ARITHMETIC SERIES

Question 1 **(**)** **non calculator**

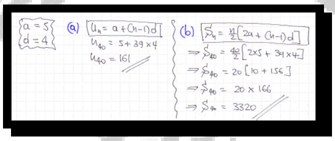
The first few terms of an arithmetic sequence are given below

$$5, 9, 13, 17, 21, \ldots$$

a) Find the fortieth term of the sequence.

b) Determine the sum of the first forty terms of the sequence.

$\boxed{u_{40} = 161}$, $\boxed{S_{40} = 3320}$



$a = 5$, $d = 4$

(a) $\boxed{u_n = a + (n-1)d}$

$u_{40} = 5 + 39 \times 4$

$u_{40} = 161$

(b) $\boxed{S_n = \frac{n}{2}\left[2a + (n-1)d\right]}$

$\Rightarrow S_{40} = \frac{40}{2}\left[2 \times 5 + 39 \times 4\right]$

$\Rightarrow S_{40} = 20\left[10 + 156\right]$

$\Rightarrow S_{40} = 20 \times 166$

$\Rightarrow S_{40} = 3320$

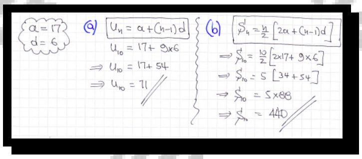
Question 2 **(**)** **non calculator**

The first term of an arithmetic progression is 17 and the common difference is 6.

a) Find the tenth term of the progression.

b) Determine the sum of the first ten terms of the progression.

$\boxed{u_{10} = 71}$, $\boxed{S_{10} = 440}$



$a = 17$, $d = 6$

(a) $\boxed{u_n = a + (n-1)d}$

$u_{10} = 17 + 9 \times 6$

$\Rightarrow u_{10} = 17 + 54$

$\Rightarrow u_{10} = 71$

(b) $\boxed{S_n = \frac{n}{2}\left[2a + (n-1)d\right]}$

$\Rightarrow S_{10} = \frac{10}{2}\left[2 \times 17 + 9 \times 6\right]$

$\Rightarrow S_{10} = 5\left[34 + 54\right]$

$\Rightarrow S_{10} = 5 \times 88$

$\Rightarrow S_{10} = 440$

Question 3 (**) non calculator

The first term of an arithmetic series is 51 and the eighth term is 100.

a) Find the twentieth term of the series.

b) Determine the sum of the first twenty terms of the series.

$\boxed{u_{20} = 184}$, $\boxed{S_{20} = 2350}$

$a = 51$
$u_8 = 100$

(a) $\boxed{u_n = a + (n-1)d}$
$100 = 51 + 7d$
$49 = 7d$
$d = 7$
$\therefore u_{20} = 51 + 19 \times 7$
$u_{20} = 51 + 133$
$u_{20} = 184$

(b) $\boxed{S_n = \frac{n}{2}\left[2a + (n-1)d\right]}$
$\Rightarrow S_{20} = \frac{20}{2}\left[2 \times 51 + 19 \times 7\right]$
$\Rightarrow S_{20} = 10\left[102 + 133\right]$
$\Rightarrow S_{20} = 2350$
(OR BETTER HERE $S_n = \frac{n}{2}\left[a + L\right]$)

Question 4 (**) non calculator

Evaluate the following expression, showing clearly all the relevant workings.

$$\sum_{r=1}^{20} (13r + 4).$$

$\boxed{}$, $\boxed{2810}$

$\sum_{r=1}^{20} (13r+4) = \underbrace{17 + 30 + 43 + 56 + \dots + 264}_{\text{this is an A.P. with}}$
$a = 17$
$d = 13$
$n = 20$

$\boxed{S_n = \frac{n}{2}\left[2a + (n-1)d\right]}$
$\Rightarrow S_{20} = \frac{20}{2}\left[2 \times 17 + 19 \times 13\right]$
$\Rightarrow S_{20} = 10\left[34 + 247\right]$
$\Rightarrow S_{20} = 2810$

OR

$\boxed{S_n = \frac{n}{2}\left[a + L\right]}$
$S_{20} = \frac{20}{2}\left[17 + 264\right]$
$S_{20} = 10 \times 281$
$S_{20} = 2810$

Question 5 (**) non calculator

Evaluate the following expression, showing clearly all the steps in the calculation.

$$\sum_{r=1}^{20}(3r+10).$$

$\boxed{}$, $\boxed{830}$

$$\sum_{r=1}^{20}(3r+10) = \underbrace{13+16+19+\dots+70}$$

This is an A.P. with $a=13$, $d=3$, $n=20$

$\boxed{S_n = \frac{n}{2}\left[2a+(n-1)d\right]}$

$\Rightarrow S_{20} = \frac{20}{2}\left[2\times13+19\times3\right]$

$\Rightarrow S_{20} = 10\left[26+57\right]$

$\Rightarrow S_{20} = 830$

OR

$\boxed{S_n = \frac{n}{2}\left[a+L\right]}$

$\Rightarrow S_{20} = \frac{20}{2}\left[13+70\right]$

$\Rightarrow S_{20} = 10\times83$

$\Rightarrow S_{20} = 830$

Question 6 (**) non calculator

A ball bearing is rolling down an inclined groove.

It rolls down by 1 cm during the first second of its motion, and in each subsequent second it rolls down by an extra 3 cm than in the previous second.

Given it takes 12 seconds for the ball bearing to roll down the groove, find in metres the length of the groove.

$\boxed{2.1 \text{ m}}$

1cm, 4cm, 7cm, 10cm, ...

THIS IS AN A.P WITH $a=1$, $d=3$, $n=12$

$S_n = \frac{n}{2}\left[2a+(n-1)d\right]$

$S_{12} = \frac{12}{2}\left[2\times1+11\times3\right]$

$S_{12} = 6(2+33)$

$S_{12} = 6\times35$

$S_{12} = 210$

$\therefore$ 210 cm OR 2.1 m

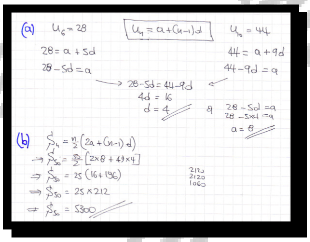
Question 7 (+) non calculator**

The sixth and the tenth term of an arithmetic progression are 28 and 44, respectively.

a) Determine, in any order, the first term and the common difference of the progression.

b) Calculate the sum of the first fifty terms of the progression.

$\boxed{a=8}$, $\boxed{d=4}$, $\boxed{S_{50}=5300}$



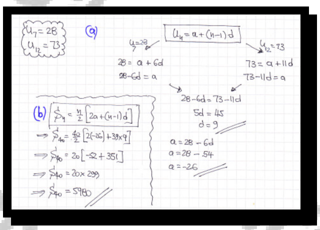
Question 8 (+) non calculator**

The seventh term and the twelfth term of an arithmetic progression are 28 and 73, respectively.

a) Find the first term and the common difference of the progression.

b) Calculate the sum of the first forty terms of the progression.

$\boxed{a=-26}$, $\boxed{d=9}$, $\boxed{S_{40}=5980}$



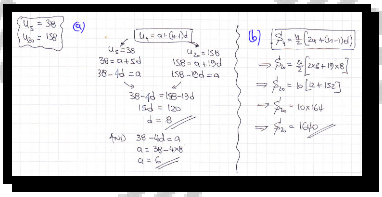
Question 9 (+) non calculator**

The fifth and the twentieth term of an arithmetic series are 38 and 158, respectively.

a) Find the first term and the common difference of the series.

b) Determine the sum of the first twenty terms of the series.

$\boxed{}$, $\boxed{a=6}$, $\boxed{d=8}$, $\boxed{S_{20}=1640}$



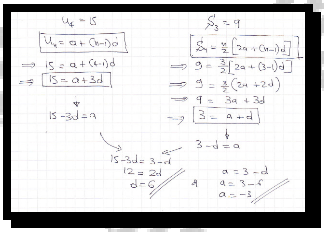
Question 10 (+) non calculator**

The fourth term of an arithmetic series is 15.

The sum of its first three terms is 9.

Find the first term and the common difference of the series.

$\boxed{a=-3}$, $\boxed{d=6}$



Question 11 **(**+)** **non calculator**

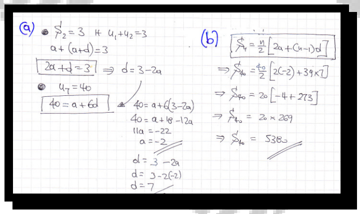
The sum of the first two terms of an arithmetic series is 3.

The seventh term of the series is 40.

a) Find the first term and the common difference of the series.

b) Determine the sum of the first forty terms of the series.

$\boxed{a=-2}$, $\boxed{d=7}$, $\boxed{S_{40}=5380}$



Question 12 **(**+)** **non calculator**

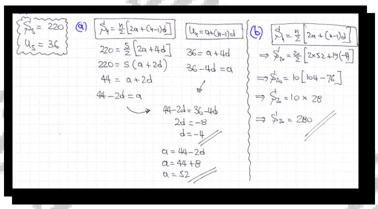
The sum of the first five terms of an arithmetic series is 220.

The fifth term of the series is 36.

a) Find the first term and the common difference of the series.

b) Determine the sum of the first twenty terms of the series.

$\boxed{a=52}$, $\boxed{d=-4}$, $\boxed{S_{20}=280}$



Question 13 (+) non calculator**

Evaluate the following sum, showing clearly all the steps in the calculation

$$\sum_{r=1}^{50}(180-7r).$$

$\boxed{75}$

$\sum_{r=1}^{50}(180-7r) = \underbrace{173+166+159+\dots+(-170)}$

$\longrightarrow$ A.P. with $a=173$, $d=-7$, $n=50$

Hence $\boxed{S_n = \frac{n}{2}[2a+(n-1)d]}$

$S_{50} = \frac{50}{2}[2\times173+49\times(-7)]$

$S_{50} = 25\times3$

$S_{50} = 75$

Question 14 (+) non calculator**

The 12^{th} term of an arithmetic progression is twice as large as the 4^{th} term.

a) Given that the 14^{th} term of the progression is 27, show that the first term of the progression is 7.5.

b) Find the sum of the first 20 terms of the progression.

$\boxed{435}$

(a) • $u_{12} = 2u_4$ • $u_{14}=27$

$u_n = a+(n-1)d$

$a+11d = 2(a+3d)$

$a+11d = 2a+6d$

$\boxed{5d=a}$

$\boxed{27 = a+13d}$

$27 = 5d+13d$

$27 = 18d$

$d = \frac{27}{18} = \frac{3}{2}$

$\therefore a = 5d$

$a = 5\times\frac{3}{2}$

$a = \frac{15}{2} = 7.5$ AS REQUIRED

(b) $a=7.5$, $d=1.5$, $n=20$

$S_n = \frac{n}{2}[2a+(n-1)d]$

$S_{20} = \frac{20}{2}[2\times7.5+19\times1.5]$

$S_{20} = 10[15+19\times1.5]$

$S_{20} = 150+19\times15$

$S_{20} = 150+285$

$S_{20} = 435$

Question 15 (+) non calculator**

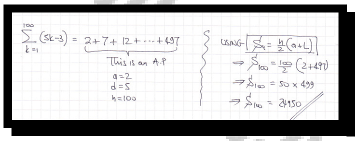
The k^{th} term of a sequence is given by

$$a_k = 5k - 3.$$

By showing clearly all the steps in the calculations, evaluate the sum

$$\sum_{k=1}^{100} a_k .$$

$\boxed{}$, $\boxed{24950}$



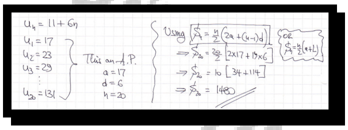
Question 16 (***) non calculator

The n^{th} term of an arithmetic series is given by

$$u_n = 11 + 6n .$$

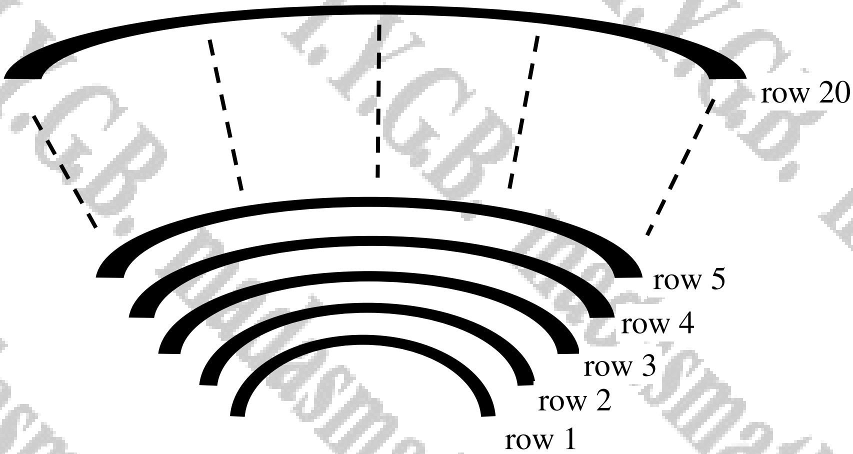
Find the sum of the first twenty terms of the series.

$\boxed{S_{20} = 1480}$



Question 17 (**+)

Seats in a theatre are arranged in rows. The number of seats in this theatre form the terms of an arithmetic series.



The sixth row has 23 seats and the fifteenth row has 50 seats.

a) Find the number of seats in the first row.

The theatre has 20 rows of seats in total.

b) Find the number of seats in this theatre.

$\boxed{}$, $\boxed{8}$, $\boxed{730}$

a) USING STANDARD SEQUENCE/SERIES FORMULA $u_n = a + (n-1)d$

- $u_6 = 23$
  $a + 5d = 23$
  $a = 23 - 5d$
- $u_{15} = 50$
  $a + 14d = 50$
  $a = 50 - 14d$

$23 - 5d = 50 - 14d$
$9d = 27$
$d = 3$ & $a = 8$

$\therefore$ THE FIRST ROW HAS 8 SEATS

b) SUMMING UP THE FIRST 20 TERMS OF AN A.P WITH $a=8$, $d=3$

$\Longrightarrow S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

$\Longrightarrow S_{20} = \frac{20}{2}\left[2 \times 8 + 19 \times 3\right]$

$\Longrightarrow S_{20} = 10\left[16 + 57\right]$

$\Longrightarrow S_{20} = 730$

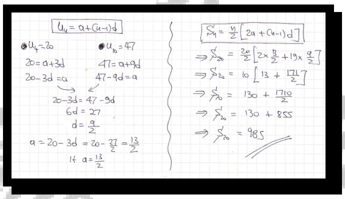
$\therefore$ A TOTAL OF 730 SEATS

Question 18 (***) non calculator

The fourth term and the tenth term of an arithmetic series are 20 and 47, respectively.

Calculate the sum of the first twenty terms of the series.

$\boxed{S_{20} = 985}$



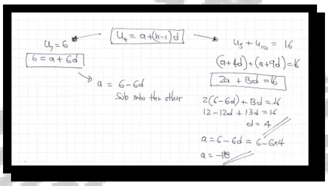
Question 19 (***) non calculator

The seventh term of an arithmetic series is 6.

The sum of its fifth term and its tenth term is 16.

Find the first term and the common difference of the series.

$\boxed{a = -18}$, $\boxed{d = 4}$



Question 20 **(***)** **non calculator**

The sum of the first 20 terms of an arithmetic series is 1070.

The sum of its fifth term and its tenth term is 65.

a) Find the first term and the common difference of the series.

b) Calculate the sum of the first 30 terms of the series.

$\boxed{a = -13}$, $\boxed{d = 7}$, $\boxed{2655}$

(a) $\boxed{S_n = \frac{n}{2}\left[2a + (n-1)d\right]}$

$\Rightarrow 1070 = \frac{20}{2}\left[2a + 19d\right]$

$\Rightarrow 1070 = 10\left[2a + 19d\right]$

$\Rightarrow \boxed{107 = 2a + 19d}$

$u_5 + u_{10} = 65$

$(a+4d) + (a+9d) = 65$

$\boxed{2a + 13d = 65}$

$2a = 107 - 19d$

$2a = 65 - 13d$

$107 - 19d = 65 - 13d$

$42 = 6d$

$d = 7$

& $2a = 65 - 13d$

$2a = 65 - 13 \times 7$

$2a = 65 - 91$

$2a = -26$

$a = -13$

(b) $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

$\Rightarrow S_{30} = \frac{30}{2}\left[2 \times (-13) + 29 \times 7\right]$

$\Rightarrow S_{30} = 15\left[-26 + 203\right]$

$\Rightarrow S_{30} = 15 \times 177$

$\Rightarrow S_{30} = 2655$

$$\begin{array}{r} 1770 \\ 885 \\ \hline 2655 \end{array}$$

Question 21 **(***)** **non calculator**

The sixteenth term of an arithmetic series is 6.

The sum of the first sixteen terms is 456.

a) Find the first term and the common difference of the series.

The sum of the first k terms of the series is zero.

b) Determine the value of k.

$\boxed{}$, $\boxed{a=51}$, $\boxed{d=-3}$, $\boxed{k=35}$

a) FORMING TWO EQUATIONS USING STANDARD FORMULAS

- $u_{16} = 6$

$\Rightarrow u_n = a+(n-1)d$

$\Rightarrow 6 = a+15d$

- $S_{16} = 456$

$\Rightarrow S_n = \frac{n}{2}(a+L)$ u_{16}

$\Rightarrow 456 = \frac{16}{2}(a+6)$

$\Rightarrow 456 = 8(a+6)$

$\Rightarrow 57 = a+6$

$\Rightarrow a = 51$

&

$\Rightarrow 6 = 51+15d$

$\Rightarrow -45 = 15d$

$\Rightarrow d = -3$

b) USING $S_n = \frac{n}{2}[2a+(n-1)d]$

$\Rightarrow 0 = \frac{k}{2}[2\times51+(k-1)\times(-3)]$

$\Rightarrow 0 = \frac{k}{2}[102-3(k-1)]$

$\Rightarrow 0 = \frac{k}{2}[102-3k+3]$

$\Rightarrow 0 = \frac{1}{2}k(105-3k)$

$\therefore k = \cancel{0}$ $k \neq 0$, 35

Question 22 **(***)** **non calculator**

An arithmetic progression has first term -8 and common difference 2.

The sum of the first n terms of the progression is 220.

Use algebra to find the value of n.

$\boxed{n=20}$

$a=-8$
$d=2$
$S_n=220$

$\boxed{S_n = \frac{n}{2}[2a+(n-1)d]}$

$\Rightarrow 220 = \frac{n}{2}[2(-8)+(n-1)\times2]$

$\Rightarrow 220 = n(-8+(n-1))$

$\Rightarrow 220 = n(n-9)$

BY INSPECTION $n=20$

Question 23 **(***)** **non calculator**

$$107+114+121+128+\ldots+1500.$$

The above series has 200 terms.

Find the sum of the **last** 40 terms of the series.

$\boxed{}$, $\boxed{54540}$

METHOD A

$$107+114+121+128+\ldots+1500 = \underbrace{1500+1493+1486+\ldots}_{\text{40 TERMS}}+128+121+114+107$$

$a = 1500$
$d = -7$
$n = 40$

$$S_n = \frac{n}{2}\left[2a+(n-1)d\right]$$
$$S_{40} = \frac{40}{2}\left[2\times1500+39(-7)\right]$$
$$S_{40} = 20\left[3000-273\right]$$
$$S_{40} = 20\times2727$$
$$S_{40} = 54540$$

METHOD B

$$\underbrace{107+114+121+128+\ldots}_{\text{160 TERMS}}\ \underbrace{\overset{\text{161ST TERM}}{\downarrow}\ldots+1500}_{\text{40 TERMS}}$$

- $u_n = a+(n-1)d$
  $u_{161} = 107+160\times7$
  $u_{161} = 107+1120$
  $u_{161} = 1227$
- HENCE u_{161} IS THE "FIRST TERM" OF THE LAST FORTY TERMS

$$\left\{\begin{array}{l} a=1227 \\ l=1500 \\ n=40 \end{array}\right\}$$

$$S_n = \frac{n}{2}\left[a+l\right]$$
$$S_{40} = \frac{40}{2}\left[1227+1500\right]$$
$$S_{40} = 20\times\ 727$$
$$S_{40} = 54540$$

AS BEFORE

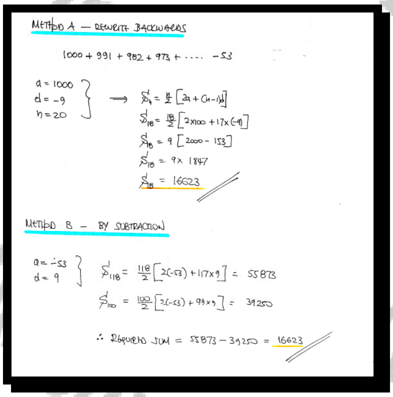
Question 24 **(***)**

$$-53 - 44 - 35 - 26 - \ \ldots \ + 1000 .$$

The above series has 118 terms.

Find the sum of the **last** 18 terms of the series.

$\boxed{\quad}$, $\boxed{16623}$



METHOD A — REWRITE BACKWARDS

$1000 + 991 + 982 + 973 + \ldots\ldots -53$

$\left.\begin{array}{l} a = 1000 \\ d = -9 \\ n = 20 \end{array}\right\} \Longrightarrow S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

$S_{18} = \frac{18}{2}\left[2\times 100 + 17\times(-9)\right]$

$S_{18} = 9\left[2000 - 153\right]$

$S_{18} = 9 \times 1847$

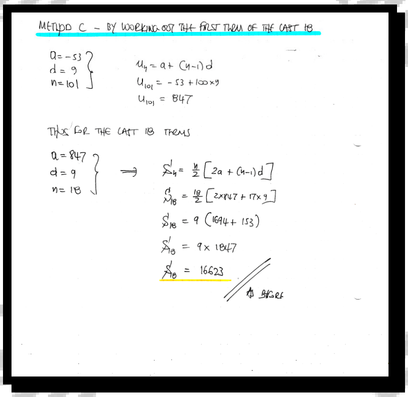
$S_{18} = 16623$

METHOD B — BY SUBTRACTION

$\left.\begin{array}{l} a = -53 \\ d = 9 \end{array}\right\}$ $S_{118} = \frac{118}{2}\left[2(-53) + 117\times 9\right] = 55873$

$S_{100} = \frac{100}{2}\left[2(-53) + 99\times 9\right] = 39250$

$\therefore$ REQUIRED SUM $= 55873 - 39250 = 16623$



METHOD C — BY WORKING OUT THE FIRST TERM OF THE LAST 18

$\left.\begin{array}{l} a = -53 \\ d = 9 \\ n = 101 \end{array}\right\}$ $u_n = a + (n-1)d$

$u_{101} = -53 + 100\times 9$

$u_{101} = 847$

THUS FOR THE LAST 18 TERMS

$\left.\begin{array}{l} a = 847 \\ d = 9 \\ n = 18 \end{array}\right\} \Longrightarrow S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

$S_{18} = \frac{18}{2}\left[2\times 847 + 17\times 9\right]$

$S_{18} = 9\left(1694 + 153\right)$

$S_{18} = 9 \times 1847$

$S_{18} = 16623$

AS BEFORE

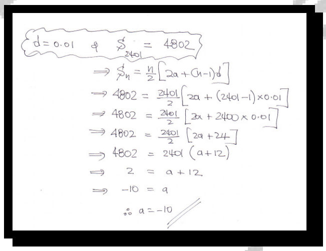
Question 25 **(***)** **non calculator**

The common difference of an arithmetic progression is 0.01.

The sum of the first 2401 terms the progression is 4802.

Find the first term of the progression.

$\boxed{a = -10}$



$d = 0.01$ & $S_{2401} = 4802$

$\Longrightarrow S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

$\Longrightarrow 4802 = \frac{2401}{2}\left[2a + (2401-1)\times 0.01\right]$

$\Longrightarrow 4802 = \frac{2401}{2}\left[2a + 2400\times 0.01\right]$

$\Longrightarrow 4802 = \frac{2401}{2}\left[2a + 24\right]$

$\Longrightarrow 4802 = 2401\left(a + 12\right)$

$\Longrightarrow 2 = a + 12$

$\Longrightarrow -10 = a$

$\therefore a = -10$

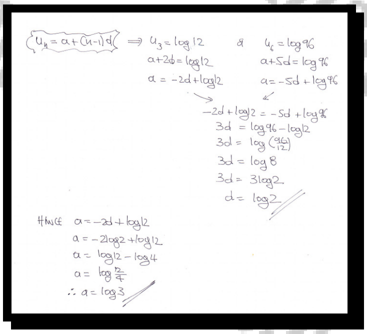
Question 26 **(***) non calculator**

The first term of an arithmetic series is a and the common difference is d.

The third term of the series is $\log 12$ and the sixth term is $\log 96$.

Find the exact values of a and d.

$\boxed{a = \log 3}$, $\boxed{d = \log 2}$


$u_n = a + (n-1)d$ $\Rightarrow$ $u_3 = \log 12$ & $u_6 = \log 96$

$a + 2d = \log 12$ $\quad$ $a + 5d = \log 96$

$a = -2d + \log 12$ $\quad$ $a = -5d + \log 96$

$-2d + \log 12 = -5d + \log 96$

$3d = \log 96 - \log 12$

$3d = \log\left(\frac{96}{12}\right)$

$3d = \log 8$

$3d = 3\log 2$

$d = \log 2$

Hence $a = -2d + \log 12$

$a = -2\log 2 + \log 12$

$a = \log 12 - \log 4$

$a = \log \frac{12}{4}$

$\therefore a = \log 3$

Question 27 **(***)**

The first term of an arithmetic series is a and the common difference is d.

The sum of the first 21 terms the series is 735.

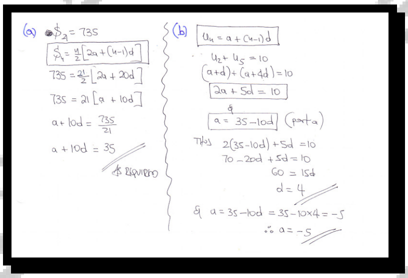
a) Show clearly that

$$a + 10d = 35.$$

The sum of the second and the fifth term is 10.

b) Find the value of a and the value of d.

$\boxed{a = -5}$, $\boxed{d = 4}$


(a) $S_{21} = 735$

$\boxed{S_n = \frac{n}{2}[2a + (n-1)d]}$

$735 = \frac{21}{2}[2a + 20d]$

$735 = 21[a + 10d]$

$a + 10d = \frac{735}{21}$

$a + 10d = 35$

AS REQUIRED

(b) $\boxed{u_n = a + (n-1)d}$

$u_2 + u_5 = 10$

$(a+d) + (a+4d) = 10$

$\boxed{2a + 5d = 10}$

&

$\boxed{a = 35 - 10d}$ (part a)

Thus $2(35 - 10d) + 5d = 10$

$70 - 20d + 5d = 10$

$60 = 15d$

$d = 4$

& $a = 35 - 10d = 35 - 10 \times 4 = -5$

$\therefore a = -5$

Question 28 (***)

Arnold is planning to save for the next 48 months in order to raise a deposit to buy a flat. He plans to save £300 this month and each successive month thereafter, to save an extra £5 compared to the previous month.

a) Find the amount he will save on the twelfth month.

b) Find the total amount he will save at the end of the 48 months.

Franco is also planning to save for the next 48 months in order to buy a car.

He plans to save £a this month and each successive month thereafter, to save an extra £15 compared to the previous month.

c) Find the value of a, if Franco saves the same amount of money as Arnold does in the next 48 months.

$\boxed{}$, $\boxed{£355}$, $\boxed{£20040}$, $\boxed{a=65}$

(a) $a = 300$, $d = 5$

$u_n = a + (n-1)d$

$u_{12} = 300 + 11 \times 5$

$u_{12} = 355$

(b) $S_n = \frac{n}{2}\left[2a + (n-1)d\right] \Rightarrow S_{48} = \frac{48}{2}\left[2 \times 300 + 47 \times 5\right]$

$\Rightarrow S_{48} = 24 \times (600 + 235)$

$\Rightarrow S_{48} = 24 \times 835$

$\Rightarrow S_{48} = 20040$

(c) $S_{48} = 20040$, $d = 15$, $n = 48$, $a = ?$

$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

$20040 = \frac{48}{2}\left[2a + 47 \times 15\right]$

$20040 = 24\left[2a + 705\right]$

$835 = 2a + 705$

$130 = 2a$

$a = 65$

$24 \times 835 = 20040$

Question 29 (***) non calculator

The first three terms of an arithmetic series are 26.1, 25.2 and 24.3.

Find the smallest value of n for which the sum of the first n terms of the series is negative.

$\boxed{n=60}$

$a = 26.1$, $d = -0.9$

$\boxed{S_n = \frac{n}{2}\left[2a + (n-1)d\right]}$

$\Rightarrow 0 = \frac{n}{2}\left[2 \times 26.1 + (n-1) \times (-0.9)\right]$

$\Rightarrow 0 = \frac{n}{2}\left[52.2 - 0.9n + 0.9\right]$

$\Rightarrow 0 = \frac{n}{2}\left[53.1 - 0.9n\right]$

$\Rightarrow 0 = 53.1 - 0.9n \quad (n \neq 0)$

$\Rightarrow 0.9n = 53.1$

$\Rightarrow n = 59$

$\therefore$ SMALLEST $n = 60$

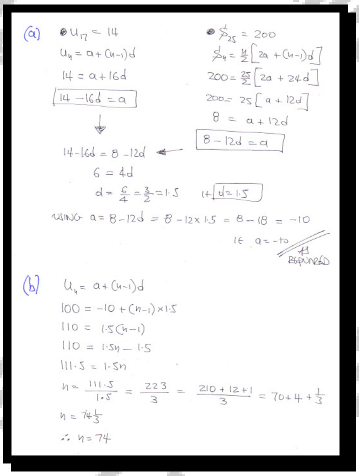
Question 30 **(***)** **non calculator**

The 17^{th} term of an arithmetic progression is 14 and the sum of the first 25 terms of the progression is 200.

a) Show that the first term of the progression is -10.

b) Find the number of terms in the progression that are less than 100.

$\boxed{74}$



Question 31 **(***)** **non calculator**

An arithmetic series has first term 5 and common difference 4.

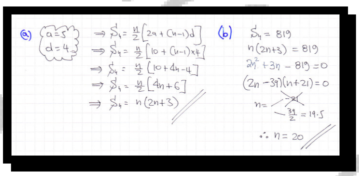
a) Show that the sum of the first n terms of the series is given by

$$n(2n+3).$$

b) Find the smallest value of n for which the sum of the first n terms of the series exceeds 819.

(You may find the fact $21 \times 39 = 819$, useful)

$\boxed{n = 20}$



Question 32 (***)

The eleventh term of an arithmetic progression is 42 and the sum of its third and eighth term is 40.

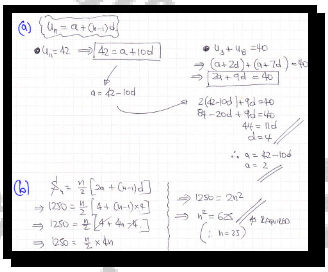
a) Find the first term and the common difference of the progression.

The first n terms of this progression add to 1250.

b) Show clearly that

$$n^2 = 625.$$

$\boxed{a = 2}$, $\boxed{d = 4}$



Question 33 (***)

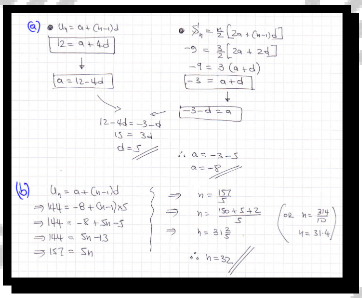
The fifth term of an arithmetic series is 12 and the sum of its first three terms is -9.

a) Find the first term and the common difference of the series.

The n^{th} term of the series exceeds 144.

b) Determine smallest value of n.

$\boxed{a = -8}$, $\boxed{d = 5}$, $\boxed{n = 32}$

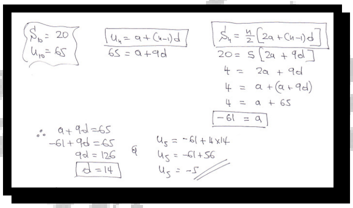


Question 34 (***)

The sum of the first ten terms of an arithmetic progression is 20 and the tenth term of the progression is 65.

Find the fifth term of the progression.

$\boxed{u_5 = -5}$



Question 35 (***)

Andrew is planning to pay money into a pension scheme for the next 40 years.

He plans to pay into the pension scheme £800 in the first year and each successive year thereafter, an extra £100 compared to the previous year.

a) Calculate the amount Andrew will pay into the scheme on the tenth year.

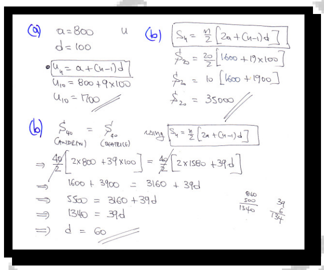
b) Find the total amount Andrew will have paid into the scheme after 20 years.

Beatrice is also planning to pay money into a pension scheme for the next 40 years.

She plans to pay £1580 in the first year and each successive year thereafter, to pay an extra £d compared to the previous year.

c) Find the value of d, if both Andrew and Beatrice paid into their pension schemes the same amount of money over the next 40 years.

$\boxed{}$, $\boxed{£1700}$, $\boxed{£35000}$, $\boxed{d = 60}$



Question 36 (***)

The n^{th} term of a sequence is given by

$$a_n = 8n + 5.$$

Show clearly that

$$\sum_{n=1}^{k} a_n = k(4k+9).$$

$\boxed{\text{proof}}$

- $a_n = 8n+5$
  THE SEQUENCE IS
  13, 21, 29, 37, ...
  IT A.P. with $a=13$
  $d=8$

- $\boxed{S_n = \frac{n}{2}[2a+(n-1)d]}$
  $\Rightarrow S_k = \frac{k}{2}[2\times13+(k-1)\times8]$
  $\Rightarrow \sum_{n=1}^{k} a_n = \frac{k}{2}(26+8k-8)$
  $\Rightarrow \sum_{n=1}^{k} a_n = \frac{k}{2}(8k+18)$
  $\Rightarrow \sum_{n=1}^{k} a_n = k(4k+9)$ // AS REQUIRED

Question 37 (***)

A novelist is planning to write a new book.

He plans to write 15 pages in the first week, 17 pages in the second week, 19 pages in the third week, and so on, so that he writes an extra two pages each week compared with the previous week.

a) Find the number of pages he plans to write in the tenth week.

b) Determine how many pages he plans to write in the first ten weeks.

The novelist sticks to his plan and produces a book with 480 pages, after n weeks.

c) Use algebra to determine the value of n.

$\boxed{33}$, $\boxed{240}$, $\boxed{n=16}$

(a) $\boxed{u_n = a+(n-1)d}$
$u_{10} = 15 + 9\times2$
$u_{10} = 15+18$
$u_{10} = 33$ //

(b) $\boxed{S_n = \frac{n}{2}[2a+(n-1)d]}$
$\Rightarrow S_{10} = \frac{10}{2}[2\times15+9\times2]$
$\Rightarrow S_{10} = 5[30+18]$
$\Rightarrow S_{10} = 240$ //

(c) $\boxed{S_n = \frac{n}{2}[2a+(n-1)d]}$
$480 = \frac{n}{2}[30+(n-1)\times2]$
$480 = \frac{n}{2}[30+2n-2]$
$480 = \frac{n}{2}[2n+28]$
$480 = n(n+14)$

SINCE SOLUTION IS A POSITIVE INTEGER
IF $n=15$ $15\times(29) = \ldots 435$
$n=16$ $16\times30 = \ldots 480$

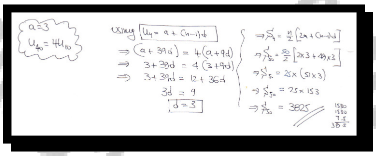
$\therefore n=16$ //

Question 38 (***)

An arithmetic series has first term 3 and its 40^{th} term is 4 times as large as its 10^{th} term.

Find the sum of the first 50 terms of the series.

$\boxed{S_{50} = 3825}$



Question 39 (***+)

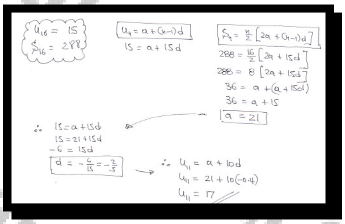
An athlete is training for a long distance race.

He is preparing by running on 16 consecutive days so that his daily running distances form an arithmetic sequence.

The athlete ran for 15 km on the 16^{th} day of his training and the total distance run over the 16 day training period was 288 km.

Find the distance the athlete ran on the 11^{th} day of his training.

$\boxed{}$, $\boxed{17 \text{ km}}$



Question 40 **(***+)** **non calculator**

On his 1^{st} birthday, Anthony was given £50 as a present by his godmother Cleo.

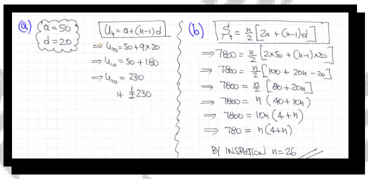
For every birthday ever since, Cleo gave Anthony £20 more than on his previous birthday. This money was saved by Anthony's mother until Anthony was n years old.

a) Find the amount of money Anthony received as a birthday present on his tenth birthday.

After Anthony's n^{th} birthday his mother gave him Cleo's presents, which was £7800 in total.

b) Determine the value of n.

$\boxed{£230}$, $\boxed{n = 26}$

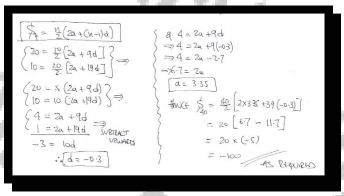


Question 41 **(***+)**

The sum of the first ten terms of an arithmetic series is 20 and the sum of its first twenty terms is 10.

Show that the sum of the first forty terms of the series is -100.

$\boxed{\text{proof}}$



Question 42 **(***+)**

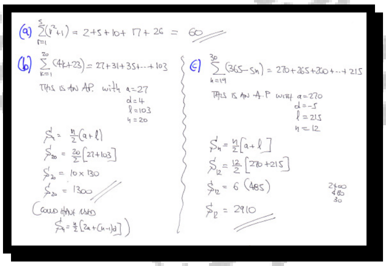
Evaluate each of the following sums.

a) $\displaystyle\sum_{r=1}^{5}\left(r^2+1\right)$.

b) $\displaystyle\sum_{k=1}^{20}\left(4k+23\right)$.

c) $\displaystyle\sum_{n=19}^{30}\left(365-5n\right)$.

$\boxed{60}$, $\boxed{1300}$, $\boxed{2910}$



(a) $\sum_{r=1}^{5}(r^2+1) = 2+5+10+17+26 = 60$

(b) $\sum_{k=1}^{20}(4k+23) = 27+31+35+\dots+103$

THIS IS AN A.P. with $a=27$, $d=4$, $l=103$, $n=20$

$S_n = \frac{n}{2}(a+l)$

$S_{20} = \frac{20}{2}[27+103]$

$S_{20} = 10 \times 130$

$S_{20} = 1300$

(COULD HAVE USED $S_n = \frac{n}{2}[2a+(n-1)d]$)

(c) $\sum_{n=19}^{30}(365-5n) = 270+265+260+\dots+215$

THIS IS AN A.P WITH $a=270$, $d=-5$, $l=215$, $n=12$

$S_n = \frac{n}{2}[a+l]$

$S_{12} = \frac{12}{2}[270+215]$

$S_{12} = 6(485)$

$S_{12} = 2910$

2400
480
30

Question 43 (***+)

The first three terms of an arithmetic series are

$$(k-2),\ (2k+5) \text{ and } (4k+1) \text{ respectively,}$$

where k is a constant.

a) Show clearly that $k = 11$.

b) Find the 41^{st} term of the series.

The sum of the first n terms of the series is denoted by S_n.

c) Show that S_n is always a square number.

$\boxed{\text{C1A}}$, $\boxed{729}$

(a) ARITHMETIC $\Rightarrow u_3 - u_2 = u_2 - u_1$

$\Rightarrow (4k+1)-(2k+5) = (2k+5)-(k-2)$

$\Rightarrow 2k-4 = k+7$

$\Rightarrow k = 11$ AS REQUIRED

(b) THUS $u_1 = 11-2 = 9$, $u_2 = 2\times11+5 = 27$, $u_3 = 4\times11+1 = 45$ $\Rightarrow a = 9$, $d = 18$

USING $\boxed{u_n = a+(n-1)d}$

$u_{41} = 9 + 40\times18$

$u_{41} = 9 + 720$

$u_{41} = 729$

(c) $\boxed{S_n = \frac{n}{2}\left[2a+(n-1)d\right]}$

$\Rightarrow S_n = \frac{n}{2}\left[2\times9+(n-1)\times18\right]$

$\Rightarrow S_n = \frac{n}{2}\left[\cancel{18}+18n-\cancel{18}\right]$

$\Rightarrow S_n = \frac{n}{2}\times18n$

$\Rightarrow S_n = 9n^2$

$\Rightarrow S_n = (3n)^2$ $\therefore$ ALWAYS A SQUARE NUMBER

Question 44 (***+)

The n^{th} term of an arithmetic series is denoted by u_n.

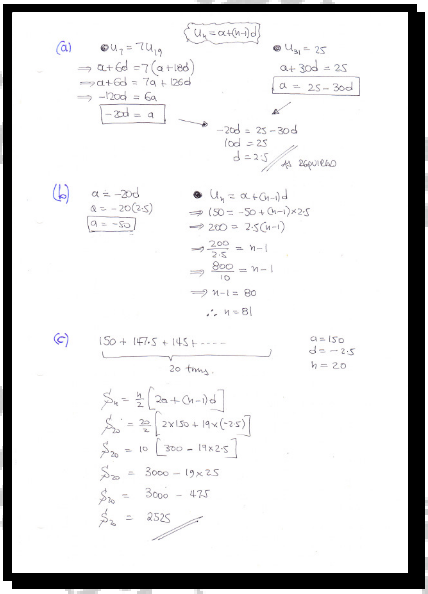
a) Given that $u_7 = 7u_{19}$ and $u_{31} = 25$, show that the common difference of the series is 2.5.

The last term of the series is 150.

b) Determine the number of terms in the series.

c) Find the sum of the last 20 terms of the series.

$\boxed{81}$, $\boxed{2525}$



$\boxed{u_n = a+(n-1)d}$

(a) • $u_7 = 7u_{19}$

$\Rightarrow a+6d = 7(a+18d)$

$\Rightarrow a+6d = 7a+126d$

$\Rightarrow -120d = 6a$

$\boxed{-20d = a}$

• $u_{31} = 25$

$a + 30d = 25$

$\boxed{a = 25 - 30d}$

$-20d = 25 - 30d$

$10d = 25$

$d = 2.5$ // AS REQUIRED

(b) $a = -20d$

$a = -20(2.5)$

$\boxed{a = -50}$

• $u_n = a + (n-1)d$

$\Rightarrow 150 = -50 + (n-1)\times 2.5$

$\Rightarrow 200 = 2.5(n-1)$

$\Rightarrow \frac{200}{2.5} = n-1$

$\Rightarrow \frac{800}{10} = n-1$

$\Rightarrow n-1 = 80$

$\therefore n = 81$

(c) $\underbrace{150 + 147.5 + 145 + \dots}_{\text{20 terms.}}$

$a = 150$
$d = -2.5$
$n = 20$

$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

$S_{20} = \frac{20}{2}\left[2\times 150 + 19\times(-2.5)\right]$

$S_{20} = 10\left[300 - 19\times 2.5\right]$

$S_{20} = 3000 - 19\times 25$

$S_{20} = 3000 - 475$

$S_{20} = 2525$ //

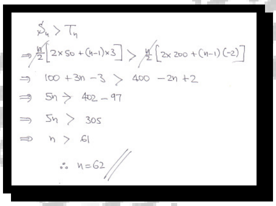
Question 45 (***+)

The sum of the first n terms of the sequence 50, 53, 56, 59, ... is denoted by S_n.

The sum of the first n terms of the sequence 200, 198, 196, 194, ... is denoted by T_n.

Find the smallest value of n so that $S_n > T_n$.

$\boxed{}$, $\boxed{n = 62}$



$S_n > T_n$

$\Rightarrow \cancel{\frac{n}{2}}\left[2\times 50+(n-1)\times 3\right] > \cancel{\frac{n}{2}}\left[2\times 200+(n-1)(-2)\right]$

$\Rightarrow 100+3n-3 > 400-2n+2$

$\Rightarrow 5n > 402-97$

$\Rightarrow 5n > 305$

$\Rightarrow n > 61$

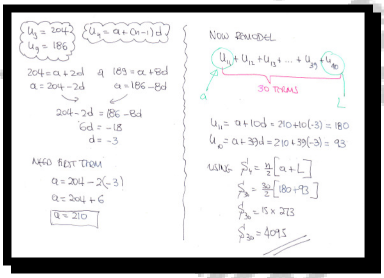
$\therefore n = 62$

Question 46 (***+)

The third term of an arithmetic series is 204 and the ninth term of the same arithmetic series is 186.

Find the sum of the eleventh to the fortieth term of the series, inclusive.

$\boxed{4095}$



$u_3 = 204$
$u_9 = 186$

$u_n = a+(n-1)d$

$204 = a+2d$ & $189 = a+8d$

$a = 204-2d$ $\quad$ $a = 186-8d$

$204-2d = 186-8d$

$6d = -18$

$d = -3$

NEED FIRST TERM

$a = 204-2(-3)$

$a = 204+6$

$\boxed{a = 210}$

NOW REMODEL

$u_{11}+u_{12}+u_{13}+\dots+u_{39}+u_{40}$

30 TERMS

a $\quad$ L

$u_{11} = a+10d = 210+10(-3) = 180$

$u_{40} = a+39d = 210+39(-3) = 93$

USING $S'_n = \frac{n}{2}\left[a+L\right]$

$S'_{30} = \frac{30}{2}\left[180+93\right]$

$S'_{30} = 15\times 273$

$S'_{30} = 4095$

Question 47 (***+)

A new gym opened and during its first trading month 26 people joined its membership.

A business forecast for the gym membership is drafted for the next twelve months.

It assumes that every month an extra x number of members will join, so that next month $(26+x)$ members will be added, the following month $(26+2x)$ members will be added, and so on.

Taking $x=15$, find …

a) … the number of members that will join in the twelfth month.

b) … the total number of members that will join during the first twelve months.

The business plan recognises that in order for the business to succeed in the long term, it needs a total membership of at least 1500 during its first twelve months.

c) Using the same model, find the required value of x in order to achieve a twelve month membership target of 1500.

$\boxed{191}$, $\boxed{1302}$, $\boxed{x=18}$

(a) $\boxed{u_n = a+(n-1)d}$

$u_{12} = 26 + 11\times 15$

$u_{12} = 26 + 165$

$u_{12} = 191$

(b) $S_n = \frac{n}{2}(2a+(n-1)d)$ or $\frac{n}{2}(a+l)$

$S_{12} = \frac{12}{2}[2\times 26 + 11\times 15]$

$S_{12} = 6[52+165]$

$S_{12} = 6\times 217$

$S_{12} = 1302$

(c) $S_n = \frac{n}{2}[2a+(n-1)d]$

$1500 = \frac{12}{2}[2\times 26 + 11\times d]$

$1500 = 6(52+11d)$

$250 = 52 + 11d$

$198 = 11d$

$d = \frac{198}{11} = \frac{99+99}{11}$

$d = 18$

Question 48 (***+)

The first few terms of an arithmetic sequence is given below

$$5, 11, 17, 23, 29, \ldots$$

Find, by using an algebraic method …

a) … the eleventh term of the sequence.

b) … the sum of the first eleven terms of the sequence.

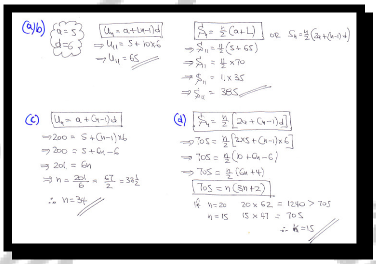
The n^{th} term of the sequence exceeds 200.

c) Determine the smallest value of n.

The sum of the first k terms of the sequence is 705.

d) Determine the value of k.

$\boxed{u_{11} = 65}$, $\boxed{S_{11} = 385}$, $\boxed{n = 34}$, $\boxed{k = 15}$


(a)/(b) $a=5$, $d=6$

$\boxed{u_n = a + (n-1)d}$

$\Rightarrow u_{11} = 5 + 10\times 6$

$\Rightarrow u_{11} = 65$

$\boxed{S_n = \frac{n}{2}(a+L)}$ OR $S_n = \frac{n}{2}(2a+(n-1)d)$

$\Rightarrow S_{11} = \frac{11}{2}(5+65)$

$\Rightarrow S_{11} = \frac{11}{2}\times 70$

$\Rightarrow S_{11} = 11\times 35$

$\Rightarrow S_{11} = 385$

(c) $\boxed{u_n = a + (n-1)d}$

$\Rightarrow 200 = 5 + (n-1)\times 6$

$\Rightarrow 200 = 5 + 6n - 6$

$\Rightarrow 201 = 6n$

$\Rightarrow n = \frac{201}{6} = \frac{67}{2} = 33\frac{1}{2}$

$\therefore n = 34$

(d) $\boxed{S_n = \frac{n}{2}[2a + (n-1)d]}$

$\Rightarrow 705 = \frac{n}{2}[2\times 5 + (n-1)\times 6]$

$\Rightarrow 705 = \frac{n}{2}(10 + 6n - 6)$

$\Rightarrow 705 = \frac{n}{2}(6n+4)$

$\boxed{705 = n(3n+2)}$

If $n=20$ $20\times 62 = 1240 > 705$

$n = 15$ $15\times 47 = 705$

$\therefore k = 15$

Question 49 **(***+)**

The n^{th} term of an arithmetic series is given by

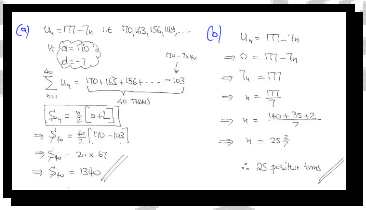
$$u_n = 177 - 7n .$$

Calculate, showing full workings, …

a) … $\displaystyle\sum_{n=1}^{40} u_n$.

b) … the number of positive terms or the series.

$\boxed{\displaystyle\sum_{n=1}^{40} u_n = 1340}$, $\boxed{25}$



(a) $u_n = 177 - 7n$ i.e. $170, 163, 156, 149, \ldots$

i.e. $a = 170$, $d = -7$

$\displaystyle\sum_{n=1}^{40} u_n = \underbrace{170 + 163 + 156 + \cdots - 103}_{40 \text{ TERMS}}$ ($170 - 7 \times 40$)

$\boxed{S_n = \frac{n}{2}[a + L]}$

$\Rightarrow S_{40} = \frac{40}{2}[170 - 103]$

$\Rightarrow S_{40} = 20 \times 67$

$\Rightarrow S_{40} = 1340$

(b) $u_n = 177 - 7n$

$\Rightarrow 0 = 177 - 7n$

$\Rightarrow 7n = 177$

$\Rightarrow n = \frac{177}{7}$

$\Rightarrow n = \frac{140 + 35 + 2}{7}$

$\Rightarrow n = 25\frac{2}{7}$

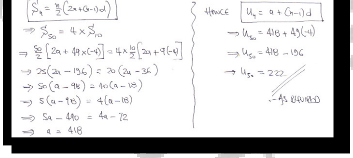
$\therefore$ 25 positive terms

Question 50 **(***+)**

An arithmetic series has common difference -4 and the sum of its first 50 terms is four times as large as the sum of its first 10 terms.

Show that the 50^{th} term of the series is 222.

$\boxed{\text{proof}}$



$\boxed{S_n = \frac{n}{2}(2a + (n-1)d)}$

$\Rightarrow S_{50} = 4 \times S_{10}$

$\Rightarrow \frac{50}{2}[2a + 49 \times (-4)] = 4 \times \frac{10}{2}[2a + 9(-4)]$

$\Rightarrow 25(2a - 196) = 20(2a - 36)$

$\Rightarrow 50(a - 98) = 40(a - 18)$

$\Rightarrow 5(a - 98) = 4(a - 18)$

$\Rightarrow 5a - 490 = 4a - 72$

$\Rightarrow a = 418$

HENCE $\boxed{u_n = a + (n-1)d}$

$\Rightarrow u_{50} = 418 + 49(-4)$

$\Rightarrow u_{50} = 418 - 196$

$\Rightarrow u_{50} = 222$

AS REQUIRED

Question 51 (***+)

A non regular polygon has 9 sides whose lengths, in cm, form an arithmetic sequence with common difference d.

The longest side of the polygon is 6 cm and the perimeter of the polygon is 45 cm.

Find in any order …

a) … the length of the shortest side of the polygon.

b) … the value of d.

$\boxed{4\text{cm}}$, $\boxed{d = 0.25}$

$n = 9$
$u_9 = 6 \leftarrow$ last term
$S_9 = 45$

$\boxed{S_n = \frac{n}{2}\left[a + L\right]}$
$\Rightarrow 45 = \frac{9}{2}\left[a + 6\right]$
$\Rightarrow 90 = 9(a + 6)$
$\Rightarrow 10 = a + 6$
$\Rightarrow a = 4$
4
shortest side

Hence $u_n = a + (n-1)d$
$\Rightarrow 6 = 4 + 8d$
$\Rightarrow 2 = 8d$
$\Rightarrow d = \frac{1}{4}$

Question 52 (***+)

Use algebra to show that

$$\sum_{k=10}^{30} (4k + 11) = 1911.$$

$\boxed{}$, $\boxed{\text{proof}}$

$\sum_{k=10}^{30}(4k+11) = \underbrace{51 + 55 + 59 + \dots + 131}_{21 \text{ terms}}$

$a = 51$
$d = 4$
$n = 21$
$L = 131$

$\boxed{S_n = \frac{n}{2}\left[a + L\right]}$
$\Rightarrow S_{21} = \frac{21}{2}\left[51 + 131\right]$
$\Rightarrow S_{21} = \frac{21}{2} \times 182$
$\Rightarrow S_{21} = 21 \times 91$
$\Rightarrow S_{21} = 1911$

910
910
91
1911

Question 53 (***+)

William started receiving his annual allowance on his 13^{th} birthday. His first allowance was £750 and this amount was increased in each successive birthday by £150 .

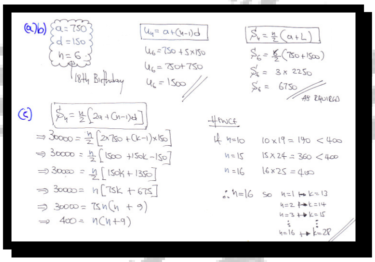
a) Use algebra to find the amount William received on his 18^{th} birthday.

b) Show that the total amount of allowances William received up and including his 18^{th} birthday was £6750 .

When William turned k years old he received his last allowance. The total amount of his allowances up and including that of his k^{th} birthday was £30000 .

c) Find the value of k .

$\boxed{£1500}$, $\boxed{k = 28}$


(a)b) $a = 750$, $d = 150$, $n = 6$ — 18th Birthday

$\boxed{u_n = a + (n-1)d}$

$u_6 = 750 + 5 \times 150$

$u_6 = 750 + 750$

$u_6 = 1500$

$\boxed{S_n = \frac{n}{2}(a + L)}$

$S_6 = \frac{6}{2}(750 + 1500)$

$S_6 = 3 \times 2250$

$S_6 = 6750$ AS REQUIRED

(c) $\boxed{S_n = \frac{n}{2}\left[2a + (n-1)d\right]}$

$\Rightarrow 30000 = \frac{n}{2}\left[2 \times 750 + (k-1) \times 150\right]$

$\Rightarrow 30000 = \frac{n}{2}\left[1500 + 150k - 150\right]$

$\Rightarrow 30000 = \frac{n}{2}\left[150k + 1350\right]$

$\Rightarrow 30000 = n\left[75k + 675\right]$

$\Rightarrow 30000 = 75n(n + 9)$

$\Rightarrow 400 = n(n+9)$

Hence

If $n = 10$ $10 \times 19 = 190 < 400$

$n = 15$ $15 \times 24 = 360 < 400$

$n = 16$ $16 \times 25 = 400$

$\therefore n = 16$ so $n = 1 \mapsto k = 13$, $n = 2 \mapsto k = 14$, $n = 3 \mapsto k = 15$, $\vdots$, $n = 16 \mapsto k = 28$

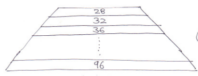
Question 54 (***+)

The roof of a museum has a sloping shape with the roof tiles arranged neatly in horizontal rows. There are 28 roof tiles in the top row and each row below the top row has an extra 4 tiles than the row above it.

The bottom row has 96 tiles.

Show that there are 1116 tiles on the roof of the museum.

$\boxed{}$, $\boxed{\text{proof}}$



TOTAL = ?

$a = 28$
$d = 4$
$u_n = L = 96$

- $u_n = a + (n-1)d$
  $96 = \dots + (n-1)4$
  $96 = 28 + 4n - 4$
  $96 = 24 + 4n$
  $72 = 4n$
  $\boxed{18 = n}$

- $S_n = \frac{n}{2}(a+L)$
  $S_{18} = \frac{18}{2}(28+96)$
  $S_{18} = 9 \times 124$
  $S_{18} = 900 + 180 + 36$
  $S_{18} = 1116$

  AS REQUIRED.

Question 55 (***+)

It is given that for all positive integers

$$\sum_{r=1}^{n} u_r = 7 + 3n^2 .$$

a) Evaluate $\displaystyle\sum_{r=1}^{4} u_r$.

b) Hence find the value of u_5 .

$\boxed{}$, $\boxed{\displaystyle\sum_{r=1}^{4} u_r = 55}$, $\boxed{u_5 = 27}$

a) $\displaystyle\sum_{r=1}^{n} u_r = 7 + 3n^2$

$\displaystyle\sum_{r=1}^{4} u_r = 7 + 3 \times 4^2 = 7 + 3 \times 16 = 7 + 48 = 55$

b) $\displaystyle u_5 = \sum_{r=1}^{5} u_r - \sum_{r=1}^{4} u_r = [7 + 3 \times 5^2] - 55 = [7 + 75] - 55 = 27$

Question 56 (***+)

$$19,\ 23,\ 27,\ 31,\ 35,\ \ldots$$

For the above arithmetic sequence, find …

a) … the thirtieth term.

b) … the sum of its first thirty terms.

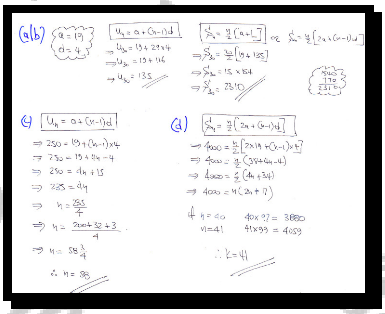
The n^{th} term of this sequence is less than 250.

c) Determine the largest value of n.

The sum of the first k terms of this sequence exceeds 4000.

d) Calculate the smallest value of k.

$\boxed{}$, $\boxed{u_{30} = 135}$, $\boxed{S_{30} = 2310}$, $\boxed{n = 58}$, $\boxed{k = 41}$



(a)(b) $a = 19$, $d = 4$

$\boxed{u_n = a + (n-1)d}$

$\Rightarrow u_{30} = 19 + 29 \times 4$

$\Rightarrow u_{30} = 19 + 116$

$\Rightarrow u_{30} = 135$

$\boxed{S_n = \frac{n}{2}[a + L]}$ or $S_n = \frac{n}{2}[2a + (n-1)d]$

$\Rightarrow S_{30} = \frac{30}{2}[19 + 135]$

$\Rightarrow S_{30} = 15 \times 154$

$\Rightarrow S_{30} = 2310$

1540
770
2310

(c) $\boxed{u_n = a + (n-1)d}$

$\Rightarrow 250 = 19 + (n-1) \times 4$

$\Rightarrow 250 = 19 + 4n - 4$

$\Rightarrow 250 = 4n + 15$

$\Rightarrow 235 = 4n$

$\Rightarrow n = \frac{235}{4}$

$\Rightarrow n = \frac{200 + 32 + 3}{4}$

$\Rightarrow n = 58\frac{3}{4}$

$\therefore n = 58$

(d) $\boxed{S_n = \frac{n}{2}[2a + (n-1)d]}$

$\Rightarrow 4000 = \frac{n}{2}[2 \times 19 + (n-1) \times 4]$

$\Rightarrow 4000 = \frac{n}{2}(38 + 4n - 4)$

$\Rightarrow 4000 = \frac{n}{2}(4n + 34)$

$\Rightarrow 4000 = n(2n + 17)$

if $n = 40$ $\quad 40 \times 97 = 3880$

$n = 41$ $\quad 41 \times 99 = 4059$

$\therefore k = 41$

Question 57 (***+)

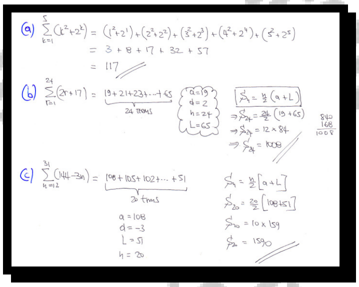
Evaluate, showing a clear method, each of the following sums.

a) $\displaystyle\sum_{k=1}^{5}\left(k^2+2^k\right)$.

b) $\displaystyle\sum_{r=1}^{24}(2r+17)$.

c) $\displaystyle\sum_{n=12}^{31}u_n$, where $u_n = 144-3n$.

$\boxed{117}$, $\boxed{1008}$, $\boxed{1590}$



(a) $\displaystyle\sum_{k=1}^{5}(k^2+2^k) = (1^2+2^1)+(2^2+2^2)+(3^2+2^3)+(4^2+2^4)+(5^2+2^5)$
$= 3+8+17+32+57$
$= 117$

(b) $\displaystyle\sum_{r=1}^{24}(2r+17) = \underbrace{19+21+23+\dots+65}_{24\text{ terms}}$

$a=19$, $d=2$, $n=24$, $L=65$

$\boxed{S_n = \frac{n}{2}(a+L)}$
$\Rightarrow S_{24} = \frac{24}{2}(19+65)$
$\Rightarrow S_{24} = 12\times 84$
$\Rightarrow S_{24} = 1008$

840
168
1008

(c) $\displaystyle\sum_{n=12}^{31}(144-3n) = \underbrace{108+105+102+\dots+51}_{20\text{ terms}}$

$a=108$, $d=-3$, $L=51$, $n=20$

$S_n = \frac{n}{2}[a+L]$
$S_{20} = \frac{20}{2}[108+51]$
$S_{20} = 10\times 159$
$S_{20} = 1590$

Question 58 **(***+)**

A non regular polygon has 10 sides whose lengths, in cm, form an arithmetic sequence with common difference d.

The longest side of the polygon is twice as long as the shortest side.

Given that the perimeter of the polygon is 405 cm, find in any order …

a) … the length of the shortest side of the polygon.

b) … the value of d.

$\boxed{27\,\text{cm}}$, $\boxed{d=3}$

$n=10$
$S_{10}=405$
$u_{10}=2a=L$

$\boxed{S_n=\frac{n}{2}(a+L)}$
$\Rightarrow 405=\frac{10}{2}(a+2a)$
$\Rightarrow 405=5(3a)$
$\Rightarrow 405=15a$
$\Rightarrow 3a=81$
$\Rightarrow a=27$

$\boxed{u_n=a+(n-1)d}$
$\Rightarrow u_{10}=a+9d$
$\Rightarrow 2a=a+9d$
$\Rightarrow a=9d$
$\Rightarrow 27=9d$
$\Rightarrow d=3$

Question 59 **(***+)**

Show clearly that

$$\sum_{r=6}^{25}(5r-1)=1530.$$

$\boxed{\text{proof}}$

$\sum_{r=6}^{25}(5r-1)=\underbrace{29+34+39+\ldots+124}_{20\text{ terms}}$

A.P with $a=29$
$d=5$
$n=20$
$L=124$

$\boxed{S_n=\frac{n}{2}(a+L)}$
$\Rightarrow S_{20}=\frac{20}{2}[29+124]$
$\Rightarrow S_{20}=10[153]$
$\Rightarrow S_{20}=1530$

Question 60 (***+)

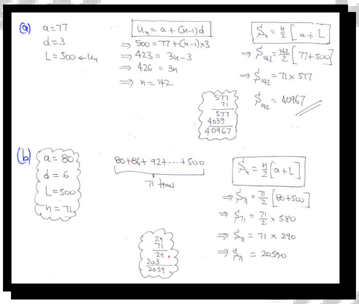
Consider the arithmetic series below

$$77 + 80 + 83 + \ldots + 500.$$

a) Find the sum of the arithmetic series.

b) Calculate the sum of the even terms of the series.

$\boxed{}$, $\boxed{\text{sum} = 40967}$, $\boxed{\text{sum of evens} = 20590}$



Question 61 (***+)

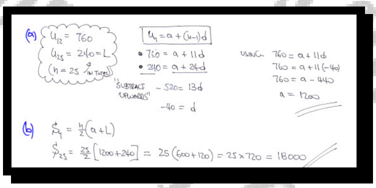
The council of Broxbourne undertook a housing development scheme which started in the year 2001 and is to finish in the year 2025. Under this scheme the council will build 760 houses in 2012 and 240 houses in 2025.

The number of houses the council builds every year, forms an arithmetic sequence.

a) Determine the number of houses built in 2001.

b) Calculate the total number of houses that will be built under this scheme.

$\boxed{}$, $\boxed{1200}$, $\boxed{18000}$



Question 62 (***+)

The first three terms of an arithmetic series are

$$-p\,, \quad (2p-5) \quad \text{and} \quad (3p-2) \quad \text{respectively,}$$

where p is a constant.

a) Show clearly that $p = 4$.

b) Find the sum of the first twenty terms of the series.

The k^{th} term of the series is over 1000.

c) Determine the smallest value of k.

$\boxed{}$, $\boxed{S_{20} = 1250}$, $\boxed{k = 145}$

(a) $u_1 = -p$, $u_2 = 2p-5$, $u_3 = 3p-2$

A.P $\Rightarrow u_2 - u_1 = u_3 - u_2$

$\Rightarrow p-1-(-p) = 3p-2-(2p-5)$

$\Rightarrow 2p-1 = p+3$

$\Rightarrow p = 4$

AS REQUIRED

(b) $u_1 = -4$, $u_2 = 3$, $u_3 = 10$ $\therefore a = -4$, $d = 7$

$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

$S_{20} = \frac{20}{2}\left[2 \times (-4) + 19 \times 7\right]$

$S_{10} = 10\left[-8 + 10 + 63\right]$

$S_{10} = 10\left[125\right]$

$S_{10} = 1250$

(c) $u_n = a + (n-1)d$

$1000 = -4 + (n-1) \times 7$

$1000 = -4 + 7n - 7$

$1000 = 7n - 11$

$1011 = 7n$

$n = \frac{1011}{7}$

$n = \frac{700 + 280 + 28 + 3}{7} = 100 + 40 + 4 + \frac{3}{7} = 144\frac{3}{7}$ $\therefore k = 145$

Question 63 (***+)

Show clearly that

$$\sum_{k=1}^{n}\left(\frac{k+5}{3}\right) \equiv \frac{1}{6}n(n+11).$$

$\boxed{\text{proof}}$

$$\sum_{k=1}^{n} \frac{k+5}{3} = \underbrace{\frac{6}{3}+\frac{7}{3}+\frac{8}{3}+\frac{9}{3}+\dots}_{\text{A.P. with } a=2,\ d=\frac{1}{3},\ n=n}$$

$\Rightarrow S_n = \frac{n}{2}\left[2a+(n-1)d\right]$

$\Rightarrow S_n = \frac{n}{2}\left[2\times 2+(n-1)\times\frac{1}{3}\right]$

$\Rightarrow S_n = \frac{n}{2}\left[4+\frac{1}{3}n-\frac{1}{3}\right]$

$\Rightarrow S_n = \frac{n}{2}\left[\frac{1}{3}n+\frac{11}{3}\right]$

$\Rightarrow S_n = \frac{n}{2}\times\frac{1}{3}\times(n+11)$

$\Rightarrow S_n = \frac{1}{6}n(n+11)$ AS REQUIRED

Question 64 (***+)

Find the sum of all the integers between -25 and 75 inclusive.

$\boxed{2525}$

$$\underbrace{(-25)+(-24)+(-23)+\dots+74+75}_{\text{AP with } a=-25,\ d=1,\ n=101}$$

$S_n = \frac{n}{2}\left[2a+(n-1)d\right]$

$S_{101} = \frac{101}{2}\left[2(-25)+100\times 1\right]$

$S_{101} = \frac{101}{2}\times 50$

$S_{101} = 2525$

OR $S_n = \frac{n}{2}\left[a+L\right]$ GIVES $\frac{101}{2}\left[-25+50\right] = 2525$

ALTERNATIVE

$$(-25)+(-24)+(-23)+\dots+74+75 = \underbrace{26+27+28+\dots+74+75}_{\text{AP with } a=26,\ d=1,\ n=50}$$

USING $S_n = \frac{n}{2}\left[2a+(n-1)d\right]$ GIVES $S_{50} = \frac{50}{2}\left[2\times 26+49\times 1\right]$

or $S_n = \frac{n}{2}\left[a+L\right]$

$S_{50} = 25\times 101$

$S_{50} = 2525$

Question 65 (***+)

Osama starts his new job on an annual salary of £18000. His contract promises a pay rise of £1800 in each subsequent year until his salary reaches £36000. When the salary reaches £36000 Osama will receive **no more** pay rises. Osama's salary first reaches the maximum salary of £36000 in year N.

a) Determine the value of N.

b) Find Osama's total salary earnings during the first N years of his employment.

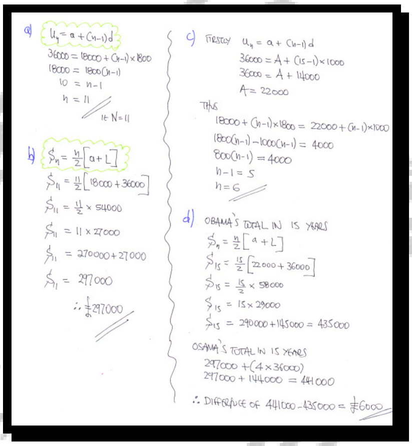
Obama starts his new job at the same time as Osama on an annual salary of £A.

His contract promises a pay rise of £1000 in each subsequent year until his salary reaches £36000. When the salary reaches £36000 Obama will receive **no more** pay rises. Obama's salary first reaches the maximum salary of £36000 in year 15.

c) Find the year when both Osama and Obama have the same annual salary.

d) Calculate the difference in the total salary earnings between Osama and Obama in the first 15 years of their employment.

$\boxed{}$, $\boxed{N=11}$, $\boxed{S_N=297000}$, $\boxed{n=6}$, $\boxed{d=6000}$


a) $u_n = a+(n-1)d$
$36000 = 18000 + (n-1)\times 1800$
$18000 = 1800(n-1)$
$10 = n-1$
$n = 11$
$\therefore N = 11$

b) $S_n = \frac{n}{2}[a+L]$
$S_{11} = \frac{11}{2}[18000+36000]$
$S_{11} = \frac{11}{2}\times 54000$
$S_{11} = 11\times 27000$
$S_{11} = 270000+27000$
$S_{11} = 297000$
$\therefore$ £297000

c) FIRSTLY $u_n = a+(n-1)d$
$36000 = A+(15-1)\times 1000$
$36000 = A + 14000$
$A = 22000$
THUS
$18000+(n-1)\times 1800 = 22000+(n-1)\times 1000$
$1800(n-1) - 1000(n-1) = 4000$
$800(n-1) = 4000$
$n-1 = 5$
$n = 6$

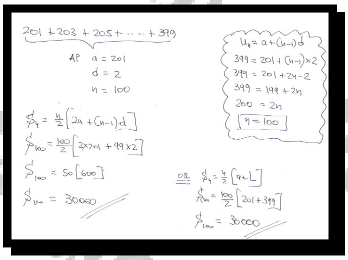
d) OBAMA'S TOTAL IN 15 YEARS
$S_n = \frac{n}{2}[a+L]$
$S_{15} = \frac{15}{2}[22000+36000]$
$S_{15} = \frac{15}{2}\times 58000$
$S_{15} = 15\times 29000$
$S_{15} = 290000+145000 = 435000$
OSAMA'S TOTAL IN 15 YEARS
$297000+(4\times 36000)$
$297000+144000 = 441000$
$\therefore$ DIFFERENCE OF $441000-435000 =$ £6000

Question 66 (*+)**

Evaluate the sum

$$201+203+205+\ldots+399.$$

$\boxed{30000}$


$\underbrace{201+203+205+\ldots+399}$

AP $a=201$, $d=2$, $n=100$

$S_n=\frac{n}{2}\left[2a+(n-1)d\right]$

$S_{100}=\frac{100}{2}\left[2\times201+99\times2\right]$

$S_{100}=50\left[600\right]$

$S_{100}=30000$

$u_n=a+(n-1)d$

$399=201+(n-1)\times2$

$399=201+2n-2$

$399=199+2n$

$200=2n$

$\boxed{n=100}$

OR $S_n=\frac{n}{2}\left[a+L\right]$

$S_{100}=\frac{100}{2}\left[201+399\right]$

$S_{100}=30000$

Question 67 (*+)**

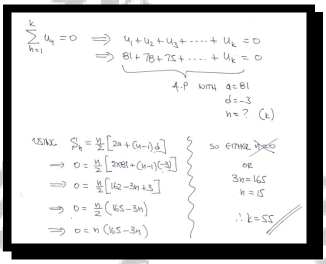
The n^{th} term of a sequence is given by

$$u_n=84-3n.$$

Find the value of k given that

$$\sum_{n=1}^{k}u_n=0.$$

$\boxed{k=55}$


$\sum_{n=1}^{k}u_n=0 \Rightarrow u_1+u_2+u_3+\ldots+u_k=0$

$\Rightarrow \underbrace{81+78+75+\ldots+u_k}=0$

A.P WITH $a=81$, $d=-3$, $n=?$ (k)

USING $S_n=\frac{n}{2}\left[2a+(n-1)d\right]$

$\Rightarrow 0=\frac{n}{2}\left[2\times81+(n-1)(-3)\right]$

$\Rightarrow 0=\frac{n}{2}\left[162-3n+3\right]$

$\Rightarrow 0=\frac{n}{2}(165-3n)$

$\Rightarrow 0=n(165-3n)$

SO EITHER $\cancel{n=0}$

OR

$3n=165$

$n=55$

$\therefore k=55$

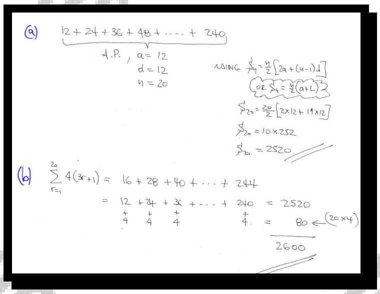
Question 68 **(***+)**

a) Find the sum of the multiples of twelve between 1 and 250.

b) Hence, or otherwise, determine the value of

$$\sum_{r=1}^{20} 4(3r+1).$$

$\boxed{}$, $\boxed{2520}$, $\boxed{2600}$



(a) $\underbrace{12 + 24 + 36 + 48 + \dots + 240}$

A.P., $a = 12$, $d = 12$, $n = 20$

USING $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

(OR $S_n = \frac{n}{2}(a+L)$)

$S_{20} = \frac{20}{2}\left[2 \times 12 + 19 \times 12\right]$

$S_{20} = 10 \times 252$

$S_{20} = 2520$

(b) $\sum_{r=1}^{20} 4(3r+1) = 16 + 28 + 40 + \dots + 244$

$= 12 + 24 + 36 + \dots + 240 = 2520$

$+4 \quad +4 \quad +4 \quad \dots \quad +4 \; = \; 80 \leftarrow (20 \times 4)$

2600

Question 69 (***+)

The first three terms of an arithmetic series are

$$8-k\,,\quad 2k+1 \text{ and } 4k-1 \text{ respectively,}$$

where k is a constant.

a) Show clearly that $k=5$.

b) Find the sum of the first fifteen terms of the series.

c) Determine how many terms of the series have a value less than 400.

$\boxed{S_{15}=885}$, $\boxed{50 \text{ terms}}$

(a) ARITHMETIC $\Rightarrow$ $u_3-u_2=u_2-u_1$

$(4k-1)-(2k+1)=(2k+1)-(8-k)$

$2k-2=3k-7$

$5=k$

(b) $u_1=8-k=3$, $u_2=2k+1=11$, $u_3=4k-1=19$ (each $+8$)

$\therefore a=3$, $d=8$, $n=15$

$S_n=\frac{n}{2}\left[2a+(n-1)d\right]$

$S_{15}=\frac{15}{2}\left[2\times3+14\times8\right]$

$S_{15}=\frac{15}{2}\times118$

$S_{15}=885$

(c) $u_n=a+(n-1)d$

$400=3+(n-1)\times8$

$400=3+8n-8$

$405=8n$

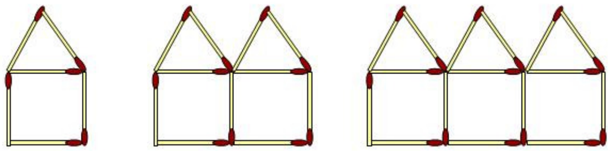
$n=50.625$

$\therefore n=50$

IE 50 TERMS

Question 70 (***+)

Thomas is making patterns using sticks. He uses 6 sticks for the first pattern, 11 sticks for the second pattern, 16 sticks for the third pattern and so on.



a) Find how many sticks Thomas uses to make the tenth pattern.

b) Show clearly that Thomas uses 285 sticks to make the first ten patterns.

Thomas has a box with 1200 sticks. Thomas can make k **complete** patterns with the sticks in his box.

c) Show further that k satisfies the inequality

$$k(5k+7) \le 2400 .$$

d) Hence find the value of k .

$\boxed{}$, $\boxed{51}$, $\boxed{k=21}$

(a) $6, 11, 16, \ldots$ i.e. $a=6$, $d=5$

$\boxed{u_n = a + (n-1)d}$

$u_{10} = 6 + 9\times 5$

$u_{10} = 51$

(b) $\boxed{S_n = \frac{n}{2}\left[2a + (n-1)d\right]}$

$\Rightarrow S_{10} = \frac{10}{2}\left[2\times 6 + 9\times 5\right]$

$\Rightarrow S_{10} = 5\left[12 + 45\right]$

$\Rightarrow S_{10} = 5 \times 57$

$\Rightarrow S_{10} = 285$

AS REQUIRED

(c) $S_k \le 1200$

$\frac{k}{2}\left[2\times 6 + (k-1)\times 5\right] \le 1200$

$\frac{k}{2}\left[12 + 5k - 5\right] \le 1200$

$\frac{k}{2}(5k+7) \le 1200$

$k(5k+7) \le 2400$

AS REQUIRED

(d) BY INSPECTION

- $k=10$ $10 \times 57 = 570 < 2400$
- $k=15$ $15 \times 82 = 1430 < 2400$
- $k=20$ $20 \times 107 = 2140 < 2400$
- $k=21$ $21 \times 112 = 2352 < 2400$
- $k=22$ $22 \times 117 = 2574 > 2400$

$\therefore k = 21$

Question 71 (***+)

The sum of the third, sixth and ninth term of an arithmetic progression is 90.

The sum of its first twelve terms is 408.

Determine the first term and the common difference of the progression.

$\boxed{}$, $\boxed{a=-10}$, $\boxed{d=8}$

- $u_3 + u_6 + u_9 = 90$

$\boxed{u_n = a + (n-1)d}$

$\Rightarrow (a+2d) + (a+5d) + (a+8d) = 90$

$\Rightarrow 3a + 15d = 90$

$\boxed{a + 5d = 30}$

- $S_{12} = 408$

$\boxed{S_n = \frac{n}{2}\left[2a + (n-1)d\right]}$

$\Rightarrow 408 = \frac{12}{2}\left[2a + 11d\right]$

$\Rightarrow 408 = 6(2a + 11d)$

$\Rightarrow \boxed{68 = 2a + 11d}$

$a = 30 - 5d$

$68 = 2(30-5d) + 11d$

$68 = 60 - 10d + 11d$

$d = 8$

& $a = 30 - 5d$

$a = 30 - 5 \times 8$

$a = -10$

Question 72 (***+)

$$T = 240 - 5 + 237 - 5 + 234 - 5 + 231 - \ \dots + 6 - 5 + 3 - 5.$$

Show clearly that $T = 9320$.

$\boxed{}$, $\boxed{\text{proof}}$

REMODEL INTO AN ARITHMETIC PROGRESSION

$T = 240 - 5 + 237 - 5 + 234 - 5 + 231 - 5 + \dots 6 - 5 + 3 - 5$

$T = (240-5) + (237-5) + (234-5) + (231-5) + \dots + (6-5) + (3-5)$

$T = 235 + 232 + 229 + 226 + \dots + 1 + (-2)$

THIS AN A.P. WITH $a = 235$ & $d = -3$

$u_n = a + (n-1)d$

$-2 = 235 + (n-1)(-3)$

$-2 = 235 - 3n + 3$

$3n = 240$

$n = 80$

USING $S_n = \frac{n}{2}(a+L)$

$\Rightarrow S_{40} = \frac{80}{2}\left[235 - 2\right]$

$\Rightarrow S_{40} = 40 \times 233$

$\Rightarrow S_{40} = 9320$

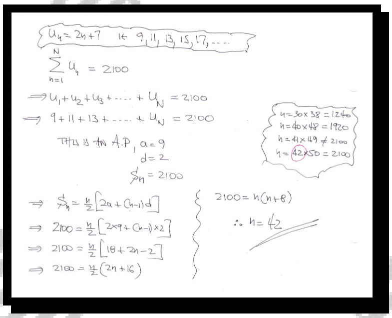
Question 73 **(***+)**

The n^{th} term of an arithmetic progression is denoted by u_n, and given by

$$u_n = 2n + 7.$$

Determine the value of N given that $\sum_{n=1}^{N} u_n = 2100$.

$\boxed{42}$


$u_n = 2n+7$ IE $9, 11, 13, 15, 17, \ldots$

$\sum_{n=1}^{N} u_n = 2100$

$\Rightarrow u_1 + u_2 + u_3 + \ldots + u_N = 2100$

$\Rightarrow 9 + 11 + 13 + \ldots + u_N = 2100$

THIS IS AN A.P, $a = 9$, $d = 2$, $S_n = 2100$

$\Rightarrow S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

$\Rightarrow 2100 = \frac{n}{2}\left[2\times 9 + (n-1)\times 2\right]$

$\Rightarrow 2100 = \frac{n}{2}\left[18 + 2n - 2\right]$

$\Rightarrow 2100 = \frac{n}{2}(2n+16)$

$2100 = n(n+8)$

$\therefore n = 42$

$n = 30 \times 38 = 1240$

$n = 40 \times 48 = 1920$

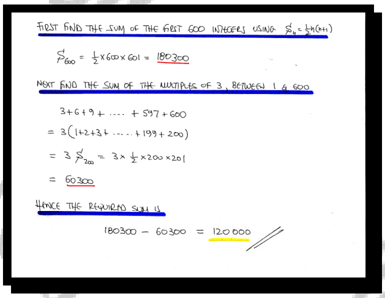
$n = 41 \times 49 \neq 2100$

$n = 42 \times 50 = 2100$

Question 74 **(***+)**

Show by using algebra, that the sum of the integers between 1 and 600 inclusive, that are **not** divisible by 3, is 120000.

$\boxed{}$, $\boxed{\text{proof}}$


FIRST FIND THE SUM OF THE FIRST 600 INTEGERS USING $S_n = \frac{1}{2}n(n+1)$

$S_{600} = \frac{1}{2}\times 600 \times 601 = 180300$

NEXT FIND THE SUM OF THE MULTIPLES OF 3, BETWEEN 1 & 600

$3 + 6 + 9 + \ldots + 597 + 600$

$= 3(1 + 2 + 3 + \ldots + 199 + 200)$

$= 3S_{200} = 3 \times \frac{1}{2} \times 200 \times 201$

$= 60300$

HENCE THE REQUIRED SUM IS

$180300 - 60300 = 120000$

Question 75 (***+)

The first three terms of an arithmetic series are

$$(m+1),\ (m^2+m) \text{ and } (3m^2-m-4), \text{ respectively,}$$

where m is a constant.

a) Find the 21^{st} term of the series.

The sum of the first n terms of the series is denoted by S_n.

b) Show that S_n is always a square number.

$\boxed{164}$

(a)

$u_1 = m+1$
$u_2 = m^2+m$
$u_3 = 3m^2-m-4$

- $u_3 - u_2 = u_2 - u_1$

$\Rightarrow (3m^2-m-4)-(m^2+m) = (m^2+\cancel{m})-(\cancel{m}+1)$

$\Rightarrow 2m^2-2m-4 = m^2-1$

$\Rightarrow m^2-2m-3=0$

$\Rightarrow (m-3)(m+1)=0$

$\Rightarrow m = 3, -1$

- if $m=-1$ $\quad u_1=0,\ u_2=(-1)^2-1=1-1=0,\ u_3=3(-1)^2-(-1)-4$

$u_3 = 3+1-4$

$u_3=0$

$\therefore m \neq -1$

- if $m=3$

$$\left.\begin{array}{l} u_1=4 \\ u_2=3^2+3=12 \\ u_3=3\times3^2-3-4=20 \end{array}\right\} \Rightarrow \begin{array}{l} a=4 \\ d=8 \end{array}$$

$u_n = a+(n-1)d$

$u_{21} = 4+20\times8$

$u_{21} = 164$

(b)

$\Rightarrow S_n = \frac{n}{2}\left[2a+(n-1)d\right]$

$\Rightarrow S_n = \frac{n}{2}\left[2\times4+(n-1)\times8\right]$

$\Rightarrow S_n = \frac{n}{2}\left[\cancel{8}+8n-\cancel{8}\right]$

$\Rightarrow S_n = \frac{n}{2}\times8n$

$\Rightarrow S_n = 4n^2$

$\Rightarrow S_n = (2n)^2$ $\quad$ IE ALWAYS A SQUARE NUMBER

Question 76 (***+)

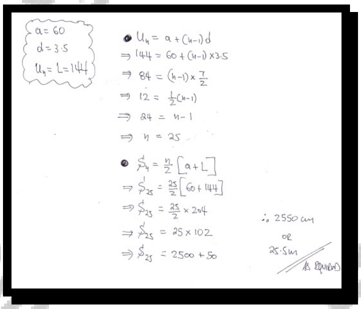
A length of rope is wrapped neatly around a circular pulley.

The length of the rope in the first coil (the nearest to the pulley) is 60 cm, and each successive coil of rope (outwards) is 3.5 cm longer than the previous one.

The outer coil has a length of 144 cm.

Show that total length of the rope is 25.5 metres.

$\boxed{}$, $\boxed{\text{proof}}$



Question 77 (***+)

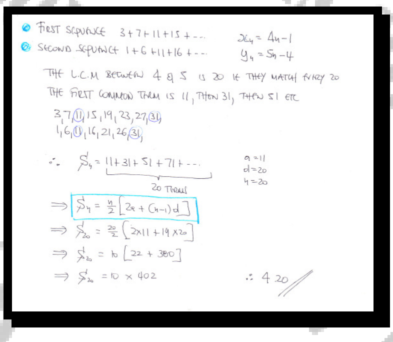
Consider the terms of the sequences

$$x_n = 4n - 1 \qquad \text{and} \qquad y_n = 5n - 4,$$

where $n = 1, 2, 3, 4, 5, \ldots$.

Determine the sum of the first 20 terms, **common** to both sequences.

$\boxed{4020}$



Question 78 (***+)

Consider the first few terms of the arithmetic progression

$$(2p+3),\ (4p+5),\ (6p+7),\ (8p+9),\ \ldots$$

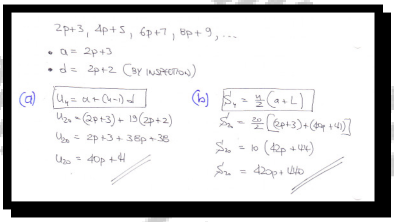
where p is a non zero constant.

Find simplified expressions, in terms of p, for …

a) … the twentieth term of the progression

b) … the sum of the first twenty terms of the progression.

$\boxed{u_{20} = 40p + 41}$, $\boxed{S_{20} = 420p + 440}$


$2p+3,\ 4p+5,\ 6p+7,\ 8p+9,\ \ldots$

- $a = 2p+3$
- $d = 2p+2$ (BY INSPECTION)

(a) $\boxed{u_n = a + (n-1)d}$

$u_{20} = (2p+3) + 19(2p+2)$

$u_{20} = 2p+3+38p+38$

$u_{20} = 40p+41$

(b) $\boxed{S_n = \frac{n}{2}(a+L)}$

$S_{20} = \frac{20}{2}\left[(2p+3)+(40p+41)\right]$

$S_{20} = 10(42p+44)$

$S_{20} = 420p+440$

Question 79 (****)

The first two terms of an arithmetic progression are

$$\log_2 a^2 \text{ and } 2\log_2 ab,\ a>0,\ b>0.$$

Given further that $ab^2 = 8$, show clearly that the sum of the first 5 terms of the progression is 30.

$\boxed{\text{proof}}$

- $d = u_2 - u_1 = 2\log_2(ab) - \log_2 a^2 = \log_2(a^2b^2) - \log_2 a^2$
  $= \log_2\left(\frac{a^2b^2}{a^2}\right) = \log_2 b^2$
- $S_n = \frac{n}{2}[2a + (n-1)d]$

$\Rightarrow S_5 = \frac{5}{2}[2\times\log_2 a^2 + 4\times\log_2 b^2]$

$\Rightarrow S_5 = 5[\log_2 a^2 + 2\log_2 b^2]$

$\Rightarrow S_5 = 5[\log_2 a^2 + \log_2 b^4]$

$\Rightarrow S_5 = 5[\log_2(a^2\times b^4)]$

$\Rightarrow S_5 = 5\log_2(ab^2)^2$

$\Rightarrow S_5 = 10\log_2(ab^2)$

$\Rightarrow S_5 = 10\times\log_2 8$

$\Rightarrow S_5 = 10\times\log_2 2^3$

$\Rightarrow S_5 = 30\log_2 2$

$\Rightarrow S_5 = 30$

AS REQUIRED

Question 80 (****)

The sum of the first eight terms of an arithmetic series is 124.

The sum of its first twenty terms of is 910.

The series has k terms.

Given the last term of the series is 193 find the value of k.

$\boxed{\quad}$, $\boxed{k=40}$

$\boxed{S_n = \frac{n}{2}[2a + (n-1)d]}$

- $S_8 = 124$

$124 = \frac{8}{2}[2a + 7d]$

$124 = 4(2a + 7d)$

$\boxed{31 = 2a + 7d}$

- $S_{20} = 910$

$910 = \frac{20}{2}[2a + 19d]$

$910 = 10(2a + 19d)$

$\boxed{91 = 2a + 19d}$

$\left.\begin{array}{l} 2a = 31 - 7d \\ 2a = 91 - 19d \end{array}\right\} \Rightarrow 31 - 7d = 91 - 19d$

$12d = 60$

$\boxed{d = 5}$ &

$2a = 31 - 7\times5$

$2a = 31 - 35$

$2a = -4$

$\boxed{a = -2}$

Now $\boxed{u_n = a + (n-1)d}$

$\Rightarrow 193 = -2 + (k-1)\times5$

$\Rightarrow 195 = 5(k-1)$

$\Rightarrow 195 = 5k - 5$

$\Rightarrow 5k = 200$

$\therefore k = 40$

Question 81 (****)

The first term of an arithmetic series is a and the common difference is d.

The 25^{th} term the series is 100.

The 5^{th} term the series is 8 times larger than the 35^{th} term the series.

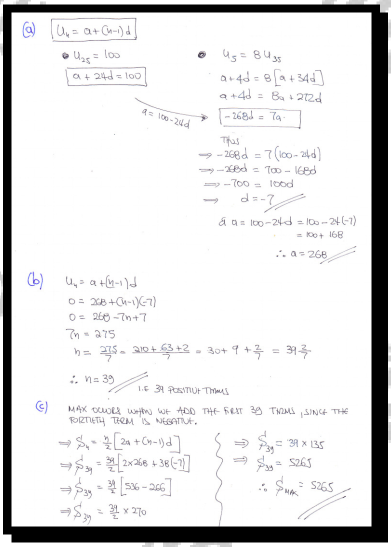
a) Find the value of a and the value of d.

b) Determine how many terms of the series are positive.

The sum of the first n terms of the series is denoted by S_n.

c) Calculate the maximum value of S_n.

$\boxed{a = 268}$, $\boxed{d = -7}$, $\boxed{39 \text{ terms}}$, $\boxed{S_{\max} = 5265}$



(a) $\boxed{u_n = a + (n-1)d}$

- $u_{25} = 100$

$\boxed{a + 24d = 100}$

- $u_5 = 8u_{35}$

$a + 4d = 8[a + 34d]$

$a + 4d = 8a + 272d$

$\boxed{-268d = 7a}$

$a = 100 - 24d$

Thus

$\Rightarrow -268d = 7(100 - 24d)$

$\Rightarrow -268d = 700 - 168d$

$\Rightarrow -700 = 100d$

$\Rightarrow d = -7$

& $a = 100 - 24d = 100 - 24(-7)$
$= 100 + 168$

$\therefore a = 268$

(b) $u_n = a + (n-1)d$

$0 = 268 + (n-1)(-7)$

$0 = 268 - 7n + 7$

$7n = 275$

$n = \frac{275}{7} = \frac{210 + 63 + 2}{7} = 30 + 9 + \frac{2}{7} = 39\frac{2}{7}$

$\therefore n = 39$

I.E 39 POSITIVE TERMS

(c) MAX OCCURS WHEN WE ADD THE FIRST 39 TERMS, SINCE THE FORTIETH TERM IS NEGATIVE.

$\Rightarrow S_n = \frac{n}{2}[2a + (n-1)d]$

$\Rightarrow S_{39} = \frac{39}{2}[2 \times 268 + 38(-7)]$

$\Rightarrow S_{39} = \frac{39}{2}[536 - 266]$

$\Rightarrow S_{39} = \frac{39}{2} \times 270$

$\Rightarrow S_{39} = 39 \times 135$

$\Rightarrow S_{39} = 5265$

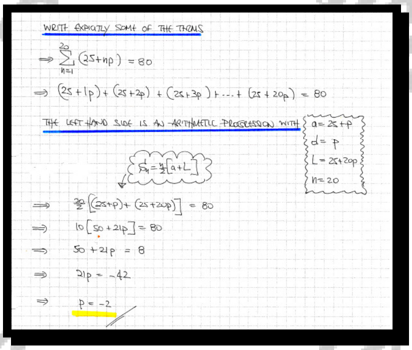
$\therefore S_{MAX} = 5265$

Question 82 (****)

Find the value of the constant p, so that

$$\sum_{n=1}^{20}(25+np)=80.$$

$\boxed{}$, $\boxed{p=-2}$



WRITE EXPLICITLY SOME OF THE TERMS

$\Rightarrow \sum_{n=1}^{20}(25+np)=80$

$\Rightarrow (25+1p)+(25+2p)+(25+3p)+\dots+(25+20p)=80$

THE LEFT HAND SIDE IS AN ARITHMETIC PROGRESSION WITH

$a=25+p$

$d=p$

$L=25+20p$

$n=20$

$S_n=\frac{n}{2}[a+L]$

$\Rightarrow \frac{20}{2}[(25+p)+(25+20p)]=80$

$\Rightarrow 10[50+21p]=80$

$\Rightarrow 50+21p=8$

$\Rightarrow 21p=-42$

$\Rightarrow p=-2$

Question 83 (****)

The n^{th} term of an arithmetic series is given by

$$u_n = \frac{5}{2}(5n+28).$$

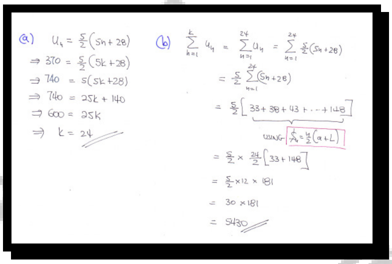
The k^{th} term of the series is 370.

a) Find the value of k.

b) Evaluate the sum

$$\sum_{n=1}^{k} u_n.$$

$\boxed{k=24}$, $\boxed{\sum_{n=1}^{k} u_n = 5430}$



(a) $u_n = \frac{5}{2}(5n+28)$

$\Rightarrow 370 = \frac{5}{2}(5k+28)$

$\Rightarrow 740 = 5(5k+28)$

$\Rightarrow 740 = 25k + 140$

$\Rightarrow 600 = 25k$

$\Rightarrow k = 24$

(b) $\sum_{n=1}^{k} u_n = \sum_{n=1}^{24} u_n = \sum_{n=1}^{24} \frac{5}{2}(5n+28)$

$= \frac{5}{2}\sum_{n=1}^{24}(5n+28)$

$= \frac{5}{2}\Big[\underbrace{33+38+43+\dots+148}_{\text{USING } \boxed{S_n = \frac{n}{2}(a+L)}}\Big]$

$= \frac{5}{2} \times \frac{24}{2}\left[33+148\right]$

$= \frac{5}{2} \times 12 \times 181$

$= 30 \times 181$

$= 5430$

Question 84 (****)

Find the value of x that satisfies the equation

$$\sum_{r=1}^{20}(2r+x)=280 .$$

$\boxed{}$, $\boxed{x=-7}$

GENERATE TERMS TO SEE THE PATTERN

$$\sum_{r=1}^{20}(2r+x) = (2+x)+(4+x)+(6+x)+\ldots(40+x)$$

... AN ARITHMETIC PROGRESSION WITH ...

- $a = 2+x$
- $d = 2$
- $L = 40+x$
- $n = 20$

USING $S_n = \frac{n}{2}[a+L]$

$$S_{20} = \frac{20}{2}\left[(2+x)+(40+x)\right]$$

$$S_{20} = 10(2x+42)$$

FINALLY SOLVE THE EQUATION

$$10(2x+42) = 280$$
$$2x+42 = 28$$
$$2x = -14$$
$$x = -7$$

Question 85 **(****)** **non calculator**

A farmer has difficulty persuading strawberry pickers to work for the entire 40 day strawberry picking season. He devises a wage plan to make the pay of the workers more attractive the more days they work.

He pays $\pounds a$ on the first day, $\pounds(a+d)$ on the second day, $\pounds(a+2d)$ on the third day, and so on, increasing the daily wages by $\pounds d$ every day.

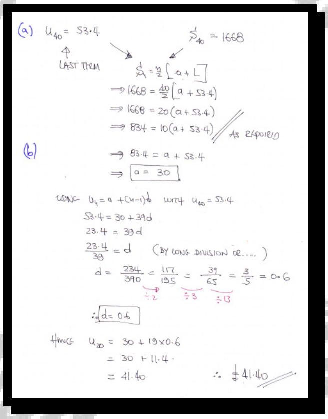
A strawberry picker that worked for forty days got paid £53.40 on the last day and earned $\pounds 1668$ in total.

a) Show clearly that

$$10(a+53.4)=834.$$

b) Calculate the wages that this strawberry picker received on the twentieth day.

$\boxed{\pounds 41.40}$


(a) $u_{40} = 53.4$ $S_{40} = 1668$

LAST TERM

$S_n = \frac{n}{2}[a+L]$

$\Rightarrow 1668 = \frac{40}{2}[a+53.4)$

$\Rightarrow 1668 = 20(a+53.4)$

$\Rightarrow 834 = 10(a+53.4)$ AS REQUIRED

(b) $\Rightarrow 83.4 = a + 53.4$

$\Rightarrow \boxed{a = 30}$

USING $u_n = a + (n-1)d$ WITH $u_{40} = 53.4$

$53.4 = 30 + 39d$

$23.4 = 39d$

$\frac{23.4}{39} = d$ (BY LONG DIVISION OR....)

$d = \underbrace{\frac{234}{390}}_{\div 2} = \underbrace{\frac{117}{195}}_{\div 3} = \underbrace{\frac{39}{65}}_{\div 13} = \frac{3}{5} = 0.6$

$\therefore \boxed{d = 0.6}$

HENCE $u_{20} = 30 + 19 \times 0.6$

$= 30 + 11.4$

$= 41.40$ $\therefore \pounds 41.40$

Question 86 (****)

Consider the multiples of seven between 1 and 1000.

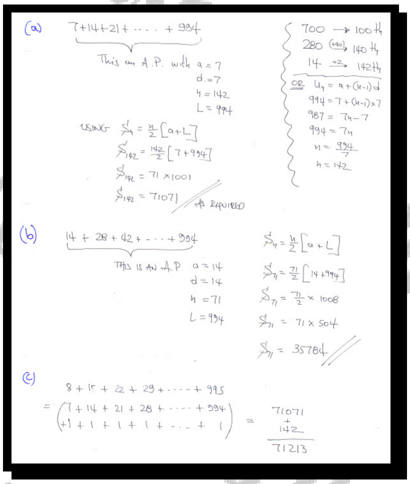
a) Show that the sum of the multiples of seven between 1 and 1000 is 71071

b) Hence find the sum of the multiples of fourteen between 1 and 1000.

c) Use the answer of part (a) to find

$$8+15+22+29+...+995.$$

$\boxed{35784}$, $\boxed{71213}$



(a) $\underbrace{7+14+21+\dots+994}$

This an A.P. with $a=7$, $d=7$, $n=142$, $L=994$

USING $S_n=\frac{n}{2}[a+L]$

$S_{142}=\frac{142}{2}[7+994]$

$S_{142}=71\times 1001$

$S_{142}=71071$ // AS REQUIRED

$700 \to 100$th
$280 \xrightarrow{(+40)} 140$th
$14 \xrightarrow{+2} 142$th

OR $u_n=a+(n-1)d$
$994=7+(n-1)\times 7$
$987=7n-7$
$994=7n$
$n=\frac{994}{7}$
$n=142$

(b) $\underbrace{14+28+42+\dots+994}$

THIS IS AN A.P $a=14$, $d=14$, $n=71$, $L=994$

$S_n=\frac{n}{2}[a+L]$

$S_{71}=\frac{71}{2}[14+994]$

$S_{71}=\frac{71}{2}\times 1008$

$S_{71}=71\times 504$

$S_{71}=35784$ //

(c) $8+15+22+29+\dots+995$

$=\begin{pmatrix}7+14+21+28+\dots+994\\+1+1+1+1+\dots+1\end{pmatrix}=71071+142=71213$

Question 87 **(****)**

The fifth term of an arithmetic series is 5 and the sum of its first five terms is $\frac{125}{4}$.

a) Show that the common difference of the series is $-\frac{5}{8}$.

The k^{th} term of the series is zero.

b) Find the value of k.

c) Show that maximum sum of this series is $\frac{195}{4}$.

$\boxed{k = 13}$

(a) • $u_n = a + (n-1)d$

$\boxed{5 = a + 4d}$

• $S_n = \frac{n}{2}[a + L]$

$\Rightarrow \frac{125}{4} = \frac{5}{2}(a + 5)$

$\Rightarrow 125 = 10(a + 5)$

$\Rightarrow 12.5 = a + 5$

$\Rightarrow \boxed{a = 7.5}$

$5 = 7.5 + 4d$

$-2.5 = 4d$

$4d = -\frac{5}{2}$

$d = -\frac{5}{8}$ // AS REQUIRED

(b) $u_n = a + (n-1)d$

$\Rightarrow 0 = \frac{15}{2} + (n-1)\left(-\frac{5}{8}\right)$

$\Rightarrow \frac{5}{8}(n-1) = \frac{15}{2}$

$\Rightarrow 5(n-1) = \frac{15}{2} \times 8$

$\Rightarrow 5(n-1) = 15 \times 4$

$\Rightarrow 5(n-1) = 60$

$\Rightarrow n - 1 = 12$

$\Rightarrow n = 13$ // IE $k = 13$

(c) MAX SUM IS S_{12} OR (S_{13})

$S_n = \frac{n}{2}[2a + (n-1)d]$

$S_{12} = \frac{12}{2}\left[2 \times 7.5 + 11\left(-\frac{5}{8}\right)\right]$

$S_{12} = 6\left(15 - \frac{55}{8}\right)$

$S_{12} = 6\left(\frac{120}{8} - \frac{55}{8}\right)$

$S_{12} = 6 \times \frac{65}{8}$

$S_{12} = 3 \times \frac{65}{4}$

$S_{12} = \frac{195}{4}$ // AS REQUIRED

Question 88 (****)

Tyler is repaying a loan over a period of n months in such a way so that his monthly repayments form an arithmetic series.

He repays £350 in the first month, £340 in the second month, £330 in the third month and so on until the full loan is repaid.

a) Assuming it takes more than 12 months to repay his loan find …

i. … the amount he pays on the twelfth month.

ii. … the total amount of his repayments in the first twelve months.

Tyler pays back his loan of £6200 after n months.

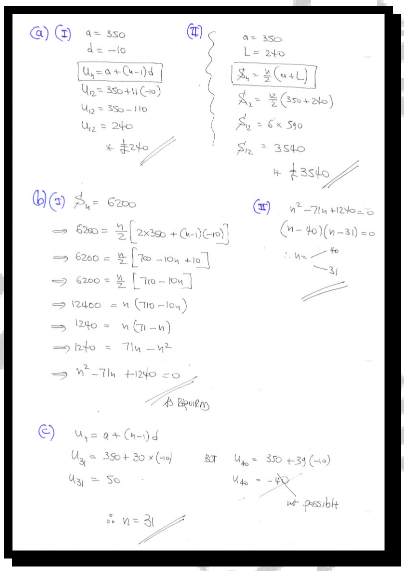
b) Show clearly that …

i. … $n^2 - 71n + 1240 = 0$

ii. … $n = 40$ is one of the solutions of this equation and find the other.

c) Determine, with a valid reason, which of the two values of n represents the actual number of months it takes Tyler to repay his loan.

$\boxed{£240}$, $\boxed{£3540}$, $\boxed{n = 31}$, $\boxed{31 \text{ months}}$


(a) (I) $a = 350$, $d = -10$

$\boxed{u_n = a + (n-1)d}$

$u_{12} = 350 + 11(-10)$

$u_{12} = 350 - 110$

$u_{12} = 240$

i.e. £240

(II) $a = 350$, $L = 240$

$\boxed{S_n = \frac{n}{2}(a + L)}$

$S_{12} = \frac{12}{2}(350 + 240)$

$S_{12} = 6 \times 590$

$S_{12} = 3540$

i.e. £3540

(b) (I) $S_n = 6200$

$\Rightarrow 6200 = \frac{n}{2}[2 \times 350 + (n-1)(-10)]$

$\Rightarrow 6200 = \frac{n}{2}[700 - 10n + 10]$

$\Rightarrow 6200 = \frac{n}{2}[710 - 10n]$

$\Rightarrow 12400 = n(710 - 10n)$

$\Rightarrow 1240 = n(71 - n)$

$\Rightarrow 1240 = 71n - n^2$

$\Rightarrow n^2 - 71n + 1240 = 0$

As required

(II) $n^2 - 71n + 1240 = 0$

$(n - 40)(n - 31) = 0$

$\therefore n = 40, 31$

(c) $u_n = a + (n-1)d$

$u_{31} = 350 + 30 \times (-10)$

$u_{31} = 50$

BUT $u_{40} = 350 + 39(-10)$

$u_{40} = -40$ not possible

$\therefore n = 31$

Question 89 (****)

An oil company is drilling for oil.

It costs £5000 to drill for the first 10 metres into the ground.

For the next 10 metres it costs an extra £1200 compared with the first 10 metres, thus it costs £6200. Each successive 10 metres drilled into the ground costs an extra £1200, compared with the cost of drilling the previous 10 metres.

a) Find the cost of drilling 200 metres into the ground.

The company has a budget of £15,000,000.

b) Determine the maximum depth, in metres, that can be reached on this budget.

$\boxed{}$, $\boxed{£328,000}$, $\boxed{1540\text{ m}}$

a) FORM A MODEL

DEPTH	COST
10	£5000
20	£6200
30	£7400
⋮	
200	?

20 PARTS OF 10 m

$\Longrightarrow S_n = \frac{n}{2}\left[2a+(n-1)d\right]$

$\Longrightarrow S_{20} = \frac{20}{2}\left[2\times5000+19\times1200\right]$

$\Longrightarrow S_{20} = 10\left[10000+22800\right]$

$\Longrightarrow S_{20} = 328000$

i.e. £328,000

b) "WORKING BACKWARDS"

$S_n = \frac{n}{2}\left[2a+(n-1)d\right]$

$\Longrightarrow 15000000 = \frac{n}{2}\left[2\times5000+(n-1)\times1200\right]$

$\Longrightarrow 15000000 = \frac{n}{2}\left[10000+1200n-1200\right]$

$\Longrightarrow 15000000 = \frac{n}{2}\left[8800+1200n\right]$

$\Longrightarrow 15000000 = 4400n+600n^2$

$\Longrightarrow 600n^2+4400n+15000000=0$ $\quad \div 100$

$\Longrightarrow 6n^2+44n-150000=0$ $\quad \div 2$

$\Longrightarrow 3n^2+22n-75000=0$

BY THE QUADRATIC FORMULA

$n = \frac{-22\pm\sqrt{484+900000}}{6} = \begin{cases} \cancel{-161.823\ldots} \\ 154.489\ldots \approx 154 \end{cases}$

$\therefore$ DEPTH OF 1540 m (ACCEPT 1544 OR 1545)

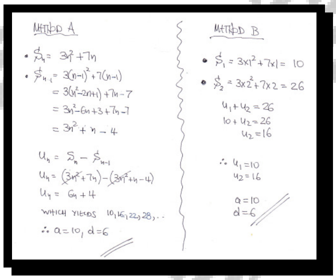
Question 90 (****)

The sum, S_n, of the first n terms of an arithmetic series is given by

$$S_n = 3n^2 + 7n .$$

Find the first term and the common difference of the series.

$\boxed{a = 10}$, $\boxed{d = 6}$



Question 91 (****)

In the TV game "Extra Fifty" contestants answer a series of questions.

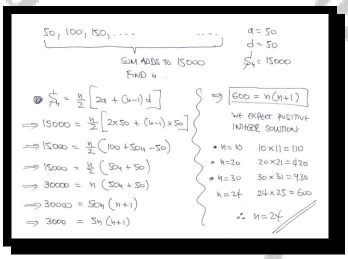
Contestants win £50 for answering the 1st question correctly, £100 for answering the 2nd question correctly, £150 for answering the 3rd question correctly, and so on.

Once an incorrect answer is given the game ends but the contestant keeps the winnings up to that point.

A contestant wins £15000 .

Determine, showing all parts in the calculation, the number of the questions he or she answered correctly.

$\boxed{24}$



Question 92 (****)

A company agrees to pay a loan back in monthly instalments, starting with £1500 .

The agreement states that the company will pay back
£$(1500-x)$ in the 2^{nd} month,
£$(1500-2x)$ in the 3^{rd} month,
£$(1500-3x)$ in the 4^{th} month,
and so on, with the repayments decreasing by £x every month.

a) Given that in the first year the company repaid a total of £15360 , find the value of x .

b) Show that the total money T_n , repaid in n months, is given by

$$T_n = 20n(76-n).$$

The total value of the loan was £26000 .

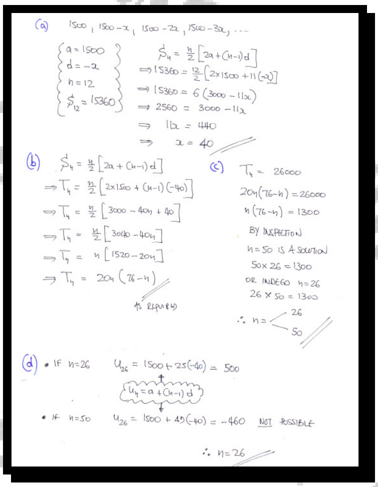
c) Show that the equation

$$T_n = 26000$$

is satisfied by two different values of n .

d) Determine, with a valid reason, which of the two values of n represents the actual number of months it takes for the company to repay the loan.

$\boxed{}$, $\boxed{x=40}$, $\boxed{n=26,50}$, $\boxed{n=26}$


(a) 1500 , 1500 − x , 1500 − 2x , 1500 − 3x , ...

a = 1500, d = −x, n = 12, S_{12} = 15360

$S_n = \frac{n}{2}[2a+(n-1)d]$
⟹ $15360 = \frac{12}{2}[2\times1500+11(-x)]$
⟹ $15360 = 6(3000-11x)$
⟹ $2560 = 3000-11x$
⟹ $11x = 440$
⟹ $x = 40$

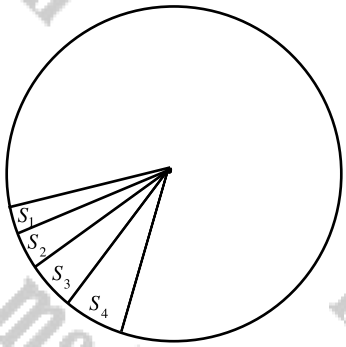
(b) $S_n = \frac{n}{2}[2a+(n-1)d]$
⟹ $T_n = \frac{n}{2}[2\times1500+(n-1)(-40)]$
⟹ $T_n = \frac{n}{2}[3000-40n+40]$
⟹ $T_n = \frac{n}{2}[3040-40n]$
⟹ $T_n = n[1520-20n]$
⟹ $T_n = 20n(76-n)$
AS REQUIRED

(c) $T_n = 26000$
$20n(76-n) = 26000$
$n(76-n) = 1300$
BY INSPECTION
n = 50 IS A SOLUTION
50 × 26 = 1300
OR INDEED n = 26
26 × 50 = 1300
∴ n = 26, 50

(d) • IF n = 26 $u_{26} = 1500+25(-40) = 500$
$u_n = a+(n-1)d$
• IF n = 50 $u_{26} = 1500+49(-40) = -460$ NOT POSSIBLE
∴ n = 26

Question 93 (****)

A machine cuts a circular sheet of plastic into **exactly** n sectors, S_1, S_2, S_3, …, S_n.

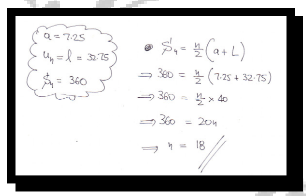


The angle that each sector subtends at the centre of the circle forms an arithmetic series.

The smallest sector and the largest sector subtend angles at the centre of $7.25°$ and $32.75°$, respectively.

Find the value of n.

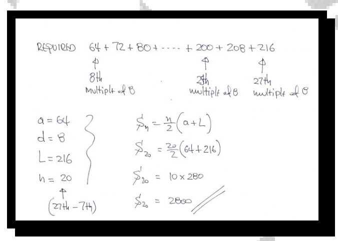
$\boxed{n = 18}$



Question 94 (****)

Use an algebraic method to show that the sum of all the integers between 60 and 220 which are divisible by 8, is 2800.

$\boxed{\text{proof}}$



Question 95 (****)

A company offers two pay schemes for its employees.

<u>Scheme One</u>

- Annual salary in Year 1 is £X .
- Annual salary increases every subsequent year by £$(2Y)$, forming an arithmetic series.

<u>Scheme Two</u>

- Annual salary in Year 1 is £$(X+2000)$.
- Annual salary increases every subsequent year by £Y , forming an arithmetic series.

a) Show that the total salary received by an employee under Scheme One, over a nine year period is

$$9(X+8Y).$$

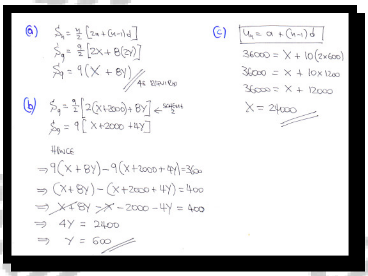
After nine years, the total salary received by an employee under Scheme One is £3600 larger than the total salary received by an employee under Scheme Two.

b) Show clearly that

$$Y = 600.$$

c) Given further that an employee under the Scheme One earns £36000 in the eleventh year of his employment, determine the value of X .

$\boxed{\quad}$, $\boxed{X = 24000}$


(a) $S_n = \frac{n}{2}[2a+(n-1)d]$
$S_9 = \frac{9}{2}[2X+8(2Y)]$
$S_9 = 9(X+8Y)$ AS REQUIRED

(b) $S_9 = \frac{9}{2}[2(X+2000)+8Y] \leftarrow$ SCHEME 2
$S_9 = 9[X+2000+4Y]$

HENCE
$\Rightarrow 9(X+8Y)-9(X+2000+4Y)=3600$
$\Rightarrow (X+8Y)-(X+2000+4Y)=400$
$\Rightarrow \cancel{X}+8Y-\cancel{X}-2000-4Y=400$
$\Rightarrow 4Y = 2400$
$\Rightarrow Y = 600$

(c) $\boxed{u_n = a+(n-1)d}$
$36000 = X + 10(2\times 600)$
$36000 = X + 10\times 1200$
$36000 = X + 12000$
$X = 24000$

Question 96 (****)

An arithmetic series has first term a, last term L and common difference d.

a) Show that the sum of the first n terms of the series is given by

$$\frac{1}{2}(L+a)\left(\frac{L-a}{d}+1\right).$$

b) Hence, or otherwise, find the sum of all the multiples of 11 between 549 and 1101.

$\boxed{}$, $\boxed{42075}$

(a) $S_n = \frac{n}{2}[a+L]$, $L = a+(n-1)d$ $\Rightarrow$

So $n-1 = \frac{L-a}{d}$

$\boxed{n = 1+\frac{L-a}{d}}$

So $S_n = \frac{1}{2}\left(1+\frac{L-a}{d}\right)(a+L)$

$S_n = \frac{1}{2}(L+a)\left(\frac{L-a}{d}+1\right)$

AS REQUIRED

(b) 549 to 1101

$\underset{a}{550} + 561 + 572 + \dots + \underset{L}{1100}$

$\therefore S = \frac{1}{2}(550+1100)\left(\frac{1100-550}{11}+1\right)$

$S = \frac{1}{2}\times 1650 \times (50+1)$

$S = 825\times 51$

$S = 42075$

Question 97 (****)

$$100+101+102+\dots+200.$$

a) Find the value of the above sum.

b) Hence, or otherwise, determine the sum of all the integers between 500 and 1000, inclusive, which are divisible by 5.

$\boxed{15150}$, $\boxed{75750}$

(a) $\underbrace{100+101+102+\dots+200}$

A.P with $a=100$, $l=200$, $n=101$

200 − 99

$S_n = \frac{n}{2}[a+L]$

$S_{101} = \frac{101}{2}[100+200]$

$S_{101} = \frac{101}{2}\times 300$

$S_{101} = 101\times 150$

$S_{101} = 15150$

(b) $500+505+510+\dots+1000$

$= 5(100+101+102+\dots+200)$

$= 5\times 15150$

$= 75750$

Question 98 **(****)**

Consider the arithmetic progression

$$t + 2t + 3t + 4t + \ldots + 50,$$

where t is a factor of 50.

Show clearly that the sum of the terms of this progression is

$$25 + \frac{1250}{t}.$$

$\boxed{}$, $\boxed{\text{proof}}$

$a = t$
$d = t$
$u_n = L = 50$

- $u_n = a + (n-1)d$

$\Rightarrow 50 = t + (n-1)t$

$\Rightarrow 50 = \cancel{t} + nt - \cancel{t}$

$\Rightarrow 50 = nt$

$\Rightarrow \boxed{n = \frac{50}{t}}$

- $S_n = \frac{n}{2}(a + L)$

$\Rightarrow S_{\frac{50}{t}} = \frac{\frac{50}{t}}{2} \times [t + 50]$

$\Rightarrow S_{\frac{50}{t}} = \frac{50}{2t}(t + 50)$

$\Rightarrow S_{\frac{50}{t}} = \frac{50t}{2t} + \frac{50 \times 50}{2t}$

$\Rightarrow S_{\frac{50}{t}} = 25 + \frac{1250}{t}$

Question 99 **(****)**

The second term of an arithmetic progression is 49 and the fifth term is 67.

a) Determine the value of the twentieth term of the progression.

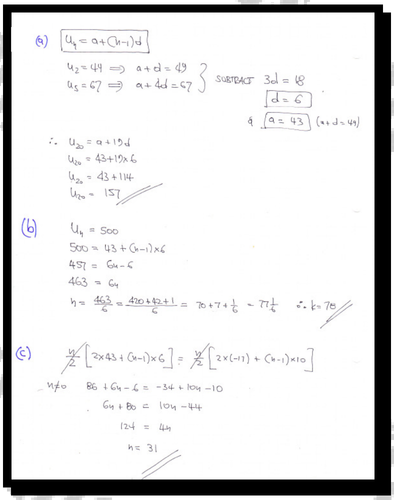
The k^{th} term of the progression is greater than 500.

b) Find the least value of k.

A different arithmetic progression has first term -17 and its common difference is 10.

c) Given that the sum of the first n terms of these two progressions are equal, determine the value of n.

$\boxed{u_{20} = 157}$, $\boxed{k = 78}$, $\boxed{n = 31}$



(a) $\boxed{u_n = a+(n-1)d}$

$u_2 = 49 \Rightarrow a+d=49$
$u_5 = 67 \Rightarrow a+4d=67$ $\Big\}$ SUBTRACT $3d=18$

$\boxed{d=6}$

& $\boxed{a=43}$ $(a+d=49)$

$\therefore u_{20} = a+19d$
$u_{20} = 43+19\times 6$
$u_{20} = 43+114$
$u_{20} = 157$

(b) $u_n = 500$
$500 = 43+(n-1)\times 6$
$457 = 6n-6$
$463 = 6n$
$n = \frac{463}{6} = \frac{420+42+1}{6} = 70+7+\frac{1}{6} = 77\frac{1}{6}$ $\therefore k=78$

(c) $\cancel{\frac{n}{2}}\left[2\times 43+(n-1)\times 6\right] = \cancel{\frac{n}{2}}\left[2\times(-17)+(n-1)\times 10\right]$

$n \neq 0$ $86+6n-6 = -34+10n-10$
$6n+80 = 10n-44$
$124 = 4n$
$n = 31$

Question 100 (****)

The second term of an arithmetic progression is $2k$ and the sum of its first six terms is $11k-2$, where k is a constant.

a) Show clearly that …

i. … the first term of the progression is $\frac{1}{9}(19k+2)$.

ii. … the common difference of the progression is $-\frac{1}{9}(k+2)$.

The eleventh term of the progression is 5.

b) Find the value of k.

c) Calculate the sum of the first 56 terms of the progression.

$\boxed{k=7}$, $\boxed{S_{56}=-700}$

(a)

- $u_n = a+(n-1)d$
- $u_2 = 2k$
- $\boxed{a+d=2k}$
- $\boxed{d=2k-a}$

- $S_n = \frac{n}{2}(2a+(n-1)d)$
- $S_6 = \frac{6}{2}(2a+5d)$
- $11k-2 = 3(2a+5d)$
- $\boxed{11k-2 = 6a+15d}$

$11k-2 = 6a+15(2k-a)$
$11k-2 = 6a+30k-15a$
$9a = 19k+2$
$a = \frac{1}{9}(19k+2)$

$d = 2k-a = 2k-\frac{1}{9}(19k+2) = 2k-\frac{19}{9}k-\frac{2}{9} = \frac{18k}{9}-\frac{19k}{9}-\frac{2}{9}$

$\therefore d = -\frac{1}{9}k-\frac{2}{9} = -\frac{1}{9}(k+2)$

(b) $u_n = a+(n-1)d$
$\Rightarrow 5 = \frac{1}{9}(19k+2)+10\times\frac{-1}{9}(k+2)$
$\Rightarrow 45 = 19k+2-10(k+2)$
$\Rightarrow 45 = 19k+2-10k-20$
$\Rightarrow 45 = 9k-18$
$\Rightarrow 9k = 63$
$\Rightarrow k = 7$

(c) Now $a = \frac{1}{9}(19\times7+2) = \frac{1}{9}\times135$
$\boxed{a=15}$
& $d = -\frac{1}{9}(7+2) = -1$
$\boxed{d=-1}$
$\Rightarrow S_n = \frac{n}{2}[2a+(n-1)d]$
$\Rightarrow S_{56} = \frac{56}{2}[2\times15+55(-1)]$
$\Rightarrow S_{56} = 28(30-55)$
$\Rightarrow S_{56} = 28(-25)$
$\Rightarrow S_{56} = -700$

Question 101 (****)

A sequence is defined as

$$u_{r+1} = u_r - 3 \, , \ u_1 = 117 \, , \ n \geq 1 .$$

Solve the equation

$$\sum_{r=1}^{n} u_r = 0 .$$

$\boxed{n = 79}$

$\sum_{r=1}^{n} u_r = 0$

$\Rightarrow u_1 + u_2 + u_3 + \dots + u_n = 0$

$\Rightarrow \frac{n}{2}\left[2a + (n-1)d\right] = 0$

$\Rightarrow \frac{n}{2}\left[2 \times 117 + (n-1)(-3)\right] = 0$

$\Rightarrow \frac{n}{2}\left[234 - 3n + 3\right] = 0$

$\Rightarrow n(237 - 3n) = 0$

$n = \cancel{0}$, 79

$u_{r+1} = u_r - 3$

$u_1 = 117$

$u_2 = u_1 - 3 = 117 - 3 = 114$

$u_3 = u_2 - 3 = 114 - 3 = 111$

$\therefore 117, 114, 111, \dots$

$a = 117$

$d = -3$

Question 102 (****+)

Ladan is repaying an interest free loan of £6200 over a period of n months, in such a way so that her monthly repayments form an arithmetic series.

She repays £350 in the first month, £340 in the second month, £330 in the third month and so on until the full loan is repaid.

Determine, showing a full algebraic method, the value of n .

$\boxed{}$, $\boxed{n = 31}$

THE ARITHMETIC SERIES IS

$$\underbrace{350 + 340 + 330 + \cdots\cdots + (\ ?\ )}_{n \text{ terms}} = 6200$$

n terms, WHERE n IS A POSITIVE INTEGER

HERE WE HAVE $a = 350$, $d = -10$ & $S_n = 6200$

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$
$$6200 = \frac{n}{2}\left[2 \times 350 + (n-1)(-10)\right]$$
$$6200 = \frac{n}{2}\left[700 - 10n + 10\right]$$
$$6200 = \frac{n}{2}\left[710 - 10n\right]$$
$$620 = \frac{n}{2}(71 - n)$$
$$1240 = n(71 - n)$$
$$1240 = 71n - n^2$$
$$n^2 - 71n + 1240 = 0$$

BY THE QUADRATIC FORMULA OR FACTORIZATION

$$n = \frac{71 \pm \sqrt{(-71)^2 - 4 \times 1 \times 1240}}{2 \times 1} = \frac{71 \pm 9}{2} = \begin{cases} 40 \\ 31 \end{cases}$$

TO CHECK WHICH SOLUTION IS VALID USE $u_n = a + (n-1)d$

- $n = 31$: $u_{31} = 350 + (31-1)(-10)$, $u_{31} = 350 - 300$, $u_{31} = 50$
- $n = 40$: $u_{40} = 350 + (40-1)(-10)$, $u_{40} = 350 - 390$, $u_{40} = \cancel{-40}$

$\therefore n = 31$

Question 103 (****+)

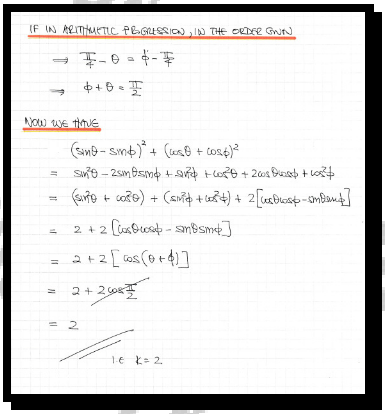
It is given that the angles θ, $\frac{\pi}{4}$ and φ, in that order, are in arithmetic progression.

Show that

$$(\sin\theta - \sin\varphi)^2 + (\cos\theta + \cos\varphi)^2 = k ,$$

where k is a constant to be found.

$\boxed{}$, $\boxed{k = 2}$



IF IN ARITHMETIC PROGRESSION, IN THE ORDER GIVEN

$\Rightarrow \frac{\pi}{4} - \theta = \phi - \frac{\pi}{4}$

$\Rightarrow \phi + \theta = \frac{\pi}{2}$

NOW WE HAVE

$(\sin\theta - \sin\phi)^2 + (\cos\theta + \cos\phi)^2$

$= \sin^2\theta - 2\sin\theta\sin\phi + \sin^2\phi + \cos^2\theta + 2\cos\theta\cos\phi + \cos^2\phi$

$= (\sin^2\theta + \cos^2\theta) + (\sin^2\phi + \cos^2\phi) + 2[\cos\theta\cos\phi - \sin\theta\sin\phi]$

$= 2 + 2[\cos\theta\cos\phi - \sin\theta\sin\phi]$

$= 2 + 2[\cos(\theta + \phi)]$

$= 2 + \cancel{2\cos\frac{\pi}{2}}$

$= 2$

i.e. $k = 2$

Question 104 (****+)

The first four terms of an arithmetic series are

$$2,\ (2b+3c),\ (b-3c+1) \text{ and } (4b+5c),$$

respectively, where b and c are a constants.

Show that the sum of the first thirty terms of the series is 1365.

$\boxed{}$, $\boxed{\text{proof}}$

- $\boxed{u_2-u_1=u_3-u_2}$

$\Rightarrow (2b+3c)-2=(b-3c+1)-(2b+3c)$
$\Rightarrow 2b+3c-2=b-3c+1-2b-3c$
$\Rightarrow 3b+9c=3$
$\boxed{b+3c=1}$

- $\boxed{u_4-u_3=u_3-u_2}$

$\Rightarrow (4b+5c)-(b-3c+1)=(b-3c+1)-(2b+3c)$
$\Rightarrow 3b+8c-1=-b-6c+1$
$\Rightarrow 4b+14c=2$
$\boxed{2b+7c=1}$

$b=1-3c$

$\Rightarrow 2(1-3c)+7c=1$
$\Rightarrow 2-6c+7c=1$
$\Rightarrow \boxed{c=-1}$
& $\boxed{b=4}$

$\therefore\ a=2$
$u_2=2\times4+3(-1)=5$
$u_3=4-3(-1)+1=8$
$u_4=4\times4+5(-1)=11$

$\Rightarrow$ A.P WITH $a=2$, $d=3$, $n=30$

$S_n=\frac{n}{2}\left[2a+(n-1)d\right]$
$S_{30}=\frac{30}{2}\left[2\times2+29\times3\right]$
$S_{30}=15(4+87)$
$S_{30}=15\times91$
$S_{30}=910+455$
$S_{30}=1365$ // AS REQUIRED

Question 105 (****+)

The common difference of an arithmetic series is denoted by d and the sum of its first n terms is denoted by S_n.

Show clearly that

$$d = S_{n+2} - 2S_{n+1} + S_n .$$

$\boxed{}$, $\boxed{\text{proof}}$

USING THE FOLLOWING APPROACH

$$S_{n+2} - 2S_{n+1} + S_n$$
$$= \left[S_n + u_{n+1} + u_{n+2}\right] - 2\left[S_n + u_{n+1}\right] + S_n$$
$$= \cancel{S_n} + u_{n+1} + u_{n+2} - \cancel{2S_n} - 2u_{n+1} + \cancel{S_n}$$
$$= u_{n+2} - u_{n+1}$$
$$= \left[u_{n+1} + d\right] - u_{n+1}$$
$$= d$$

AS REQUIRED

ALTERNATIVE

$$S_{n+2} - S_{n+1} = u_{n+2} = a + (n+1)d = a + nd + d$$
$$S_{n+1} - S_n = u_{n+1} = a + (n)d = a + nd$$

SUBTRACTING THE ABOVE SIDE BY SIDE

$$\Rightarrow S_{n+2} - 2S_{n+1} + S_n = d$$

AS REQUIRED

LONGER ALTERNATIVE

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right] = \frac{n}{2}\left[2a + nd - d\right]$$
$$S_{n+1} = \frac{n+1}{2}\left[2a + nd\right] = \left(\frac{n}{2} + \frac{1}{2}\right)(2a + nd) = \frac{n}{2}(2a + nd) + \frac{1}{2}(2a + nd)$$
$$S_{n+2} = \frac{n+2}{2}\left[2a + (n+1)d\right] = \left(\frac{n}{2} + 1\right)(2a + nd + d) = \frac{n}{2}(2a + nd + d) + (2a + nd + d)$$

TIDY EACH EXPRESSION FURTHER

$$S_n = \frac{n}{2}(2a + nd - d) = \frac{n}{2}(2a + nd) - \frac{1}{2}nd$$
$$S_{n+1} = \frac{n}{2}(2a + nd) + \frac{1}{2}(2a + nd) = \frac{n}{2}(2a + nd) + a + \frac{1}{2}nd$$
$$S_{n+2} = \frac{n}{2}(2a + nd + d) + (2a + nd + d) = \frac{n}{2}(2a + nd) + \frac{1}{2}nd + 2a + nd + d$$
$$= \frac{n}{2}(2a + nd) + 2a + \frac{3}{2}nd + d$$

FINALLY TIDYING UP

$$S_{n+2} - 2S_{n+1} + S_n = \cancel{\frac{n}{2}(2a + nd)} + \cancel{2a} + \cancel{\frac{3}{2}nd} + d$$
$$\cancel{- n(2a + nd)} \cancel{- 2a} \cancel{- nd}$$
$$\cancel{\frac{n}{2}(2a + nd)} \cancel{- \frac{1}{2}nd}$$

$$d$$

AS REQUIRED

Question 106 (****+)

The sum of the first 25 terms of an arithmetic series is 1050 and its 25^{th} term is 72.

a) Find the first term and the common difference of the series.

The n^{th} term of the series is denoted by u_n.

b) Given further that

$$117\left[\sum_{n=1}^{25} u_n - \sum_{n=1}^{k} u_n\right] = 233\sum_{n=1}^{k} u_n .$$

determine the value of k.

$\boxed{}$, $\boxed{a=12}$, $\boxed{d=2.5}$, $\boxed{k=13}$

a) SETTING UP TWO EQUATIONS

- $S_n = \frac{n}{2}[a+L]$

$S_{25} = 1050$

$\frac{25}{2}(a+L) = 1050$

$25(a+72) = 2100$

$a+72 = \frac{2100}{25}$

$a+72 = \frac{4200}{50}$

$a+72 = \frac{8400}{100}$

$a+72 = 84$

$a = 12$ //

- $u_n = a + (n-1)d$

$u_{25} = 72$

$a + 24d = 72$

$12 + 24d = 72$

$24d = 60$

$d = \frac{60}{24}$

$d = \frac{5}{2}$ //

b) LET US NOTE THAT $\sum_{n=1}^{25} u_n = 1050$ (GIVEN IN PART a)

$\Rightarrow 117\left[1050 - T_k\right] = 233\,T_k$ $\quad T_k = \sum_{n=1}^{k} u_n$

$\Rightarrow 117 \times 1050 - 117\,T_k = 233\,T_k$

$\Rightarrow 117 \times 1050 = 350\,T_k$

$\Rightarrow T_k = \frac{117 \times \cancel{1050}^{\,3}}{\cancel{350}_{\,1}}$

$\Rightarrow T_k = 351$

$\Rightarrow \sum_{n=1}^{k} u_n = 351$

USING $a=12$ & $d=5/2$

$\Rightarrow \frac{k}{2}\left[2\times12 + (k-1)\times\frac{5}{2}\right] = 351$

$\Rightarrow \frac{k}{2}\left[24 + \frac{5}{2}(k-1)\right] = 351$

$\Rightarrow k\left[12 + \frac{5}{4}(k-1)\right] = 351$

$\Rightarrow 4k\left[12 + \frac{5}{4}(k-1)\right] = 1404$ $\quad \times 4$

$\Rightarrow k\left[48 + 5(k-1)\right] = 1404$

$\Rightarrow k\left[5k+43\right] = 1404$

NOW BY TRIAL & IMPROVEMENT, NOTING THAT $k<24$

IF $k=10 \Rightarrow 10\times 93 < 1404$

IF $k=15 \Rightarrow 15\times 118 > 1404$

IF $k=13 \Rightarrow 13\times 108 = 1404$

$\therefore k=13$ //

ALTERNATIVE

$\frac{k}{2}\left[2\times12 + (k-1)\times\frac{5}{2}\right] = 351$

$\frac{k}{2}\left[24 + \frac{5}{2}k - \frac{5}{2}\right] = 351$

$12k + \frac{5}{4}k^2 - \frac{5}{4}k = 351$

$48k + 5k^2 - 5k = 1404$

$5k^2 + 43k - 1404 = 0$

$k = \frac{-43 \pm \sqrt{43^2 - 4\times5\times(-1404)}}{2\times5} = \begin{cases} \frac{-43+173}{10} = 13 \\ \frac{-43-173}{10} = \cancel{-21.6} \end{cases}$

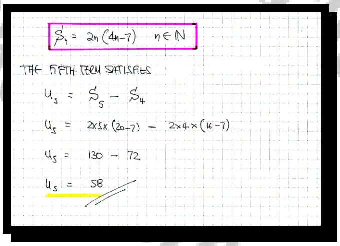
Question 107 (****+)

The sum, S_n, of the first n terms of an arithmetic series is given by

$$S_n = 2n(4n-7).$$

Find the fifth term of the series.

$\boxed{}$, $\boxed{u_5 = 58}$



$\boxed{S_n = 2n(4n-7) \quad n \in \mathbb{N}}$

THE FIFTH TERM SATISFIES

$u_5 = S_5 - S_4$

$u_5 = 2 \times 5 \times (20-7) - 2 \times 4 \times (16-7)$

$u_5 = 130 - 72$

$u_5 = 58$

Question 108 (****+)

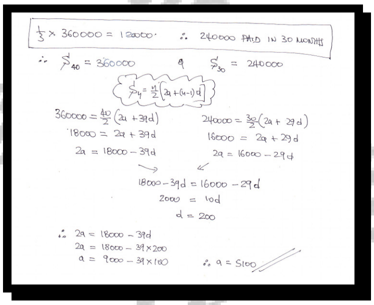
A company arranges to pay a debt of £360,000 by 40 monthly instalments.

These monthly instalments form an arithmetic series.

After 30 of these instalments were paid, the company declared themselves bankrupt leaving one third of their debt unpaid.

Find the value of the first instalment.

$\boxed{}$, $\boxed{£5100}$



$\boxed{\frac{1}{3} \times 360000 = 120000 \quad \therefore \; 240000 \text{ PAID IN 30 MONTHS}}$

$\therefore \; S_{40} = 360000 \quad \text{\&} \quad S_{30} = 240000$

$\boxed{S_n = \frac{n}{2}\left[2a + (n-1)d\right]}$

$360000 = \frac{40}{2}(2a + 39d)$ $\qquad$ $240000 = \frac{30}{2}(2a + 29d)$

$18000 = 2a + 39d$ $\qquad$ $16000 = 2a + 29d$

$2a = 18000 - 39d$ $\qquad$ $2a = 16000 - 29d$

$18000 - 39d = 16000 - 29d$

$2000 = 10d$

$d = 200$

$\therefore \; 2a = 18000 - 39d$

$2a = 18000 - 39 \times 200$

$a = 9000 - 39 \times 100$

$\therefore \; a = 5100$

Question 109 (****+)

A gym has 125 members and in order to meet its outgoings it needs 600 members.

A Public Relations company is hired to re-launch the gym and increase its membership thereafter, using a variety of marketing strategies.

A preliminary model for the recruitment of new members is as follows.

It is expected that 10 new members will join in the week following the gym's re-launch, 12 new members in the second week, 14 in the third week and so on with 2 new members joining the gym in each subsequent week.

a) Find according to this preliminary model …

i. … the number of the new members that will join in the 12^{th} week.

ii. … the **total** number of members after 12 weeks.

The model is refined to allow for the gym losing members at the constant rate of 3 members per week. The gym **reaches** the desired target of 600 members in N weeks.

b) Determine the value of N.

$\boxed{}$, $\boxed{32}$, $\boxed{377}$, $\boxed{19 \text{ weeks}}$

(a) (I) $u_n = a + (n-1)d$

$u_{12} = 10 + 11 \times 2$

$u_{12} = 32$

(II) $S_n = \frac{n}{2}(a+L)$

$S_{12} = \frac{12}{2}[10+32]$

$S_{12} = 6 \times 42$

$S_{12} = 252$

HENCE $252 + 125 = 377$

(b)
- WEEK 1 $+10 \ -3 = +7$
- WEEK 2 $+12 \ -3 = +9$
- WEEK 3 $+14 \ -3 = +11$ ETC...

SO $a = 7$, $d = 2$

HENCE $125 + S_n = 600$

$S_n = 475$

$\frac{n}{2}[2 \times 7 + (n-1) \times 2] = 475$

$\frac{n}{2}(14 + 2n - 2) = 475$

$\frac{n}{2}(2n + 12) = 475$

$n(n+6) = 475$

BY INSPECTION

$n = 15$, $15 \times 21 = 315$

$n = 19$, $19 \times 25 = 475$

$\therefore N = 19$

Question 110 (****+)

Five numbers are consecutive terms of an arithmetic progression.

The arithmetic mean of these numbers is 7, while the arithmetic mean of the **squares** of these numbers is 67.

Determine these five numbers.

$\boxed{}$, $\boxed{1,\ 4,\ 7,\ 10,\ 13}$

MODELLING AS FOLLOWS – LET THE "MIDDLE" TERM BE x

$$u_{n-2},\quad u_{n-1},\quad u_n,\quad u_{n+1},\quad u_{n+2}$$
$$x-2d\quad x-d\quad x\quad x+d\quad x+2d$$

THE ARITHMETIC MEAN IS 7

$$\Rightarrow \frac{(x-2d)+(x-d)+x+(x+d)+(x+2d)}{5}=7$$
$$\Rightarrow \frac{5x}{5}=7$$
$$\Rightarrow x=7$$

NEXT THE ARITHMETIC MEAN OF THE SQUARES IS 67

$$\Rightarrow \frac{(x-2d)^2+(x-d)^2+x^2+(x+d)^2+(x+2d)^2}{5}=67$$
$$\Rightarrow (7-2d)^2+(7-d)^2+7^2+(7+d)^2+(7+2d)^2=67\times5$$
$$\Rightarrow 49-\cancel{28d}+4d^2+49-\cancel{14d}+d^2+49+49+\cancel{14d}+d^2+49+\cancel{28d}+4d^2=335$$
$$\Rightarrow 10d^2+49\times5=335$$
$$\Rightarrow 2d^2+49=67$$
$$\Rightarrow 2d^2=18$$
$$\Rightarrow d^2=9$$
$$\Rightarrow d=\pm3$$

$\therefore$ IF $d=3$ THE NUMBERS ARE $1, 4, 7, 10, 13$

(IF $d=-3$ THE NUMBERS ARE $13, 10, 7, 4, 1$)

Question 111 (****+)

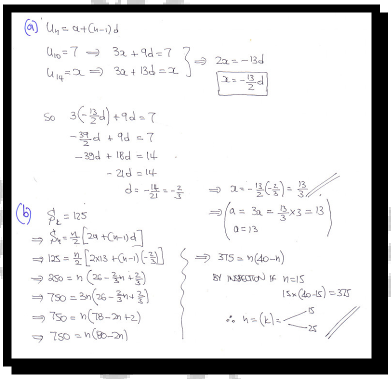
The 1^{st} term of an arithmetic series is $3x$, the 10^{th} term is 7 and the 14^{tn} term is x.

a) Show clearly that $x = \dfrac{13}{3}$

The sum of the first k terms of the series is 125.

b) Find the possible values of k.

$\boxed{d = -\dfrac{2}{13}x}$, $\boxed{k = 15, 25}$



(a) $u_n = a + (n-1)d$

$u_{10} = 7 \Rightarrow 3x + 9d = 7$

$u_{14} = x \Rightarrow 3x + 13d = x$

$\Rightarrow 2x = -13d$

$\boxed{x = -\frac{13}{2}d}$

So $3\left(-\frac{13}{2}d\right) + 9d = 7$

$-\frac{39}{2}d + 9d = 7$

$-39d + 18d = 14$

$-21d = 14$

$d = -\frac{14}{21} = -\frac{2}{3}$ $\Rightarrow x = -\frac{13}{2}\left(-\frac{2}{3}\right) = \frac{13}{3}$

$\Rightarrow \left(a = 3x = \frac{13}{3} \times 3 = 13 \quad a = 13\right)$

(b) $S_k = 125$

$\Rightarrow S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

$\Rightarrow 125 = \frac{n}{2}\left[2 \times 13 + (n-1)\left(-\frac{2}{3}\right)\right]$

$\Rightarrow 250 = n\left(26 - \frac{2}{3}n + \frac{2}{3}\right)$

$\Rightarrow 750 = 3n\left(26 - \frac{2}{3}n + \frac{2}{3}\right)$

$\Rightarrow 750 = n(78 - 2n + 2)$

$\Rightarrow 750 = n(80 - 2n)$

$\Rightarrow 375 = n(40 - n)$

BY INSPECTION IF $n = 15$

$15 \times (40 - 15) = 375$

$\therefore n = (k) = 15, 25$

Question 112 (****+)

A pension broker gets paid £15 commission **per week** for every pension scheme he sells. Each week he sells a new pension scheme so that …

In the 1st week he gets paid £15 commission for the pension he just sold.

In the 2nd week he gets paid £30, £15 for the pension sold in the 1st week plus £15 for pension he sold in the 2nd week.

In the 3rd week he gets paid £45, £15 for the pension sold in the 1st week plus £15 for pension he sold in the 2nd week, plus £15 for the pension he sold in the 3rd week, and so on.

a) Find the commission he gets paid on the last week of the year.

b) Find his annual earnings after one year in this job.

His commission increases to £20 for new pension schemes sold during the 2nd year but decreases to £10 for the schemes he sold in the 1st year.

The broker continues to sell at the rate of one new pension scheme every week.

c) Find his annual earnings in the 2nd year.

MP2-I , £780 , £20670 , £54600

a) LOOKING AT THE PATTERN

WEEK:	1	2	3	....	52
COMM.:	15	15 + 15 = 30	15 + 15 + 15 = 45		?

$a = 15$, $d = 15$, $n = 52$

$\Rightarrow u_n = a + (n-1)d$

$\Rightarrow u_{52} = 15 + 51 \times 15$

$\Rightarrow u_{52} = 780$ i.e. £780

b) SUMMING THE COMMISSIONS USING PART (a)

$S_n = \frac{n}{2}[a+L]$

$S_{52} = \frac{52}{2}[15+780] = 26 \times 795 = 20670$

i.e. £20670

c) CONTINUING THE PATTERN & LINKING WITH THE FIRST YEAR

WEEK	COMMISSION	
51	£765	FIRST YEAR
52	£780	
1	$(52 \times 10) + 20$	$a = 540$
2	$(52 \times 10) + 20 + 20$	SECOND YEAR
3	$(52 \times 10) + 20 + 20 + 20$	
⋮	⋮	
52	$(52 \times 10) + 20 \times 52$	$L = 1560$

SUMMING USING $S_n = \frac{n}{2}[a+L]$

$S_{52} = \frac{52}{2}[540 + 1560] = 54600$ i.e. £54600

Question 113 (****+)

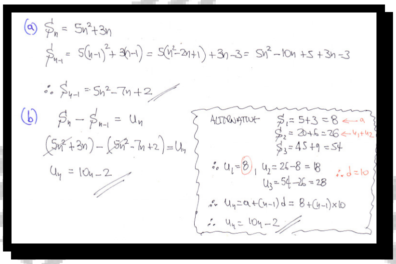
The sum, S_n, of the first n terms of an arithmetic series is given by

$$S_n = 5n^2 + 3n.$$

a) Find a simplified expression for S_{n-1}.

b) Hence, or otherwise, find a simplified expression for the n^{th} term of the series, denoted by u_n.

$\boxed{S_{n-1} = 5n^2 - 7n + 2}$, $\boxed{u_n = 10n - 2}$



Question 114 (****+)

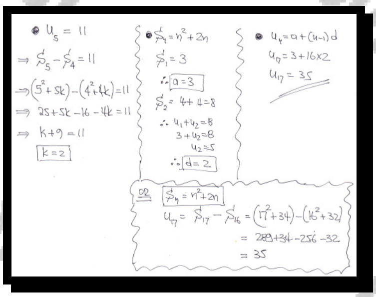
The sum, S_n, of the first n terms of an arithmetic series is given by

$$S_n = n^2 + kn,$$

where k is a non zero constant.

Given that the 5^{th} term of the series is 11, find the 17^{th} term of the series.

$\boxed{u_{17} = 35}$



Question 115 (****+)

An arithmetic progression has first term -10 and common difference 4.

The n^{th} term of the progression is denoted by u_n.

Determine the value of k given that

$$\sum_{n=1}^{2k} u_n - \sum_{n=1}^{k} u_n = 1728.$$

$\boxed{}$, $\boxed{k = 18}$

USING THE SUMMATION FORMULA FOR AN A.P. WITH $a = -10$, $d = 4$

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

$$\sum_{n=1}^{k} u_n = S_k = \frac{k}{2}\left[2(-10) + (k-1)\times 4\right] = \frac{k}{2}\left[-20 + 4k - 4\right] = k(2k-12)$$

$$\sum_{n=1}^{2k} u_n = S_{2k} = \frac{2k}{2}\left[2(-10) + (2k-1)\times 4\right] = k\left[-20 + 8k - 4\right] = k(8k-24)$$

THUS WE CAN WRITE

$$\sum_{n=1}^{2k} u_n - \sum_{n=1}^{k} u_n = 1728$$

$$k(8k-24) - k(2k-12) = 1728$$

$\div 2$

$$k(4k-12) - k(k-6) = 864$$

$$k(4k-12-k+6) = 864$$

$$k(3k-6) = 864$$

$\div 3$

$$k(k-2) = 288$$

BY INSPECTION AS WE ARE LOOKING FOR A POSITIVE INTEGER OR THE QUADRATIC FORMULA

$$k^2 - 2k - 288 = 0$$

$$(k+16)(k-18) = 0$$

$$k = 18$$

Question 116 (****+)

The r^{th} term of an arithmetic progression is given by $u_r = 120 - 3r$.

Determine the value of N given that

$$\sum_{r=N}^{3N} u_r = 444 .$$

$\boxed{}$, $\boxed{N = 18}$

PROCEED AS FOLLOWS NOTING $u_r = 120 - 3r$, $a = 117$, $d = -3$

$$\Rightarrow \sum_{r=N}^{3N} u_r = 444$$

$$\Rightarrow \sum_{r=1}^{3N} u_r - \sum_{r=1}^{N-1} u_r = 444$$

$$\Rightarrow \frac{3N}{2}\left[2 \times 117 + (3N-1)(-3)\right] - \frac{N-1}{2}\left[2 \times 117 + (N-2)(-3)\right] = 444$$

TIDYING UP YIELDS

$$\Rightarrow \frac{3N}{2}(234 - 9N + 3) - \frac{1}{2}(N-1)(234 - 3N + 6) = 444$$

$$\Rightarrow \frac{3N}{2}(237 - 9N) - \frac{1}{2}(N-1)(240 - 3N) = 444$$

$$\Rightarrow 3N(237 - 9N) - (N-1)(240 - 3N) = 888$$

$$\Rightarrow 3N(237 - 9N) + (N-1)(3N - 240) = 888$$

DIVIDE THROUGH BY 3

$$\Rightarrow N(237 - 9N) + (N-1)(N-80) = 296$$

$$\Rightarrow 237N - 9N^2 + N^2 - 80N - N + 80 = 296$$

$$\Rightarrow 0 = 8N^2 - 156N + 216$$

$$\Rightarrow 2N^2 - 39N + 54 = 0$$

$$\Rightarrow (N - 18)(2N - 3) = 0$$

$$\Rightarrow N = \begin{cases} 18 \\ 3/2 \end{cases}$$

$\therefore N = 18$

Question 117 (****+)

Find in simplified form, in terms of n, the value of

$$\sum_{r=1}^{2n}\left[(3r-2)(-1)^r\right].$$

$\boxed{}$, $\boxed{3n}$

- $\displaystyle\sum_{r=1}^{2n}\left[(-1)^r(3r-2)\right] = -1+4-7+10-13+16-\dots-(3(2n-1)-2)+(6n-2)$

  $= -1+4-7+10-13+16-\dots-(6n-5)+(6n-2)$

- NOW PROCEED BY GROUPING

  $= \underbrace{(-1+4)+(-7+10)+(-13+16)+\dots+(-6n+5+6n-2)}_{n \text{ terms}}$

  $= 3n$

ALTERNATIVE/REGROUP AS FOLLOWS

$= \underbrace{\left[4+10+16+\dots+(6n-2)\right]}_{n \text{ terms}} - \underbrace{\left[1+7+13+\dots+6n-5\right]}_{n \text{ terms}}$

$a=4$, $d=3$, $L=6n-2$; $a=1$, $d=3$, $L=6n-5$

$= \frac{n}{2}\left[4+6n-2\right] - \frac{n}{2}\left[1+6n-5\right]$

$= \frac{n}{2}(6n+2) - \frac{n}{2}(6n-4)$

$= \cancel{3n^2}+n-\cancel{3n^2}+2n$

$= 3n$

AS BEFORE

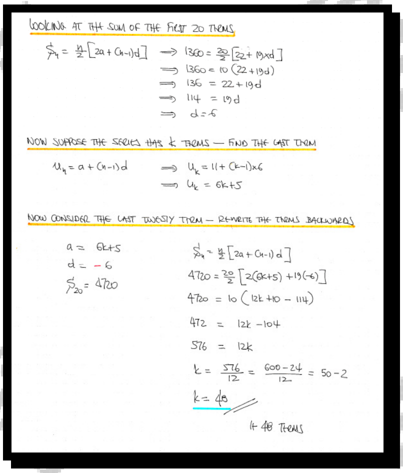
Question 118 (*****)

An arithmetic progression has first term 11.

The sum of its **first** 20 terms is 1360, and the sum of its **last** 20 terms is 4720.

Determine the number of terms in the progression.

$\boxed{\text{SYN}}$, $\boxed{48}$



LOOKING AT THE SUM OF THE FIRST 20 TERMS

$S_n = \frac{n}{2}[2a + (n-1)d] \implies 1360 = \frac{20}{2}[22 + 19 \times d]$

$\implies 1360 = 10(22 + 19d)$

$\implies 136 = 22 + 19d$

$\implies 114 = 19d$

$\implies d = 6$

NOW SUPPOSE THE SERIES HAS k TERMS — FIND THE LAST TERM

$u_n = a + (n-1)d \implies u_k = 11 + (k-1) \times 6$

$\implies u_k = 6k + 5$

NOW CONSIDER THE LAST TWENTY TERM — REWRITE THE TERMS BACKWARDS

$a = 6k + 5$

$d = -6$

$S_{20} = 4720$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$4720 = \frac{20}{2}[2(6k+5) + 19(-6)]$

$4720 = 10(12k + 10 - 114)$

$472 = 12k - 104$

$576 = 12k$

$k = \frac{576}{12} = \frac{600 - 24}{12} = 50 - 2$

$k = 48$

i.e. 48 TERMS

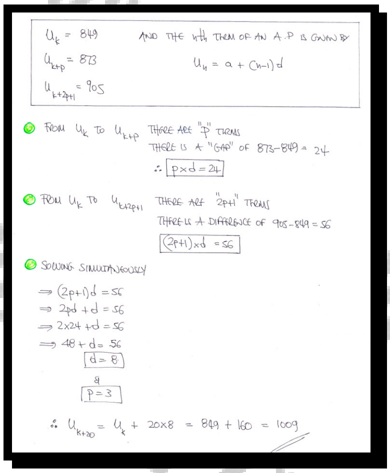
Question 119 (*****)

The k^{th} of an arithmetic progression is 849, where k is a positive integer.

The $(k+p)^{\text{th}}$ term and the $(k+2p+1)^{\text{th}}$ term of the same arithmetic progression are 873 and 905 respectively, where p is a positive integer.

Find the value of the $(k+20)^{\text{th}}$ term of the progression.

$\boxed{}$, $\boxed{(k+20)^{\text{th}} = 1009}$



$u_k = 849$ AND THE nth TERM OF AN A.P IS GIVEN BY
$u_{k+p} = 873$ $u_n = a + (n-1)d$
$u_{k+2p+1} = 905$

FROM u_k TO u_{k+p} THERE ARE "p" TERMS
THERE IS A "GAP" OF $873 - 849 = 24$
$\therefore \boxed{p \times d = 24}$

FROM u_k TO u_{k+2p+1} THERE ARE "2p+1" TERMS
THERE IS A DIFFERENCE OF $905 - 849 = 56$
$\boxed{(2p+1) \times d = 56}$

SOLVING SIMULTANEOUSLY

$\Rightarrow (2p+1)d = 56$
$\Rightarrow 2pd + d = 56$
$\Rightarrow 2 \times 24 + d = 56$
$\Rightarrow 48 + d = 56$
$\boxed{d = 8}$
&
$\boxed{p = 3}$

$\therefore u_{k+20} = u_k + 20 \times 8 = 849 + 160 = 1009$

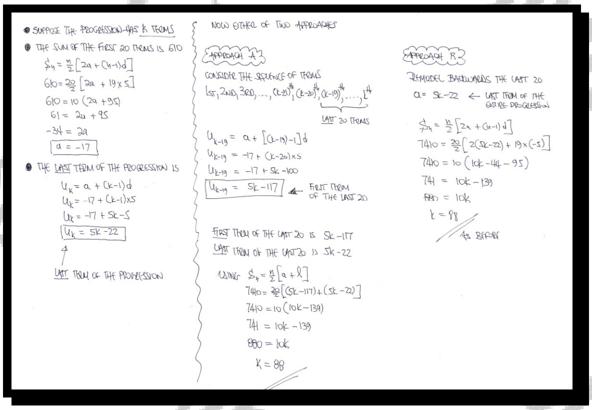
Question 120 (*****)

An arithmetic progression has common difference 5.

The sum of its **first** 20 terms is 610, and the sum of its **last** 20 terms is 7410.

Determine the number of terms in the progression.

$\boxed{}$, $\boxed{88}$

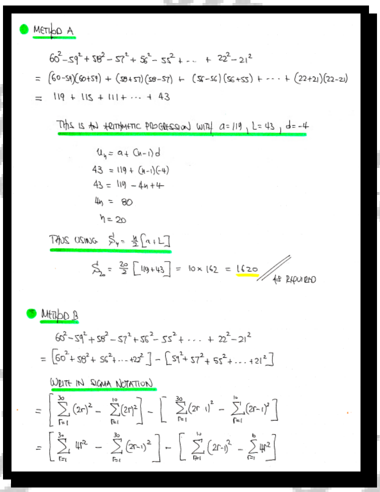


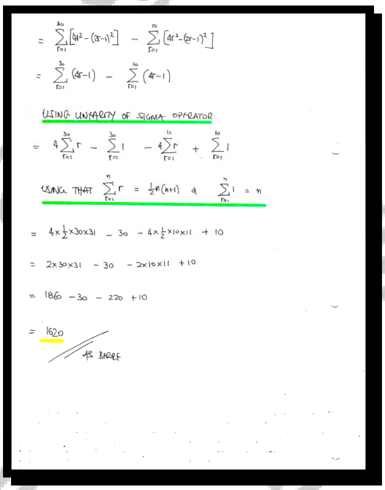
Question 121 (*****)

Show by a suitable algebraic method that

$$60^2 - 59^2 + 58^2 - 57^2 + \ldots + 22^2 - 21^2 = 1620.$$

$\boxed{}$, $\boxed{\text{proof}}$





Question 122 (*****) non calculator

The sum of the first k terms of an arithmetic progression is 110.

The sum of the first $2k$ terms of the same arithmetic progression is 946.

Given further that $k \neq 1$, determine the first term and the common difference of the arithmetic progression.

$\boxed{}$, $\boxed{a=-20}$, $\boxed{d=6}$

USING THE SUMMATION FORMULA FOR AN ARITHMETIC SERIES

$$\boxed{S_n = \frac{n}{2}\left[2a+(n-1)d\right]}$$

AND FORMING TWO EQUATIONS

$$\left.\begin{array}{l} S_k = 110 \\ S_{2k} = 946 \end{array}\right\} \Rightarrow \left.\begin{array}{l} \frac{k}{2}\left[2a+(k-1)d\right] = 110 \\ \frac{2k}{2}\left[2a+(2k-1)d\right] = 946 \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l} \frac{k}{2}\left[2a+kd-d\right] = 110 \\ k\left[2a+2kd-d\right] = 946 \end{array}\right\} \Rightarrow \begin{array}{l} 2a+kd-d = \frac{220}{k} \\ 2a+2kd-d = \frac{946}{k} \end{array}$$

SUBTRACTING THE EQUATIONS WE OBTAIN

$$kd = \frac{726}{k}$$

$$\boxed{k^2 d = 726}$$

NOW k IS A POSITIVE INTEGER $\Rightarrow$ k^2 IS ALSO A POSITIVE INTEGER

$\Rightarrow$ d IS ALSO A POSITIVE INTEGER

$d = \frac{726}{k^2}$ WITH $k = 1, 2, 3, 4, \dots$ OR $k^2 = 1, 4, 9, \dots, 25, \dots$

726 IS DIVISIBLE BY 2 & BY 3

$$\frac{726}{6} = \frac{600+120+6}{6} = 100+20+1 = 121$$

$$726 = \underbrace{2\times3}\times\underbrace{11\times11}$$

$\therefore$ $k^2 = 121$ & $d = 6$

AND SINCE $k=11$ $d=6$

$$2a + kd - d = \frac{220}{k}$$

$$2a + 66 - 6 = \frac{220}{11}$$

$$2a + 60 = 20$$

$$2a = -40$$

$$a = -20$$

Question 123 (*****) non calculator

An arithmetic series has an even number of terms.

The sum of its odd numbered terms, $u_1 + u_3 + u_5 + u_7 + \ldots$, is 752.

The sum of its even numbered terms, $u_2 + u_4 + u_6 + u_8 + \ldots$, is 800.

Given further that the difference between the last and the first term of the series is 93, use an algebraic method to find the number of terms of the series.

$\boxed{}$, $\boxed{N = 32}$

LET THE SERIES HAVE N TERMS, WHERE $N = 2n$

$$u_1 + u_3 + u_5 + \ldots + u_{2n-1} = 752$$
$$u_2 + u_4 + u_6 + \ldots + u_{2n} = 800$$

SUBTRACTING "UPWARDS", I.E. $u_2 - u_1$, $u_4 - u_3$, $\ldots$ $u_{2n} - u_{2n-1}$ GIVES THE COMMON DIFFERENCE d, OF THE SERIES

$$\Rightarrow (u_2 - u_1) + (u_4 - u_3) + \ldots + (u_{2n} - u_{2n-1}) = 800 - 752$$
$$\Rightarrow d \times \frac{N}{2} = 48$$
$$\Rightarrow \boxed{dN = 96}$$

ALSO "LAST TERM" – "FIRST TERM" IS 93

$$\Rightarrow [a + (N-1)d] - a = 93$$
$$\Rightarrow \boxed{(N-1)d = 93}$$

SOLVING THE EQUATIONS

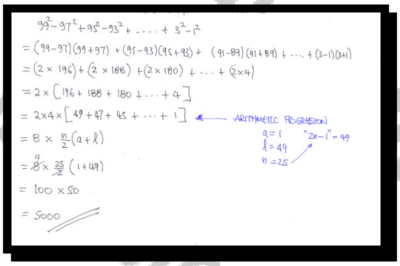
$$\left.\begin{array}{r} dN = 96 \\ dN - d = 93 \end{array}\right\} \Rightarrow 96 - d = 93$$
$$\boxed{d = 3}$$
$$\Rightarrow dN = 96$$
$$\Rightarrow 3N = 96$$
$$\Rightarrow N = 32$$

Question 124 (*****)

By considering a suitable arithmetic series, evaluate

$$99^2 - 97^2 + 95^2 - 93^2 + \ldots + 3^2 - 1^2$$

$\boxed{5000}$



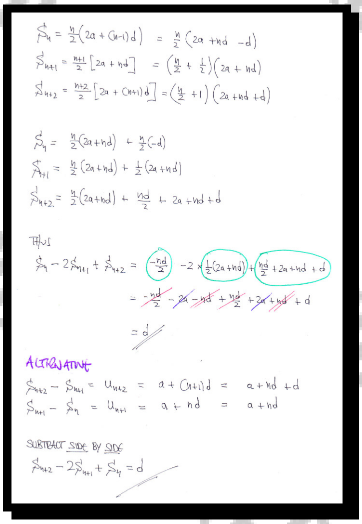
Question 125 (*****)

The sum of the first n terms of an arithmetic series with first term a and common difference d, is denoted by S_n.

Simplify fully

$$S_n - 2S_{n+1} + S_{n+2}.$$

$\boxed{}$, $\boxed{S_n - 2S_{n+1} + S_{n+2} = d}$



Question 126 (*****)

The r^{th} term of an arithmetic progression is denoted by u_r and satisfies

$$u_r = 4r - 7 .$$

Solve the simultaneous equations

$$\sum_{r=K+1}^{N} u_r - \sum_{r=1}^{K} u_r = 400$$

$$u_N - u_K = 40 .$$

$\boxed{}$, $\boxed{N = 20}$, $\boxed{K = 10}$

- $u_r = 4r - 7$ YIELDS $-3, 1, 5, 9, 13, \ldots$ so $\begin{cases} a = -3 \\ d = 4 \end{cases}$

- $u_N - u_K = 40 \implies \left[-3 + (N-1)\times 4\right] - \left[-3 + (k-1)\times 4\right] = 40$

$\implies (\cancel{-3} + 4N \cancel{-4}) - (\cancel{-3} + 4k \cancel{-4}) = 40$

$\implies 4N - 4k = 40$

$\implies N - k = 10$

$\implies \boxed{N = k + 10}$

- $\displaystyle\sum_{r=k+1}^{N} u_r - \sum_{r=1}^{k} u_r = 400$

$\implies \left[\displaystyle\sum_{r=1}^{N} u_r - \sum_{r=1}^{k} u_r\right] - \sum_{r=1}^{k} u_r = 400$

$\implies S_N - 2S_k = 400$

$\implies \frac{N}{2}\left[2(-3) + (N-1)\times 4\right] - 2\times\frac{k}{2}\left[2(-3) + (k-1)\times 4\right] = 400$

$\implies \frac{N}{2}\left[-6 + 4N - 4\right] - k\left[-6 + 4k - 4\right] = 400$

$\implies \frac{N}{2}\left[4N - 10\right] - k\left[4k - 10\right] = 400$

$\implies 2N^2 - 5N - 4k^2 + 10k = 400$

$\implies 2(k+10)^2 - 5(k+10) - 4k^2 + 10k = 400$

$\implies 2k^2 + 40k + 200 - 5k - 50 - 4k^2 + 10k = 400$

$\implies -2k^2 + 45k - 250 = 0$

$\implies 2k^2 - 45k + 250 = 0$

$\implies (2k - 25)(k - 10) = 0$

$k = \cancel{\frac{25}{2}}$ or 10 $\implies N = 20$

Question 127 (*****)

The coefficients of x^n, x^{n+1} and x^{n+2} in the binomial expansion of $(1+x)^{23}$ are in arithmetic progression.

Determine the possible values of n.

$\boxed{n = 8,\ 13}$

- $\binom{23}{n}$, $\binom{23}{n+1}$ & $\binom{23}{n+2}$ ARE IN ARITHMETIC PROGRESSION

$$\Rightarrow \binom{23}{n+2} - \binom{23}{n+1} = \binom{23}{n+1} - \binom{23}{n}$$

$$\Rightarrow \binom{23}{n+2} + \binom{23}{n} = 2\binom{23}{n+1}$$

$$\Rightarrow \frac{\cancel{23!}}{(n+2)!\,(23-n-2)!} + \frac{\cancel{23!}}{n!\,(23-n)!} = 2 \times \frac{\cancel{23!}}{(n+1)!\,(23-n-1)!}$$

$$\Rightarrow \frac{1}{(n+2)!\,(21-n)!} + \frac{1}{n!\,(23-n)!} = \frac{2}{(n+1)!\,(22-n)!}$$

- MULTIPLY THE EQUATION THROUGH BY $(n+2)!$

$$\Rightarrow \frac{1}{(21-n)!} + \frac{(n+2)(n+1)}{(23-n)!} = \frac{2(n+2)}{(22-n)!}$$

- MANIPULATE THE FACTORIALS FURTHER

$$\Rightarrow \frac{1}{\cancel{(21-n)!}} + \frac{(n+2)(n+1)}{(23-n)(22-n)\cancel{(21-n)!}} = \frac{2(n+2)}{(22-n)\cancel{(21-n)!}}$$

$$\Rightarrow 1 + \frac{(n+2)(n+1)}{(23-n)(22-n)} = \frac{2(n+2)}{22-n}$$

$$\Rightarrow (23-n)(22-n) + (n+2)(n+1) = 2(n+2)(23-n)$$

$$\Rightarrow (n-22)(n-23) + (n+2)(n+1) + 2(n+2)(n-23) = 0$$

$$\Rightarrow n^2 - 45n + 22\times 23 + n^2 + 3n + 2 + 2n^2 - 42n - 92 = 0$$

$$\left.\begin{matrix}230\\230\\46\end{matrix}\right\} = 506$$

$$\Rightarrow \left\{\begin{matrix} n^2 - 45n + 506 \\ n^2 + 3n + 2 \\ 2n^2 - 42n - 92 \end{matrix}\right. = 0$$

$$\Rightarrow 4n^2 - 84n + 416 = 0$$

$$\Rightarrow n^2 - 21n + 104 = 0 \qquad \left\{\begin{matrix} 2\times 52 \\ 4\times 26 \\ 8\times 13 \end{matrix}\right\}$$

$$\Rightarrow (n-8)(n-13) = 0$$

$$\Rightarrow n = \begin{cases} 8 \\ 13 \end{cases}$$

Question 128 (*****)

The sum of the first n terms of an arithmetic series is m, $m \in \mathbb{N}$.

The sum of the first m terms of the same arithmetic series is n.

Use algebra to show that the sum of the first $(m+n)$ terms of the series is $-m-n$.

$\boxed{}$, $\boxed{\text{proof}}$

START BY FINDING THE FIRST TERM & COMMON DIFFERENCE

$$\sum_{i=1}^{n} u_i = m \qquad \text{AND} \qquad \sum_{i=1}^{m} u_i = n$$

$$\frac{n}{2}\left[2a + (n-1)d\right] = m \qquad \qquad \frac{m}{2}\left[2a + (m-1)d\right] = n$$

$$2a + (n-1)d = \frac{2m}{n} \qquad \qquad 2a + (m-1)d = \frac{2n}{m}$$

SUBTRACT

$$\Rightarrow (n-1)d - (m-1)d = \frac{2m}{n} - \frac{2n}{m}$$

$$\Rightarrow (n-m)d = 2\left(\frac{m}{n} - \frac{n}{m}\right)$$

$$\Rightarrow (n-m)d = 2\left(\frac{m^2-n^2}{mn}\right)$$

$$\Rightarrow (n-m)d = \frac{2(m-n)(m+n)}{mn}$$

$$\Rightarrow \cancel{(n-m)}d = \frac{-2(m+n)\cancel{(n-m)}}{mn} \qquad n \neq m$$

$$\Rightarrow d = -\frac{2(m+n)}{mn}$$

TO FIND THE FIRST TERM WE HAVE

$$\Rightarrow 2a + (n-1)d = \frac{2m}{n}$$

$$\Rightarrow 2a + (n-1)\left[\frac{-2(m+n)}{mn}\right] = \frac{2m}{n}$$

$$\Rightarrow a = \frac{m}{n} + \frac{(n-1)(m+n)}{mn}$$

$$\Rightarrow a = \frac{m^2 + (n-1)(m+n)}{mn}$$

NOW WE REQUIRE THE SUM OF THE FIRST $m+n$ TERMS

$$\Rightarrow S_{n+m} = \frac{m+n}{2}\left[2 \times \frac{m^2+(n-1)(m+n)}{mn} + (m+n-1)\left[-\frac{2(m+n)}{mn}\right]\right]$$

$$\Rightarrow S_{n+m} = (m+n)\left[\frac{m^2+(n-1)(m+n)}{mn} - \frac{(m+n)(m+n-1)}{mn}\right]$$

$$\Rightarrow S_{m+n} = (m+n)\left[\frac{m^2}{mn} + \frac{(m+n)(n-1)}{mn} - \frac{(m+n)(m+n-1)}{mn}\right]$$

$$\Rightarrow S_{m+n} = (m+n)\left[\frac{m^2}{mn} + \frac{m+n}{mn}\left[\cancel{n}\cancel{-1} - m - \cancel{n}\cancel{+1}\right]\right]$$

$$\Rightarrow S_{m+n} = (m+n)\left[\frac{m^2}{mn} + \frac{(m+n)(-m)}{mn}\right]$$

$$\Rightarrow S_{m+n} = (m+n)\left[\frac{\cancel{m^2} - \cancel{m^2} - nm}{mn}\right]$$

$$\Rightarrow S_{m+n} = (m+n)(-1)$$

$$\Rightarrow S_{m+n} = -m-n$$