RESIDUES and APPLICATIONS in SERIES SUMMATION

The Residue Theorem can often be used to sum various types of series.

The following results are valid under some restrictions on $f(z)$, which more often than not are satisfied when the series converges.

$$\sum_{r=-\infty}^{\infty} f(r)$$

use $\displaystyle\oint_{\Gamma_n} f(z)\pi\cot\pi z\,dz$, where Γ_n is the square with vertices at $\left(n+\frac{1}{2}\right)(\pm1\pm\mathrm{i})$

$$\sum_{r=-\infty}^{\infty} (-1)^r f(r)$$

use $\displaystyle\oint_{\Gamma_n} f(z)\pi\operatorname{cosec}\pi z\,dz$, where Γ_n is the square with vertices at $\left(n+\frac{1}{2}\right)(\pm1\pm\mathrm{i})$

$$\sum_{r=-\infty}^{\infty} f\left(\frac{2r+1}{2}\right)$$

use $\displaystyle\oint_{\Gamma_n} f(z)\pi\tan\pi z\,dz$, where Γ_n is the square with vertices at $n(\pm1\pm\mathrm{i})$

$$\sum_{r=-\infty}^{\infty} (-1)^r f\left(\frac{2r+1}{2}\right)$$

use $\displaystyle\oint_{\Gamma_n} f(z)\pi\sec\pi z\,dz$, where Γ_n is the square with vertices at $n(\pm1\pm\mathrm{i})$

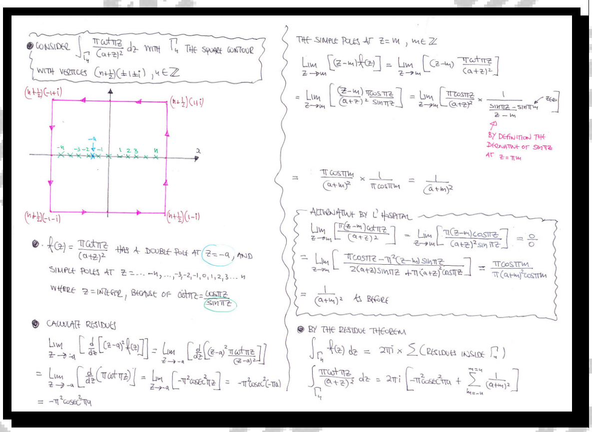
Question 1

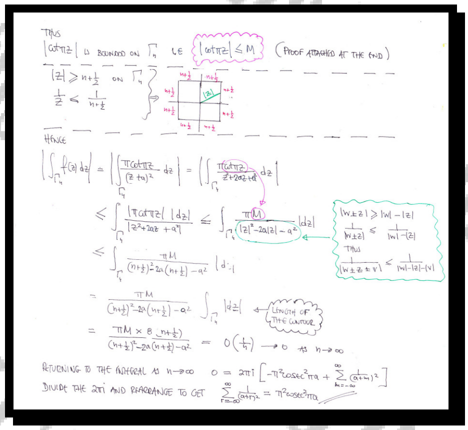
$$f(z) = \frac{\pi \cot \pi z}{(a+z)^2}, \quad z \in \mathbb{C}.$$

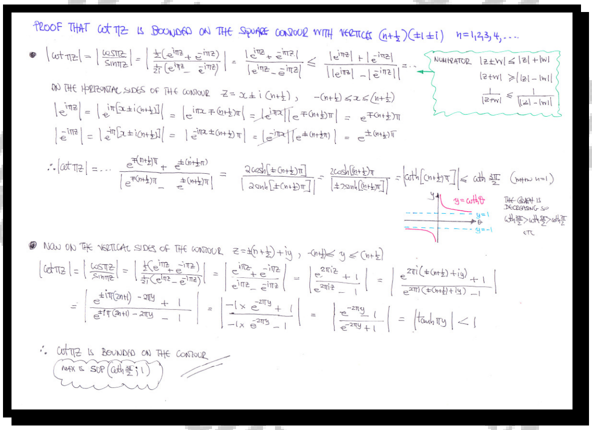
By integrating $f(z)$ over a suitable contour Γ, show that

$$\sum_{r=-\infty}^{\infty} \frac{1}{(a+r)^2} = \pi^2 \operatorname{cosec}^2(\pi a), \quad a \notin \mathbb{Z}.$$

proof







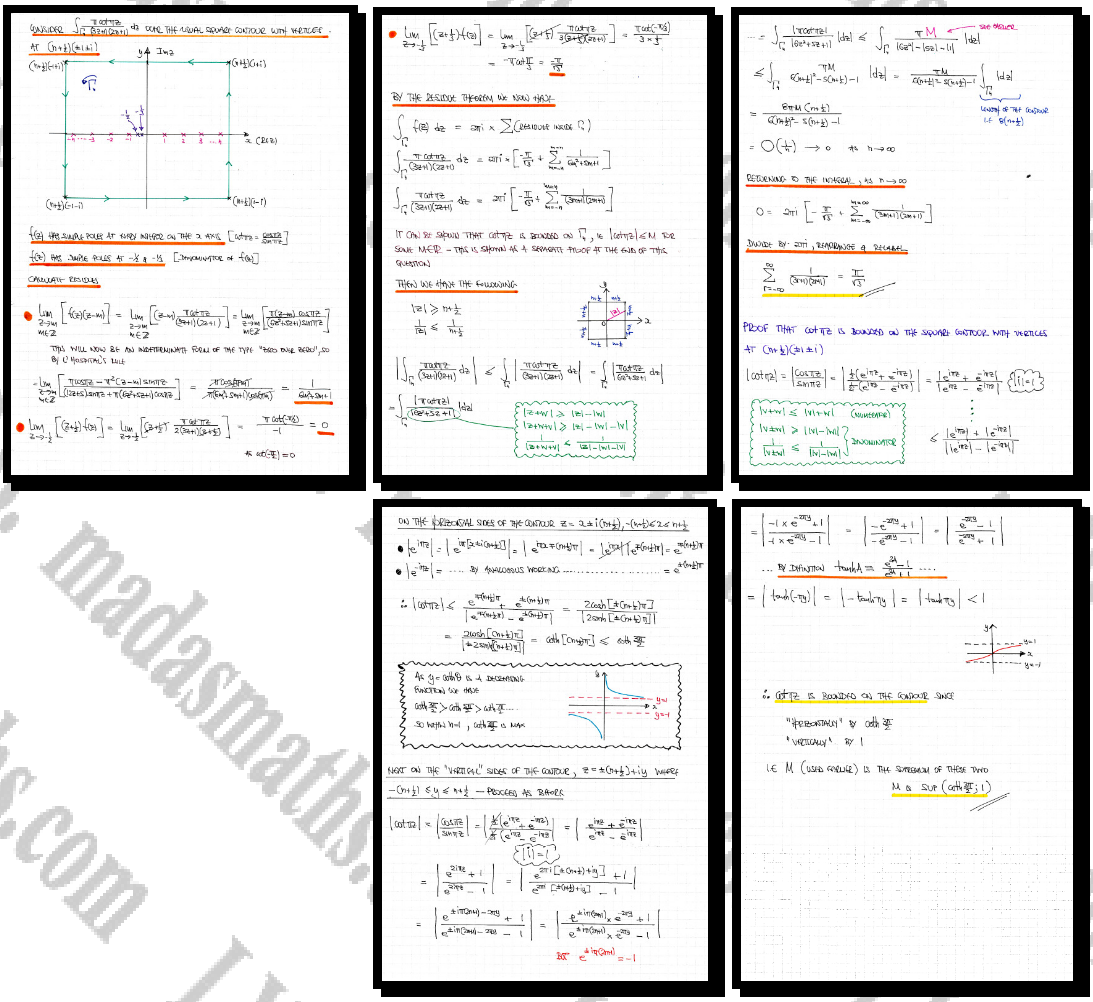
Question 2

$$f(z) = \frac{\pi \cot \pi z}{(3z+1)(2z+1)}, \quad z \in \mathbb{C}.$$

By integrating $f(z)$ over a suitable contour Γ, show that

$$\sum_{r=-\infty}^{\infty} \frac{1}{(3r+1)(2r+1)} = \frac{\pi}{\sqrt{3}}.$$

$\boxed{}$, $\boxed{\text{proof}}$



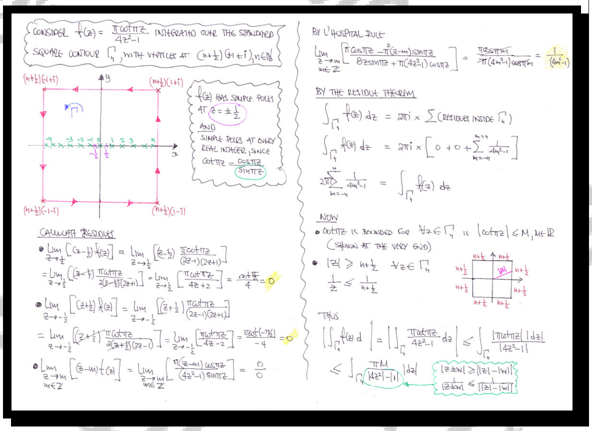
Question 3

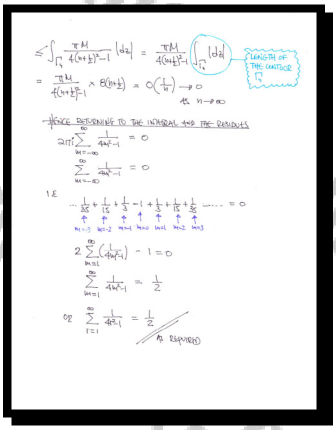
$$f(z) = \frac{\pi \cot \pi z}{4z^2 - 1}, \quad z \in \mathbb{C}.$$

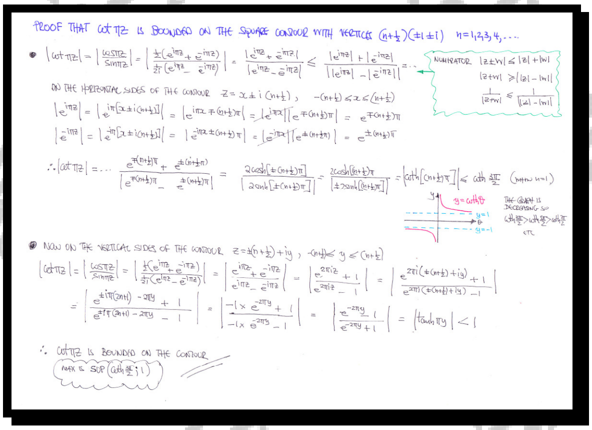
By integrating $f(z)$ over a suitable contour Γ, show that

$$\sum_{r=1}^{\infty} \frac{1}{4r^2 - 1} = \frac{1}{2}.$$

$\boxed{\text{proof}}$







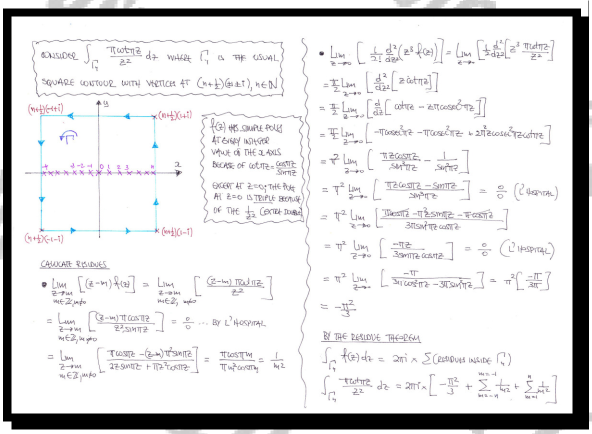
Question 4

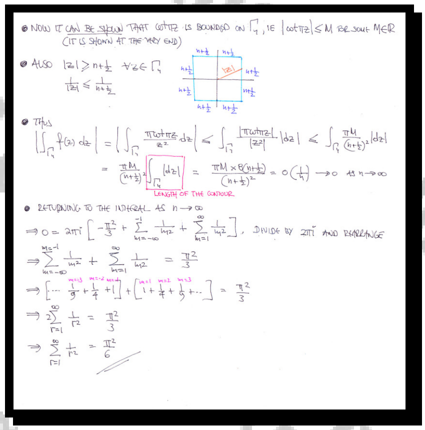
$$f(z) = \frac{\pi \cot \pi z}{z^2}, \quad z \in \mathbb{C}.$$

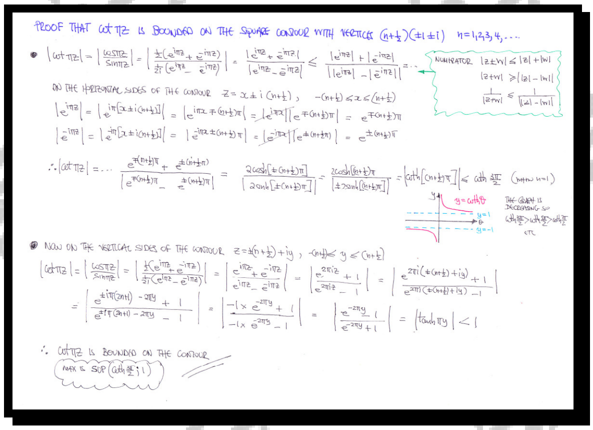
By integrating $f(z)$ over a suitable contour Γ, show that

$$\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6}.$$

proof







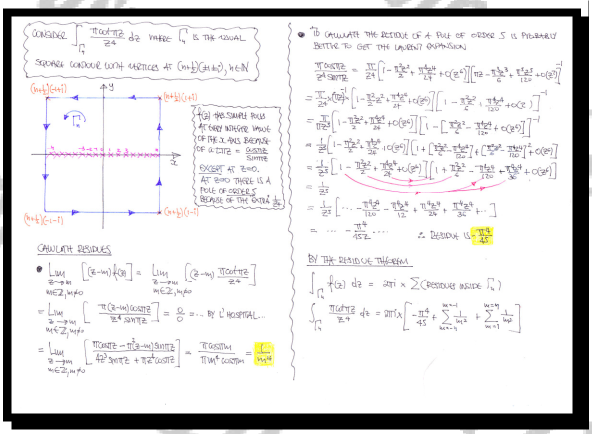
Question 5

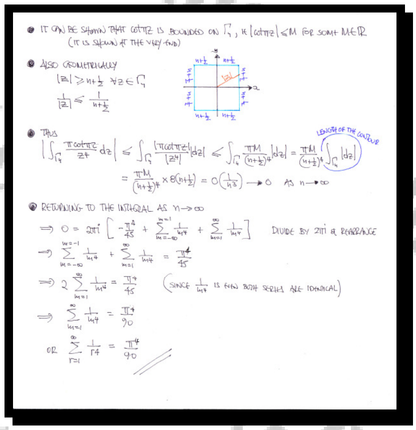
$$f(z) = \frac{\pi \cot \pi z}{z^4}, \quad z \in \mathbb{C}.$$

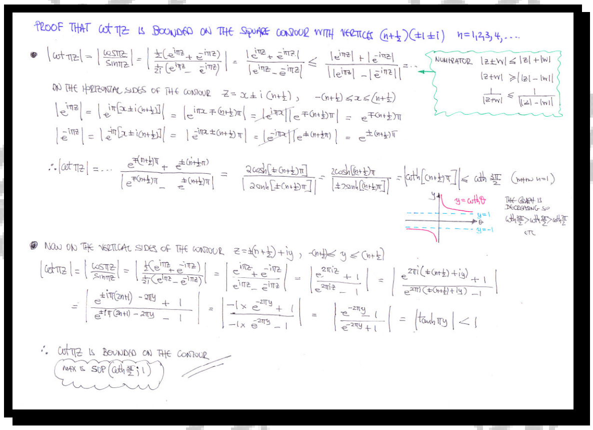
By integrating $f(z)$ over a suitable contour Γ, show that

$$\sum_{r=1}^{\infty} \frac{1}{r^4} = \frac{\pi^4}{90}.$$

proof







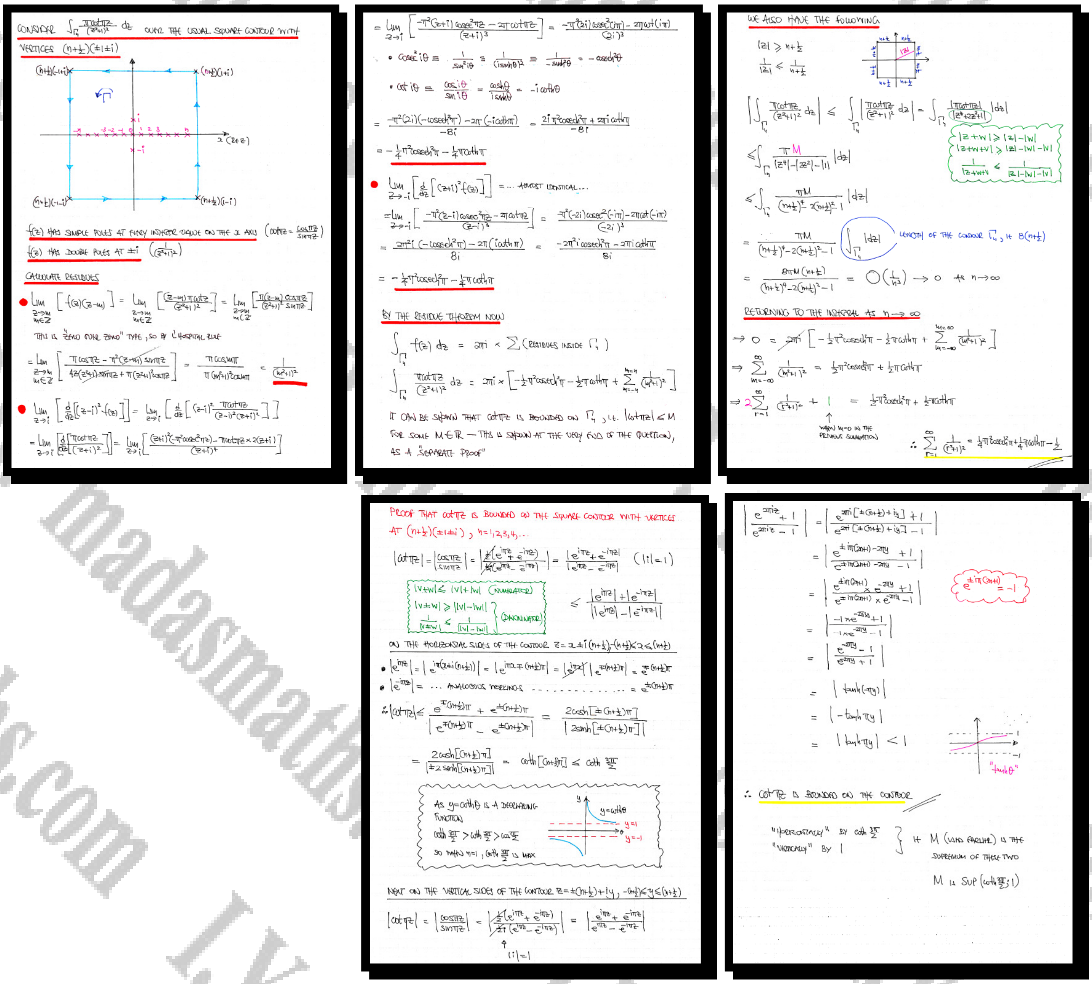
Question 6

$$f(z) = \frac{\pi \cot \pi z}{\left(z^2+1\right)^2}, \quad z \in \mathbb{C}.$$

By integrating $f(z)$ over a suitable contour Γ, show that

$$\sum_{r=1}^{\infty} \frac{1}{\left(r^2+1\right)^2} = \frac{1}{4}\pi^2 \operatorname{cosech}^2 \pi + \frac{1}{4}\pi \coth \pi - \frac{1}{2}.$$

proof



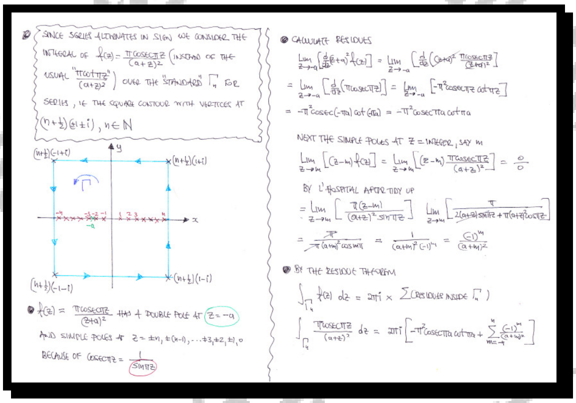
Question 7

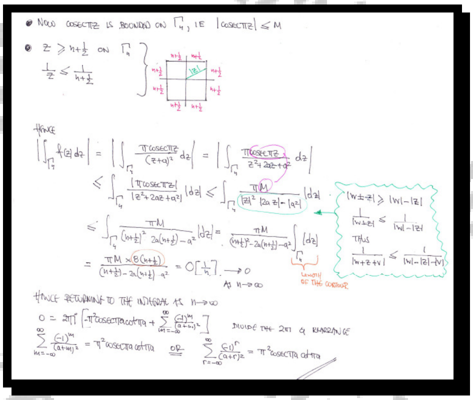
$$f(z) = \frac{\pi \operatorname{cosec} \pi z}{(a+z)^2}, \quad z \in \mathbb{C}.$$

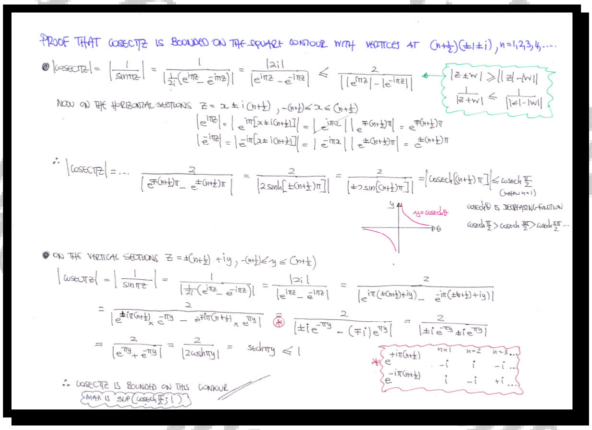
By integrating $f(z)$ over a suitable contour Γ, show that

$$\sum_{r=-\infty}^{\infty} \frac{(-1)^r}{(a+r)^2} = \pi^2 \operatorname{cosec}(\pi a)\cot(\pi a), \quad a \notin \mathbb{Z}.$$

$\boxed{\text{proof}}$







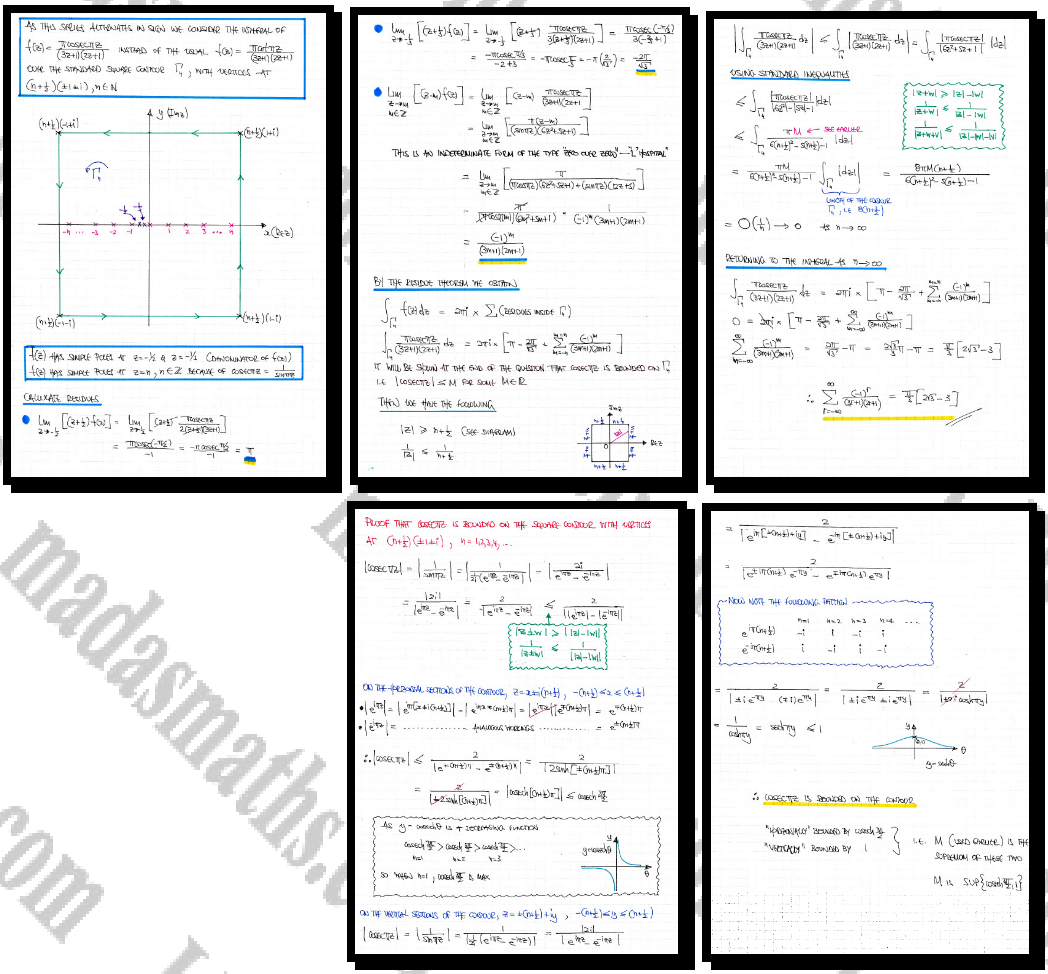
Question 8

$$f(z) = \frac{\pi \operatorname{cosec} \pi z}{(2z+1)(3z+1)}, \quad z \in \mathbb{C}.$$

By integrating $f(z)$ over a suitable contour Γ, show that

$$\sum_{r=-\infty}^{\infty} \frac{(-1)^r}{(2r+1)(3r+1)} = \frac{\pi}{3}\left(2\sqrt{3}-3\right).$$

$\boxed{}$, $\boxed{\text{proof}}$



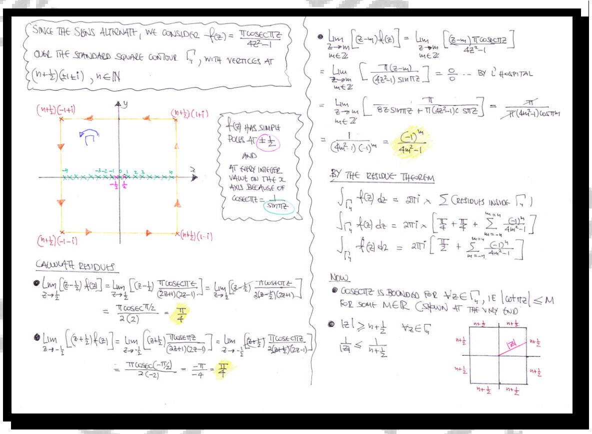
Question 9

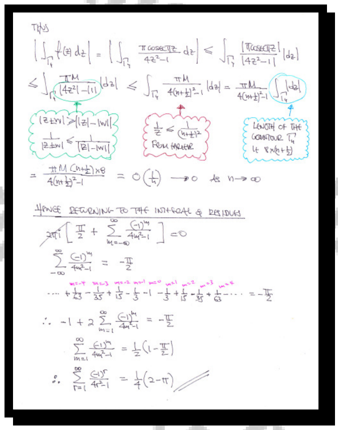
$$f(z) = \frac{\pi \operatorname{cosec} \pi z}{4z^2 - 1}, \quad z \in \mathbb{C}.$$

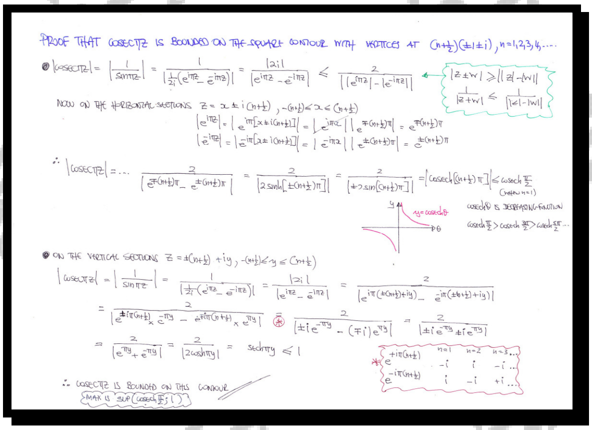
By integrating $f(z)$ over a suitable contour Γ, show that

$$\sum_{r=1}^{\infty} \frac{(-1)^r}{4r^2 - 1} = \frac{1}{4}(2 - \pi).$$

$\boxed{\text{proof}}$







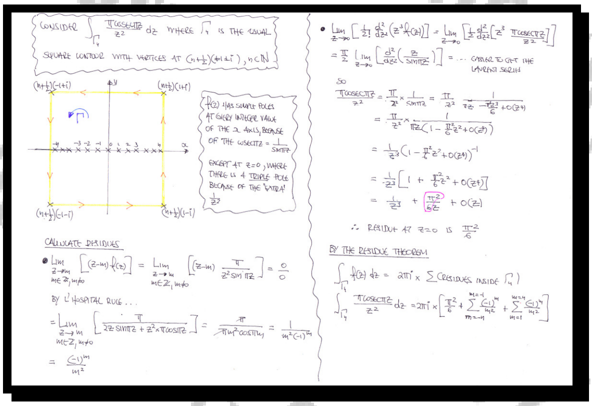
Question 10

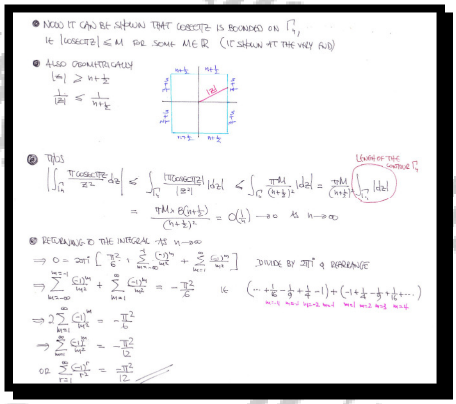
$$f(z) = \frac{\pi \operatorname{cosec} \pi z}{z^2}, \quad z \in \mathbb{C}.$$

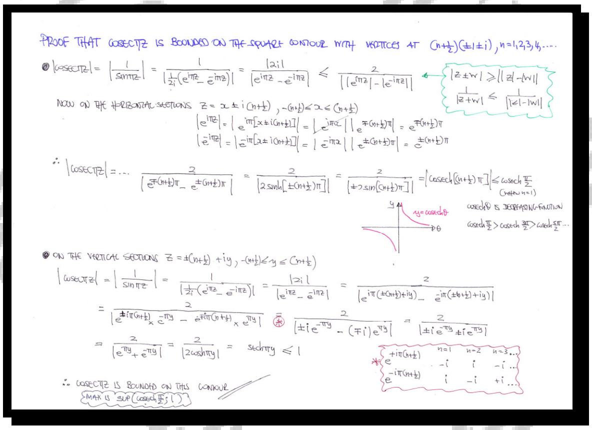
By integrating $f(z)$ over a suitable contour Γ, show that

$$\sum_{r=1}^{\infty} \frac{(-1)^r}{r^2} = -\frac{1}{12}\pi^2.$$

proof







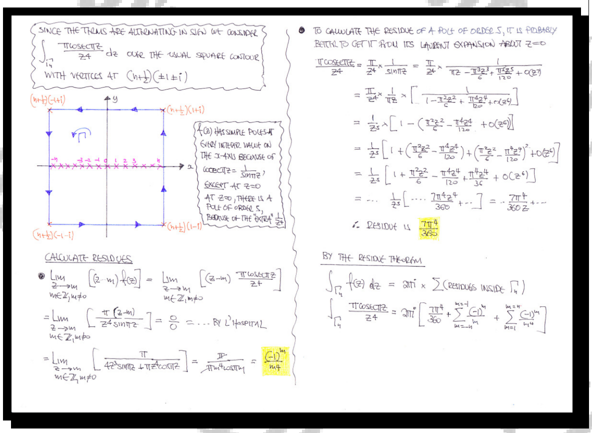
Question 11

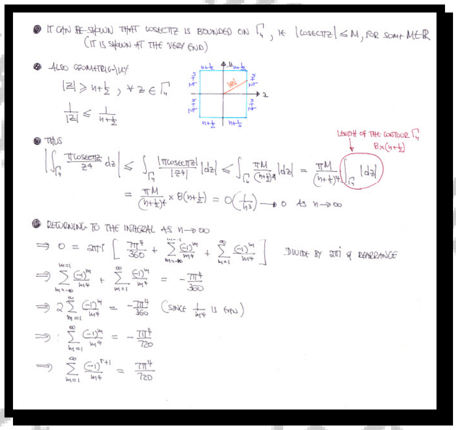
$$f(z) = \frac{\pi \operatorname{cosec} \pi z}{z^4}, \quad z \in \mathbb{C}.$$

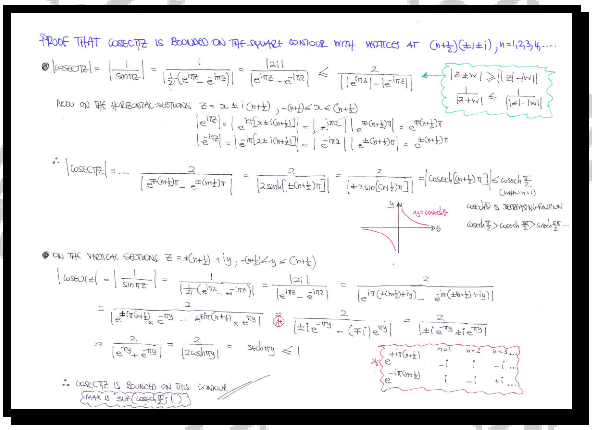
By integrating $f(z)$ over a suitable contour Γ, show that

$$\sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r^4} = \frac{7\pi^4}{720}.$$

proof







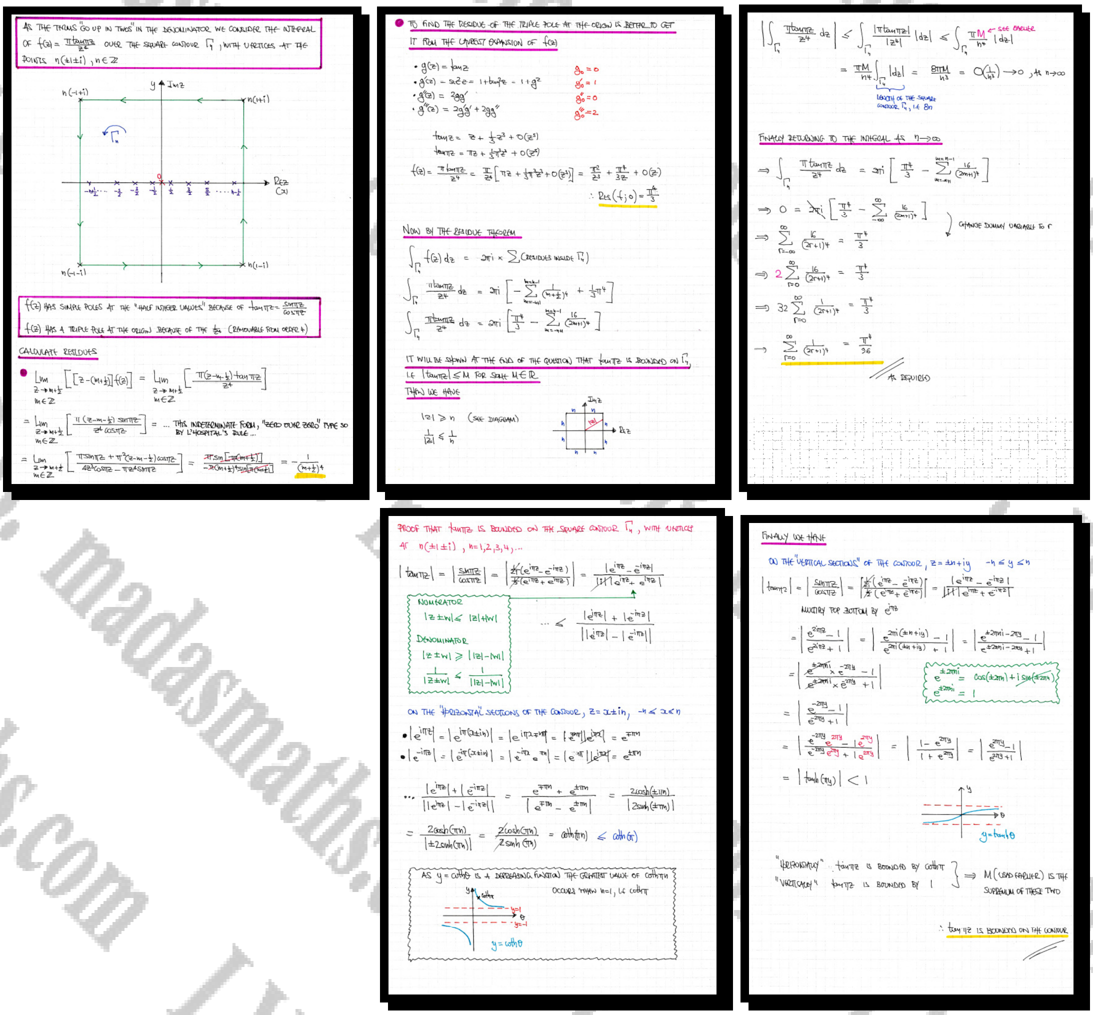
Question 12

$$f(z) = \frac{\pi \tan \pi z}{z^4}, \quad z \in \mathbb{C}.$$

By integrating $f(z)$ over a suitable contour Γ, show that

$$\sum_{r=0}^{\infty} \frac{1}{(2r+1)^4} = \frac{\pi^4}{96}.$$

$\boxed{}$, $\boxed{\text{proof}}$



Question 13

$$f(z) = \frac{\pi \sec \pi z}{z^3}, \quad z \in \mathbb{C}.$$

By integrating $f(z)$ over a suitable contour Γ, show that

$$\sum_{r=0}^{\infty} \frac{(-1)^r}{(2r+1)^3} = \frac{\pi^3}{32}.$$

$\boxed{}$, $\boxed{\text{proof}}$