \times e^{0}\right) + \frac{1}{\pi}\int_{0}^{\infty} \frac{e^{-ky}\sin kx}{k} \, dk$$

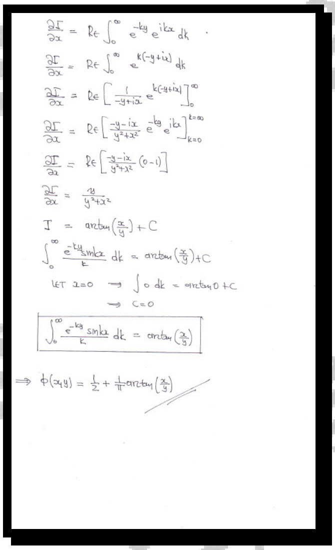
$$\Rightarrow \phi(x,y) = \frac{1}{2} + \frac{1}{\pi}\int_{0}^{\infty} \frac{e^{-ky}\sin kx}{k} \, dk$$

TO FIND THIS INTEGRAL CARRY OUT DIFFERENTIATION UNDER THE INTEGRAL SIGN (WITH RESPECT TO x OR y)

$$I = \int_{0}^{\infty} \frac{e^{-ky}\sin kx}{k} \, dk$$

$$\frac{\partial I}{\partial x} = \frac{\partial}{\partial x}\int_{0}^{\infty} \frac{e^{-ky}}{k} \sin kx \, dk = \int_{0}^{\infty} \frac{e^{-ky}}{k}\frac{\partial}{\partial x}(\sin kx) \, dk$$

$$\frac{\partial I}{\partial x} = \int_{0}^{\infty} e^{-ky} \cos kx \, dk$$



$$\frac{\partial I}{\partial x} = \mathrm{Re}\int_{0}^{\infty} e^{-ky} e^{ikx} \, dk$$

$$\frac{\partial I}{\partial x} = \mathrm{Re}\int_{0}^{\infty} e^{k(-y+ix)} \, dk$$

$$\frac{\partial I}{\partial x} = \mathrm{Re}\left[\frac{1}{-y+ix} e^{k(-y+ix)}\right]_{0}^{\infty}$$

$$\frac{\partial I}{\partial x} = \mathrm{Re}\left[\frac{-y-ix}{y^2+x^2} e^{-ky} e^{ikx}\right]_{k=0}^{k=\infty}$$

$$\frac{\partial I}{\partial x} = \mathrm{Re}\left[\frac{-y-ix}{y^2+x^2}(0-1)\right]$$

$$\frac{\partial I}{\partial x} = \frac{y}{y^2+x^2}$$

$$I = \arctan\left(\frac{x}{y}\right) + C$$

$$\int_{0}^{\infty} \frac{e^{-ky}\sin kx}{k} \, dk = \arctan\left(\frac{x}{y}\right) + C$$

LET $x = 0 \Rightarrow \int 0 \, dk = \arctan 0 + C$

$\Rightarrow C = 0$

$$\boxed{\int_{0}^{\infty} \frac{e^{-ky}\sin kx}{k} \, dk = \arctan\left(\frac{x}{y}\right)}$$

$$\Rightarrow \phi(x,y) = \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{x}{y}\right)$$

Question 13

The temperature $\theta(x,t)$ in a semi-infinite thin rod satisfies the heat equation

$$\frac{\partial^2\theta}{\partial x^2}=\frac{\partial\theta}{\partial t}, \quad x \geq 0, \quad t \geq 0.$$

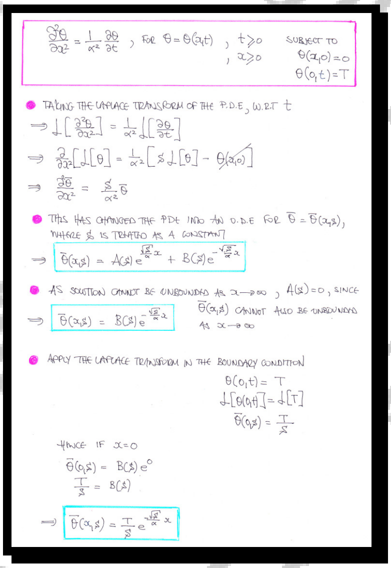
The initial temperature of the rod is 0 °C, and for $t>0$ the endpoint at $x=0$ is maintained at T °C.

Assuming the rod is insulated along its length, use Laplace transforms to find an expression for $\theta(x,t)$.

You may assume that

- $\mathcal{L}^{-1}\left[\frac{\mathrm{e}^{-\sqrt{s}}}{s}\right]=\operatorname{erfc}\left(\frac{1}{2\sqrt{t}}\right)$
- $\mathcal{L}^{-1}\left[\bar{f}(k\,s)\right]=\frac{1}{k}f\left(\frac{t}{k}\right)$, where k is a constant.

$$\boxed{\theta(x,t)=\frac{2T}{\sqrt{\pi}}\int_{\frac{x}{2\alpha\sqrt{t}}}^{\infty}\mathrm{e}^{-u^2}\,du \;=\; T\operatorname{erfc}\left(\frac{x}{2\alpha\sqrt{t}}\right)}$$



$$\boxed{\frac{\partial^2\theta}{\partial x^2}=\frac{1}{\alpha^2}\frac{\partial\theta}{\partial t}, \text{ FOR } \theta=\theta(x,t), \; t\geqslant 0, \; x\geqslant 0 \quad \text{SUBJECT TO } \theta(x,0)=0, \; \theta(0,t)=T}$$

- TAKING THE LAPLACE TRANSFORM OF THE P.D.E, W.R.T t

$\Rightarrow \mathcal{L}\left[\frac{\partial^2\theta}{\partial x^2}\right]=\frac{1}{\alpha^2}\mathcal{L}\left[\frac{\partial\theta}{\partial t}\right]$

$\Rightarrow \frac{\partial}{\partial x^2}\left[\mathcal{L}[\theta]\right]=\frac{1}{\alpha^2}\left[s\mathcal{L}[\theta]-\cancel{\theta(x,0)}\right]$

$\Rightarrow \frac{\partial^2\bar{\theta}}{\partial x^2}=\frac{s}{\alpha^2}\bar{\theta}$

- THIS HAS CHANGED THE PDE INTO AN O.D.E FOR $\bar{\theta}=\bar{\theta}(x,s)$, WHERE s IS TREATED AS A CONSTANT

$\Rightarrow \boxed{\bar{\theta}(x,s)=A(s)\mathrm{e}^{\frac{\sqrt{s}}{\alpha}x}+B(s)\mathrm{e}^{-\frac{\sqrt{s}}{\alpha}x}}$

- AS SOLUTION CANNOT BE UNBOUNDED AS $x\to\infty$, $A(s)=0$, SINCE $\bar{\theta}(x,s)$ CANNOT ALSO BE UNBOUNDED AS $x\to\infty$

$\Rightarrow \boxed{\bar{\theta}(x,s)=B(s)\mathrm{e}^{-\frac{\sqrt{s}}{\alpha}x}}$

- APPLY THE LAPLACE TRANSFORM IN THE BOUNDARY CONDITION

$\theta(0,t)=T$

$\mathcal{L}[\theta(0,t)]=\mathcal{L}[T]$

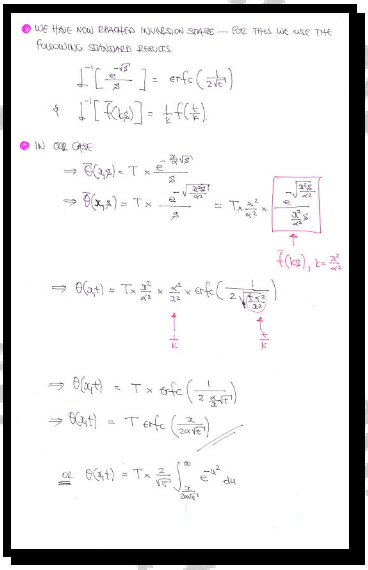
$\bar{\theta}(0,s)=\frac{T}{s}$

HENCE IF $x=0$

$\bar{\theta}(0,s)=B(s)\mathrm{e}^{0}$

$\frac{T}{s}=B(s)$

$\Rightarrow \boxed{\bar{\theta}(x,s)=\frac{T}{s}\mathrm{e}^{-\frac{\sqrt{s}}{\alpha}x}}$



- WE HAVE NOW REACHED INVERSION STAGE — FOR THIS WE USE THE FOLLOWING STANDARD RESULTS

$\mathcal{L}^{-1}\left[\frac{\mathrm{e}^{-\sqrt{s}}}{s}\right]=\operatorname{erfc}\left(\frac{1}{2\sqrt{t}}\right)$

& $\mathcal{L}^{-1}\left[\bar{f}(ks)\right]=\frac{1}{k}f\left(\frac{t}{k}\right)$.

- IN OUR CASE

$\Rightarrow \bar{\theta}(x,s)=T\times\frac{\mathrm{e}^{-\frac{x}{\alpha}\sqrt{s}}}{s}$

$\Rightarrow \bar{\theta}(x,s)=T\times\frac{\mathrm{e}^{-\sqrt{\frac{x^2s}{\alpha^2}}}}{s}=T\times\frac{x^2}{\alpha^2}\times\underbrace{\boxed{\frac{\mathrm{e}^{-\sqrt{\frac{x^2s}{\alpha^2}}}}{\frac{x^2}{\alpha^2}s}}}_{\bar{f}(ks),\; k=\frac{x^2}{\alpha^2}}$

$\Rightarrow \theta(x,t)=T\times\underbrace{\frac{x^2}{\alpha^2}\times\frac{\alpha^2}{x^2}}_{\frac{1}{k}}\times\operatorname{erfc}\left(\frac{1}{2\sqrt{\underbrace{\frac{t\alpha^2}{x^2}}_{\frac{t}{k}}}}\right)$

$\Rightarrow \theta(x,t)=T\times\operatorname{erfc}\left(\frac{1}{2\frac{\alpha}{x}\sqrt{t}}\right)$

$\Rightarrow \theta(x,t)=T\operatorname{erfc}\left(\frac{x}{2\alpha\sqrt{t}}\right)$ //

OR $\theta(x,t)=T\times\frac{2}{\sqrt{\pi}}\int_{\frac{x}{2\alpha\sqrt{t}}}^{\infty}\mathrm{e}^{-u^2}\,du$

Question 14

The function $u = u(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad -\infty < x < \infty, \quad 0 < y < 1.$$

It is further given that

- $u(x,0) = 0$
- $u(x,1) = f(x)$

  where $f(-x) = f(x)$ and $f(x) \to 0$ as $x \to \infty$

a) Use Fourier transforms to show that

$$u(x, y) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{\hat{f}(k) \cos kx \sinh ky}{\sinh k} \, dk, \quad \hat{f}(k) = \mathcal{F}\left[f(x) \right].$$

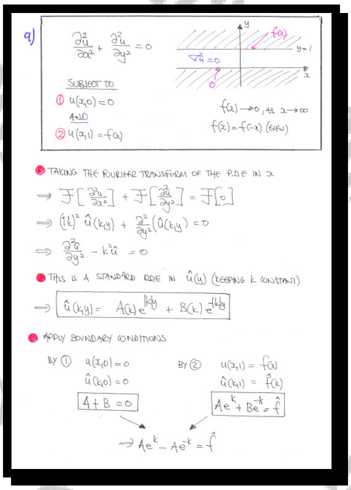
b) Given that $f(x) = \delta(x)$ show further that

$$u(x, y) = \frac{\sin \pi y}{2[\cosh \pi x + \cos \pi y]}.$$

You may assume without proof

$$\int_0^{\infty} \frac{\cos Au \sinh Bu}{\sinh Cu} \, du = \frac{\pi}{2C} \left[\frac{\sin(B\pi/C)}{\cosh(A\pi/C) + \cos(B\pi/C)} \right], \quad 0 \le B < C.$$

proof



a) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

SUBJECT TO

① $u(x,0) = 0$

AND

② $u(x,1) = f(x)$

$f(x) \to 0$, AS $x \to \infty$

$f(x) = f(-x)$ (EVEN)

- TAKING THE FOURIER TRANSFORM OF THE P.D.E IN x

$\Rightarrow \mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] + \mathcal{F}\left[\frac{\partial^2 u}{\partial y^2}\right] = \mathcal{F}[0]$

$\Rightarrow (ik)^2 \hat{u}(k,y) + \frac{\partial^2}{\partial y^2}(\hat{u}(k,y) = 0$

$\Rightarrow \frac{\partial^2 \hat{u}}{\partial y^2} - k^2 \hat{u} = 0$

- THIS IS A STANDARD O.D.E. IN $\hat{u}(y)$ (KEEPING k CONSTANT)

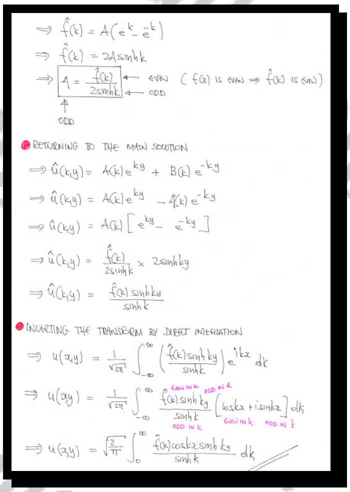
$\Rightarrow \boxed{\hat{u}(k,y) = A(k) e^{|k|y} + B(k) e^{-|k|y}}$

- APPLY BOUNDARY CONDITIONS

BY ① $u(x,0) = 0$
$\hat{u}(k,0) = 0$
$\boxed{A + B = 0}$

BY ② $u(x,1) = f(x)$
$\hat{u}(k,1) = \hat{f}(k)$
$\boxed{Ae^{k} + Be^{-k} = \hat{f}}$

$\Rightarrow Ae^{k} - Ae^{-k} = \hat{f}$



$\Rightarrow \hat{f}(k) = A(e^{k} - e^{-k})$

$\Rightarrow \hat{f}(k) = 2A\sinh k$

$\Rightarrow \boxed{A = \frac{\hat{f}(k)}{2\sinh k}}$ ← EVEN ($f(x)$ IS EVEN $\Rightarrow \hat{f}(k)$ IS EVEN) ← ODD

↑ ODD

- RETURNING TO THE MAIN SOLUTION

$\Rightarrow \hat{u}(k,y) = A(k) e^{ky} + B(k) e^{-ky}$

$\Rightarrow \hat{u}(k,y) = A(k) e^{ky} - A(k) e^{-ky}$

$\Rightarrow \hat{u}(k,y) = A(k)\left[e^{ky} - e^{-ky}\right]$

$\Rightarrow \hat{u}(k,y) = \frac{\hat{f}(k)}{2\sinh k} \times 2\sinh ky$

$\Rightarrow \hat{u}(k,y) = \frac{\hat{f}(k)\sinh ky}{\sinh k}$

- INVERTING THE TRANSFORM BY DIRECT INTEGRATION

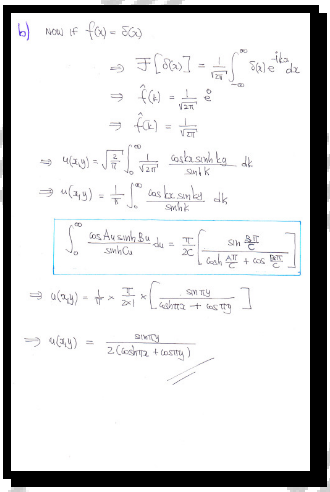
$\Rightarrow u(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{\hat{f}(k)\sinh ky}{\sinh k}\right) e^{ikx} \, dk$

EVEN IN k ODD IN k

$\Rightarrow u(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\hat{f}(k)\sinh ky}{\sinh k} \left[\cos kx + i\sin kx\right] dk$

ODD IN k EVEN IN k ODD IN k

$\Rightarrow u(x,y) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \frac{\hat{f}(k)\cos kx \sinh ky}{\sinh k} \, dk$



b) NOW IF $f(x) = \delta(x)$

$\Rightarrow \mathcal{F}[\delta(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x) e^{-ikx} \, dx$

$\Rightarrow \hat{f}(k) = \frac{1}{\sqrt{2\pi}} e^{0}$

$\Rightarrow \hat{f}(k) = \frac{1}{\sqrt{2\pi}}$

$\Rightarrow u(x,y) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{\cos kx \sinh ky}{\sinh k} \, dk$

$\Rightarrow u(x,y) = \frac{1}{\pi} \int_{0}^{\infty} \frac{\cos kx \sin ky}{\sinh k} \, dk$

$\boxed{\int_0^{\infty} \frac{\cos Au \sinh Bu}{\sinh Cu} \, du = \frac{\pi}{2C}\left[\frac{\sin \frac{B\pi}{C}}{\cosh \frac{A\pi}{C} + \cos \frac{B\pi}{C}}\right]}$

$\Rightarrow u(x,y) = \frac{1}{\pi} \times \frac{\pi}{2 \times 1} \times \left[\frac{\sin \pi y}{\cosh \pi x + \cos \pi y}\right]$

$\Rightarrow u(x,y) = \frac{\sin \pi y}{2(\cosh \pi x + \cos \pi y)}$

Question 15

The function $\psi = \psi(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0,$$

in the part of the x-y plane for which $y \geq 0$.

It is further given that

- $\psi(x,0) = f(x)$
- $\psi(x, y) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$

c) Use Fourier transforms to convert the above partial differential equation into an ordinary differential equation and hence show that

$$\psi(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(u)}{(x-u)^2 + y^2} \, du.$$

d) Evaluate the above integral for …

i. … $f(x) = 1$.

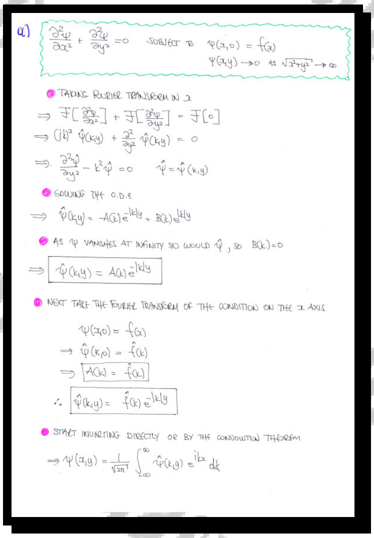
ii. … $f(x) = \operatorname{sgn} x$

iii. … $f(x) = \mathrm{H}(x)$

commenting further whether these answers are consistent.

$\boxed{\psi(x, y) = 1}$, $\boxed{\psi(x, y) = \frac{2}{\pi}\arctan\left(\frac{x}{y}\right)}$, $\boxed{\psi(x, y) = \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{x}{y}\right)}$

[solution overleaf]



a) $$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0 \quad \text{SUBJECT TO} \quad \psi(x,0)=f(x)$$
$$\psi(x,y)\to 0 \text{ AS } \sqrt{x^2+y^2}\to\infty$$

- TAKING FOURIER TRANSFORM IN x

$$\Rightarrow \mathcal{F}\left[\frac{\partial^2\psi}{\partial x^2}\right]+\mathcal{F}\left[\frac{\partial^2\psi}{\partial y^2}\right]=\mathcal{F}[0]$$
$$\Rightarrow (ik)^2\,\hat{\psi}(k,y)+\frac{\partial^2}{\partial y^2}\hat{\psi}(k,y)=0$$
$$\Rightarrow \frac{\partial^2\hat{\psi}}{\partial y^2}-k^2\hat{\psi}=0 \qquad \hat{\psi}=\hat{\psi}(k,y)$$

- SOLVING THE O.D.E

$$\Rightarrow \hat{\psi}(k,y)=A(k)e^{-|k|y}+B(k)e^{|k|y}$$

- AS ψ VANISHES AT INFINITY SO WOULD $\hat{\psi}$, SO $B(k)=0$

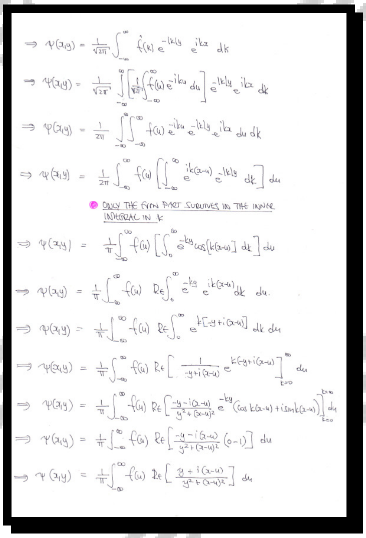
$$\Rightarrow \boxed{\hat{\psi}(k,y)=A(k)e^{-|k|y}}$$

- NEXT TAKE THE FOURIER TRANSFORM OF THE CONDITION ON THE x AXIS

$$\psi(x,0)=f(x)$$
$$\Rightarrow \hat{\psi}(k,0)=\hat{f}(k)$$
$$\Rightarrow \boxed{A(k)=\hat{f}(k)}$$
$$\therefore \boxed{\hat{\psi}(k,y)=\hat{f}(k)e^{-|k|y}}$$

- START INVERTING DIRECTLY OR BY THE CONVOLUTION THEOREM

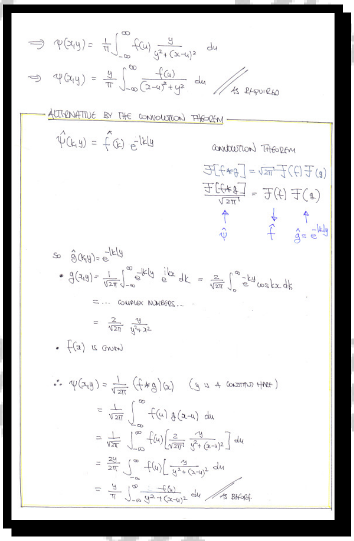
$$\Rightarrow \psi(x,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{\psi}(k,y)\,e^{ikx}\,dk$$



$$\Rightarrow \psi(x,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(k)\,e^{-|k|y}\,e^{ikx}\,dk$$
$$\Rightarrow \psi(x,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(u)\,e^{-iku}\,du\right]e^{-|k|y}e^{ikx}\,dk$$
$$\Rightarrow \psi(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(u)\,e^{-iku}\,e^{-|k|y}e^{ikx}\,du\,dk$$
$$\Rightarrow \psi(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(u)\left[\int_{-\infty}^{\infty}e^{ik(x-u)}\,e^{-|k|y}\,dk\right]du$$

- ONLY THE EVEN PART SURVIVES IN THE INNER INTEGRAL IN k

$$\Rightarrow \psi(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(u)\left[\int_0^{\infty}e^{-ky}\cos[k(x-u)]\,dk\right]du$$
$$\Rightarrow \psi(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(u)\ \mathrm{Re}\int_0^{\infty}e^{-ky}\,e^{ik(x-u)}\,dk\ du.$$
$$\Rightarrow \psi(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(u)\ \mathrm{Re}\int_0^{\infty}e^{k[-y+i(x-u)]}\,dk\,du$$
$$\Rightarrow \psi(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(u)\ \mathrm{Re}\left[\frac{1}{-y+i(x-u)}e^{k(-y+i(x-u))}\right]_{k=0}^{\infty}du$$
$$\Rightarrow \psi(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(u)\ \mathrm{Re}\left[\frac{-y-i(x-u)}{y^2+(x-u)^2}e^{-ky}(\cos k(x-u)+i\sin k(x-u))\right]_{k=0}^{k=\infty}du$$
$$\Rightarrow \psi(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(u)\ \mathrm{Re}\left[\frac{-y-i(x-u)}{y^2+(x-u)^2}(0-1)\right]du$$
$$\Rightarrow \psi(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(u)\ \mathrm{Re}\left[\frac{y+i(x-u)}{y^2+(x-u)^2}\right]du$$



$$\Rightarrow \psi(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(u)\frac{y}{y^2+(x-u)^2}\,du$$
$$\Rightarrow \psi(x,y)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(u)}{(x-u)^2+y^2}\,du \quad \text{// AS REQUIRED}$$

ALTERNATIVE BY THE CONVOLUTION THEOREM

$$\hat{\psi}(k,y)=\hat{f}(k)\,e^{-|k|y}$$

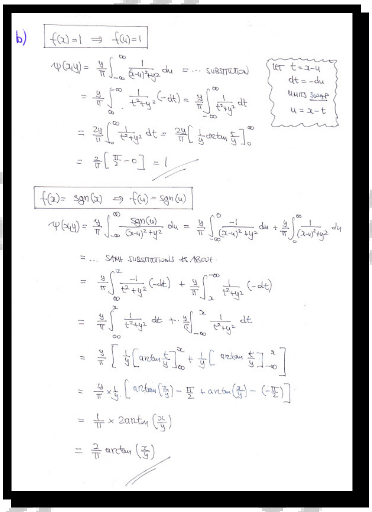
CONVOLUTION THEOREM

$$\mathcal{F}[f*g]=\sqrt{2\pi}\,\mathcal{F}(f)\,\mathcal{F}(g)$$
$$\frac{\mathcal{F}[f*g]}{\sqrt{2\pi}}=\mathcal{F}(f)\,\mathcal{F}(g)$$
$$\uparrow \hat{\psi} \qquad \downarrow \hat{f} \qquad \uparrow \hat{g}=e^{-|k|y}$$

SO $\hat{g}(k,y)=e^{-|k|y}$

- $$g(x,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-|k|y}\,e^{ikx}\,dk=\frac{2}{\sqrt{2\pi}}\int_0^{\infty}e^{-ky}\cos kx\,dk$$
$$=\dots \text{ COMPLEX NUMBERS} \dots$$
$$=\frac{2}{\sqrt{2\pi}}\,\frac{y}{y^2+x^2}$$
- $f(x)$ IS GIVEN

$$\therefore \psi(x,y)=\frac{1}{\sqrt{2\pi}}(f*g)(x) \quad (y \text{ IS A CONSTANT HERE})$$
$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(u)\,g(x-u)\,du$$
$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(u)\left[\frac{2}{\sqrt{2\pi}}\,\frac{y}{y^2+(x-u)^2}\right]du$$
$$=\frac{2y}{2\pi}\int_{-\infty}^{\infty}f(u)\left[\frac{y}{y^2+(x-u)^2}\right.du$$
$$=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(u)}{y^2+(x-u)^2}\,du \quad \text{// AS BEFORE.}$$



b) $\boxed{f(x)=1 \Rightarrow f(u)=1}$

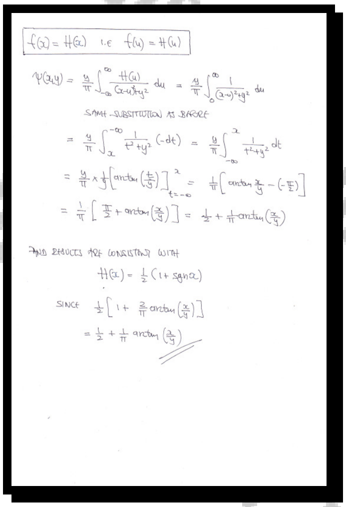
$$\psi(x,y)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{1}{(x-u)^2+y^2}\,du=\dots \text{ SUBSTITUTION}$$

LET $t=x-u$
$dt=-du$
LIMITS SWAP
$u=x-t$

$$=\frac{y}{\pi}\int_{\infty}^{-\infty}\frac{1}{t^2+y^2}(-dt)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{1}{t^2+y^2}\,dt$$
$$=\frac{2y}{\pi}\int_0^{\infty}\frac{1}{t^2+y^2}\,dt=\frac{2y}{\pi}\left[\frac{1}{y}\arctan\frac{t}{y}\right]_0^{\infty}$$
$$=\frac{2}{\pi}\left[\frac{\pi}{2}-0\right]=1 \quad \text{//}$$

$\boxed{f(x)=\mathrm{sgn}(x) \Rightarrow f(u)=\mathrm{sgn}(u)}$

$$\psi(x,y)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{sgn}(u)}{(x-u)^2+y^2}\,du=\frac{y}{\pi}\int_{-\infty}^{0}\frac{-1}{(x-u)^2+y^2}\,du+\frac{y}{\pi}\int_0^{\infty}\frac{1}{(x-u)^2+y^2}\,du$$
$$=\dots \text{ SAME SUBSTITUTIONS AS ABOVE.}$$
$$=\frac{y}{\pi}\int_{\infty}^{x}\frac{-1}{t^2+y^2}(-dt)+\frac{y}{\pi}\int_{x}^{-\infty}\frac{1}{t^2+y^2}(-dt)$$
$$=\frac{y}{\pi}\int_{\infty}^{x}\frac{1}{t^2+y^2}\,dt+\frac{y}{\pi}\int_{-\infty}^{x}\frac{1}{t^2+y^2}\,dt$$
$$=\frac{y}{\pi}\left[\frac{1}{y}\left[\arctan\frac{t}{y}\right]_{\infty}^{x}+\frac{1}{y}\left[\arctan\frac{t}{y}\right]_{-\infty}^{x}\right]$$
$$=\frac{y}{\pi}\times\frac{1}{y}\left[\arctan\left(\frac{x}{y}\right)-\frac{\pi}{2}+\arctan\left(\frac{x}{y}\right)-\left(-\frac{\pi}{2}\right)\right]$$
$$=\frac{1}{\pi}\times 2\arctan\left(\frac{x}{y}\right)$$
$$=\frac{2}{\pi}\arctan\left(\frac{x}{y}\right) \quad \text{//}$$



$\boxed{f(x)=H(x) \quad \text{I.E} \quad f(u)=H(u)}$

$$\psi(x,y)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{H(u)}{(x-u)^2+y^2}\,du=\frac{y}{\pi}\int_0^{\infty}\frac{1}{(x-u)^2+y^2}\,du$$

SAME SUBSTITUTION AS BEFORE

$$=\frac{y}{\pi}\int_{x}^{-\infty}\frac{1}{t^2+y^2}(-dt)=\frac{y}{\pi}\int_{-\infty}^{x}\frac{1}{t^2+y^2}\,dt$$
$$=\frac{y}{\pi}\times\frac{1}{y}\left[\arctan\left(\frac{t}{y}\right)\right]_{t=-\infty}^{x}=\frac{1}{\pi}\left[\arctan\frac{x}{y}-\left(-\frac{\pi}{2}\right)\right]$$
$$=\frac{1}{\pi}\left[\frac{\pi}{2}+\arctan\left(\frac{x}{y}\right)\right]=\frac{1}{2}+\frac{1}{\pi}\arctan\left(\frac{x}{y}\right)$$

AND RESULTS ARE CONSISTENT WITH

$$H(x)=\frac{1}{2}(1+\mathrm{sgn}\,x)$$

SINCE $$\frac{1}{2}\left[1+\frac{2}{\pi}\arctan\left(\frac{x}{y}\right)\right]$$
$$=\frac{1}{2}+\frac{1}{\pi}\arctan\left(\frac{x}{y}\right) \quad \text{//}$$

Question 16

The function $u = u(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 ,$$

in the part of the x-y plane for which $x \geq 0$ and $y \geq 0$.

It is further given that

- $u(0, y) = 0$
- $u(x, y) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$
- $u(x,0) = f(x)$, $f(0) = 0$, $f(x) \to 0$ as $x \to \infty$

Use Fourier transforms to show that

$$u(x, y) = \frac{y}{\pi} \int_0^{\infty} f(w) \left[\frac{1}{y^2 + (x-w)^2} - \frac{1}{y^2 + (x+w)^2} \right] dw .$$

$\boxed{\text{proof}}$

[solution overleaf]

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

SUBJECT TO

① $u(x,y)\to 0$ AS $\sqrt{x^2+y^2}\to\infty$

② $u(0,y)=0$

③ $u(x,0)=f(x)$, $f(0)=0$

$f(x)\to 0$ AS $x\to\infty$

- ALTHOUGH THE REGION IS NOT SYMMETRICAL IN x (OR IN y), EXTEND $u(x,y)$ & $f(x)$ IN THE NEGATIVE x DIRECTION, SO BOTH ARE ODD
- TAKE FOURIER TRANSFORM OF THE P.D.E IN x

$$\Rightarrow \mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right]+\mathcal{F}\left[\frac{\partial^2 u}{\partial y^2}\right]=\mathcal{F}[0]$$

$$\Rightarrow (ik)^2\hat{u}(k,y)+\frac{\partial^2}{\partial y^2}\hat{u}(k,y)=0$$

$$\Rightarrow \frac{\partial^2\hat{u}}{\partial y^2}-k^2\hat{u}=0$$

$$\Rightarrow \boxed{\hat{u}(k,y)=A(k)e^{|k|y}+B(k)e^{-|k|y}}$$

- APPLY BOUNDARY CONDITION ①

IF $u(x,y)\to 0$ AS $\sqrt{x^2+y^2}\to\infty$, THEN $\hat{u}(k,y)\to 0$ AS $\sqrt{k^2+y^2}\to\infty$

$\therefore A(k)=0$

$$\Rightarrow \boxed{\hat{u}(k,y)=B(k)e^{-|k|y}}$$

- APPLY BOUNDARY CONDITION ③

$$\Rightarrow u(x,0)=f(x)$$

$$\Rightarrow \hat{u}(k,0)=\hat{f}(k)$$

$$\Rightarrow \hat{f}(k)=B(k)e^{0}$$

$$\Rightarrow B(k)=\hat{f}(k)$$

$$\therefore \boxed{u(k,y)=\hat{f}(k)e^{-|k|y}}$$

- AS WE DO NOT KNOW $f(x)$ EXPLICITLY, WE START THE INVERSION FROM FIRST PRINCIPLES

$$\Rightarrow u(x,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{u}(k,y)e^{ikx}\,dk$$

- AS $u(x,y)$ IS ODD (WE BUILT ODD EXTENSION), $\hat{u}(k,y)$ WILL ALSO BE ODD

$$\Rightarrow u(x,y)=\frac{2}{\sqrt{2\pi}}\int_0^{\infty}\hat{u}(k,y)i\sin kx\,dk$$

$$\Rightarrow u(x,y)=\sqrt{\frac{2}{\pi}}\,i\int_0^{\infty}\hat{f}(k)\,e^{-ky}\sin kx\,dk$$

$$\Rightarrow u(x,y)=\sqrt{\frac{2}{\pi}}\,i\int_0^{\infty}e^{-ky}\sin kx\underbrace{\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(w)e^{-ikw}\,dw\right]}_{\hat{f}(k)}dk$$

- AS f IS "BUILT" ODD, ONLY THE ODD (SINE PART) SURVIVES

$$\Rightarrow u(x,y)=\sqrt{\frac{2}{\pi}}\,i\int_0^{\infty}e^{-ky}\sin kx\left[-\sqrt{\frac{2}{\pi}}\,i\int_0^{\infty}f(w)\sin kw\,dw\right]dk$$

$$\Rightarrow u(x,y)=-\frac{2}{\pi}(i)^2\int_0^{\infty}e^{-ky}\sin kx\left[\int_0^{\infty}f(w)\sin kw\,dw\right]dk$$

$$\Rightarrow u(x,y)=\frac{2}{\pi}\int_0^{\infty}\int_0^{\infty}e^{-ky}f(w)\sin kx\sin kw\,dw\,dk$$

- REVERSING THE ORDER OF INTEGRATION NOTING THAT THE LIMITS ARE UNCHANGED ("BOX" REGION FROM 0 TO ∞)

$$\Rightarrow u(x,y)=\frac{2}{\pi}\int_{w=0}^{\infty}f(w)\left[\int_{k=0}^{\infty}e^{-ky}\sin kx\sin kw\,dk\right]dw$$

NEED TO DERIVE AN IDENTITY

$$\cos(kx+kw)\equiv\cos kx\cos kw-\sin kx\sin kw$$

$$\cos(kx-kw)\equiv\cos kx\cos kw+\sin kx\sin kw$$

$$\cos(kx-kw)-\cos(kx+kw)\equiv 2\sin kx\sin kw$$

$$\Rightarrow u(x,y)=\frac{1}{\pi}\int_{w=0}^{\infty}f(w)\left[\int_{k=0}^{\infty}e^{-ky}\cos[k(x-w)]-e^{-ky}\cos[k(x+w)]\,dk\right]dw$$

- LOOKING AT EACH OF THE "INNER" INTEGRALS

$$\int_{k=0}^{\infty}e^{-ky}\cos[k(x-w)]\,dk=\mathrm{Re}\int_{k=0}^{\infty}e^{-ky}e^{ik(x-w)}\,dk$$

$$=\mathrm{Re}\int_{k=0}^{\infty}e^{k[-y+i(x-w)]}\,dk=\mathrm{Re}\left[\frac{1}{-y+i(x-w)}e^{k(-y+i(x-w))}\right]_{k=0}^{\infty}$$

$$=\mathrm{Re}\left[\frac{-y-i(x-w)}{y^2+(x-w)^2}e^{-ky}e^{ik(x-w)}\right]_{k=0}^{\infty}=\mathrm{Re}\left[\frac{-y-i(x-w)}{y^2+(x-w)^2}(0-1)\right]$$

$$=\frac{y}{y^2+(x-w)^2}$$

SIMILARLY THE OTHER INTEGRAL EXCEPT THAT WE HAVE $(x+w)$ INSTEAD OF $(x-w)$

- $$u(x,y)=\frac{1}{\pi}\int_0^{\infty}f(w)\left[\frac{y}{y^2+(x-w)^2}-\frac{y}{y^2+(x+w)^2}\right]dw$$

$$u(x,y)=\frac{y}{\pi}\int_0^{\infty}f(w)\left[\frac{1}{y^2+(x-w)^2}-\frac{1}{y^2+(x+w)^2}\right]dw$$

Question 17

The function $\theta = \theta(x,t)$ satisfies the heat equation in one spatial dimension,

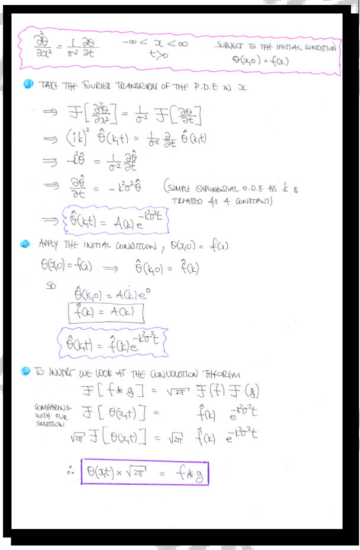
$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\sigma^2}\frac{\partial \theta}{\partial t}, \quad -\infty < x < \infty\,,\ t \ge 0\,,$$

where σ is a positive constant.

Given further that $\theta(x,0) = f(x)$, use Fourier transforms to convert the above partial differential equation into an ordinary differential equation and hence show that

$$\theta(x,t) = \frac{1}{2\sigma\sqrt{\pi t}}\int_{-\infty}^{\infty} f(x-u)\exp\left(\frac{u^2}{4t\sigma^2}\right)\,du\,.$$

proof



$$\boxed{\frac{\partial^2\theta}{\partial x^2} = \frac{1}{\sigma^2}\frac{\partial\theta}{\partial t} \qquad -\infty < x < \infty,\ t>0 \qquad \text{SUBJECT TO THE INITIAL CONDITION } \theta(x,0)=f(x)}$$

- TAKE THE FOURIER TRANSFORM OF THE P.D.E IN x

$$\Rightarrow \mathcal{F}\left[\frac{\partial^2\theta}{\partial x^2}\right] = \frac{1}{\sigma^2}\mathcal{F}\left[\frac{\partial\theta}{\partial t}\right]$$

$$\Rightarrow (ik)^2\,\hat\theta(k,t) = \frac{1}{\sigma^2}\frac{\partial}{\partial t}\hat\theta(k,t)$$

$$\Rightarrow -k^2\hat\theta = \frac{1}{\sigma^2}\frac{\partial\hat\theta}{\partial t}$$

$$\Rightarrow \frac{\partial\hat\theta}{\partial t} = -k^2\sigma^2\hat\theta$$

(SIMPLE EXPONENTIAL O.D.E AS k IS TREATED AS A CONSTANT)

$$\Rightarrow \boxed{\hat\theta(k,t) = A(k)e^{-k^2\sigma^2 t}}$$

- APPLY THE INITIAL CONDITION, $\theta(x,0)=f(x)$

$$\theta(x,0) = f(x) \Rightarrow \hat\theta(k,0) = \hat f(k)$$

SO

$$\hat\theta(k,0) = A(k)e^0$$

$$\boxed{\hat f(k) = A(k)}$$

$$\boxed{\hat\theta(k,t) = \hat f(k)e^{-k^2\sigma^2 t}}$$

- TO INVERT WE LOOK AT THE CONVOLUTION THEOREM

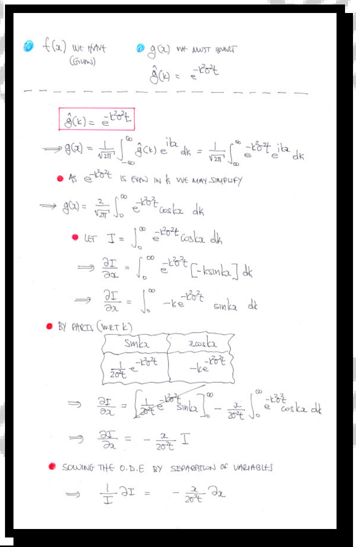
$$\mathcal{F}[f*g] = \sqrt{2\pi}\,\mathcal{F}(f)\,\mathcal{F}(g)$$

COMPARING WITH OUR SOLUTION

$$\mathcal{F}[\theta(x,t)] = \hat f(k)\ e^{-k^2\sigma^2 t}$$

$$\sqrt{2\pi}\,\mathcal{F}[\theta(x,t)] = \sqrt{2\pi}\,\hat f(k)\ e^{-k^2\sigma^2 t}$$

$$\therefore \boxed{\theta(x,t)\times\sqrt{2\pi} = f*g}$$



- $f(x)$ WE HAVE (GIVEN)
- $g(x)$ WE MUST INVERT $\hat g(k) = e^{-k^2\sigma^2 t}$

$$\boxed{\hat g(k) = e^{-k^2\sigma^2 t}}$$

$$\Longrightarrow g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat g(k)\,e^{ikx}\,dk = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty} e^{-k^2\sigma^2 t}e^{ikx}\,dk$$

- AS $e^{-k^2\sigma^2 t}$ IS EVEN IN k WE MAY SIMPLIFY

$$\Longrightarrow g(x) = \frac{2}{\sqrt{2\pi}}\int_0^\infty e^{-k^2\sigma^2 t}\cos kx\ dk$$

- LET $I = \int_0^\infty e^{-k^2\sigma^2 t}\cos kx\,dk$

$$\Longrightarrow \frac{\partial I}{\partial x} = \int_0^\infty e^{-k^2\sigma^2 t}\left[-k\sin kx\right]dk$$

$$\Longrightarrow \frac{\partial I}{\partial x} = \int_0^\infty -ke^{-k^2\sigma^2 t}\ \sin kx\ dk$$

- BY PARTS (W.R.T k)

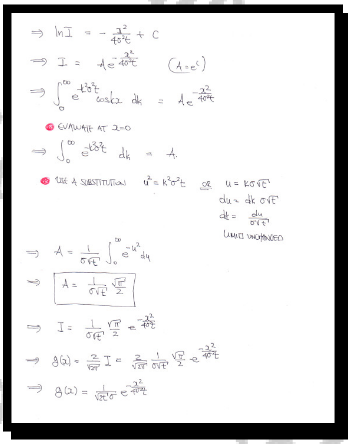
$\sin kx$	$x\cos kx$
$\frac{1}{2\sigma^2 t}e^{-k^2\sigma^2 t}$	$-ke^{-k^2\sigma^2 t}$

$$\Rightarrow \frac{\partial I}{\partial x} = \cancel{\left[\frac{1}{2\sigma^2 t}e^{-k^2\sigma^2 t}\sin kx\right]_0^\infty} - \frac{x}{2\sigma^2 t}\int_0^\infty e^{-k^2\sigma^2 t}\cos kx\,dk$$

$$\Rightarrow \frac{\partial I}{\partial x} = -\frac{x}{2\sigma^2 t}I$$

- SOLVING THE O.D.E BY SEPARATION OF VARIABLES

$$\Rightarrow \frac{1}{I}\partial I = -\frac{x}{2\sigma^2 t}\partial x$$



$$\Rightarrow \ln I = -\frac{x^2}{4\sigma^2 t} + C$$

$$\Rightarrow I = Ae^{-\frac{x^2}{4\sigma^2 t}} \qquad (A = e^C)$$

$$\Rightarrow \int_0^\infty e^{-k^2\sigma^2 t}\cos kx\ dk = Ae^{-\frac{x^2}{4\sigma^2 t}}$$

- EVALUATE AT $x=0$

$$\Rightarrow \int_0^\infty e^{-k^2\sigma^2 t}\ dk = A.$$

- USE A SUBSTITUTION $u^2 = k^2\sigma^2 t$ OR $u = k\sigma\sqrt{t}$

$du = dk\,\sigma\sqrt{t}$

$dk = \frac{du}{\sigma\sqrt{t}}$

LIMITS UNCHANGED

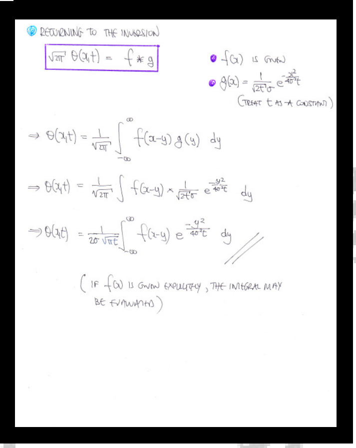
$$\Rightarrow A = \frac{1}{\sigma\sqrt{t}}\int_0^\infty e^{-u^2}du$$

$$\Rightarrow \boxed{A = \frac{1}{\sigma\sqrt{t}}\frac{\sqrt{\pi}}{2}}$$

$$\Rightarrow I = \frac{1}{\sigma\sqrt{t}}\frac{\sqrt{\pi}}{2}e^{-\frac{x^2}{4\sigma^2 t}}$$

$$\rightarrow g(x) = \frac{2}{\sqrt{2\pi}}I = \frac{2}{\sqrt{2\pi}}\frac{1}{\sigma\sqrt{t}}\frac{\sqrt{\pi}}{2}e^{-\frac{x^2}{4\sigma^2 t}}$$

$$\Rightarrow g(x) = \frac{1}{\sqrt{2t}\,\sigma}e^{-\frac{x^2}{4\sigma^2 t}}$$



- RETURNING TO THE INVERSION

$$\boxed{\sqrt{2\pi}\,\theta(x,t) = f*g}$$

- $f(x)$ IS GIVEN
- $g(x) = \frac{1}{\sqrt{2t}\,\sigma}e^{-\frac{x^2}{4\sigma^2 t}}$ (TREAT t AS A CONSTANT)

$$\Rightarrow \theta(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-y)\,g(y)\ dy$$

$$\Rightarrow \theta(x,t) = \frac{1}{\sqrt{2\pi}}\int f(x-y)\times\frac{1}{\sqrt{2t}\,\sigma}e^{\frac{-y^2}{4\sigma^2 t}}\ dy$$

$$\Rightarrow \theta(x,t) = \frac{1}{2\sigma\sqrt{\pi t}}\int_{-\infty}^{\infty} f(x-y)\,e^{\frac{-y^2}{4\sigma^2 t}}\ dy \quad //$$

(IF $f(x)$ IS GIVEN EXPLICITLY, THE INTEGRAL MAY BE EVALUATED)

Question 18

The function $T = T(x,t)$ satisfies the heat equation in one spatial dimension,

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\sigma}\frac{\partial \theta}{\partial t}, \quad x \geq 0\,,\ t \geq 0\,,$$

where σ is a positive constant.

It is further given that

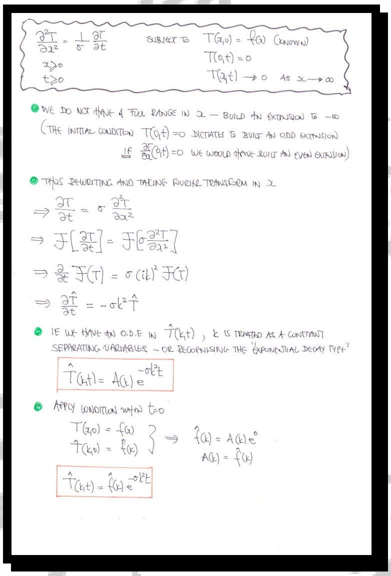
- $T(x,0) = f(x)$
- $T(0,t) = 0$
- $T(x,t) \to 0$ as $x \to \infty$

Use Fourier transforms to convert the above partial differential equation into an ordinary differential equation and hence show that

$$T(x,t) = \frac{1}{\sqrt{4\pi\sigma t}}\int_{-\infty}^{\infty} f(u)\exp\left[\frac{(x-u)^2}{4t\sigma}\right] du\,.$$

You may assume that $\mathcal{F}\left[\mathrm{e}^{ax^2}\right] = \dfrac{1}{\sqrt{2a}}\mathrm{e}^{\frac{k^2}{4a}}$.

proof



$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\sigma}\frac{\partial T}{\partial t}$ SUBJECT TO $T(x,0) = f(x)$ (KNOWN)

$x \geqslant 0$ $\qquad T(0,t) = 0$

$t \geqslant 0$ $\qquad T(x,t) \to 0$ AS $x \to \infty$

● WE DO NOT HAVE A FULL RANGE IN x — BUILD AN EXTENSION TO $-\infty$

(THE INITIAL CONDITION $T(0,t) = 0$ DICTATES TO BUILT AN ODD EXTENSION

IF $\frac{\partial T}{\partial x}(0,t) = 0$ WE WOULD HAVE BUILT AN EVEN EXTENSION)

● THUS REWRITING AND TAKING FOURIER TRANSFORM IN x

$$\Rightarrow \frac{\partial T}{\partial t} = \sigma\frac{\partial^2 T}{\partial x^2}$$

$$\Rightarrow \mathcal{F}\left[\frac{\partial T}{\partial t}\right] = \mathcal{F}\left[\sigma\frac{\partial^2 T}{\partial x^2}\right]$$

$$\Rightarrow \frac{\partial}{\partial t}\mathcal{F}(T) = \sigma(ik)^2\mathcal{F}(T)$$

$$\Rightarrow \frac{\partial \hat{T}}{\partial t} = -\sigma k^2 \hat{T}$$

● IF WE HAVE AN O.D.E IN $\hat{T}(k,t)$, k IS TREATED AS A CONSTANT

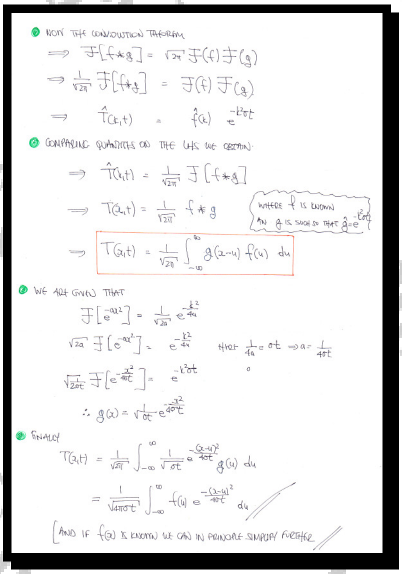
SEPARATING VARIABLES – OR RECOGNISING THE "EXPONENTIAL DECAY TYPE"

$$\boxed{\hat{T}(k,t) = A(k)e^{-\sigma k^2 t}}$$

● APPLY CONDITION WHEN $t=0$

$$\left.\begin{array}{l} T(x,0) = f(x) \\ \hat{T}(k,0) = \hat{f}(k) \end{array}\right\} \Rightarrow \begin{array}{l} \hat{f}(k) = A(k)e^{0} \\ A(k) = \hat{f}(k) \end{array}$$

$$\boxed{\hat{T}(k,t) = \hat{f}(k)e^{-\sigma k^2 t}}$$



● NOW THE CONVOLUTION THEOREM

$$\Longrightarrow \mathcal{F}[f*g] = \sqrt{2\pi}\,\mathcal{F}(f)\mathcal{F}(g)$$

$$\Longrightarrow \frac{1}{\sqrt{2\pi}}\mathcal{F}[f*g] = \mathcal{F}(f)\mathcal{F}(g)$$

$$\Longrightarrow \hat{T}(k,t) = \hat{f}(k)\ e^{-k^2\sigma t}$$

● COMPARING QUANTITIES ON THE LHS WE OBTAIN

$$\Longrightarrow \hat{T}(k,t) = \frac{1}{\sqrt{2\pi}}\mathcal{F}[f*g]$$

$$\Longrightarrow T(x,t) = \frac{1}{\sqrt{2\pi}}\,f*g$$

WHERE f IS KNOWN

AN g IS SUCH SO THAT $\hat{g} = e^{-k^2\sigma t}$

$$\Longrightarrow \boxed{T(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x-u)f(u)\,du}$$

● WE ARE GIVEN THAT

$$\mathcal{F}\left[e^{-ax^2}\right] = \frac{1}{\sqrt{2a}}e^{-\frac{k^2}{4a}}$$

$$\sqrt{2a}\,\mathcal{F}\left[e^{-ax^2}\right] = e^{-\frac{k^2}{4a}} \qquad \text{THEN } \frac{1}{4a} = \sigma t \Rightarrow a = \frac{1}{4\sigma t}$$

$$\sqrt{\frac{1}{2\sigma t}}\,\mathcal{F}\left[e^{-\frac{x^2}{4\sigma t}}\right] = e^{-k^2\sigma t}$$

$$\therefore g(x) = \sqrt{\frac{1}{\sigma t}}\,e^{-\frac{x^2}{4\sigma t}}$$

● FINALLY

$$T(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{\sigma t}}e^{-\frac{(x-u)^2}{4\sigma t}}g(u)\,du$$

$$= \frac{1}{\sqrt{4\pi\sigma t}}\int_{-\infty}^{\infty} f(u)\,e^{\frac{-(x-u)^2}{4\sigma t}}\,du$$

[AND IF $f(x)$ IS KNOWN WE CAN IN PRINCIPLE SIMPLIFY FURTHER

Question 19

The one dimensional heat equation for the temperature, $T(x,t)$, satisfies

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\sigma}\frac{\partial T}{\partial t}, \; t \geq 0,$$

where t is the time, x is a spatial dimension and σ is a positive constant.

The temperature $T(x,t)$ is subject to the following conditions.

- **i.** $\lim_{x\to\infty}\left[T(x,t)\right] = 0$
- **ii.** $T(0,t) = 1$
- **iii.** $T(x,0) = 0$

a) Use Laplace transforms to show that

$$\mathcal{L}\left[T(x,t)\right] = \bar{T}(x,s) = \frac{1}{s}\exp\left[-\sqrt{\frac{s}{\sigma}}\,x\right].$$

b) Use contour integration on the Laplace transformed temperature gradient $\frac{\partial}{\partial x}\left[\bar{T}(x,s)\right]$ to show further that

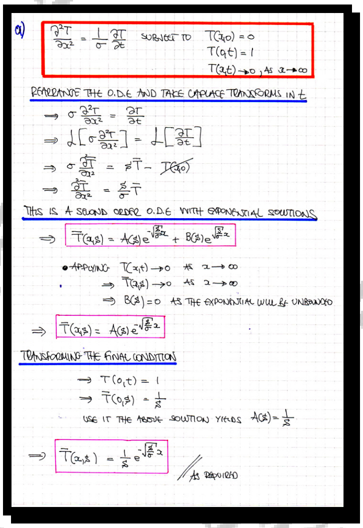
$$T(x,t) = 1 - \text{erf}\left[\frac{x}{\sqrt{4\sigma t}}\right].$$

You may assume without proof that

- $\int_0^\infty e^{-ax^2}\cos kx \; dx = \sqrt{\frac{\pi}{4a}}\,\exp\left[-\frac{k^2}{4a}\right]$
- $\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-\xi^2}\; d\xi$

, proof

[solution overleaf]



a) $$\boxed{\frac{\partial^2 T}{\partial x^2} = \frac{1}{\sigma}\frac{\partial T}{\partial t} \quad \text{SUBJECT TO} \quad \begin{matrix} T(x,0)=0 \\ T(0,t)=1 \\ T(x,t)\to 0,\ \text{AS } x\to\infty \end{matrix}}$$

REARRANGE THE O.D.E AND TAKE LAPLACE TRANSFORMS IN t

$$\Rightarrow \sigma\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

$$\Rightarrow \mathcal{L}\left[\sigma\frac{\partial^2 T}{\partial x^2}\right] = \mathcal{L}\left[\frac{\partial T}{\partial t}\right]$$

$$\Rightarrow \sigma\frac{\partial^2 \bar{T}}{\partial x^2} = s\bar{T} - \cancel{T(x,0)}$$

$$\Rightarrow \frac{\partial^2 \bar{T}}{\partial x^2} = \frac{s}{\sigma}\bar{T}$$

THIS IS A SECOND ORDER O.D.E WITH EXPONENTIAL SOLUTIONS

$$\Rightarrow \boxed{\bar{T}(x,s) = A(s)e^{-\sqrt{\frac{s}{\sigma}}x} + B(s)e^{\sqrt{\frac{s}{\sigma}}x}}$$

- APPLYING $T(x,t)\to 0$ AS $x\to\infty$
  $\Rightarrow \bar{T}(x,s)\to 0$ AS $x\to\infty$
  $\Rightarrow B(s)=0$ AS THE EXPONENTIAL WILL BE UNBOUNDED

$$\Rightarrow \boxed{\bar{T}(x,s) = A(s)e^{-\sqrt{\frac{s}{\sigma}}x}}$$

TRANSFORMING THE FINAL CONDITION

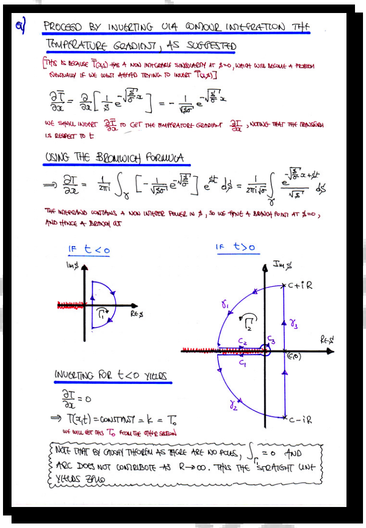
$$\Rightarrow T(0,t) = 1$$

$$\Rightarrow \bar{T}(0,s) = \frac{1}{s}$$

USE IT THE ABOVE SOLUTION YIELDS $A(s) = \frac{1}{s}$

$$\Rightarrow \boxed{\bar{T}(x,s) = \frac{1}{s}e^{-\sqrt{\frac{s}{\sigma}}x}}$$

// AS REQUIRED



a) PROCEED BY INVERTING VIA CONTOUR INTEGRATION THE TEMPERATURE GRADIENT, AS SUGGESTED

[THIS IS BECAUSE $\bar{T}(x,s)$ HAS A NON INTEGRABLE SINGULARITY AT $s=0$, WHICH WILL BECOME A PROBLEM EVENTUALLY IF WE WENT AHEAD TRYING TO INVERT $\bar{T}(x,s)$]

$$\frac{\partial \bar{T}}{\partial x} = \frac{\partial}{\partial x}\left[\frac{1}{s}e^{-\sqrt{\frac{s}{\sigma}}x}\right] = -\frac{1}{\sqrt{s\sigma}}e^{-\sqrt{\frac{s}{\sigma}}x}$$

WE SHALL INVERT $\frac{\partial \bar{T}}{\partial x}$ TO GET THE TEMPERATURE GRADIENT $\frac{\partial T}{\partial x}$, NOTING THAT THE TRANSFORM IS RESPECT TO t

USING THE BROMWICH FORMULA

$$\Rightarrow \frac{\partial T}{\partial x} = \frac{1}{2\pi i}\int_{\gamma}\left[-\frac{1}{\sqrt{s\sigma}}e^{-\sqrt{\frac{s}{\sigma}}}\right]e^{st}\,ds = \frac{1}{2\pi i\sqrt{\sigma}}\int_{\gamma}\frac{e^{-\sqrt{\frac{s}{\sigma}}x+st}}{\sqrt{s}}\,ds$$

THE INTEGRAND CONTAINS A NON INTEGER POWER IN s, SO WE HAVE A BRANCH POINT AT $s=0$, AND HENCE A BRANCH CUT

IF $t<0$

IF $t>0$

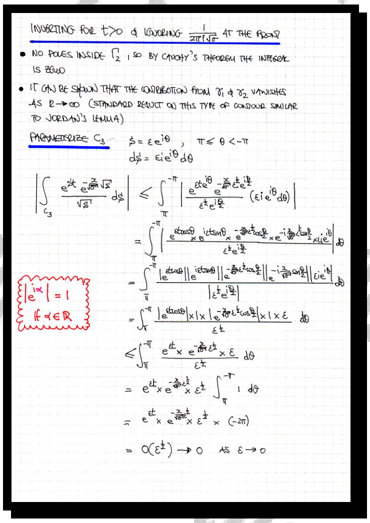
INVERTING FOR $t<0$ YIELDS

$$\frac{\partial T}{\partial x} = 0$$

$$\Rightarrow T(x,t) = \text{CONSTANT} = k = T_0$$

WE WILL GET THIS T_0 FROM THE OTHER SECTION

NOTE THAT BY CAUCHY THEOREM AS THERE ARE NO POLES, $\int_{\Gamma_1} = 0$ AND ARC DOES NOT CONTRIBUTE AS $R\to\infty$. THUS THE STRAIGHT LINE YIELDS ZERO



INVERTING FOR $t>0$ & IGNORING $\frac{1}{2\pi i\sqrt{\sigma}}$ AT THE FRONT

- NO POLES INSIDE Γ_2, SO BY CAUCHY'S THEOREM THE INTEGRAL IS ZERO
- IT CAN BE SHOWN THAT THE CONTRIBUTION FROM γ_1 & γ_2 VANISHES AS $R\to\infty$ (STANDARD RESULT ON THIS TYPE OF CONTOUR SIMILAR TO JORDAN'S LEMMA)

PARAMETERIZE C_3 $\quad s = \varepsilon e^{i\theta}, \quad \pi \leqslant \theta < -\pi$
$ds = \varepsilon i e^{i\theta}\,d\theta$

$$\left|\int_{C_3}\frac{e^{st}e^{-\frac{x}{\sqrt{\sigma}}\sqrt{s}}}{\sqrt{s}}\,ds\right| \leqslant \int_{\pi}^{-\pi}\left|\frac{e^{\varepsilon te^{i\theta}}e^{-\frac{x}{\sqrt{\sigma}}\varepsilon^{\frac{1}{2}}e^{\frac{i\theta}{2}}}}{\varepsilon^{\frac{1}{2}}e^{\frac{i\theta}{2}}}(\varepsilon ie^{i\theta}d\theta)\right|$$

$$= \int_{\pi}^{-\pi}\left|\frac{e^{\varepsilon t\cos\theta}\times e^{i\varepsilon t\sin\theta}\times e^{-\frac{x}{\sqrt{\sigma}}\varepsilon^{\frac{1}{2}}\cos\frac{\theta}{2}}\times e^{-i\frac{x}{\sqrt{\sigma}}\varepsilon^{\frac{1}{2}}\sin\frac{\theta}{2}}\times\varepsilon ie^{i\theta}}{\varepsilon^{\frac{1}{2}}e^{\frac{i\theta}{2}}}\right|d\theta$$

$$\boxed{\begin{matrix}|e^{i\alpha}| = 1 \\ \text{If } \alpha\in\mathbb{R}\end{matrix}}$$

$$= \int_{\pi}^{-\pi}\frac{|e^{\varepsilon t\cos\theta}||e^{i\varepsilon t\sin\theta}||e^{-\frac{x}{\sqrt{\sigma}}\varepsilon^{\frac{1}{2}}\cos\frac{\theta}{2}}||e^{-i\frac{x}{\sqrt{\sigma}}\sin\frac{\theta}{2}}||\varepsilon ie^{i\theta}|}{|\varepsilon^{\frac{1}{2}}e^{\frac{i\theta}{2}}|}d\theta$$

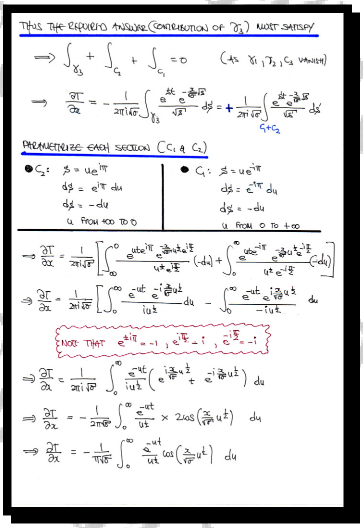
$$= \int_{\pi}^{-\pi}\frac{|e^{\varepsilon t\cos\theta}|\times 1\times|e^{-\frac{x}{\sqrt{\sigma}}\varepsilon^{\frac{1}{2}}\cos\frac{\theta}{2}}|\times 1\times\varepsilon}{\varepsilon^{\frac{1}{2}}}\,d\theta$$

$$\leqslant \int_{\pi}^{-\pi}\frac{e^{\varepsilon t}\times e^{-\frac{x}{\sqrt{\sigma}}\varepsilon^{\frac{1}{2}}}\times\varepsilon}{\varepsilon^{\frac{1}{2}}}\,d\theta$$

$$= e^{\varepsilon t}\times e^{-\frac{x}{\sqrt{\sigma}}\varepsilon^{\frac{1}{2}}}\times\varepsilon^{\frac{1}{2}}\int_{\pi}^{-\pi}1\,d\theta$$

$$= e^{\varepsilon t}\times e^{-\frac{x}{\sqrt{\sigma}}\varepsilon^{\frac{1}{2}}}\times\varepsilon^{\frac{1}{2}}\times(-2\pi)$$

$$= O(\varepsilon^{\frac{1}{2}}) \to 0 \quad \text{AS } \varepsilon\to 0$$



THUS THE REQUIRED ANSWER (CONTRIBUTION OF γ_3) MUST SATISFY

$$\Longrightarrow \int_{\gamma_3} + \int_{C_2} + \int_{C_1} = 0 \qquad (\text{AS } \gamma_1, \gamma_2, C_3 \text{ VANISH})$$

$$\Longrightarrow \frac{\partial T}{\partial x} = -\frac{1}{2\pi i\sqrt{\sigma}}\int_{\gamma_3}\frac{e^{st}e^{-\frac{x}{\sqrt{\sigma}}\sqrt{s}}}{\sqrt{s}}\,ds = +\frac{1}{2\pi i\sqrt{\sigma}}\int_{C_1+C_2}\frac{e^{st}e^{-\frac{x}{\sqrt{\sigma}}\sqrt{s}}}{\sqrt{s}}\,ds$$

PARAMETRIZE EACH SECTION (C_1 & C_2)

- C_2: $s = ue^{i\pi}$
  $ds = e^{i\pi}\,du$
  $ds = -du$
  u FROM $+\infty$ TO 0
- C_1: $s = ue^{-i\pi}$
  $ds = e^{-i\pi}\,du$
  $ds = -du$
  u FROM 0 TO $+\infty$

$$\Rightarrow \frac{\partial T}{\partial x} = \frac{1}{2\pi i\sqrt{\sigma}}\left[\int_{\infty}^{0}\frac{e^{ute^{i\pi}}e^{-\frac{x}{\sqrt{\sigma}}u^{\frac{1}{2}}e^{\frac{i\pi}{2}}}}{u^{\frac{1}{2}}e^{\frac{i\pi}{2}}}(-du) + \int_{0}^{\infty}\frac{e^{ute^{-i\pi}}e^{-\frac{x}{\sqrt{\sigma}}u^{\frac{1}{2}}e^{-\frac{i\pi}{2}}}}{u^{\frac{1}{2}}e^{-\frac{i\pi}{2}}}(-du)\right]$$

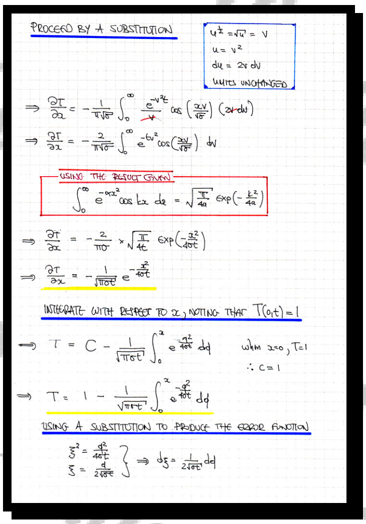
$$\Rightarrow \frac{\partial T}{\partial x} = \frac{1}{2\pi i\sqrt{\sigma}}\left[\int_{0}^{\infty}\frac{e^{-ut}e^{-i\frac{x}{\sqrt{\sigma}}u^{\frac{1}{2}}}}{iu^{\frac{1}{2}}}\,du - \int_{0}^{\infty}\frac{e^{-ut}e^{i\frac{x}{\sqrt{\sigma}}u^{\frac{1}{2}}}}{-iu^{\frac{1}{2}}}\,du\right.$$

NOTE THAT $e^{\pm i\pi} = -1$, $e^{\frac{i\pi}{2}} = i$, $e^{-\frac{i\pi}{2}} = -i$

$$\Rightarrow \frac{\partial T}{\partial x} = \frac{1}{2\pi i\sqrt{\sigma}}\int_{0}^{\infty}\frac{e^{-ut}}{iu^{\frac{1}{2}}}\left(e^{i\frac{x}{\sqrt{\sigma}}u^{\frac{1}{2}}} + e^{-i\frac{x}{\sqrt{\sigma}}u^{\frac{1}{2}}}\right)du$$

$$\Rightarrow \frac{\partial T}{\partial x} = -\frac{1}{2\pi\sqrt{\sigma}}\int_{0}^{\infty}\frac{e^{-ut}}{u^{\frac{1}{2}}}\times 2\cos\left(\frac{x}{\sqrt{\sigma}}u^{\frac{1}{2}}\right)du$$

$$\Rightarrow \frac{\partial T}{\partial x} = -\frac{1}{\pi\sqrt{\sigma}}\int_{0}^{\infty}\frac{e^{-ut}}{u^{\frac{1}{2}}}\cos\left(\frac{x}{\sqrt{\sigma}}u^{\frac{1}{2}}\right)du$$



PROCEED BY A SUBSTITUTION

$u^{\frac{1}{2}} = \sqrt{u} = v$
$u = v^2$
$du = 2v\,dv$
LIMITS UNCHANGED

$$\Rightarrow \frac{\partial T}{\partial x} = -\frac{1}{\pi\sqrt{\sigma}}\int_{0}^{\infty}\frac{e^{-v^2t}}{\cancel{v}}\cos\left(\frac{xv}{\sqrt{\sigma}}\right)(2\cancel{v}\,dv)$$

$$\Rightarrow \frac{\partial T}{\partial x} = -\frac{2}{\pi\sqrt{\sigma}}\int_{0}^{\infty}e^{-tv^2}\cos\left(\frac{xv}{\sqrt{\sigma}}\right)dv$$

USING THE RESULT GIVEN

$$\boxed{\int_{0}^{\infty}e^{-ax^2}\cos kx\,dx = \sqrt{\frac{\pi}{4a}}\exp\left(-\frac{k^2}{4a}\right)}$$

$$\Rightarrow \frac{\partial T}{\partial x} = -\frac{2}{\pi\sigma}\times\sqrt{\frac{\pi}{4t}}\exp\left(-\frac{x^2}{4\sigma t}\right)$$

$$\Rightarrow \frac{\partial T}{\partial x} = -\frac{1}{\sqrt{\pi\sigma t}}e^{-\frac{x^2}{4\sigma t}}$$

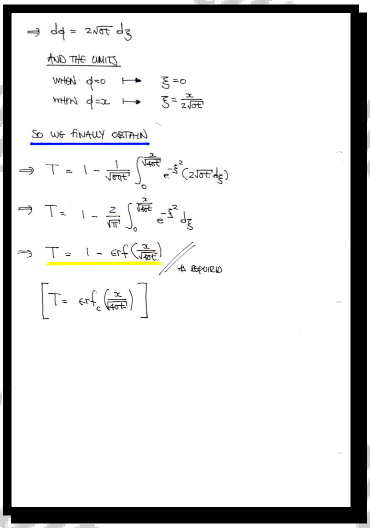
INTEGRATE WITH RESPECT TO x, NOTING THAT $T(0,t) = 1$

$$\Longrightarrow T = C - \frac{1}{\sqrt{\pi\sigma t}}\int_{0}^{x}e^{-\frac{q^2}{4\sigma t}}\,dq \qquad \text{when } x=0,\ T=1 \quad \therefore C=1$$

$$\Rightarrow T = 1 - \frac{1}{\sqrt{\pi\sigma t}}\int_{0}^{x}e^{-\frac{q^2}{4\sigma t}}\,dq$$

USING A SUBSTITUTION TO PRODUCE THE ERROR FUNCTION

$$\left.\begin{matrix}\xi^2 = \frac{q^2}{4\sigma t} \\ \xi = \frac{q}{2\sqrt{\sigma t}}\end{matrix}\right\} \Rightarrow d\xi = \frac{1}{2\sqrt{\sigma t}}\,dq$$



$$\Rightarrow dq = 2\sqrt{\sigma t}\,d\xi$$

AND THE LIMITS

WHEN $q=0 \mapsto \xi=0$
WHEN $q=x \mapsto \xi=\frac{x}{2\sqrt{\sigma t}}$

SO WE FINALLY OBTAIN

$$\Rightarrow T = 1 - \frac{1}{\sqrt{\sigma\pi t}}\int_{0}^{\frac{x}{\sqrt{4\sigma t}}}e^{-\xi^2}(2\sqrt{\sigma t}\,d\xi)$$

$$\Rightarrow T = 1 - \frac{2}{\sqrt{\pi}}\int_{0}^{\frac{x}{\sqrt{4\sigma t}}}e^{-\xi^2}\,d\xi$$

$$\Rightarrow T = 1 - \text{erf}\left(\frac{x}{\sqrt{4\sigma t}}\right)$$

// AS REQUIRED

$$\left[T = \text{erf}_c\left(\frac{x}{\sqrt{4\sigma t}}\right)\right]$$