CIRCLE COORDINATE GEOMETRY

(EXAM QUESTIONS)

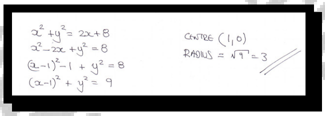
Question 1 (**)

A circle has equation

$$x^2 + y^2 = 2x + 8$$

Determine the radius and the coordinates of the centre of the circle.

$\boxed{r = 3}$, $\boxed{(1,0)}$



$x^2 + y^2 = 2x + 8$
$x^2 - 2x + y^2 = 8$
$(x-1)^2 - 1 + y^2 = 8$
$(x-1)^2 + y^2 = 9$

CENTRE $(1,0)$
RADIUS $= \sqrt{9} = 3$

Question 2 (**)

A circle C has equation

$$x^2 + y^2 - 12x + 2y + 24 = 0.$$

a) Find the coordinates of the centre of the circle and the length of its radius.

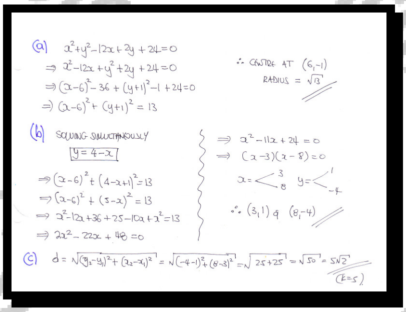
The straight line L has equation

$$x + y = 4.$$

b) Determine the coordinates of the points of intersection between C and L.

c) Show that the distance between these points of intersection is $k\sqrt{2}$, where k is an integer.

$\boxed{(6,-1),\ r = \sqrt{13}}$, $\boxed{(8,-4),\ (3,1)}$, $\boxed{k = 5}$



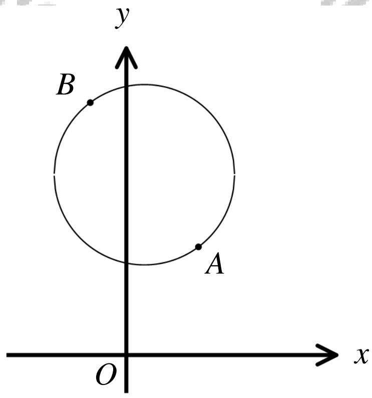
(a) $x^2 + y^2 - 12x + 2y + 24 = 0$
$\Rightarrow x^2 - 12x + y^2 + 2y + 24 = 0$
$\Rightarrow (x-6)^2 - 36 + (y+1)^2 - 1 + 24 = 0$
$\Rightarrow (x-6)^2 + (y+1)^2 = 13$

$\therefore$ CENTRE AT $(6,-1)$
RADIUS $= \sqrt{13}$

(b) SOLVING SIMULTANEOUSLY
$\boxed{y = 4 - x}$
$\Rightarrow (x-6)^2 + (4-x+1)^2 = 13$
$\Rightarrow (x-6)^2 + (5-x)^2 = 13$
$\Rightarrow x^2 - 12x + 36 + 25 - 10x + x^2 = 13$
$\Rightarrow 2x^2 - 22x + 48 = 0$
$\Rightarrow x^2 - 11x + 24 = 0$
$\Rightarrow (x-3)(x-8) = 0$
$x = 3, 8 \quad y = 1, -4$
$\therefore (3,1)$ & $(8,-4)$

(c) $d = \sqrt{(y_2-y_1)^2 + (x_2-x_1)^2} = \sqrt{(-4-1)^2 + (8-3)^2} = \sqrt{25+25} = \sqrt{50} = 5\sqrt{2}$
$(k=5)$

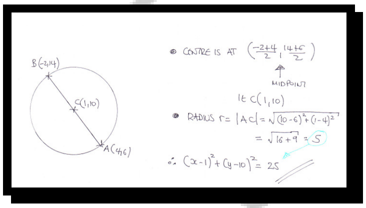
Question 3 **(**)**



The figure above shows the points $A(4,6)$ and $B(-2,14)$, which both lie on the circumference of a circle.

Given that AB is a diameter of the circle, determine an equation for the circle.

C2B , $(x-1)^2+(y-10)^2=25$



Question 4 (**)

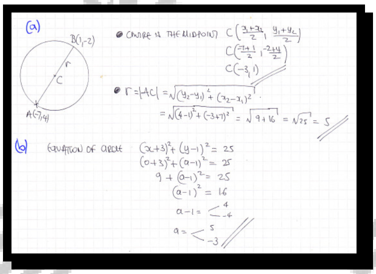
The straight line segment joining the points $A(-7,4)$ and $B(1,-2)$ is a diameter of a circle with centre at the point C and radius r.

a) Find the coordinates of C and the value of r.

The point $(0,a)$ lies on the circumference of this circle.

b) Determine the possible values of a.

$\boxed{}$, $\boxed{C(-3,1)}$, $\boxed{r=5}$, $\boxed{a=-3,\ 5}$



Question 5 (**)

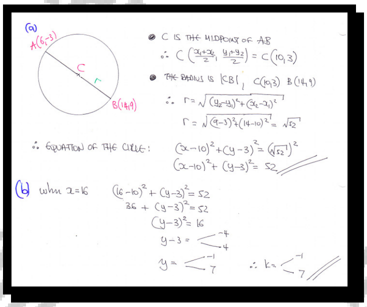
The straight line joining the points $A(6,-3)$ and $B(14,9)$ is a diameter of a circle.

a) Determine an equation for the circle.

The point $(16,k)$ lies on the circumference of the circle.

b) Find the possible values of k.

$\boxed{(x-10)^2+(y-3)^2=52}$, $\boxed{k=-1 \cup k=7}$



Question 6 (**)

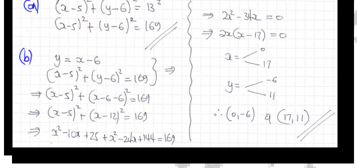
A circle is centred at $(5,6)$ and has radius 13.

a) Find an equation for this circle.

The straight line l with equation $y=x-6$ intersects the circle at the points A and B.

b) Determine the coordinates of A and the coordinates of B.

$\boxed{(x-5)^2+(y-6)^2=169}$, $\boxed{(17,11),\ (0,-6)}$



Question 7 (**+)

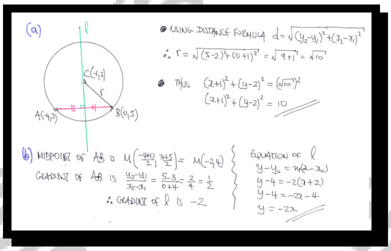
A circle has its centre at $C(-1,2)$.

The points $A(-4,3)$ and $B(0,5)$ lie on this circle.

a) Find an equation for the circle.

b) Determine an equation of the straight line which passes through C and bisects the chord AB.

$\boxed{}$, $\boxed{(x+1)^2+(y-2)^2=10}$, $\boxed{y=-2x}$



Question 8 (**+)

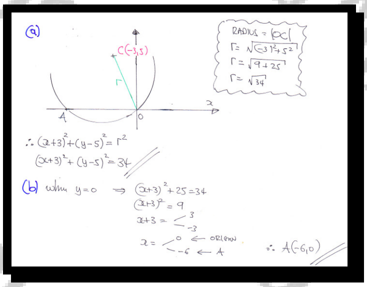
A circle has centre $C(-3,5)$ and passes through the origin.

a) Find an equation for this circle.

The circle crosses the x axis at the origin and at the point A.

b) Determine the coordinates of A.

$\boxed{(x+3)^2+(y-5)^2=34}$, $\boxed{A(-6,0)}$



Question 9 **(**+)**

A circle has equation

$$x^2 + y^2 - 20x + 8y + 16 = 0.$$

The centre of the circle is at C and its radius is r.

a) Determine …

i. … the coordinates of C.

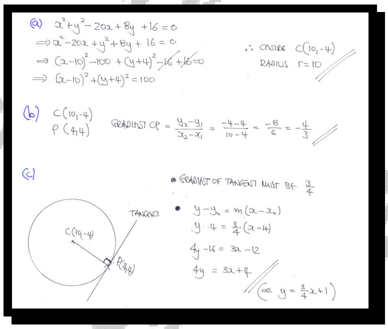
ii. … the length of r.

The point $P(4,4)$, lies on this circle.

b) Find the gradient of CP.

c) Hence find an equation of the tangent to the circle at P.

$\boxed{}$, $\boxed{C(10,-4)}$, $\boxed{r=10}$, $\boxed{-\frac{4}{3}}$, $\boxed{y=\frac{3}{4}x+1}$



Question 10 (**+)

A circle C has equation

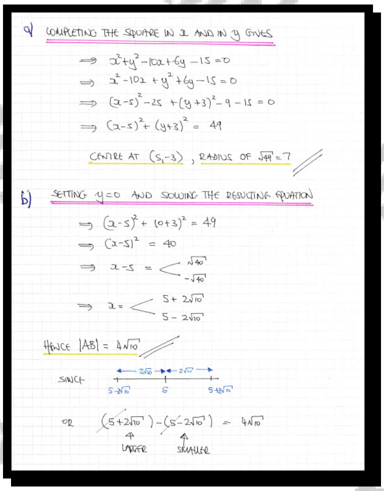
$$x^2 + y^2 - 10x + 6y - 15 = 0$$

a) Find the coordinates of the centre of C and determine the size of its radius.

The circle intersects the x axis at the points A and B.

b) Find, in exact surd form, the x coordinate of A and the x coordinate of B and hence state the distance AB.

$\boxed{}$, $\boxed{(5,-3)}$, $\boxed{r=7}$, $\boxed{x = 5 \pm 2\sqrt{10}}$, $\boxed{|AB| = 4\sqrt{10}}$


a) COMPLETING THE SQUARE IN x AND IN y GIVES

$\Rightarrow x^2+y^2-10x+6y-15=0$

$\Rightarrow x^2-10x+y^2+6y-15=0$

$\Rightarrow (x-5)^2-25+(y+3)^2-9-15=0$

$\Rightarrow (x-5)^2+(y+3)^2=49$

CENTRE AT $(5,-3)$, RADIUS OF $\sqrt{49}=7$

b) SETTING $y=0$ AND SOLVING THE RESULTING EQUATION

$\Rightarrow (x-5)^2+(0+3)^2=49$

$\Rightarrow (x-5)^2=40$

$\Rightarrow x-5 = \pm\sqrt{40}$

$\Rightarrow x = 5+2\sqrt{10}$ or $5-2\sqrt{10}$

HENCE $|AB| = 4\sqrt{10}$

SINCE $5-2\sqrt{10}$, 5, $5+2\sqrt{10}$ ($2\sqrt{10}$, $2\sqrt{10}$)

OR $(5+2\sqrt{10})-(5-2\sqrt{10}) = 4\sqrt{10}$
LARGER, SMALLER

Question 11 (**+)

A circle C has equation

$$x^2 + y^2 = 8x + 4y .$$

a) Determine the coordinates of the centre of C and the size of its radius.

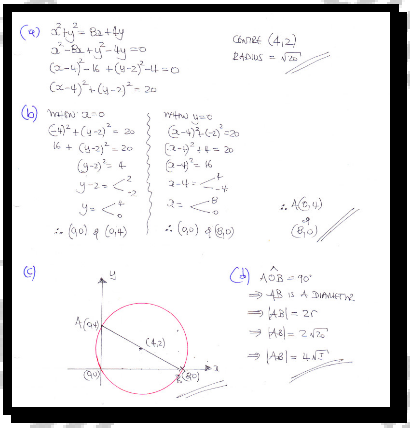
The circle meets the coordinate axes at the origin O and at two more points A and B.

b) Find the coordinates of A and B.

c) Sketch the graph of C.

d) **State** with justification but **without** any further calculations the length of AB.

$\boxed{}$, $\boxed{(4,2),\ r = \sqrt{20} = 2\sqrt{5}}$, $\boxed{(8,0),(0,4)}$, $\boxed{|AB| = 4\sqrt{5}}$



Question 12 (***)

A circle whose centre is at $(3,-5)$ has equation

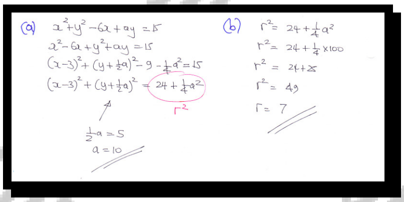
$$x^2 + y^2 - 6x + ay = 15,$$

where a is a constant.

a) Find the value of a.

b) Determine the radius of the circle.

C2, $\boxed{a=10}$, $\boxed{r=7}$


(a) $x^2+y^2-6x+ay=15$
$x^2-6x+y^2+ay=15$
$(x-3)^2+(y+\frac{1}{2}a)^2-9-\frac{1}{4}a^2=15$
$(x-3)^2+(y+\frac{1}{2}a)^2=24+\frac{1}{4}a^2$
r^2
$\frac{1}{2}a=5$
$a=10$

(b) $r^2=24+\frac{1}{4}a^2$
$r^2=24+\frac{1}{4}\times 100$
$r^2=24+25$
$r^2=49$
$r=7$

Question 13 (***)

The endpoints of a diameter of a circle are located at $A(-7,4)$ and $B(1,-2)$.

a) Find an equation for the circle.

The straight line with equation

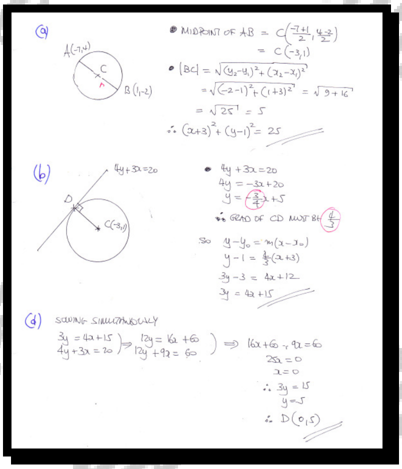
$$4y + 3x = 20$$

is a tangent to the circle at the point D.

b) Find an equation for the straight line CD, where C is the centre of the circle.

c) Determine the coordinates of D.

$\boxed{}$, $\boxed{(x+3)^2 + (y-1)^2 = 25}$, $\boxed{3y = 4x + 15}$, $\boxed{D(0,5)}$



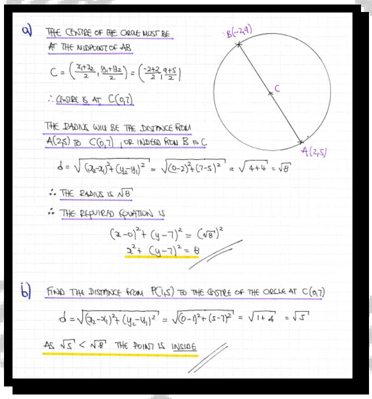
Question 14 (***)

The straight line joining the points $A(2,5)$ and $B(-2,9)$ is a diameter of a circle.

a) Find an equation for this circle.

b) Determine by calculation whether the point $P(1,5)$ lies inside or outside the above mentioned circle.

$x^2+(y-7)^2=8$



Question 15 (***)

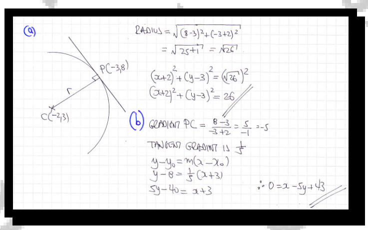
A circle has its centre at the point $C(-2,3)$ and passes through the point $P(-3,8)$.

a) Find an equation for this circle.

b) Show that an equation of the tangent to the circle at P is

$$x-5y+43=0.$$

$(x+2)^2+(y-3)^2=26$



Question 16 **(***)**

A circle C has equation

$$x^2 + y^2 - 6x - 10y + k = 0,$$

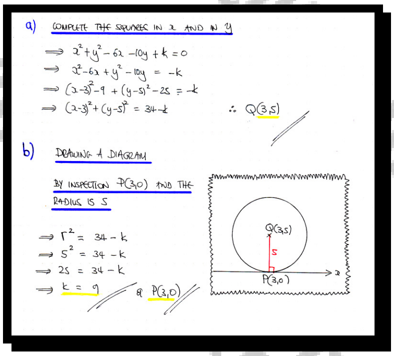
where k is a constant.

a) Determine the coordinates of the centre of C.

The x axis is a tangent to C at the point P.

b) State the coordinates of P and find the value of k.

$\boxed{}$, $\boxed{(3,5)}$, $\boxed{(3,0)}$, $\boxed{k=9}$



Question 17 (***)

$$x^2 + y^2 - 2x - 2y = 8$$

The circle with the above equation has radius r and has its centre at the point C.

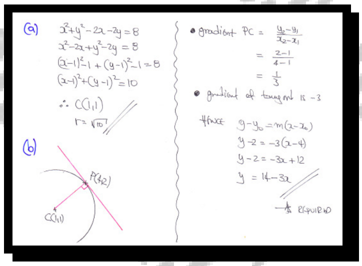
a) Determine the value of r and the coordinates of C.

The point $P(4,2)$ lies on the circle.

b) Show that an equation of the tangent to the circle at P is

$$y = 14 - 3x$$

$\boxed{}$, $\boxed{r = \sqrt{10}}$, $\boxed{C(1,1)}$



Question 18 (***)

$$x^2 + y^2 - 10x + 4y + 9 = 0$$

The circle with the above equation has radius r and has its centre at the point C.

a) Determine the value of r and the coordinates of C.

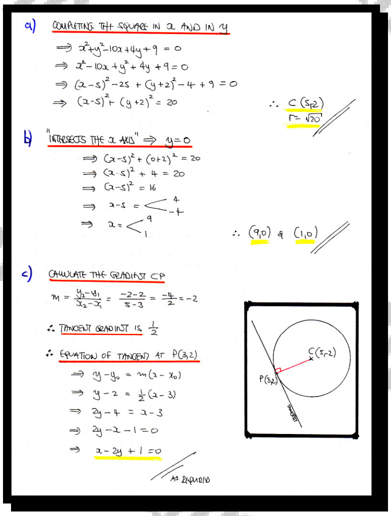
b) Find the coordinates of the points where the circle intersects the x axis.

The point $P(3,2)$ lies on the circle.

c) Show that an equation of the tangent to the circle at P is

$$x - 2y + 1 = 0.$$

$\boxed{}$, $\boxed{r = \sqrt{20}}$, $\boxed{C(5,-2)}$, $\boxed{(1,0),\ (9,0)}$


a) COMPLETING THE SQUARE IN x AND IN y

$\Rightarrow x^2+y^2-10x+4y+9=0$

$\Rightarrow x^2-10x+y^2+4y+9=0$

$\Rightarrow (x-5)^2-25+(y+2)^2-4+9=0$

$\Rightarrow (x-5)^2+(y+2)^2=20$

$\therefore C(5,-2)$, $r=\sqrt{20}$

b) "INTERSECTS THE x AXIS" $\Rightarrow y=0$

$\Rightarrow (x-5)^2+(0+2)^2=20$

$\Rightarrow (x-5)^2+4=20$

$\Rightarrow (x-5)^2=16$

$\Rightarrow x-5 = 4$ or -4

$\Rightarrow x = 9$ or 1

$\therefore (9,0)$ & $(1,0)$

c) CALCULATE THE GRADIENT CP

$m = \frac{y_2-y_1}{x_2-x_1} = \frac{-2-2}{5-3} = \frac{-4}{2} = -2$

$\therefore$ TANGENT GRADIENT IS $\frac{1}{2}$

$\therefore$ EQUATION OF TANGENT AT $P(3,2)$

$\Rightarrow y-y_0 = m(x-x_0)$

$\Rightarrow y-2 = \frac{1}{2}(x-3)$

$\Rightarrow 2y-4 = x-3$

$\Rightarrow 2y-x-1=0$

$\Rightarrow x-2y+1=0$

AS REQUIRED

Question 19 (***)

The points A and B have coordinates $(3,-1)$ and $(9,7)$, respectively.

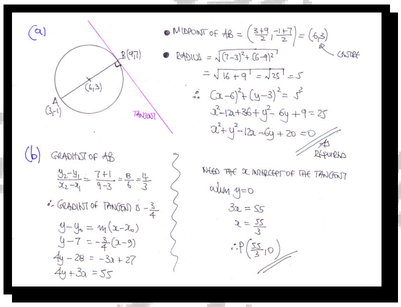
a) Show that the equation of the circle whose diameter is AB can be written as

$$x^2 + y^2 - 12x - 6y + 20 = 0 .$$

The tangent to the circle at B meets the x axis at the point P.

b) Determine the exact coordinates of P.

$$\boxed{P\left(\frac{55}{3},0\right)}$$



Question 20 (***)

A circle has equation

$$x^2 + y^2 + ax + by = 0,$$

where a and b are constants.

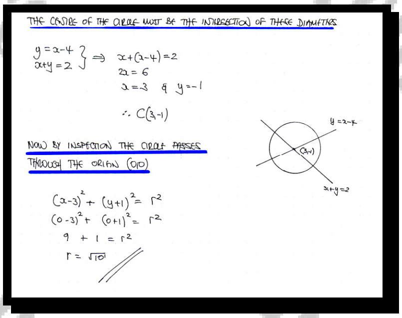
The straight lines with equations

$$y = x - 4 \quad \text{and} \quad x + y = 2$$

are both diameters of this circle.

Determine the length of the radius of the circle.

$r = \sqrt{10}$



Question 21 (***)

A circle C has its centre at the point with coordinates $(5,4)$ and its radius is $3\sqrt{2}$.

a) Find an equation for C.

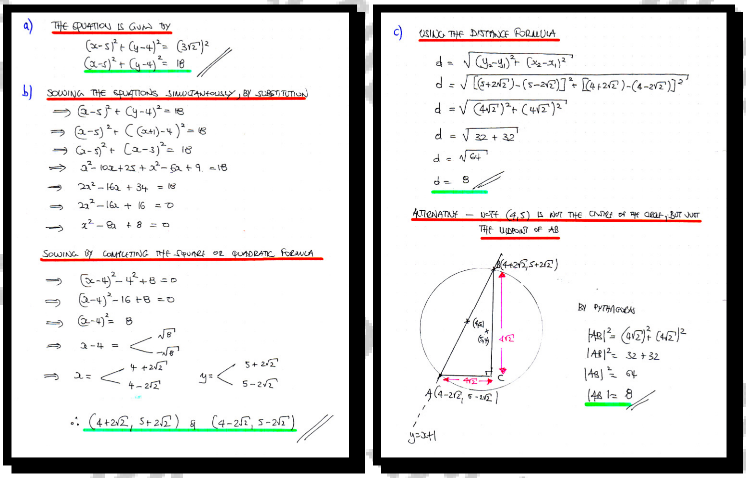
The straight line L has equation

$$y = x + 1.$$

b) Determine, as exact surds, the coordinates of the points of intersection between C and L.

c) Show that the distance between these points of intersection is 8 units.

$(x-5)^2 + (y-4)^2 = 18$, $\left(4+2\sqrt{2}, 5+2\sqrt{2}\right)$, $\left(4-2\sqrt{2}, 5-2\sqrt{2}\right)$



Question 22 (***+)

A circle has its centre at the point $C(2,5)$ and its radius is $\sqrt{10}$.

a) Show that an equation for the circle is

$$x^2 + y^2 - 4x - 10y + 19 = 0.$$

The straight line with equation

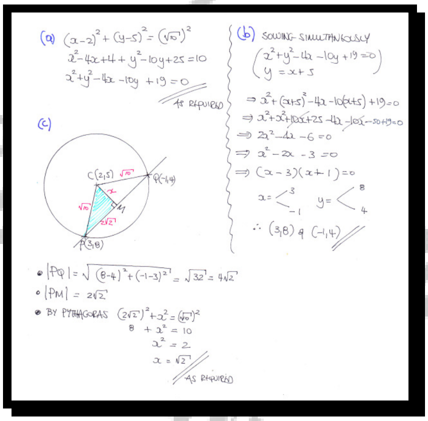
$$y = x + 5$$

meets the circle at the points P and Q.

b) Determine the coordinates of P and the coordinates of Q.

c) Show that the distance of the chord PQ from C is $\sqrt{2}$ units.

$\boxed{}$, $\boxed{(3,8),\ (-1,4)}$



Question 23 (***+)

A circle has its centre at the point $C(-3,8)$ and the length of its diameter is $\sqrt{80}$.

a) State an equation for this circle.

The straight line with equation

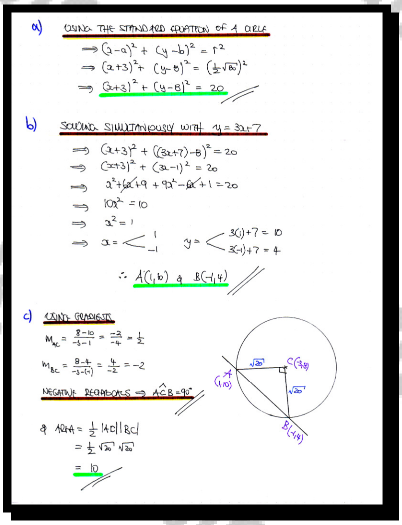
$$y = 3x + 7$$

intersects the circle at the points A and B.

b) Find the coordinates of A and the coordinates of B.

c) Show that ACB is a right angle and hence determine the area of the triangle ACB.

[ ], $\boxed{(x+3)^2 + (y-8)^2 = 20}$, $\boxed{(-1,4),\ (1,10)}$, $\boxed{\text{area} = 10}$



Question 24 **(***+)**

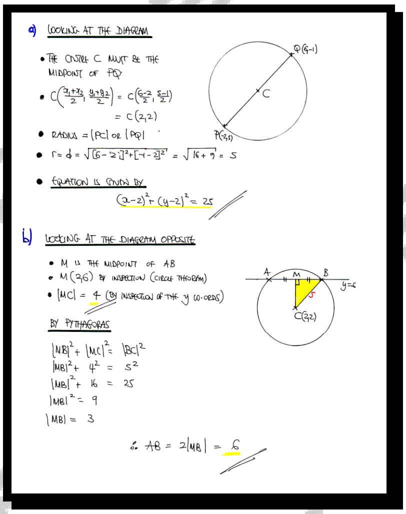
The points $P(-2,5)$ and $Q(6,-1)$ lie on a circle so that the chord PQ is a diameter of this circle.

a) Find an equation for this circle.

The straight line with equation $y=6$ intersects the circle at the points A and B.

b) Determine the shortest distance of AB from the centre of the circle and hence, or otherwise, find the distance AB.

[ ] , $(x-2)^2+(y-2)^2=25$, 4 , $|AB|=6$


a) LOOKING AT THE DIAGRAM

- THE CENTRE C MUST BE THE MIDPOINT OF PQ
- $C\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = C\left(\frac{6-2}{2}, \frac{5-1}{2}\right) = C(2,2)$
- RADIUS $= |PC|$ OR $|PQ|$
- $r = d = \sqrt{[6-2]^2+[-1-2]^2} = \sqrt{16+9} = 5$
- EQUATION IS GIVEN BY $(x-2)^2+(y-2)^2=25$

b) LOOKING AT THE DIAGRAM OPPOSITE

- M IS THE MIDPOINT OF AB
- M(2,6) BY INSPECTION (CIRCLE THEOREM)
- $|MC| = 4$ (BY INSPECTION OF THE y CO-ORDS)

BY PYTHAGORAS

$|MB|^2+|MC|^2=|BC|^2$
$|MB|^2+4^2=5^2$
$|MB|^2+16=25$
$|MB|^2=9$
$|MB|=3$

$\therefore AB = 2|MB| = 6$

Question 25 (***+)

A circle has equation

$$x^2 + y^2 - 8x + 6y = 0$$

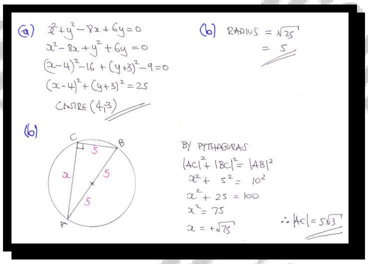
a) Find the coordinates of the centre of the circle.

b) Determine the radius of the circle.

The points A, B and C lie on the circle so that $|AB| = 10$ and $|BC| = 5$.

c) Find the distance of AC, giving the answer in the form $k\sqrt{3}$, where k is a positive integer.

$\boxed{}$, $\boxed{(4,-3)}$, $\boxed{r = 5}$, $\boxed{d = 5\sqrt{3}}$



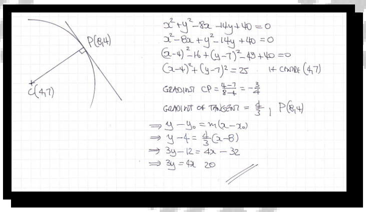
Question 26 **(***+)**

A circle has equation

$$x^2 + y^2 - 8x - 14y + 40 = 0$$

Find an equation of the tangent to the circle at the point $(8,4)$.

$$\boxed{4x - 3y = 20}$$



Question 27 (***+)

The points A, B and C have coordinates $(-3,0)$, $(-1,6)$ and $(11,2)$, respectively.

a) Show clearly that

$$\measuredangle ABC = 90°.$$

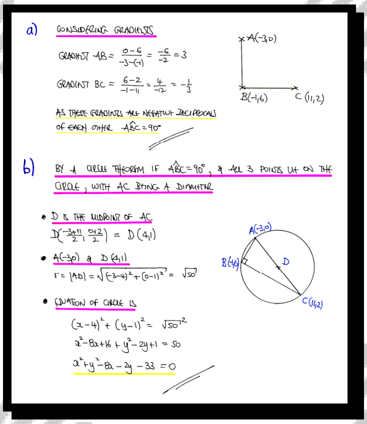
The points A, B and C lie on the circumference of a circle centred at the point D.

b) Find an equation for this circle in the form

$$x^2 + y^2 + ax + by + c = 0,$$

where a, b and c, are constants to be found.

$\boxed{}$, $\boxed{x^2 + y^2 - 8x - 2y - 33 = 0}$


a) CONSIDERING GRADIENTS

GRADIENT AB = $\frac{0-6}{-3-(-1)} = \frac{-6}{-2} = 3$

GRADIENT BC = $\frac{6-2}{-1-11} = \frac{4}{-12} = -\frac{1}{3}$

AS THESE GRADIENTS ARE NEGATIVE RECIPROCALS OF EACH OTHER $A\hat{B}C = 90°$

b) BY A CIRCLE THEOREM IF $A\hat{B}C = 90°$, ALL 3 POINTS LIE ON THE CIRCLE, WITH AC BEING A DIAMETER

- D IS THE MIDPOINT OF AC
  $D\left(\frac{-3+11}{2}, \frac{0+2}{2}\right) = D(4,1)$
- $A(-3,0)$ & $D(4,1)$
  $r = |AD| = \sqrt{(-3-4)^2 + (0-1)^2} = \sqrt{50}$
- EQUATION OF CIRCLE IS
  $(x-4)^2 + (y-1)^2 = \sqrt{50}^2$
  $x^2 - 8x + 16 + y^2 - 2y + 1 = 50$
  $x^2 + y^2 - 8x - 2y - 33 = 0$

Question 28 **(***+)**

The points A and B have coordinates $(-1,2)$ and $(1,8)$, respectively.

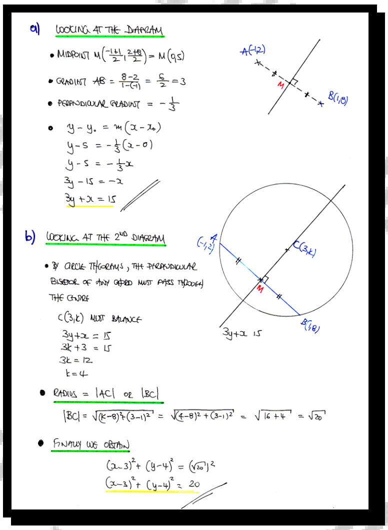
a) Show that the equation of the perpendicular bisector of AB is

$$3y + x = 15.$$

The points A and B lie on a circle whose centre is at $C(3,k)$.

b) Determine an equation for the circle.

$\boxed{}$, $\boxed{(x-3)^2 + (y-4)^2 = 20}$


a) LOOKING AT THE DIAGRAM

• MIDPOINT $M\left(\frac{-1+1}{2}, \frac{2+8}{2}\right) = M(0,5)$

• GRADIENT $AB = \frac{8-2}{1-(-1)} = \frac{6}{2} = 3$

• PERPENDICULAR GRADIENT $= -\frac{1}{3}$

• $y - y_0 = m(x - x_0)$

$y - 5 = -\frac{1}{3}(x - 0)$

$y - 5 = -\frac{1}{3}x$

$3y - 15 = -x$

$3y + x = 15$

b) LOOKING AT THE 2ND DIAGRAM

• BY CIRCLE THEOREMS, THE PERPENDICULAR BISECTOR OF ANY CHORD MUST PASS THROUGH THE CENTRE

$C(3,k)$ MUST BALANCE

$3y + x = 15$

$3k + 3 = 15$

$3k = 12$

$k = 4$

• RADIUS $= |AC|$ OR $|BC|$

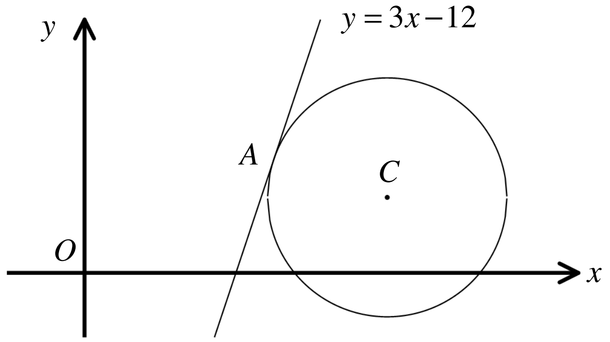
$|BC| = \sqrt{(k-8)^2 + (3-1)^2} = \sqrt{(4-8)^2 + (3-1)^2} = \sqrt{16+4} = \sqrt{20}$

• FINALLY WE OBTAIN

$(x-3)^2 + (y-4)^2 = (\sqrt{20})^2$

$(x-3)^2 + (y-4)^2 = 20$

Question 29 (***+)



The figure above shows a circle whose centre is at $C(8,k)$, where k is a constant.

The straight line with equation

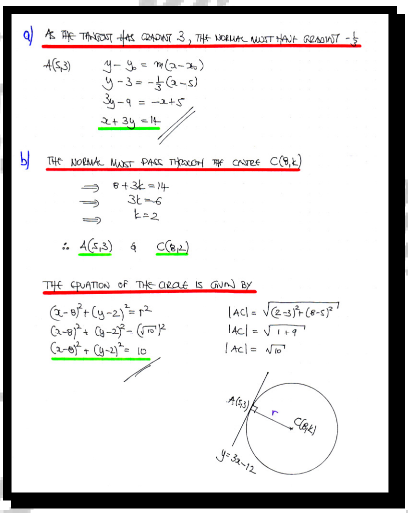
$$y = 3x - 12$$

is a tangent to the circle at the point $A(5,3)$.

a) Find an equation of the normal to the circle at A.

b) Determine an equation for the circle.

, $\boxed{x+3y=14}$, $\boxed{(x-8)^2+(y-2)^2=10}$



Question 30 **(***+)**

A circle C has equation

$$x^2 + y^2 - 14x - 14y + 49 = 0 .$$

a) Find the radius of the circle and the coordinates of its centre.

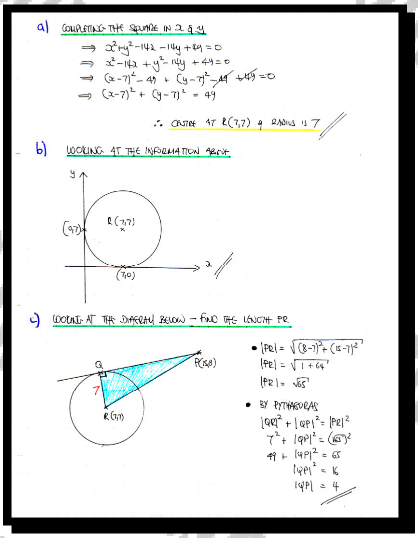
b) Sketch the circle, indicating clearly all the relevant details.

The point P has coordinates $(15,8)$.

A tangent drawn from P touches the circle at the point Q.

c) Determine the distance PQ.

$\boxed{}$, $\boxed{(7,7)}$, $\boxed{r = 7}$, $\boxed{|PQ| = 4}$



Question 31 (***+)

A circle has equation

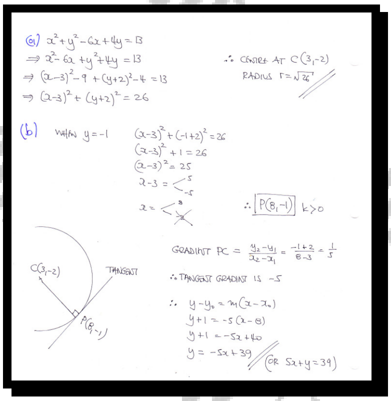
$$x^2 + y^2 - 6x + 4y = 13.$$

a) Find the coordinates of its centre and the length of its radius.

The point $P(k,-1)$, $k > 0$, lies on the circle.

b) Determine an equation for the tangent to the circle at P.

$\boxed{(3,-2),\ r = \sqrt{26}}$, $\boxed{y + 5x = 39}$



Question 32 (***+)

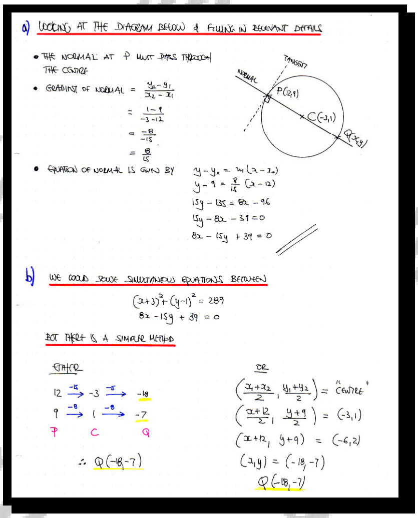
The point $P(12,9)$ lies on the circle with equation

$$(x+3)^2+(y-1)^2=289.$$

a) Find an equation of the normal to the circle at P.

b) Determine the coordinates of the point Q, where the normal to the circle at P intersects the circle again.

$\boxed{}$, $\boxed{8x-15y+39=0}$, $\boxed{Q(-18,-7)}$



Question 33 **(***+)**

A circle has equation

$$x^2 + y^2 - 10x - 8y + 21 = 0.$$

a) Find the coordinates of the centre and the radius of the circle.

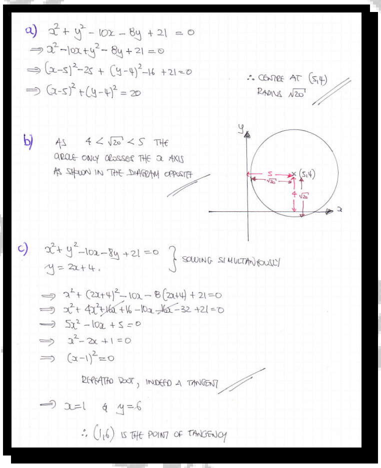
b) Determine mathematically, but without solving any equations, whether the circle crosses the coordinate axes.

c) Show that the straight line with equation

$$y = 2x + 4$$

is a tangent to the circle, and determine the coordinates of the point where the tangent meets the circle.

$\boxed{}$, $\boxed{(5,4), \quad r = 2\sqrt{5}}$, $\boxed{(1,6)}$



Question 34 (***+)

A circle C has equation

$$4x^2 + 4y^2 - 8x + 24y - 5 = 0$$

a) Find the coordinates of the centre of the circle.

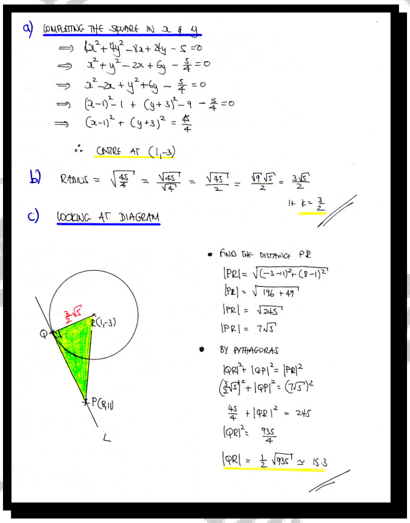
b) Determine the size of the radius of the circle, giving the answer in the form $k\sqrt{5}$, where k is a rational constant.

The point P has coordinates $(8,11)$.

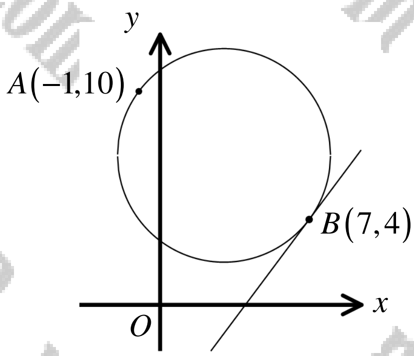
The straight line L passes through P and touches the circle at the point Q.

c) Calculate the distance PQ.

$\boxed{}$, $\boxed{(1,-3)}$, $\boxed{r = \frac{3}{2}\sqrt{5}}$, $\boxed{\frac{1}{2}\sqrt{935} \approx 15.29}$



Question 35 (***+)



The figure above shows a circle that passes through the points $A(-1,10)$ and $B(7,4)$.

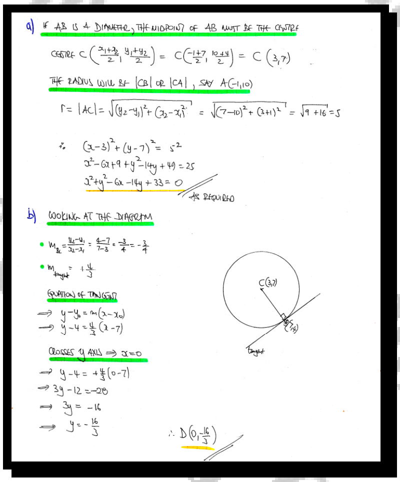
a) Given that AB is a diameter of the circle show that an equation for this circle is given by

$$x^2 + y^2 - 6x - 14y + 33 = 0.$$

The tangent to the circle at B meets the y axis at the point D.

b) Show that the coordinates of D are $\left(0,-\frac{16}{3}\right)$.

[ ], proof


a) IF AB IS A DIAMETER, THE MIDPOINT OF AB MUST BE THE CENTRE

CENTRE $C\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right) = C\left(\frac{-1+7}{2},\frac{10+4}{2}\right) = C(3,7)$

THE RADIUS WILL BE $|CB|$ OR $|CA|$, SAY $A(-1,10)$

$r = |AC| = \sqrt{(y_2-y_1)^2+(x_2-x_1)^2} = \sqrt{(7-10)^2+(3+1)^2} = \sqrt{9+16} = 5$

$\therefore (x-3)^2+(y-7)^2 = 5^2$

$x^2-6x+9+y^2-14y+49 = 25$

$x^2+y^2-6x-14y+33 = 0$ // AS REQUIRED

b) LOOKING AT THE DIAGRAM

- $m_{BC} = \frac{y_2-y_1}{x_2-x_1} = \frac{4-7}{7-3} = \frac{-3}{4} = -\frac{3}{4}$
- $m_{tangent} = +\frac{4}{3}$

EQUATION OF TANGENT

$\Rightarrow y-y_0 = m(x-x_0)$

$\Rightarrow y-4 = \frac{4}{3}(x-7)$

CROSSES y AXIS $\Rightarrow x=0$

$\Rightarrow y-4 = +\frac{4}{3}(0-7)$

$\Rightarrow 3y-12 = -28$

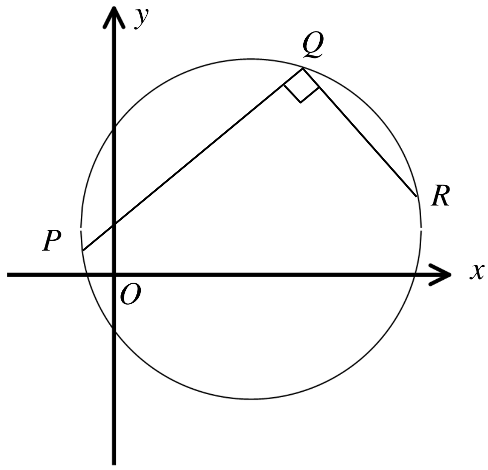
$\Rightarrow 3y = -16$

$\Rightarrow y = -\frac{16}{3}$

$\therefore D\left(0,-\frac{16}{3}\right)$

C(3,7)
(7,4)
tangent

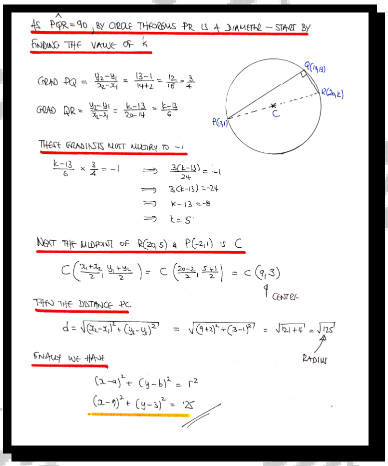
Question 36 (***+)



A circle passes through the points with coordinates $P(-2,1)$, $Q(14,13)$ and $R(20,k)$, where k is a constant.

Given that $\measuredangle PQR = 90°$, determine an equation for the circle.

$\boxed{}$, $\boxed{(x-9)^2+(y-3)^2=125}$



AS $\hat{PQR}=90$, BY CIRCLE THEOREMS PR IS A DIAMETER – START BY FINDING THE VALUE OF k

GRAD PQ $= \frac{y_2-y_1}{x_2-x_1} = \frac{13-1}{14+2} = \frac{12}{16} = \frac{3}{4}$

GRAD QR $= \frac{y_2-y_1}{x_2-x_1} = \frac{k-13}{20-14} = \frac{k-13}{6}$

THESE GRADIENTS MUST MULTIPLY TO -1

$\frac{k-13}{6} \times \frac{3}{4} = -1 \implies \frac{3(k-13)}{24} = -1$

$\implies 3(k-13) = -24$

$\implies k-13 = -8$

$\implies k = 5$

NEXT THE MIDPOINT OF $R(20,5)$ & $P(-2,1)$ IS C

$C\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = C\left(\frac{20-2}{2}, \frac{5+1}{2}\right) = C(9,3)$ ← CENTRE

THEN THE DISTANCE PC

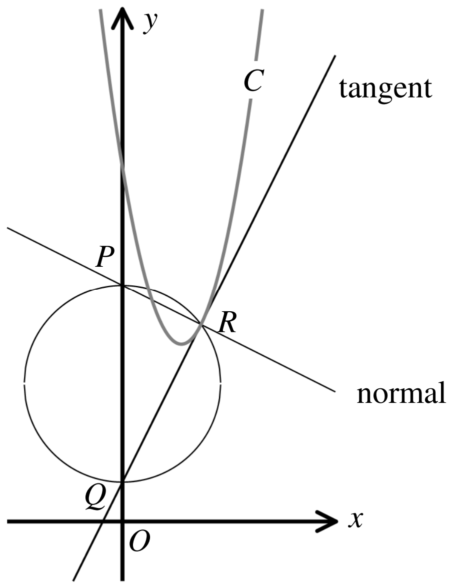
$d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = \sqrt{(9+2)^2+(3-1)^2} = \sqrt{121+4} = \sqrt{125}$ ← RADIUS

FINALLY WE HAVE

$(x-a)^2+(y-b)^2 = r^2$

$(x-9)^2+(y-3)^2 = 125$

Question 37 (***+)



The point $R(4,10)$ lies on the curve C whose equation is

$$y = x^2 - 6x + 18, \; x \in \mathbb{R}.$$

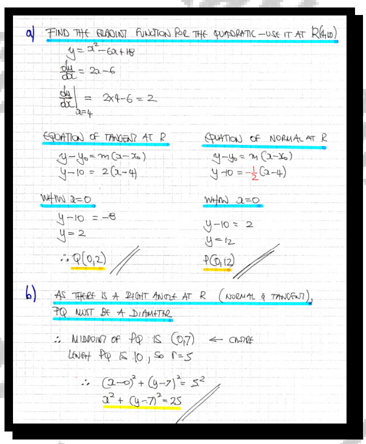
The tangent and the normal to C at R meet the y axis at the points Q and P, respectively, as shown in the figure above.

a) Find the coordinates of Q and the coordinates of P.

A circle passes through the points P, Q and R.

b) Determine an equation for the circle.

[ ], $P(0,12), Q(0,2)$, $x^2 + (y-7)^2 = 25$



a) FIND THE GRADIENT FUNCTION FOR THE QUADRATIC – USE IT AT R(4,10)

$y = x^2 - 6x + 18$

$\frac{dy}{dx} = 2x - 6$

$\left.\frac{dy}{dx}\right|_{x=4} = 2 \times 4 - 6 = 2$

EQUATION OF TANGENT AT R

$y - y_0 = m(x - x_0)$

$y - 10 = 2(x - 4)$

WHEN $x = 0$

$y - 10 = -8$

$y = 2$

$\therefore Q(0,2)$

EQUATION OF NORMAL AT R

$y - y_0 = m(x - x_0)$

$y - 10 = -\frac{1}{2}(x - 4)$

WHEN $x = 0$

$y - 10 = 2$

$y = 12$

$P(0,12)$

b) AS THERE IS A RIGHT ANGLE AT R (NORMAL & TANGENT), PQ MUST BE A DIAMETER

$\therefore$ MIDPOINT OF PQ IS (0,7) ← CENTRE

LENGTH PQ IS 10, SO $r = 5$

$\therefore (x-0)^2 + (y-7)^2 = 5^2$

$x^2 + (y-7)^2 = 25$

Question 38 (***+)

A circle C_1 has equation

$$x^2 + y^2 - 4x + 12y + 4 = 0$$

a) Determine the coordinates of the centre and the radius of C_1.

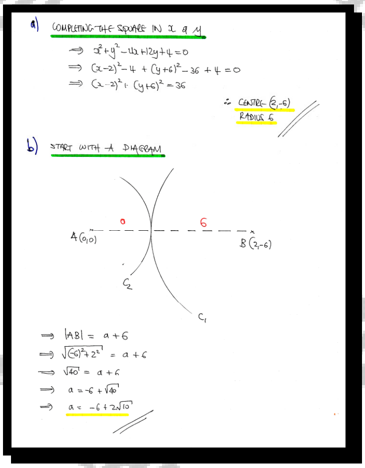
The circle C_2 with equation

$$x^2 + y^2 = a^2, \ a > 0$$

touches C_1 externally.

b) Find the value of a as an exact surd.

$(2,-6), r = 6$, $a = -6 + 2\sqrt{10}$



Question 39 (***+)

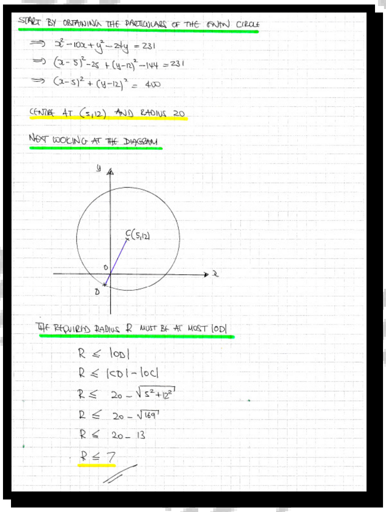
A circle has centre at the origin and radius R.

This circle fits wholly inside the circle with equation

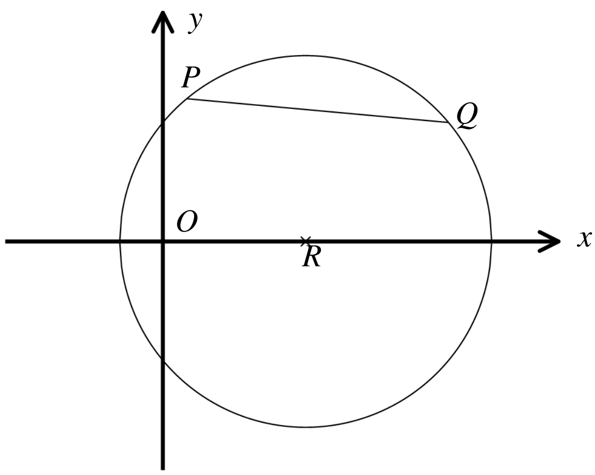
$$x^2 + y^2 - 10x - 24y = 231.$$

Determine the range of possible values of R.

[ ] , $\boxed{R \le 7}$



Question 40 (***+)



A circle C passes through the points $P(1,6)$ and $Q(12,5)$.

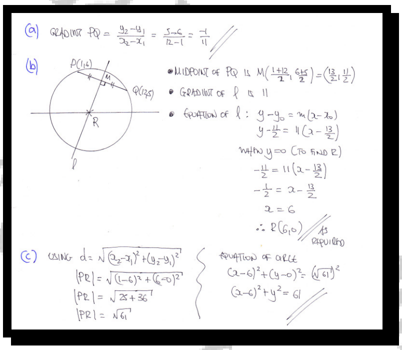
a) Find the gradient of PQ.

The centre of C is the point R which lies on the x axis.

b) Show that the coordinates of R are $(6,0)$.

c) Determine an equation for C.

$\boxed{}$, $\boxed{-\frac{1}{11}}$, $\boxed{(x-6)^2+(y-0)^2=61}$



(a) GRADIENT PQ $= \frac{y_2-y_1}{x_2-x_1} = \frac{5-6}{12-1} = \frac{-1}{11}$

(b) • MIDPOINT OF PQ IS $M\left(\frac{1+12}{2}, \frac{6+5}{2}\right) = \left(\frac{13}{2}, \frac{11}{2}\right)$

• GRADIENT OF l IS 11

• EQUATION OF l: $y - y_0 = m(x - x_0)$

$y - \frac{11}{2} = 11\left(x - \frac{13}{2}\right)$

WHEN $y = 0$ (TO FIND R)

$-\frac{11}{2} = 11\left(x - \frac{13}{2}\right)$

$-\frac{1}{2} = x - \frac{13}{2}$

$x = 6$

$\therefore R(6,0)$ AS REQUIRED

(c) USING $d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$|PR| = \sqrt{(1-6)^2+(6-0)^2}$

$|PR| = \sqrt{25+36}$

$|PR| = \sqrt{61}$

EQUATION OF CIRCLE

$(x-6)^2+(y-0)^2 = (\sqrt{61})^2$

$(x-6)^2+y^2 = 61$

Question 41 (***+)

The points A, B and C have coordinates $(-1,-4)$, $(0,3)$ and $(14,1)$, respectively.

a) Find an equation of the straight line which passes through B and C, giving the answer in the form $ax+by=c$, where a, b and c are integers.

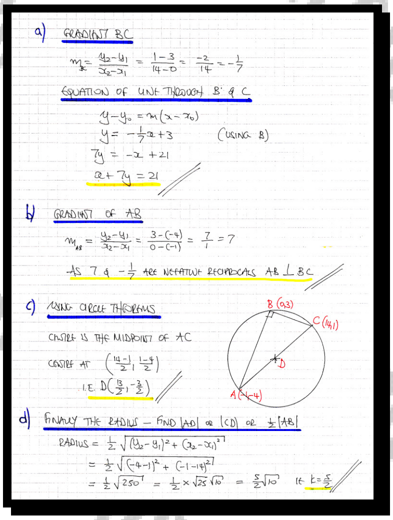
b) Show that AB is perpendicular to BC.

A circle is passing through the points A, B and C.

c) Determine the coordinates of the centre of this circle.

d) Show that the radius of this circle is $k\sqrt{10}$, where k is a rational number.

SYN-H, $7y+x=21$, $\left(\frac{13}{2},-\frac{3}{2}\right)$, $k=\frac{5}{2}$



a) GRADIENT BC

$m_{BC} = \frac{y_2-y_1}{x_2-x_1} = \frac{1-3}{14-0} = \frac{-2}{14} = -\frac{1}{7}$

EQUATION OF LINE THROUGH B & C

$y-y_0 = m(x-x_0)$

$y = -\frac{1}{7}x+3$ (USING B)

$7y = -x+21$

$x+7y=21$

b) GRADIENT OF AB

$m_{AB} = \frac{y_2-y_1}{x_2-x_1} = \frac{3-(-4)}{0-(-1)} = \frac{7}{1} = 7$

AS 7 & $-\frac{1}{7}$ ARE NEGATIVE RECIPROCALS AB ⊥ BC

c) USING CIRCLE THEOREMS

CENTRE IS THE MIDPOINT OF AC

CENTRE AT $\left(\frac{14-1}{2}, \frac{1-4}{2}\right)$

I.E. $D\left(\frac{13}{2},-\frac{3}{2}\right)$

d) FINALLY THE RADIUS – FIND |AD| OR |CD| OR $\frac{1}{2}$|AB|

RADIUS $= \frac{1}{2}\sqrt{(y_2-y_1)^2+(x_2-x_1)^2}$

$= \frac{1}{2}\sqrt{(-4-1)^2+(-1-14)^2}$

$= \frac{1}{2}\sqrt{250} = \frac{1}{2}\times\sqrt{25}\sqrt{10} = \frac{5}{2}\sqrt{10}$ IE $k=\frac{5}{2}$

Question 42 (***+)

A circle C has equation

$$x^2 + y^2 - 6x + 14y + 33 = 0$$

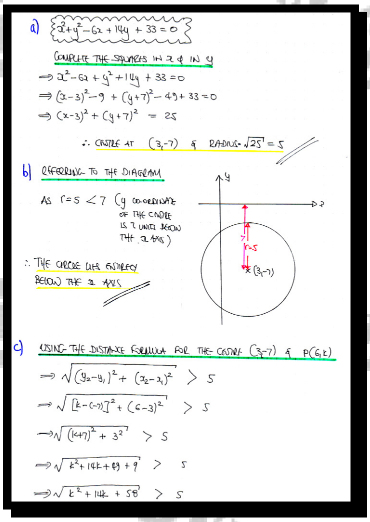
a) Determine the coordinates of the centre and the radius of C.

b) Show that the circle lies entirely below the x axis.

The point $P(6,k)$, where k is a constant, lies outside the circle.

c) By considering the distance of P from the centre of the circle, or otherwise, determine the range of the possible values of k.

, $(3,-7)$, $r=5$, $k < -11$ or $k > -3$



a) $x^2+y^2-6x+14y+33=0$

COMPLETE THE SQUARES IN x & IN y

$\Rightarrow x^2-6x+y^2+14y+33=0$

$\Rightarrow (x-3)^2-9+(y+7)^2-49+33=0$

$\Rightarrow (x-3)^2+(y+7)^2=25$

$\therefore$ CENTRE AT $(3,-7)$ & RADIUS $=\sqrt{25}=5$

b) REFERRING TO THE DIAGRAM

AS $r=5<7$ (y CO-ORDINATE OF THE CENTRE IS 7 UNITS BELOW THE x AXIS)

$\therefore$ THE CIRCLE LIES ENTIRELY BELOW THE x AXIS

c) USING THE DISTANCE FORMULA FOR THE CENTRE $(3,-7)$ & $P(6,k)$

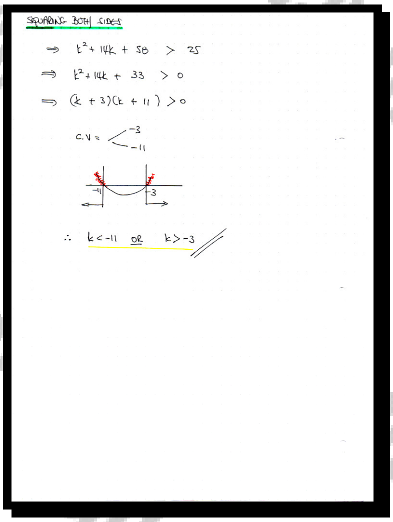
$\Rightarrow \sqrt{(y_2-y_1)^2+(x_2-x_1)^2} > 5$

$\Rightarrow \sqrt{[k-(-7)]^2+(6-3)^2} > 5$

$\Rightarrow \sqrt{(k+7)^2+3^2} > 5$

$\Rightarrow \sqrt{k^2+14k+49+9} > 5$

$\Rightarrow \sqrt{k^2+14k+58} > 5$



SQUARING BOTH SIDES

$\Rightarrow k^2+14k+58 > 25$

$\Rightarrow k^2+14k+33 > 0$

$\Rightarrow (k+3)(k+11) > 0$

C.V. $= -3, -11$

$\therefore k<-11$ OR $k>-3$

Question 43 **(***+)**

The points $A(1,0)$, $B(8,7)$ and $C(7,8)$ lie on the circumference of a circle.

a) Show that AB is perpendicular to BC.

b) Find the distance AC.

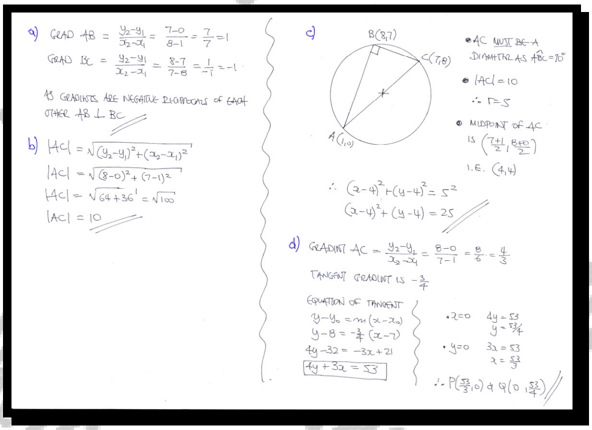
c) Show that an equation of the circle is

$$(x-4)^2+(y-4)^2=25.$$

The tangent to the circle at the point C crosses the x axis and the y axis at the points P and Q, respectively.

d) Determine the exact coordinates of P and Q.

$\boxed{|AC|=10}$, $\boxed{P\left(\frac{53}{3},0\right),\ Q\left(0,\frac{53}{4}\right)}$



Question 44 (***+)

A circle with centre at the point C has equation

$$x^2 + y^2 - 10x - 6y + 14 = 0.$$

The straight line with equation $y = k$, where k is a non zero constant, is a tangent to this circle

a) Find the possible values k, giving the answers as exact simplified surds.

The points A and B lie on the circumference of the circle and the point M is the midpoint of the chord AB.

b) Given the length of MC is 2, find the length of the chord AB.

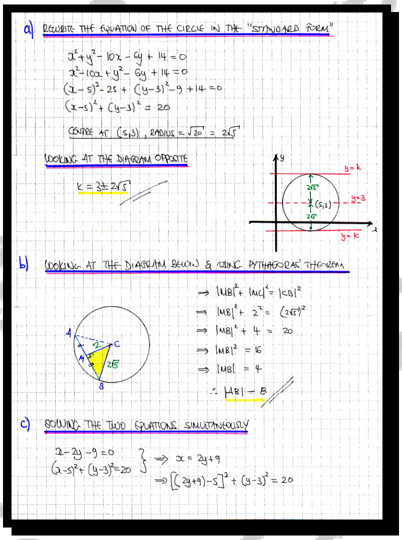
The straight line with equation

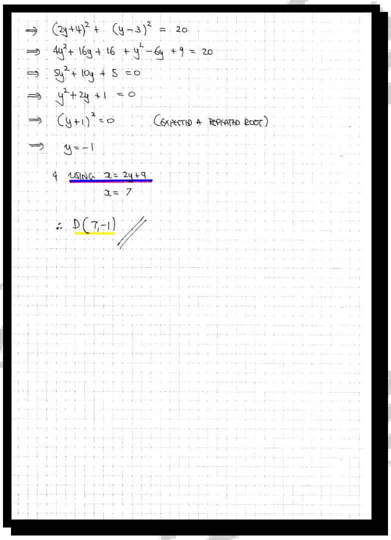
$$x - 2y - 9 = 0$$

is a tangent to the circle at the point D.

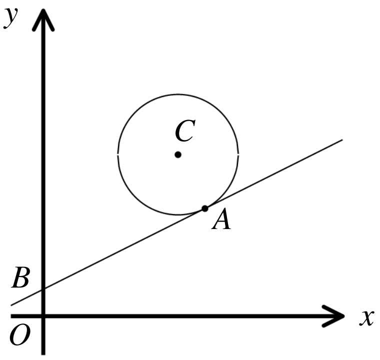
c) Determine the coordinates of D.

$\boxed{}$, $\boxed{k = 3 \pm 2\sqrt{5}}$, $\boxed{|AB| = 8}$, $\boxed{D(7,-1)}$





Question 45 (***+)



The figure above shows a circle with centre at C with equation

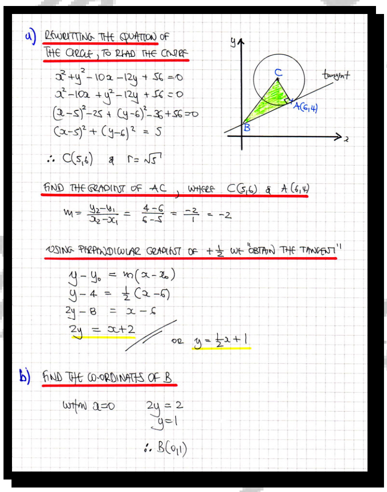
$$x^2 + y^2 - 10x - 12y + 56 = 0.$$

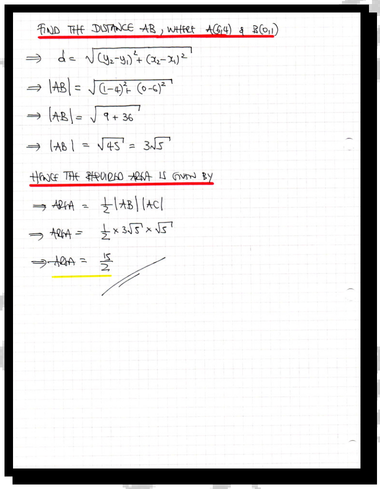
The tangent to the circle at the point $A(6,4)$ meets the y axis at the point B.

a) Find an equation of the tangent to the circle at A.

b) Determine the area of the triangle ABC.

$\boxed{}$, $\boxed{y = \frac{1}{2}x + 1}$, $\boxed{\frac{15}{2}}$





Question 46 (***+)

A circle C has equation

$$x^2 + y^2 - 12x - 2y + 33 = 0.$$

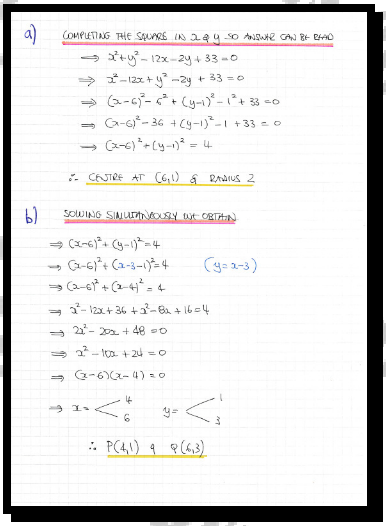
a) Find the radius of the circle and the coordinates of its centre.

The straight line with equation $y = x - 3$ intersects the circle at the points P and Q, dividing the circle into two segments.

b) Determine the coordinates of P and Q.

c) Show that the area of the minor segment is $\pi - 2$.

$\boxed{}$, $\boxed{r = 2,\ (6,1)}$, $\boxed{(4,1),(6,3)}$



a) COMPLETING THE SQUARE IN x & y SO ANSWER CAN BE READ

$\Rightarrow x^2 + y^2 - 12x - 2y + 33 = 0$

$\Rightarrow x^2 - 12x + y^2 - 2y + 33 = 0$

$\Rightarrow (x-6)^2 - 6^2 + (y-1)^2 - 1^2 + 33 = 0$

$\Rightarrow (x-6)^2 - 36 + (y-1)^2 - 1 + 33 = 0$

$\Rightarrow (x-6)^2 + (y-1)^2 = 4$

$\therefore$ CENTRE AT $(6,1)$ & RADIUS 2

b) SOLVING SIMULTANEOUSLY WE OBTAIN

$\Rightarrow (x-6)^2 + (y-1)^2 = 4$

$\Rightarrow (x-6)^2 + (x-3-1)^2 = 4$ $\quad (y = x-3)$

$\Rightarrow (x-6)^2 + (x-4)^2 = 4$

$\Rightarrow x^2 - 12x + 36 + x^2 - 8x + 16 = 4$

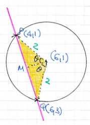
$\Rightarrow 2x^2 - 20x + 48 = 0$

$\Rightarrow x^2 - 10x + 24 = 0$

$\Rightarrow (x-6)(x-4) = 0$

$\Rightarrow x = \begin{cases} 4 \\ 6 \end{cases} \quad y = \begin{cases} 1 \\ 3 \end{cases}$

$\therefore P(4,1)$ & $Q(6,3)$



c) START BY DRAWING A DIAGRAM

- DISTANCE PQ

$|PQ| = \sqrt{(1-3)^2 + (4-6)^2}$

$= \sqrt{4+4}$

$= \sqrt{8} = 2\sqrt{2}$

- BY THE COSINE RULE OR SIMPLE TRIGONOMETRY

$\sin\theta = \frac{|MP|}{2}$

$\sin\theta = \frac{\sqrt{2}}{2}$

$\theta = 45° = \frac{\pi}{4}$

- AREA OF SECTOR IS $\frac{1}{2}r^2\theta^c$

$\frac{1}{2} \times 2^2 \times \frac{\pi}{2} = \pi$ (↑ 2θ)

- AREA OF TRIANGLE IN YELLOW

$\frac{1}{2} \times 2 \times 2 \times \sin(2\theta) = 2\sin\frac{\pi}{2} = 2$

- REQUIRED AREA = AREA OF SECTOR − AREA OF TRIANGLE

$= \pi - 2$

AS REQUIRED

Question 47 (***+)

The points A, B and C have coordinates $(2,1)$, $(4,0)$ and $(6,4)$ respectively.

a) Determine an equation of the straight line L which passes through C and is parallel to AB.

b) Show that the angle ABC is $90°$.

c) Calculate the distance AC.

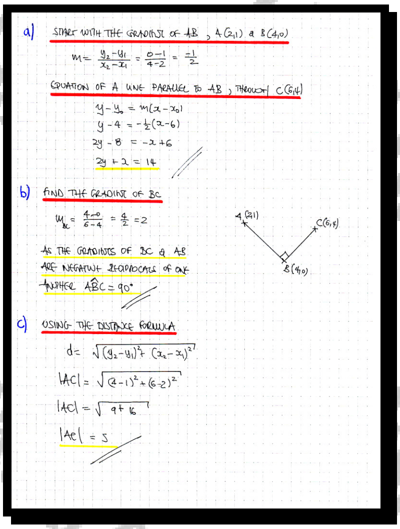
A circle passes through the points A, B and C.

d) Show that the equation of this circle is given by

$$x^2 + y^2 - 8x - 5y + 16 = 0.$$

e) Find the coordinates of the point other than the point C where L intersects the circle.

$\boxed{}$, $\boxed{x + 2y = 14}$, $\boxed{|AC| = 5}$, $\boxed{(4,5)}$


a) START WITH THE GRADIENT OF AB, A(2,1) & B(4,0)

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-1}{4-2} = \frac{-1}{2}$

EQUATION OF A LINE PARALLEL TO AB, THROUGH C(6,4)

$y - y_0 = m(x - x_0)$

$y - 4 = -\frac{1}{2}(x - 6)$

$2y - 8 = -x + 6$

$2y + x = 14$

b) FIND THE GRADIENT OF BC

$m_{BC} = \frac{4-0}{6-4} = \frac{4}{2} = 2$

AS THE GRADIENTS OF BC & AB ARE NEGATIVE RECIPROCALS OF ONE ANOTHER $\hat{ABC} = 90°$

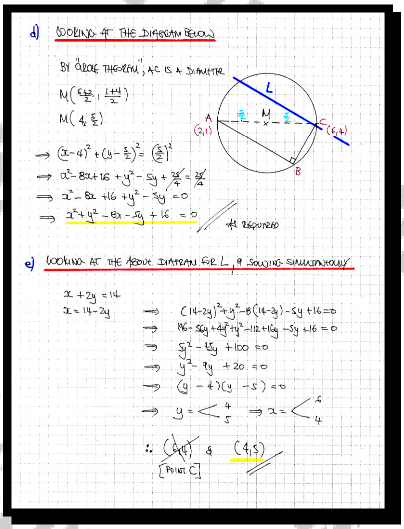
c) USING THE DISTANCE FORMULA

$d = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$

$|AC| = \sqrt{(4-1)^2 + (6-2)^2}$

$|AC| = \sqrt{9 + 16}$

$|AC| = 5$


d) LOOKING AT THE DIAGRAM BELOW

BY "CIRCLE THEOREM", AC IS A DIAMETER

$M\left(\frac{6+2}{2}, \frac{1+4}{2}\right)$

$M\left(4, \frac{5}{2}\right)$

$\Rightarrow (x-4)^2 + \left(y - \frac{5}{2}\right)^2 = \left(\frac{5}{2}\right)^2$

$\Rightarrow x^2 - 8x + 16 + y^2 - 5y + \frac{25}{4} = \frac{25}{4}$

$\Rightarrow x^2 - 8x + 16 + y^2 - 5y = 0$

$\Rightarrow x^2 + y^2 - 8x - 5y + 16 = 0$ AS REQUIRED

e) LOOKING AT THE ABOVE DIAGRAM FOR L, & SOLVING SIMULTANEOUSLY

$x + 2y = 14$

$x = 14 - 2y$

$\Rightarrow (14-2y)^2 + y^2 - 8(14-2y) - 5y + 16 = 0$

$\Rightarrow 196 - 56y + 4y^2 + y^2 - 112 + 16y - 5y + 16 = 0$

$\Rightarrow 5y^2 - 45y + 100 = 0$

$\Rightarrow y^2 - 9y + 20 = 0$

$\Rightarrow (y-4)(y-5) = 0$

$\Rightarrow y = \begin{cases} 4 \\ 5 \end{cases} \Rightarrow x = \begin{cases} 6 \\ 4 \end{cases}$

$\therefore$ ~~(6,4)~~ [POINT C] & $(4,5)$

Question 48 (***+)

The straight line l passes through the points $P(7,10)$ and $Q(17,3)$.

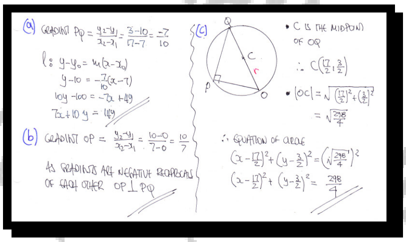
a) Find an equation of l.

b) Show that OP is perpendicular to PQ, where O is the origin.

A circle passes through O, P and Q.

c) Find an equation for this circle.

$\boxed{7x+10y=149}$, $\boxed{\left(x-\frac{17}{2}\right)^2+\left(y-\frac{3}{2}\right)^2=\frac{298}{4}}$



Question 49 (***+)

A circle C with centre at the point P and radius r, has equation

$$x^2 - 8x + y^2 - 2y = 0.$$

a) Find the value of r and the coordinates of P.

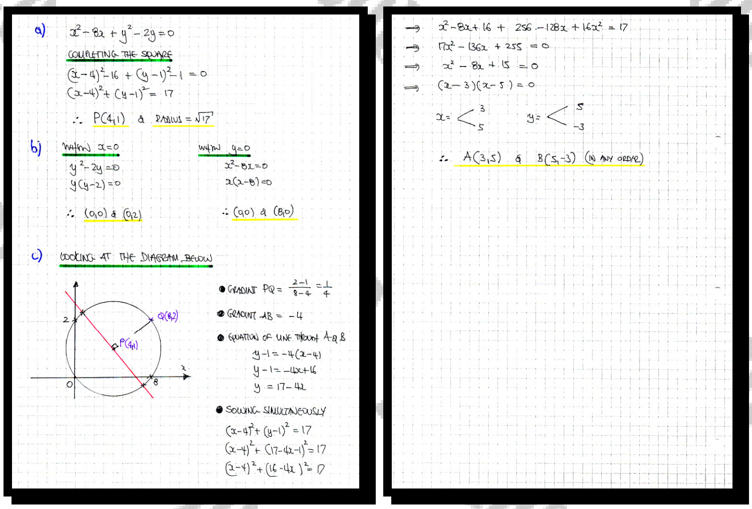
b) Determine the coordinates of the points where C meets the coordinate axes.

The points A, B and $Q(8,2)$ lie on C.

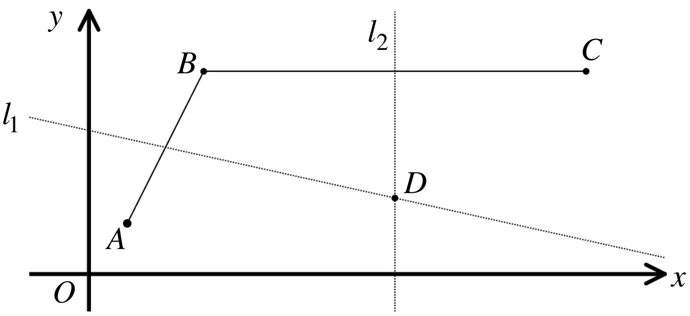
The straight line AB is diameter of the circle so that PQ is perpendicular to AB.

c) Calculate the coordinates of A and B.

[illegible], $r = \sqrt{17}$, $P(4,1)$, $(8,0),(0,0),(0,2)$, $A(3,5)$, $B(5,-3)$



Question 50 **(***+)**



The points $A(1,2)$, $B(3,8)$ and $C(13,8)$ are shown in the figure below.

The straight lines l_1 and l_2 are the perpendicular bisectors of straight line segments AB and BC, respectively.

a) Find an equation for l_1.

Give the answer in the form $ax+by=c$, where a, b and c are integers.

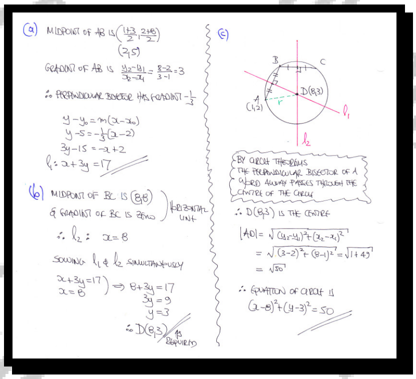
The point D is the intersection of l_1 and l_2.

b) Show that the coordinates of D are $(8,3)$.

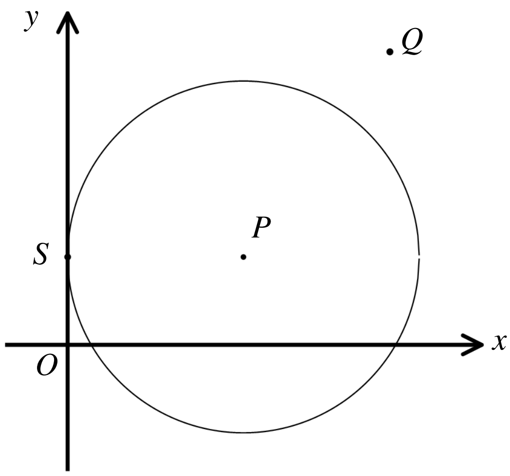
A circle passes through the points A, B and C.

c) Find an equation for this circle.

$\boxed{x+3y=17}$, $\boxed{(x-8)^2+(y-3)^2=50}$



Question 51 (****)



The figure above shows a circle with centre at P and radius of 6 units.

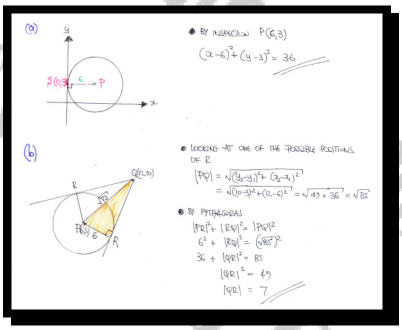
The y axis is a tangent to the circle at the point $S(0,3)$.

a) Find an equation for the circle.

A tangent to the circle is drawn from the point $Q(12,10)$ and meets the circle at the point R.

b) Determine the length of QR.

$\boxed{}$, $\boxed{(x-6)^2+(y-3)^2=36}$, $\boxed{|QR|=7}$



Question 52 (****)

The points A, B and C have coordinates $(3,5)$, $(15,11)$ and $(17,7)$, respectively.

a) Show that $\measuredangle ABC = 90°$.

All three points A, B and C lie on the circumference of a circle.

b) Find an equation for this circle in the form

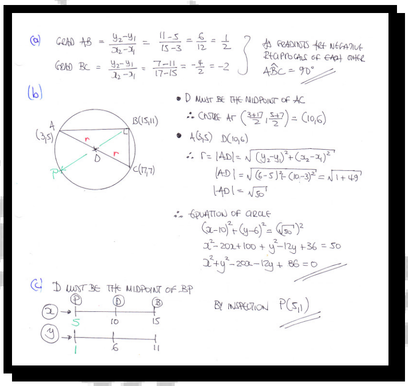
$$x^2 + y^2 + ax + by + c = 0,$$

where a, b and c, are integers to be found.

The point P also lies on this circle so that BP is a diameter of the circle.

c) Determine the coordinates of P.

$\boxed{x^2 + y^2 - 20x - 12y + 86 = 0}$, $\boxed{P(5,1)}$



Question 53 (****)

The circle C_1 has centre at $(8,4)$ and **touches** the y axis.

The circle C_2 has centre at $(16,4)$ and **touches** the x axis.

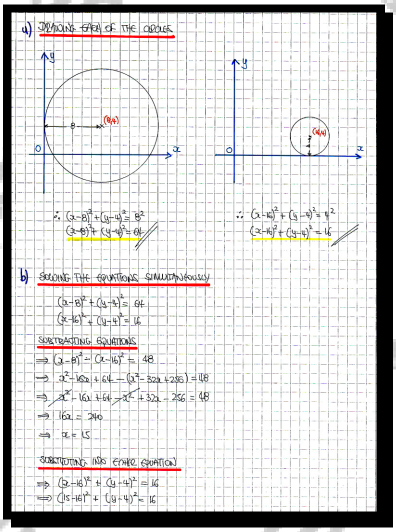
a) Find the equation of C_1 and the equation of C_2.

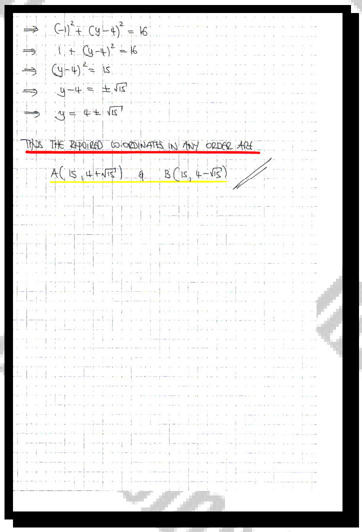
Give the answers in the form $(x-a)^2+(y-b)^2=c$, where a, b and c are constants to be found.

The two circles intersect at the points A and B.

b) Determine, in exact surd form where appropriate, the coordinates of A and the coordinates of B.

$(x-8)^2+(y-4)^2=64$, $(x-16)^2+(y-4)^2=16$, $(15, 4+\sqrt{15})$, $(15, 4-\sqrt{15})$





Question 54 (****)

The points $P(-4,1)$ and $Q(5,10)$ lie on a circle C.

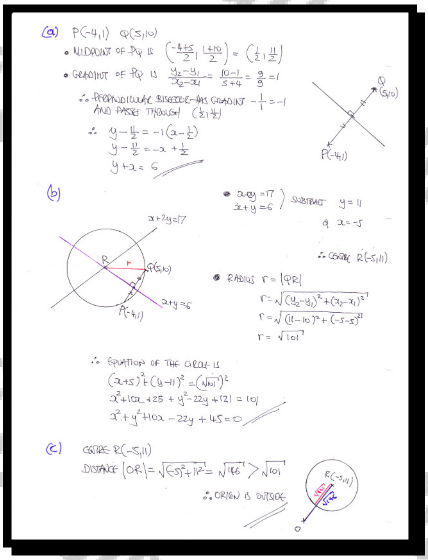
a) Find an equation of the perpendicular bisector of PQ.

b) Given that one of the diameters of C has equation $x+2y=17$, show that an equation of C is

$$x^2+y^2+10x-22y+45=0.$$

c) Determine whether the origin O lies inside the circle.

$\boxed{x+y=6}$



Question 55 (****)

A circle C has equation

$$x^2 + y^2 + ax + by + 43 = 0,$$

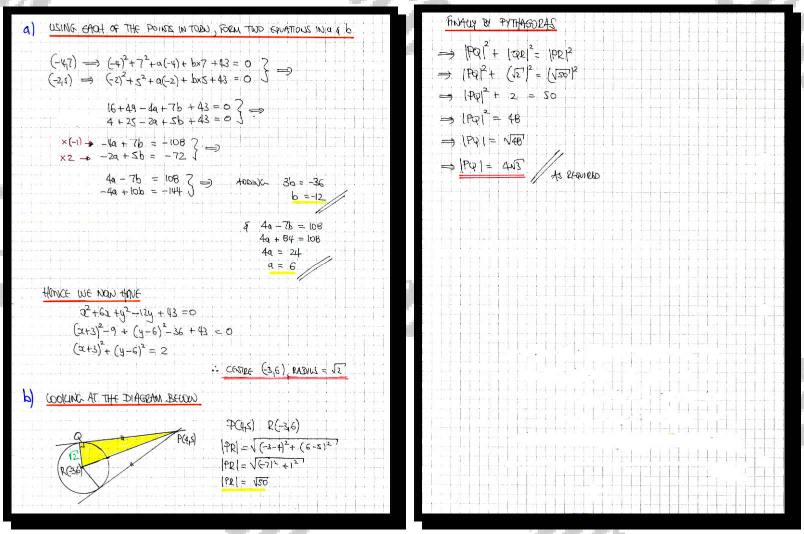
where a and b are constants.

a) Given that the points $(-4,7)$ and $(-2,5)$ lie on C, determine the coordinates of the centre of C and the size of its radius.

A straight line passes through the point $P(4,5)$ and is a tangent to C at the point Q.

b) Show that the length of PQ is $4\sqrt{3}$.

$\boxed{}$, $\boxed{(-3,6),\ r = \sqrt{2}}$



a) USING EACH OF THE POINTS IN TURN, FORM TWO EQUATIONS IN a & b

$(-4,7) \Rightarrow (-4)^2 + 7^2 + a(-4) + b\times 7 + 43 = 0$
$(-2,5) \Rightarrow (-2)^2 + 5^2 + a(-2) + b\times 5 + 43 = 0$

$16 + 49 - 4a + 7b + 43 = 0$
$4 + 25 - 2a + 5b + 43 = 0$

$\times(-1) \rightarrow -4a + 7b = -108$
$\times 2 \rightarrow -2a + 5b = -72$

$4a - 7b = 108$
$-4a + 10b = -144$ $\Rightarrow$ ADDING $3b = -36$, $b = -12$

& $4a - 7b = 108$
$4a + 84 = 108$
$4a = 24$
$a = 6$

HENCE WE NOW HAVE

$x^2 + 6x + y^2 - 12y + 43 = 0$
$(x+3)^2 - 9 + (y-6)^2 - 36 + 43 = 0$
$(x+3)^2 + (y-6)^2 = 2$

$\therefore$ CENTRE $(-3,6)$, RADIUS $= \sqrt{2}$

b) LOOKING AT THE DIAGRAM BELOW

$P(4,5)$ $R(-3,6)$
$|PR| = \sqrt{(-3-4)^2 + (6-5)^2}$
$|PR| = \sqrt{(-7)^2 + 1^2}$
$|PR| = \sqrt{50}$

FINALLY BY PYTHAGORAS

$\Rightarrow |PQ|^2 + |QR|^2 = |PR|^2$
$\Rightarrow |PQ|^2 + (\sqrt{2})^2 = (\sqrt{50})^2$
$\Rightarrow |PQ|^2 + 2 = 50$
$\Rightarrow |PQ|^2 = 48$
$\Rightarrow |PQ| = \sqrt{48}$
$\Rightarrow |PQ| = 4\sqrt{3}$ // AS REQUIRED

Question 56 (****)

A circle C_1 has equation

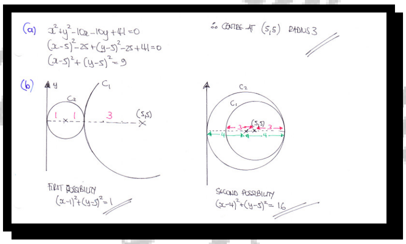
$$x^2 + y^2 - 10x - 10y + 41 = 0.$$

a) Determine the coordinates of the centre of C_1 and the size of its radius.

Another circle C_2 is such so that C_2 is **touching both** C_1 and the y axis.

b) Find the two possible equations of C_2, given further that the centres of both C_1 and C_2, have the same y coordinate.

$\boxed{}$, $\boxed{(5,5),\ r=3}$, $\boxed{(x-1)^2+(y-5)^2=1 \text{ or } (x-4)^2+(y-5)^2=16}$



Question 57 (****)

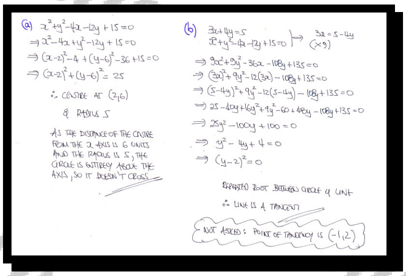
A circle C has equation

$$x^2 + y^2 - 4x - 12y + 15 = 0.$$

a) Show that C does not cross the x axis.

b) Show further that the straight line with equation $3x + 4y = 5$ is a tangent to C.

proof



(a) $x^2+y^2-4x-12y+15=0$

$\Rightarrow x^2-4x+y^2-12y+15=0$

$\Rightarrow (x-2)^2-4+(y-6)^2-36+15=0$

$\Rightarrow (x-2)^2+(y-6)^2=25$

$\therefore$ CENTRE AT $(2,6)$

& RADIUS 5

AS THE DISTANCE OF THE CENTRE FROM THE x AXIS IS 6 UNITS AND THE RADIUS IS 5, THE CIRCLE IS ENTIRELY ABOVE THE AXIS, SO IT DOESN'T CROSS

(b) $3x+4y=5$, $x^2+y^2-4x-12y+15=0$ $\mapsto$ $3x=5-4y$ $(\times 9)$

$\Rightarrow 9x^2+9y^2-36x-108y+135=0$

$\Rightarrow (3x)^2+9y^2-12(3x)-108y+135=0$

$\Rightarrow (5-4y)^2+9y^2-12(5-4y)-108y+135=0$

$\Rightarrow 25-40y+16y^2+9y^2-60+48y-108y+135=0$

$\Rightarrow 25y^2-100y+100=0$

$\Rightarrow y^2-4y+4=0$

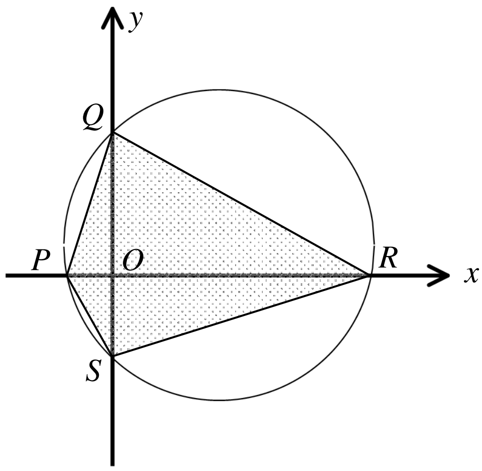
$\Rightarrow (y-2)^2=0$

REPEATED ROOT BETWEEN CIRCLE & LINE

$\therefore$ LINE IS A TANGENT

NOT ASKED: POINT OF TANGENCY IS $(-1,2)$

Question 58 (****)



The figure above shows a circle with centre at $C(7,2)$.

The circle meets the x axis at the points P and R, and the y axis at the points Q and S.

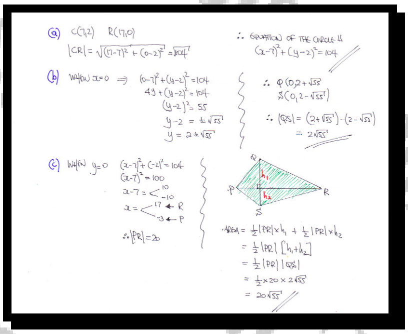
a) Given that $R(17,0)$, find an equation of this circle.

b) Show that

$$|QS| = 2\sqrt{55}.$$

c) Determine the area of the quadrilateral $PQRS$.

$\boxed{}$, $\boxed{(x-7)^2+(y-2)^2=104}$, $\boxed{20\sqrt{55}\approx 148.32}$



Question 59 (****)

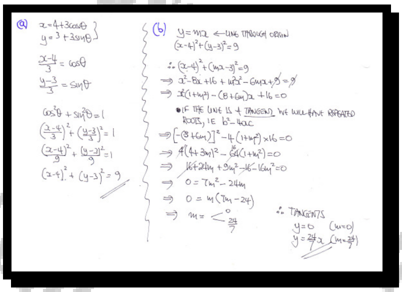
A circle is given parametrically by the equations

$$x = 4 + 3\cos\theta, \; y = 3 + 3\sin\theta, \; 0 \le \theta < 2\pi .$$

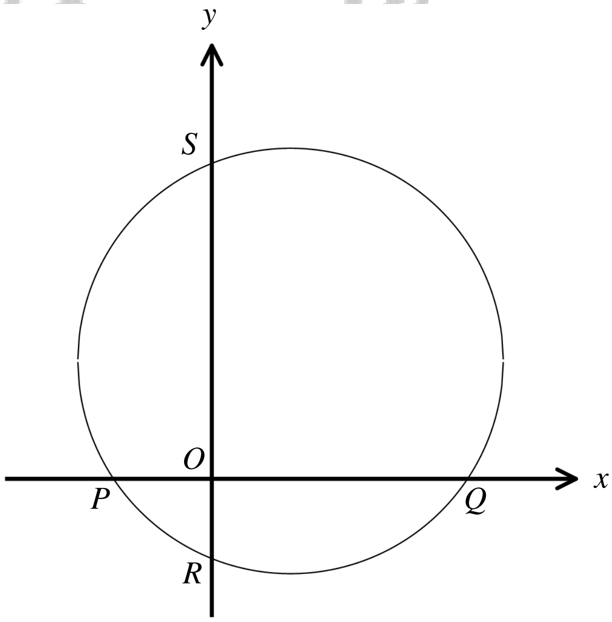
a) Find a Cartesian equation for the circle.

b) Find the equations of each of the two tangents to the circle, which pass through the origin O.

$\boxed{(x-4)^2 + (y-3)^2 = 9}$, $\boxed{y = 0 \quad \text{and} \quad y = \frac{24}{7}x}$



Question 60 (****)



The figure above shows a circle meeting the x axis at $P(-5,0)$ and $Q(13,0)$, and the y axis at $R(0,-4)$ and $S(0,16)$.

a) Show that an equation of the circle is

$$x^2 + y^2 + ax + by = c \, ,$$

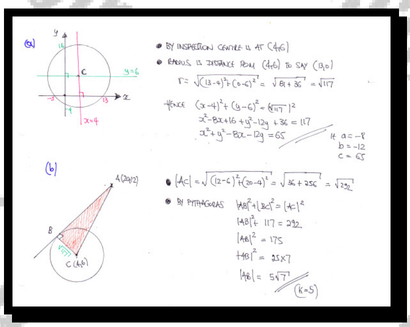
where a, b and c are constants to be found.

The point A has coordinates $(20,12)$.

A tangent drawn through A meets the circle at the point B.

b) Show that the distance AB is $k\sqrt{7}$, where k is an integer.

$\boxed{a=-8, \ b=-12, \ c=65}$, $\boxed{k=5}$



Question 61 (****)

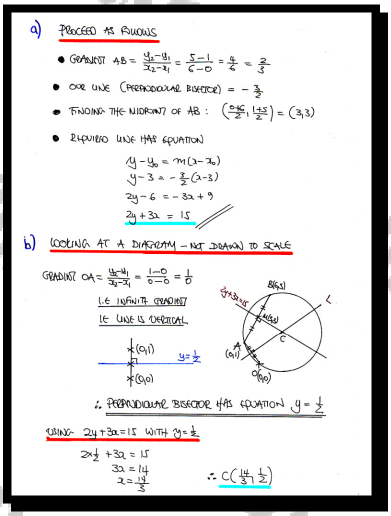
Relative to a fixed origin O, the points A and B have coordinates $(0,1)$ and $(6,5)$, respectively.

a) Find an equation of the perpendicular bisector of AB.

A circle passes through the points O, A and B.

b) Determine the coordinates of the centre of this circle.

$\boxed{}$, $\boxed{3x+2y=15}$, $\boxed{\left(\frac{14}{3},\frac{1}{2}\right)}$



a) PROCEED AS FOLLOWS

- GRADIENT $AB = \frac{y_2-y_1}{x_2-x_1} = \frac{5-1}{6-0} = \frac{4}{6} = \frac{2}{3}$
- OUR LINE (PERPENDICULAR BISECTOR) $= -\frac{3}{2}$
- FINDING THE MIDPOINT OF AB : $\left(\frac{0+6}{2}, \frac{1+5}{2}\right) = (3,3)$
- REQUIRED LINE HAS EQUATION

$$y - y_0 = m(x - x_0)$$
$$y - 3 = -\frac{3}{2}(x-3)$$
$$2y - 6 = -3x + 9$$
$$2y + 3x = 15$$

b) LOOKING AT A DIAGRAM – NOT DRAWN TO SCALE

GRADIENT $OA = \frac{y_2-y_1}{x_2-x_1} = \frac{1-0}{0-0} = \frac{1}{0}$

I.E INFINITE GRADIENT

IE LINE IS VERTICAL

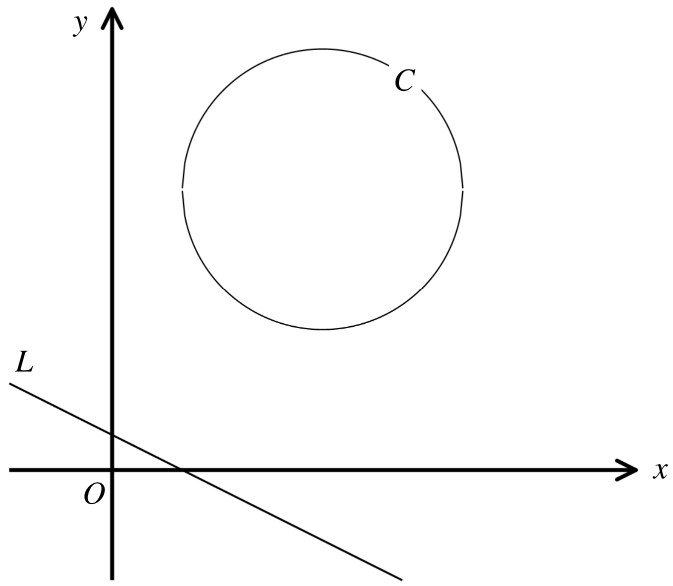
$\therefore$ PERPENDICULAR BISECTOR HAS EQUATION $y = \frac{1}{2}$

USING $2y+3x=15$ WITH $y=\frac{1}{2}$

$$2\times\tfrac{1}{2} + 3x = 15$$
$$3x = 14$$
$$x = \frac{14}{3}$$

$\therefore C\left(\frac{14}{3}, \frac{1}{2}\right)$

Question 62 (****)



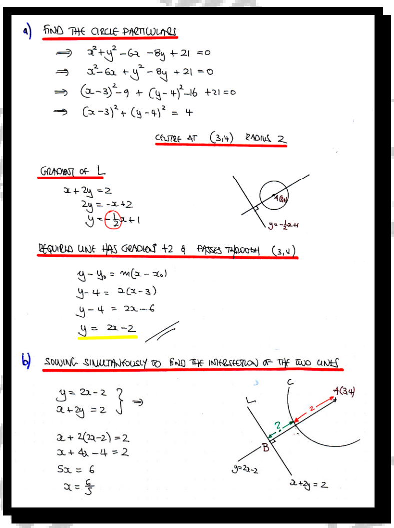
The figure above shows the circle C and the straight line L with respective equations

$$x^2 + y^2 - 6x - 8y + 21 = 0 \quad \text{and} \quad x + 2y = 2.$$

a) Find an equation of the line which passes through the centre of C and is perpendicular to L.

b) Hence determine, in exact surd form, the shortest distance between C and L.

$\boxed{}$, $\boxed{y = 2x - 2}$, $\boxed{\frac{9}{5}\sqrt{5} - 2}$



a) FIND THE CIRCLE PARTICULARS

$\Rightarrow x^2 + y^2 - 6x - 8y + 21 = 0$

$\Rightarrow x^2 - 6x + y^2 - 8y + 21 = 0$

$\Rightarrow (x-3)^2 - 9 + (y-4)^2 - 16 + 21 = 0$

$\Rightarrow (x-3)^2 + (y-4)^2 = 4$

CENTRE AT $(3,4)$ RADIUS 2

GRADIENT OF L

$x + 2y = 2$

$2y = -x + 2$

$y = -\frac{1}{2}x + 1$

REQUIRED LINE HAS GRADIENT $+2$ & PASSES THROUGH $(3,4)$

$y - y_0 = m(x - x_0)$

$y - 4 = 2(x - 3)$

$y - 4 = 2x - 6$

$y = 2x - 2$

b) SOLVING SIMULTANEOUSLY TO FIND THE INTERSECTION OF THE TWO LINES

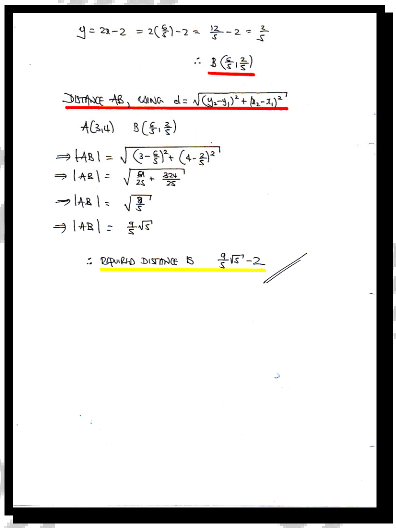
$\left.\begin{array}{l} y = 2x - 2 \\ x + 2y = 2 \end{array}\right\} \Rightarrow$

$x + 2(2x-2) = 2$

$x + 4x - 4 = 2$

$5x = 6$

$x = \frac{6}{5}$



$y = 2x - 2 = 2\left(\frac{6}{5}\right) - 2 = \frac{12}{5} - 2 = \frac{2}{5}$

$\therefore B\left(\frac{6}{5}, \frac{2}{5}\right)$

DISTANCE AB, USING $d = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$

$A(3,4) \quad B\left(\frac{6}{5}, \frac{2}{5}\right)$

$\Rightarrow |AB| = \sqrt{\left(3 - \frac{6}{5}\right)^2 + \left(4 - \frac{2}{5}\right)^2}$

$\Rightarrow |AB| = \sqrt{\frac{81}{25} + \frac{324}{25}}$

$\Rightarrow |AB| = \sqrt{\frac{81}{5}}$

$\Rightarrow |AB| = \frac{9}{5}\sqrt{5}$

$\therefore$ REQUIRED DISTANCE IS $\frac{9}{5}\sqrt{5} - 2$

Question 63 (****)

A circle has centre at $C(4,4)$ and passes through the point $P(6,8)$.

The straight line l_1 is a tangent to the circle at P.

a) Show that an equation of l_1 is

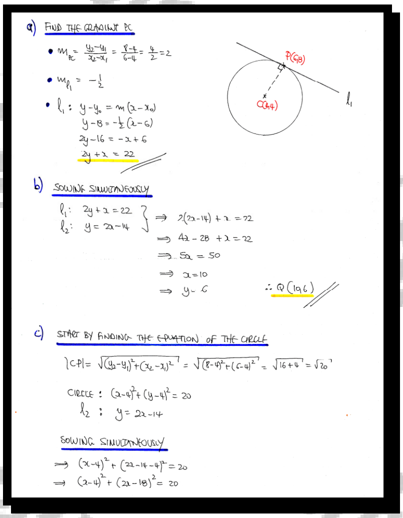
$$x+2y=22.$$

The straight line l_2 has equation $y=2x-14$ and meets l_1 at the point Q.

b) Find the coordinates of Q.

c) Show that l_2 is also a tangent to this circle at the point R, and determine the coordinates of R.

$Q(10,6)$, $R(8,2)$


a) FIND THE GRADIENT PC

- $m_{PC}=\frac{y_2-y_1}{x_2-x_1}=\frac{8-4}{6-4}=\frac{4}{2}=2$
- $m_{l_1}=-\frac{1}{2}$
- l_1: $y-y_0=m(x-x_0)$

$y-8=-\frac{1}{2}(x-6)$

$2y-16=-x+6$

$2y+x=22$

b) SOLVING SIMULTANEOUSLY

l_1: $2y+x=22$, l_2: $y=2x-14$ $\Rightarrow 2(2x-14)+x=22$

$\Rightarrow 4x-28+x=22$

$\Rightarrow 5x=50$

$\Rightarrow x=10$

$\Rightarrow y=6$

$\therefore Q(10,6)$

c) START BY FINDING THE EQUATION OF THE CIRCLE

$|CP|=\sqrt{(y_2-y_1)^2+(x_2-x_1)^2}=\sqrt{(8-4)^2+(6-4)^2}=\sqrt{16+4}=\sqrt{20}$

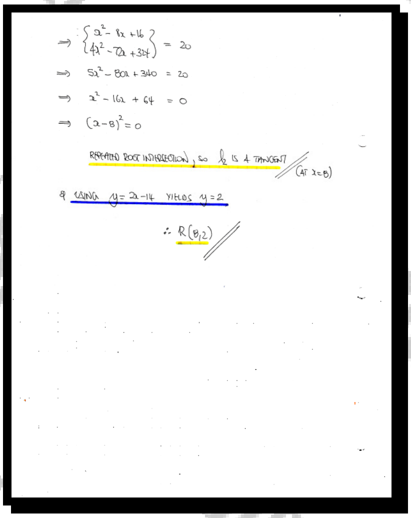
CIRCLE : $(x-4)^2+(y-4)^2=20$

l_2 : $y=2x-14$

SOLVING SIMULTANEOUSLY

$\Rightarrow (x-4)^2+(2x-14-4)^2=20$

$\Rightarrow (x-4)^2+(2x-18)^2=20$


$\Rightarrow \{x^2-8x+16\}+\{4x^2-72x+324\}=20$

$\Rightarrow 5x^2-80x+340=20$

$\Rightarrow x^2-16x+64=0$

$\Rightarrow (x-8)^2=0$

REPEATED ROOT INTERSECTION, SO l_2 IS A TANGENT (AT $x=8$)

& USING $y=2x-14$ YIELDS $y=2$

$\therefore R(8,2)$

Question 64 (****)

A circle C_1 has equation

$$x^2 + y^2 + 20x - 2y + 52 = 0.$$

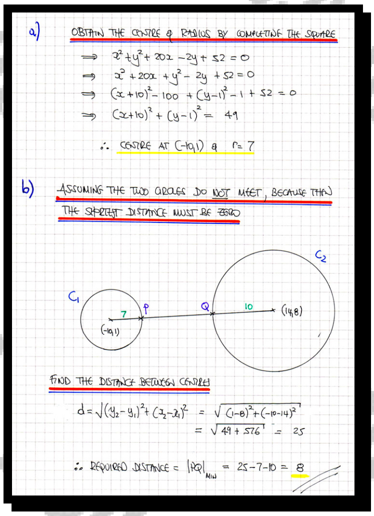
a) Determine the coordinates of the centre of C_1 and the size of its radius.

A different circle C_2 has its centre at $(14, 8)$ and the size of its radius is 10.

The point P lies on C_1 and the point Q lies on C_2.

b) Determine the minimum distance of PQ.

$\boxed{(-10,1),\ r=7}$, $\boxed{|PQ|_{\min} = 8}$


a) OBTAIN THE CENTRE & RADIUS BY COMPLETING THE SQUARE

$\Rightarrow x^2 + y^2 + 20x - 2y + 52 = 0$

$\Rightarrow x^2 + 20x + y^2 - 2y + 52 = 0$

$\Rightarrow (x+10)^2 - 100 + (y-1)^2 - 1 + 52 = 0$

$\Rightarrow (x+10)^2 + (y-1)^2 = 49$

$\therefore$ CENTRE AT $(-10,1)$ & $r = 7$

b) ASSUMING THE TWO CIRCLES DO NOT MEET, BECAUSE THEN THE SHORTEST DISTANCE MUST BE ZERO

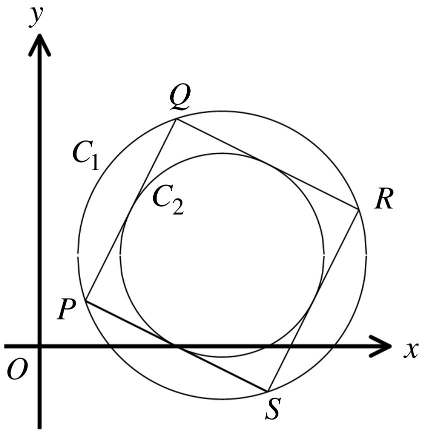
FIND THE DISTANCE BETWEEN CENTRES

$d = \sqrt{(y_2-y_1)^2 + (x_2-x_1)^2} = \sqrt{(1-8)^2 + (-10-14)^2}$

$= \sqrt{49 + 576} = 25$

$\therefore$ REQUIRED DISTANCE $= |PQ|_{\min} = 25 - 7 - 10 = 8$

Question 65 (****)



The figure above shows a square $PQRS$ with vertices at the points $P(1,1)$, $Q(3,5)$, $R(7,3)$ and $S(5,-1)$.

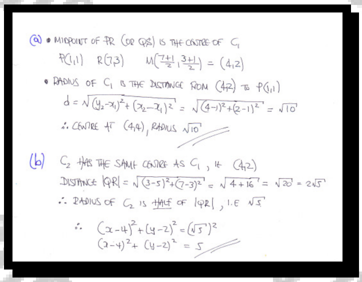
The square is circumscribed by the circle C_1.

a) Determine the coordinates of the centre of C_1 and the size of its radius.

A circle C_2 is inscribed in the square $PQRS$.

b) Find an equation of the circle C_2.

$\boxed{}$, $\boxed{r=\sqrt{10}}$, $\boxed{(4,2)}$, $\boxed{(x-4)^2+(y-2)^2=5}$



(a) • MIDPOINT OF PR (OR QS) IS THE CENTRE OF C_1

$P(1,1)$ $R(7,3)$ $M\left(\frac{7+1}{2},\frac{3+1}{2}\right)=(4,2)$

• RADIUS OF C_1 IS THE DISTANCE FROM (4,2) TO P(1,1)

$d=\sqrt{(y_2-x_1)^2+(x_2-x_1)^2}=\sqrt{(4-1)^2+(2-1)^2}=\sqrt{10}$

∴ CENTRE AT (4,4), RADIUS $\sqrt{10}$

(b) C_2 HAS THE SAME CENTRE AS C_1, IE (4,2)

DISTANCE $|QR|=\sqrt{(3-5)^2+(7-3)^2}=\sqrt{4+16}=\sqrt{20}=2\sqrt{5}$

∴ RADIUS OF C_2 IS HALF OF $|QR|$, I.E $\sqrt{5}$

∴ $(x-4)^2+(y-2)^2=(\sqrt{5})^2$

$(x-4)^2+(y-2)^2=5$

Question 66 (****)

A circle has equation

$$x^2 + y^2 - 4x - 4y + 6 = 0.$$

a) Determine the coordinates of the centre and the radius of the circle.

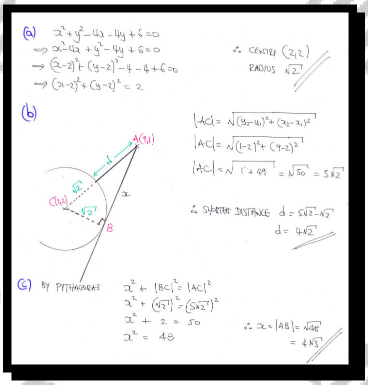
The point $A(9,1)$ lies outside the circle.

b) Find the shortest distance from A to the circle, giving the answer as a surd.

A tangent is drawn from the point A to the circle, touching the circle at the point B.

c) Determine, as an exact surd, the distance AB.

$\boxed{(2,2), r = \sqrt{2}}$, $\boxed{4\sqrt{2}}$, $\boxed{|AB| = 4\sqrt{3}}$



Question 67 (****)

A circle has equation

$$(x-2)^2+(y+2)^2=20.$$

a) Write down the coordinates of its centre the size of its radius.

b) Sketch the circle.

The sketch must include the coordinates of any points where the graph meets the coordinate axes.

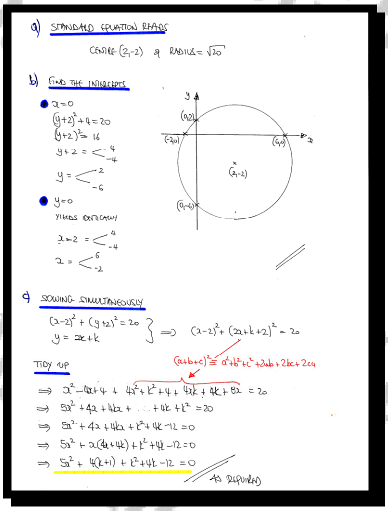
The straight line with equation $y=2x+k$, where k is a constant, meets the circle.

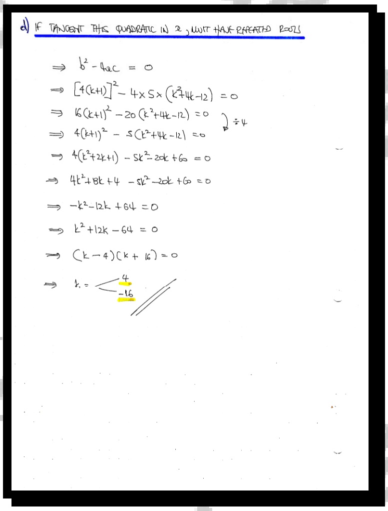
c) Show that the coordinates of any points of intersection between the line and the circle satisfies the equation

$$5x^2+4(k+1)x+k^2+4k-12=0.$$

d) Hence, find the two values of k for which the line $y=2x+k$ is a tangent to the circle.

$\boxed{}$, $\boxed{(2,-2),\ r=2\sqrt{5}}$, $\boxed{k=-16 \cup k=4}$





Question 68 (****)

A circle has centre at $C(2,3)$ and radius 6.

a) Show that an equation of the circle is

$$x^2 + y^2 - 4x - 6y = 23.$$

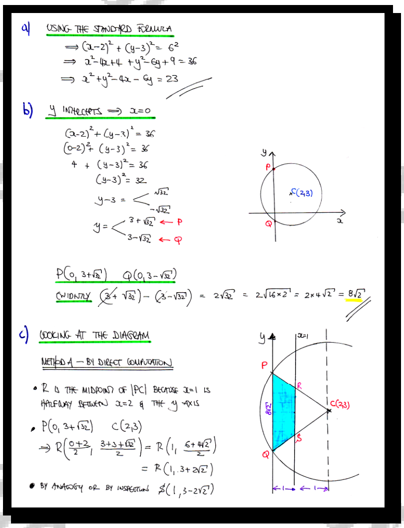
The circle crosses the y axis at the points P and Q, where the y coordinate of P is positive.

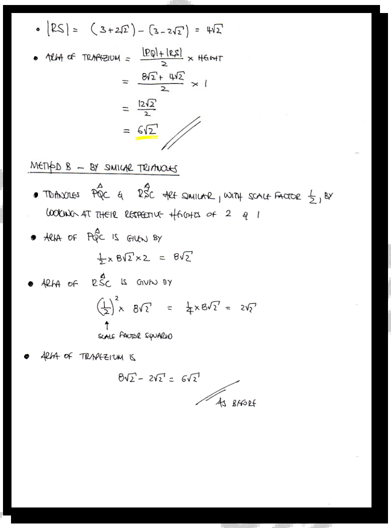
b) Find the distance PQ, giving the answer as an exact simplified surd.

The vertical straight line with equation $x = 1$ intersects the radii CP and CQ at the points R and S, respectively.

c) Determine the exact area of the trapezium $PQSR$.

$|PQ| = 8\sqrt{2}$, area $= 6\sqrt{2}$





Question 69 (****)

The points $A\left(\frac{96}{13},\frac{40}{13}\right)$ and $B\left(\frac{216}{13},\frac{90}{13}\right)$ are the endpoints of the diameter of circle.

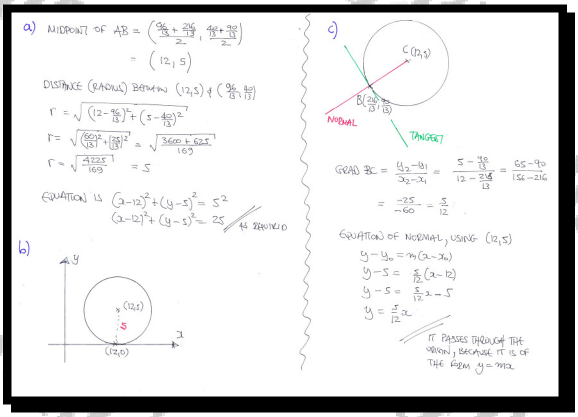
a) Show that an equation of the circle is

$$(x-12)^2+(y-5)^2=25.$$

b) Sketch the circle, indicating clearly all the relevant details.

c) Show that a normal to the circle at B passes through the origin.

proof



Question 70 **(****)**

A circle C_1 has equation

$$x^2 + y^2 - 10x + 4y = 71.$$

a) Find the coordinates of the centre of C_1 and the size of its radius.

Another circle C_2 is centred at the point with coordinates $(-4,10)$ and has radius 5.

b) Show that C_1 and C_2 **touch** each other.

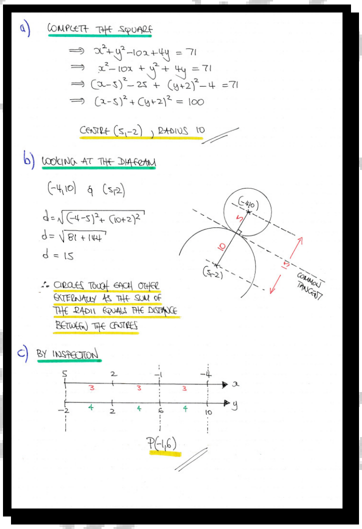
The two circles touch each other at the point P and the straight line L is the **common** tangent of C_1 and C_2.

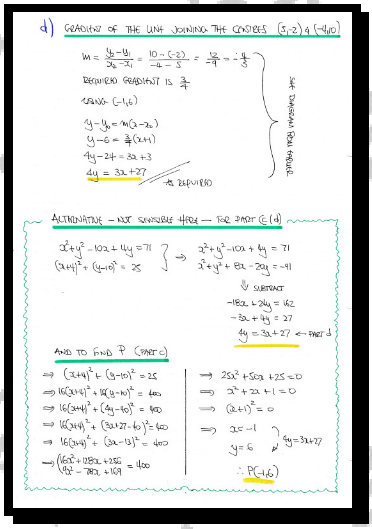
c) Determine the coordinates of P.

d) Show that an equation of L is

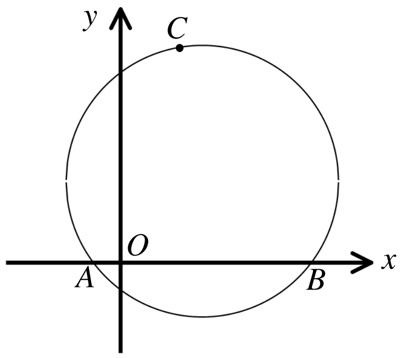
$$4y = 3x + 27.$$

$\boxed{}$, $\boxed{(5,-2)}$, $\boxed{r=10}$, $\boxed{(-1,6)}$





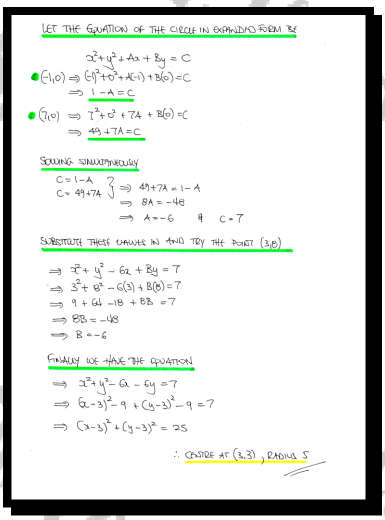
Question 71 (****)



The figure above shows a circle that crosses the x axis at the points $A(-1,0)$ and $B(7,0)$, while it passes through the point $C(3,8)$.

Determine the coordinates of the centre of the circle and the length of its radius.

$\boxed{}$, $\boxed{(3,3)}$, $\boxed{\text{radius} = 5}$



LET THE EQUATION OF THE CIRCLE IN EXPANDED FORM BE

$$x^2+y^2+Ax+By=C$$

- $(-1,0) \Rightarrow (-1)^2+0^2+A(-1)+B(0)=C$
  $\Rightarrow 1-A=C$
- $(7,0) \Rightarrow 7^2+0^2+7A+B(0)=C$
  $\Rightarrow 49+7A=C$

SOLVING SIMULTANEOUSLY

$C=1-A$, $C=49+7A$ $\Rightarrow 49+7A=1-A$
$\Rightarrow 8A=-48$
$\Rightarrow A=-6$ & $C=7$

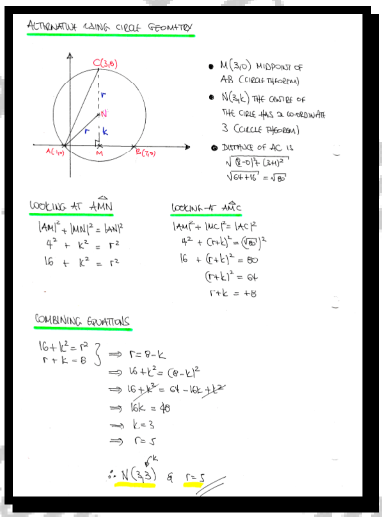
SUBSTITUTE THESE VALUES IN AND TRY THE POINT (3,8)

$\Rightarrow x^2+y^2-6x+By=7$
$\Rightarrow 3^2+8^2-6(3)+B(8)=7$
$\Rightarrow 9+64-18+8B=7$
$\Rightarrow 8B=-48$
$\Rightarrow B=-6$

FINALLY WE HAVE THE EQUATION

$\Rightarrow x^2+y^2-6x-6y=7$
$\Rightarrow (x-3)^2-9+(y-3)^2-9=7$
$\Rightarrow (x-3)^2+(y-3)^2=25$

$\therefore$ CENTRE AT (3,3), RADIUS 5



ALTERNATIVE USING CIRCLE GEOMETRY

- M(3,0) MIDPOINT OF AB (CIRCLE THEOREM)
- N(3,k) THE CENTRE OF THE CIRCLE HAS x CO-ORDINATE 3 (CIRCLE THEOREM)
- DISTANCE OF AC IS $\sqrt{(8-0)^2+(3+1)^2}$
  $\sqrt{64+16}=\sqrt{80}$

LOOKING AT $\hat{AMN}$

$|AM|^2+|MN|^2=|AN|^2$
$4^2+k^2=r^2$
$16+k^2=r^2$

LOOKING AT $\hat{AMC}$

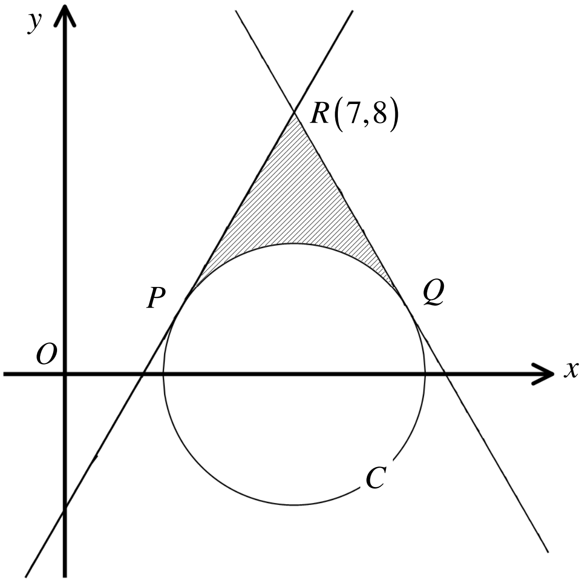
$|AM|^2+|MC|^2=|AC|^2$
$4^2+(r+k)^2=(\sqrt{80})^2$
$16+(r+k)^2=80$
$(r+k)^2=64$
$r+k=+8$

COMBINING EQUATIONS

$16+k^2=r^2$, $r+k=8$ $\Rightarrow r=8-k$
$\Rightarrow 16+k^2=(8-k)^2$
$\Rightarrow 16+k^2=64-16k+k^2$
$\Rightarrow 16k=48$
$\Rightarrow k=3$
$\Rightarrow r=5$

$\therefore N(3,3)$ & $r=5$

Question 72 (****)



The figure above shows the circle C with equation

$$x^2 + y^2 - 14x + 33 = 0.$$

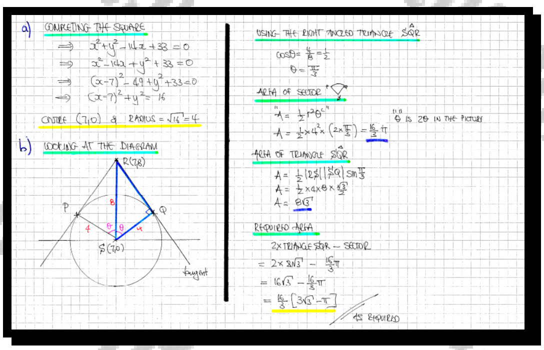
a) Determine the coordinates of the centre of C and the size of its radius.

The tangents to C from the point $R(7,8)$ meet C at the points P and Q.

b) Show that the area of the finite region bounded by C and the two tangents, shown shaded in figure, is

$$\frac{16}{3}\left[3\sqrt{3} - \pi\right].$$

$\boxed{}$, $\boxed{(7,0),\ r = 4}$



Question 73 (****)

A circle passes through the points $A(1,2)$, $B(3,-4)$ and $C(15,0)$.

a) Show that AB is perpendicular to BC.

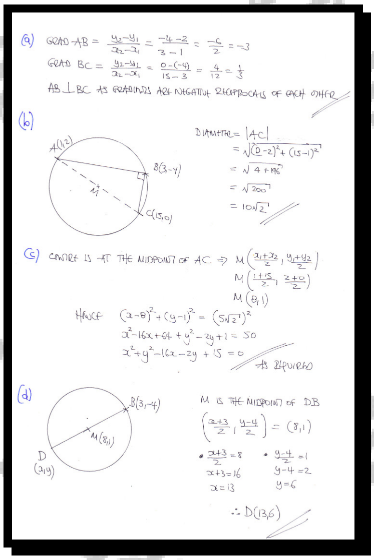
b) Hence find the exact length of the diameter of the circle.

c) Show that an equation of the circle is

$$x^2 + y^2 - 16x - 2y + 15 = 0.$$

d) Determine the coordinates of the point which lies on the circle and is furthest away from the point B.

$\boxed{\text{diameter} = 10\sqrt{2}}$, $\boxed{(13,6)}$



(a) GRAD AB $= \frac{y_2-y_1}{x_2-x_1} = \frac{-4-2}{3-1} = \frac{-6}{2} = -3$

GRAD BC $= \frac{y_2-y_1}{x_2-x_1} = \frac{0-(-4)}{15-3} = \frac{4}{12} = \frac{1}{3}$

AB ⊥ BC AS GRADIENTS ARE NEGATIVE RECIPROCALS OF EACH OTHER

(b)

DIAMETER $= |AC|$

$= \sqrt{(0-2)^2 + (15-1)^2}$

$= \sqrt{4+196}$

$= \sqrt{200}$

$= 10\sqrt{2}$

(c) CENTRE IS AT THE MIDPOINT OF AC $\Rightarrow M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

$M\left(\frac{1+15}{2}, \frac{2+0}{2}\right)$

$M(8,1)$

HENCE $(x-8)^2 + (y-1)^2 = (5\sqrt{2})^2$

$x^2 - 16x + 64 + y^2 - 2y + 1 = 50$

$x^2 + y^2 - 16x - 2y + 15 = 0$

AS REQUIRED

(d)

M IS THE MIDPOINT OF DB

$\left(\frac{x+3}{2}, \frac{y-4}{2}\right) = (8,1)$

• $\frac{x+3}{2} = 8$, $x+3 = 16$, $x = 13$

• $\frac{y-4}{2} = 1$, $y-4 = 2$, $y = 6$

$\therefore D(13,6)$

Question 74 (****)

A circle has equation

$$x^2 + y^2 = 8y .$$

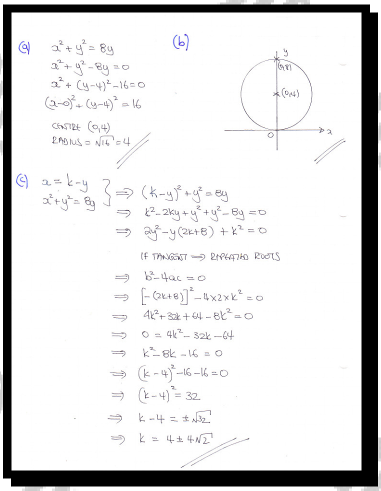
a) Find the coordinates of the centre of the circle and the size of its radius.

b) Sketch the circle.

The line with equation $x + y = k$, where k is a constant, is a tangent to this circle.

c) Determine, as exact surds, the possible values of k.

$\boxed{}$, $\boxed{(0,4)}$, $\boxed{r=4}$, $\boxed{k = 4 \pm 4\sqrt{2}}$


(a) $x^2 + y^2 = 8y$

$x^2 + y^2 - 8y = 0$

$x^2 + (y-4)^2 - 16 = 0$

$(x-0)^2 + (y-4)^2 = 16$

CENTRE $(0,4)$

RADIUS $= \sqrt{16} = 4$

(b)

(c) $x = k - y$, $x^2 + y^2 = 8y$ $\Rightarrow (k-y)^2 + y^2 = 8y$

$\Rightarrow k^2 - 2ky + y^2 + y^2 - 8y = 0$

$\Rightarrow 2y^2 - y(2k+8) + k^2 = 0$

IF TANGENT $\Rightarrow$ REPEATED ROOTS

$\Rightarrow b^2 - 4ac = 0$

$\Rightarrow [-(2k+8)]^2 - 4 \times 2 \times k^2 = 0$

$\Rightarrow 4k^2 + 32k + 64 - 8k^2 = 0$

$\Rightarrow 0 = 4k^2 - 32k - 64$

$\Rightarrow k^2 - 8k - 16 = 0$

$\Rightarrow (k-4)^2 - 16 - 16 = 0$

$\Rightarrow (k-4)^2 = 32$

$\Rightarrow k - 4 = \pm\sqrt{32}$

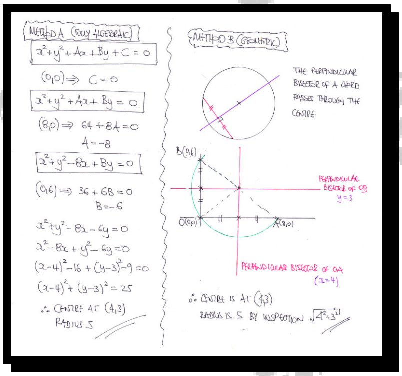
$\Rightarrow k = 4 \pm 4\sqrt{2}$

Question 75 (****)

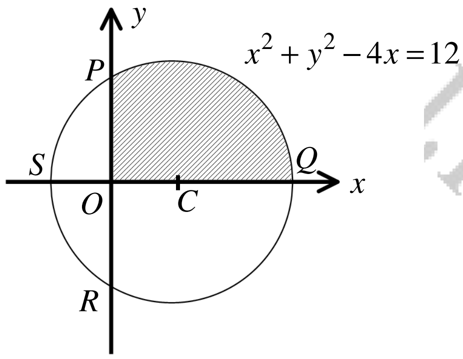
A circle passes through the points $(0,0)$, $(8,0)$ and $(0,6)$.

Determine the coordinates of the centre of the circle and the size of its radius.

$\boxed{}$, $\boxed{(4,3)}$, $\boxed{r=5}$



Question 76 **(****)**



The figure above shows the circle with equation

$$x^2 + y^2 - 4x = 12 .$$

The circle has centre at C and radius r

a) Find the coordinates of C and the value of r.

The circle crosses the coordinate axes at the points P, Q, R and S, as shown in the figure above.

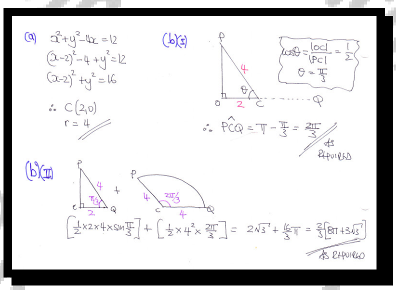
b) Show that …

i. … $\measuredangle PCQ = \dfrac{2\pi}{3}$.

ii. … the area of the shaded region bounded by the circle and the **positive sections** of the coordinate axes is

$$\tfrac{2}{3}\left(8\pi + 3\sqrt{3}\right).$$

[ ], $(2,0),\ r = 4$



Question 77 (****)

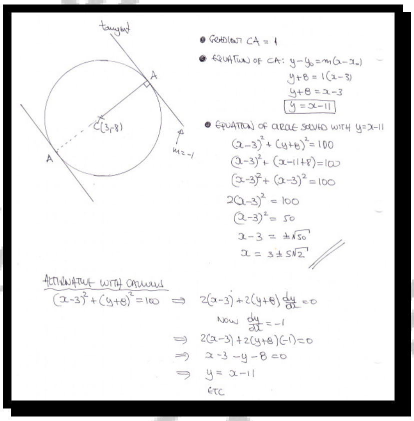
A circle has centre at $C(3,-8)$ and radius of 10 units.

The tangent to the circle at the point A has gradient -1.

Determine, as exact surds, the possible x coordinates of A.

*You **may not** use a calculus method in this question*

$\boxed{}$, $\boxed{x = 3 \pm 5\sqrt{2}}$



Question 78 (****)

A circle C has equation

$$x^2 + y^2 + 4x - 10y + 9 = 0 .$$

a) Find the coordinates of the centre of C and the size of its radius.

A tangent to the circle T, passes through the point with coordinates $(0,-1)$ and has gradient m, where $m < 0$.

b) Show that m is a solution of the equation

$$2m^2 - 3m - 2 = 0 .$$

The tangent T meets C at the point P.

c) Determine the coordinates of P.

$\boxed{}$, $\boxed{(-2,5),\ r = \sqrt{20}}$, $\boxed{P(-4,1)}$

a) COMPLETING THE SQUARE IN x & y

$x^2 + y^2 + 4x - 10y + 9 = 0$
$x^2 + 4x + y^2 - 10y + 9 = 0$
$(x+2)^2 - 4 + (y-5)^2 - 25 + 9 = 0$
$(x+2)^2 + (y-5)^2 = 20$

$\therefore$ CENTRE AT $(-2,5)$
RADIUS $\sqrt{20} = 2\sqrt{5}$

b) THE EQUATION OF THE LINE MUST BE $y = mx - 1$

$\left.\begin{array}{l} y = mx - 1 \\ (x+2)^2 + (y-5)^2 = 20 \end{array}\right\} \Rightarrow (x+2)^2 + (mx-1-5)^2 = 20$
$\Rightarrow (x+2)^2 + (mx-6)^2 = 20$
$\Rightarrow x^2 + 4x + 4 + m^2x^2 - 12mx + 36 = 20$
$\Rightarrow (1+m^2)x^2 + (4-12m)x + 20 = 0$

IF A TANGENT THIS EQUATION MUST HAVE REPEATED ROOTS

$b^2 - 4ac = 0 \Rightarrow (4-12m)^2 - 4(1+m^2)(+20) = 0$
$\Rightarrow 4^2(1-3m)^2 - 80(1+m^2) = 0$ $\quad \div 16$
$\Rightarrow (1-3m)^2 - 5(1+m^2) = 0$
$\Rightarrow 1 - 6m + 9m^2 - 5 - 5m^2 = 0$
$\Rightarrow 4m^2 - 6m - 4 = 0$
$\Rightarrow 2m^2 - 3m - 2 = 0$

AS REQUIRED

c) SOLVING THE EQUATION IN m

$\Rightarrow (2m+1)(m-2) = 0$
$\Rightarrow m = 2, \ -\frac{1}{2}$

BUT IT IS GIVEN THAT $m < 0 \Rightarrow m = -\frac{1}{2}$

$\Rightarrow (1+m^2)x^2 + (4-12m)x + 20 = 0$
$\Rightarrow (1+\frac{1}{4})x^2 + [4 - 12\times(-\frac{1}{2})]x + 20 = 0$
$\Rightarrow \frac{5}{4}x^2 + 10x + 20 = 0$
$\Rightarrow 5x^2 + 40x + 80 = 0$
$\Rightarrow x^2 + 8x + 16 = 0$
$\Rightarrow (x+4)^2 = 0$
$\Rightarrow x = -4$

& USING $y = -\frac{1}{2}x - 1$
$y = 2 - 1$
$y = 1$

$\therefore P(-4,1)$

Question 79 (****)

A circle has equation

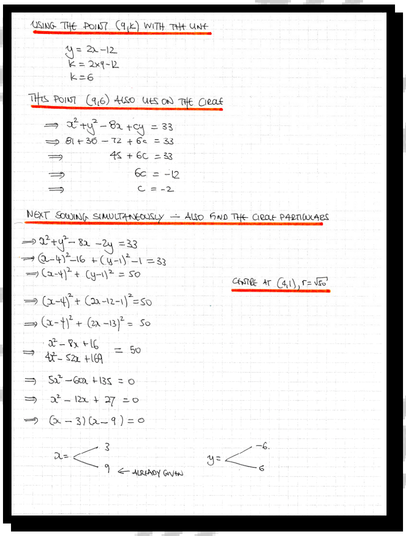
$$x^2 + y^2 - 8x + cy = 33,$$

where c is a positive constant.

The straight line L, with equation $y = 2x - 12$, intersects the circle at the point with coordinates $(9, k)$.

Find, as an exact surd, the perpendicular distance of L from the centre of the circle.

[SYN], [$D = \sqrt{5}$]



USING THE POINT $(9,k)$ WITH THE LINE

$y = 2x - 12$

$k = 2 \times 9 - 12$

$k = 6$

THIS POINT $(9,6)$ ALSO LIES ON THE CIRCLE

$\Rightarrow x^2 + y^2 - 8x + cy = 33$

$\Rightarrow 81 + 36 - 72 + 6c = 33$

$\Rightarrow 45 + 6c = 33$

$\Rightarrow 6c = -12$

$\Rightarrow c = -2$

NEXT SOLVING SIMULTANEOUSLY — ALSO FIND THE CIRCLE PARTICULARS

$\Rightarrow x^2 + y^2 - 8x - 2y = 33$

$\Rightarrow (x-4)^2 - 16 + (y-1)^2 - 1 = 33$

$\Rightarrow (x-4)^2 + (y-1)^2 = 50$

CENTRE AT $(4,1)$, $r = \sqrt{50}$

$\Rightarrow (x-4)^2 + (2x-12-1)^2 = 50$

$\Rightarrow (x-4)^2 + (2x-13)^2 = 50$

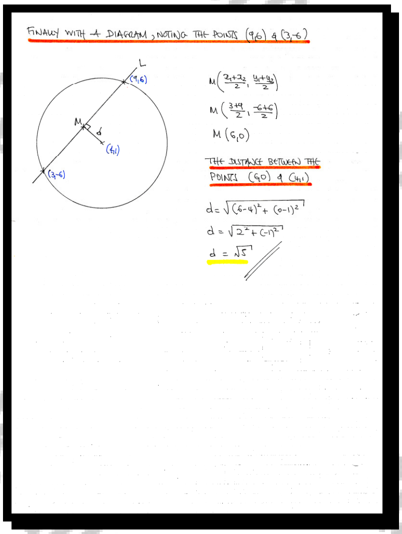
$\Rightarrow \begin{matrix} x^2 - 8x + 16 \\ 4x^2 - 52x + 169 \end{matrix} = 50$

$\Rightarrow 5x^2 - 60x + 135 = 0$

$\Rightarrow x^2 - 12x + 27 = 0$

$\Rightarrow (x-3)(x-9) = 0$

$x = 3$ or 9 ← ALREADY GIVEN $y = -6$ or 6



$M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

$M\left(\frac{3+9}{2}, \frac{-6+6}{2}\right)$

$M(6,0)$

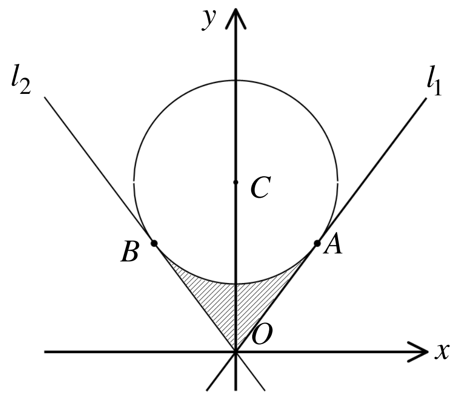
THE DISTANCE BETWEEN THE POINTS $(6,0)$ & $(4,1)$

$d = \sqrt{(6-4)^2 + (0-1)^2}$

$d = \sqrt{2^2 + (-1)^2}$

$d = \sqrt{5}$

Question 80 (****)



The figure above shows the circle with equation

$$9x^2 + (3y - 25)^2 = 225$$

whose centre is at C and its radius is r.

a) Determine the coordinates of C and the value of r.

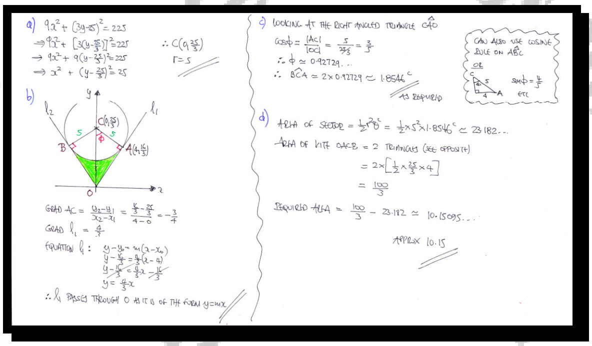
The points $A\left(4, \frac{16}{3}\right)$ and $B\left(-4, \frac{16}{3}\right)$ lie on the circle. The straight lines l_1 and l_2 are tangents to the circle at A and B, respectively.

b) Show that l_1 passes through the origin O.

c) Show further that the angle BCA is approximately 1.8546 radians.

d) Calculate the area of the shaded region, bounded by the circle, l_1 and l_2.

$\boxed{}$, $\boxed{C\left(0, \frac{25}{3}\right)}$, $\boxed{r = 5}$, $\boxed{\text{area} \approx 10.15...}$



Question 81 (****)

The circles C_1 and C_2 have respective equations

$$x^2 + y^2 - 6x - 2y = 15$$

$$x^2 + y^2 - 18x + 14y = 95.$$

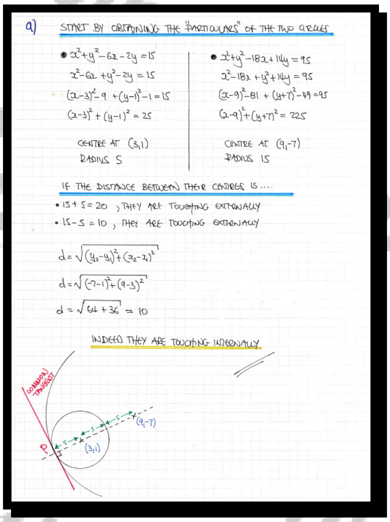
a) By considering the coordinates of the centres and the lengths of the radii of C_1 and C_2, show that C_1 and C_2 touch internally at some point P.

b) Determine the coordinates of P.

c) Show that the equation of the common tangent to the circles at P is given by

$$3x - 4y + 20 = 0.$$

$\boxed{}$, $\boxed{P(0,5)}$


a) START BY OBTAINING THE "PARTICULARS" OF THE TWO CIRCLES

- $x^2+y^2-6x-2y=15$
  $x^2-6x+y^2-2y=15$
  $(x-3)^2-9+(y-1)^2-1=15$
  $(x-3)^2+(y-1)^2=25$
  CENTRE AT $(3,1)$
  RADIUS 5
- $x^2+y^2-18x+14y=95$
  $x^2-18x+y^2+14y=95$
  $(x-9)^2-81+(y+7)^2-49=95$
  $(x-9)^2+(y+7)^2=225$
  CENTRE AT $(9,-7)$
  RADIUS 15

IF THE DISTANCE BETWEEN THEIR CENTRES IS ....

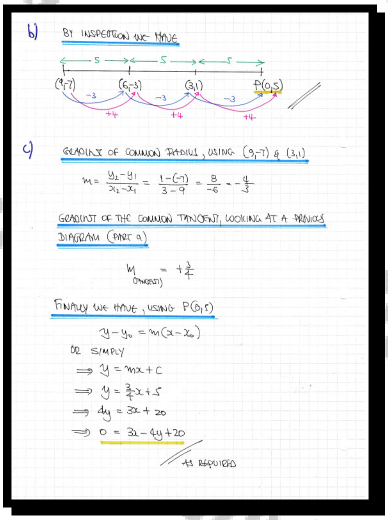
- $15+5=20$, THEY ARE TOUCHING EXTERNALLY
- $15-5=10$, THEY ARE TOUCHING EXTERNALLY

$d=\sqrt{(y_2-y_1)^2+(x_2-x_1)^2}$

$d=\sqrt{(-7-1)^2+(9-3)^2}$

$d=\sqrt{64+36}=10$

INDEED THEY ARE TOUCHING INTERNALLY


b) BY INSPECTION WE HAVE

c) GRADIENT OF COMMON RADIUS, USING $(9,-7)$ & $(3,1)$

$m=\frac{y_2-y_1}{x_2-x_1}=\frac{1-(-7)}{3-9}=\frac{8}{-6}=-\frac{4}{3}$

GRADIENT OF THE COMMON TANGENT, LOOKING AT A PREVIOUS DIAGRAM (PART a)

$m_{(\text{TANGENT})}=+\frac{3}{4}$

FINALLY WE HAVE, USING $P(0,5)$

$y-y_0=m(x-x_0)$

OR SIMPLY

$\Rightarrow y=mx+c$

$\Rightarrow y=\frac{3}{4}x+5$

$\Rightarrow 4y=3x+20$

$\Rightarrow 0=3x-4y+20$

AS REQUIRED

Question 82 (****)

A circle has equation

$$x^2 + y^2 - 4x - 6y + 8 = 0.$$

The straight line T_1 is a tangent to the circle at the point $P(4,4)$.

a) Find an equation of T_1.

The tangent T_1 passes through the point $Q(2,8)$.

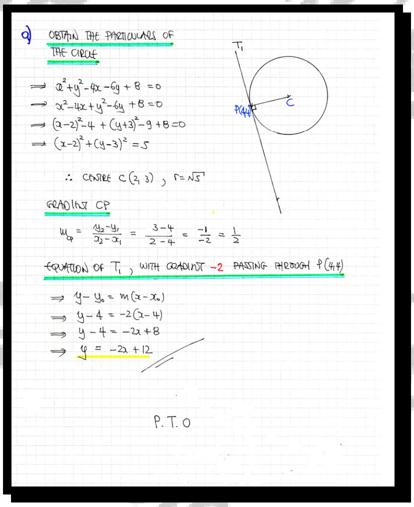
The straight line T_2 is a tangent to the circle at the point R and it also passes through the point Q.

b) Determine in any order

i. … the coordinates of R.

ii. … an equation of T_2.

$\boxed{}$, $\boxed{y = -2x + 12}$, $\boxed{R(0,4)}$, $\boxed{y = 2x + 4}$,


a) OBTAIN THE PARTICULARS OF THE CIRCLE

$\Rightarrow x^2+y^2-4x-6y+8=0$

$\Rightarrow x^2-4x+y^2-6y+8=0$

$\Rightarrow (x-2)^2-4+(y-3)^2-9+8=0$

$\Rightarrow (x-2)^2+(y-3)^2=5$

$\therefore$ CENTRE $C(2,3)$, $r=\sqrt{5}$

GRADIENT CP

$m_{CP} = \frac{y_2-y_1}{x_2-x_1} = \frac{3-4}{2-4} = \frac{-1}{-2} = \frac{1}{2}$

EQUATION OF T_1, WITH GRADIENT -2 PASSING THROUGH $P(4,4)$

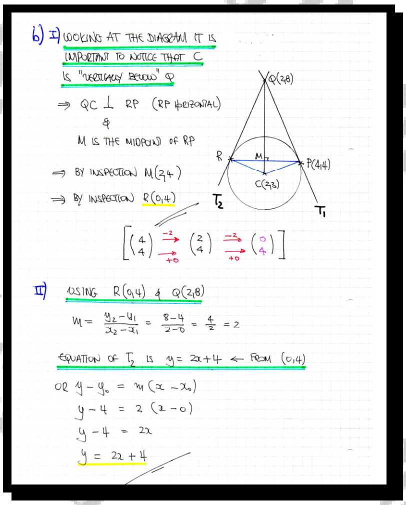
$\Rightarrow y-y_0=m(x-x_0)$

$\Rightarrow y-4=-2(x-4)$

$\Rightarrow y-4=-2x+8$

$\Rightarrow y=-2x+12$

P.T.O


b) I) LOOKING AT THE DIAGRAM IT IS IMPORTANT TO NOTICE THAT C IS "VERTICALLY BELOW" Q

$\Rightarrow QC \perp RP$ (RP HORIZONTAL) & M IS THE MIDPOINT OF RP

$\Rightarrow$ BY INSPECTION $M(2,4)$

$\Rightarrow$ BY INSPECTION $R(0,4)$

II) USING $R(0,4)$ & $Q(2,8)$

$m = \frac{y_2-y_1}{x_2-x_1} = \frac{8-4}{2-0} = \frac{4}{2} = 2$

EQUATION OF T_2 IS $y=2x+4$ ← FROM $(0,4)$

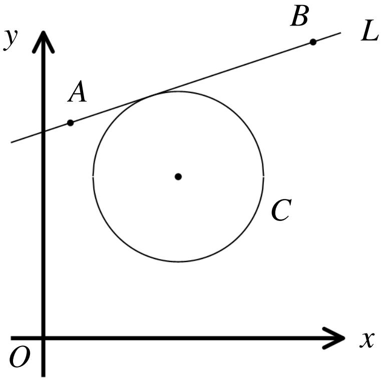
OR $y-y_0=m(x-x_0)$

$y-4=2(x-0)$

$y-4=2x$

$y=2x+4$

Question 83 (****+)

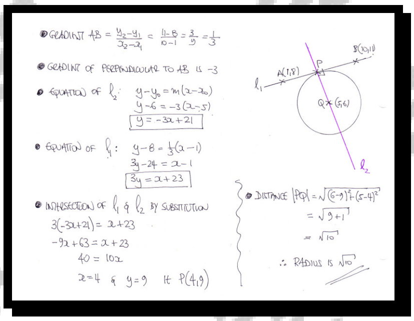


The figure above shows a circle C centred at the point with coordinates $(5,6)$, and the straight line L which passes though the points $A(1,8)$ and $B(10,11)$.

Given that L is a tangent to C, determine the radius of C.

[*You **may not** use a standard formula which finds the shortest distance of a point from a line*]

$\boxed{}$, $\boxed{r=\sqrt{10}}$



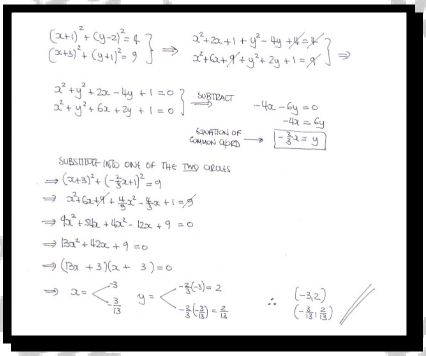
- GRADIENT $AB = \frac{y_2-y_1}{x_2-x_1} = \frac{11-8}{10-1} = \frac{3}{9} = \frac{1}{3}$
- GRADIENT OF PERPENDICULAR TO AB IS -3
- EQUATION OF l_2: $y-y_0 = m(x-x_0)$
  $y-6 = -3(x-5)$
  $\boxed{y=-3x+21}$
- EQUATION OF l_1: $y-8 = \frac{1}{3}(x-1)$
  $3y-24 = x-1$
  $\boxed{3y = x+23}$
- INTERSECTION OF l_1 & l_2 BY SUBSTITUTION
  $3(-3x+21) = x+23$
  $-9x+63 = x+23$
  $40 = 10x$
  $x=4$ & $y=9$ IE $P(4,9)$
- DISTANCE $|PQ| = \sqrt{(6-9)^2+(5-4)^2}$
  $= \sqrt{9+1}$
  $= \sqrt{10}$
  $\therefore$ RADIUS IS $\sqrt{10}$

Question 84 (****+)

Determine the exact coordinates of the points of intersection between the circles with equations

$$(x+1)^2+(y-2)^2=4$$
$$(x+3)^2+(y+1)^2=9$$

$$\boxed{(-3,2),\ \left(-\frac{3}{13},\frac{2}{13}\right)}$$



$(x+1)^2+(y-2)^2=4$, $(x+3)^2+(y+1)^2=9$ $\Rightarrow$ $x^2+2x+1+y^2-4y+4=4$, $x^2+6x+9+y^2+2y+1=9$ $\Rightarrow$

$x^2+y^2+2x-4y+1=0$, $x^2+y^2+6x+2y+1=0$ SUBTRACT $\Rightarrow$ $-4x-6y=0$

$-4x=6y$

EQUATION OF COMMON CHORD $\rightarrow$ $\boxed{-\frac{2}{3}x=y}$

SUBSTITUTE INTO ONE OF THE TWO CIRCLES

$\Rightarrow (x+3)^2+\left(-\frac{2}{3}x+1\right)^2=9$

$\Rightarrow x^2+6x+9+\frac{4}{9}x^2-\frac{4}{3}x+1=9$

$\Rightarrow 9x^2+54x+4x^2-12x+9=0$

$\Rightarrow 13x^2+42x+9=0$

$\Rightarrow (13x+3)(x+3)=0$

$\Rightarrow x = -3$ or $-\frac{3}{13}$, $y = -\frac{2}{3}(-3)=2$ or $-\frac{2}{3}\left(-\frac{3}{13}\right)=\frac{2}{13}$

$\therefore (-3,2)$, $\left(-\frac{3}{13},\frac{2}{13}\right)$

Question 85 **(****+)**

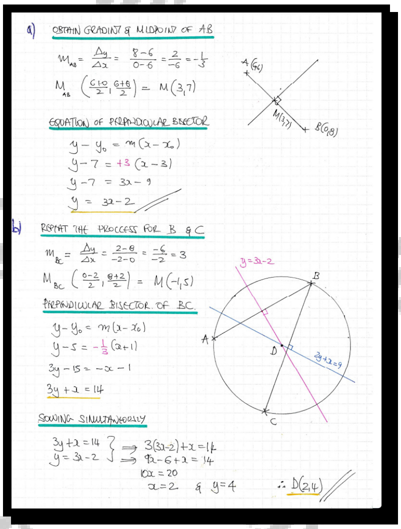
The points A, B and C have coordinates $(6,6)$, $(0,8)$ and $(-2,2)$, respectively.

a) Find an equation of the perpendicular bisector of AB.

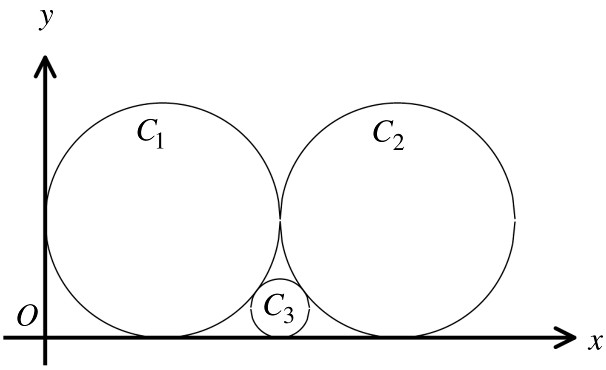
The points A, B and C lie on the circumference of a circle whose centre is located at the point D.

b) Determine the coordinates of D.

$\boxed{}$, $\boxed{y = 3x - 2}$, $\boxed{D(2,4)}$



Question 86 (****+)



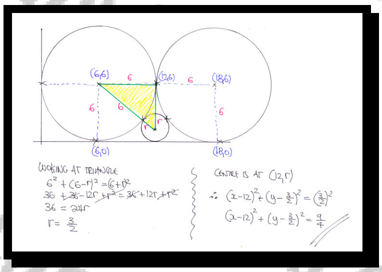
The figure above shows three circles C_1, C_2 and C_3.

The coordinates of the centres of all three circles are positive.

- The circle C_1 has centre at $(6,6)$ and **touches** both the x axis and the y axis.
- The circle C_2 has the same size radius as C_1 and **touches** the x axis.
- The circle C_3 **touches** the x axis and **both** C_1 and C_2.

Determine an equation of C_3.

, $(x-12)^2+\left(y-\frac{3}{2}\right)^2=\frac{9}{4}$



Question 87 (****+)

The circles C_1 and C_2 have respective equations

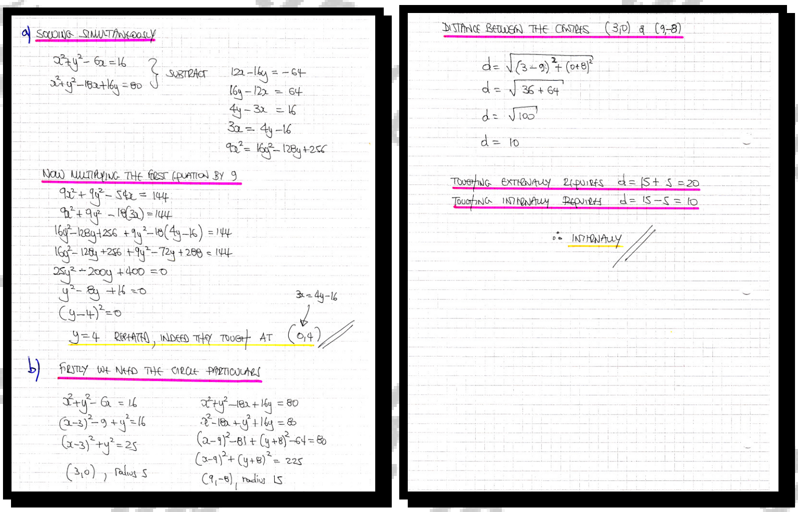
$$x^2 + y^2 - 6x = 16$$

$$x^2 + y^2 - 18x + 16y = 80.$$

a) By solving these equations simultaneously show that C_1 and C_2 touch at a point P and determine its coordinates.

b) Determine further whether C_1 and C_2 touch internally or externally.

MP1 , $P(0,4)$



a) SOLVING SIMULTANEOUSLY

$x^2+y^2-6x=16$
$x^2+y^2-18x+16y=80$ } SUBTRACT

$12x-16y=-64$
$16y-12x=64$
$4y-3x=16$
$3x=4y-16$
$9x^2=16y^2-128y+256$

NOW MULTIPLYING THE FIRST EQUATION BY 9

$9x^2+9y^2-54x=144$
$9x^2+9y^2-18(3x)=144$
$16y^2-128y+256+9y^2-18(4y-16)=144$
$16y^2-128y+256+9y^2-72y+288=144$
$25y^2-200y+400=0$
$y^2-8y+16=0$
$(y-4)^2=0$

$y=4$ REPEATED, INDEED THEY TOUCH AT $(0,4)$ [$3x=4y-16$]

b) FIRSTLY WE NEED THE CIRCLE PARTICULARS

$x^2+y^2-6x=16$
$(x-3)^2-9+y^2=16$
$(x-3)^2+y^2=25$
$(3,0)$, radius 5

$x^2+y^2-18x+16y=80$
$x^2-18x+y^2+16y=80$
$(x-9)^2-81+(y+8)^2-64=80$
$(x-9)^2+(y+8)^2=225$
$(9,-8)$, radius 15

DISTANCE BETWEEN THE CENTRES $(3,0)$ & $(9,-8)$

$d=\sqrt{(3-9)^2+(0+8)^2}$
$d=\sqrt{36+64}$
$d=\sqrt{100}$
$d=10$

TOUCHING EXTERNALLY REQUIRES $d=15+5=20$
TOUCHING INTERNALLY REQUIRES $d=15-5=10$

$\therefore$ INTERNALLY

Question 88 (****+)

Two circles have equations

$$x^2 + y^2 - 8x - 2y + 13 = 0,$$

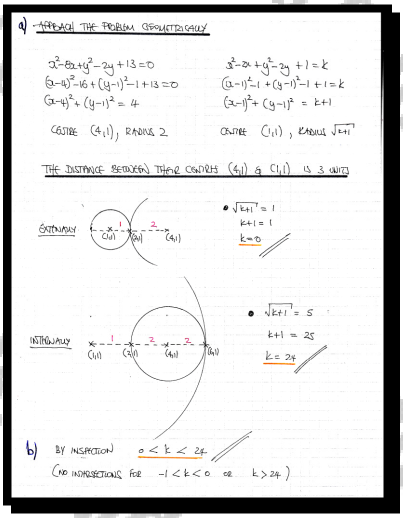
$$x^2 + y^2 - 2x - 2y + 1 = k,$$

where k is a constant.

a) Find the values of k, for which the two circles touch each other.

b) Hence state the range of values of k, for which the two circles intersect each other at exactly two points.

[ ] , $\boxed{k = 0, \quad k = 24}$, $\boxed{0 < k < 24}$



a) APPROACH THE PROBLEM GEOMETRICALLY

$x^2 - 8x + y^2 - 2y + 13 = 0$
$(x-4)^2 - 16 + (y-1)^2 - 1 + 13 = 0$
$(x-4)^2 + (y-1)^2 = 4$

CENTRE $(4,1)$, RADIUS 2

$x^2 - 2x + y^2 - 2y + 1 = k$
$(x-1)^2 - 1 + (y-1)^2 - 1 + 1 = k$
$(x-1)^2 + (y-1)^2 = k+1$

CENTRE $(1,1)$, RADIUS $\sqrt{k+1}$

THE DISTANCE BETWEEN THEIR CENTRES $(4,1)$ & $(1,1)$ IS 3 UNITS

EXTERNALLY

- $\sqrt{k+1} = 1$
  $k+1 = 1$
  $k = 0$

INTERNALLY

- $\sqrt{k+1} = 5$
  $k+1 = 25$
  $k = 24$

b) BY INSPECTION $0 < k < 24$

(NO INTERSECTIONS FOR $-1 < k < 0$ OR $k > 24$)

Question 89 (****+)

A circle has centre at $C(6,2)$ and radius of 4 units.

The point $P\left(6+2\sqrt{2},k\right)$ lies on this circle, where k is a positive constant.

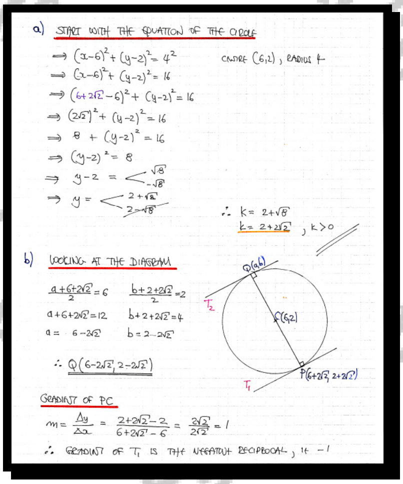
a) Determine the exact value of k.

The straight line T_1 is the tangent to the circle at the point P.

The straight line T_2 is another tangent to the circle so that T_2 is parallel to T_1.

b) Determine the equations of T_1 and T_2.

$\boxed{}$, $\boxed{k=2+2\sqrt{2}}$, $\boxed{T_1 : x+y=8+4\sqrt{2}}$, $\boxed{T_2 : x+y=8-4\sqrt{2}}$


a) START WITH THE EQUATION OF THE CIRCLE

$\Rightarrow (x-6)^2+(y-2)^2=4^2$ CENTRE $(6,2)$, RADIUS 4

$\Rightarrow (x-6)^2+(y-2)^2=16$

$\Rightarrow (6+2\sqrt{2}-6)^2+(y-2)^2=16$

$\Rightarrow (2\sqrt{2})^2+(y-2)^2=16$

$\Rightarrow 8+(y-2)^2=16$

$\Rightarrow (y-2)^2=8$

$\Rightarrow y-2 = \pm\sqrt{8}$

$\Rightarrow y = 2+\sqrt{8}$ or $2-\sqrt{8}$

$\therefore k=2+\sqrt{8}$

$k=2+2\sqrt{2}$, $k>0$

b) LOOKING AT THE DIAGRAM

$\frac{a+6+2\sqrt{2}}{2}=6$ $\qquad$ $\frac{b+2+2\sqrt{2}}{2}=2$

$a+6+2\sqrt{2}=12$ $\qquad$ $b+2+2\sqrt{2}=4$

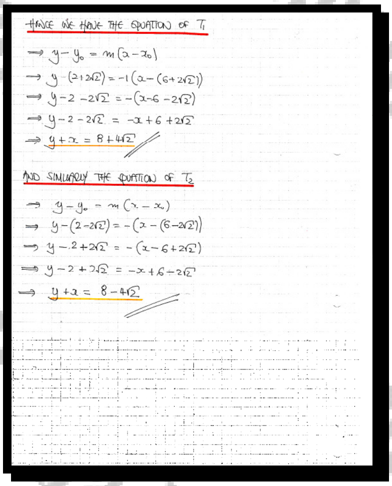
$a=6-2\sqrt{2}$ $\qquad$ $b=2-2\sqrt{2}$

$\therefore Q(6-2\sqrt{2}, 2-2\sqrt{2})$

GRADIENT OF PC

$m=\frac{\Delta y}{\Delta x}=\frac{2+2\sqrt{2}-2}{6+2\sqrt{2}-6}=\frac{2\sqrt{2}}{2\sqrt{2}}=1$

$\therefore$ GRADIENT OF T_1 IS THE NEGATIVE RECIPROCAL, IE -1


HENCE WE HAVE THE EQUATION OF T_1

$\Rightarrow y-y_0=m(x-x_0)$

$\Rightarrow y-(2+2\sqrt{2})=-1(x-(6+2\sqrt{2}))$

$\Rightarrow y-2-2\sqrt{2}=-(x-6-2\sqrt{2})$

$\Rightarrow y-2-2\sqrt{2}=-x+6+2\sqrt{2}$

$\Rightarrow y+x=8+4\sqrt{2}$

AND SIMILARLY THE EQUATION OF T_2

$\Rightarrow y-y_0=m(x-x_0)$

$\Rightarrow y-(2-2\sqrt{2})=-(x-(6-2\sqrt{2}))$

$\Rightarrow y-2+2\sqrt{2}=-(x-6+2\sqrt{2})$

$\Rightarrow y-2+2\sqrt{2}=-x+6-2\sqrt{2}$

$\Rightarrow y+x=8-4\sqrt{2}$

Question 90 (****+)

A circle has equation

$$x^2 + y^2 - 4x - 2y = 13.$$

a) Find the coordinates of the centre of the circle and the size of its radius.

The points A and B lie on the circle such that the length of AB is 6 units.

b) Show that $\measuredangle ACB = 90°$, where C is the centre of the circle.

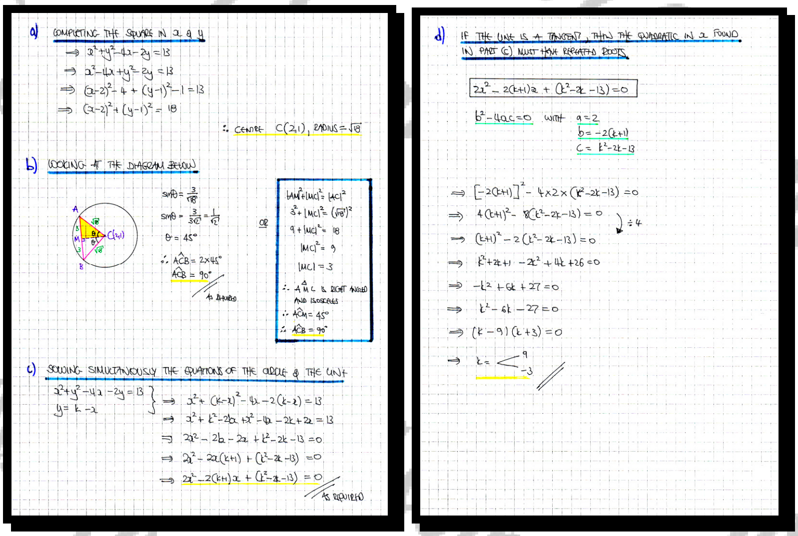
A tangent to the circle has equation $y = k - x$, where k is a constant.

c) Show clearly that

$$2x^2 - 2(k+1)x + k^2 - 2k - 13 = 0.$$

d) Determine the possible values of k.

$\boxed{}$, $\boxed{(2,1),\ r = \sqrt{18}}$, $\boxed{k = -3,\ k = 9}$



Question 91 **(****+)**

The straight line passing through the points $P(1,9)$ and $Q(5,5)$ is a tangent to a circle with centre at $C(6,8)$.

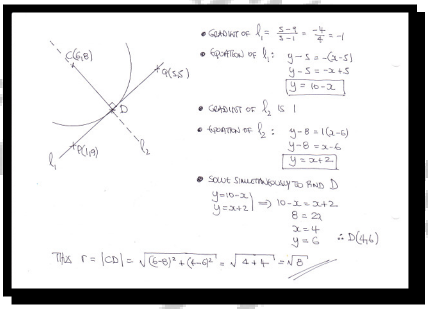
Determine, in exact surd form, the radius of the circle.

In this question you may **not** use …

… a standard formula which determines the shortest distance of a point from a straight line.

… any form of calculus.

$\boxed{}$, $\boxed{r=\sqrt{8}}$



Question 92 (****+)

The straight line with equation $y = 2x - 3$ is a tangent to a circle with centre at the point $C(2,-3)$.

Determine, in exact surd form, the radius of the circle.

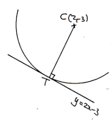
In this question you may **not** use …

… a standard formula which determines the shortest distance of a point from a straight line.

… any form of calculus.

$\boxed{}$, $\boxed{r = \frac{4}{5}\sqrt{5}}$

METHOD A – BY CO-ORDINATE GEOMETRY



- GRADIENT OF THE TANGENT IS 2
- GRADIENT CT MUST BE $-\frac{1}{2}$
- EQUATION OF LINE THROUGH C & T

$y-(-3) = -\frac{1}{2}(x-2)$

$y+3 = -\frac{1}{2}(x-2)$

$2y+6 = -x+2$

$2y+x+4=0$

- SOLVING SIMULTANEOUSLY WITH $y=2x-3$

$2(2x-3)+x+4=0$

$5x-2=0$

$x=\frac{2}{5}$

AND $y = 2\left(\frac{2}{5}\right)-3 = \frac{4}{5}-3 = -\frac{11}{5}$ $\therefore \left(\frac{2}{5},-\frac{11}{5}\right)$

- DISTANCE CT FINALLY, $C(2,-3)$ & $T\left(\frac{2}{5},-\frac{11}{5}\right)$

$d = \sqrt{(y_2-y_1)^2+(x_2-x_1)^2}$

$|CT| = \sqrt{\left(-3+\frac{11}{5}\right)^2+\left(\frac{2}{5}-2\right)^2}$

$r = \sqrt{\left(-\frac{4}{5}\right)^2+\left(-\frac{8}{5}\right)^2} = \sqrt{\frac{16}{25}+\frac{64}{25}} = \sqrt{\frac{80}{25}} = \frac{4}{5}\sqrt{5}$

METHOD B – USING DISCRIMINANTS

- LET THE CIRCLE HAVE EQUATION

$(x-2)^2+(y+3)^2=r^2$

- SOLVING SIMULTANEOUSLY WITH $y=2x-3$ TO "FIND" T

$\Rightarrow (x-2)^2+(2x-3+3)^2=r^2$

$\Rightarrow (x-2)^2+(2x)^2=r^2$

$\Rightarrow x^2-4x+4+4x^2=r^2$

$\Rightarrow 5x^2-4x+(4-r^2)=0$

- THIS EQUATION MUST PRODUCE REPEATED ROOTS AS THE POINT T IS A POINT OF TANGENCY

$b^2-4ac=0 \Rightarrow (-4)^2-4\times5\times(4-r^2)=0$

$\Rightarrow 16-20(4-r^2)=0$

$\Rightarrow 16-80+20r^2=0$

$\Rightarrow 20r^2=64$

$\Rightarrow r^2=\frac{64}{20}=\frac{64}{100}\times5$

$\Rightarrow r=\frac{4}{5}\sqrt{5}$ AS BEFORE

METHOD C – BY MINIMIZATION (COMPLETING THE SQUARE)

- CONSIDER A POINT ON THE LINE $y=2x-3$, IE $(x,2x-3)$
- THE DISTANCE FROM $(x,2x-3)$ TO THE CENTRE $(2,-3)$ IS GIVEN BY

$\Rightarrow d=\sqrt{(x-2)^2+[2x-3-(-3)]^2}$

$\Rightarrow d=\sqrt{(x-2)^2+4x^2}$

$\Rightarrow d^2=x^2-4x+4+4x^2$

$\Rightarrow d^2=5x^2-4x+4$

$\Rightarrow d^2=5\left[x^2-\frac{4}{5}x+\frac{4}{5}\right]$

$\Rightarrow d^2=5\left[\left(x-\frac{2}{5}\right)^2-\frac{4}{25}+\frac{4}{5}\right]$

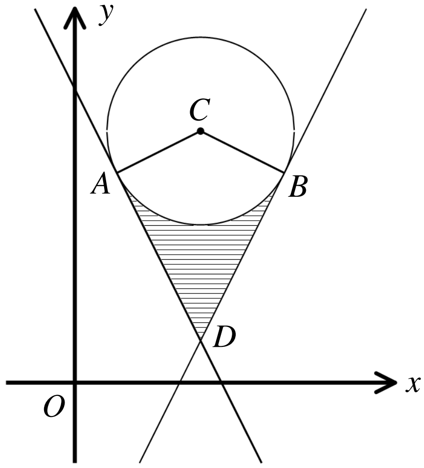
$\Rightarrow d^2=5\left(x-\frac{2}{5}\right)^2-\frac{4}{5}+4$

$\Rightarrow d^2=5\left(x-\frac{2}{5}\right)^2+\frac{16}{5}$

$\therefore$ MINIMUM VALUE OF d^2 IS $\frac{16}{5}$ (OCCURS AT $x=\frac{2}{5}$)

$\therefore d_{MIN}=r=\sqrt{\frac{16}{5}}=\sqrt{\frac{16\times5}{25}}=\frac{4}{5}\sqrt{5}$

Question 93 **(****+)**



The figure above shows a circle with centre at $C(3,6)$. The points $A(1,5)$ and $B(p,q)$ lie on the circle. The straight lines AD and BD are tangents to the circle.

The kite $CADB$ is symmetrical about the straight line with equation $x=3$.

a) Calculate the radius of the circle.

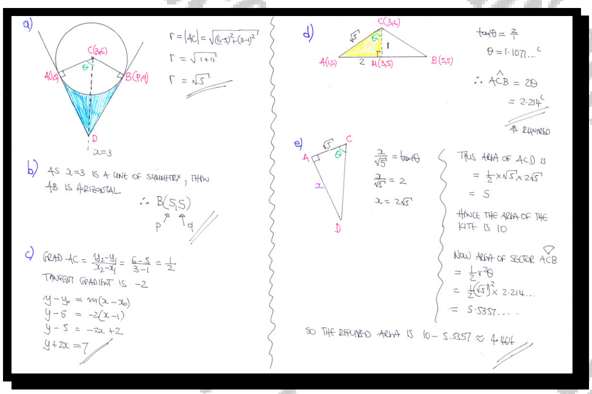
b) State the value of p and the value of q.

c) Find an equation of the tangent to the circle at A.

d) Show that the angle ACB is approximately 2.214 radians.

e) Hence determine, to three significant figures, the area of the shaded region bounded by the circle and its tangents at A and B.

[ ], $\boxed{r=\sqrt{5}}$, $\boxed{p=q=5}$, $\boxed{y=7-2x}$, $\boxed{\text{area} \approx 4.46}$



Question 94 (****+)

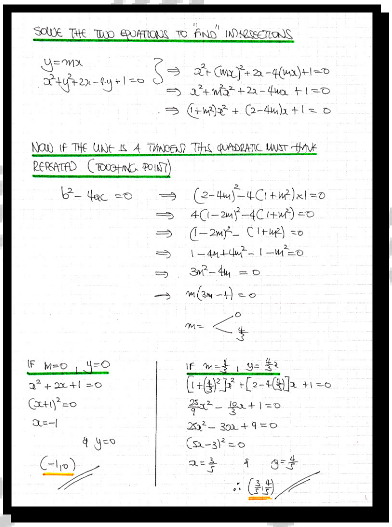
A circle C has equation

$$x^2 + y^2 + 2x - 4y + 1 = 0$$

The straight line L with equation $y = mx$ is a tangent to C.

Find the possible values of m and hence determine the possible coordinates at which L meets C.

$\boxed{}$, $\boxed{m = 0,\ m = \frac{4}{3}}$, $\boxed{(-1,0),\ \left(\frac{3}{5},\frac{4}{5}\right)}$


SOLVE THE TWO EQUATIONS TO "FIND" INTERSECTIONS

$y = mx$
$x^2 + y^2 + 2x - 4y + 1 = 0$ $\Rightarrow x^2 + (mx)^2 + 2x - 4(mx) + 1 = 0$
$\Rightarrow x^2 + m^2x^2 + 2x - 4mx + 1 = 0$
$\Rightarrow (1+m^2)x^2 + (2-4m)x + 1 = 0$

NOW IF THE LINE IS A TANGENT THIS QUADRATIC MUST HAVE REPEATED (TOUCHING POINT)

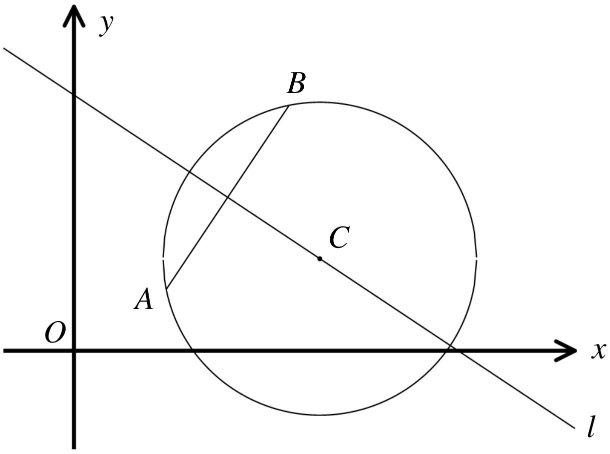
$b^2 - 4ac = 0 \Rightarrow (2-4m)^2 - 4(1+m^2)\times 1 = 0$
$\Rightarrow 4(1-2m)^2 - 4(1+m^2) = 0$
$\Rightarrow (1-2m)^2 - (1+m^2) = 0$
$\Rightarrow 1 - 4m + 4m^2 - 1 - m^2 = 0$
$\Rightarrow 3m^2 - 4m = 0$
$\Rightarrow m(3m-4) = 0$

$m = 0$ or $\frac{4}{3}$

IF $m = 0$, $y = 0$
$x^2 + 2x + 1 = 0$
$(x+1)^2 = 0$
$x = -1$ & $y = 0$
$(-1, 0)$

IF $m = \frac{4}{3}$, $y = \frac{4}{3}x$
$\left[1 + \left(\frac{4}{3}\right)^2\right]x^2 + \left[2 - 4\left(\frac{4}{3}\right)\right]x + 1 = 0$
$\frac{25}{9}x^2 - \frac{10}{3}x + 1 = 0$
$25x^2 - 30x + 9 = 0$
$(5x-3)^2 = 0$
$x = \frac{3}{5}$ & $y = \frac{4}{5}$
$\therefore \left(\frac{3}{5}, \frac{4}{5}\right)$

Question 95 (*****)



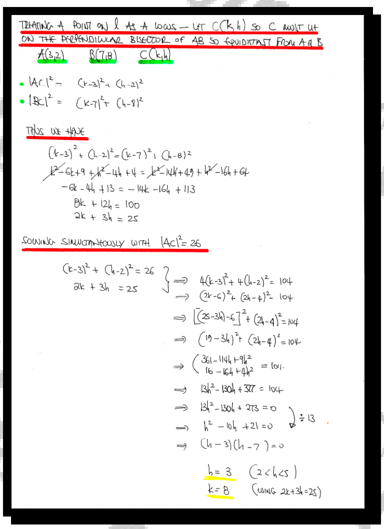
The figure above shows a circle whose centre is located at the point $C(k,h)$, where k and h are constants such that $2<h<5$.

The points $A(3,2)$ and $B(7,8)$ lie on this circle.

The straight line l passes through C and the midpoint of AB.

Given that the radius of the circle is $\sqrt{26}$, find an equation for l, the value of k and the value of h.

$\boxed{}$, $\boxed{2x+3y=25}$, $\boxed{k=8}$, $\boxed{h=3}$



TREATING A POINT ON l AS A LOCUS – LET $C(k,h)$ SO C MUST LIE ON THE PERPENDICULAR BISECTOR OF AB SO EQUIDISTANT FROM A & B

$A(3,2)$ $B(7,8)$ $C(k,h)$

- $|AC|^2 = (k-3)^2+(h-2)^2$
- $|BC|^2 = (k-7)^2+(h-8)^2$

THUS WE HAVE

$(k-3)^2+(h-2)^2=(k-7)^2+(h-8)^2$

$k^2-6k+9+h^2-4h+4=k^2-14k+49+h^2-16h+64$

$-6k-4h+13=-14k-16h+113$

$8k+12h=100$

$2k+3h=25$

SOLVING SIMULTANEOUSLY WITH $|AC|^2=26$

$(k-3)^2+(h-2)^2=26$, $2k+3h=25$

$\Rightarrow 4(k-3)^2+4(h-2)^2=104$

$\Rightarrow (2k-6)^2+(2h-4)^2=104$

$\Rightarrow [(25-3h)-6]^2+(2h-4)^2=104$

$\Rightarrow (19-3h)^2+(2h-4)^2=104$

$\Rightarrow 361-114h+9h^2+16-16h+4h^2=104$

$\Rightarrow 13h^2-130h+377=104$

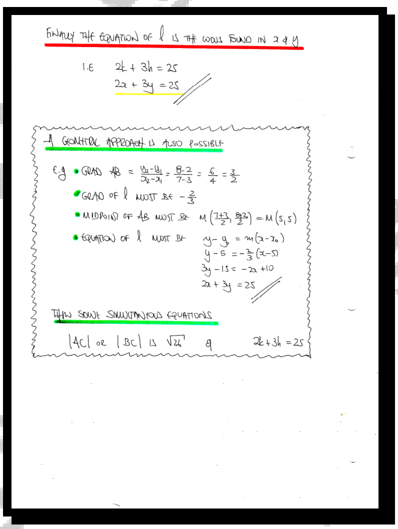
$\Rightarrow 13h^2-130h+273=0$ $\div 13$

$\Rightarrow h^2-10h+21=0$

$\Rightarrow (h-3)(h-7)=0$

$h=3$ $(2<h<5)$

$k=8$ (USING $2k+3h=25$)



FINALLY THE EQUATION OF l IS THE LOCUS FOUND IN x & y

I.E $2k+3h=25$

$2x+3y=25$

A GEOMETRIC APPROACH IS ALSO POSSIBLE

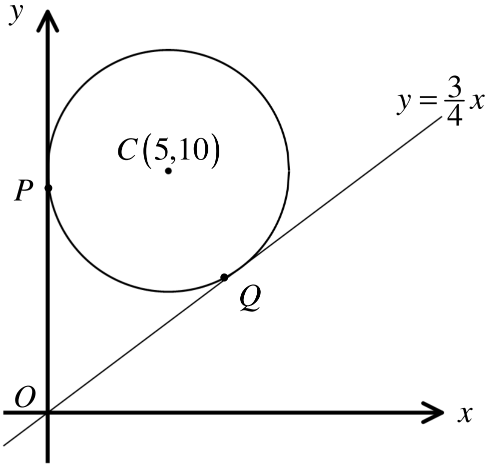
E.g.
- GRAD $AB=\frac{y_2-y_1}{x_2-x_1}=\frac{8-2}{7-3}=\frac{6}{4}=\frac{3}{2}$
- GRAD OF l MUST BE $-\frac{2}{3}$
- MIDPOINT OF AB MUST BE $M\left(\frac{7+3}{2},\frac{8+2}{2}\right)=M(5,5)$
- EQUATION OF l MUST BE $y-y_0=m(x-x_0)$
  $y-5=-\frac{2}{3}(x-5)$
  $3y-15=-2x+10$
  $2x+3y=25$

THEN SOLVE SIMULTANEOUS EQUATIONS

$|AC|$ OR $|BC|$ IS $\sqrt{26}$ & $2k+3h=25$

Question 96 (*****)

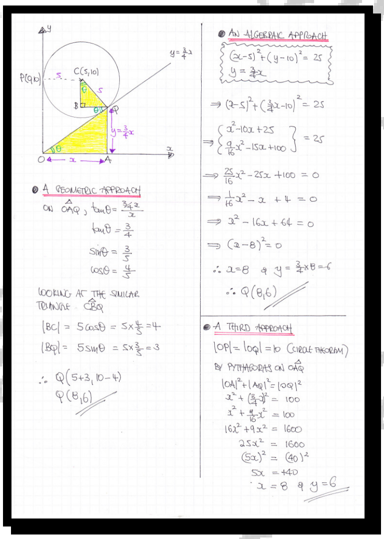
The figure below shows the circle with centre at $C(5,10)$ and radius 5.

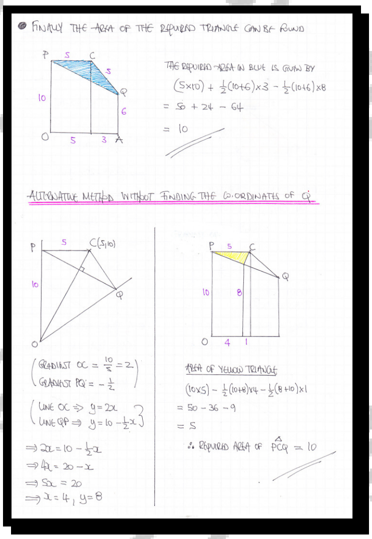


The straight lines with equations, $x = 0$ and $y = \frac{3}{4}x$ are tangents to the circle at the points P and Q respectively.

Show that the area of the triangle PCQ is 10 square units.

proof





Question 97 (*****)

The circle C_1 has equation

$$x^2 + y^2 - 4x - 4y + 6 = 0.$$

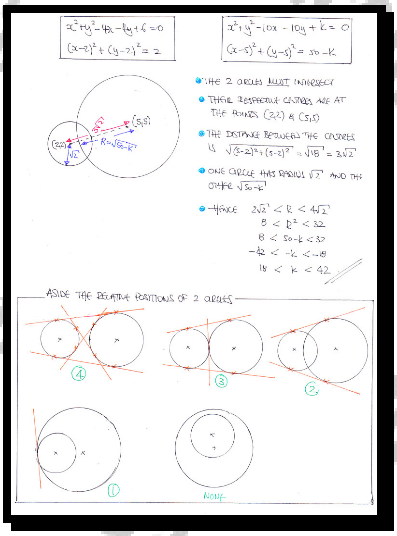
The circle C_2 has equation

$$x^2 + y^2 - 10x - 10y + k = 0,$$

where k is a constant.

Given that C_1 and C_2 have exactly two common tangents, determine the range of possible values of k.

$\boxed{}$, $\boxed{18 < k < 42}$



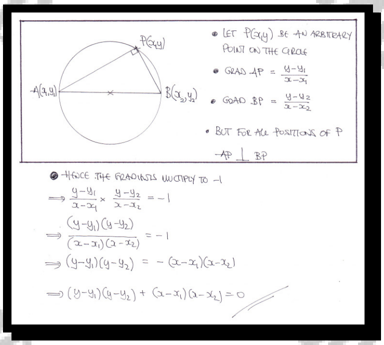
Question 98 (*****)

A circle passes through the points $A(x_1, y_1)$ and $A(x_2, y_2)$.

Given that AB is a diameter of the circle, show that the equation of the circle is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0.$$

[ ], proof

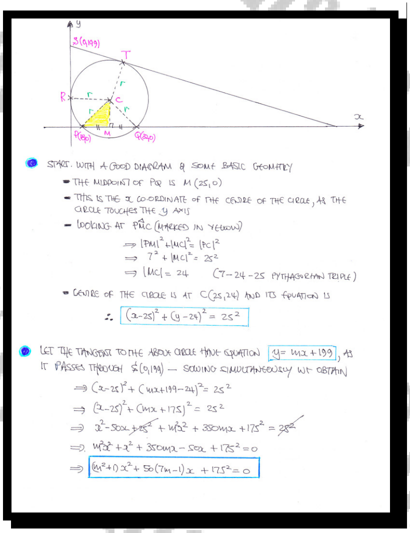


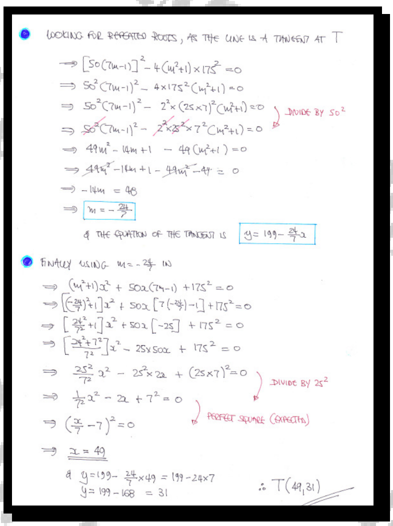
Question 99 (*****)

A circle passes through the points $P(18,0)$ and $Q(32,0)$. A tangent to this circle passes through the point $S(0,199)$ and touches the circle at the point T.

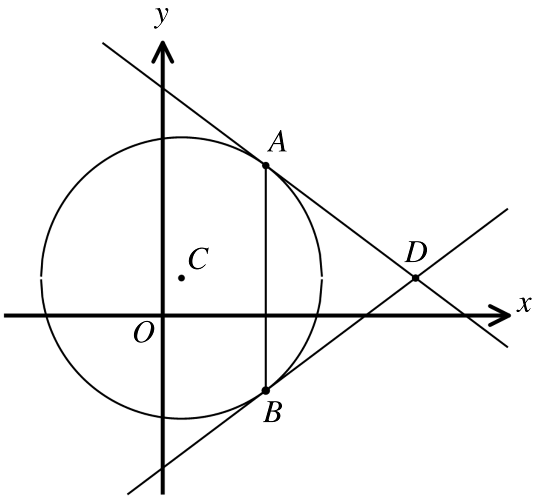
Given that the y axis is a tangent to this circle, determine the coordinates of T

[ ], $(49,31)$





Question 100 (*****) **non calculator**



The figure above shows the circle with equation

$$x^2 + y^2 - 4x - 8y = 205,$$

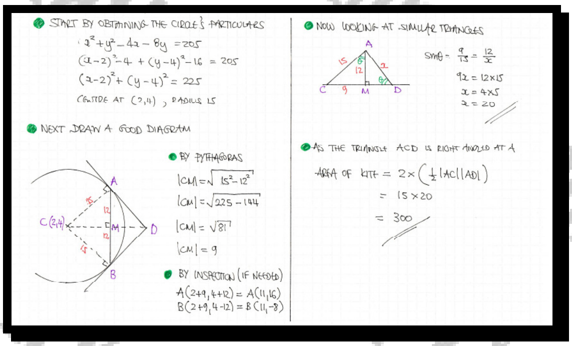
with centre at the point C and radius r.

The straight line AB is parallel to the y axis and has length 24 units.

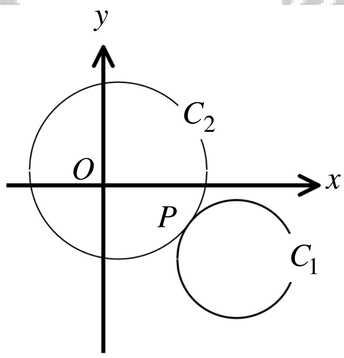
The tangents to the circle at A and B meet at the point D.

Find the length of AD and hence deduce the area of the kite $CADB$.

$\boxed{}$, $\boxed{|AD| = 20}$, $\boxed{\text{area} = 300}$



Question 101 (*****)



The figure above shows a circle C_1 with equation

$$x^2 + y^2 - 18x + ky + 90 = 0,$$

where k is a positive constant.

a) Determine, in terms of k, the coordinates of the centre of C_1 and the size of its radius.

Another circle C_2 has equation

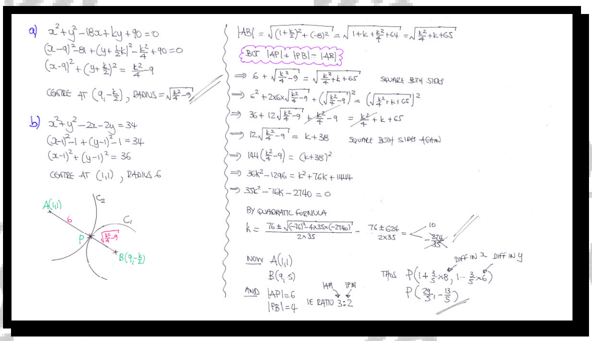
$$x^2 + y^2 - 2x - 2y = 34.$$

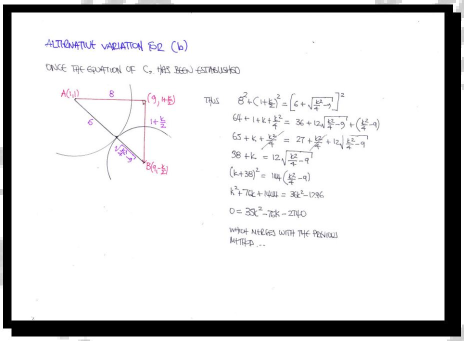
b) Given that C_1 and C_2 are **touching externally** at the point P, find …

i. … the value of k.

ii. … the coordinates of P.

$\boxed{}$, $\boxed{\left(9, -\frac{1}{2}k\right),\ r = \sqrt{\frac{k^2}{4} - 9}}$, $\boxed{k = 10}$, $\boxed{P\left(\frac{29}{5}, -\frac{13}{5}\right)}$





Question 102 (*****)

The curve C has equation

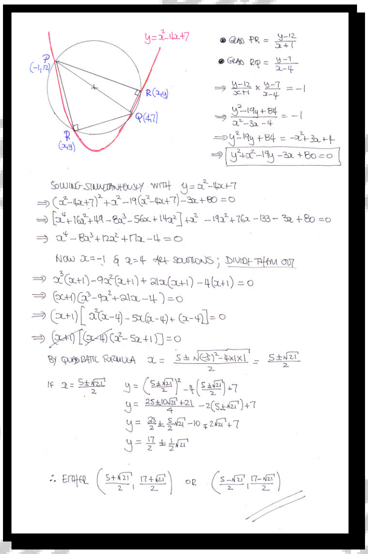
$$y = x^2 - 4x + 7.$$

The points $P(-1,12)$ and $Q(4,7)$ lie on C.

The point R also lies on C so that $\measuredangle PRQ = 90°$.

Determine, as exact surds, the possible coordinates of R.

$\left(\frac{5+\sqrt{21}}{2}, \frac{17+\sqrt{21}}{2}\right)$ or $\left(\frac{5-\sqrt{21}}{2}, \frac{17-\sqrt{21}}{2}\right)$



Question 103 (*****)

A circle C is centred at (a,a) and has radius a, where a is a positive constant.

The straight line L has equation

$$4x-3y+4=0.$$

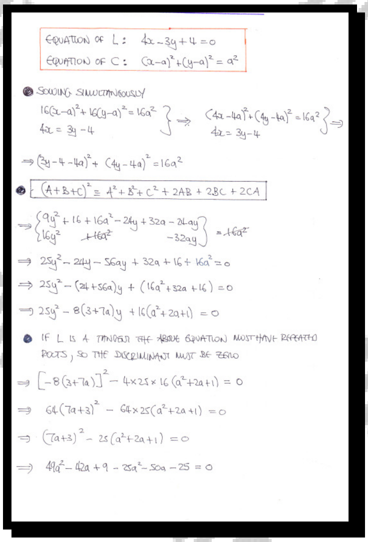
Given that L is tangent to C at the point P, determine …

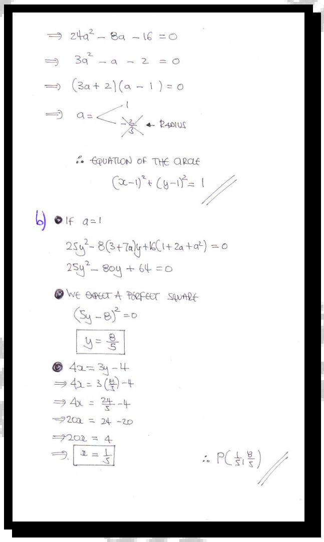
a) … an equation of C.

b) … the coordinates of P.

*You may **not** use a standard formula which determines the shortest distance of a point from a straight line in this question.*

$\boxed{}$, $\boxed{(x-1)^2+(y-1)^2=1}$, $\boxed{P\left(\frac{1}{5},\frac{8}{5}\right)}$





Question 104 (*****)

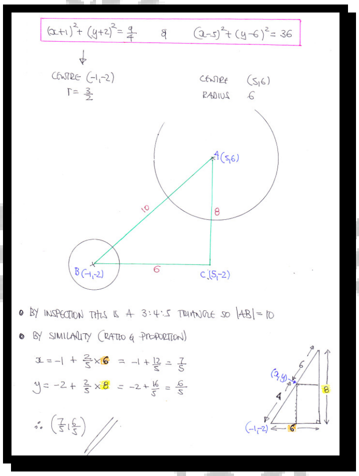
The circles C_1 and C_2 have respective equations

$$(x+1)^2+(y+2)^2=\frac{9}{4} \qquad \text{and} \qquad (x-5)^2+(y-6)^2=36.$$

The point P lies on C_2 so that the distance of P from C_1 is least.

Determine the exact coordinates of P.

$\boxed{}$, $\boxed{P\left(\frac{7}{5},\frac{6}{5}\right)}$



Question 105 (*****)

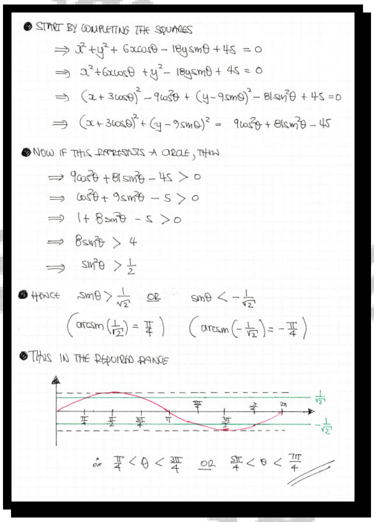
A curve in the x-y plane has equation

$$x^2 + y^2 + 6x\cos\theta - 18y\sin\theta + 45 = 0,$$

where θ is a parameter such that $0 \le \theta < 2\pi$.

Given that curve represents a circle determine the range of possible values of θ.

$\boxed{}$, $\boxed{\left\{\frac{1}{4}\pi < \theta < \frac{3}{4}\pi\right\} \cup \left\{\frac{5}{4}\pi < \theta < \frac{7}{4}\pi\right\}}$



- START BY COMPLETING THE SQUARES

$\Rightarrow x^2+y^2+6x\cos\theta-18y\sin\theta+45=0$

$\Rightarrow x^2+6x\cos\theta+y^2-18y\sin\theta+45=0$

$\Rightarrow (x+3\cos\theta)^2-9\cos^2\theta+(y-9\sin\theta)^2-81\sin^2\theta+45=0$

$\Rightarrow (x+3\cos\theta)^2+(y-9\sin\theta)^2 = 9\cos^2\theta+81\sin^2\theta-45$

- NOW IF THIS REPRESENTS A CIRCLE, THEN

$\Rightarrow 9\cos^2\theta+81\sin^2\theta-45>0$

$\Rightarrow \cos^2\theta+9\sin^2\theta-5>0$

$\Rightarrow 1+8\sin^2\theta-5>0$

$\Rightarrow 8\sin^2\theta>4$

$\Rightarrow \sin^2\theta>\frac{1}{2}$

- HENCE $\sin\theta>\frac{1}{\sqrt{2}}$ OR $\sin\theta<-\frac{1}{\sqrt{2}}$

$\left(\arcsin\left(\frac{1}{\sqrt{2}}\right)=\frac{\pi}{4}\right)$ $\left(\arcsin\left(-\frac{1}{\sqrt{2}}\right)=-\frac{\pi}{4}\right)$

- THUS IN THE REQUIRED RANGE

$\therefore \frac{\pi}{4}<\theta<\frac{3\pi}{4}$ OR $\frac{5\pi}{4}<\theta<\frac{7\pi}{4}$

Question 106 (*****)

Two circles, C_1 and C_2, have respective radii of 4 units and 1 unit and are touching each other externally at the point A.

The coordinates axes are tangents to C_1, whose centre P lies in the first quadrant.

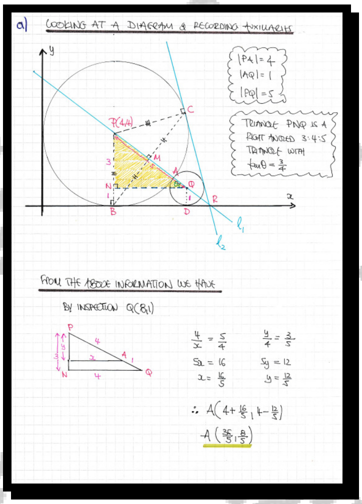
The x axis is a tangent to C_2, whose centre Q also lies in the first quadrant.

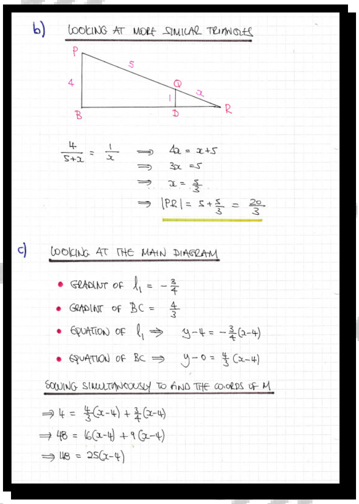
The straight line l_1, passes through P and Q, and meets the x axis at the point R.

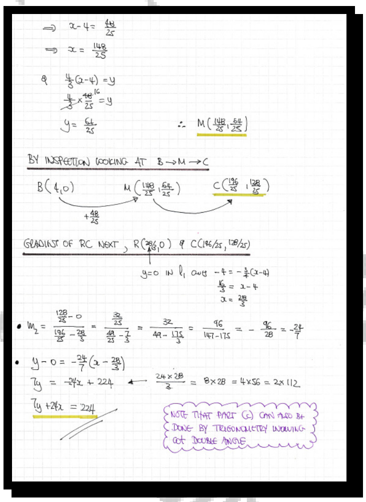
The straight line l_2 has negative gradient, passes through R and is a common tangent to C_1 and C_2.

Determine, in any order and in exact form where appropriate, the coordinates of A, the length of PR and an equation of l_2.

$\boxed{}$, $\boxed{A\left(\frac{36}{5},\frac{8}{5}\right)}$, $\boxed{|PR|=\frac{20}{3}}$, $\boxed{24x+7y=224}$







Question 107 (*****)

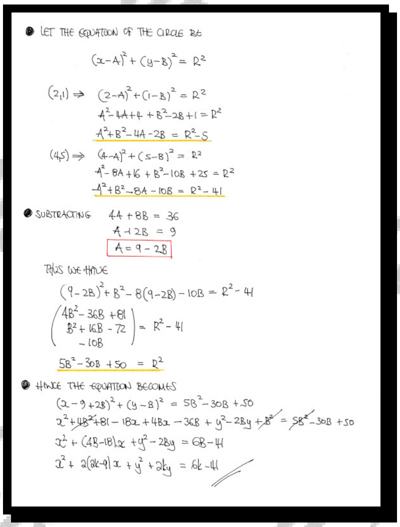
A family of circles is passing through the points with coordinates $(2,1)$ and $(4,5)$

Show that the equation of every such circle has equation

$$x^2 + y^2 + 2x(2k-9) + 2ky = 6k - 41,$$

where k is a parameter.

[ ], proof



- LET THE EQUATION OF THE CIRCLE BE

$$(x-A)^2 + (y-B)^2 = R^2$$

$(2,1) \Rightarrow (2-A)^2 + (1-B)^2 = R^2$

$A^2 - 4A + 4 + B^2 - 2B + 1 = R^2$

$A^2 + B^2 - 4A - 2B = R^2 - 5$

$(4,5) \Rightarrow (4-A)^2 + (5-B)^2 = R^2$

$A^2 - 8A + 16 + B^2 - 10B + 25 = R^2$

$A^2 + B^2 - 8A - 10B = R^2 - 41$

- SUBTRACTING $4A + 8B = 36$

$A + 2B = 9$

$A = 9 - 2B$

THUS WE HAVE

$(9-2B)^2 + B^2 - 8(9-2B) - 10B = R^2 - 41$

$\begin{pmatrix} 4B^2 - 36B + 81 \\ B^2 + 16B - 72 \\ -10B \end{pmatrix} = R^2 - 41$

$5B^2 - 30B + 50 = R^2$

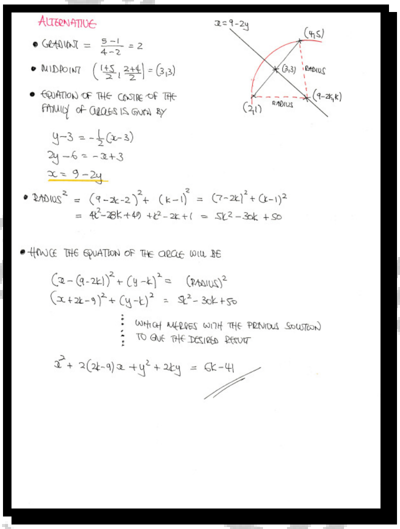
- HENCE THE EQUATION BECOMES

$(x - 9 + 2B)^2 + (y-B)^2 = 5B^2 - 30B + 50$

$x^2 + 4B^2 + 81 - 18x + 4Bx - 36B + y^2 - 2By + B^2 = 5B^2 - 30B + 50$

$x^2 + (4B-18)x + y^2 - 2By = 6B - 41$

$x^2 + 2(2k-9)x + y^2 + 2ky = 6k - 41$



ALTERNATIVE

- GRADIENT $= \frac{5-1}{4-2} = 2$
- MIDPOINT $\left(\frac{1+5}{2}, \frac{2+4}{2}\right) = (3,3)$
- EQUATION OF THE CENTRE OF THE FAMILY OF CIRCLES IS GIVEN BY

$y - 3 = -\frac{1}{2}(x-3)$

$2y - 6 = -x + 3$

$x = 9 - 2y$

- $\text{RADIUS}^2 = (9-2k-2)^2 + (k-1)^2 = (7-2k)^2 + (k-1)^2$

$= 4k^2 - 28k + 49 + k^2 - 2k + 1 = 5k^2 - 30k + 50$

- HENCE THE EQUATION OF THE CIRCLE WILL BE

$(x - (9-2k))^2 + (y-k)^2 = (\text{RADIUS})^2$

$(x + 2k - 9)^2 + (y-k)^2 = 5k^2 - 30k + 50$

⋮ WHICH MERGES WITH THE PREVIOUS SOLUTION TO GIVE THE DESIRED RESULT

$x^2 + 2(2k-9)x + y^2 + 2ky = 6k - 41$

Question 108 (*****)

The straight line with equation

$$y = t(x-2),$$

where t is a parameter, crosses the circle with equation

$$x^2 + y^2 = 1$$

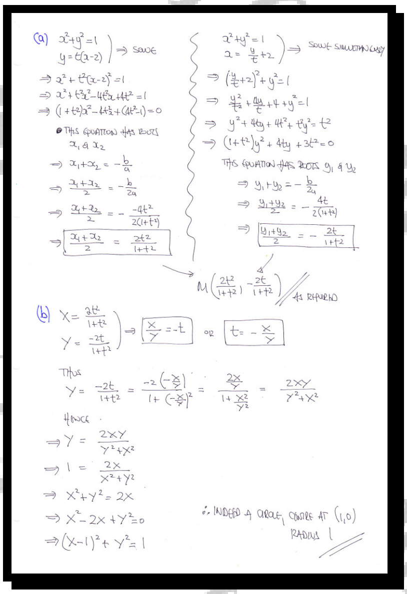
at two distinct points A and B.

a) Show that the coordinates of the midpoint of AB are given by

$$M\left(\frac{2t^2}{1+t^2}, -\frac{2t}{1+t^2}\right).$$

b) Hence show that the locus of M as t varies is a circle, stating its radius and the coordinates of its centre.

$\boxed{}$, $\boxed{(x-1)^2 + y^2 = 1}$



(a) $x^2+y^2=1$, $y=t(x-2)$ ⇒ SOLVE

$\Rightarrow x^2 + t^2(x-2)^2 = 1$

$\Rightarrow x^2 + t^2x^2 - 4t^2x + 4t^2 = 1$

$\Rightarrow (1+t^2)x^2 - 4t^2x + (4t^2-1) = 0$

• THIS EQUATION HAS ROOTS x_1 & x_2

$\Rightarrow x_1 + x_2 = -\frac{b}{a}$

$\Rightarrow \frac{x_1+x_2}{2} = -\frac{b}{2a}$

$\Rightarrow \frac{x_1+x_2}{2} = -\frac{-4t^2}{2(1+t^2)}$

$\Rightarrow \boxed{\frac{x_1+x_2}{2} = \frac{2t^2}{1+t^2}}$

$x^2+y^2=1$, $x = \frac{y}{t}+2$ ⇒ SOLVE SIMULTANEOUSLY

$\Rightarrow \left(\frac{y}{t}+2\right)^2 + y^2 = 1$

$\Rightarrow \frac{y^2}{t^2} + \frac{4y}{t} + 4 + y^2 = 1$

$\Rightarrow y^2 + 4ty + 4t^2 + t^2y^2 = t^2$

$\Rightarrow (1+t^2)y^2 + 4ty + 3t^2 = 0$

THIS EQUATION HAS ROOTS y_1 & y_2

$\Rightarrow y_1 + y_2 = -\frac{b}{2a}$

$\Rightarrow \frac{y_1+y_2}{2} = -\frac{4t}{2(1+t^2)}$

$\Rightarrow \boxed{\frac{y_1+y_2}{2} = -\frac{2t}{1+t^2}}$

$M\left(\frac{2t^2}{1+t^2}, -\frac{2t}{1+t^2}\right)$ // AS REQUIRED

(b) $X = \frac{2t^2}{1+t^2}$, $Y = \frac{-2t}{1+t^2}$ ⇒ $\boxed{\frac{X}{Y} = -t}$ OR $\boxed{t = -\frac{X}{Y}}$

THUS

$Y = \frac{-2t}{1+t^2} = \frac{-2\left(-\frac{X}{Y}\right)}{1+\left(-\frac{X}{Y}\right)^2} = \frac{\frac{2X}{Y}}{1+\frac{X^2}{Y^2}} = \frac{2XY}{Y^2+X^2}$

HENCE

$\Rightarrow Y = \frac{2XY}{Y^2+X^2}$

$\Rightarrow 1 = \frac{2X}{X^2+Y^2}$

$\Rightarrow X^2+Y^2 = 2X$

$\Rightarrow X^2 - 2X + Y^2 = 0$

$\Rightarrow (X-1)^2 + Y^2 = 1$

∴ INDEED A CIRCLE, CENTRE AT $(1,0)$, RADIUS 1 //

Question 109 (*****)

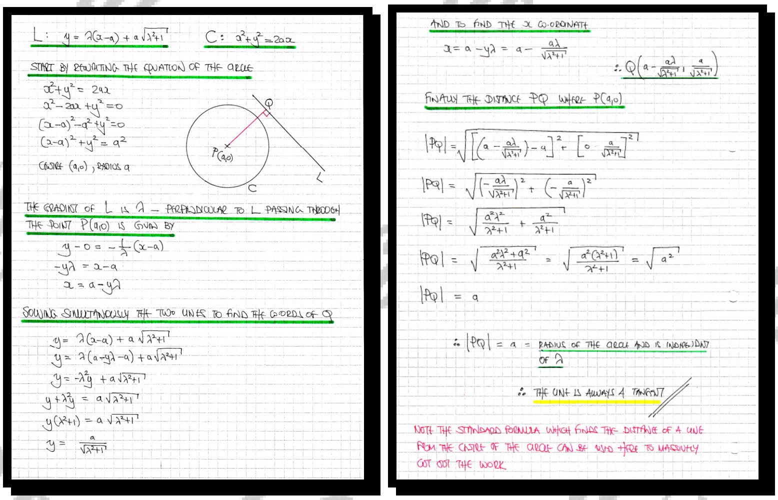
The straight line L and the circle C, have respective equations

$$L : y = \lambda(x-a) + a\sqrt{\lambda^2+1} \quad \text{and} \quad C : x^2 + y^2 = 2ax,$$

where a is a positive constant and λ is a parameter.

Show that for all values of λ, L is a tangent to C.

[ ] , [proof]


$L: y = \lambda(x-a) + a\sqrt{\lambda^2+1}$ $\quad C: x^2+y^2 = 2ax$

START BY REWRITING THE EQUATION OF THE CIRCLE

$x^2+y^2 = 2ax$

$x^2 - 2ax + y^2 = 0$

$(x-a)^2 - a^2 + y^2 = 0$

$(x-a)^2 + y^2 = a^2$

CENTRE $(a,0)$, RADIUS a

THE GRADIENT OF L IS λ – PERPENDICULAR TO L PASSING THROUGH THE POINT $P(a,0)$ IS GIVEN BY

$y - 0 = -\frac{1}{\lambda}(x-a)$

$-y\lambda = x - a$

$x = a - y\lambda$

SOLVING SIMULTANEOUSLY THE TWO LINES TO FIND THE COORDS OF Q

$y = \lambda(x-a) + a\sqrt{\lambda^2+1}$

$y = \lambda(a - y\lambda - a) + a\sqrt{\lambda^2+1}$

$y = -\lambda^2 y + a\sqrt{\lambda^2+1}$

$y + \lambda^2 y = a\sqrt{\lambda^2+1}$

$y(\lambda^2+1) = a\sqrt{\lambda^2+1}$

$y = \frac{a}{\sqrt{\lambda^2+1}}$

AND TO FIND THE x CO-ORDINATE

$x = a - y\lambda = a - \frac{a\lambda}{\sqrt{\lambda^2+1}}$

$\therefore Q\left(a - \frac{a\lambda}{\sqrt{\lambda^2+1}}, \frac{a}{\sqrt{\lambda^2+1}}\right)$

FINALLY THE DISTANCE PQ WHERE $P(a,0)$

$|PQ| = \sqrt{\left[\left(a - \frac{a\lambda}{\sqrt{\lambda^2+1}}\right) - a\right]^2 + \left[0 - \frac{a}{\sqrt{\lambda^2+1}}\right]^2}$

$|PQ| = \sqrt{\left(-\frac{a\lambda}{\sqrt{\lambda^2+1}}\right)^2 + \left(-\frac{a}{\sqrt{\lambda^2+1}}\right)^2}$

$|PQ| = \sqrt{\frac{a^2\lambda^2}{\lambda^2+1} + \frac{a^2}{\lambda^2+1}}$

$|PQ| = \sqrt{\frac{a^2\lambda^2+a^2}{\lambda^2+1}} = \sqrt{\frac{a^2(\lambda^2+1)}{\lambda^2+1}} = \sqrt{a^2}$

$|PQ| = a$

$\therefore |PQ| = a$ = RADIUS OF THE CIRCLE AND IS INDEPENDENT OF λ

$\therefore$ THE LINE IS ALWAYS A TANGENT

NOTE THE STANDARD FORMULA WHICH FINDS THE DISTANCE OF A LINE FROM THE CENTRE OF THE CIRCLE CAN BE USED HERE TO MASSIVELY CUT OUT THE WORK

Question 110 (*****)

Two parallel straight lines, L_1 and L_2, have respective equations

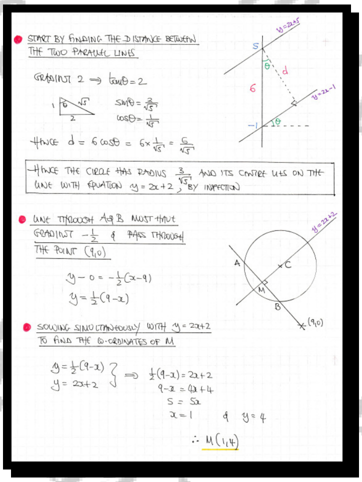
$$y = 2x + 5 \qquad \text{and} \qquad y = 2x - 1 .$$

L_1 and L_2, are tangents to a circle centred at the point C.

A third line L_3 is perpendicular to L_1 and L_2, and meets the circle in two distinct points, A and B.

Given that L_3 passes through the point $(9,0)$, find, in exact simplified surd form, the coordinates of C.

$$C\left[\frac{1}{10}\left(5+\sqrt{61}\right), \frac{1}{5}\left(15+\sqrt{61}\right)\right]$$



START BY FINDING THE DISTANCE BETWEEN THE TWO PARALLEL LINES

GRADIENT 2 $\Rightarrow \tan\theta = 2$

$\sin\theta = \frac{2}{\sqrt{5}}$, $\cos\theta = \frac{1}{\sqrt{5}}$

HENCE $d = 6\cos\theta = 6 \times \frac{1}{\sqrt{5}} = \frac{6}{\sqrt{5}}$

HENCE THE CIRCLE HAS RADIUS $\frac{3}{\sqrt{5}}$ AND ITS CENTRE LIES ON THE LINE WITH EQUATION $y = 2x+2$, BY INSPECTION

LINE THROUGH A & B MUST HAVE GRADIENT $-\frac{1}{2}$ & PASS THROUGH THE POINT $(9,0)$

$y - 0 = -\frac{1}{2}(x-9)$

$y = \frac{1}{2}(9-x)$

SOLVING SIMULTANEOUSLY WITH $y = 2x+2$ TO FIND THE CO-ORDINATES OF M

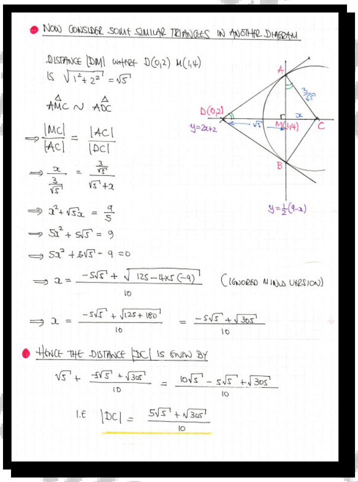
$y = \frac{1}{2}(9-x)$, $y = 2x+2$ $\Rightarrow \frac{1}{2}(9-x) = 2x+2$

$9 - x = 4x + 4$

$5 = 5x$

$x = 1$ & $y = 4$

$\therefore M(1,4)$



NOW CONSIDER SOME SIMILAR TRIANGLES IN ANOTHER DIAGRAM

DISTANCE $|DM|$ WHERE $D(0,2)$ $M(1,4)$ IS $\sqrt{1^2+2^2} = \sqrt{5}$

$\hat{AMC} \sim \hat{ADC}$

$\Rightarrow \frac{|MC|}{|AC|} = \frac{|AC|}{|DC|}$

$\Rightarrow \frac{x}{\frac{3}{\sqrt{5}}} = \frac{\frac{3}{\sqrt{5}}}{\sqrt{5}+x}$

$\Rightarrow x^2 + \sqrt{5}x = \frac{9}{5}$

$\Rightarrow 5x^2 + 5\sqrt{5} = 9$

$\Rightarrow 5x^2 + 5\sqrt{5} - 9 = 0$

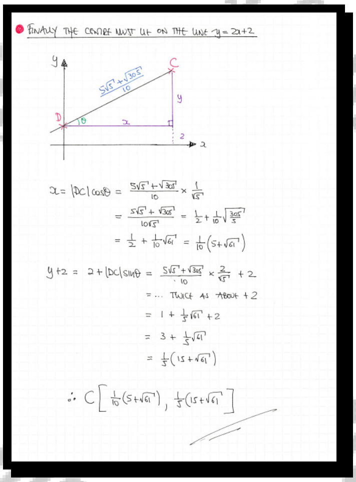
$\Rightarrow x = \frac{-5\sqrt{5} + \sqrt{125 - 4\times5(-9)}}{10}$ (IGNORED MINUS VERSION)

$\Rightarrow x = \frac{-5\sqrt{5} + \sqrt{125+180}}{10} = \frac{-5\sqrt{5}+\sqrt{305}}{10}$

HENCE THE DISTANCE $|DC|$ IS GIVEN BY

$\sqrt{5} + \frac{-5\sqrt{5}+\sqrt{305}}{10} = \frac{10\sqrt{5} - 5\sqrt{5} + \sqrt{305}}{10}$

I.E $|DC| = \frac{5\sqrt{5}+\sqrt{305}}{10}$



FINALLY THE CENTRE MUST LIE ON THE LINE $y = 2x+2$

$x = |DC|\cos\theta = \frac{5\sqrt{5}+\sqrt{305}}{10} \times \frac{1}{\sqrt{5}}$

$= \frac{5\sqrt{5}+\sqrt{305}}{10\sqrt{5}} = \frac{1}{2} + \frac{1}{10}\sqrt{\frac{305}{5}}$

$= \frac{1}{2} + \frac{1}{10}\sqrt{61} = \frac{1}{10}(5+\sqrt{61})$

$y+2 = 2 + |DC|\sin\theta = \frac{5\sqrt{5}+\sqrt{305}}{10} \times \frac{2}{\sqrt{5}} + 2$

$= \dots$ TWICE AS ABOVE $+2$

$= 1 + \frac{1}{5}\sqrt{61} + 2$

$= 3 + \frac{1}{5}\sqrt{61}$

$= \frac{1}{5}(15+\sqrt{61})$

$\therefore C\left[\frac{1}{10}(5+\sqrt{61}), \frac{1}{5}(15+\sqrt{61})\right]$

Question 111 (*****)

Three circles, C_1, C_2 and C_3, have their centres at A, B and C, respectively, so that $|AB| = 5$, $|AC| = 4$ and $|BC| = 3$.

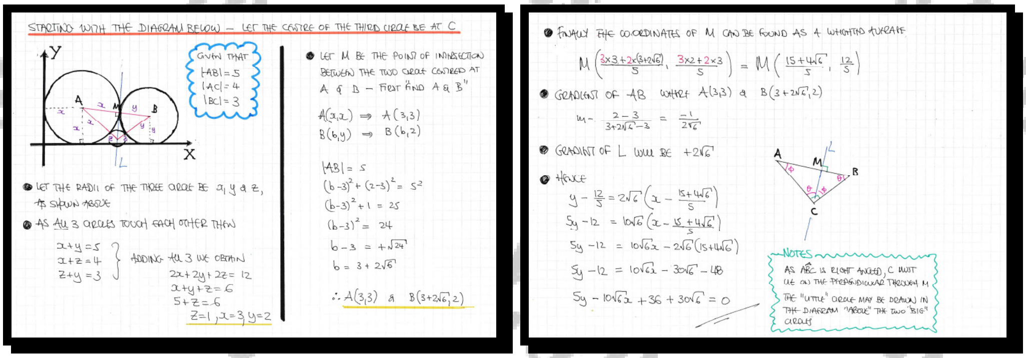
The positive x and y axis are tangents to C_1.

The positive x axis is a tangent to C_2.

C_1 and C_2 touch each other externally at the point M.

Given further that C_3 touches externally both C_1 and C_2, find, in exact simplified form, an equation of the straight line which passes through M and C.

, $\boxed{5y - 10\sqrt{6}\,x + 36 + 30\sqrt{6} = 0}$



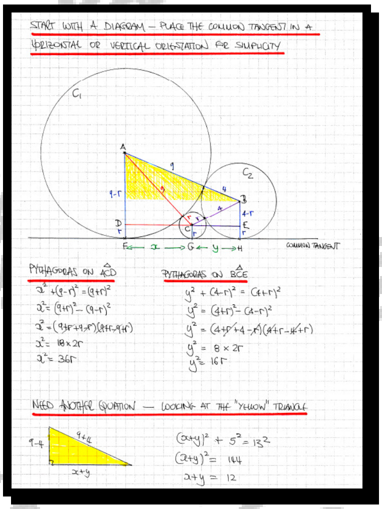
Question 112 (*****)

Two circles, C_1 and C_2, are touching each other **externally**, and have respective radii of 9 and 4 units.

A third circle C_3, of radius r, touches C_1 **and** C_2 **externally**.

Given further that all three circles have a common tangent, determine the value of r.

$\boxed{}$, $\boxed{r = \dfrac{36}{25} = 1.44}$


START WITH A DIAGRAM – PLACE THE COMMON TANGENT IN A HORIZONTAL OR VERTICAL ORIENTATION FOR SIMPLICITY

PYTHAGORAS ON $\hat{ACD}$

$x^2 + (9-r)^2 = (9+r)^2$

$x^2 = (9+r)^2 - (9-r)^2$

$x^2 = (9+r+9-r)(9+r-9+r)$

$x^2 = 18 \times 2r$

$x^2 = 36r$

PYTHAGORAS ON $\hat{BCE}$

$y^2 + (4-r)^2 = (4+r)^2$

$y^2 = (4+r)^2 - (4-r)^2$

$y^2 = (4+r+4-r)(4+r-4+r)$

$y^2 = 8 \times 2r$

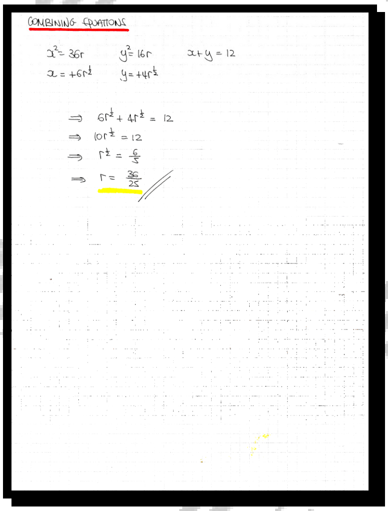
$y^2 = 16r$

NEED ANOTHER EQUATION — LOOKING AT THE "YELLOW" TRIANGLE

$(x+y)^2 + 5^2 = 13^2$

$(x+y)^2 = 144$

$x+y = 12$


COMBINING EQUATIONS

$x^2 = 36r$ $\quad y^2 = 16r$ $\quad x+y = 12$

$x = +6r^{\frac{1}{2}}$ $\quad y = +4r^{\frac{1}{2}}$

$\Rightarrow 6r^{\frac{1}{2}} + 4r^{\frac{1}{2}} = 12$

$\Rightarrow 10r^{\frac{1}{2}} = 12$

$\Rightarrow r^{\frac{1}{2}} = \frac{6}{5}$

$\Rightarrow r = \frac{36}{25}$

Question 113 (*****)

The point $A(6,-1)$ lies on the circle with equation

$$x^2 + y^2 - 4x + 6y = 7.$$

The tangent to the circle at A passes through the point P, so that the distance of P from the centre of the circle is $\sqrt{65}$.

Another tangent to the circle, at some point B, also passes through P.

Determine in any order the two sets of the possible coordinates of P and B.

$$P(3,5) \cap B\left(-\frac{1}{13},-\frac{18}{13}\right) \cup P(9,-7) \cap B\left(\frac{30}{13},-\frac{97}{13}\right)$$