VECTOR INTEGRALS

TYPE $\int_C \varphi \, \mathbf{dr}$

Question 1

$$V(x,y,z) = 60xyz^2 .$$

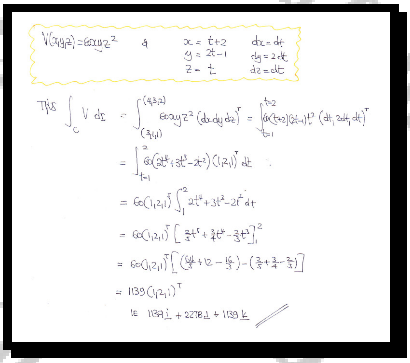
Evaluate the following integral along C, from $(3,1,1)$ to $(4,3,2)$,

$$\int_C V \, \mathbf{dr} , \qquad \mathbf{dr} = (dx, dy, dz)^{\mathrm{T}} ,$$

where C is the curve with parametric equations

$$x = t+2 , \quad y = 2t-1 , \quad z = t .$$

$\boxed{1139\mathbf{i} + 2278\mathbf{j} + 1139\mathbf{k}}$



Question 2

$$\varphi(x, y, z) \equiv 3x + 2y + z .$$

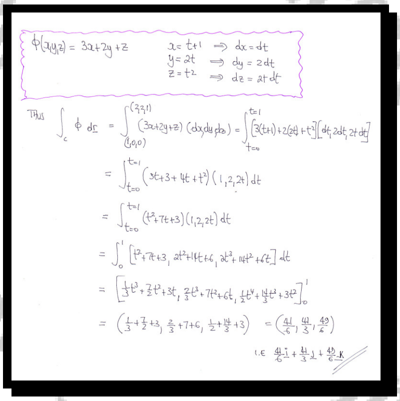
Evaluate the following integral along C, from $(1,0,0)$ to $(2,2,1)$,

$$\int_C \varphi \, \mathbf{dr} , \qquad \mathbf{dr} = (dx, dy, dz)^{\mathrm{T}} ,$$

where C is the curve with parametric equations

$$x = t + 1 , \quad y = 2t , \quad z = t^2 .$$

$$\boxed{\frac{41}{6}\mathbf{i} + \frac{41}{3}\mathbf{j} + \frac{49}{6}\mathbf{k}}$$



$\phi(x,y,z) = 3x+2y+z$

$x = t+1 \Rightarrow dx = dt$
$y = 2t \Rightarrow dy = 2dt$
$z = t^2 \Rightarrow dz = 2t\,dt$

Thus $\int_C \phi \, d\underline{r} = \int_{(1,0,0)}^{(2,2,1)} (3x+2y+z)(dx,dy,dz) = \int_{t=0}^{t=1} [3(t+1)+2(2t)+t^2][dt, 2dt, 2t\,dt]$

$= \int_{t=0}^{t=1} (3t+3+4t+t^2)(1,2,2t)\,dt$

$= \int_{t=0}^{t=1} (t^2+7t+3)(1,2,2t)\,dt$

$= \int_0^1 [t^2+7t+3, 2t^2+14t+6, 2t^3+14t^2+6t]\,dt$

$= \left[\frac{1}{3}t^3+\frac{7}{2}t^2+3t, \frac{2}{3}t^3+7t^2+6t, \frac{1}{2}t^4+\frac{14}{3}t^3+3t^2\right]_0^1$

$= \left(\frac{1}{3}+\frac{7}{2}+3, \frac{2}{3}+7+6, \frac{1}{2}+\frac{14}{3}+3\right) = \left(\frac{41}{6}, \frac{41}{3}, \frac{49}{6}\right)$

i.e. $\frac{41}{6}\underline{i} + \frac{41}{3}\underline{j} + \frac{49}{6}\underline{k}$

Question 3

$$F(x,y,z) = xyz \,.$$

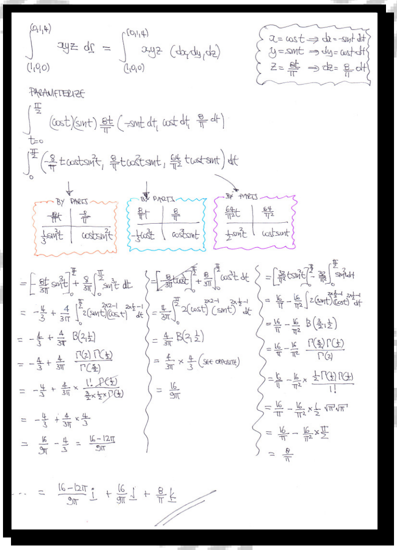
Evaluate the following integral along C, from $(1,0,0)$ to $(0,1,4)$,

$$\int_C F \, \mathbf{dr} \,, \qquad \mathbf{dr} = (dx, dy, dz)^{\mathrm{T}} \,,$$

where C is the curve with parametric equations

$$x = \cos t \,, \qquad y = \sin t \,, \qquad z = \frac{8t}{\pi} \,.$$

$$\boxed{\frac{16-12\pi}{9\pi}\mathbf{i} + \frac{16}{9\pi}\mathbf{j} + \frac{8}{\pi}\mathbf{k}}$$



TYPE $\int_V \mathbf{F}\, dV$

Question 1

$$\mathbf{F}(x, y, z) \equiv xy\mathbf{i} + z\mathbf{j} - x^2\mathbf{k}\,.$$

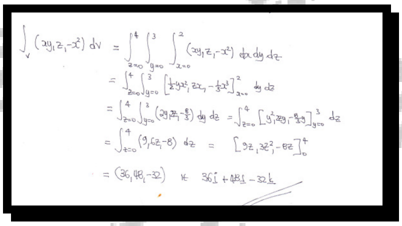
Evaluate the vector integral

$$\int_V \mathbf{F}\, dV\,,$$

where V is the finite region in the first octant bounded by the planes with equations

$$x = 2,\ \ y = 3 \ \text{ and } \ z = 4\,.$$

$$\boxed{36\mathbf{i} + 48\mathbf{j} - 32\mathbf{k}}$$


$$\int_V (xy, z, -x^2)\, dV = \int_{z=0}^{4}\int_{y=0}^{3}\int_{x=0}^{2} (xy, z, -x^2)\, dx\, dy\, dz$$
$$= \int_{z=0}^{4}\int_{y=0}^{3} \left[\tfrac{1}{2}yx^2, zx, -\tfrac{1}{3}x^3\right]_{x=0}^{2} dy\, dz$$
$$= \int_{z=0}^{4}\int_{y=0}^{3} \left(2y, 2z, -\tfrac{8}{3}\right) dy\, dz = \int_{z=0}^{4} \left[y^2, 2zy, -\tfrac{8}{3}y\right]_{y=0}^{3} dz$$
$$= \int_{z=0}^{4} (9, 6z, -8)\, dz = \left[9z, 3z^2, -8z\right]_0^4$$
$$= (36, 48, -32) \equiv 36\mathbf{i} + 48\mathbf{j} - 32\mathbf{k}$$

Question 2

$$\mathbf{F}(x, y, z) \equiv z\mathbf{i} + \mathbf{j} + y\mathbf{k}\ .$$

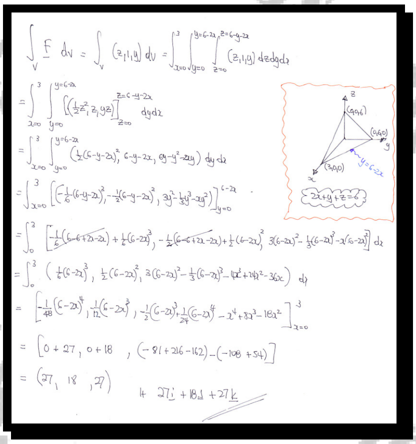
Evaluate the vector integral

$$\int_V \mathbf{F}\ dV\ ,$$

where V is the finite region in the first octant bounded by the plane with equation

$$2x + y + z = 6\ .$$

$\boxed{27\mathbf{i} + 18\mathbf{j} + 27\mathbf{k}}$



Question 3

$$\mathbf{F}(x, y, z) \equiv \mathbf{i} + 2z\mathbf{j} + y\mathbf{k} .$$

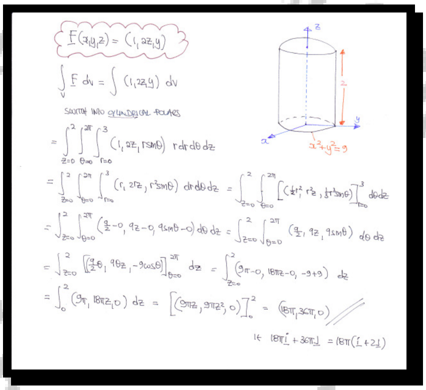
Evaluate the vector integral

$$\int_V \mathbf{F}\ dV ,$$

where V is the finite region enclosed by the cylinder with equation

$$x^2 + y^2 = 9 ,\ 0 \le z \le 2 .$$

$\boxed{18\pi(\mathbf{i}+2\mathbf{j})}$



Question 4

$$\mathbf{F}(x,y,z) \equiv \frac{1}{6\pi}\mathbf{i} + \frac{z}{18\pi}\mathbf{j} + \frac{y}{9\pi}\mathbf{k}\,.$$

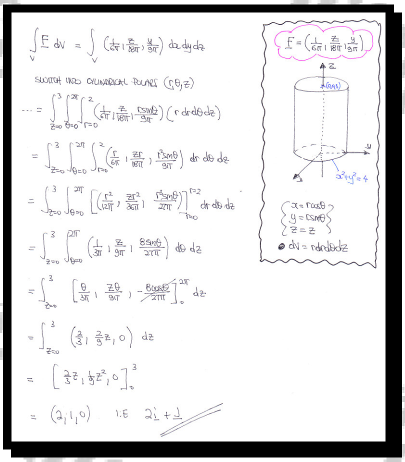
Evaluate the vector integral

$$\int_V \mathbf{F}\; dV\,,$$

where V is the finite region enclosed by the cylinder with equation

$$x^2 + y^2 = 4\,,\ 0 \le z \le 3\,.$$

$\boxed{2\mathbf{i} + \mathbf{j}}$



Question 5

$$\mathbf{F}(x, y, z) \equiv 3\mathbf{i} + -y\mathbf{j} + 6x\mathbf{k}.$$

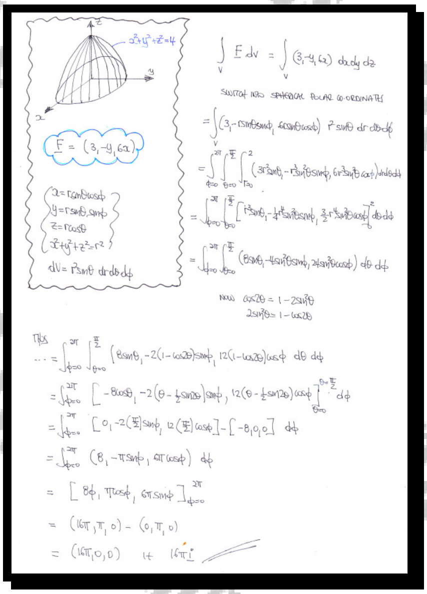
Evaluate the vector integral

$$\int_V \mathbf{F} \, dV ,$$

where V is the finite region enclosed by the hemisphere with equation

$$x^2 + y^2 + z^2 = 4, \; z \geq 0.$$

$\boxed{16\pi\mathbf{i}}$



Question 6

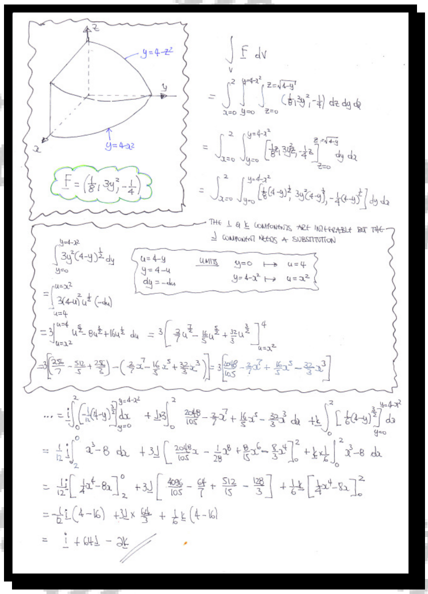
The finite region V in the first octant, is bounded by the surfaces with equations

$$y = 4 - x^2 \quad \text{and} \quad y = 4 - z^2 .$$

Given that $\mathbf{F} = \frac{1}{8}\mathbf{i} + 3y^2\mathbf{j} - \frac{1}{4}\mathbf{k}$ determine

$$\int_V \mathbf{F} \ dv .$$

$\boxed{\mathbf{i} + 64\mathbf{j} - 2\mathbf{k}}$



TYPE $\int_S F \, \mathbf{dS}$

Question 1

$$F(x,y,z) \equiv x+y+z \,.$$

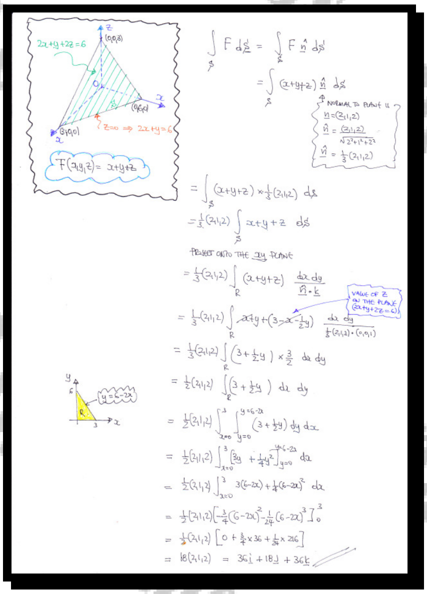
Evaluate the integral

$$\int_S F \,\mathbf{dS} \,,$$

where S is the plane surface with equation

$$2x+y+2z=6 \,,\ x \geq 0 \,,\ y \geq 0 \,,\ z \geq 0 \,.$$

$\boxed{36\mathbf{i}+18\mathbf{j}+36\mathbf{k}}$



Question 2

$$\varphi(x, y, z) \equiv \frac{3}{4}xyz\,.$$

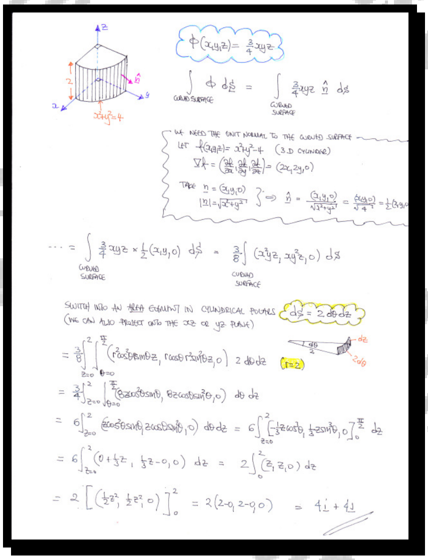
Evaluate the integral

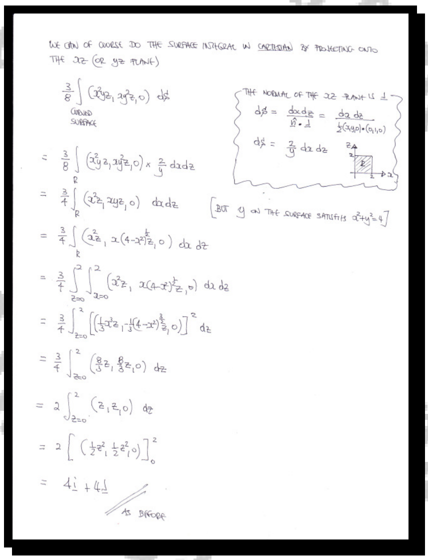
$$\int_S \varphi \, \mathbf{dS}\,,$$

where S is the curved surface of the cylinder with equation

$$x^2 + y^2 = 4,\ x \geq 0,\ y \geq 0,\ 0 \leq z \leq 2\,.$$

$\boxed{4\mathbf{i} + 4\mathbf{j}}$





Question 3

$$\varphi(x, y, z) \equiv \tfrac{1}{2} xyz^2 .$$

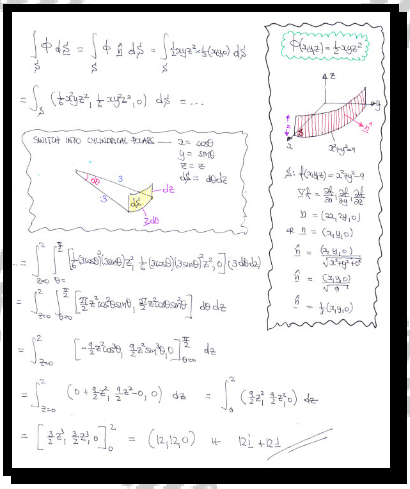
Evaluate the integral

$$\int_S \varphi \, \mathbf{dS} ,$$

where S is the curved surface of the cylinder with equation

$$x^2 + y^2 = 9 , \; x \geq 0 , \; y \geq 0 , \; 0 \leq z \leq 2 .$$

$\boxed{12\mathbf{i} + 12\mathbf{j}}$



Question 4

$$\varphi(x,y,z) \equiv 2x + 2y \,.$$

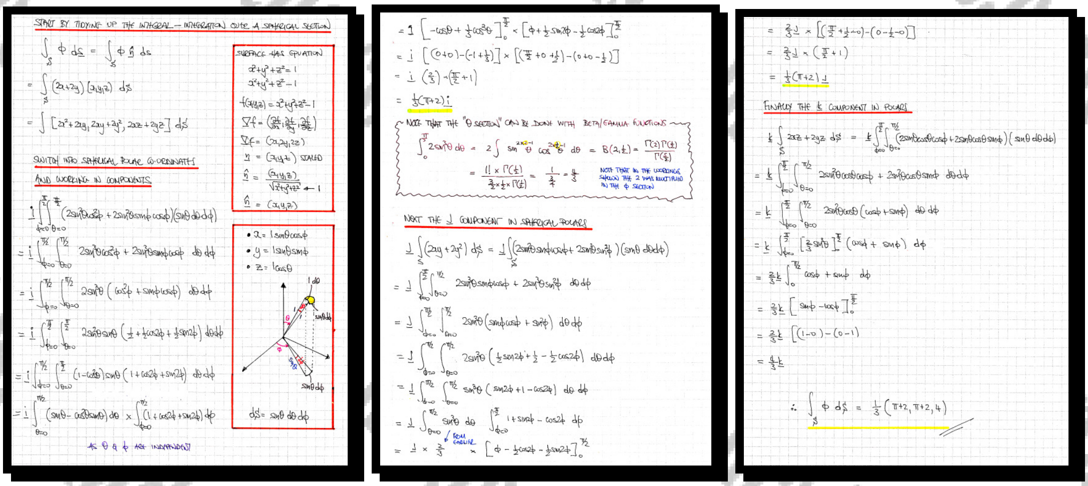
Evaluate the integral

$$\int_S \varphi \, \mathbf{dS} \,,$$

where S is the curved surface of the sphere with equation

$$x^2 + y^2 + z^2 = 1, \; x \ge 0, \; y \ge 0, \; z \ge 0 \,.$$

$$\boxed{}, \; \boxed{\frac{1}{3}\big[(\pi+2)\mathbf{i} + (\pi+2)\mathbf{j} + 4\mathbf{k}\big]}$$



TYPE $\int_S \mathbf{F} \, dS$

Question 1

The Cartesian equation of a surface S is

$$z = x^2 + y^2, \quad z \le 1.$$

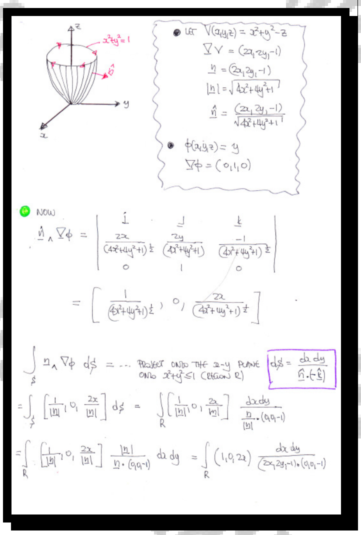
Evaluate the surface integral

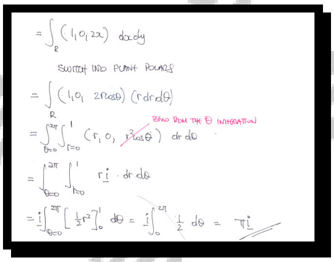
$$\int_S \hat{\mathbf{n}} \wedge \nabla \varphi \, dS,$$

where $\hat{\mathbf{n}}$ is an outward normal unit vector field to S, and φ is the function with Cartesian equation

$$\varphi(x, y, z) = y.$$

$\boxed{\pi \mathbf{i}}$





Question 2

The Cartesian equation of a surface S is

$$z = 1 - x^2 - y^2, \quad z \geq 0.$$

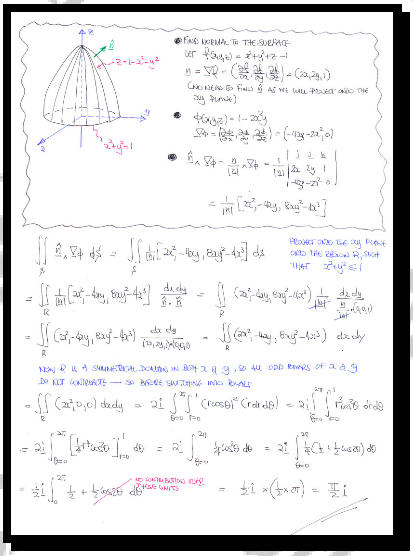
Evaluate the surface integral

$$\int_S \hat{\mathbf{n}} \wedge \nabla \varphi \, dS,$$

where $\hat{\mathbf{n}}$ is an outward unit normal vector field to S, and φ is the function with Cartesian equation

$$\varphi(x, y, z) = 1 - 2x^2 y.$$

$$\boxed{\frac{\pi}{2}\mathbf{i}}$$



TYPE $\int_S \mathbf{F} \cdot d\mathbf{S}$

Question 1

Evaluate the surface integral

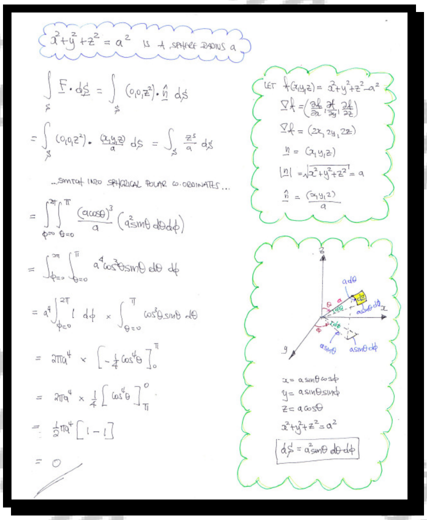
$$\int_S \mathbf{F} \cdot \mathbf{dS} ,$$

where S is the surface with equation

$$x^2 + y^2 + z^2 = a^2 , \; a > 0 ,$$

and $\mathbf{F} = z^2\mathbf{k}$.

$\boxed{0}$



Question 2

$$\mathbf{F}(x, y, z) \equiv x^2\mathbf{i} - 2y\mathbf{j} - 2z\mathbf{k} .$$

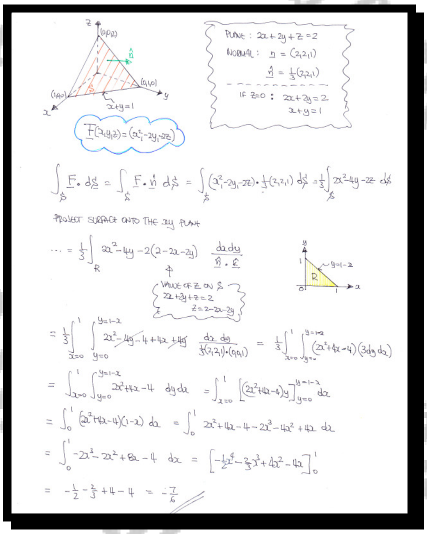
Evaluate the surface integral

$$\int_S \mathbf{F} \cdot \mathbf{dS} ,$$

where S is the plane surface with equation

$$2x + 2y + z = 2 ,\ x \geq 0 ,\ y \geq 0 ,\ z \geq 0 .$$

$$\boxed{-\frac{7}{6}}$$



Question 3

$$\mathbf{F}(x,y,z) \equiv 4y\mathbf{i} + \mathbf{j} + 2\mathbf{k}\,.$$

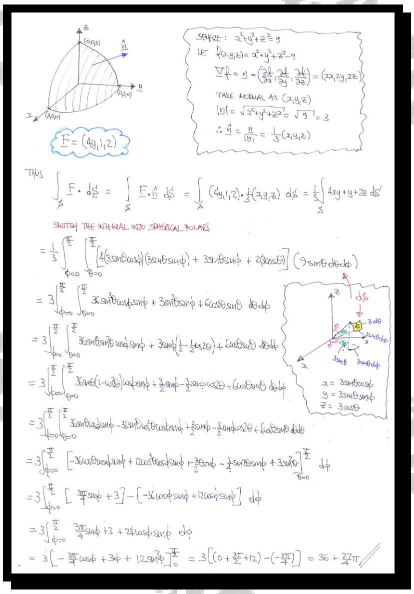
Evaluate the surface integral

$$\int_S \mathbf{F}\cdot\mathbf{dS}\,,$$

where S is the surface with equation

$$x^2 + y^2 + z^2 = 9\,,\ x \ge 0\,,\ y \ge 0\,,\ z \ge 0\,.$$

$$\boxed{36 + \frac{9}{4}\pi}$$



SPHERE: $x^2+y^2+z^2=9$

LET $f(x,y,z) = x^2+y^2+z^2-9$

$\nabla f = \underline{n} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (2x, 2y, 2z)$

TAKE NORMAL AS (x,y,z)

$|\underline{n}| = \sqrt{x^2+y^2+z^2} = \sqrt{9} = 3$

$\therefore \hat{\underline{n}} = \frac{\underline{n}}{|\underline{n}|} = \frac{1}{3}(x,y,z)$

$\underline{F} = (4y, 1, 2)$

THUS

$$\int_S \underline{F}\cdot d\underline{S} = \int_S \underline{F}\cdot\hat{\underline{n}}\,dS = \int_S (4y,1,2)\cdot\tfrac{1}{3}(x,y,z)\,dS = \frac{1}{3}\int_S 4xy + y + 2z\,dS$$

SWITCH THE INTEGRAL INTO SPHERICAL POLARS

$$= \frac{1}{3}\int_{\phi=0}^{\frac{\pi}{2}}\int_{\theta=0}^{\frac{\pi}{2}} \left[4(3\sin\theta\cos\phi)(3\sin\theta\sin\phi) + 3\sin\theta\sin\phi + 2(3\cos\theta)\right](9\sin\theta\,d\theta\,d\phi)$$

$$= 3\int_{\phi=0}^{\frac{\pi}{2}}\int_{\theta=0}^{\frac{\pi}{2}} 36\sin^3\theta\cos\phi\sin\phi + 3\sin^2\theta\sin\phi + 6\cos\theta\sin\theta\,d\theta\,d\phi$$

$$= 3\int_{\phi=0}^{\frac{\pi}{2}}\int_{\theta=0}^{\frac{\pi}{2}} 36\sin\theta\sin^2\theta\cos\phi\sin\phi + 3\sin\phi\left(\tfrac{1}{2} - \tfrac{1}{2}\cos2\theta\right) + 6\cos\theta\sin\theta\,d\theta\,d\phi$$

$$= 3\int_{\phi=0}^{\frac{\pi}{2}}\int_{\theta=0}^{\frac{\pi}{2}} 36\sin\theta(1-\cos^2\theta)\cos\phi\sin\phi + \tfrac{3}{2}\sin\phi - \tfrac{3}{2}\sin\phi\cos2\theta + 6\cos\theta\sin\theta\,d\theta\,d\phi$$

$$= 3\int_{\phi=0}^{\frac{\pi}{2}}\int_{\theta=0}^{\frac{\pi}{2}} 36\sin\theta\cos\phi\sin\phi - 36\sin\theta\cos^2\theta\cos\phi\sin\phi + \tfrac{3}{2}\sin\phi - \tfrac{3}{2}\sin\phi\cos2\theta + 6\cos\theta\sin\theta\,d\theta\,d\phi$$

$$= 3\int_{\phi=0}^{\frac{\pi}{2}} \left[-36\cos\theta\cos\phi\sin\phi + 12\cos^3\theta\cos\phi\sin\phi + \tfrac{3}{2}\theta\sin\phi - \tfrac{3}{4}\sin2\theta\sin\phi + 3\sin^2\theta\right]_{\theta=0}^{\frac{\pi}{2}} d\phi$$

$$= 3\int_{\phi=0}^{\frac{\pi}{2}} \left[\tfrac{3\pi}{4}\sin\phi + 3\right] - \left[-36\cos\phi\sin\phi + 12\cos\phi\sin\phi\right] d\phi$$

$$= 3\int_{\phi=0}^{\frac{\pi}{2}} \tfrac{3\pi}{4}\sin\phi + 3 + 24\cos\phi\sin\phi\,d\phi$$

$$= 3\left[-\tfrac{3\pi}{4}\cos\phi + 3\phi + 12\sin^2\phi\right]_0^{\frac{\pi}{2}} = 3\left[\left(0 + \tfrac{3\pi}{2} + 12\right) - \left(-\tfrac{3\pi}{4}\right)\right] = 36 + \tfrac{27}{4}\pi$$

Question 4

Evaluate the surface integral

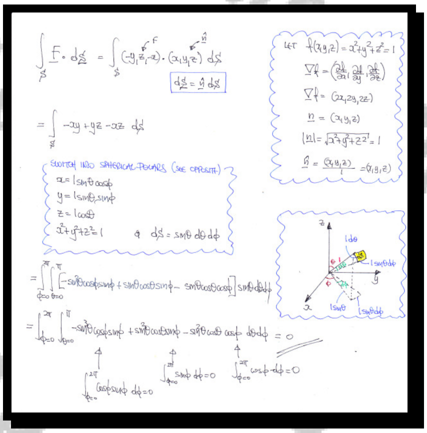
$$\int_S \mathbf{F} \cdot \mathbf{dS},$$

where S is the surface with equation

$$x^2 + y^2 + z^2 = 1,$$

and $\mathbf{F} = -y\mathbf{i} + z\mathbf{j} - x\mathbf{k}$.

$\boxed{0}$



Question 5

$$\mathbf{F}(x, y, z) \equiv \mathbf{i} + \frac{1}{2} y\mathbf{j} + z^2\mathbf{k} .$$

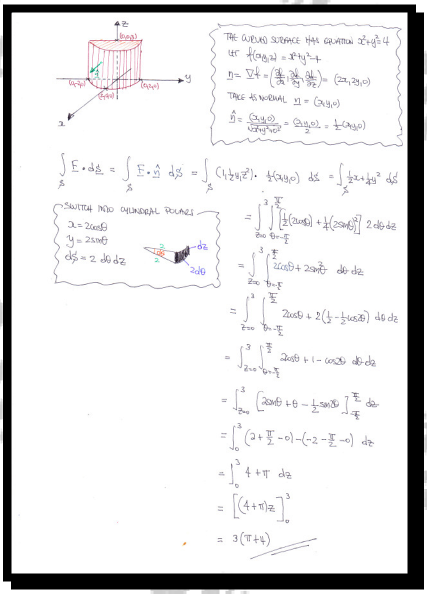
Evaluate the surface integral

$$\int_S \mathbf{F} \cdot \mathbf{dS} ,$$

where S is the curved cylindrical surface with equation

$$x^2 + y^2 = 4 ,\ x \geq 0 ,\ 0 \leq z \leq 3 .$$

$\boxed{3\pi + 12}$



Question 6

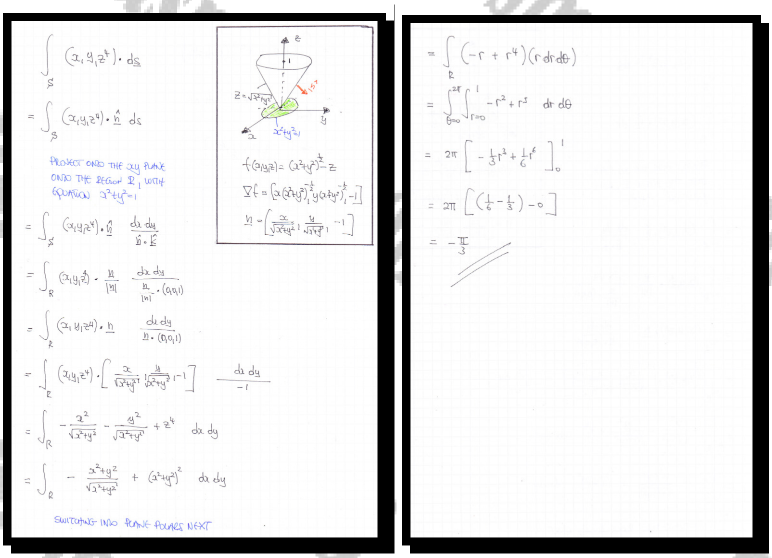
$$\mathbf{F}(x, y, z) \equiv x\,\mathbf{i} + y\,\mathbf{j} + z^4\,\mathbf{k}\,.$$

Calculate the flux of $\mathbf{F}$ through the open surface with equation

$$z = \sqrt{x^2 + y^2}\,,\ z \le 1\,,$$

in the direction of z decreasing.

$\boxed{-\frac{1}{3}\pi}$



Question 7

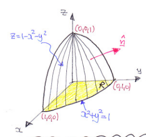
The surface S has Cartesian equation

$$z = 1 - x^2 - y^2, \; x \geq 0, \; y \geq 0, \; z \geq 0.$$

Evaluate the surface integral

$$\int_S 15z\mathbf{i} \cdot \mathbf{dS}.$$

$\boxed{4}$



$dS = \frac{dx\,dy}{\hat{n}\cdot\hat{k}} = \frac{dx\,dy}{\frac{n\cdot k}{|n|}}$

$z = 1-x^2-y^2$
$z+x^2+y^2-1=0$
LET $f(x,y,z) = x^2+y^2+z-1$
$n = \nabla f = (2x,2y,1)$
$|n| = \sqrt{4x^2+4y^2+1} = \sqrt{4(x^2+y^2)+1} = \sqrt{4(1-z)+1} = \sqrt{5-z}$
$\hat{n} = \frac{(2x,2y,1)}{\sqrt{5-z}}$

THUS

$\int_S 15z\mathbf{i}\cdot d\mathbf{S} = \int_S 15z\mathbf{i}\cdot\hat{n}\,dS$ ... PROJECT ONTO THE xy PLANE I.E REGION R IN YELLOW

$= \int_R 15z\mathbf{i}\cdot\frac{n}{|n|}\frac{|n|\,dx\,dy}{n\cdot k} = \int_R \frac{15z\mathbf{i}\cdot n}{n\cdot k}\,dx\,dy$

$= \int_R \frac{15(z,0,0)\cdot(2x,2y,1)}{(2x,2y,1)\cdot(0,0,1)}\,dx\,dy = \int_R 30xz\,dx\,dy$

SWITCH INTO PLANE POLARS ... NOTE THAT $z = 1-x^2-y^2 = 1-r^2$

$= \int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{1} 30r\cos\theta(1-r^2)(r\,dr\,d\theta) = \left[\int_{\theta=0}^{\frac{\pi}{2}}\cos\theta\,d\theta\right]\left[\int_{r=0}^{1} 30r^2-30r^4\,dr\right]$

$= \left[\sin\theta\right]_0^{\frac{\pi}{2}}\left[10r^3-6r^5\right]_0^1 = 1\times(10-6) = 4$

Question 8

$$\mathbf{F}(x, y, z) \equiv -y\mathbf{i} + x\mathbf{j} + 3z\mathbf{k}.$$

Evaluate the surface integral

$$\int_S \mathbf{F} \cdot \mathbf{dS},$$

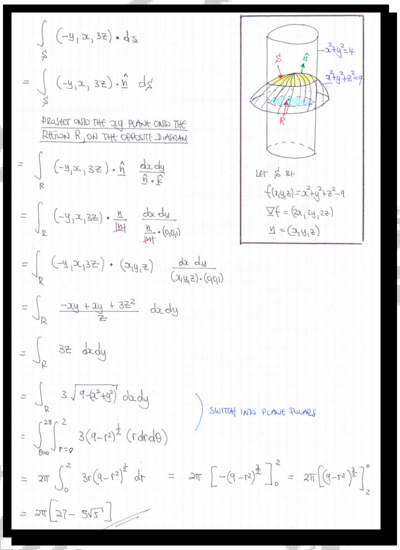
where S is the surface of the hemisphere with equation

$$x^2 + y^2 + z^2 = 9, \ z \geq 0,$$

contained inside the cylinder with equation

$$x^2 + y^2 = 4, \ z \geq 0,$$

$$\boxed{2\pi\left[27 - 5\sqrt{5}\right]}$$



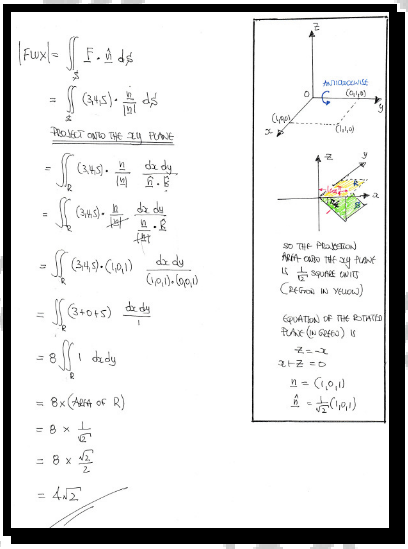
Question 9

Space is filled uniformly by the constant vector field $3\mathbf{i}+4\mathbf{j}+5\mathbf{k}$.

A square lamina whose vertices are at $(0,0,0)$, $(1,0,0)$, $(1,1,0)$ and $(0,1,0)$ is rotated by $\frac{1}{4}\pi$, anticlockwise, about the y axis.

determine the magnitude of the flux of the field through the rotated lamina.

$\boxed{4\sqrt{2}}$



Question 10

The surface S has Cartesian equation

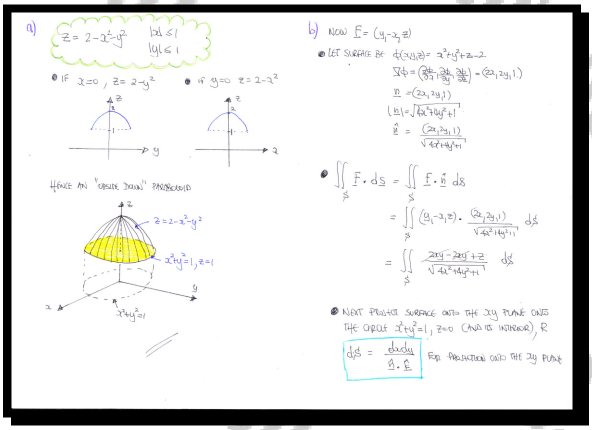
$$z = 2 - x^2 - y^2, \; x^2 + y^2 \le 1 .$$

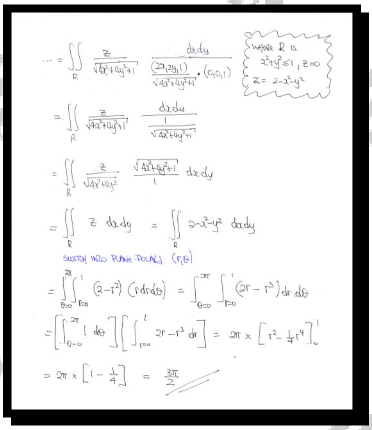
a) Sketch the graph of S.

b) Given that $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$, evaluate the integral

$$\int_S \mathbf{F} \cdot \mathbf{dS} .$$

$\boxed{\frac{3\pi}{2}}$





Question 11

The surface S has Cartesian equation

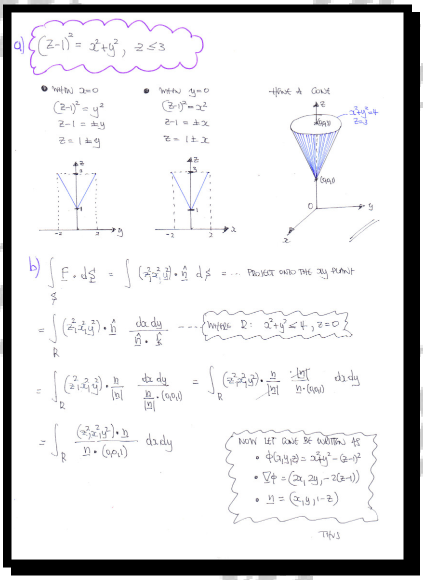
$$(z-1)^2 = x^2 + y^2, \; 1 \le z \le 3.$$

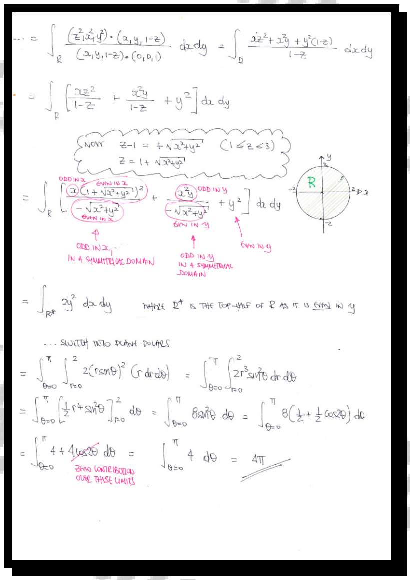
a) Sketch the graph of S.

b) Given that $\mathbf{F} = z^2\mathbf{i} + x^2\mathbf{j} + y^2\mathbf{k}$, evaluate the integral

$$\int_S \mathbf{F} \cdot \mathbf{dS}.$$

$\boxed{4\pi}$





Question 12

$$\mathbf{F}(x, y, z) \equiv 3x\,\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k} .$$

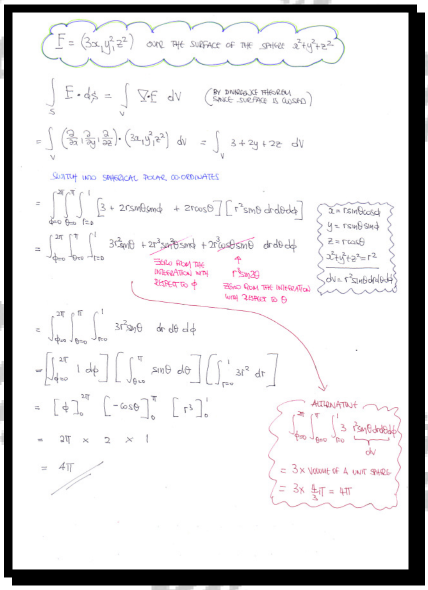
Evaluate the surface integral

$$\int_S \mathbf{F} \cdot \mathbf{dS} ,$$

where S is the surface with Cartesian equation

$$x^2 + y^2 + z^2 = 1 .$$

$\boxed{4\pi}$



Question 13

$$\mathbf{F}(x,y,z) \equiv (x+y)\mathbf{i} + (x-y)\mathbf{j} + (x+z)\mathbf{k}\ .$$

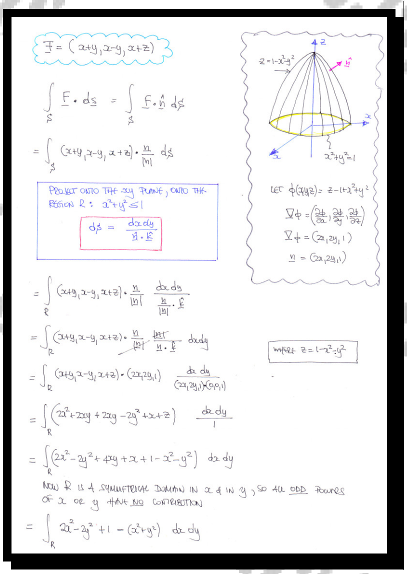
Evaluate the surface integral

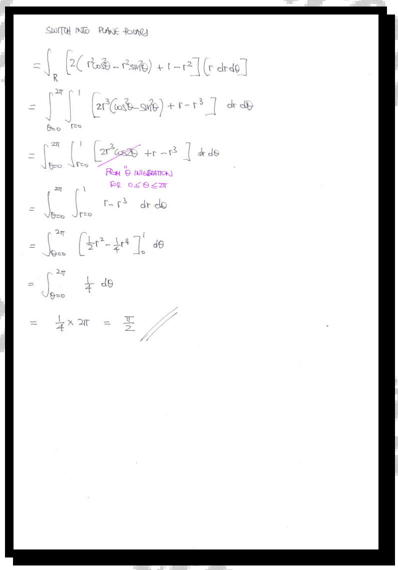
$$\int_S \mathbf{F} \cdot \mathbf{dS}\ ,$$

where S is the surface with Cartesian equation

$$z = 1 - x^2 - y^2\ ,\ z \geq 0\ .$$

$$\boxed{\frac{\pi}{2}}$$





Question 14

$$\mathbf{F}(x,y,z) \equiv (x+z+xy)\mathbf{i} + \left(z^2 - 2xz - y\right)\mathbf{j} + \mathbf{k}.$$

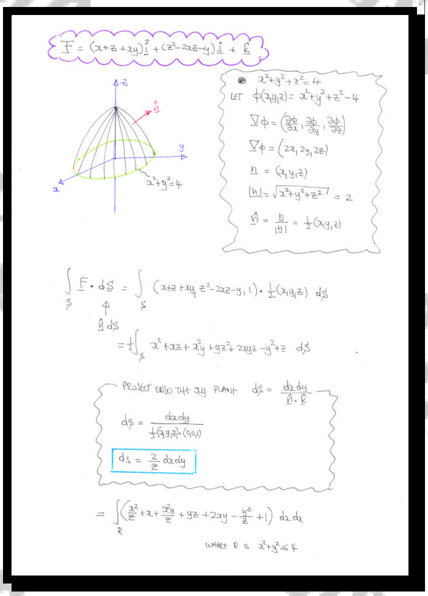
Evaluate the surface integral

$$\int_S \mathbf{F} \cdot \mathbf{dS},$$

where S is the surface with Cartesian equation

$$x^2 + y^2 + z^2 = 4, \ z \geq 0.$$

$\boxed{4\pi}$



$\underline{F} = (x+z+xy)\hat{\underline{i}} + (z^2-2xz-y)\hat{\underline{j}} + \hat{\underline{k}}$

$x^2+y^2+z^2=4$

LET $\phi(x,y,z) = x^2+y^2+z^2-4$

$\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$

$\nabla\phi = (2x, 2y, 2z)$

$\underline{n} = (x,y,z)$

$|\underline{n}| = \sqrt{x^2+y^2+z^2} = 2$

$\hat{\underline{n}} = \frac{\underline{n}}{|\underline{n}|} = \frac{1}{2}(x,y,z)$

$\int_S \underline{F} \cdot d\underline{S} = \int_S (x+z+xy, z^2-2xz-y, 1) \cdot \frac{1}{2}(x,y,z)\, dS$ (where $d\underline{S} = \hat{\underline{n}}\, dS$)

$= \frac{1}{2}\int_S x^2+xz+x^2y+yz^2-2xyz-y^2+z \, dS$

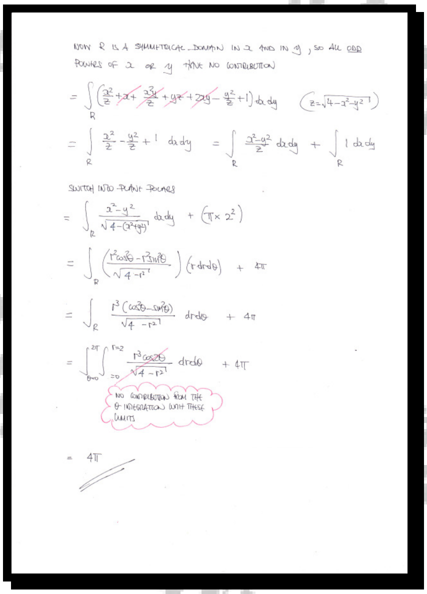
PROJECT ONTO THE xy PLANE $dS = \frac{dx\,dy}{\hat{\underline{n}} \cdot \hat{\underline{k}}}$

$dS = \frac{dx\,dy}{\frac{1}{2}(x,y,z)\cdot(0,0,1)}$

$\boxed{dS = \frac{2}{z}\,dx\,dy}$

$= \int_R \left(\frac{x^2}{z} + x + \frac{x^2y}{z} + yz + 2xy - \frac{y^2}{z} + 1\right) dx\,dy$

WHERE R IS $x^2+y^2 \leq 4$



NOW R IS A SYMMETRICAL DOMAIN IN x AND IN y, SO ALL ODD POWERS OF x OR y HAVE NO CONTRIBUTION

$= \int_R \left(\frac{x^2}{z} + x + \frac{x^2y}{z} + yz + 2xy - \frac{y^2}{z} + 1\right) dx\,dy \quad \left(z = \sqrt{4-x^2-y^2}\right)$

$= \int_R \frac{x^2}{z} - \frac{y^2}{z} + 1 \, dx\,dy = \int_R \frac{x^2-y^2}{z}\,dx\,dy + \int_R 1\,dx\,dy$

SWITCH INTO PLANE POLARS

$= \int_R \frac{x^2-y^2}{\sqrt{4-(x^2+y^2)}}\,dx\,dy + (\pi \times 2^2)$

$= \int_R \left(\frac{r^2\cos^2\theta - r^2\sin^2\theta}{\sqrt{4-r^2}}\right)(r\,dr\,d\theta) + 4\pi$

$= \int_R \frac{r^3(\cos^2\theta - \sin^2\theta)}{\sqrt{4-r^2}}\,dr\,d\theta + 4\pi$

$= \int_{\theta=0}^{2\pi}\int_{r=0}^{r=2} \frac{r^3\cos 2\theta}{\sqrt{4-r^2}}\,dr\,d\theta + 4\pi$

NO CONTRIBUTION FROM THE θ INTEGRATION WITH THESE LIMITS

$= 4\pi$

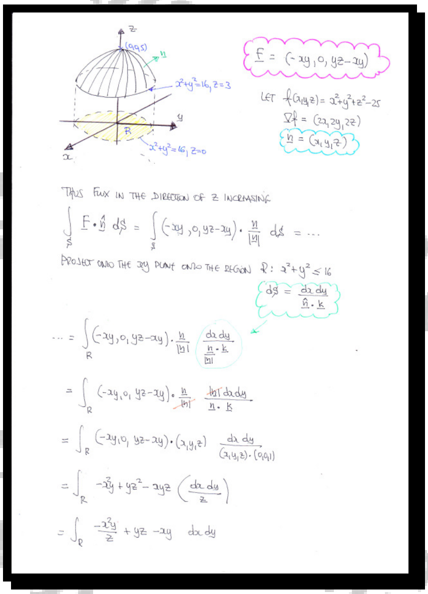
Question 15

$$\mathbf{F}(x, y, z) \equiv -xy\,\mathbf{i} + (yz - xy)\,\mathbf{k}\,.$$

Show that there is zero net flux of $\mathbf{F}$ through the surface with Cartesian equation

$$x^2 + y^2 + z^2 = 25\,,\ z \geq 3\,.$$

proof



$\underline{F} = (-xy, 0, yz-xy)$

LET $f(x,y,z) = x^2+y^2+z^2-25$

$\nabla f = (2x, 2y, 2z)$

$\underline{n} = (x, y, z)$

THUS FLUX IN THE DIRECTION OF z INCREASING

$$\int_S \underline{F} \cdot \hat{\underline{n}}\, dS = \int_S (-xy, 0, yz-xy) \cdot \frac{\underline{n}}{|\underline{n}|}\, dS = \dots$$

PROJECT ONTO THE xy PLANE ONTO THE REGION R: $x^2+y^2 \leq 16$

$$dS = \frac{dx\,dy}{\hat{\underline{n}} \cdot \underline{k}}$$

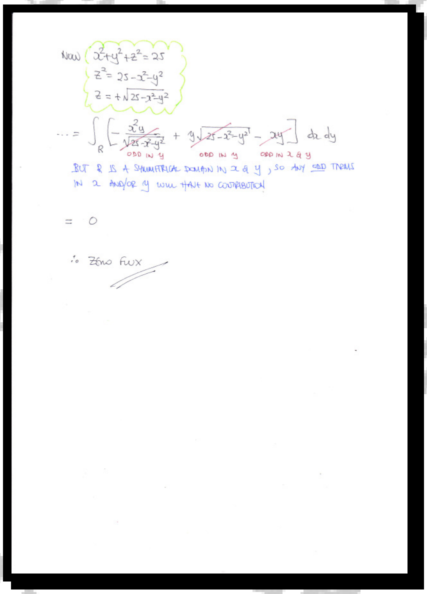
$$\dots = \int_R (-xy, 0, yz-xy) \cdot \frac{\underline{n}}{|\underline{n}|} \frac{dx\,dy}{\frac{\underline{n}\cdot\underline{k}}{|\underline{n}|}}$$

$$= \int_R (-xy, 0, yz-xy) \cdot \frac{\underline{n}}{|\underline{n}|} \frac{|\underline{n}|\,dx\,dy}{\underline{n}\cdot\underline{k}}$$

$$= \int_R (-xy, 0, yz-xy) \cdot (x, y, z) \frac{dx\,dy}{(x,y,z)\cdot(0,0,1)}$$

$$= \int_R -x^2y + yz^2 - xyz \left(\frac{dx\,dy}{z}\right)$$

$$= \int_R \frac{-x^2y}{z} + yz - xy \quad dx\,dy$$



NOW $x^2+y^2+z^2 = 25$

$z^2 = 25 - x^2 - y^2$

$z = +\sqrt{25-x^2-y^2}$

$$\dots = \int_R \left[-\frac{x^2y}{\sqrt{25-x^2-y^2}} + y\sqrt{25-x^2-y^2} - xy\right] dx\,dy$$

ODD IN y — ODD IN y — ODD IN x & y

BUT R IS A SYMMETRICAL DOMAIN IN x & y, SO ANY ODD TERMS IN x AND/OR y WILL HAVE NO CONTRIBUTION

$= 0$

$\therefore$ ZERO FLUX

Question 16

$$\mathbf{F}(x, y, z) \equiv x\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}.$$

a) Given that S is the surface with Cartesian equation

$$x^2 + y^2 + z^2 = 1, \ z \geq 0,$$

show that

$$\int_S \mathbf{F} \cdot \mathbf{dS} = 4 \iint_R \left[\frac{x^2}{\sqrt{1 - x^2 - y^2}} + 1 - x^2 - y^2 \right] dx\,dy,$$

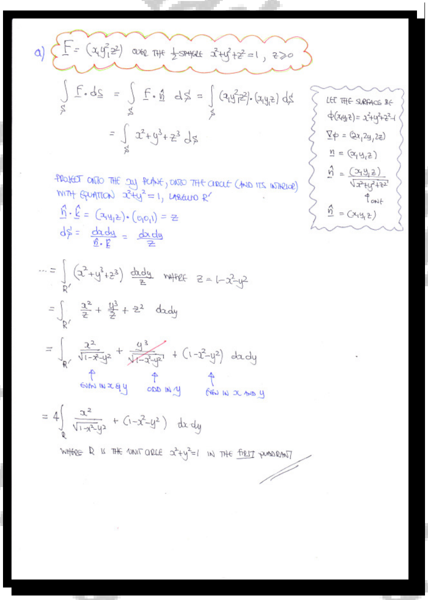
where R is the region in the first quadrant with Cartesian equation

$$x^2 + y^2 \leq 1, \ x \geq 0, \ y \geq 0.$$

b) Evaluate the surface integral

$$\int_S \mathbf{F} \cdot \mathbf{dS}.$$

$\boxed{\frac{7}{6}\pi}$



a) $\underline{F} = (x, y^2, z^2)$ over the $\frac{1}{2}$-sphere $x^2+y^2+z^2=1$, $z \geq 0$

$\int_S \underline{F} \cdot d\underline{S} = \int_S \underline{F} \cdot \hat{\underline{n}}\, dS = \int_S (x, y^2, z^2) \cdot (x, y, z)\, dS$

$= \int_S x^2 + y^3 + z^3\, dS$

LET THE SURFACE BE
$\phi(x,y,z) = x^2+y^2+z^2-1$
$\underline{\nabla}\phi = (2x, 2y, 2z)$
$\underline{n} = (x, y, z)$
$\hat{\underline{n}} = \frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}}$ ← UNIT
$\hat{\underline{n}} = (x, y, z)$

PROJECT ONTO THE xy PLANE, ONTO THE CIRCLE (AND ITS INTERIOR) WITH EQUATION $x^2+y^2=1$, LABELLED R'

$\hat{\underline{n}} \cdot \hat{\underline{k}} = (x,y,z) \cdot (0,0,1) = z$

$dS = \frac{dx\,dy}{\hat{\underline{n}} \cdot \hat{\underline{k}}} = \frac{dx\,dy}{z}$

$\dots = \int_{R'} (x^2+y^3+z^3) \frac{dx\,dy}{z}$ WHERE $z = 1 - x^2 - y^2$

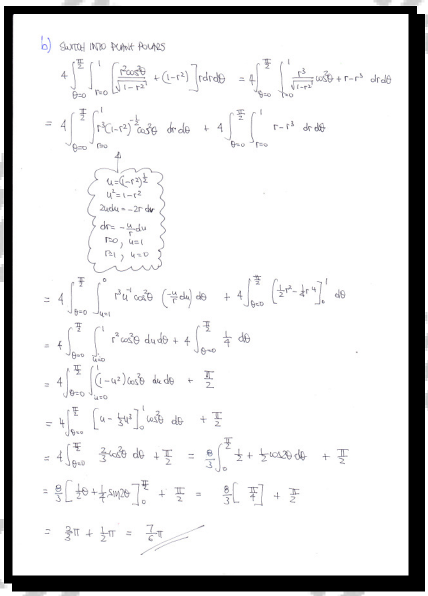
$= \int_{R'} \frac{x^2}{z} + \frac{y^3}{z} + z^2\, dx\,dy$

$= \int_{R'} \frac{x^2}{\sqrt{1-x^2-y^2}} + \frac{y^3}{\sqrt{1-x^2-y^2}} + (1-x^2-y^2)\, dx\,dy$

EVEN IN x & y — ODD IN y — EVEN IN x AND y

$= 4\int_R \frac{x^2}{\sqrt{1-x^2-y^2}} + (1-x^2-y^2)\, dx\,dy$

WHERE R IS THE UNIT CIRCLE $x^2+y^2=1$ IN THE FIRST QUADRANT



b) SWITCH INTO PLANE POLARS

$4\int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{1} \left[\frac{r^2\cos^2\theta}{\sqrt{1-r^2}} + (1-r^2)\right] r\,dr\,d\theta = 4\int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{1} \frac{r^3}{\sqrt{1-r^2}}\cos^2\theta + r - r^3\, dr\,d\theta$

$= 4\int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{1} r^3(1-r^2)^{-\frac{1}{2}}\cos^2\theta\, dr\,d\theta + 4\int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{1} r - r^3\, dr\,d\theta$

$u = (1-r^2)^{\frac{1}{2}}$
$u^2 = 1 - r^2$
$2u\,du = -2r\,dr$
$dr = -\frac{u}{r}du$
$r=0,\ u=1$
$r=1,\ u=0$

$= 4\int_{\theta=0}^{\frac{\pi}{2}} \int_{u=1}^{0} r^3 u^{-1}\cos^2\theta \left(-\frac{u}{r}du\right) d\theta + 4\int_{\theta=0}^{\frac{\pi}{2}} \left[\frac{1}{2}r^2 - \frac{1}{4}r^4\right]_0^1 d\theta$

$= 4\int_{\theta=0}^{\frac{\pi}{2}} \int_{u=0}^{1} r^2\cos^2\theta\, du\,d\theta + 4\int_{\theta=0}^{\frac{\pi}{2}} \frac{1}{4}\, d\theta$

$= 4\int_{\theta=0}^{\frac{\pi}{2}} \int_{u=0}^{1} (1-u^2)\cos^2\theta\, du\,d\theta + \frac{\pi}{2}$

$= 4\int_{\theta=0}^{\frac{\pi}{2}} \left[u - \frac{1}{3}u^3\right]_0^1 \cos^2\theta\, d\theta + \frac{\pi}{2}$

$= 4\int_{\theta=0}^{\frac{\pi}{2}} \frac{2}{3}\cos^2\theta\, d\theta + \frac{\pi}{2} = \frac{8}{3}\int_0^{\frac{\pi}{2}} \frac{1}{2} + \frac{1}{2}\cos 2\theta\, d\theta + \frac{\pi}{2}$

$= \frac{8}{3}\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{2}} + \frac{\pi}{2} = \frac{8}{3}\left[\frac{\pi}{4}\right] + \frac{\pi}{2}$

$= \frac{2}{3}\pi + \frac{1}{2}\pi = \frac{7}{6}\pi$

Question 17

$$\mathbf{F} = x^2 y^3 \mathbf{i} + z\mathbf{j} + x\mathbf{k}\,.$$

Show by **direct** evaluation that

$$\int_S \nabla \wedge \mathbf{F} \cdot \hat{\mathbf{n}} \, dS = 0\,,$$

where S is the sphere with equation

$$x^2 + y^2 + z^2 = 1\,,$$

and $\hat{\mathbf{n}}$ is an outward unit normal to S.

proof

$\underline{F} = (x^2y^3, z, x)$

- $\nabla \wedge \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y^3 & z & x \end{vmatrix} = [0-1, 0-1, 0-3x^2y^2] = [-1,-1,-3x^2y^2]$
- $\nabla(x^2+y^2+z^2-1) = (2x, 2y, 2z)$
- $\underline{n} = (x,y,z)$
- $\hat{\underline{n}} = \frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}} = \frac{(x,y,z)}{1} = (x,y,z)$

THUS INTEGRATING OVER THE SURFACE OF THE SPHERE $x^2+y^2+z^2=1$

$$\int_S \nabla \wedge \underline{F} \cdot d\underline{S} = \int_S \nabla \wedge \underline{F} \cdot \hat{\underline{n}} \, dS = \int_S (-1,-1,-3x^2y^2)\cdot(x,y,z) \, dS$$

$$= \int_S -x-y-3x^2y^2z \, dS = -\int_S x+y+3x^2y^2z \, dS$$

SWITCH INTO SPHERICAL POLARS

$x = 1\sin\theta\cos\phi$

$y = 1\sin\theta\sin\phi$

$z = 1\cos\theta$

$x^2+y^2+z^2 = 1^2$

$dS = 1^2\sin\theta \, d\theta \, d\phi$

$0 \leq \theta \leq \pi$

$0 \leq \phi \leq 2\pi$

$$\dots = -\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \left[(\sin\theta\cos\phi) + (\sin\theta\sin\phi) + 3(\sin\theta\cos\phi)^2(\sin\theta\sin\phi)^2\cos\theta\right]\sin\theta \, d\theta \, d\phi$$

$$= -\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \sin^2\theta\cos\phi + \sin^2\theta\sin\phi + 3\sin^5\theta\cos\theta\cos^2\phi\sin^2\phi \, d\theta \, d\phi$$

$$= -\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \sin^2\theta\cos\phi + \sin^2\theta\sin\phi + 3\sin^5\theta\cos\theta\cos^2\phi\sin^2\phi \, d\theta \, d\phi$$

LOOKING AT THESE INTEGRATIONS WITHOUT PHYSICALLY CARRYING THEM OUT

- $\int_0^{2\pi} \cos\phi \, d\phi = 0$
- $\int_0^{2\pi} \sin\phi \, d\phi = 0$
- $\int_0^{\pi} 3\sin^5\theta\cos\theta \, d\theta = \left[\frac{1}{2}\sin^6\theta\right]_0^{\pi} = 0$

$= 0$

AS REQUIRED

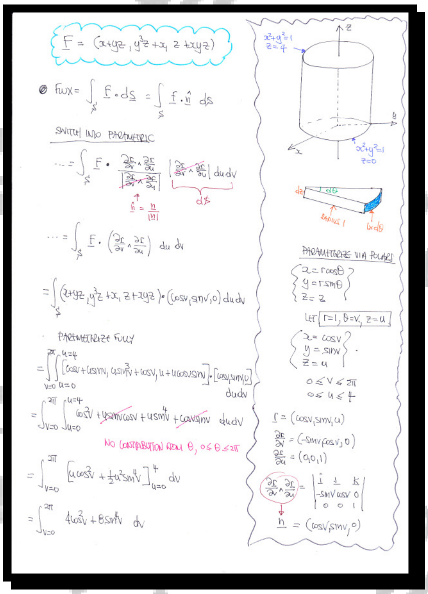
Question 18

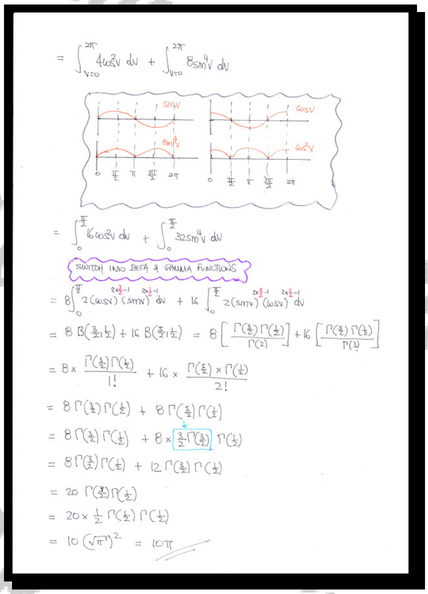
$$\mathbf{F}(x,y,z) \equiv (x+yz)\mathbf{i} + \left(y^3 z + x\right)\mathbf{j} + (z+xyz)\mathbf{k}\,.$$

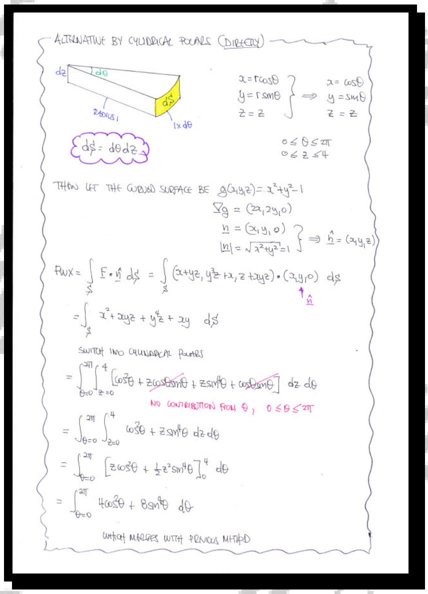
Calculate the magnitude of the flux of **F** through the open cylindrical surface with equation

$$x^2+y^2=1,\; 0 \le z \le 4\,.$$

$\boxed{10\pi}$







Question 19

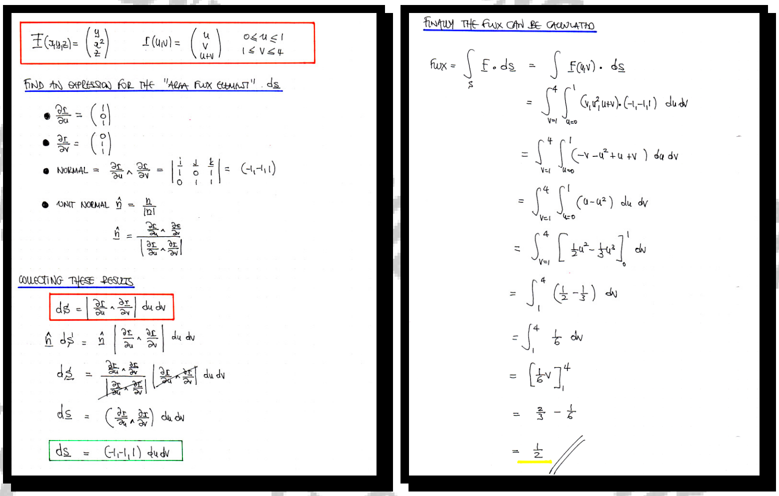
$$\mathbf{F}(x, y, z) \equiv y\,\mathbf{i} + x^2\mathbf{j} + z\,\mathbf{k}\ .$$

Find the magnitude of the flux through the surface with parametric equations

$$\mathbf{r}(u,v) = u\,\mathbf{i} + v\,\mathbf{j} + (u+v)\,\mathbf{k}\ , \quad 0 \le u \le 1,\ 1 \le v \le 4\ .$$

All integrations must be carried out in parametric.

$\boxed{}$, $\boxed{\frac{1}{2}}$



Question 20

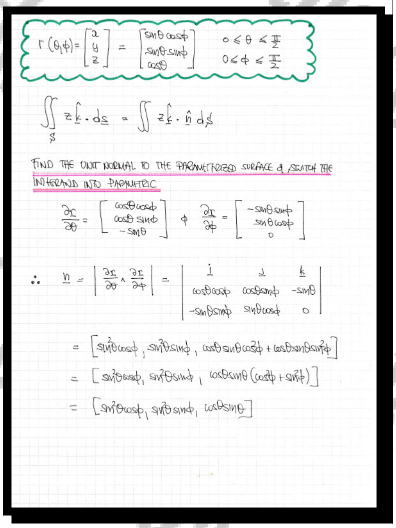
Evaluate the surface integral

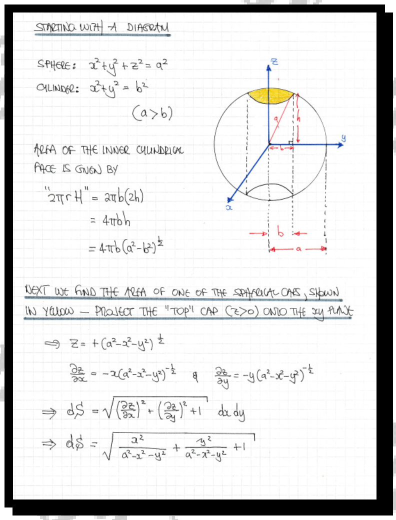
$$\int_S z\,\mathbf{k}\cdot d\mathbf{S}\,,$$

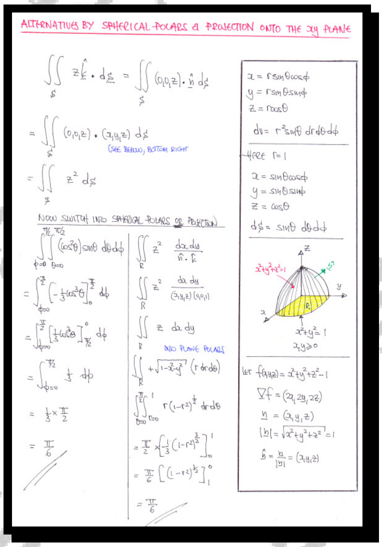
where S is the surface represented parametrically by

$$\mathbf{r}(\theta,\varphi)=\begin{bmatrix}\sin\theta\cos\varphi\\ \sin\theta\sin\varphi\\ \cos\theta\end{bmatrix},\ 0\le\theta\le\frac{1}{2}\pi\,,\ 0\le\varphi\le\frac{1}{2}\pi\,.$$

$\boxed{\frac{1}{6}\pi}$







Question 21

Evaluate the surface integral

$$\int_S \mathbf{F} \cdot \mathbf{dS},$$

where S is the surface represented parametrically by

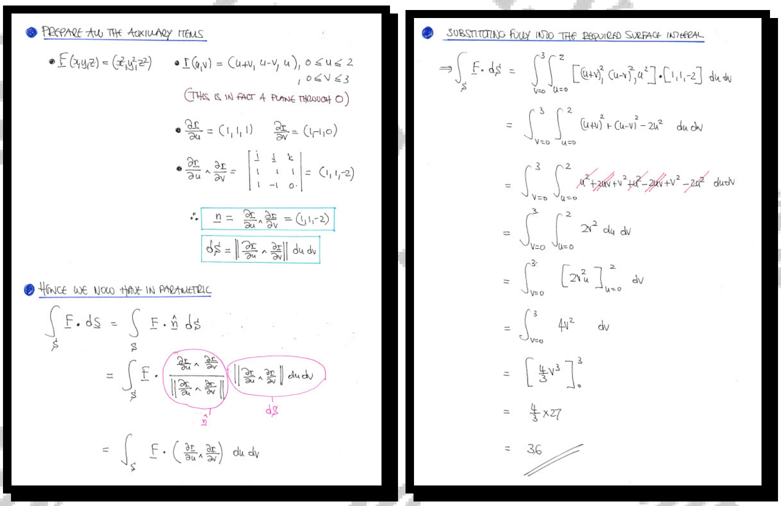
$$\mathbf{r}(u,v) = \begin{bmatrix} u+v \\ u-v \\ u \end{bmatrix}, \quad 0 \le u \le 2, \quad 0 \le v \le 3,$$

and $\mathbf{F}$ is the vector field

$$x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}.$$

All integrations must be carried out in parametric.

$\boxed{}$, $\boxed{36}$


PREPARE ALL THE AUXILIARY ITEMS

$\mathbf{F}(x,y,z) = (x^2, y^2, z^2)$ • $\mathbf{r}(u,v) = (u+v, u-v, u)$, $0 \le u \le 2$, $0 \le v \le 3$

(THIS IS IN FACT A PLANE THROUGH O)

$\frac{\partial \mathbf{r}}{\partial u} = (1,1,1)$ $\quad \frac{\partial \mathbf{r}}{\partial v} = (1,-1,0)$

$\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = (1,1,-2)$

$\therefore \ \mathbf{n} = \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} = (1,1,-2)$

$dS = \left\| \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right\| du\,dv$

HENCE WE NOW HAVE IN PARAMETRIC

$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \int_S \mathbf{F} \cdot \frac{\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v}}{\left\| \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right\|} \left\| \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right\| du\,dv = \int_S \mathbf{F} \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right) du\,dv$

SUBSTITUTING FULLY INTO THE REQUIRED SURFACE INTEGRAL

$\Rightarrow \int_S \mathbf{F} \cdot d\mathbf{S} = \int_{v=0}^{3} \int_{u=0}^{2} \left[(u+v)^2, (u-v)^2, u^2 \right] \cdot [1,1,-2] \, du\,dv$

$= \int_{v=0}^{3} \int_{u=0}^{2} (u+v)^2 + (u-v)^2 - 2u^2 \, du\,dv$

$= \int_{v=0}^{3} \int_{u=0}^{2} u^2 + 2uv + v^2 + u^2 - 2uv + v^2 - 2u^2 \, du\,dv$

$= \int_{v=0}^{3} \int_{u=0}^{2} 2v^2 \, du\,dv$

$= \int_{v=0}^{3} \left[2v^2 u \right]_{u=0}^{2} dv$

$= \int_{v=0}^{3} 4v^2 \, dv$

$= \left[\frac{4}{3} v^3 \right]_0^3$

$= \frac{4}{3} \times 27$

$= 36$

Question 22

$$\mathbf{F}(x,y,z) \equiv x\,\mathbf{i} + y\,\mathbf{j} + 2z\,\mathbf{k}\,.$$

Find the magnitude of the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S}\,,$$

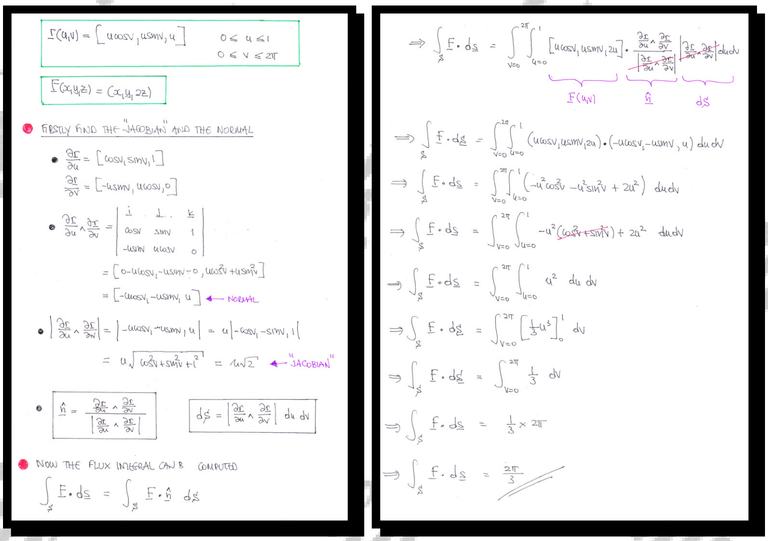
where S is the surface with parametric equations

$$\mathbf{r}(u,v) = (u\cos v)\,\mathbf{i} + (u\sin v)\,\mathbf{j} + u\,\mathbf{k}\,,$$

such that $0 \le u \le 1$, $0 \le v \le 2\pi$.

All integrations must be carried out in parametric.

$\boxed{\frac{2}{3}\pi}$



$\underline{r}(u,v) = [u\cos v, u\sin v, u]$ $\quad 0 \le u \le 1$, $0 \le v \le 2\pi$

$\underline{F}(x,y,z) = (x, y, 2z)$

FIRSTLY FIND THE "JACOBIAN" AND THE NORMAL

- $\frac{\partial \underline{r}}{\partial u} = [\cos v, \sin v, 1]$, $\quad \frac{\partial \underline{r}}{\partial v} = [-u\sin v, u\cos v, 0]$
- $\frac{\partial \underline{r}}{\partial u} \wedge \frac{\partial \underline{r}}{\partial v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = [0 - u\cos v, -u\sin v - 0, u\cos^2 v + u\sin^2 v] = [-u\cos v, -u\sin v, u]$ ← NORMAL
- $\left|\frac{\partial \underline{r}}{\partial u} \wedge \frac{\partial \underline{r}}{\partial v}\right| = |-u\cos v, -u\sin v, u| = u|-\cos v, -\sin v, 1| = u\sqrt{\cos^2 v + \sin^2 v + 1^2} = u\sqrt{2}$ ← "JACOBIAN"
- $\hat{\underline{n}} = \dfrac{\frac{\partial \underline{r}}{\partial u} \wedge \frac{\partial \underline{r}}{\partial v}}{\left|\frac{\partial \underline{r}}{\partial u} \wedge \frac{\partial \underline{r}}{\partial v}\right|}$ $\qquad dS = \left|\frac{\partial \underline{r}}{\partial u} \wedge \frac{\partial \underline{r}}{\partial v}\right| du\, dv$

NOW THE FLUX INTEGRAL CAN B COMPUTED

$$\int_S \underline{F} \cdot d\underline{S} = \int_S \underline{F} \cdot \hat{\underline{n}}\, dS$$

$$\Rightarrow \int_S \underline{F} \cdot d\underline{S} = \int_{v=0}^{2\pi}\int_{u=0}^{1} \underbrace{[u\cos v, u\sin v, 2u]}_{\underline{F}(u,v)} \cdot \underbrace{\frac{\frac{\partial \underline{r}}{\partial u} \wedge \frac{\partial \underline{r}}{\partial v}}{\left|\frac{\partial \underline{r}}{\partial u} \wedge \frac{\partial \underline{r}}{\partial v}\right|}}_{\hat{\underline{n}}} \underbrace{\left|\frac{\partial \underline{r}}{\partial u} \wedge \frac{\partial \underline{r}}{\partial v}\right| du\, dv}_{dS}$$

$$\Rightarrow \int_S \underline{F} \cdot d\underline{S} = \int_{v=0}^{2\pi}\int_{u=0}^{1} (u\cos v, u\sin v, 2u) \cdot (-u\cos v, -u\sin v, u)\, du\, dv$$

$$\Rightarrow \int_S \underline{F} \cdot d\underline{S} = \int_{v=0}^{2\pi}\int_{u=0}^{1} \left(-u^2\cos^2 v - u^2\sin^2 v + 2u^2\right) du\, dv$$

$$\Rightarrow \int_S \underline{F} \cdot d\underline{S} = \int_{v=0}^{2\pi}\int_{u=0}^{1} -u^2(\cos^2 v + \sin^2 v) + 2u^2\, du\, dv$$

$$\Rightarrow \int_S \underline{F} \cdot d\underline{S} = \int_{v=0}^{2\pi}\int_{u=0}^{1} u^2\, du\, dv$$

$$\Rightarrow \int_S \underline{F} \cdot d\underline{S} = \int_{v=0}^{2\pi} \left[\tfrac{1}{3}u^3\right]_0^1 dv$$

$$\Rightarrow \int_S \underline{F} \cdot d\underline{S} = \int_{v=0}^{2\pi} \tfrac{1}{3}\, dv$$

$$\Rightarrow \int_S \underline{F} \cdot d\underline{S} = \tfrac{1}{3} \times 2\pi$$

$$\Rightarrow \int_S \underline{F} \cdot d\underline{S} = \frac{2\pi}{3}$$

Question 23

$$\mathbf{F}(x,y,z) \equiv x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}\,.$$

Find the magnitude of the surface integral

$$\int_S \mathbf{F} \cdot \mathbf{dS}\,,$$

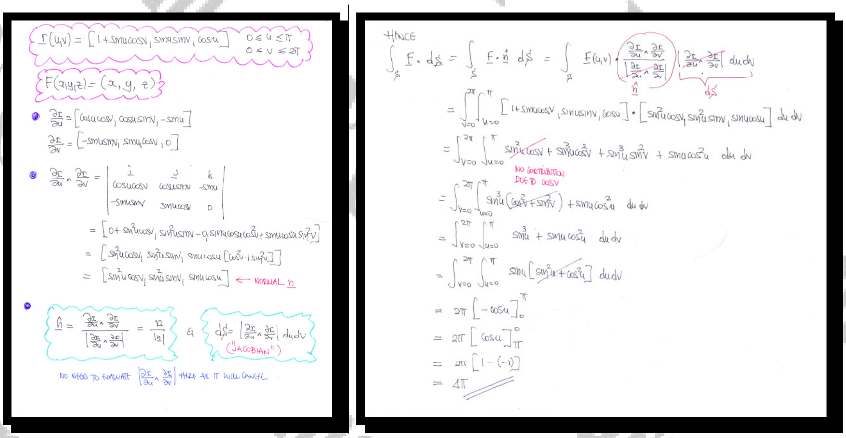
where S is the surface with parametric equations

$$\mathbf{r}(u,v) = (1+\sin u \cos v)\,\mathbf{i} + (\sin u \sin v)\,\mathbf{j} + (\cos u)\,\mathbf{k}\,,$$

such that $0 \le u \le \pi$, $0 \le v \le 2\pi$.

All integrations must be carried out in parametric.

$\boxed{4\pi}$



Question 24

$$\mathbf{F}(x, y, z) \equiv x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}\,.$$

Find the magnitude of the surface integral

$$\int_S \mathbf{F} \cdot \mathbf{dS}\,,$$

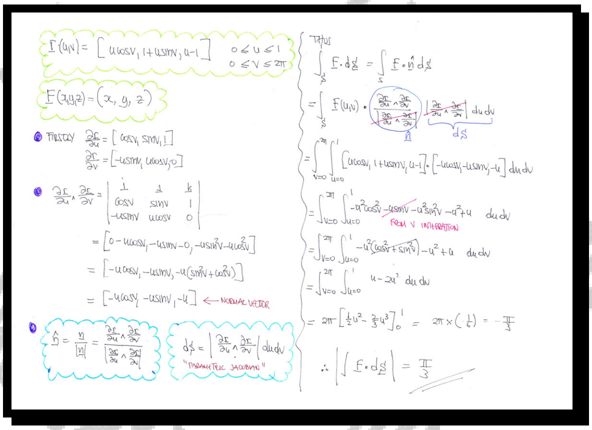
where S is the surface with parametric equations

$$\mathbf{r}(u,v) = (u\cos v)\mathbf{i} + (1 + u\sin v)\mathbf{j} + (u-1)\mathbf{k}\,,$$

such that $0 \le u \le 1,\ \ 0 \le v \le 2\pi$.

All integrations must be carried out in parametric.

$\boxed{\frac{1}{3}\pi}$



Question 25

$$\mathbf{F}(x,y,z) \equiv x\,\mathbf{i} + y\,\mathbf{j} + 2z\,\mathbf{k} .$$

Find the magnitude of the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S} ,$$

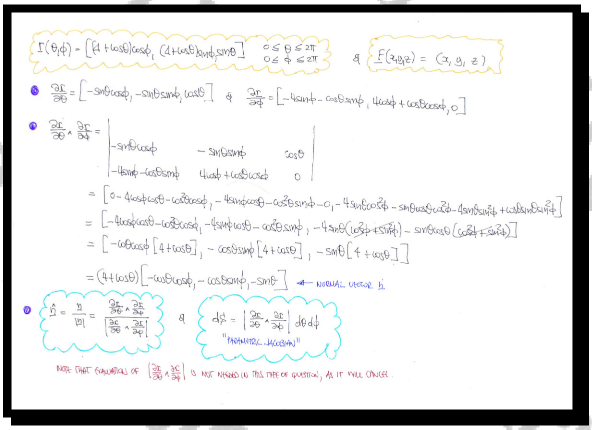
where S is the surface with parametric equations

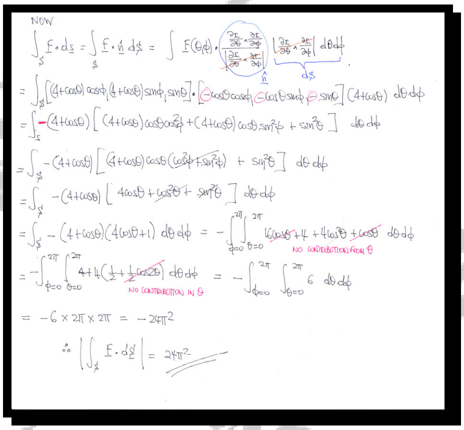
$$\mathbf{r}(\theta,\varphi) = \left[(4+\cos\theta)\cos\varphi\right]\mathbf{i} + \left[(4+\cos\theta)\sin\varphi\right]\mathbf{j} + (\sin\theta)\mathbf{k} ,$$

such that $0 \le \theta \le 2\pi$, $0 \le \varphi \le 2\pi$.

All integrations must be carried out in parametric.

$\boxed{24\pi^2}$





Question 26

It is given that the vector field $\mathbf{F}$ satisfies

$$\mathbf{F} = 2y\,\mathbf{i} - 2x\,\mathbf{j} + \mathbf{k} .$$

Find the magnitude of the surface integral

$$\int_S \mathbf{F} \cdot \mathbf{dS} ,$$

where S is the surface with Cartesian equation

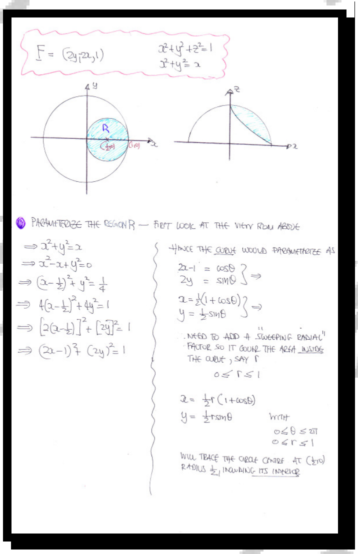
$$x^2 + y^2 + z^2 = 1, \quad z \geq 0 ,$$

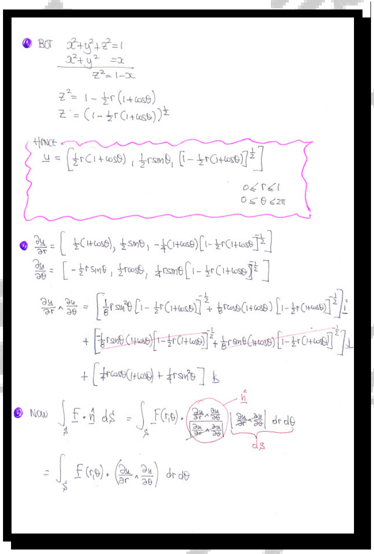
cut off by the cylinder with cartesian equation

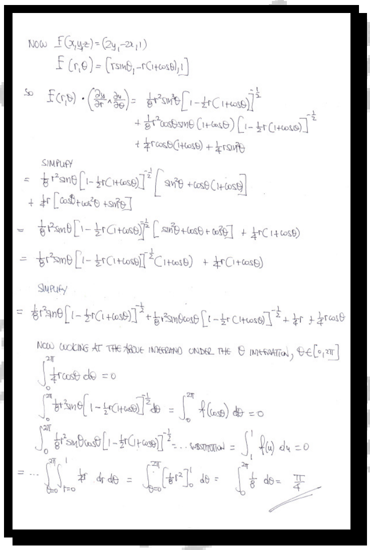
$$x^2 + y^2 = x .$$

You **must** find a suitable parameterization for S, and carry out the **integration in parametric**, without using any integral theorems.

$$\boxed{\frac{\pi}{4}}$$







TYPE $\oiint_S \mathbf{F} \cdot \mathbf{dS}$

Question 1

$$\mathbf{F}(x, y, z) \equiv xy\mathbf{i} + y\mathbf{j} + 4\mathbf{k}.$$

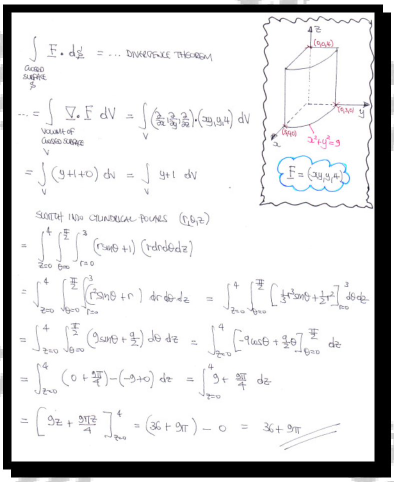
Evaluate the integral

$$\oiint_S \mathbf{F} \cdot \mathbf{dS},$$

where S is the **closed** surface enclosing the finite region V, defined by

$$x^2 + y^2 \le 9, \quad x \ge 0, \quad y \ge 0, \quad 0 \le z \le 4.$$

$\boxed{9\pi + 36}$



Question 2

$$\mathbf{F}(x,y,z) \equiv \left(x+y^2\right)\mathbf{i} + (2y+xz)\mathbf{j} + (3z+xyz)\mathbf{k}\,.$$

Evaluate the integral

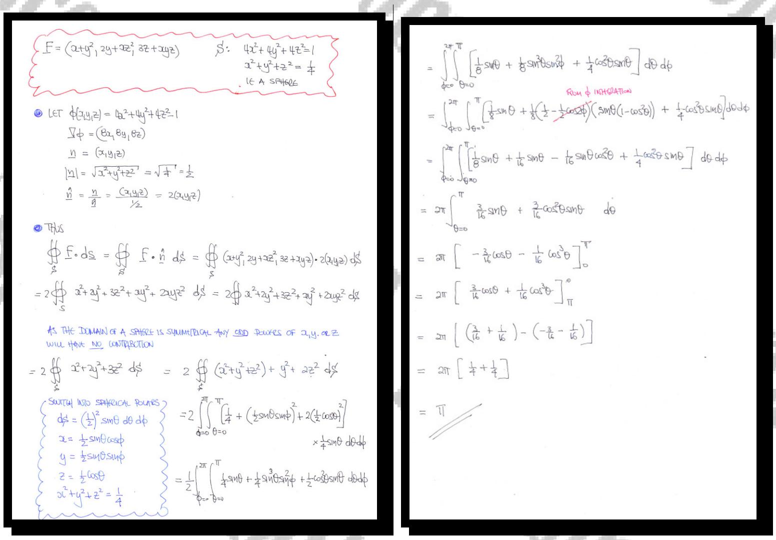
$$\oiint_S \mathbf{F}\cdot\mathbf{dS}\,,$$

where S is the surface with Cartesian equation

$$4x^2+4y^2+4z^2=1\,.$$

You may not use the Divergence Theorem in this question.

$\boxed{\pi}$



Question 3

It is given that

$$\mathbf{F}(x, y, z) \equiv \mathbf{k} \wedge \mathbf{r}\,, \quad \text{where} \quad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\,.$$

Show by direct integration that

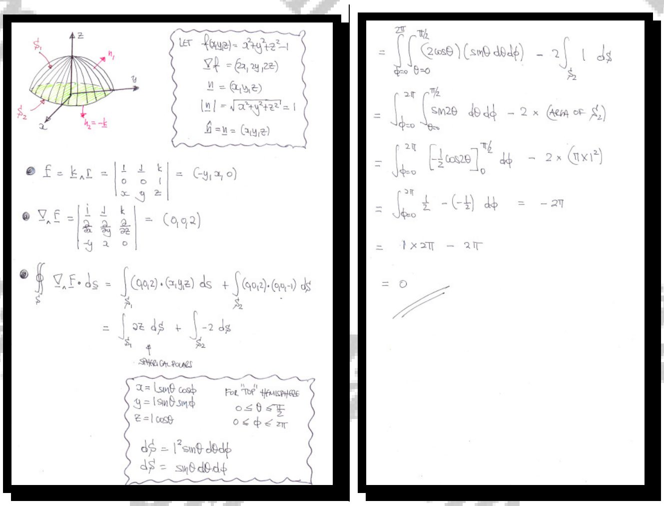
$$\oiint_S \nabla \wedge \mathbf{F} \cdot \mathbf{dS} = 0\,,$$

where S is the **closed** surface enclosing the finite region V, defined by

$$x^2 + y^2 + z^2 \le 1\,, \quad z \ge 0\,, \quad \text{and} \quad x^2 + y^2 \le 1\,.$$

You may not use any Integral Theorems in this question.

proof



Question 4

$$\mathbf{F}(x,y,z) \equiv (4yz)\mathbf{i} + \left(2y^2\right)\mathbf{j} + \left(5xyz + 6z^2 + 3z\right)\mathbf{k}.$$

Evaluate the integral

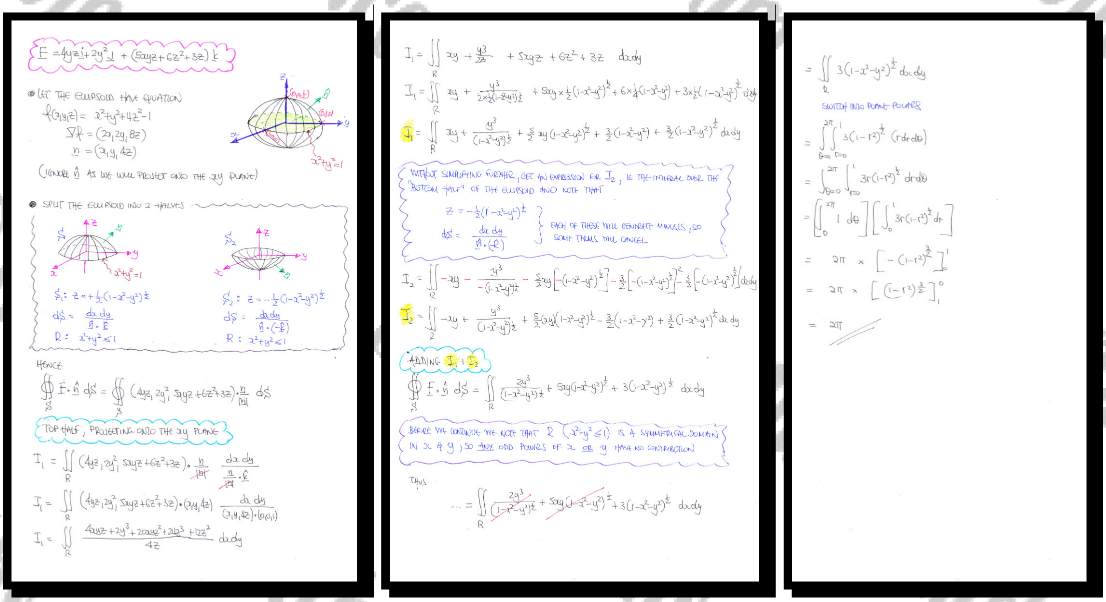
$$\oiint_S \mathbf{F}\cdot\mathbf{dS},$$

where S is the surface with Cartesian equation

$$x^2 + y^2 + 4z^2 = 1.$$

You may not use the Divergence Theorem in this question.

$\boxed{2\pi}$



Question 5

The surface Ω is the sphere with Cartesian equation

$$(x-1)^2+(y-1)^2+(z-1)^2=1$$

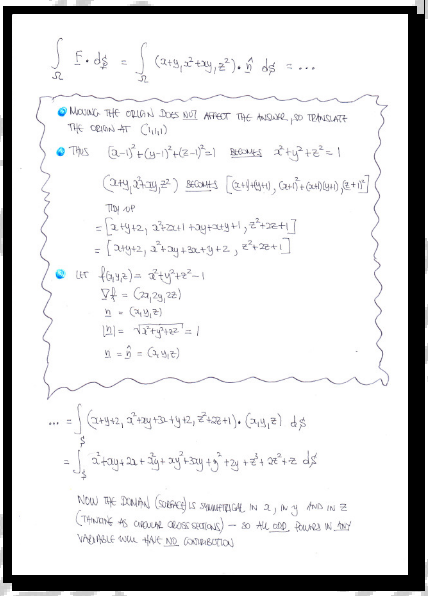
Evaluate the surface integral

$$\oiint_{\Omega}\left[(x+y)\mathbf{i}+\left(x^2+xy\right)\mathbf{j}+z^2\mathbf{k}\right]\cdot\mathbf{dS},$$

where $\mathbf{dS}$ is a unit surface element on Ω.

You may not use the Divergence Theorem in this question.

$\boxed{\frac{16}{3}\pi}$



$\int_{\Omega} \underline{F}\cdot d\underline{S} = \int_{\Omega} (x+y, x^2+xy, z^2)\cdot\hat{\underline{n}}\; dS = \ldots$

- MOVING THE ORIGIN DOES NOT AFFECT THE ANSWER, SO TRANSLATE THE ORIGIN AT $(1,1,1)$
- THUS $(x-1)^2+(y-1)^2+(z-1)^2=1$ BECOMES $x^2+y^2+z^2=1$

  $(x+y, x^2+xy, z^2)$ BECOMES $[(x+1)+(y+1), (x+1)^2+(x+1)(y+1), (z+1)^2]$

  TIDY UP

  $= [x+y+2, x^2+2x+1+xy+x+y+1, z^2+2z+1]$

  $= [x+y+2, x^2+xy+3x+y+2, z^2+2z+1]$
- LET $f(x,y,z)=x^2+y^2+z^2-1$

  $\nabla f=(2x,2y,2z)$

  $\underline{n}=(x,y,z)$

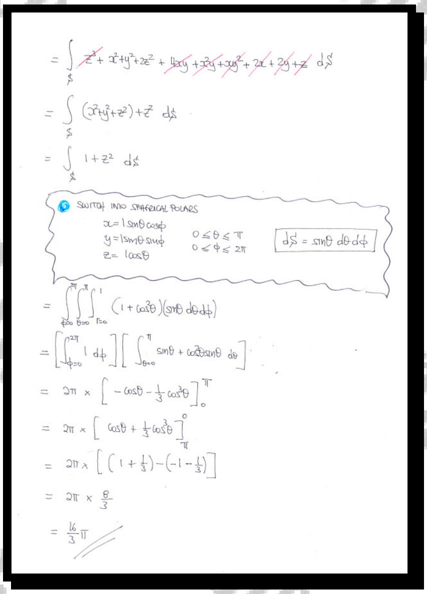
  $|\underline{n}|=\sqrt{x^2+y^2+z^2}=1$

  $\underline{n}=\hat{\underline{n}}=(x,y,z)$

$\ldots = \int_S (x+y+2, x^2+xy+3x+y+2, z^2+2z+1)\cdot(x,y,z)\; dS$

$= \int_S x^2+xy+2x+x^2y+xy^2+3xy+y^2+2y+z^3+2z^2+z\; dS$

NOW THE DOMAIN (SURFACE) IS SYMMETRICAL IN x, IN y AND IN z (THINKING AS CIRCULAR CROSS SECTIONS) — SO ALL ODD POWERS IN ANY VARIABLE WILL HAVE NO CONTRIBUTION



$= \int_S z^3 + x^2+y^2+2z^2 + 4xy + x^2y + xy^2 + 2x + 2y + z\; dS$

$= \int_S (x^2+y^2+z^2)+z^2\; dS$

$= \int_S 1+z^2\; dS$

- SWITCH INTO SPHERICAL POLARS

  $x = 1\sin\theta\cos\phi$, $y=1\sin\theta\sin\phi$, $z=1\cos\theta$, $0\le\theta\le\pi$, $0\le\phi\le 2\pi$, $dS=\sin\theta\, d\theta\, d\phi$

$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{1} (1+\cos^2\theta)(\sin\theta\, d\theta\, d\phi)$

$= \left[\int_{\phi=0}^{2\pi} 1\, d\phi\right]\left[\int_{\theta=0}^{\pi} \sin\theta+\cos^2\theta\sin\theta\, d\theta\right]$

$= 2\pi\times\left[-\cos\theta-\tfrac{1}{3}\cos^3\theta\right]_0^{\pi}$

$= 2\pi\times\left[\cos\theta+\tfrac{1}{3}\cos^3\theta\right]_{\pi}^{0}$

$= 2\pi\times\left[\left(1+\tfrac{1}{3}\right)-\left(-1-\tfrac{1}{3}\right)\right]$

$= 2\pi\times\tfrac{8}{3}$

$= \tfrac{16}{3}\pi$