# 55 DISCRIMINANT EXAM QUESTIONS

# 7

# BASIC QUESTIONS

**Question 1** (**)

Show by using the discriminant that the graph of the curve with equation

$$y = x^2 - 4x + 10,$$

does not cross the $x$ axis.

proof

$y = x^2 - 4x + 10$

$b^2 - 4ac = (-4)^2 - 4 \times 1 \times 10$

$= 16 - 40$

$= -24 < 0$

$\therefore$ GRAPH IS ENTIRELY ABOVE THE $x$ AXIS

**Question 2** (**)

Show that the quadratic equation

$$x^2 + (2k+3)x + k^2 + 3k + 1 = 0$$

has two distinct real roots in $x$, for all values of the constant $k$.

, proof

$x^2 + (2k+3)x + (k^2+3k+1) = 0$

USING THE DISCRIMINANT OF THE QUADRATIC

$\Longrightarrow b^2 - 4ac = (2k+3)^2 - 4 \times 1 \times (k^2+3k+1)$

$= (2k+3)^2 - 4(k^2+3k+1)$

$= \cancel{4k^2} + \cancel{12k} + 9 - \cancel{4k^2} - \cancel{12k} - 4$

$= 5$

AS THE DISCRIMINANT IS POSITIVE, THE EQUATION WILL ALWAYS HAVE 2 DISTINCT REAL ROOTS

**Question 3** (**+)

Find the range of values of the constant $k$ so that the equation

$$x^2 + kx + 16 = 0,$$

has no real roots.

$\boxed{-8 < k < 8}$

$x^2 + kx + 16 = 0$

NO REAL ROOTS $\Rightarrow b^2 - 4ac < 0$

$\Rightarrow k^2 - 4 \times 1 \times 16 < 0$

$\Rightarrow k^2 - 64 < 0$

$\Rightarrow (k-8)(k+8) < 0$

C.V $= 8, -8$

$-8 < k < 8$

**Question 4** (**+)

Find the range of values of the constant $k$ so that the graph of the curve with equation

$$y = x^2 + 5x + k,$$

does not **cross** the $x$ axis.

$\boxed{x \geq \frac{25}{4}}$

$y = x^2 + 5x + k$

DOESN'T CROSS $x$ AXIS $\Rightarrow$ NO REAL ROOTS, OR REPEATED ROOT

$\Rightarrow b^2 - 4ac \leq 0$

$\Rightarrow 5^2 - 4 \times 1 \times k \leq 0$

$\Rightarrow 25 - 4k \leq 0$

$-4k \leq -25$

$k \geq \frac{25}{4}$

**Question 5** (**+)

Use an algebraic method to show that the graphs

$$y = 1 - x \quad \text{and} \quad y = x^2 - 6x + 10,$$

do **not** intersect.

C1 , proof

$\left.\begin{array}{l} y = 1 - x \\ y = x^2 - 6x + 10 \end{array}\right\} \Rightarrow x^2 - 6x + 10 = 1 - x$

$\Rightarrow x^2 - 5x + 9 = 0$

$b^2 - 4ac = (-5)^2 - 4 \times 1 \times 9 = 25 - 36 = -11 < 0$

NO REAL SOLUTIONS

NO INTERSECTIONS BETWEEN THE GRAPHS

**Question 6** (***)

Find the range of values of the constant $m$ so that the graph of the curve with equation

$$y = 2x^2 + mx + 2,$$

does not **cross** the $x$ axis.

$\boxed{-4 \le m \le 4}$

$y = 2x^2 + mx + 2$

DOESN'T CROSS $\Rightarrow$ NO ROOTS OR "ONE" ROOT

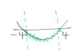
$\Rightarrow b^2 - 4ac \le 0$

$\Rightarrow m^2 - 4 \times 2 \times 2 \le 0$

$\Rightarrow m^2 - 16 \le 0$

$\Rightarrow (m - 4)(m + 4) \le 0$

$\Rightarrow$ C.V. $= -4, \ 4$




$-4 \le m \le 4$

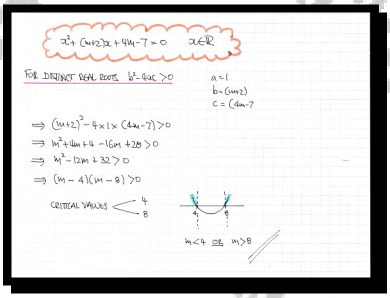
**Question 7** (***)

The following quadratic equation, where $m$ is a constant, has two distinct real roots.

$$x^2 + (m+2)x + 4m - 7 = 0, \; x \in \mathbb{R} \; .$$

Determine the range of the possible values of $m$.

$\boxed{\phantom{xxxx}}$, $\boxed{m < 4 \;\cup\; m > 8}$



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# STANDARD QUESTIONS

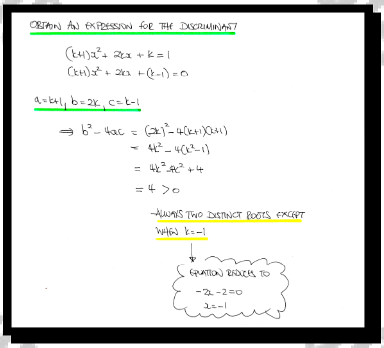
**Question 1** (***+)

Show that the quadratic equation

$$(k+1)x^2 + 2kx + k = 1$$

has two distinct real roots for all real values of the constant $k$, except for one value which must be stated.

$\boxed{\phantom{xxxx}}$, $\boxed{k \neq -1}$



OBTAIN AN EXPRESSION FOR THE DISCRIMINANT

$(k+1)x^2 + 2kx + k = 1$
$(k+1)x^2 + 2kx + (k-1) = 0$

$a = k+1,\ b = 2k,\ c = k-1$

$\Rightarrow b^2 - 4ac = (2k)^2 - 4(k+1)(k+1)$
$= 4k^2 - 4(k^2-1)$
$= 4k^2 - 4k^2 + 4$
$= 4 > 0$

ALWAYS TWO DISTINCT ROOTS EXCEPT WHEN $k = -1$

EQUATION REDUCES TO
$-2x - 2 = 0$
$x = -1$

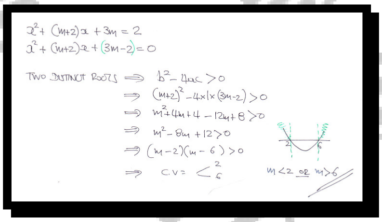
**Question 2** (***+)

Find the range of the values of the constant $m$ so that the equation

$$x^2 + (m+2)x + 3m = 2,$$

has two distinct real roots.

$\boxed{\phantom{xxxx}}$, $\boxed{m < 2 \text{ or } m > 6}$



$x^2 + (m+2)x + 3m = 2$
$x^2 + (m+2)x + (3m-2) = 0$

TWO DISTINCT ROOTS $\Rightarrow b^2 - 4ac > 0$
$\Rightarrow (m+2)^2 - 4 \times 1 \times (3m-2) > 0$
$\Rightarrow m^2 + 4m + 4 - 12m + 8 > 0$
$\Rightarrow m^2 - 8m + 12 > 0$
$\Rightarrow (m-2)(m-6) > 0$
$\Rightarrow$ C.V. $= <^2_6$



$m < 2$ or $m > 6$

**Question 3** (***+)

The straight line $L$ and the curve $C$ have respective equations

$$L:\ 2y = 7x + 10\,,$$

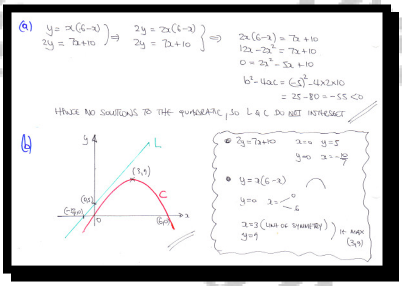
$$C:\ y = x(6-x)\,.$$

**a)** Show that $L$ and $C$ do not intersect.

**b)** Find the coordinates of the maximum point of $C$.

**c)** Sketch on the same diagram the graph of $L$ and the graph of $C$, showing clearly the coordinates of any points where each of the graphs meet the coordinate axes.

$\boxed{\phantom{xx}}$, $\boxed{\text{max}\,(3,9)}$



**Question 4** (***+)

The quadratic curves with equations

$$y = x^2 - 4x + 5 \quad \text{and} \quad y = m + 2x - x^2,$$

where $m$ is a constant, **touch** each other at the point $P$.

Determine the coordinates of $P$.

$\boxed{P\left(\frac{3}{2}, \frac{5}{4}\right)}$

$y = x^2 - 4x + 5$
$y = m + 2x - x^2$

$\Rightarrow x^2 - 4x + 5 = m + 2x - x^2$
$\Rightarrow 2x^2 - 6x + (5 - m) = 0$

LOOKING FOR REPEATED ROOTS

$b^2 - 4ac = 0$
$\Rightarrow (-6)^2 - 4 \times 2 \times (5 - m) = 0$
$\Rightarrow 36 - 8(5 - m) = 0$
$\Rightarrow 36 = 8(5 - m)$
$\Rightarrow \frac{9}{2} = 5 - m$
$\Rightarrow \boxed{m = \frac{1}{2}}$

IF $m = \frac{1}{2}$
$\Rightarrow 2x^2 - 6x + (5 - m) = 0$
$\Rightarrow 2x^2 - 6x + (5 - \frac{1}{2}) = 0$
$\Rightarrow 2x^2 - 6x + \frac{9}{2} = 0$
$\Rightarrow 4x^2 - 12x + 9 = 0$
$\Rightarrow (2x - 3)^2 = 0$
$\boxed{x = \frac{3}{2}}$
$\Rightarrow y = x^2 - 4x + 5$
$\Rightarrow y = \frac{9}{4} - 4\left(\frac{3}{2}\right) + 5$
$\Rightarrow y = \frac{9}{4} - 6 + 5$
$\boxed{y = \frac{5}{4}}$

$\therefore P\left(\frac{3}{2}, \frac{5}{4}\right)$

**Question 5** (***+)

Use the discriminant of a suitable quadratic equation to show that the graphs of the curves with equations

$$y = 2 - \frac{1}{x} \quad \text{and} \quad y = \frac{1}{2 - x},$$

**touch** each other.

C4Q, $\boxed{\text{proof}}$

$y = 2 - \frac{1}{x}$
$y = \frac{1}{2 - x}$
SOLVE SIMULTANEOUSLY

$2 - \frac{1}{x} = \frac{1}{2 - x}$ $\times x$
$2x - 1 = \frac{x}{2 - x}$ $\times (2 - x)$
$(2x - 1)(2 - x) = x$
$4x - 2x^2 - 2 + \cancel{x} = \cancel{x}$
$0 = 2x^2 - 4x + 2$
$\boxed{x^2 - 2x + 1 = 0}$

NOW $b^2 - 4ac = (-2)^2 - 4 \times 1 \times 1 = 4 - 4 = 0$
$\therefore$ REPEATED ROOT
$\therefore$ ONE SOLUTION
$\therefore$ GRAPHS TOUCH

**Question 6** (***+)

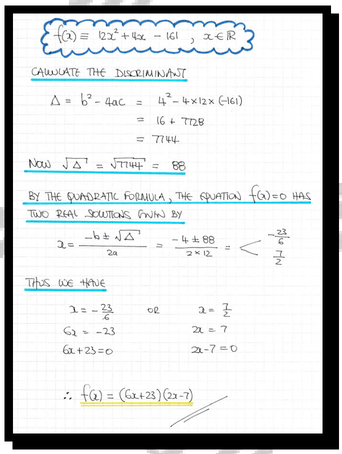
A quadratic curve has equation

$$f(x) \equiv 12x^2 + 4x - 161, \; x \in \mathbb{R}.$$

Express the above equation as the product of two linear factors.

***A detailed method must be shown in this question.***

$\boxed{\phantom{xx}}$, $\boxed{f(x) \equiv (6x+23)(2x-7)}$



$f(x) \equiv 12x^2 + 4x - 161, \; x \in \mathbb{R}$

CALCULATE THE DISCRIMINANT

$\Delta = b^2 - 4ac = 4^2 - 4 \times 12 \times (-161)$

$= 16 + 7728$

$= 7744$

NOW $\sqrt{\Delta} = \sqrt{7744} = 88$

BY THE QUADRATIC FORMULA, THE EQUATION $f(x) = 0$ HAS TWO REAL SOLUTIONS GIVEN BY

$x = \dfrac{-b \pm \sqrt{\Delta}}{2a} = \dfrac{-4 \pm 88}{2 \times 12} = \begin{cases} -\frac{23}{6} \\ \frac{7}{2} \end{cases}$

THUS WE HAVE

$x = -\frac{23}{6}$ OR $x = \frac{7}{2}$

$6x = -23$ | $2x = 7$

$6x + 23 = 0$ | $2x - 7 = 0$

$\therefore f(x) = (6x+23)(2x-7)$

**Question 7** (***+)

Find the possible solutions of the quadratic equation

$$x^2 + (k-1)x + k + 2 = 0,$$

where $k$ is a constant, given that the equation has repeated roots.

$\boxed{\phantom{xx}}$, $\boxed{x = 1 \cup x = -3}$

$\boxed{x^2 + (k-1)x + (k+2) = 0 \quad , k \in \mathbb{R}}$

REPEATED ROOTS $\Rightarrow b^2 - 4ac = 0$

$\Rightarrow (k-1)^2 - 4 \times 1 \times (k+2) = 0$

$\Rightarrow (k-1)^2 - 4(k+2) = 0$

$\Rightarrow k^2 - 2k + 1 - 4k - 8 = 0$

$\Rightarrow k^2 - 6k - 7 = 0$

$\Rightarrow (k+1)(k-7) = 0$

$\Rightarrow k = \begin{cases} -1 \\ 7 \end{cases}$

THERE ARE NOW TWO CASES TO CONSIDER

- IF $k = -1$

  $x^2 - 2x + 1 = 0$

  $(x-1)^2 = 0$

  $x = 1$

- IF $k = 7$

  $x^2 + 6x + 9 = 0$

  $(x+3)^2 = 0$

  $x = -3$

**Question 8** (****)

The quadratic curves with equations

$$y = k\left(2x^2 + 1\right) \quad \text{and} \quad y = x^2 - 2x,$$

where $k$ is a constant, **touch** each other.

Determine the possible values of $k$.

$\boxed{\phantom{xx}}$, $\boxed{k = -\frac{1}{2}, \ k = 1}$

$\left.\begin{array}{l} y = k(2x^2+1) \\ y = x^2 - 2x \end{array}\right\} \Rightarrow$ LOOK FOR INTERSECTIONS

$\Rightarrow k(2x^2+1) = x^2 - 2x$

$\Rightarrow 2kx^2 + k = x^2 - 2x$

$\Rightarrow 2kx^2 - x^2 + 2x + k = 0$

$\Rightarrow \boxed{(2k-1)x^2 + 2x + k = 0}$

IF THEY TOUCH, WE MUST HAVE REPEATED ROOTS

$b^2 - 4ac = 0$

HENCE $2^2 - 4 \times (2k-1) \times k = 0$

$\Rightarrow 4 - 4k(2k-1) = 0$

$\Rightarrow 1 - k(2k-1) = 0$

$\Rightarrow 1 - 2k^2 + k = 0$

$\Rightarrow 0 = 2k^2 - k - 1$

$\Rightarrow (2k+1)(k-1) = 0$

$k = \begin{cases} 1 \\ -\frac{1}{2} \end{cases}$

**Question 9** (****)

Find the range of values that the constant $k$ can take so that

$$2x^2+(k+2)x+k=0$$

has two distinct real roots.

$\boxed{\phantom{MP1-Y}}$, $\boxed{k\in\mathbb{R},\ k\neq 2}$

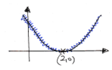
TWO DISTINCT REAL ROOTS $\Rightarrow b^2-4ac>0$

$\Rightarrow (k+2)^2-4\times2\times k>0$

$\Rightarrow k^2+4k+4-8k>0$

$\Rightarrow k^2-4k+4>0$

$\Rightarrow (k-2)^2>0$




$\therefore$ ALL REAL NUMBER EXCEPT 2

$\therefore k\in\mathbb{R},\ k\neq 2$

**Question 10** (****)

Find the possible solutions of the quadratic equation

$$x^2+(3-m)x+5=m^2,$$

where $m$ is a constant, given that the equation has repeated roots.

$\boxed{\phantom{MP1-Y}}$, $\boxed{x=-2\ \cup\ x=-\frac{2}{5}}$

REARRANGE THE EQUATION

$\Rightarrow x^2+(3-m)x+5=m^2$

$\Rightarrow x^2+(3-m)x+(5-m^2)=0$

AS THE EQUATION HAS REPEATED ROOTS $b^2-4ac=0$

$\Rightarrow (3-m)^2-4\times1\times(5-m^2)=0$

$\Rightarrow 9-6m+m^2-20+4m^2=0$

$\Rightarrow 5m^2-6m-11=0$

$\Rightarrow (5m-11)(m+1)=0$

$\Rightarrow m = -1$ or $\frac{11}{5}$

NOW CONSIDER AND SOLVE EACH CASE SEPARATELY

IF $m=-1$

$x^2+[3-(-1)]x+[5-(-1)^2]=0$

$x^2+4x+4=0$

$(x+2)^2=0$

$x=-2$

IF $m=\frac{11}{5}$

$x^2+\left[3-\frac{11}{5}\right]x+\left[5-\left(\frac{11}{5}\right)^2\right]=0$

$x^2+\frac{4}{5}x+\frac{4}{25}=0$

$\left(x+\frac{2}{5}\right)^2=0$

$x=-\frac{2}{5}$

**Question 11** (****)

$$f(x) = px^2 + 4x(p+3) + 5p,$$

where $p$ is a non zero constant.

The equation $f(x) = -19$ has two distinct real roots.

Find the range of the possible values of $p$.

$\boxed{\phantom{xx}}$, $\boxed{-4 < p < 0 \quad \cup \quad 0 < p < 9}$

FORMING THE EQUATION & TIDY TO A 3-TERM QUADRATIC

$\Rightarrow px^2 + 4x(p+3) + 5p = -19$

$\Rightarrow \underbrace{p}_{a}x^2 + \underbrace{4(p+3)}_{b}x + \underbrace{(5p+19)}_{c} = 0$

TWO DISTINCT ROOTS IMPLY $b^2 - 4ac > 0$

$\Rightarrow [4(p+3)]^2 - 4 \times p \times (5p+19) > 0$

$\Rightarrow 16(p+3)^2 - 4p(5p+19) > 0$

$\Rightarrow 4(p+3)^2 - p(5p+19) > 0$

$\Rightarrow 4(p^2+6p+9) - 5p^2 - 19p > 0$

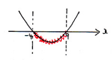
$\Rightarrow 4p^2 + 24p + 36 - 5p^2 - 19p > 0$

$\Rightarrow -p^2 + 5p + 36 > 0$

$\Rightarrow p^2 - 5p - 36 < 0$

$\Rightarrow (p-9)(p+4) < 0$

CRITICAL VALUES ARE 9 & −4




SO $-4 < p < 9$

BUT $p \neq 0$, OTHERWISE NO QUADRATIC

$-4 < p < 0 \quad \cup \quad 0 < p < 9$

**Question 12** **(****)**

$$x^2 - 4ax + 2b + 1 = 0.$$

The above quadratic equation, where $a$ and $b$ are constants, has no real solutions.

Show clearly that

$$b > \frac{1}{2}(2a+1)(2a-1).$$

$\boxed{\phantom{xx}}$, $\boxed{\text{proof}}$

$x^2 - 4ax + 2b + 1 = 0$

(CHOOSE CAPITALS FOR DISCRIMINANT TO AVOID CONFUSION)

NO REAL SOLUTIONS $\Rightarrow B^2 - 4AC < 0$

$\Rightarrow (-4a)^2 - 4 \times 1 \times (2b+1) < 0$

$\Rightarrow 16a^2 - 4(2b+1) < 0$

$\Rightarrow 4a^2 - (2b+1) < 0$

$\Rightarrow -(2b+1) < -4a^2$

$\Rightarrow 2b+1 > 4a^2$

$\Rightarrow 2b > 4a^2 - 1$

$\Rightarrow 2b > (2a-1)(2a+1)$

$\Rightarrow b > \frac{1}{2}(2a-1)(2a+1)$

**Question 13** (****)

The curve $C$ has equation

$$y = 4x^2 - 7x + 11.$$

The straight line $L$ has equation

$$y = 5x + k,$$

where $k$ is a constant.

Given that $C$ and $L$ intersect at two distinct points, show that $k > 2$.

$\boxed{\phantom{SYN}}$, $\boxed{\text{proof}}$

SOLVING THE EQUATIONS SIMULTANEOUSLY

$$\left.\begin{array}{l} y = 4x^2 - 7x + 11 \\ y = 5x + k \end{array}\right\} \Rightarrow 4x^2 - 7x + 11 = 5x + k$$

$$\Rightarrow 4x^2 - 12x + (11 - k) = 0$$

IF THERE ARE TWO DISTINCT POINTS OF INTERSECTION, THE DISCRIMINANT OF THE ABOVE QUADRATIC MUST BE POSITIVE

$$\Rightarrow \text{"}b^2 - 4ac\text{"} > 0$$

$$\Rightarrow (-12)^2 - 4 \times 4 \times (11 - k) > 0$$

$$\Rightarrow 144 - 16(11 - k) > 0$$

$$\Rightarrow 144 - 176 + 16k > 0$$

$$\Rightarrow 16k > 32$$

$$\Rightarrow k > 2$$

// AS REQUIRED

**Question 14** **(****)**

The straight line $L$ has equation

$$y = kx - 9,$$

where $k$ is a constant.

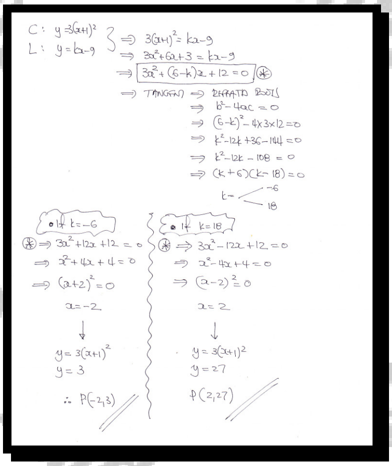
The curve $C$ has equation

$$y = 3(x+1)^2.$$

It is further given that $L$ is a tangent to $C$ at the point $P$.

Determine the possible coordinates of $P$.

$\boxed{\phantom{xx}}$, $\boxed{(-2,3) \text{ or } (2,27)}$



$C: y = 3(x+1)^2$
$L: y = kx - 9$
$\Rightarrow 3(x+1)^2 = kx - 9$
$\Rightarrow 3x^2 + 6x + 3 = kx - 9$
$\Rightarrow \boxed{3x^2 + (6-k)x + 12 = 0}$ (*)
$\Rightarrow$ TANGENT $\Rightarrow$ REPEATED ROOTS
$\Rightarrow b^2 - 4ac = 0$
$\Rightarrow (6-k)^2 - 4 \times 3 \times 12 = 0$
$\Rightarrow k^2 - 12k + 36 - 144 = 0$
$\Rightarrow k^2 - 12k - 108 = 0$
$\Rightarrow (k+6)(k-18) = 0$
$k = -6$ or $18$

• If $k = -6$
(*) $\Rightarrow 3x^2 + 12x + 12 = 0$
$\Rightarrow x^2 + 4x + 4 = 0$
$\Rightarrow (x+2)^2 = 0$
$x = -2$
$y = 3(x+1)^2$
$y = 3$
$\therefore P(-2,3)$

• If $k = 18$
(*) $\Rightarrow 3x^2 - 12x + 12 = 0$
$\Rightarrow x^2 - 4x + 4 = 0$
$\Rightarrow (x-2)^2 = 0$
$x = 2$
$y = 3(x+1)^2$
$y = 27$
$P(2,27)$

**Question 15** **(****)**

The curve $C$ has equation

$$y = 3x^2 - 4x + 7 .$$

The straight line $L$ has equation

$$y = 2x + k ,$$

where $k$ is a constant.

Given that $C$ and $L$ do not intersect, show that $k < 4$

proof

$$\left.\begin{array}{l} y = 3x^2 - 4x + 7 \\ y = 2x + k \end{array}\right\} \Rightarrow 3x^2 - 4x + 7 = 2x + k$$

$\Rightarrow 3x^2 - 6x + (7 - k) = 0$

$b^2 - 4ac < 0$ (NO SOLUTION SINCE THERE ARE NO INTERSECTIONS)

$\Rightarrow (-6)^2 - 4 \times 3 \times (7 - k) < 0$

$\Rightarrow 36 - 12(7 - k) < 0$

$\Rightarrow 3 - (7 - k) < 0$

$\Rightarrow -4 + k < 0$

$\Rightarrow k < 4$

AS REQUIRED

**Question 16** (****)

The straight line with equation

$$y = 2x + c$$

is a tangent to the curve with equation

$$y = x^2 + 6x + 7 .$$

By **using the discriminant** of a suitable quadratic, determine the value of the constant $c$ and find the point of contact between the tangent and the curve.

$\boxed{\text{[illegible]}}$, $\boxed{c = 3}$, $\boxed{(-2,-1)}$

SOLVING SIMULTANEOUSLY

$$\left.\begin{array}{l} y = x^2+6x+7 \\ y = 2x+c \end{array}\right\} \Rightarrow x^2+6x+7 = 2x+c$$

$$\Rightarrow x^2+4x+7-c=0$$

IF A TANGENT THE ABOVE QUADRATIC MUST HAVE REPEATED ROOTS, IE $b^2-4ac=0$

$$\Rightarrow 4^2 - 4\times1\times(7-c) = 0$$
$$\Rightarrow 16 - 4(7-c) = 0$$
$$\Rightarrow 16 - 28 + 4c = 0$$
$$\Rightarrow -12 + 4c = 0$$
$$\Rightarrow 4c = 12$$
$$\Rightarrow c = 3$$

FINALLY IF $c=3$ WE HAVE

$$\Rightarrow x^2+4x+7-c=0$$
$$\Rightarrow x^2+4x+4=0$$
$$\Rightarrow (x+2)^2=0$$
$$\Rightarrow x=-2$$

& USING $y = 2x+3$

$y = 2(-2)+3$

$y = -1$

$\therefore (-2,-1)$

**Question 17** (****)

A circle has equation

$$x^2 + y^2 = 8y .$$

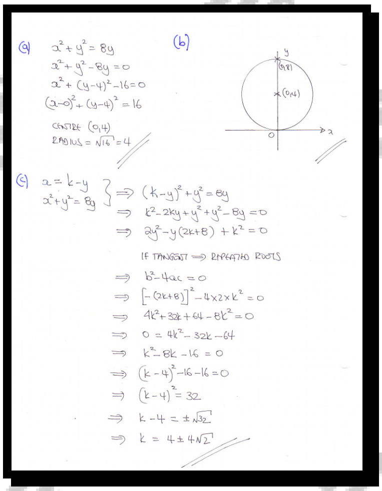
**a)** Find the coordinates of the centre of the circle and the size of its radius.

**b)** Sketch the circle.

The line with equation $x + y = k$, where $k$ is a constant, is a tangent to this circle.

**c)** Determine, as exact surds, the possible values of $k$.

$\boxed{(0,4)}$, $\boxed{r=4}$, $\boxed{k=4\pm 4\sqrt{2}}$



(a) $x^2+y^2=8y$
$x^2+y^2-8y=0$
$x^2+(y-4)^2-16=0$
$(x-0)^2+(y-4)^2=16$
CENTRE $(0,4)$
RADIUS $=\sqrt{16}=4$

(b)



(c) $\left.\begin{array}{l} x=k-y \\ x^2+y^2=8y \end{array}\right\} \Rightarrow (k-y)^2+y^2=8y$
$\Rightarrow k^2-2ky+y^2+y^2-8y=0$
$\Rightarrow 2y^2-y(2k+8)+k^2=0$

IF TANGENT $\Rightarrow$ REPEATED ROOTS

$\Rightarrow b^2-4ac=0$
$\Rightarrow [-(2k+8)]^2-4\times 2\times k^2=0$
$\Rightarrow 4k^2+32k+64-8k^2=0$
$\Rightarrow 0=4k^2-32k-64$
$\Rightarrow k^2-8k-16=0$
$\Rightarrow (k-4)^2-16-16=0$
$\Rightarrow (k-4)^2=32$
$\Rightarrow k-4=\pm\sqrt{32}$
$\Rightarrow k=4\pm 4\sqrt{2}$

**Question 18** (****)

$$f(n) = n^2 - 2kn + k + 12, \; n \in \mathbb{N},$$

where $k$ is a constant.

Given that $f(n) = n^2 - 2kn + k + 12$ is a square number for all values of $n$, determine the possible values of the constant $k$.

$\boxed{\phantom{k=0}}$ , $\boxed{k = -3 \cup k = 4}$

IF $f(n)$ IS TO BE A SQUARE NUMBER, THEN IT MUST BE A PERFECT SQUARE IE REPEATED ROOTS

$f(n) = n^2 - 2kn + k + 12$

$\Rightarrow b^2 - 4ac = 0$

$\Rightarrow (-2k)^2 - 4 \times 1 \times (k+12) = 0$

$\Rightarrow 4k^2 - 4(k+12) = 0$

$\Rightarrow k^2 - (k+12) = 0$

$\Rightarrow k^2 - k - 12 = 0$

$\Rightarrow (k-4)(k+3) = 0$

$\therefore k = -3, \; 4$

**Question 19** (****)

The straight line with equation

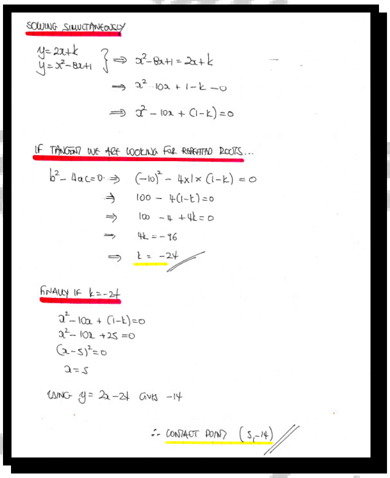
$$y = 2x + k,$$

where $k$ is constant, is a tangent to the curve with equation

$$y = x^2 - 8x + 1.$$

By **using the discriminant** of a suitable quadratic, determine the value of the constant $k$ and hence find the point of contact between the tangent and the curve.

$\boxed{\phantom{xx}}$, $\boxed{k = -24}$, $\boxed{(5, -14)}$



SOLVING SIMULTANEOUSLY

$$\left.\begin{array}{l} y = 2x + k \\ y = x^2 - 8x + 1 \end{array}\right\} \Rightarrow x^2 - 8x + 1 = 2x + k$$
$$\Rightarrow x^2 - 10x + 1 - k = 0$$
$$\Rightarrow x^2 - 10x + (1 - k) = 0$$

IF TANGENT WE ARE LOOKING FOR REPEATED ROOTS...

$$b^2 - 4ac = 0 \Rightarrow (-10)^2 - 4 \times 1 \times (1 - k) = 0$$
$$\Rightarrow 100 - 4(1 - k) = 0$$
$$\Rightarrow 100 - 4 + 4k = 0$$
$$\Rightarrow 4k = -96$$
$$\Rightarrow k = -24$$

FINALLY IF $k = -24$

$$x^2 - 10x + (1 - k) = 0$$
$$x^2 - 10x + 25 = 0$$
$$(x - 5)^2 = 0$$
$$x = 5$$

USING $y = 2x - 24$ GIVES $-14$

$\therefore$ CONTACT POINT $(5, -14)$

## Question 20 (****)

Find, in surd form, the range of values of $m$ for which the quadratic equation

$$x^2 + (3-m)x + 10 = 3$$

has no real roots.

$$\boxed{3-2\sqrt{7} < m < 3+2\sqrt{7}}$$

$x^2+(3-m)x+10=3$

$x^2+(3-m)x+7=0$

• NO REAL ROOTS

$b^2-4ac<0$

$\Rightarrow (3-m)^2-4\times1\times7<0$

$\Rightarrow (3-m)^2<28$

$\Rightarrow -\sqrt{28}<3-m<\sqrt{28}$

$\Rightarrow -3-\sqrt{28}<-m<-3+\sqrt{28}$

NOW $-m<-3+\sqrt{28}$

$m>3-\sqrt{28}$

AND $-m>-3-\sqrt{28}$

$m<3+\sqrt{28}$

$\therefore 3-2\sqrt{7}<m<3+2\sqrt{7}$

ALTERNATIVE SOLUTION

$(3-m)^2<28$

CONSIDER CORRESPONDING EQUATION

$(3-m)^2=28$

$(m-3)^2=28$

$m-3=\pm\sqrt{28}$

$m=3\pm2\sqrt{7}$

$-3+2\sqrt{7}$ $3+2\sqrt{7}$

$\therefore 3-2\sqrt{7}<m<3+2\sqrt{7}$

## Question 21 (****)

Find the possible roots of the following quadratic equation

$$mx^2 - 4x + m = 3,$$

where $m$ is a non zero constant, given that it has repeated roots.

$\boxed{\phantom{m=-1,4}}$ , $\boxed{x=-2,\frac{1}{2}}$

STARTING BY FINDING THE VALUES OF $m$ FOR REPEATED ROOTS

$\Rightarrow b^2-4ac=0$ ← $[mx^2-4x+(m-3)=0]$

$\Rightarrow (-4)^2-4\times m\times(m-3)=0$

$\Rightarrow 16-4m(m-3)=0$

$\Rightarrow 16-4m^2+12m=0$

$\Rightarrow 0=4m^2-12m-16$

$\Rightarrow m^2-3m-4=0$

$\Rightarrow (m+1)(m-4)=0$

$\Rightarrow m = -1, 4$

NOW SOLVE $mx^2-4x+m=3$ FOR EACH OF THE TWO VALUES OF $m$

IF $m=-1$

$-x^2-4x-1=3$

$-x^2-4x-4=0$

$x^2+4x+4=0$

$(x+2)^2=0$

$x=-2$

IF $m=4$

$4x^2-4x+4=3$

$4x^2-4x+1=0$

$(2x-1)^2=0$

$x=\frac{1}{2}$

**Question 22** (****)

The cubic curve with equation

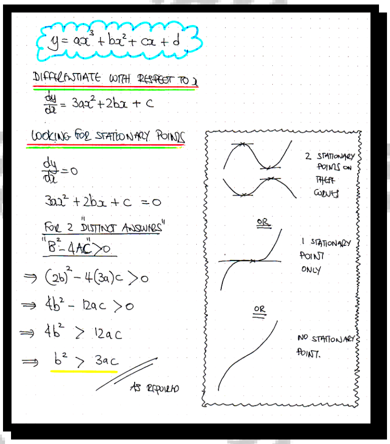
$$y = ax^3 + bx^2 + cx + d,$$

where $a$, $b$, $c$ are non zero constants and $d$ is a constant, has one local maximum and one local minimum.

Show clearly that

$$b^2 > 3ac$$

$\boxed{\phantom{xxx}}$, $\boxed{\text{proof}}$



$y = ax^3 + bx^2 + cx + d$

DIFFERENTIATE WITH RESPECT TO $x$

$\frac{dy}{dx} = 3ax^2 + 2bx + c$

LOOKING FOR STATIONARY POINTS

$\frac{dy}{dx} = 0$

$3ax^2 + 2bx + c = 0$

FOR 2 "DISTINCT ANSWERS"

"$B^2 - 4AC > 0$"

$\Rightarrow (2b)^2 - 4(3a)c > 0$

$\Rightarrow 4b^2 - 12ac > 0$

$\Rightarrow 4b^2 > 12ac$

$\Rightarrow b^2 > 3ac$

AS REQUIRED

**Question 23** (****)

The straight line with equation

$$y = k(4x - 17),$$

does **not** intersect with the quadratic with equation

$$y = 13 - 8x - x^2.$$

Find the range of possible values of $k$.

$\boxed{-\frac{29}{4} < k < -1}$

SOLVING SIMULTANEOUSLY

$\left.\begin{array}{l} y = k(4x-17) \\ y = 13-8x-x^2 \end{array}\right\} \Rightarrow k(4x-17) = 13-8x-x^2$

$\Rightarrow x^2+8x-13+4kx-17k = 0$

$\Rightarrow x^2+(8+4k)x+(-13-17k) = 0$

NO INTERSECTIONS, NO REAL ROOTS

$\Rightarrow b^2-4ac < 0$

$\Rightarrow (8+4k)^2 - 4\times 1\times(-13-17k) < 0$

$\Rightarrow 16(2+k)^2 + 4(13+17k) < 0$ $\div 4$

$\Rightarrow 4(k+2)^2 + (13+17k) < 0$

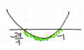
$\Rightarrow 4(k^2+4k+4) + 17k + 13 < 0$

$\Rightarrow 4k^2+16k+16+17k+13 < 0$

$\Rightarrow 4k^2+33k+29 < 0$

FACTORIZE TO OBTAIN CRITICAL VALUES

$\Rightarrow (4k+29)(k+1) < 0$




$-\frac{29}{4} < k < -1$

**Question 24** (****)

A straight line crosses the $y$ axis at $(0, -5)$ and **does not cross** the curve $y = 3x^2 - 2$.

Find the range of the possible values of the **gradient** of the line.

$\boxed{-6 < \text{gradient} < 6}$

$\left.\begin{array}{l} y = mx-5 \\ y = 3x^2-2 \end{array}\right\} \Rightarrow$

$\Rightarrow 3x^2-2 = mx-5$

$\Rightarrow 3x^2-mx+3 = 0$

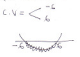
"DOESN'T INTERSECT" $\Rightarrow b^2-4ac < 0$

$\Rightarrow m^2-4\times 3\times 3 < 0$

$\Rightarrow m^2-36 < 0$

$\Rightarrow (m-6)(m+6) < 0$

$\Rightarrow$ C.V $= \Big< \begin{matrix} -6 \\ 6 \end{matrix}$




$-6 < \underset{\text{(GRADIENT)}}{m} < 6$

**Question 25** (****)

The straight line with equation

$$y = 3(2x+1)$$

meets the curve with equation

$$y = k\left(x^2+2\right).$$

By **using the discriminant** of a suitable quadratic, determine the range of the possible values of the constant $k$.

$$\boxed{k \le -\frac{3}{2} \text{ or } k \ge 3}$$

$$\left.\begin{array}{l} y = 3(2x+1) \\ y = k(x^2+2) \end{array}\right\} \Rightarrow k(x^2+2) = 3(2x+1)$$

THUS

$kx^2 + 2k = 6x + 3$

$kx^2 - 6x + 2k - 3 = 0$

"MEETS" $\Rightarrow$ REAL ROOTS $\Rightarrow b^2 - 4ac \geqslant 0$

$\Rightarrow (-6)^2 - 4k(2k-3) \geqslant 0$

$\Rightarrow 36 - 8k^2 + 12k \geqslant 0$

$\Rightarrow -8k^2 + 12k + 36 \geqslant 0$

$\Rightarrow 8k^2 - 12k - 36 \leqslant 0$

$\Rightarrow 2k^2 - 3k - 9 \leqslant 0$

$\Rightarrow (2k+3)(k-3) \geqslant 0$

HENCE C.V $= \begin{cases} 3 \\ -\frac{3}{2} \end{cases}$

$k \le -\frac{3}{2}$ OR $k \ge 3$

**Question 26** (****)

Show that the curves with equations

$$y = x^4 - 4 \quad \text{and} \quad y = kx^2$$

intersect for all values of the constant $k$.

$$\boxed{\text{proof}}$$

$$\left.\begin{array}{l} y = x^4 - 4 \\ y = kx^2 \end{array}\right\} \Rightarrow x^4 - 4 = kx^2$$

THUS

$\Rightarrow x^4 - 4 = kx^2$

$\Rightarrow x^4 - kx^2 - 4 = 0$

THIS IS A QUADRATIC IN $x^2$

$b^2 - 4ac = (-k)^2 - 4 \times 1 \times (-4)$

$= k^2 + 16 > 0$

$\therefore$ THERE WILL ALWAYS BE SOLUTIONS

SO

CURVES INTERSECT FOR ALL $k$

**Question 27** **(****)**

$$(x-a)(x-b)=m^2,$$

where $a$, $b$ and $m$ are constants.

By using discriminant considerations, show that the above quadratic equation will always have real solutions.

[ ], [proof]

$(x-a)(x-b)=m^2$

$x^2-bx-ax+ab=m^2$

$x^2-(a+b)x+ab-m^2=0$

NOW CONSIDER THE DISCRIMINANT

$[-(a+b)]^2-4\times1\times(ab-m^2)$

$= a^2+2ab+b^2-4ab+4m^2$

$= a^2-2ab+b^2+4m^2$

$= (a-b)^2+4m^2 \geqslant 0$

all quantities are positive or squared

$\therefore$ EQUATION WILL ALWAYS HAVE REAL ROOTS

**Question 28** **(****)**

The curve $C$ and the straight line $L$ have respective equations

$$x^2-\frac{y^2}{2}=1 \qquad \text{and} \qquad y=x+c,$$

where $c$ is a constant.

Show that $C$ and $L$, intersect for all values of $c$.

[ ], [proof]

SOLVING SIMULTANEOUSLY

$x^2-\frac{1}{2}y^2=1 \qquad y=x+c$

$x^2-\frac{1}{2}(x+c)^2=1$

$\Rightarrow 2x^2-(x+c)^2=2$

$\Rightarrow 2x^2-(x^2+2cx+c^2)-2=0$

$\Rightarrow 2x^2-x^2-2cx-c^2-2=0$

$\Rightarrow x^2-2cx-(c^2+2)=0$

NOW USING THE DISCRIMINANT WITH $A=1$, $B=-2c$, $C=-(c^2+2)$

$\Rightarrow B^2-4AC = (-2c)^2-4\times1\times[-c^2-2]$

$= 4c^2+4c^2+8$

$= 8c^2+8$

$= \geqslant 8$

AS THE DISCRIMINANT IS POSITIVE (AT LEAST 8) FOR ALL VALUES OF C THE CURVE & LINE INTERSECT REGARDLESS

**Question 29** (****)

A curve $C$ has equation

$$y = \frac{1}{x-1}, \quad x \neq 1.$$

**a)** Sketch the graph of $C$, clearly labelling its asymptotes and the coordinates of any point where $C$ meets the coordinate axes.

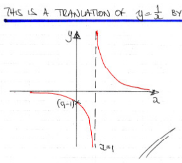
The line with equation $y = a - 2x$, where $a$ is a constant, does not meet $C$.

**b)** Show clearly that

$$2 - 2\sqrt{2} < a < 2 + 2\sqrt{2}.$$

$\boxed{\phantom{graph}}$, $\boxed{\text{asymptotes } x = 1, \ y = 0}$, $\boxed{(0,-1)}$

a) THIS IS A TRANSLATION OF $y = \frac{1}{x}$ BY 1 UNIT TO THE RIGHT




ASYMPTOTES $y = 0$ & $x = 1$

b) LOOKING FOR "INTERSECTIONS", ALTHOUGH WE ARE TOLD THERE ARE NOT ANY

$$\left.\begin{array}{l} y = \frac{1}{x-1} \\ y = a - 2x \end{array}\right\} \Rightarrow \frac{1}{x-1} = a - 2x$$

$$\Rightarrow 1 = (x-1)(a-2x)$$

$$\Rightarrow 1 = ax - 2x^2 - a + 2x$$

$$\Rightarrow 2x^2 - ax - 2x + 1 + a = 0$$

$$\Rightarrow 2x^2 - (a+2)x + (a+1) = 0$$

BUT THIS EQUATION HAS NO REAL ROOTS (NO INTERSECTIONS)

$$\Rightarrow B^2 - 4AC < 0$$

$$\Rightarrow [-(a+2)]^2 - 4 \times 2 \times (a+1) < 0$$

$$\Rightarrow (a+2)^2 - 8(a+1) < 0$$

$$\Rightarrow a^2 + 4a + 4 - 8a - 8 < 0$$

$$\Rightarrow a^2 - 4a - 4 < 0$$

$$\Rightarrow (a-2)^2 - 8 < 0$$

$$\Rightarrow (a-2)^2 < 8$$

$$\Rightarrow -\sqrt{8} < a - 2 < \sqrt{8}$$

$$\Rightarrow 2 - \sqrt{8} < a < 2 + \sqrt{8}$$

$$\therefore \ 2 - 2\sqrt{2} < a < 2 + 2\sqrt{2}$$

**Question 30** (****)

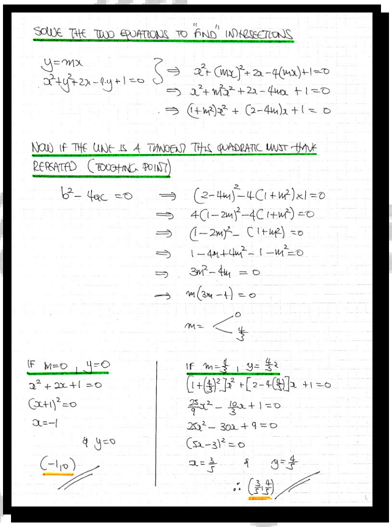
A circle $C$ has equation

$$x^2 + y^2 + 2x - 4y + 1 = 0$$

The straight line $L$ with equation $y = mx$ is a tangent to $C$.

Find the possible values of $m$ and hence determine the possible coordinates at which $L$ meets $C$.

$\boxed{\phantom{xxxx}}$, $\boxed{m = 0, \ m = \frac{4}{3}}$, $\boxed{(-1,0), \ \left(\frac{3}{5}, \frac{4}{5}\right)}$



SOLVE THE TWO EQUATIONS TO "FIND" INTERSECTIONS

$$\left.\begin{array}{l} y = mx \\ x^2+y^2+2x-4y+1=0 \end{array}\right\} \Rightarrow x^2 + (mx)^2 + 2x - 4(mx) + 1 = 0$$

$$\Rightarrow x^2 + m^2x^2 + 2x - 4mx + 1 = 0$$

$$\Rightarrow (1+m^2)x^2 + (2-4m)x + 1 = 0$$

NOW IF THE LINE IS A TANGENT THIS QUADRATIC MUST HAVE REPEATED (TOUCHING POINT)

$$b^2 - 4ac = 0 \Rightarrow (2-4m)^2 - 4(1+m^2)\times 1 = 0$$

$$\Rightarrow 4(1-2m)^2 - 4(1+m^2) = 0$$

$$\Rightarrow (1-2m)^2 - (1+m^2) = 0$$

$$\Rightarrow 1 - 4m + 4m^2 - 1 - m^2 = 0$$

$$\Rightarrow 3m^2 - 4m = 0$$

$$\Rightarrow m(3m-4) = 0$$

$$m = \begin{cases} 0 \\ \frac{4}{3} \end{cases}$$

IF $m=0$, $y=0$

$x^2 + 2x + 1 = 0$

$(x+1)^2 = 0$

$x = -1$

& $y = 0$

$(-1,0)$

IF $m = \frac{4}{3}$, $y = \frac{4}{3}x$

$\left[1 + \left(\frac{4}{3}\right)^2\right]x^2 + \left[2 - 4\left(\frac{4}{3}\right)\right]x + 1 = 0$

$\frac{25}{9}x^2 - \frac{10}{3}x + 1 = 0$

$25x^2 - 30x + 9 = 0$

$(5x-3)^2 = 0$

$x = \frac{3}{5}$ & $y = \frac{4}{5}$

$\therefore \left(\frac{3}{5}, \frac{4}{5}\right)$

# 9 HARD QUESTIONS

**Question 1** **(****+)**

The straight line $L$ crosses the $y$ axis at $(0,-1)$.

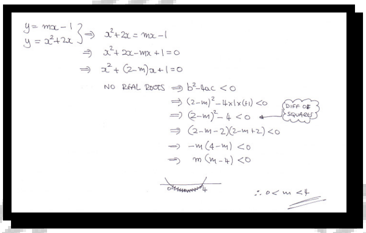
The curve with equation

$$y = x^2 + 2x$$

has **no intersections** with $L$.

Determine the range of the possible values of the gradient of $L$.

$\boxed{\phantom{00}}$, $\boxed{0 < m < 4}$



$$\left.\begin{array}{l} y = mx - 1 \\ y = x^2 + 2x \end{array}\right\} \Rightarrow x^2 + 2x = mx - 1$$

$\Rightarrow x^2 + 2x - mx + 1 = 0$

$\Rightarrow x^2 + (2-m)x + 1 = 0$

NO REAL ROOTS $\Rightarrow b^2 - 4ac < 0$

$\Rightarrow (2-m)^2 - 4 \times 1 \times (+1) < 0$

$\Rightarrow (2-m)^2 - 4 < 0$ DIFF OF SQUARES

$\Rightarrow (2-m-2)(2-m+2) < 0$

$\Rightarrow -m(4-m) < 0$

$\Rightarrow m(m-4) < 0$

$\therefore 0 < m < 4$

**Question 2** (****+)

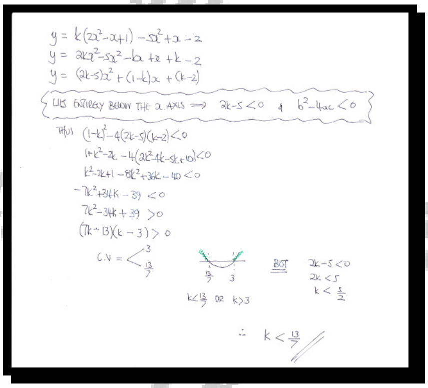
The equation of a quadratic curve $C$ is

$$y = k\left(2x^2 - x + 1\right) - 5x^2 + x - 2,$$

where $k$ is a constant.

Given that the graph of $C$ lies below the $x$ axis, determine the range of the possible values of $k$.

$\boxed{\phantom{xx}}$, $\boxed{k < \frac{13}{7}}$



$y = k(2x^2 - x + 1) - 5x^2 + x - 2$

$y = 2kx^2 - 5x^2 - kx + x + k - 2$

$y = (2k-5)x^2 + (1-k)x + (k-2)$

LIES ENTIRELY BELOW THE $x$ AXIS $\Longrightarrow$ $2k-5<0$ & $b^2-4ac<0$

THUS $(1-k)^2 - 4(2k-5)(k-2) < 0$

$1 + k^2 - 2k - 4(2k^2 - 4k - 5k + 10) < 0$

$k^2 - 2k + 1 - 8k^2 + 36k - 40 < 0$

$-7k^2 + 34k - 39 < 0$

$7k^2 - 34k + 39 > 0$

$(7k-13)(k-3) > 0$

C.V. $= \left\langle \begin{matrix} 3 \\ \frac{13}{7} \end{matrix} \right.$

$\frac{13}{7}$ $3$

$k < \frac{13}{7}$ OR $k > 3$

BUT $2k - 5 < 0$

$2k < 5$

$k < \frac{5}{2}$

$\therefore$ $k < \frac{13}{7}$

**Question 3** (****+)

A quadratic equation has two real roots differing by $k$, where $k$ is a positive constant.

Determine, in terms of $k$, an exact simplified expression for the discriminant of this quadratic.

You may assume that the coefficient of the quadratic term of the equation is one.

$\boxed{\phantom{xx}}$, $\boxed{b^2 - 4ac = k^2}$

LET THE TWO ROOTS BE $\alpha$ & $\alpha + k$

THEN
$$(x-\alpha)\left[x-(\alpha+k)\right] = 0$$
$$x^2 - (\alpha+k)x - \alpha x + \alpha(\alpha+k) = 0$$
$$x^2 - (2\alpha+k)x + (\alpha^2 + k\alpha) = 0$$

THUS $b = -(2\alpha+k)$ & $c = \alpha^2 + \alpha k$

$$\Rightarrow b^2 - 4ac = \left[-(2\alpha+k)\right]^2 - 4 \times 1 \times (\alpha^2 + \alpha k)$$
$$= (2\alpha+k)^2 - 4(\alpha^2 + \alpha k)$$
$$= \cancel{4\alpha^2} + \cancel{4\alpha k} + k^2 - \cancel{4\alpha^2} - \cancel{4\alpha k}$$
$$= k^2$$

**Question 4** (****+)

$$f(x) = k + 12x - 4x^2,$$

where $k$ is a constant.

It is further given that $f(x) > 5$ for **some** values of $x$.

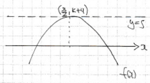
Show by suitable discriminant calculations, or otherwise, that

$$k > -4.$$

[ ], [proof]

METHOD A (PROBABLY THE BEST APPROACH BY CALCULUS)

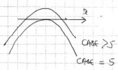
- $f(x) = k + 12x - 4x^2$
  $f'(x) = 12 - 8x$
- SOLVE FOR ZERO
  $0 = 12 - 8x$
  $8x = 12$
  $x = \frac{3}{2}$
- $f\left(\frac{3}{2}\right) = k + 12\left(\frac{3}{2}\right) - 4\left(\frac{3}{2}\right)^2$
  $f\left(\frac{3}{2}\right) = k + 18 - 9$
  $f\left(\frac{3}{2}\right) = k + 9$
- LOOKING AT A SKETCH OF $f(x)$




- WE REQUIRE THAT $k + 9 > 5$
  $\therefore k > -4$

METHOD B (DISCRIMINANT AS SUGGESTED)

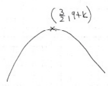
- $f(x) > 5$
  $k + 12x - 4x^2 > 5$
  $-4x^2 + 12x + k - 5 > 0$
- LOOKING AT THE SKETCH OF THIS




- THE DISCRIMINANT OF THIS INEQUALITY MUST PRODUCE 2 DISTINCT ROOTS
  (THIS WILL STILL BE THE SAME IF WE MODIFIED THIS TO $4x^2 - 12x - k + 5 < 0$)
- "$b^2 - 4ac$" $> 0$
  $144 - 4 \times (-4)(k-5) > 0$
  $144 + 16(k-5) > 0$
  $16(k-5) > -144$
  $k - 5 > -9$
  $k > -4$

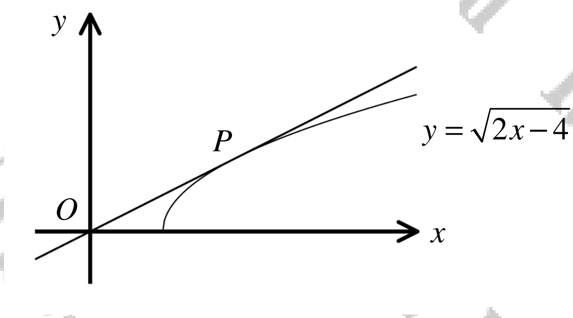
METHOD C – BY COMPLETING THE SQUARE

- $f(x) = k + 12x - 4x^2$
  $f(x) - k = 12x - 4x^2$
  $-f(x) + k = 4x^2 - 12x$
  $-f(x) + k = 4\left[x^2 - 3x\right]$
  $-f(x) + k = 4\left[\left(x - \frac{3}{2}\right)^2 - \frac{9}{4}\right]$
  $-f(x) + k = 4\left(x - \frac{3}{2}\right)^2 - 9$
  $f(x) - k = 9 - 4\left(x - \frac{3}{2}\right)^2$
  $f(x) = 9 + k - 4\left(x - \frac{3}{2}\right)^2$
- LOOKING AT A SKETCH




- WE REQUIRE $9 + k > 5$
  $k > -4$
  AS BEFORE

**Question 5** (****+)




The figure above shows the graph of the curve $C$ with equation

$$y = \sqrt{2x-4}\,,\ \ x \geq 2\,.$$

The point $P$ lies on $C$, so that the tangent to $C$ at $P$ passes through the origin $O$.

Determine the coordinates of $P$.

**You may not use calculus in this question**

$\boxed{P(4,2)}$

- A LINE THROUGH THE ORIGIN WILL BE OF THE FORM $y = mx,\ m>0$
- SOLVE SIMULTANEOUSLY WITH THE EQUATION OF THE CURVE

$$\left.\begin{array}{l} y = mx \\ y = (2x-4)^{\frac{1}{2}} \end{array}\right\} \Rightarrow mx = (2x-4)^{\frac{1}{2}}$$
$$\Rightarrow m^2x^2 = 2x-4$$
$$\Rightarrow \boxed{m^2x^2 - 2x + 4 = 0}$$

- LOOKING FOR REPEATED ROOTS (TANGENT)

$$\Rightarrow b^2 - 4ac = 0$$
$$\Rightarrow (-2)^2 - 4m^2 \times 4 = 0$$
$$\Rightarrow 4 - 16m^2 = 0$$
$$\Rightarrow 4 = 16m^2$$
$$\Rightarrow m^2 = \frac{1}{4}$$
$$\Rightarrow m = +\frac{1}{2} \quad (m>0)$$

- RETURNING TO THE QUADRATIC WITH $m = \frac{1}{2}$, AND EXPECT A PERFECT SQUARE

$$\Rightarrow \left(\frac{1}{2}\right)^2 x^2 - 2x + 4 = 0$$
$$\Rightarrow \frac{1}{4}x^2 - 2x + 4 = 0$$
$$\Rightarrow x^2 - 8x + 16 = 0$$
$$\Rightarrow (x-4)^2 = 0$$
$$\Rightarrow x = 4$$

AND $y = \frac{1}{2}x = \frac{1}{2} \times 4 = 2$

$\therefore P(4,2)$

**Question 6** **(****+)**

A curve $C$ has equation

$$y = 2x^2 + 4(p+2)x + 8p + q + 8,$$

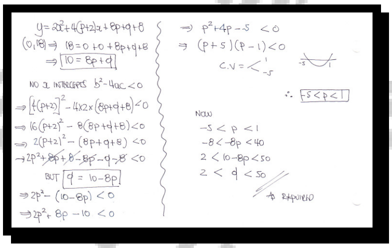
where $p$ and $q$ are constants.

The curve meets the $y$ axis at $y = 18$.

Given further that $C$ has no $x$ intercepts, show that

$$2 < q < 50.$$

$\boxed{\phantom{C}}$, $\boxed{\text{proof}}$



$y = 2x^2 + 4(p+2)x + 8p + q + 8$

$(0,18) \Rightarrow 18 = 0 + 0 + 8p + q + 8$

$\Rightarrow \boxed{10 = 8p + q}$

NO $x$ INTERCEPTS $b^2 - 4ac < 0$

$\Rightarrow [4(p+2)]^2 - 4 \times 2 \times (8p + q + 8) < 0$

$\Rightarrow 16(p+2)^2 - 8(8p + q + 8) < 0$

$\Rightarrow 2(p+2)^2 - (8p + q + 8) < 0$

$\Rightarrow 2p^2 + \cancel{8p} + \cancel{8} - \cancel{8p} - q - \cancel{8} < 0$

BUT $\boxed{q = 10 - 8p}$

$\Rightarrow 2p^2 - (10 - 8p) < 0$

$\Rightarrow 2p^2 + 8p - 10 < 0$

$\Rightarrow p^2 + 4p - 5 < 0$

$\Rightarrow (p+5)(p-1) < 0$

C.V $= \begin{cases} 1 \\ -5 \end{cases}$

$-5$ $1$

$\therefore \boxed{-5 < p < 1}$

NOW

$-5 < p < 1$

$-8 < -8p < 40$

$2 < 10 - 8p < 50$

$2 < q < 50$

AS REQUIRED

**Question 7** **(****+)**

The curve $C$ has equation

$$y = \frac{x+1}{x^2+3}, \quad x \in \mathbb{R}.$$

By considering the discriminant of a suitable quadratic equation, determine the range of the possible values of $y$.

$\boxed{\phantom{xx}}$, $\boxed{-\frac{1}{6} < y < \frac{1}{2}}$

START BY WRITING THE EQUATION AS A QUADRATIC IN $x$

$\Rightarrow y = \frac{x+1}{x^2+3}$

$\Rightarrow y(x^2+3) = x+1$

$\Rightarrow yx^2 + 3y = x+1$

$\Rightarrow yx^2 - x + (3y-1) = 0$

FOR REAL $x$, $b^2 - 4ac \geqslant 0$

$\Rightarrow (-1)^2 - 4 \times y \times (3y-1) \geqslant 0$

$\Rightarrow 1 - 4y(3y-1) \geqslant 0$

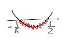
$\Rightarrow 1 - 12y^2 + 4y \geqslant 0$

$\Rightarrow -12y^2 + 4y + 1 \geqslant 0$

$\Rightarrow 12y^2 - 4y - 1 \leqslant 0$

$\Rightarrow (6y+1)(2y-1) \leqslant 0$

CRITICAL VALUES $< \begin{matrix} \frac{1}{2} \\ -\frac{1}{6} \end{matrix}$




$\therefore -\frac{1}{6} \leqslant y \leqslant \frac{1}{2}$

**Question 8** (****+)

The curve $C$ has equation

$$y = 1 - \frac{3x}{x^2 - 2x + 4}, \quad x \in \mathbb{R}.$$

Use a non differentiation method to find the coordinates of the stationary points of $C$.

$\boxed{\phantom{xxxx}}$, $\boxed{\left(-2, \frac{3}{2}\right), \left(2, -\frac{1}{2}\right)}$

- IF WE ARE LOOKING FOR STATIONARY POINTS THEN THE INTERSECTION OF THE CURVE & THE HORIZONTAL LINE $y = k$ MUST PRODUCE REPEATED ROOTS

$$y = 1 - \frac{3x}{x^2-2x+4} \quad \& \quad y = k$$

$$\Rightarrow k = 1 - \frac{3x}{x^2-2x+4}$$

$$\Rightarrow \frac{3x}{x^2-2x+4} = 1-k$$

$$\Rightarrow (1-k)x^2 - 2(1-k)x + 4(1-k) = 3x$$

$$\Rightarrow (1-k)x^2 + (2k-2)x + (4-4k) = 3x$$

$$\Rightarrow (1-k)x^2 + (2k-5)x + (4-4k) = 0$$

- LOOKING FOR REPEATED ROOTS, SO $b^2 - 4ac = 0$

$$\Rightarrow (2k-5)^2 - 4(1-k)(4-4k) = 0$$

$$\Rightarrow (2k-5)^2 - 4(k-1)(4k-4) = 0$$

$$\Rightarrow 4k^2 - 20k + 25 - 4(4k^2 - 8k + 4) = 0$$

$$\Rightarrow 4k^2 - 20k + 25 - 16k^2 + 32k - 16 = 0$$

$$\Rightarrow -12k^2 + 12k + 9 = 0$$

$$\Rightarrow 12k^2 - 12k - 9 = 0$$

$$\Rightarrow 4k^2 - 4k - 3 = 0$$

$$\Rightarrow (2k-3)(2k+1) = 0$$

$$\Rightarrow k = \begin{cases} -\frac{1}{2} \\ \frac{3}{2} \end{cases}$$

THESE ARE THE $y$ CO-ORDINATES

- FINALLY LOOKING AT THE EQUATION $(1-k)x^2 + (2k-5)x + (4-4k) = 0$

IF $k = -\frac{1}{2}$

$$\Rightarrow \frac{3}{2}x^2 - 6x + 6 = 0$$

$$\Rightarrow 3x^2 - 12x + 12 = 0$$

$$\Rightarrow x^2 - 4x + 4 = 0$$

$$\Rightarrow (x-2)^2 = 0$$

$$\Rightarrow x = 2$$

$$\therefore \left(2, -\frac{1}{2}\right)$$

IF $k = \frac{3}{2}$

$$\Rightarrow -\frac{1}{2}x^2 - 2x - 2 = 0$$

$$\Rightarrow \frac{1}{2}x^2 + 2x + 2 = 0$$

$$\Rightarrow x^2 + 4x + 4 = 0$$

$$\Rightarrow (x+2)^2$$

$$\Rightarrow x = -2$$

$$\therefore \left(-2, \frac{3}{2}\right)$$

**Question 9** (****+)

A quadratic curve has equation

$$f(x) \equiv 2x^2 + (4k+3)x + (2k-1)(k+2), \; x \in \mathbb{R},$$

where $k$ is a constant.

**a)** Evaluate the discriminant of $f(x)$.

**b)** Express $f(x)$ as the product of two linear factors.

$\boxed{\phantom{xx}}$, $\boxed{b^2 - 4ac = 25}$, $\boxed{f(x) \equiv (2x + 2k - 1)(x + k + 2)}$

$$\boxed{f(x) \equiv 2x^2 + (4k+3)x + (2k-1)(k+2), \; x \in \mathbb{R}}$$

a) CALCULATE THE DISCRIMINANT OF THE QUADRATIC

$$\Delta = b^2 - 4ac = (4k+3)^2 - 4 \times 2 \times (2k-1)(k+2)$$
$$= 16k^2 + 24k + 9 - 8(2k^2 + 3k - 2)$$
$$= \cancel{16k^2} + \cancel{24k} + 9 - \cancel{16k^2} - \cancel{24k} + 16$$
$$= 25$$

b) THE EQUATION $f(x) = 0$, HAS TWO DISTINCT SOLUTIONS WHICH CAN BE FOUND BY THE QUADRATIC FORMULA

$$x = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-(4k+3) \pm \sqrt{25}}{2 \times 2} = \frac{-4k - 3 \pm 5}{4}$$

THUS WE HAVE TWO POSSIBILITIES

- $x = \dfrac{-4k+2}{4}$
  $x = \dfrac{-2k+1}{2}$
  $2x = -2k + 1$
  $2x + 2k - 1 = 0$
- $x = \dfrac{-4k-8}{4}$
  $x = -k - 2$
  $x + k + 2 = 0$

$\therefore \; f(x) = (2x + 2k - 1)(x + k + 2)$

# 9
# ENRICHMENT QUESTIONS

**Question 1** (*****)

$$x^2+2x+1+k=0,\ x\in\mathbb{R},$$

where $k$ is a real constant.

Given that the above equation has distinct real roots, determine the nature of the roots of the following equation

$$(k+2)\left(x^2+2x+1+k\right)=2k\left(x^2+1\right).$$

$\boxed{\text{SP-M}}$ , $\boxed{\text{no real solutions}}$

$$x^2+2x+1+k=0 \quad x\in\mathbb{R}$$

- FIND THE CONDITION ON $k$, SO THE ABOVE EQUATION HAS DISTINCT REAL ROOTS

$$b^2-4ac>0$$
$$2^2-4\times1\times(1+k)>0$$
$$4-4(1+k)>0$$
$$1-(1+k)>0$$
$$\cancel{1}-\cancel{1}-k>0$$
$$\boxed{k<0}$$

- NOW WE NEED TO CONSIDER THE DISCRIMINANT OF THE FOLLOWING QUADRATIC IN $x$

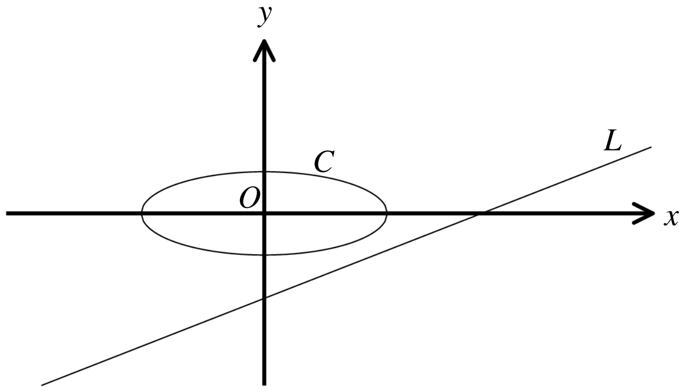
$$\Rightarrow (k+2)(x^2+2x+1+k)=2k(x^2+1)$$
$$\Rightarrow (k+2)x^2+2(k+2)x+(k+2)(k+1)-2kx^2-2k=0$$
$$\Rightarrow (2-k)x^2+2(k+2)x+(k^2+3k+2-2k)=0$$
$$\Rightarrow (2-k)x^2+2(k+2)x+(k^2+k+2)=0$$

- CALCULATING THE DISCRIMINANT

$$\Rightarrow \left[2(k+2)\right]^2-4(2-k)(k^2+k+2)$$
$$=4(k+2)^2+4(k-2)(k^2+k+2)$$
$$=4\left[\begin{array}{l} k^3+k^2+2k \\ \quad -2k^2-2k-4 \\ \quad\ \ k^2+4k+4 \end{array}\right]$$
$$=4(k^3+4k)$$
$$=4k(k^2+4)$$

- NOW IF $k<0$ THE ABOVE EXPRESSION IS NEGATIVE, WHICH IMPLIES THAT THE QUADRATIC HAS NO REAL SOLUTIONS

**Question 2** (*****)




The figure above shows the graph of the curve $C$ and the straight line $L$ with respective equations

$$\frac{x^2}{5}+\frac{y^2}{4}=1 \quad \text{and} \quad y=x-5.$$

When $C$ is translated in the positive $x$ direction, $L$ becomes a tangent to $C$, at some point $P$.

Determine the exact coordinates of $P$.

$\boxed{P\left(\frac{11}{3},-\frac{4}{3}\right) \quad \text{or} \quad P\left(\frac{19}{3},\frac{4}{3}\right)}$

$\frac{x^2}{5}+\frac{y^2}{4}=1$, $y=x-5$ TRANSLATION TO THE "RIGHT" $x \to x-k$

THUS

$\frac{(x-k)^2}{5}+\frac{y^2}{4}=1$ & $y=x-5$

$\Rightarrow \frac{(x-k)^2}{5}+\frac{(x-5)^2}{4}=1$

$\Rightarrow \frac{x^2-2kx+k^2}{5}+\frac{x^2-10x+25}{4}=1$

$\Rightarrow 4x^2-8kx+4k^2+5x^2-50x+125=20$

$\Rightarrow 9x^2-(8k+50)x+(105+4k^2)=0$

IF TANGENT $b^2-4ac=0$

$\Rightarrow 64k^2+800k+2500-4\times9\times(105+4k^2)=0$

$\Rightarrow 64k^2+800k+2500-3780-144k^2=0$

$\Rightarrow 0=80k^2-800k+1280$

$\Rightarrow 0=k^2-10k+16$

$\Rightarrow 0=(k-2)(k-8)$

$k = 2$ or $8$

HENCE
- IF $k=2$
  $9x^2-66x+121=0$
  $(3x-11)^2=0$
  $x=\frac{11}{3}$
  $y=\frac{11}{3}-5=-\frac{4}{3}$
  $\therefore \left(\frac{11}{3},-\frac{4}{3}\right)$
- IF $k=8$
  $9x^2-114x+361=0$
  $(3x-19)^2=0$
  $x=\frac{19}{3}$
  $y=\frac{19}{3}-5=\frac{4}{3}$
  $\therefore \left(\frac{19}{3},\frac{4}{3}\right)$

**Question 3** (*****)

$$ax^3 + ax^2 + ax + b = 0,$$

where $a$ and $b$ are non zero real constants.

Given that $x = b$ is a root of the above equation, determine the range of the possible values of $a$.

$\boxed{\phantom{xx}}$, $\boxed{-\frac{4}{3} \le a < 0}$

$x=b$ IS A SOLUTION OF THE EQUATION

$\Rightarrow \quad ax^3+ax^2+ax+b=0$

$\Rightarrow \quad ab^3+ab^2+ab+b=0$ ) $\div b$ (as $b \neq 0$)

$\Rightarrow \quad ab^2+ab+a+1=0$

THIS IS A QUADRATIC IN $b$

$\Rightarrow \quad ab^2+ab+(a+1)=0$

IT MUST HAVE REAL SOLUTIONS IN $b$

$\Rightarrow \quad B^2 \; 4AC \geqslant 0$

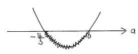
$\Rightarrow \quad a^2-4 \times a \times (a+1) \geqslant 0$

$\Rightarrow \quad a^2-4a^2-4a \geqslant 0$

$\Rightarrow \quad -3a^2 \; 4a \geqslant 0$

$\Rightarrow \quad 3a^2+4a \leqslant 0$

$\Rightarrow \quad a(3a+4) \leqslant 0$




$-\frac{4}{3} \leqslant a \leqslant 0$

**Question 4** (*****)

$$\sqrt{x+2+\sqrt{x+2+\sqrt{x+2+\sqrt{x+2+\sqrt{x+2+\dots}}}}}\ ,$$

It is given that the above nested radical converges to a limit $L$, $L \in \mathbb{R}$.

Determine the range of possible values of $x$.

$\boxed{\phantom{x}}$, $\boxed{x \geq -\frac{9}{4}}$

LET $\boxed{L = \sqrt{x+2+\sqrt{x+2+\sqrt{x+2+\sqrt{x+2+\dots}}}}}$

$\Rightarrow L = \sqrt{x+2+L}$

$\Rightarrow L^2 = x+2+L$

$\Rightarrow L^2 - L - x - 2 = 0$

● LIMIT WILL ONLY EXIST IF $b^2 - 4ac \geqslant 0$

$\Rightarrow (-1)^2 - 4 \times 1 \times (-x-2) \geqslant 0$

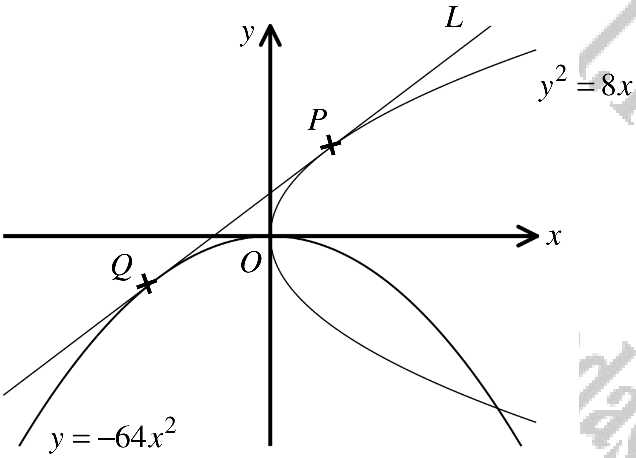
$\Rightarrow 1 + 4(x+2) \geqslant 0$

$\Rightarrow 1 + 4x + 8 \geqslant 0$

$4x \geqslant -9$

$x \geqslant -\frac{9}{4}$ //

**Question 5** (*****)




The straight line $L$ is a tangent at the point $P$ to the curve with equation

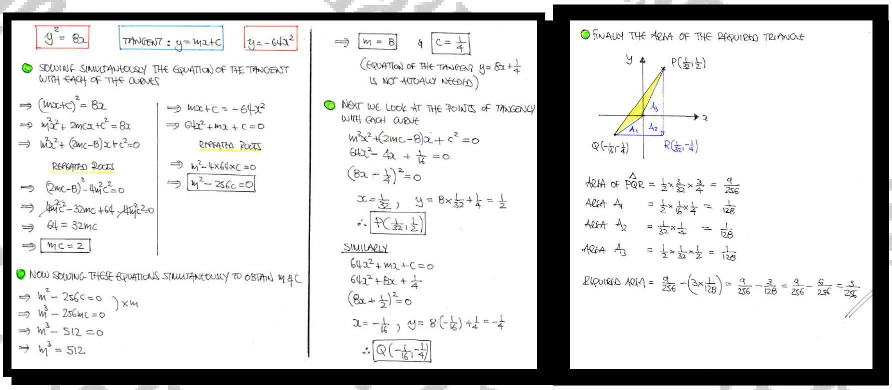
$$y^2 = 8x.$$

The straight line $L$ is also a tangent at the point $Q$ to the curve with equation

$$y = -64x^2.$$

Determine the exact area of the triangle $POQ$, where $O$ is the origin.

, $\boxed{\text{area} = \frac{3}{256}}$



**Question 6** (*****)

Find in exact form the equations of the common tangents to the curves with equations

$$(x-2)^2+(y+1)^2=4 \quad \text{and} \quad y=x^2-4x+11.$$

$\boxed{y=2\sqrt{2}(x-2)+5}$, $\boxed{y=-2\sqrt{2}(x-2)+5}$, $\boxed{y=2\sqrt{30}(x-2)-23}$, $\boxed{y=-2\sqrt{30}(x-2)-23}$

$(x-2)^2+(y+1)^2=4$ & $y=x^2-4x+11$

- FIRSTLY TRANSLATE THE "SYSTEM" BY PLACING THE CENTRE OF THE CIRCLE ONTO THE ORIGIN (FOR SIMPLICITY)

$x \mapsto x+2$
$y \mapsto y-1$

$[(x+2)-2]^2+[(y-1)+1]^2=4$
$\boxed{x^2+y^2=4}$

$(y-1)=(x+2)^2-4(x+2)+11$
$y-1=x^2+4x+4-x-8+11$
$\boxed{y=x^2+8}$

- LET THE EQUATION OF THE COMMON TANGENT BE $y=mx+c$

$\Rightarrow x^2+(mx+c)^2=4$
$\Rightarrow x^2+m^2x^2+2mcx+c^2=4$
$\Rightarrow (1+m^2)x^2+(2mc)x+(c^2-4)=0$

FOR REPEATED ROOTS

$\Rightarrow 4m^2c^2-4(1+m^2)(c^2-4)=0$
$\Rightarrow m^2c^2-(m^2+1)(c^2-4)=0$
$\Rightarrow m^2c^2-m^2c^2+4m^2-c^2+4=0$
$\Rightarrow 4m^2-c^2=-4$
$\Rightarrow \boxed{c^2-4m^2=4}$

$\Rightarrow (mx+c)=x^2+8$
$\Rightarrow mx+c=x^2+8$
$\Rightarrow 0=x^2-mx+(8-c)$

FOR REPEATED ROOTS

$\Rightarrow (-m)^2-4\times1\times(8-c)=0$
$\Rightarrow m^2-4(8-c)=0$
$\Rightarrow m^2+4c-32=0$
$\Rightarrow m^2=-4c+32$
$\Rightarrow \boxed{-4m^2=16c-128}$

- SOLVING SIMULTANEOUSLY

$\Rightarrow c^2+16c-128=4$
$\Rightarrow c^2+16c-132=0$ $\quad (m^2=32-4c)$
$\Rightarrow (c+22)(c-6)=0$

$c = 6, -22 \qquad m^2 = 8,\ 32+88=120 \qquad m = \pm2\sqrt{2},\ \pm2\sqrt{30}$

- HENCE $y=\pm2\sqrt{2}x+6$
$y=\pm2\sqrt{30}x-22$

- REVERSING THE TRANSFORMATIONS (INITIAL TRANSLATIONS)

$x \mapsto x-2$
$y \mapsto y+1$

$$\left\{\begin{array}{l} y+1=2\sqrt{2}(x-2)+6 \\ y+1=-2\sqrt{2}(x-2)+6 \\ y+1=2\sqrt{30}(x-2)-22 \\ y+1=-2\sqrt{30}(x-2)-22 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} y=2\sqrt{2}(x-2)+5 \\ y=-2\sqrt{2}(x-2)+5 \\ y=2\sqrt{30}(x-2)-23 \\ y=-2\sqrt{30}(x-2)-23 \end{array}\right.$$

**Question 7** (*****)

The following quadratic in $x$ is given below

$$x^2 + 3kx + k^2 = 7x + 3k,$$

where $k$ is a constant.

Show that the above quadratic has real solutions whose difference is at least $2$.

$\boxed{\phantom{xx}}$, $\boxed{\text{proof}}$

$$\boxed{x^2 + 3kx + k^2 = 7x + 3k}$$

- REGROUP THE TERMS AS A QUADRATIC IN $x$

$$\Rightarrow x^2 + 3kx - 7x + k^2 - 3k = 0$$
$$\Rightarrow x^2 + (3k-7)x + (k^2-3k) = 0$$

- CALCULATE THE DISCRIMINANT IN TERMS OF $k$

$$\Rightarrow \Delta = b^2 - 4ac = (3k-7)^2 - 4 \times 1 \times (k^2-3k)$$
$$= 9k^2 - 42k + 49 - 4k^2 + 12k$$
$$= 5k^2 - 30k + 49$$
$$= 5(k^2 - 6k) + 49$$
$$= 5[(k-3)^2 - 9] + 49$$
$$= 5(k-3)^2 - 45 + 49$$
$$= 5(k-3)^2 + 4 \geqslant 4 > 0$$

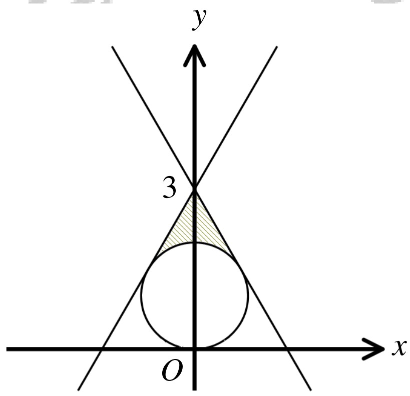
$\therefore$ ALWAYS REAL ROOTS

- FINALLY THE DIFFERENCE OF THE ROOTS IS GIVEN BY

$$x_2 - x_1 = \frac{-b+\sqrt{\Delta}}{2a} - \frac{-b-\sqrt{\Delta}}{2a} = \frac{-b+\sqrt{\Delta}+b+\sqrt{\Delta}}{2a}$$
$$= \frac{2\sqrt{\Delta}}{\cancel{2a}} = \frac{\sqrt{5(k-3)^2+4}}{1} = \sqrt{5(k-3)^2+4} \geqslant 2$$

(OCCURS WHEN $k=3$)

**Question 8** (*****)




A circle with equation

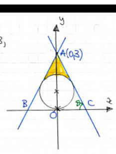
$$x^2 + (y-1)^2 = 1.$$

Two tangents to the circle are drawn so both are passing though the point $(0,3)$.

Determine in exact simplified form the value of the finite region between the circle and the two tangents, shown shaded in the figure above.

$\boxed{\phantom{xx}}$, $\boxed{\text{area} = \frac{1}{3}\left(3\sqrt{3} - \pi\right)}$

- LET THE TANGENT HAVE EQUATION $y = mx + 3$, AND SOLVE SIMULTANEOUSLY WITH THE CIRCLE




$$\left.\begin{array}{l} y = mx + 3 \\ x^2 + (y-1)^2 = 1 \end{array}\right\} \Rightarrow$$

$\Rightarrow x^2 + (mx + 3 - 1)^2 = 1$
$\Rightarrow x^2 + (mx + 2)^2 = 1$
$\Rightarrow x^2 + m^2x^2 + 4mx + 4 = 1$
$\Rightarrow (1 + m^2)x^2 + 4mx + 3 = 0$

FOR REPEATED ROOTS (TANGENTS), $b^2 - 4ac = 0$

$\Rightarrow (4m)^2 - 4(1 + m^2) \times 3 = 0$
$\Rightarrow 16m^2 - 12m^2 - 12 = 0$
$\Rightarrow 4m^2 = 12$
$\Rightarrow m^2 = 3$
$\Rightarrow m = \pm\sqrt{3}$

- HENCE THE TANGENT "ON THE RIGHT" HAS EQUATION $y = 3 - \sqrt{3}x$

$0 = 3 - \sqrt{3}x$
$\sqrt{3}x = 3$
$3x = 3\sqrt{3}$
$x = \sqrt{3}$ $\quad\therefore$ $C(\sqrt{3}, 0)$ & BY SYMMETRY $B(-\sqrt{3}, 0)$

- NEXT USING TRIGONOMETRY ON $O\hat{A}C$

$\tan\theta = \dfrac{|OA|}{|OC|} \Rightarrow \tan\theta = \dfrac{3}{\sqrt{3}} = \dfrac{3\sqrt{3}}{3}$
$\Rightarrow \tan\theta = \sqrt{3}$
$\Rightarrow \theta = \dfrac{\pi}{3}$

$\therefore$ ABC IS EQUILATERAL

- AREA OF TRIANGLE $= \frac{1}{2}|AB||AC|\sin\frac{\pi}{3}$
$= \frac{1}{2}\, 2\sqrt{3} \times 2\sqrt{3} \times \frac{\sqrt{3}}{2}$
$= 3\sqrt{3}$

- AREA OF THE CIRCLE $= \pi r^2$
$= \pi \times 1^2$
$= \pi$

- REQUIRED AREA BY SYMMETRY $= \dfrac{3\sqrt{3} - \pi}{3}$
$= \frac{1}{3}\left(3\sqrt{3} - \pi\right)$

**Question 9** (*****)

The points $P$ and $Q$ are the points of tangency of the common tangent to each of the curves with equations

$$y^2 = 4ax \quad \text{and} \quad ay = 2x^2,$$

where $a$ is a positive constant.

Show that $|PQ|$ is $7\frac{1}{2}$ times the distance of the common tangent from the origin $O$.

$\boxed{\phantom{xx}}$ , $\boxed{\text{proof}}$

$y^2 = 4ax \quad \& \quad ay = 2x^2$

- LET THE COMMON TANGENT HAVE EQUATION $y = mx + c$
- SOLVING SIMULTANEOUSLY WITH EACH CURVE & LOOK FOR REPEATED ROOTS

$\left.\begin{array}{l} y^2 = 4ax \\ y = mx + c \end{array}\right\} \Rightarrow$

$\Rightarrow (mx+c)^2 = 4ax$

$\Rightarrow m^2x^2 + 2mcx + c^2 = 4ax$

$\Rightarrow m^2x^2 + (2mc - 4a)x + c^2 = 0$

REPEATED ROOTS $B^2 - 4AC = 0$

$\Rightarrow (2mc - 4a)^2 - 4m^2c^2 = 0$

$\Rightarrow \cancel{4}(mc - 2a)^2 - \cancel{4}m^2c^2 = 0$

$\Rightarrow m^2c^2 - 4mca + 4a^2 - m^2c^2 = 0$

$\Rightarrow 4a^2 = 4mca$

$\Rightarrow a = mc$

$\left.\begin{array}{l} ay = 2x^2 \\ y = mx + c \end{array}\right\} \Rightarrow$

$\Rightarrow a(mx + c) = 2x^2$

$\Rightarrow amx + ac = 2x^2$

$\Rightarrow 0 = 2x^2 - amx - ac$

REPEATED ROOTS $B^2 - 4AC$

$\Rightarrow (-am)^2 - 4 \times 2 \times (-ac) = 0$

$\Rightarrow a^2m^2 + 8ac = 0$

$\Rightarrow am^2 + 8c = 0$

- SOLVING SIMULTANEOUSLY THE LAST TWO EQUATIONS

$\left.\begin{array}{l} a = mc \\ am^2 + 8c = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} a^2 = m^2c^2 \\ am^2 + 8c = 0 \end{array}\right\} \Rightarrow a\left(\frac{a^2}{c^2}\right) + 8c = 0$

$\Rightarrow a^3 + 8c^3 = 0$

$\Rightarrow c^3 = -\frac{a^3}{8}$

$\Rightarrow c = -\frac{a}{2}$

& $a = mc$

$a = m\left(\frac{-a}{2}\right)$

$m = -2$

- HENCE WE CAN FIND EACH OF THE POINTS OF TANGENCY

$\Rightarrow m^2x^2 + (2mc - 4a)x + c^2 = 0$

$\Rightarrow 4x^2 + \left[2\times(-2)\left(-\frac{a}{2}\right) - 4a\right]x + \frac{a^2}{4} = 0$

$\Rightarrow 4x^2 + [2a - 4a]x + \frac{a^2}{4} = 0$

$\Rightarrow 16x^2 - 8ax + a^2 = 0$

$\Rightarrow (4x - a)^2 = 0$

$\Rightarrow x = \frac{a}{4}$

& USING $y^2 = 4ax$

$\Rightarrow y^2 = 4ax$

$\Rightarrow y^2 = 4a \times \frac{a}{4}$

$\Rightarrow y^2 = a^2$

$\Rightarrow y = -a$ (SEE DIAGRAM)

$\Rightarrow 2x^2 - max - ac = 0$

$\Rightarrow 2x^2 - (-2)ax - a\left(-\frac{a}{2}\right) = 0$

$\Rightarrow 2x^2 + 2ax + \frac{a^2}{2} = 0$

$\Rightarrow 4x^2 + 4ax + a^2 = 0$

$\Rightarrow (2x + a)^2 = 0$

$\Rightarrow x = -\frac{a}{2}$

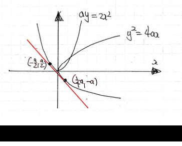
& USING $ay = 2x^2$

$\Rightarrow ay = 2\left(-\frac{a}{2}\right)^2$

$\Rightarrow ay = 2\left(\frac{a^2}{4}\right)$

$\Rightarrow ay = \frac{1}{2}a^2$

$\Rightarrow y = \frac{1}{2}a$




- NOW THE DISTANCE BETWEEN $\left(-\frac{1}{2}a, \frac{a}{2}\right)$ & $\left(\frac{a}{4}, -a\right)$ IS

$d = \sqrt{\left(-\frac{a}{2} - \frac{a}{4}\right)^2 + \left(\frac{a}{2} + a\right)^2} = \sqrt{\frac{9a^2}{16} + \frac{9a^2}{4}}$

$d = \sqrt{\frac{9a^2}{16} + \frac{36a^2}{16}} = \sqrt{\frac{45a^2}{16}} = \frac{3}{4}\sqrt{5}\,a$

- THE EQUATION OF THE COMMON TANGENT WAS FOUND AS

$y = -2x - \frac{a}{2}$

- THE PERPENDICULAR THROUGH O, WILL BE

$y = \frac{1}{2}x$

- SOLVING SIMULTANEOUSLY

$\Rightarrow \frac{1}{2}x = -2x - \frac{a}{2}$

$\Rightarrow x = -4x - a$

$\Rightarrow 5x = -a$

$\Rightarrow x = -\frac{a}{5}$ & HENCE $y = -\frac{a}{10}$

- DISTANCE OF $\left(-\frac{a}{5}, -\frac{a}{10}\right)$ FROM O IS GIVEN BY

$\sqrt{\left(-\frac{a}{5}\right)^2 + \left(-\frac{a}{10}\right)^2} = \sqrt{\frac{a^2}{25} + \frac{a^2}{100}} = \sqrt{\frac{4a^2}{100} + \frac{a^2}{100}}$

$= \sqrt{\frac{5a^2}{100}} = \frac{\sqrt{5}}{10}a$

- FINALLY: "$7\frac{1}{2}$" $\times \frac{\sqrt{5}}{10}a = \frac{15}{2} \times \frac{\sqrt{5}}{10}a = \frac{3}{4}\sqrt{5}\,a$ FOUND ABOVE