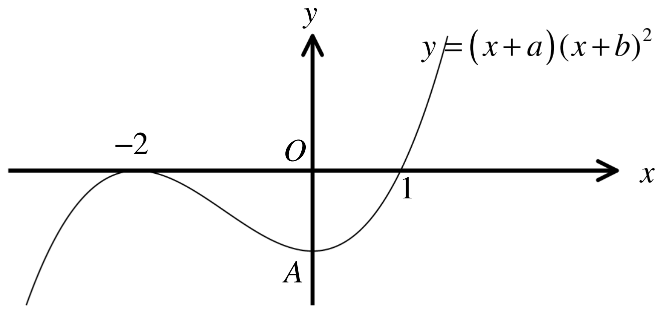
CUBICS EXAM QUESTIONS

Question 1 (**)



The diagram above shows the graph of the curve with equation

$$y=(x+a)(x+b)^2,$$

where a and b are constants.

The curve crosses the x axis at $(1,0)$ and touches the x axis at $(-2,0)$.

Determine, showing full workings the coordinates of the point A, where A is the point where the curve crosses the y axis.

$\boxed{A(0,-4)}$

$y=(x+2)^2(x-1)$

TOUCHING POINTS

CROSSING POINT

WHEN $x=0$

$y=(0+2)^2(0-1)$

$y=-4$

$\therefore A(0,-4)$

Question 2 (**)

A cubic graph is defined in terms of a constant k as

$$f(x) \equiv x^3 - 19x + k, \quad x \in \mathbb{R}.$$

Find the value k, if the graph of $f(x)$ …

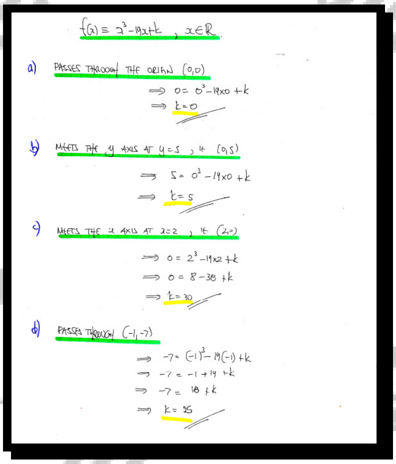
a) … passes through the origin.

b) … meets the y axis at $y = 5$.

c) … meets the x axis at $x = 2$.

d) … passes through the point $(-1, -7)$.

$\boxed{}$, $\boxed{k=0}$, $\boxed{k=5}$, $\boxed{k=30}$, $\boxed{k=25}$



$f(x) \equiv x^3 - 19x + k, \quad x \in \mathbb{R}$

a) PASSES THROUGH THE ORIGIN $(0,0)$

$\Rightarrow 0 = 0^3 - 19 \times 0 + k$

$\Rightarrow k = 0$

b) MEETS THE y AXIS AT $y = 5$, IE $(0,5)$

$\Rightarrow 5 = 0^3 - 19 \times 0 + k$

$\Rightarrow k = 5$

c) MEETS THE x AXIS AT $x = 2$, IE $(2,0)$

$\Rightarrow 0 = 2^3 - 19 \times 2 + k$

$\Rightarrow 0 = 8 - 38 + k$

$\Rightarrow k = 30$

d) PASSES THROUGH $(-1,-7)$

$\Rightarrow -7 = (-1)^3 - 19(-1) + k$

$\Rightarrow -7 = -1 + 19 + k$

$\Rightarrow -7 = 18 + k$

$\Rightarrow k = 25$

Question 3 (***)

A cubic graph is defined as

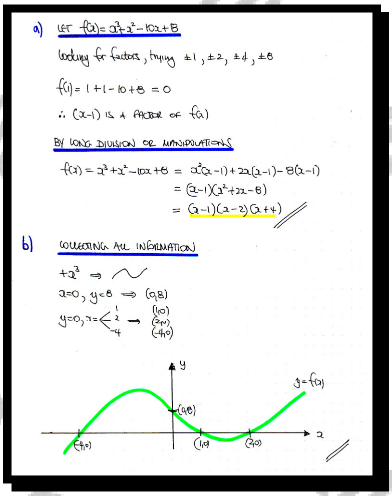
$$f(x) \equiv x^3 + x^2 - 10x + 8, \ x \in \mathbb{R}.$$

a) By considering the factors of 8, or otherwise, express $f(x)$ as the product of three linear factors.

b) Sketch the graph of $f(x)$.

The sketch must include the coordinates of any points where the graph of $f(x)$ meets the coordinate axes.

$\boxed{}$, $\boxed{f(x) = (x-2)(x-1)(x+4)}$



Question 4 (***)

The curve C has equation

$$y = x^3 - 9x.$$

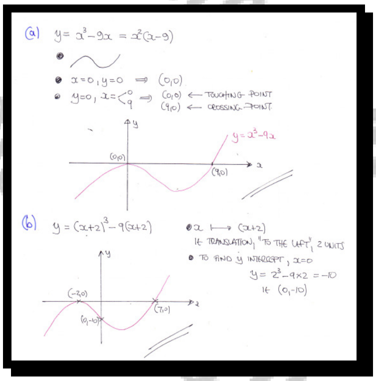
a) Sketch the graph of C.

b) Hence sketch on a separate diagram the graph of

$$y = (x+2)^3 - 9(x+2).$$

Each of the two sketches must include the coordinates of all the points where the curve meets the coordinate axes.

$\boxed{}$, $\boxed{\text{graph}}$



Question 5 (***)

A cubic polynomial is defined as

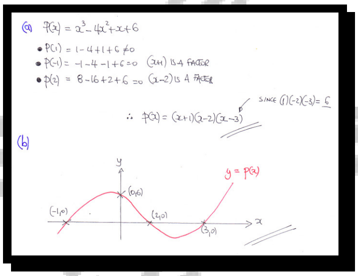
$$p(x) \equiv x^3 - 4x^2 + x + 6, \quad x \in \mathbb{R}.$$

a) By considering the factors of 6, or otherwise, express $p(x)$ as the product of three linear factors.

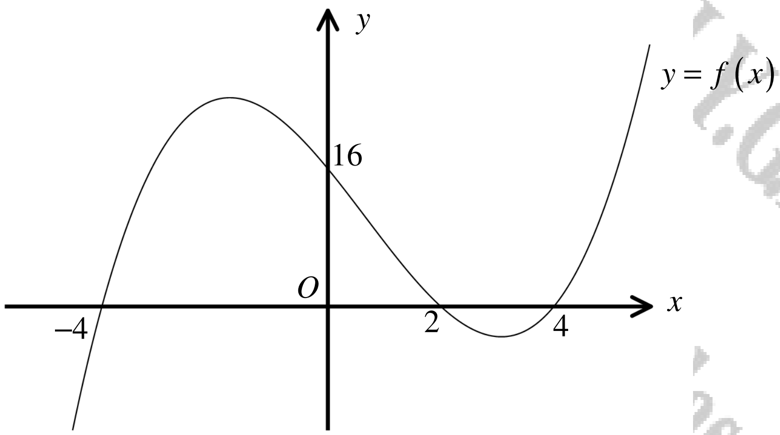
b) Sketch the graph of $p(x)$.

The sketch must include the coordinates of any points where the graph of $p(x)$ meets the coordinate axes.

$$\boxed{p(x) = (x-3)(x-2)(x+1)}$$



Question 6 (***)



The figure above shows the graph of the curve with equation $y = f(x)$.

The curve crosses the x axis at the points $(-4,0)$, $(2,0)$ and $(4,0)$, and the y axis at the point $(0,16)$.

Determine the equation of $f(x)$ in the form

$$f(x) \equiv ax^3 + bx^2 + cx + d\ ,$$

where a, b, c and d are constants.

$\boxed{}$, $\boxed{f(x) = \frac{1}{2}x^3 - x^2 - 8x + 16}$

FROM THE x INTERCEPTS WE OBTAIN

$y = k(x+4)(x-2)(x-4)$

THIS IS TO MATCH THE y INTERCEPT

So $k \times 4 \times (-2) \times (-4) = 16$

$32k = 16$

$k = \frac{1}{2}$

$y = \frac{1}{2}(x+4)(x-4)(x-2)$

$y = \frac{1}{2}(x^2-16)(x-2)$

$y = \frac{1}{2}[x^3 - 2x^2 - 16x + 32]$

$y = \frac{1}{2}x^3 - x^2 - 8x + 16$

Question 7 (***)

A cubic curve C_1 has equation

$$y=(x-8)\left(x^2-4x+3\right).$$

A quadratic curve C_2 has equation

$$y=(2x-3)(8-x).$$

a) Sketch on separate set of axes the graphs of C_1 and C_2.

The sketches must contain the coordinates of the points where each of the curves meet the coordinate axes.

b) Hence find the solutions of the following equation.

$$(x-8)\left(x^2-4x+3\right)=(2x-3)(8-x).$$

$\boxed{}$, $\boxed{x=0,\quad x=2,\quad x=8}$

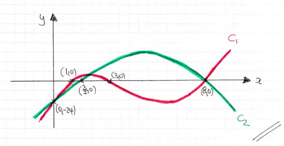
a) C_1: $y=(x-8)(x^2-4x+3)$

$y=(x-8)(x-3)(x-1)$

$(1,0)$, $(3,0)$, $(8,0)$, $(0,-24)$

C_2: $y=(2x-3)(8-x)$

$\left(\frac{3}{2},0\right)$, $(8,0)$, $(0,-24)$



b) SOLVING AS FOLLOWS

$\Rightarrow (x-8)(x^2-4x+3)=(2x-3)(8-x)$

$\Rightarrow (x-8)(x^2-4x+3)-(2x-3)(8-x)=0$

$\Rightarrow (x-8)(x^2-4x+3)+(x-8)(2x-3)=0$

$\Rightarrow (x-8)\left[(x^2-4x+\cancel{3})+(2x-\cancel{3})\right]=0$

$\Rightarrow (x-8)(x^2-2x)=0$

$\Rightarrow x(x-2)(x-8)$

$\Rightarrow x = 0,\ 2,\ 8$

ALTERNATIVE SOLUTION TO (b)

$\Rightarrow (x-8)(x^2-4x+3)=(2x-3)(8-x)$

$\Rightarrow x^3-4x^2+3x-8x^2+32x-24 = 16x-2x^2-24+3x$

$\Rightarrow x^3-12x^2+35x-\cancel{24} = -2x^2+19x-\cancel{24}$

$\Rightarrow x^3-10x^2+16x=0$

$\Rightarrow x(x^2-10x+16)=0$

$\Rightarrow x(x-2)(x-8)=0$

$\Rightarrow x = 0,\ 2,\ 8$

Question 8 (***+)

The curve C has equation

$$y = 5x - 4x^2 - x^3.$$

a) Express y as a product of linear factors.

b) Sketch the graph of C.

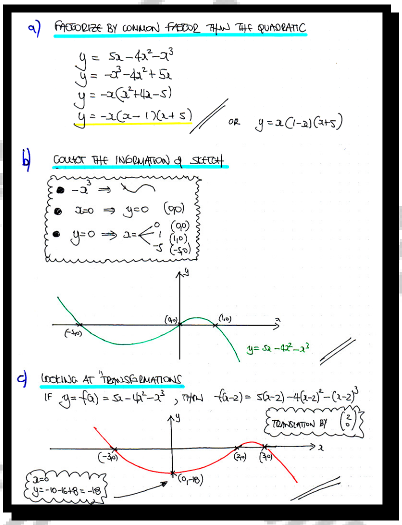
The sketch must include the coordinates of all the points where the curve meets the coordinate axes.

c) Hence sketch the curve with equation

$$y = 5(x-2) - 4(x-2)^2 - (x-2)^3,$$

clearly showing the coordinates of all the points where the curve meets the coordinate axes.

$\boxed{y = x(x+5)(1-x)}$



Question 9 (***+)

The curve C has equation

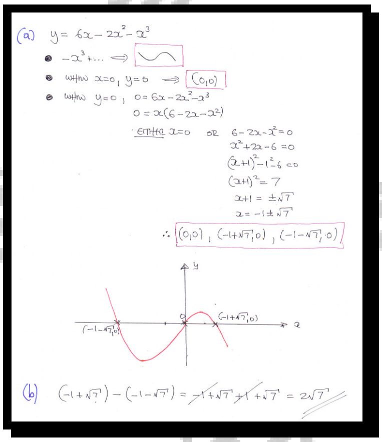
$$y = 6x - 2x^2 - x^3 .$$

a) Sketch the graph of C.

The sketch must include the **exact** coordinates of all the points where the curve meets the coordinate axes.

b) Determine, as an exact surd, the **greatest** distance between the x intercepts of C.

$\boxed{}$, $\boxed{2\sqrt{7}}$


(a) $y = 6x - 2x^2 - x^3$

- $-x^3 + \dots \Rightarrow$
- when $x=0$, $y=0 \Rightarrow \boxed{(0,0)}$
- when $y=0$, $0 = 6x - 2x^2 - x^3$

$0 = x(6 - 2x - x^2)$

EITHER $x=0$ OR $6 - 2x - x^2 = 0$

$x^2 + 2x - 6 = 0$

$(x+1)^2 - 1^2 - 6 = 0$

$(x+1)^2 = 7$

$x + 1 = \pm\sqrt{7}$

$x = -1 \pm \sqrt{7}$

$\therefore \boxed{(0,0),\ (-1+\sqrt{7}, 0),\ (-1-\sqrt{7}, 0)}$

(b) $(-1+\sqrt{7}) - (-1-\sqrt{7}) = \cancel{-1} + \sqrt{7} \cancel{+1} + \sqrt{7} = 2\sqrt{7}$

Question 10 (***+)

$$f(x) \equiv 2x^3 - 9x^2 - 11x + 30.$$

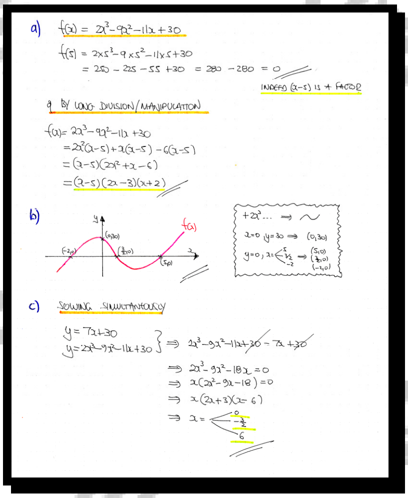
a) Show, by using the factor theorem, that $(x-5)$ is a factor of $f(x)$ and hence factorize $f(x)$ into product of three linear factors.

b) Sketch the graph of $f(x)$.

The sketch must include the coordinates of all the points where the graph meets the coordinate axes.

c) Find the x coordinates of the points where the line with equation $y = 7x + 30$ meets the graph of $f(x)$.

$\boxed{}$, $\boxed{f(x) \equiv (x-5)(2x-3)(x+2)}$, $\boxed{x = -\frac{3}{2} \cup x = 0 \cup x = 6}$



Question 11 (***+)

Sketch on separate diagrams the curve with equation …

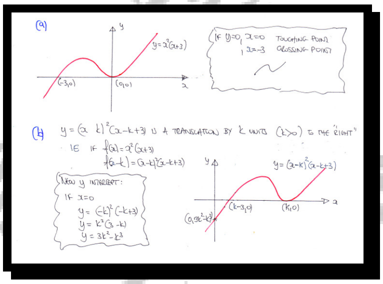
a) … $y = x^2(x+3)$.

b) … $y = (x-k)^2(x-k+3)$,

where k is a constant such that $k > 3$.

Both sketches must include the coordinates, in terms of k where appropriate, of any points where each of the curves meets the coordinate axes.

[ ], [graph]



Question 12 (***+)

A cubic graph is defined by

$$f(x) \equiv x^3 - 3x^2 - 4x + 12, \quad x \in \mathbb{R}.$$

a) Show, by using the factor theorem, that $(x-3)$ is a factor of $f(x)$ and hence factorize $f(x)$ into product of three linear factors.

b) Sketch the graph of $f(x)$.

The sketch must include the coordinates of any points where the graph of $f(x)$ meets the coordinate axes.

Another cubic graph is defined as

$$g(x) \equiv (x-2)(x-4)^2, \quad x \in \mathbb{R}.$$

The two graphs meet at the points P and Q.

c) Determine the x coordinates of P and the x coordinates of Q.

$\boxed{}$, $\boxed{f(x) = (x-2)(x+2)(x-3)}$, $\boxed{x = 2, \frac{22}{7}}$

a) SUBSTITUTE $x=3$ INTO THE FUNCTION

$f(x) = x^3 - 3x^2 - 4x + 12$

$f(3) = 3^3 - 3\times3^2 - 4\times3 + 12$

$f(3) = 27 - 27 - 12 + 12$

$f(3) = 0$

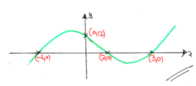
$\therefore (x-3)$ IS A FACTOR

BY LONG DIVISION OR INSPECTION

$f(x) = x^3 - 3x^2 - 4x + 12 = x^2(x-3) - 4(x-3) = (x-3)(x^2-4)$

$f(x) = (x-3)(x-2)(x+2)$

b)



$+x^3 \Rightarrow$

$y=0 \Rightarrow x = 2, -2, 3$

$x=0 \Rightarrow y=12$

c) SOLVING SIMULTANEOUSLY $f(x) = g(x)$

$\Rightarrow (x-3)(x-2)(x+2) = (x-2)(x-4)^2$

$\Rightarrow (x-3)(x+2) = (x-4)^2$

DIVIDE BY $(x-2)$ & NOTE $x=2$ IS A SOLUTION

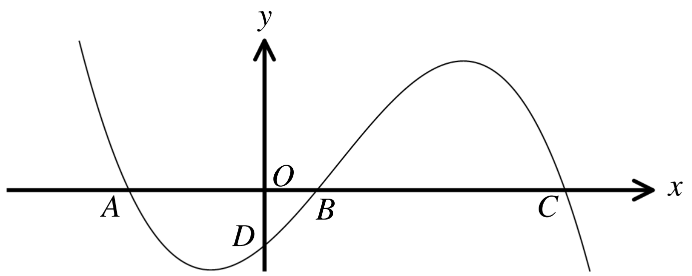
$\Rightarrow \cancel{x^2} - 3x + 2x - 6 = \cancel{x^2} - 8x + 16$

$\Rightarrow 7x = 22$

$\Rightarrow x = \frac{22}{7}$

$\therefore x = 2, \frac{22}{7}$

Question 13 (***+)



The figure above shows the graph of a cubic polynomial $f(x)$ given by

$$f(x) = -x^3 + 5x^2 + 17x - 21, \ x \in \mathbb{R}.$$

The graph meets the coordinate axes at four distinct points, labelled A, B, C and D.

Given that the coordinates of the point A are $(-3,0)$, determine the coordinates of the points B, C and D.

$\boxed{B(1,0)}$, $\boxed{C(7,0)}$, $\boxed{D(0,-21)}$

AS $A(-3,0)$, $x+3$ MUST BE ONE OF THE FACTORS

$\Rightarrow f(x) = -x^3 + 5x^2 + 17x - 21$

$\Rightarrow -f(x) = x^3 - 5x^2 - 17x + 21$

BY LONG DIVISION OR MANIPULATIONS

$\Rightarrow -f(x) = x^2(x+3) - 8x(x+3) + 7(x+3)$

$\Rightarrow -f(x) = (x+3)(x^2 - 8x + 7)$

$\Rightarrow -f(x) = (x+3)(x-1)(x-7)$

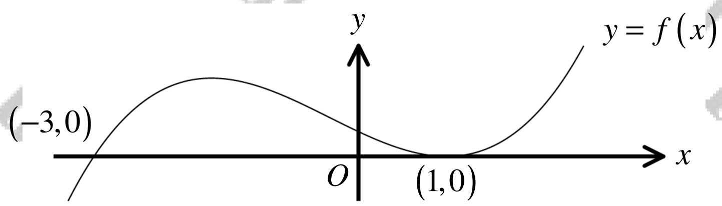
$\Rightarrow f(x) = (x+3)(1-x)(x-7)$

$A(-3,0)$ $B(1,0)$ $C(7,0)$

AND WHEN $x=0$ $y=-21$, IE $D(0,-21)$

$\therefore A(-3,0), B(1,0), C(7,0), D(0,-21)$

Question 14 (***+)



The figure above shows the graph of the curve C with equation

$$f(x) = x^3 + ax^2 + bx + c,$$

where a, b and c are constants.

The curve crosses the x axis at $(-3,0)$ and touches the x axis at $(1,0)$.

a) Find the value of a, b and c.

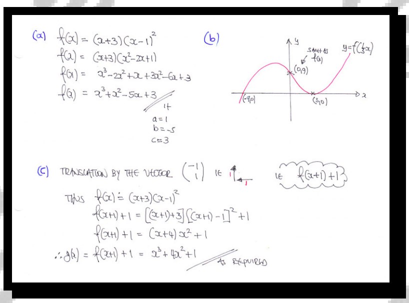
b) Sketch the graph of $y = f\left(\frac{1}{3}x\right)$, clearly marking the coordinates of any points of intersection with the coordinate axes.

The graph of C is translated by the vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ to give the graph of $g(x)$.

c) Show clearly that

$$g(x) = x^3 + 4x + 1.$$

$\boxed{}$, $\boxed{a=1}$, $\boxed{b=-5}$, $\boxed{c=3}$



Question 15 (***+)

$$f(x) \equiv x^3 - 3x^2 - 6x + 8, \; x \in \mathbb{R}.$$

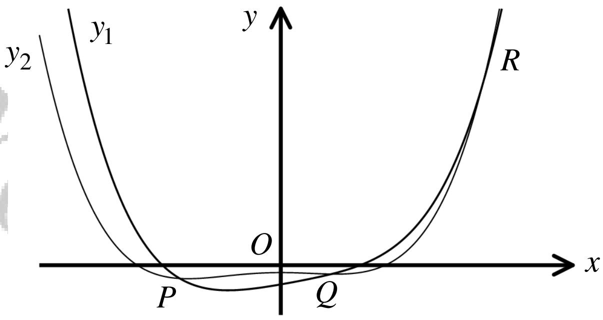
a) Show that $(x-1)$ is a factor of $f(x)$.

b) Hence factorize $f(x)$ into three linear factors.

c) Sketch the graph of $f(x)$.
The sketch must include the coordinates of any points where the graph of $f(x)$ meets the coordinate axes.

The figure below shows the graphs of the curves with equations

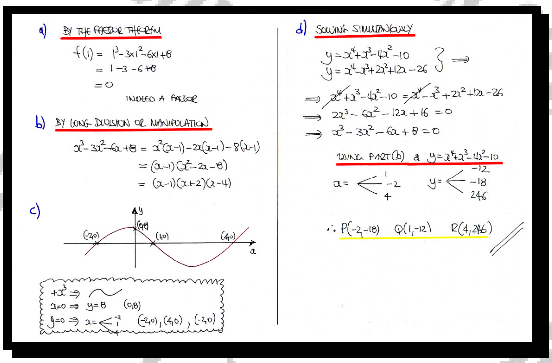
$$y_1 = x^4 + x^3 - 4x^2 - 10 \qquad \text{and} \qquad y_2 = x^4 - x^3 + 2x^2 + 12x - 26.$$



The two graphs meet at the points P, Q and R.

d) Determine the coordinates of P, Q and R.

$\boxed{}$, $\boxed{f(x) = (x-1)(x+2)(x-4)}$, $\boxed{P(-2,-18)}$, $\boxed{Q(1,-12)}$, $\boxed{R(4,246)}$



Question 16 (***+)

The curve C has equation

$$y = x^3 - 6x^2 + 9x .$$

a) Express y as a product of linear factors.

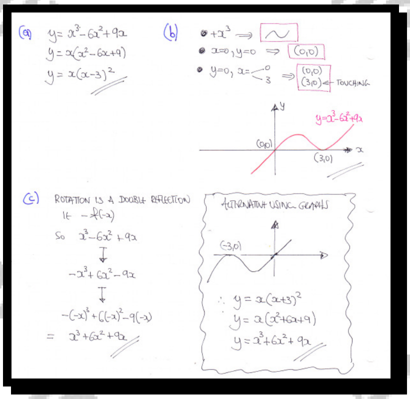
b) Sketch the graph of C.

The sketch must include the coordinates of all the points where the curve meets the coordinates axes.

The graph of C is rotated by $180°$ about the origin onto another curve C'.

c) Determine the equation of C'.

$\boxed{\text{C.P.}}$, $\boxed{y = x(x-3)^2}$, $\boxed{y = x^3 + 6x^2 + 9x}$



Question 17 (***+)

$$f(x) = x^2(x-4), \ x \in \mathbb{R}.$$

$$g(x) = x(10-x), \ x \in \mathbb{R}.$$

a) Determine the coordinates of the points of intersection between the graph of $f(x)$ and the graph of $g(x)$.

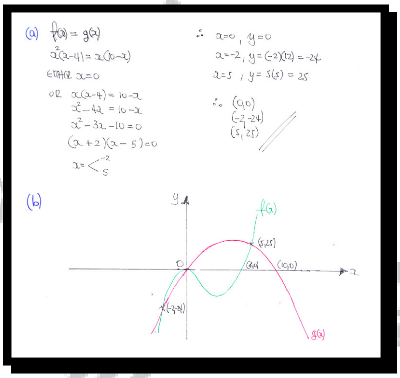
b) Sketch the graph of $f(x)$ and the graph of $g(x)$ in the same diagram.

The sketch must include …

… the coordinates of any points where either of the two graphs meet the coordinate axes.

… the coordinates of the points of intersection between the graph of $f(x)$ and the graph of $g(x)$.

$\boxed{}$, $\boxed{(0,0),(-2,-24),(5,25)}$



Question 18 (***+)

A cubic curve C_1 has equation

$$y = (x-7)\left(x^2+2x-3\right).$$

A quadratic curve C_2 has equation

$$y = (2x+5)(7-x).$$

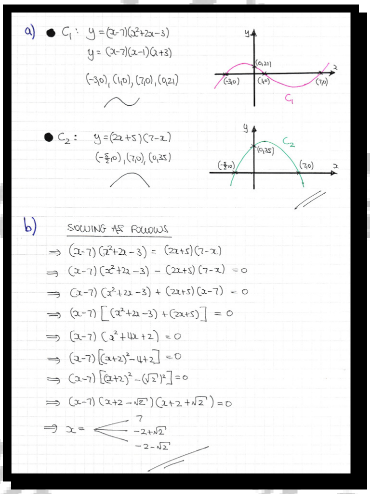
a) Sketch on separate set of axes the graphs of C_1 and C_2.

The sketches must contain the coordinates of the points where each of the curves meet the coordinate axes.

b) Hence, find in exact form where appropriate, the three solutions of the following equation.

$$(x-7)\left(x^2+2x-3\right) = (2x+5)(7-x).$$

$\boxed{}$, $\boxed{x=7, \quad x=-2+\sqrt{2}, \quad x=-2-\sqrt{2}}$



Question 19 (****)

$$f(x) = x^3 - 9x^2 + 13x + 2, \; x \in \mathbb{R}.$$

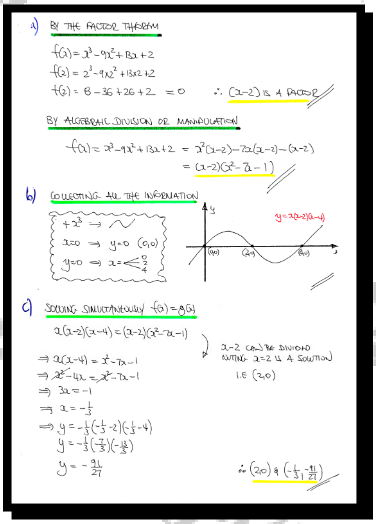
a) Show, by using the factor theorem, that $(x-2)$ is a factor of $f(x)$ and hence express $f(x)$ as a product of a linear and a quadratic factor.

$$g(x) = x(x-2)(x-4), \; x \in \mathbb{R}.$$

b) Sketch the graph of $g(x)$, indicating clearly the coordinates of any points where the graph of $g(x)$ meets the coordinate axes.

c) Determine the exact coordinates, where appropriate, of the points of intersection between the graph of $f(x)$ and the graph of $g(x)$.

$\boxed{}$, $\boxed{f(x) = (x-2)\left(x^2 - 7x - 1\right)}$, $\boxed{(2,0), \left(-\frac{1}{3}, -\frac{91}{27}\right)}$



a) BY THE FACTOR THEOREM

$f(x) = x^3 - 9x^2 + 13x + 2$

$f(2) = 2^3 - 9 \times 2^2 + 13 \times 2 + 2$

$f(2) = 8 - 36 + 26 + 2 = 0$ $\quad \therefore (x-2)$ IS A FACTOR

BY ALGEBRAIC DIVISION OR MANIPULATION

$f(x) = x^3 - 9x^2 + 13x + 2 = x^2(x-2) - 7x(x-2) - (x-2)$

$= (x-2)(x^2 - 7x - 1)$

b) COLLECTING ALL THE INFORMATION

$+x^3 \Rightarrow$

$x=0 \Rightarrow y=0 \quad (0,0)$

$y=0 \Rightarrow x = 0, 2, 4$

c) SOLVING SIMULTANEOUSLY $f(x) = g(x)$

$x(x-2)(x-4) = (x-2)(x^2 - 7x - 1)$

$x-2$ CAN BE DIVIDED NOTING $x=2$ IS A SOLUTION I.E $(2,0)$

$\Rightarrow x(x-4) = x^2 - 7x - 1$

$\Rightarrow \cancel{x^2} - 4x = \cancel{x^2} - 7x - 1$

$\Rightarrow 3x = -1$

$\Rightarrow x = -\frac{1}{3}$

$\Rightarrow y = -\frac{1}{3}\left(-\frac{1}{3} - 2\right)\left(-\frac{1}{3} - 4\right)$

$y = -\frac{1}{3}\left(-\frac{7}{3}\right)\left(-\frac{13}{3}\right)$

$y = -\frac{91}{27}$

$\therefore (2,0) \;\&\; \left(-\frac{1}{3}, -\frac{91}{27}\right)$

Question 20 (****)

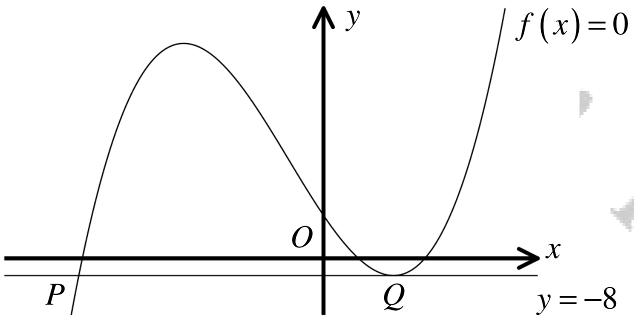
$$f(x) \equiv x^3 + 3x^2 - 24x + 20, \quad x \in \mathbb{R}.$$

a) Show that $(x-1)$ is a factor of $f(x)$.

b) Hence factorize $f(x)$ as the product of a linear and a quadratic factor.

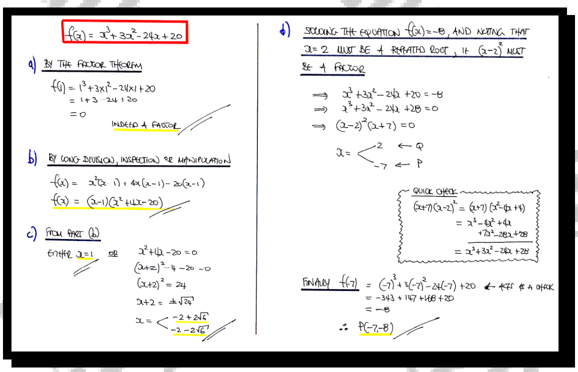
c) Find, in exact form where appropriate, the solutions of the equation $f(x) = 0$.

The straight line with equation $y = -8$ **touches** the graph of $f(x)$ at the point $Q(2,-8)$ and crosses the graph of $f(x)$ at the point P, as shown in the figure below.



d) Determine the coordinates of P.

$\boxed{}$, $\boxed{f(x) = (x-1)(x^2+4x-20)}$, $\boxed{x = 1,\ -2 \pm 2\sqrt{6}}$, $\boxed{P(-7,-8)}$



Question 21 (****)

$$f(x) \equiv x^3 - 3x + 2, \; x \in \mathbb{R}.$$

a) Express $f(x)$ as the product of three linear factors.

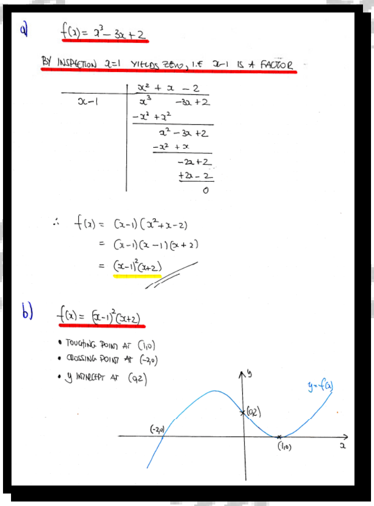
b) Sketch the graph of $f(x)$.

The sketch must include the coordinates of any points where the graph of $f(x)$ meets the coordinate axes.

c) Solve the equation

$$f(x) = (x-1)^2.$$

$\boxed{f(x) = (x+2)(x-1)^2}$, $\boxed{x = \pm 1}$



a) $f(x) = x^3 - 3x + 2$

BY INSPECTION $x=1$ YIELDS ZERO, I.E $x-1$ IS A FACTOR

$$\begin{array}{r|l} & x^2 + x - 2 \\ \hline x-1 & x^3 \qquad -3x+2 \\ & -x^3 + x^2 \\ & \quad x^2 - 3x + 2 \\ & \quad -x^2 + x \\ & \qquad -2x + 2 \\ & \qquad +2x - 2 \\ & \qquad\qquad 0 \end{array}$$

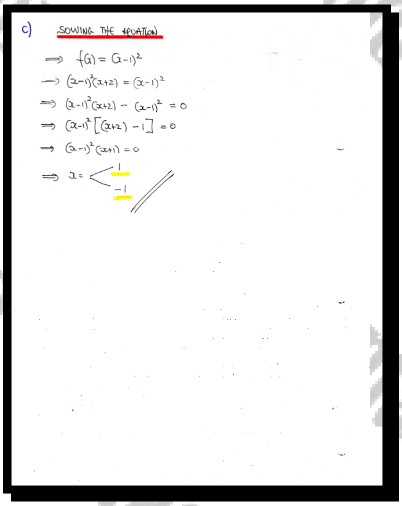
$\therefore \; f(x) = (x-1)(x^2+x-2)$

$= (x-1)(x-1)(x+2)$

$= (x-1)^2(x+2)$

b) $f(x) = (x-1)^2(x+2)$

- TOUCHING POINT AT $(1,0)$
- CROSSING POINT AT $(-2,0)$
- y INTERCEPT AT $(0,2)$



c) SOLVING THE EQUATION

$\Longrightarrow f(x) = (x-1)^2$

$\Longrightarrow (x-1)^2(x+2) = (x-1)^2$

$\Longrightarrow (x-1)^2(x+2) - (x-1)^2 = 0$

$\Longrightarrow (x-1)^2\left[(x+2) - 1\right] = 0$

$\Longrightarrow (x-1)^2(x+1) = 0$

$\Longrightarrow x = \begin{cases} 1 \\ -1 \end{cases}$

Question 22 (****)

A cubic curve C has equation

$$y=(3-x)(4+x)^2.$$

a) Sketch the graph of C.

The sketch must include any points where the graph meets the coordinate axes.

b) Sketch in separate diagrams the graph of …

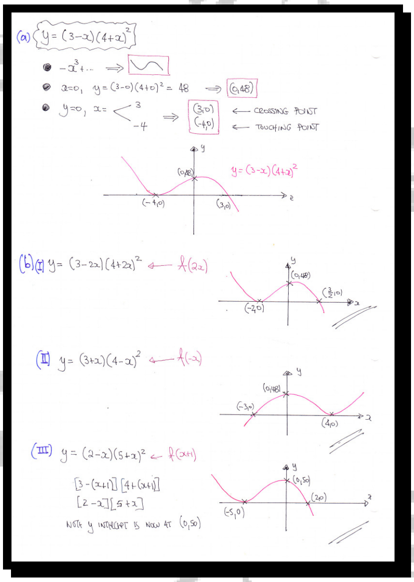
i. … $y=(3-2x)(4+2x)^2$.

ii. … $y=(3+x)(4-x)^2$.

iii. … $y=(2-x)(5+x)^2$.

Each of the sketches must include any points where the graph meets the coordinate axes.

$\boxed{}$, $\boxed{\text{graph}}$



Question 23 **(****)**

The cubic curve with equation

$$y = ax^3 + bx^2 + cx + d \, ,$$

where a, b, c are non zero constants and d is a constant, has one local maximum and one local minimum.

Show clearly that

$$b^2 > 3ac$$

$$\boxed{y = ax^3 + bx^2 + cx + d}$$

DIFFERENTIATE WITH RESPECT TO x

$$\frac{dy}{dx} = 3ax^2 + 2bx + c$$

LOOKING FOR STATIONARY POINTS

$$\frac{dy}{dx} = 0$$

$$3ax^2 + 2bx + c = 0$$

FOR 2 "DISTINCT ANSWERS"

"$B^2 - 4AC$" > 0

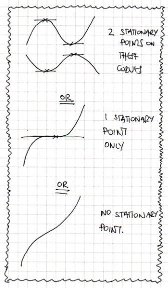
$$\Rightarrow (2b)^2 - 4(3a)c > 0$$

$$\Rightarrow 4b^2 - 12ac > 0$$

$$\Rightarrow 4b^2 > 12ac$$

$$\Rightarrow b^2 > 3ac$$

AS REQUIRED



Question 24 (****)

A polynomial $f(x)$ is defined in terms of the constants a, b and c as

$$f(x) = 2x^3 + ax^2 + bx + c, \quad x \in \mathbb{R}.$$

It is further given that

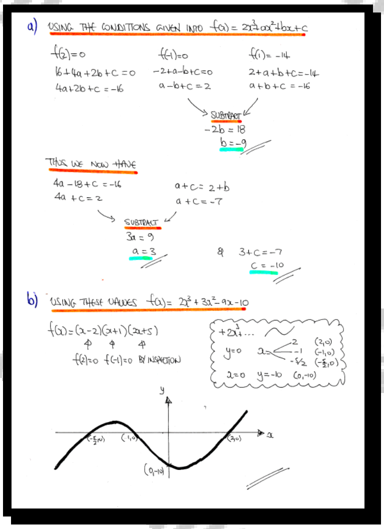
$$f(2) = f(-1) = 0 \quad \text{and} \quad f(1) = -14.$$

a) Find the value of a, b and c.

b) Sketch the graph of $f(x)$.

The sketch must include any points where the graph of $f(x)$ meets the coordinate axes.

$\boxed{a = 3, \ b = -9, \ c = -10}$



a) USING THE CONDITIONS GIVEN INTO $f(x) = 2x^3 + ax^2 + bx + c$

$f(2) = 0$
$16 + 4a + 2b + c = 0$
$4a + 2b + c = -16$

$f(-1) = 0$
$-2 + a - b + c = 0$
$a - b + c = 2$

$f(1) = -14$
$2 + a + b + c = -14$
$a + b + c = -16$

SUBTRACT
$-2b = 18$
$b = -9$

THUS WE NOW HAVE

$4a - 18 + c = -16$
$4a + c = 2$

$a + c = 2 + b$
$a + c = -7$

SUBTRACT
$3a = 9$
$a = 3$

& $3 + c = -7$
$c = -10$

b) USING THESE VALUES $f(x) = 2x^3 + 3x^2 - 9x - 10$

$f(x) = (x-2)(x+1)(2x+5)$
$f(2) = 0$ $f(-1) = 0$ BY INSPECTION

$+2x^3 \dots$
$y = 0$ $x = 2$ $(2,0)$; $x = -1$ $(-1,0)$; $x = -5/2$ $(-\frac{5}{2},0)$
$x = 0$ $y = -10$ $(0,-10)$

Question 25 (****)

A cubic curve C_1 has equation

$$y = (2x-3)^3 .$$

A quadratic curve C_2 has equation

$$y = (2x-1)^2 - 4 .$$

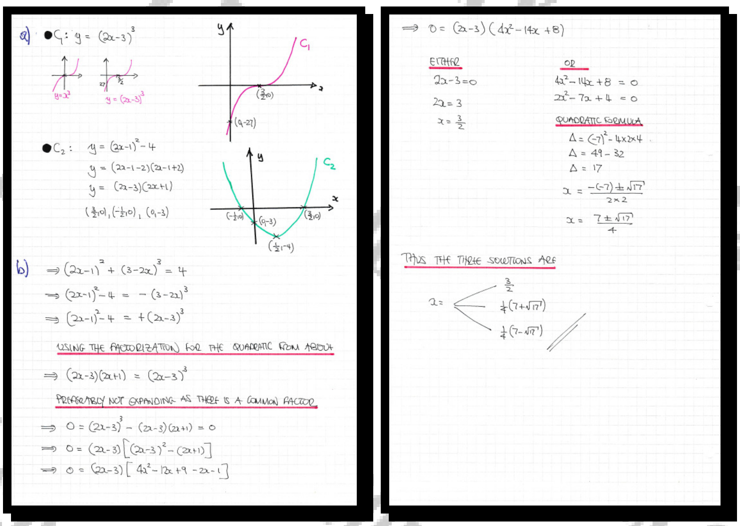
a) Sketch on separate set of axes the graphs of C_1 and C_2.

The sketches must contain the coordinates of the points where each of the curves meet the coordinate axes.

b) Hence, find in exact form, the solutions of the following equation.

$$(2x-1)^2 + (3-2x)^3 = 4 .$$

$\boxed{\text{MP1-Z}}$, $\boxed{x = \frac{3}{2} \cup x = \frac{1}{4}\left(7+\sqrt{17}\right) \cup x = \frac{1}{4}\left(7-\sqrt{17}\right)}$



Question 26 (****+)

The cubic equation

$$x^3 + kx + 4 = 0,$$

where k is a constant has 2 distinct real roots.

Determine the exact value of k.

$\boxed{}$, $\boxed{k = -3\sqrt[3]{4}}$

START WITH AN IDENTITY

$$x^2 + kx + 4 \equiv (x-A)^2(x-B)$$
$$\equiv (x-B)(x^2-2Ax+A^2)$$
$$\equiv x^3 - 2Ax^2 + A^2x - Bx^2 + 2ABx - A^2B$$
$$\equiv x^3 - (2A+B)x^2 + (A^2+2AB)x - A^2B$$

EQUATING COEFFICIENTS WE OBTAIN

$$2A + B = 0$$
$$\underline{\underline{B = -2A}}$$

THEN

$$\Rightarrow -A^2B = 4$$
$$\Rightarrow -A^2(-2A) = 4$$
$$\Rightarrow 2A^3 = 4$$
$$\Rightarrow A^3 = 2$$
$$\Rightarrow \underline{\underline{A = 2^{\frac{1}{3}}}}$$

& $B = -2(2^{\frac{1}{3}})$

$$\underline{\underline{B = -2^{\frac{4}{3}}}}$$

$$\therefore k = A^2 + 2AB = (2^{\frac{1}{3}})^2 + 2 \times 2^{\frac{1}{3}} \times (-2^{\frac{4}{3}}) = 2^{\frac{2}{3}} - 2^{\frac{8}{3}}$$
$$= 2^{\frac{2}{3}}(1 - 2^2) = -3 \times 2^{\frac{2}{3}}$$

Question 27 (*****)

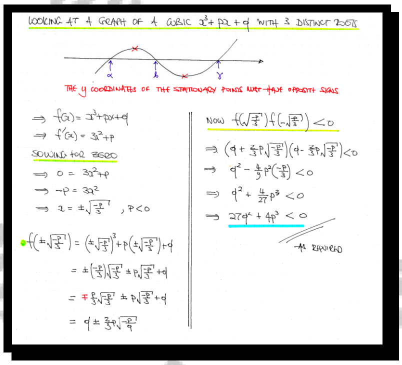
The cubic equation

$$x^3 + px + q = 0\,,\ \ p < 0$$

has 3 distinct real roots.

Show that $27q^2 + 4p^3 < 0$.

$\boxed{}$, $\boxed{\text{proof}}$


LOOKING AT A GRAPH OF A CUBIC x^3+px+q WITH 3 DISTINCT ROOTS

THE y COORDINATES OF THE STATIONARY POINTS MUST HAVE OPPOSITE SIGNS

$\Rightarrow f(x) = x^3+px+q$

$\Rightarrow f'(x) = 3x^2+p$

SOLVING FOR ZERO

$\Rightarrow 0 = 3x^2+p$

$\Rightarrow -p = 3x^2$

$\Rightarrow x = \pm\sqrt{\frac{-p}{3}}\ ,\ p<0$

$f\left(\pm\sqrt{\frac{-p}{3}}\right) = \left(\pm\sqrt{\frac{-p}{3}}\right)^3 + p\left(\pm\sqrt{\frac{-p}{3}}\right)+q$

$= \pm\left(\frac{-p}{3}\right)\sqrt{\frac{-p}{3}} \pm p\sqrt{\frac{-p}{3}}+q$

$= \mp\frac{p}{3}\sqrt{\frac{-p}{3}} \pm p\sqrt{\frac{-p}{3}}+q$

$= q \pm \frac{2}{3}p\sqrt{\frac{-p}{9}}$

NOW $f\left(\sqrt{\frac{-p}{3}}\right)f\left(-\sqrt{\frac{-p}{3}}\right) < 0$

$\Rightarrow \left(q+\frac{2}{3}p\sqrt{\frac{-p}{3}}\right)\left(q-\frac{2}{3}p\sqrt{\frac{-p}{3}}\right)<0$

$\Rightarrow q^2 - \frac{4}{9}p^2\left(\frac{-p}{3}\right) < 0$

$\Rightarrow q^2 + \frac{4}{27}p^3 < 0$

$\Rightarrow 27q^2 + 4p^3 < 0$

AS REQUIRED

Question 28 (*****)

A function is defined as

$$f(x) \equiv x^3 - \tfrac{1}{2}\lambda x + \lambda - 8, \; x \in \mathbb{R},$$

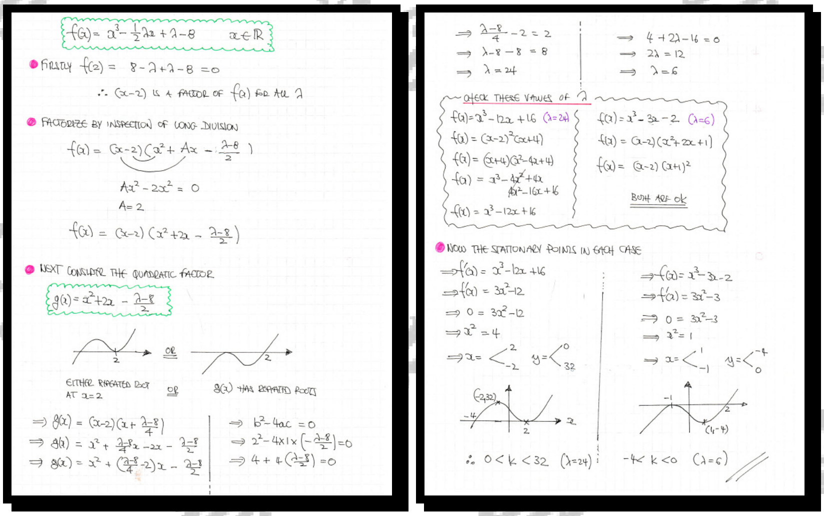
where λ is a non zero constant.

The equation $f(x) = 0$ has exactly two real distinct roots.

The equation $f(x) = k$, where k is a constant, has three distinct real roots.

By considering $f(2)$, or otherwise, determine the range of values of k.

$\boxed{}$, $\boxed{-4 < k < 0 \;\cup\; 0 < k < 32}$



Question 29 (*****)

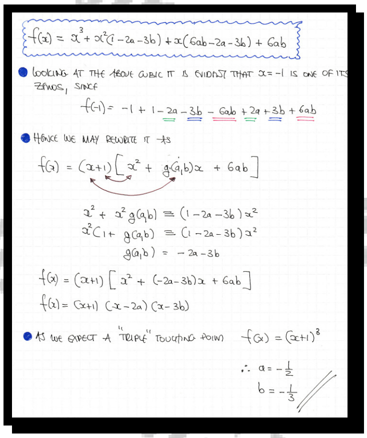
A cubic curve has equation

$$f(x) \equiv x^3 + x^2(1-2a-3b) + x(6ab-2a-b) + 6ab, \quad x \in \mathbb{R},$$

where a and b are constants.

Given that the equation $f(x) = 0$ has 3 equal real roots, determine the value of a and the value of b.

$\boxed{\text{SP-I}}$, $\boxed{a = -\frac{1}{2}}$, $\boxed{a = -\frac{1}{3}}$



$f(x) = x^3 + x^2(1-2a-3b) + x(6ab-2a-3b) + 6ab$

- LOOKING AT THE ABOVE CUBIC IT IS EVIDENT THAT $x = -1$ IS ONE OF ITS ZEROS, SINCE

$f(-1) = -1 + 1 - 2a - 3b - 6ab + 2a + 3b + 6ab$

- HENCE WE MAY REWRITE IT AS

$f(x) = (x+1)\left[x^2 + g(a,b)x + 6ab\right]$

$x^2 + x^2 g(a,b) \equiv (1-2a-3b)x^2$

$x^2(1 + g(a,b)) \equiv (1-2a-3b)x^2$

$g(a,b) = -2a-3b$

$f(x) = (x+1)\left[x^2 + (-2a-3b)x + 6ab\right]$

$f(x) = (x+1)(x-2a)(x-3b)$

- AS WE EXPECT A "TRIPLE" TOUCHING POINT $f(x) = (x+1)^3$

$\therefore a = -\frac{1}{2}$

$b = -\frac{1}{3}$

Question 30 (*****)

Find an equation of a cubic function, with integer coefficients, whose graph crosses the x axis at the point $\left(3+2^{\frac{2}{3}}+2^{\frac{5}{3}}, 0\right)$.

$\boxed{}$, $\boxed{f(x)=x^3-9x^2+3x-3}$

LET THE EQUATION OF THE CUBIC BE

$$y = ax^3+bx^2+cx+d$$
$$y = x^3+\frac{b}{a}x^2+\frac{c}{a}x+\frac{d}{a}$$
$$y = x^3+Ax^2+Bx+C$$

NOW $x = 3+2^{\frac{4}{3}}+2^{\frac{5}{3}} = 3+2\times2^{\frac{1}{3}}+2\times2^{\frac{2}{3}}$

$$x^2 = \left(3+2\times2^{\frac{1}{3}}+2\times2^{\frac{2}{3}}\right)^2 \qquad (a+b+c)^2 \equiv a^2+b^2+c^2+2ab+2ac+2bc$$
$$x^2 = 9+4\times2^{\frac{2}{3}}+4\times2^{\frac{4}{3}}+2\times3\times2\times2^{\frac{1}{3}}+2\times3\times2\times2^{\frac{2}{3}}+2\times2\times2^{\frac{1}{3}}\times2\times2^{\frac{2}{3}}$$
$$x^2 = 9+4\times2^{\frac{2}{3}}+8\times2^{\frac{1}{3}}+12\times2^{\frac{1}{3}}+12\times2^{\frac{2}{3}}+16$$
$$x^2 = 25+20\times2^{\frac{1}{3}}+16\times2^{\frac{2}{3}}$$

$$x^3 = \left(3+2\times2^{\frac{1}{3}}+2\times2^{\frac{2}{3}}\right)\left(25+20\times2^{\frac{1}{3}}+16\times2^{\frac{2}{3}}\right)$$
$$\begin{aligned} x^3 = {} & 75+60\times2^{\frac{1}{3}}+48\times2^{\frac{2}{3}} \\ & 50\times2^{\frac{1}{3}}+40\times2^{\frac{2}{3}}+64 \\ & +50\times2^{\frac{2}{3}}+80+32\times2^{\frac{4}{3}} \end{aligned}$$
$$x^3 = 75+110\times2^{\frac{1}{3}}+138\times2^{\frac{2}{3}}+144+64\times2^{\frac{1}{3}}$$
$$x^3 = 219+174\times2^{\frac{1}{3}}+138\times2^{\frac{2}{3}}$$

SUBSTITUTE INTO THE CUBIC & COMPARE COEFFICIENTS

$$\begin{aligned} x^3 &= 219+174\sqrt[3]{2}+138\sqrt[3]{4} \\ +Ax^2 &= 25A+20A\sqrt[3]{2}+16A\sqrt[3]{4} \\ +Bx &= 3B+2B\sqrt[3]{2}+2B\sqrt[3]{4} \\ +C &= C \end{aligned}$$

$[\sqrt[3]{2}]$: $174+20A+2B=0$
$[\sqrt[3]{4}]$: $138+16A+2B=0$ SUBTRACTING

$36+4A=0$

$A=-9$

$174+20(-9)+2B=0$
$2B=6$
$B=3$

$219+25A+3B+C=0$
$219+25(-9)+3\times3+C=0$
$219-225+9+C=0$
$C=-3$

$\therefore\ y = x^3-9x^2+3x-3$

Question 31 (*****)

A cubic curve has equation

$$f(x) \equiv x^3 - x^2(1+\sin t + 2\cos t) + x(2\sin t\cos t + 2\cos t + \sin t) - 2\sin t\cos t,$$

where $x \in \mathbb{R}$ and t is a constant such that $0 \le t < 2\pi$.

Given that the equation $f(x)=0$ has exactly 2 equal real roots, determine the possible values of t.

$\boxed{t = \frac{1}{3}\pi, \ \ t = \arctan 2, \ \ t = \frac{1}{2}\pi, \ \ t = \pi + \arctan 2, \ \ t = \frac{5}{3}\pi}$

$\boxed{y = x^3 - x^2(1+\sin t + 2\cos t) + x(2\sin t\cos t + 2\cos t + \sin t) - 2\sin t\cos t}$

- BY INSPECTION IF $x=1$, THE ABOVE EQUATION BECOMES

$y = 1 - 1 - \cancel{\sin t} - \cancel{2\cos t} + \cancel{2\sin t\cos t} + \cancel{2\cos t} + \cancel{\sin t} - \cancel{2\sin t\cos t} = 0$

$\therefore (x-1)$ IS A

$\therefore y = (x-1)(x^2 + A(t)x + 2\sin t\cos t)$

$-x^2 + A(t)x^2 \equiv (-1-\sin t - 2\cos t)x^2$

$-x^2 + A(t)x^2 \equiv -x^2 + (-\sin t - 2\cos t)x^2$

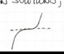
$\therefore A(t) = -\sin t - 2\cos t$

- HENCE WE HAVE

$\Rightarrow y = (x-1)(x^2 - (\sin t + 2\cos t)x + 2\sin t\cos t)$

$\Rightarrow y = (x-1)(x - \sin t)(x - 2\cos t)$

- NOW THE CURVE TOUCHES THE x AXIS — ONE OF THE x INTERCEPTS MUST BE $x=1$
- AS $\sin t = 1$ & $2\cos t = 1$ HAVE NO COMMON SOLUTIONS, THE CURVE DOES NOT HAVE A "TRIPLE" TOUCHING POINT, IE



- THUS EITHER

$\sin t = 1$	OR $2\cos t = 1$	OR $\sin t = 2\cos t$
	$\cos t = \frac{1}{2}$	$\tan t = 2$
$t = \frac{\pi}{2}$	$t = \frac{\pi}{3}, \frac{5\pi}{3}$	$t = \arctan 2, \pi + \arctan 2$

$\therefore t = \frac{\pi}{3}, \ \arctan 2, \ \frac{\pi}{2}, \ \pi + \arctan 2, \ \frac{5\pi}{3}$