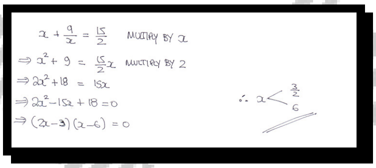
EQUATIONS EXAM QUESTIONS

Question 1 (**)

Solve the following equation

$$x + \frac{9}{x} = \frac{15}{2}, \quad x \neq 0 .$$

$\boxed{x = \frac{3}{2}, 6}$



$x + \frac{9}{x} = \frac{15}{2}$ MULTIPLY BY x

$\Rightarrow x^2 + 9 = \frac{15}{2}x$ MULTIPLY BY 2

$\Rightarrow 2x^2 + 18 = 15x$

$\Rightarrow 2x^2 - 15x + 18 = 0$

$\Rightarrow (2x-3)(x-6) = 0$

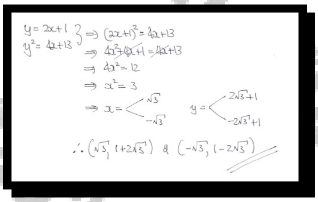
$\therefore x \begin{matrix} \nearrow \frac{3}{2} \\ \searrow 6 \end{matrix}$

Question 2 (**)

Find as exact simplified surds the coordinates of the points of intersection between the graphs of

$$y = 2x + 1 \quad \text{and} \quad y^2 = 4x + 13 .$$

$\boxed{\left(\sqrt{3}, 1 + 2\sqrt{3}\right), \left(-\sqrt{3}, 1 - 2\sqrt{3}\right)}$



$\left.\begin{matrix} y = 2x+1 \\ y^2 = 4x+13 \end{matrix}\right\} \Rightarrow (2x+1)^2 = 4x+13$

$\Rightarrow 4x^2 + \cancel{4x} + 1 = \cancel{4x} + 13$

$\Rightarrow 4x^2 = 12$

$\Rightarrow x^2 = 3$

$\Rightarrow x = \begin{matrix} \nearrow \sqrt{3} \\ \searrow -\sqrt{3} \end{matrix} \qquad y = \begin{matrix} \nearrow 2\sqrt{3}+1 \\ \searrow -2\sqrt{3}+1 \end{matrix}$

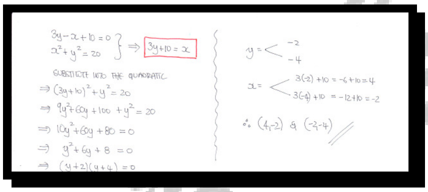
$\therefore \left(\sqrt{3}, 1+2\sqrt{3}\right)$ & $\left(-\sqrt{3}, 1-2\sqrt{3}\right)$

Question 3 **(**)**

Solve the simultaneous equations

$$3y - x + 10 = 0$$
$$x^2 + y^2 = 20$$

$\boxed{(4,-2) \ \& \ (-2,-4)}$



$$\left.\begin{array}{l} 3y - x + 10 = 0 \\ x^2 + y^2 = 20 \end{array}\right\} \Rightarrow \boxed{3y + 10 = x}$$

SUBSTITUTE INTO THE QUADRATIC

$\Rightarrow (3y+10)^2 + y^2 = 20$

$\Rightarrow 9y^2 + 60y + 100 + y^2 = 20$

$\Rightarrow 10y^2 + 60y + 80 = 0$

$\Rightarrow y^2 + 6y + 8 = 0$

$\Rightarrow (y+2)(y+4) = 0$

$y = \Big\langle \begin{array}{l} -2 \\ -4 \end{array}$

$x = \Big\langle \begin{array}{l} 3(-2) + 10 = -6 + 10 = 4 \\ 3(-4) + 10 = -12 + 10 = -2 \end{array}$

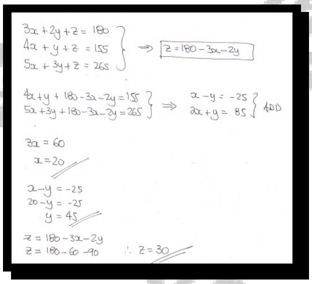
$\therefore (4,-2) \ \& \ (-2,-4)$

Question 4 **(**+)**

Solve the following system of simultaneous equations

$$\begin{array}{llll} 3x & +2y & +z & =180 \\ 4x & +y & +z & =155 \\ 5x & +3y & +z & =265 \end{array}$$

$\boxed{x = 20}$, $\boxed{y = 45}$, $\boxed{z = 30}$



$$\left.\begin{array}{l} 3x + 2y + z = 180 \\ 4x + y + z = 155 \\ 5x + 3y + z = 265 \end{array}\right\} \Rightarrow \boxed{z = 180 - 3x - 2y}$$

$$\left.\begin{array}{l} 4x + y + 180 - 3x - 2y = 155 \\ 5x + 3y + 180 - 3x - 2y = 265 \end{array}\right\} \Rightarrow \left.\begin{array}{l} x - y = -25 \\ 2x + y = 85 \end{array}\right\} \text{ADD}$$

$3x = 60$

$x = 20$

$x - y = -25$

$20 - y = -25$

$y = 45$

$z = 180 - 3x - 2y$

$z = 180 - 60 - 90$

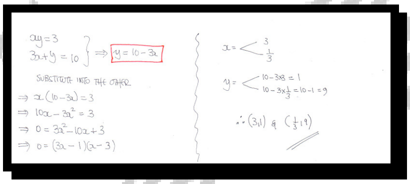
$\therefore z = 30$

Question 5 **(**+)**

Solve the following simultaneous equations

$$xy = 3$$
$$3x + y = 10$$

$\boxed{}$, $\boxed{(3,1)\ \&\ \left(\frac{1}{3},9\right)}$



$$\left.\begin{array}{l} xy=3 \\ 3x+y=10 \end{array}\right\} \Rightarrow \boxed{y=10-3x}$$

SUBSTITUTE INTO THE OTHER

$$\Rightarrow x(10-3x)=3$$
$$\Rightarrow 10x-3x^2=3$$
$$\Rightarrow 0=3x^2-10x+3$$
$$\Rightarrow 0=(3x-1)(x-3)$$

$$x = \left\langle \begin{array}{l} 3 \\ \frac{1}{3} \end{array}\right.$$

$$y = \left\langle \begin{array}{l} 10-3\times 3=1 \\ 10-3\times\frac{1}{3}=10-1=9 \end{array}\right.$$

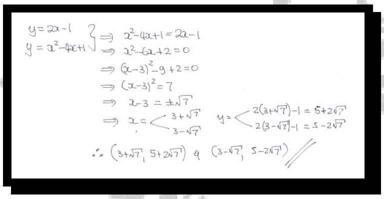
$\therefore (3,1)$ & $\left(\frac{1}{3},9\right)$ //

Question 6 **(**+)**

Find as exact simplified surds the coordinates of the points of intersection between the graphs of

$$y = 2x - 1 \text{ and } y = x^2 - 4x + 1.$$

$$\boxed{\left(3+\sqrt{7}, 5+2\sqrt{7}\right), \left(3-\sqrt{7}, 5-2\sqrt{7}\right)}$$



$$\left.\begin{array}{l} y=2x-1 \\ y=x^2-4x+1 \end{array}\right\} \Rightarrow x^2-4x+1=2x-1$$
$$\Rightarrow x^2-6x+2=0$$
$$\Rightarrow (x-3)^2-9+2=0$$
$$\Rightarrow (x-3)^2=7$$
$$\Rightarrow x-3=\pm\sqrt{7}$$
$$\Rightarrow x = \left\langle \begin{array}{l} 3+\sqrt{7} \\ 3-\sqrt{7} \end{array}\right. \quad y = \left\langle \begin{array}{l} 2(3+\sqrt{7})-1=5+2\sqrt{7} \\ 2(3-\sqrt{7})-1=5-2\sqrt{7} \end{array}\right.$$

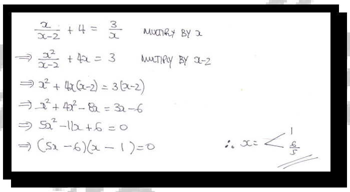
$\therefore \left(3+\sqrt{7}, 5+2\sqrt{7}\right)$ & $\left(3-\sqrt{7}, 5-2\sqrt{7}\right)$ //

Question 7 (+)**

Solve the following equation

$$\frac{x}{x-2}+4=\frac{3}{x},\ x\neq 0\,.$$

$\boxed{x=1,\frac{6}{5}}$



$\frac{x}{x-2}+4=\frac{3}{x}$ MULTIPLY BY x

$\Rightarrow \frac{x^2}{x-2}+4x=3$ MULTIPLY BY $x-2$

$\Rightarrow x^2+4x(x-2)=3(x-2)$

$\Rightarrow x^2+4x^2-8x=3x-6$

$\Rightarrow 5x^2-11x+6=0$

$\Rightarrow (5x-6)(x-1)=0$

$\therefore x = \begin{cases}1\\ \frac{6}{5}\end{cases}$

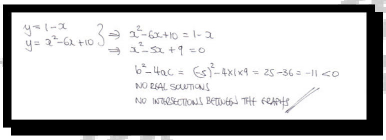
Question 8 (+)**

Use an algebraic method to show that the graphs

$$y=1-x \quad\text{and}\quad y=x^2-6x+10\,,$$

do **not** intersect.

$\boxed{\text{proof}}$



$\left.\begin{matrix}y=1-x\\ y=x^2-6x+10\end{matrix}\right\} \Rightarrow x^2-6x+10=1-x$

$\Rightarrow x^2-5x+9=0$

$b^2-4ac=(-5)^2-4\times1\times9=25-36=-11<0$

NO REAL SOLUTIONS

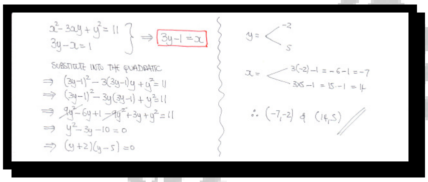
NO INTERSECTIONS BETWEEN THE GRAPHS

Question 9 **(**+)**

Solve the following simultaneous equations

$$x^2 - 3xy + y^2 = 11$$
$$3y - x = 1$$

$\boxed{}$, $\boxed{(14,5) \ \& \ (-7,-2)}$



$$\left.\begin{aligned} x^2 - 3xy + y^2 &= 11 \\ 3y - x &= 1 \end{aligned}\right\} \Rightarrow \boxed{3y-1=x}$$

SUBSTITUTE INTO THE QUADRATIC

$\Rightarrow (3y-1)^2 - 3(3y-1)y + y^2 = 11$

$\Rightarrow (3y-1)^2 - 3y(3y-1) + y^2 = 11$

$\Rightarrow \cancel{9y^2} - 6y + 1 \cancel{- 9y^2} + 3y + y^2 = 11$

$\Rightarrow y^2 - 3y - 10 = 0$

$\Rightarrow (y+2)(y-5) = 0$

$y = \begin{cases} -2 \\ 5 \end{cases}$

$x = \begin{cases} 3(-2) - 1 = -6 - 1 = -7 \\ 3 \times 5 - 1 = 15 - 1 = 14 \end{cases}$

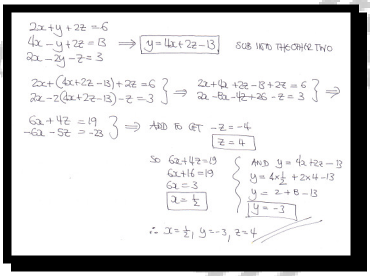
$\therefore (-7,-2)$ & $(14,5)$ //

Question 10 **(***)**

Solve the following simultaneous equations

$$2x + y + 2z = 6$$
$$4x - y + 2z = 13$$
$$2x - 2y - z = 3$$

$\boxed{x = \frac{1}{2}, \ y = -3, \ z = 4}$



$2x + y + 2z = 6$

$4x - y + 2z = 13 \Longrightarrow \boxed{y = 4x + 2z - 13}$ SUB INTO THE OTHER TWO

$2x - 2y - z = 3$

$$\left.\begin{aligned} 2x + (4x + 2z - 13) + 2z &= 6 \\ 2x - 2(4x + 2z - 13) - z &= 3 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} 2x + 4x + 2z - 13 + 2z &= 6 \\ 2x - 8x - 4z + 26 - z &= 3 \end{aligned}\right\} \Rightarrow$$

$$\left.\begin{aligned} 6x + 4z &= 19 \\ -6x - 5z &= -23 \end{aligned}\right\} \Rightarrow \text{ADD TO GET } -z = -4$$

$\boxed{z = 4}$

SO $6x + 4z = 19$

$6x + 16 = 19$

$6x = 3$

$\boxed{x = \frac{1}{2}}$

AND $y = 4x + 2z - 13$

$y = 4 \times \frac{1}{2} + 2 \times 4 - 13$

$y = 2 + 8 - 13$

$\boxed{y = -3}$

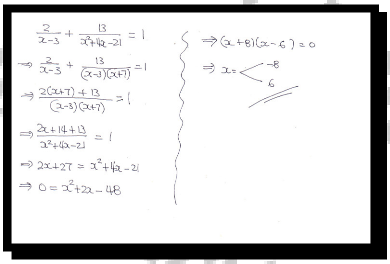
$\therefore x = \frac{1}{2}, y = -3, z = 4$ //

Question 11 **(***)**

Solve the following equation

$$\frac{2}{x-3}+\frac{13}{x^2+4x-21}=1,\ x \neq 3,\ x \neq 7 .$$

$$\boxed{x=-8,6}$$



$\frac{2}{x-3}+\frac{13}{x^2+4x-21}=1$

$\Rightarrow \frac{2}{x-3}+\frac{13}{(x-3)(x+7)}=1$

$\Rightarrow \frac{2(x+7)+13}{(x-3)(x+7)}=1$

$\Rightarrow \frac{2x+14+13}{x^2+4x-21}=1$

$\Rightarrow 2x+27=x^2+4x-21$

$\Rightarrow 0=x^2+2x-48$

$\Rightarrow (x+8)(x-6)=0$

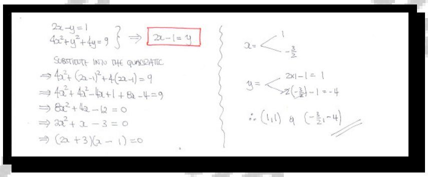
$\Rightarrow x = \begin{cases} -8 \\ 6 \end{cases}$

Question 12 **(***)**

Solve the following simultaneous equations

$$2x-y=1$$
$$4x^2+y^2+4y=9$$

$$\boxed{(1,1)\ \&\ \left(-\frac{3}{2},-4\right)}$$



$\left.\begin{array}{l} 2x-y=1 \\ 4x^2+y^2+4y=9 \end{array}\right\} \Rightarrow \boxed{2x-1=y}$

SUBSTITUTE INTO THE QUADRATIC

$\Rightarrow 4x^2+(2x-1)^2+4(2x-1)=9$

$\Rightarrow 4x^2+4x^2-4x+1+8x-4=9$

$\Rightarrow 8x^2+4x-12=0$

$\Rightarrow 2x^2+x-3=0$

$\Rightarrow (2x+3)(x-1)=0$

$x = \begin{cases} 1 \\ -\frac{3}{2} \end{cases}$

$y = \begin{cases} 2\times1-1=1 \\ 2\left(-\frac{3}{2}\right)-1=-4 \end{cases}$

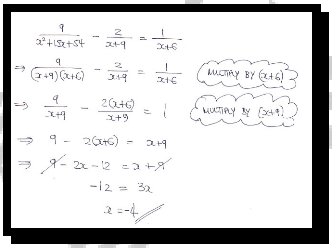
$\therefore (1,1)\ \&\ \left(-\frac{3}{2},-4\right)$

Question 13 **(***)**

Solve the following equation

$$\frac{9}{x^2+15x+54}-\frac{2}{x+9}=\frac{1}{x+6}\,,\ x\neq -6\,,\ x\neq -9\,.$$

$$\boxed{x=-4}$$



$$\frac{9}{x^2+15x+54}-\frac{2}{x+9}=\frac{1}{x+6}$$

$$\Rightarrow \frac{9}{(x+9)(x+6)}-\frac{2}{x+9}=\frac{1}{x+6}$$ MULTIPLY BY $(x+6)$

$$\Rightarrow \frac{9}{x+9}-\frac{2(x+6)}{x+9}=1$$ MULTIPLY BY $(x+9)$

$$\Rightarrow 9-2(x+6)=x+9$$

$$\Rightarrow \cancel{9}-2x-12=x+\cancel{9}$$

$$-12=3x$$

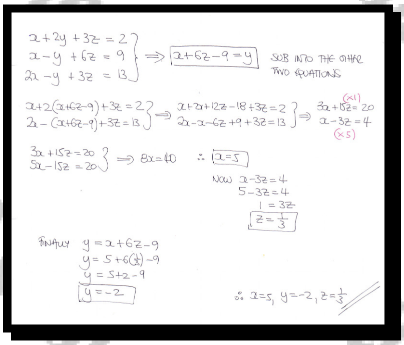
$$x=-4$$

Question 14 **(***)**

Solve the following simultaneous equations

$$\begin{aligned} x+2y+3z&=2\\ x\ \ -y+6z&=9\\ 2x\ \ -y+3z&=13 \end{aligned}$$

$$\boxed{x=5,\ y=-2,\ z=\frac{1}{3}}$$



$$\left.\begin{aligned} x+2y+3z&=2\\ x-y+6z&=9\\ 2x-y+3z&=13 \end{aligned}\right\}\Rightarrow \boxed{x+6z-9=y}$$ SUB INTO THE OTHER TWO EQUATIONS

$$\left.\begin{aligned} x+2(x+6z-9)+3z&=2\\ 2x-(x+6z-9)+3z&=13 \end{aligned}\right\}\Rightarrow \left.\begin{aligned} x+2x+12z-18+3z&=2\\ 2x-x-6z+9+3z&=13 \end{aligned}\right\}\Rightarrow \begin{aligned} &\overset{(\times 1)}{3x+15z=20}\\ &\underset{(\times 5)}{x-3z=4} \end{aligned}$$

$$\left.\begin{aligned} 3x+15z&=20\\ 5x-15z&=20 \end{aligned}\right\}\Rightarrow 8x=40 \quad \therefore \boxed{x=5}$$

NOW $x-3z=4$

$$5-3z=4$$

$$1=3z$$

$$\boxed{z=\tfrac{1}{3}}$$

FINALLY $y=x+6z-9$

$$y=5+6(\tfrac{1}{3})-9$$

$$y=5+2-9$$

$$\boxed{y=-2}$$

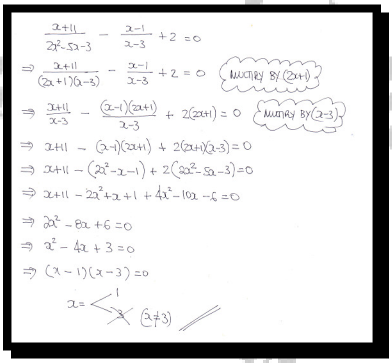
$$\therefore x=5,\ y=-2,\ z=\tfrac{1}{3}$$

Question 15 **(***)**

Solve the following equation

$$\frac{x+11}{2x^2-5x-3}-\frac{x-1}{x-3}+2=0\,,\ x\neq -\frac{1}{2}\,,\ x\neq 3\,.$$

$$\boxed{x=1}$$



$$\frac{x+11}{2x^2-5x-3}-\frac{x-1}{x-3}+2=0$$

$$\Rightarrow \frac{x+11}{(2x+1)(x-3)}-\frac{x-1}{x-3}+2=0$$ MULTIPLY BY $(2x+1)$

$$\Rightarrow \frac{x+11}{x-3}-\frac{(x-1)(2x+1)}{x-3}+2(2x+1)=0$$ MULTIPLY BY $(x-3)$

$$\Rightarrow x+11-(x-1)(2x+1)+2(2x+1)(x-3)=0$$

$$\Rightarrow x+11-(2x^2-x-1)+2(2x^2-5x-3)=0$$

$$\Rightarrow x+11-2x^2+x+1+4x^2-10x-6=0$$

$$\Rightarrow 2x^2-8x+6=0$$

$$\Rightarrow x^2-4x+3=0$$

$$\Rightarrow (x-1)(x-3)=0$$

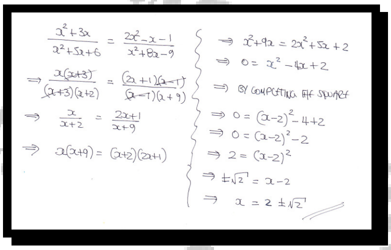
$$x=\begin{cases}1\\ \cancel{3}\end{cases}\ (x\neq 3)$$

Question 16 **(***)**

Find, in exact surd form, the roots of the equation

$$\frac{x^2+3x}{x^2+5x+6}=\frac{2x^2-x-1}{x^2+8x-9}\,,\ x\neq -3\,,\ x\neq 1\,.$$

$$\boxed{x=2\pm\sqrt{2}}$$



$$\frac{x^2+3x}{x^2+5x+6}=\frac{2x^2-x-1}{x^2+8x-9}$$

$$\Rightarrow \frac{x\cancel{(x+3)}}{\cancel{(x+3)}(x+2)}=\frac{(2x+1)\cancel{(x-1)}}{\cancel{(x-1)}(x+9)}$$

$$\Rightarrow \frac{x}{x+2}=\frac{2x+1}{x+9}$$

$$\Rightarrow x(x+9)=(x+2)(2x+1)$$

$$\Rightarrow x^2+9x=2x^2+5x+2$$

$$\Rightarrow 0=x^2-4x+2$$

$\Rightarrow$ BY COMPLETING THE SQUARE

$$\Rightarrow 0=(x-2)^2-4+2$$

$$\Rightarrow 0=(x-2)^2-2$$

$$\Rightarrow 2=(x-2)^2$$

$$\Rightarrow \pm\sqrt{2}=x-2$$

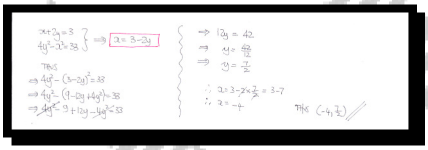
$$\Rightarrow x=2\pm\sqrt{2}$$

Question 17 **(***)**

Solve the following simultaneous equations

$$x+2y=3$$
$$4y^2-x^2=33$$

$\boxed{\text{C A}}$, $\boxed{\left(-4,\frac{7}{2}\right)}$



$\left.\begin{array}{l} x+2y=3 \\ 4y^2-x^2=33 \end{array}\right\} \Rightarrow \boxed{x=3-2y}$

THUS

$\Rightarrow 4y^2-(3-2y)^2=33$

$\Rightarrow 4y^2-(9-12y+4y^2)=33$

$\Rightarrow \cancel{4y^2}-9+12y-\cancel{4y^2}=33$

$\Rightarrow 12y=42$

$\Rightarrow y=\frac{42}{12}$

$\Rightarrow y=\frac{7}{2}$

$\therefore x=3-\cancel{2}\times\frac{7}{\cancel{2}}=3-7$

$\therefore x=-4$

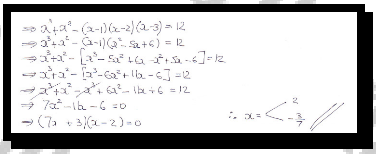
THUS $\left(-4,\frac{7}{2}\right)$ //

Question 18 **(***)**

Solve the following equation

$$x^3+x^2-(x-1)(x-2)(x-3)=12.$$

$\boxed{x=-\frac{3}{7},2}$



$\Rightarrow x^3+x^2-(x-1)(x-2)(x-3)=12$

$\Rightarrow x^3+x^2-(x-1)(x^2-5x+6)=12$

$\Rightarrow x^3+x^2-[x^3-5x^2+6x-x^2+5x-6]=12$

$\Rightarrow x^3+x^2-[x^3-6x^2+11x-6]=12$

$\Rightarrow \cancel{x^3}+x^2-\cancel{x^3}+6x^2-11x+6=12$

$\Rightarrow 7x^2-11x-6=0$

$\Rightarrow (7x+3)(x-2)=0$

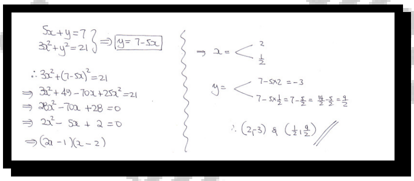
$\therefore x=\left\langle\begin{array}{l} 2 \\ -\frac{3}{7} \end{array}\right.$ //

Question 19 (***)

Solve the following simultaneous equations

$$5x + y = 7$$
$$3x^2 + y^2 = 21$$

$\boxed{}$, $\boxed{(2,-3) \ \& \ \left(\frac{1}{2},\frac{9}{2}\right)}$



$$\left.\begin{array}{l}5x+y=7\\3x^2+y^2=21\end{array}\right\} \Rightarrow \boxed{y=7-5x}$$

$\therefore 3x^2+(7-5x)^2=21$

$\Rightarrow 3x^2+49-70x+25x^2=21$

$\Rightarrow 28x^2-70x+28=0$

$\Rightarrow 2x^2-5x+2=0$

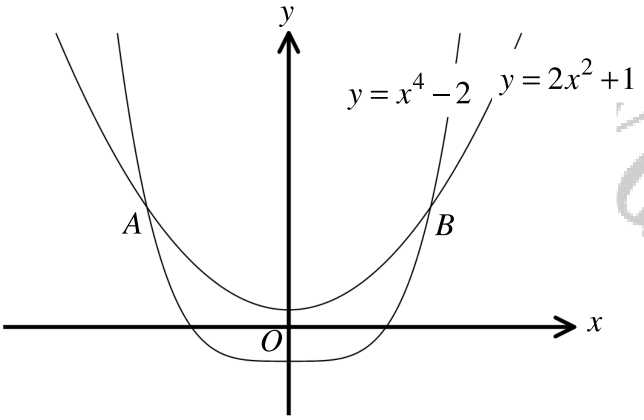
$\Rightarrow (2x-1)(x-2)$

$\Rightarrow x = \begin{cases}2\\ \frac{1}{2}\end{cases}$

$y = \begin{cases}7-5\times2=-3\\ 7-5\times\frac{1}{2}=7-\frac{5}{2}=\frac{14}{2}-\frac{5}{2}=\frac{9}{2}\end{cases}$

$\therefore (2,-3)$ & $\left(\frac{1}{2},\frac{9}{2}\right)$ //

Question 20 (***+)



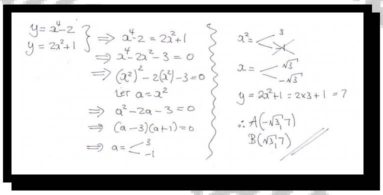
The figure above shows the graphs of the curves with equations

$$y = x^4 - 2 \quad \text{and} \quad y = 2x^2 + 1.$$

The two curves intersect at the points A and B.

Find the exact coordinates of A and B.

$$\boxed{A\left(-\sqrt{3},7\right), B\left(\sqrt{3},7\right)}$$



$\left.\begin{array}{l} y = x^4 - 2 \\ y = 2x^2 + 1 \end{array}\right\} \Rightarrow x^4 - 2 = 2x^2 + 1$

$\Rightarrow x^4 - 2x^2 - 3 = 0$

$\Rightarrow (x^2)^2 - 2(x^2) - 3 = 0$

LET $a = x^2$

$\Rightarrow a^2 - 2a - 3 = 0$

$\Rightarrow (a-3)(a+1) = 0$

$\Rightarrow a = \begin{cases} 3 \\ -1 \end{cases}$

$x^2 = \begin{cases} 3 \\ \times \end{cases}$

$x = \begin{cases} \sqrt{3} \\ -\sqrt{3} \end{cases}$

$y = 2x^2 + 1 = 2 \times 3 + 1 = 7$

$\therefore A(-\sqrt{3}, 7)$
$B(\sqrt{3}, 7)$

Question 21 (***+)

Three students are on the same tariff from a certain mobile company.

All calls cost X pence per minute, every text message costs Y pence each and every picture message costs Z pence each.

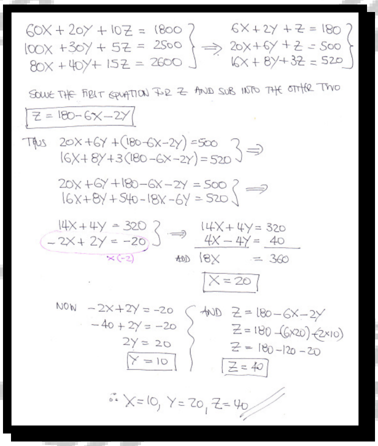
Abbie made 60 minutes of calls, sent 20 text messages and sent 10 picture messages. Her monthly bill came to £18.00 .

Beth made 100 minutes of calls, sent 30 text messages and sent 5 picture messages. Her monthly bill came to £25.00 .

Chiara made 80 minutes of calls, sent 40 text messages and sent 15 picture messages. Her monthly bill came to £26.00 .

Find the values of X , Y and Z .

$$\boxed{X = 20,\ Y = 10,\ Z = 40}$$



$$\left.\begin{array}{l} 60X + 20Y + 10Z = 1800 \\ 100X + 30Y + 5Z = 2500 \\ 80X + 40Y + 15Z = 2600 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 6X + 2Y + Z = 180 \\ 20X + 6Y + Z = 500 \\ 16X + 8Y + 3Z = 520 \end{array}\right\}$$

SOLVE THE FIRST EQUATION FOR Z AND SUB INTO THE OTHER TWO

$$\boxed{Z = 180 - 6X - 2Y}$$

THUS
$$\left.\begin{array}{l} 20X + 6Y + (180 - 6X - 2Y) = 500 \\ 16X + 8Y + 3(180 - 6X - 2Y) = 520 \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l} 20X + 6Y + 180 - 6X - 2Y = 500 \\ 16X + 8Y + 540 - 18X - 6Y = 520 \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l} 14X + 4Y = 320 \\ -2X + 2Y = -20 \end{array}\right\} \Rightarrow \begin{array}{l} 14X + 4Y = 320 \\ 4X - 4Y = 40 \\ \hline 18X = 360 \end{array}$$

$\times(-2)$ ADD

$$\boxed{X = 20}$$

NOW
$$\left.\begin{array}{l} -2X + 2Y = -20 \\ -40 + 2Y = -20 \\ 2Y = 20 \\ \boxed{Y = 10} \end{array}\right\} \text{AND} \begin{array}{l} Z = 180 - 6X - 2Y \\ Z = 180 - (6\times20) - (2\times10) \\ Z = 180 - 120 - 20 \\ \boxed{Z = 40} \end{array}$$

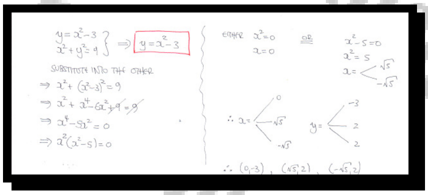
$$\therefore X = 10,\ Y = 20,\ Z = 40$$

Question 22 **(***+)**

Solve the following simultaneous equations

$$y = x^2 - 3$$
$$x^2 + y^2 = 9$$

$\boxed{(0,-3) \text{ \& } \left(\sqrt{5},2\right) \text{ \& } \left(-\sqrt{5},2\right)}$



$y = x^2 - 3$
$x^2 + y^2 = 9$ $\Rightarrow \boxed{y = x^2 - 3}$

SUBSTITUTE INTO THE OTHER

$\Rightarrow x^2 + (x^2-3)^2 = 9$

$\Rightarrow x^2 + x^4 - 6x^2 \cancel{+9} = \cancel{9}$

$\Rightarrow x^4 - 5x^2 = 0$

$\Rightarrow x^2(x^2-5) = 0$

EITHER $x^2 = 0$ OR $x^2 - 5 = 0$

$x = 0$

$x^2 = 5$

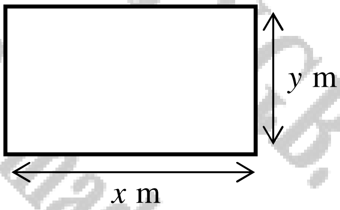
$x = \left\langle \begin{matrix} \sqrt{5} \\ -\sqrt{5} \end{matrix} \right.$

$\therefore x = \left\langle \begin{matrix} 0 \\ \sqrt{5} \\ -\sqrt{5} \end{matrix} \right.$ $\quad y = \left\langle \begin{matrix} -3 \\ 2 \\ 2 \end{matrix} \right.$

$\therefore (0,-3)\ ,\ (\sqrt{5},2)\ ,\ (-\sqrt{5},2)$

Question 23 **(***+)**

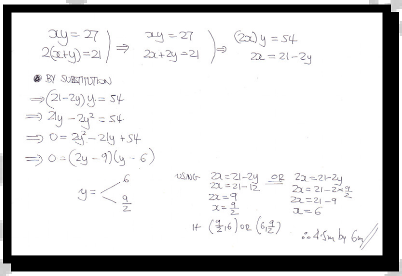
The figure below shows the plan of the floor of a room with a length of x m and a width of y m.



The floor has an area of $27\,\text{m}^2$ and a perimeter of 21 m.

Determine the measurements of the room.

$\boxed{\text{6 by 4.5}}$



$xy = 27$
$2(x+y) = 21$ $\Rightarrow$ $xy = 27$
$2x + 2y = 21$ $\Rightarrow$ $(2x)y = 54$
$2x = 21 - 2y$

- BY SUBSTITUTION

$\Rightarrow (21-2y)y = 54$

$\Rightarrow 21y - 2y^2 = 54$

$\Rightarrow 0 = 2y^2 - 21y + 54$

$\Rightarrow 0 = (2y-9)(y-6)$

$y = \left\langle \begin{matrix} 6 \\ \frac{9}{2} \end{matrix} \right.$

USING $2x = 21 - 2y$ OR $2x = 21 - 2y$

$2x = 21 - 12$ | $2x = 21 - 2 \times \frac{9}{2}$

$2x = 9$ | $2x = 21 - 9$

$x = \frac{9}{2}$ | $x = 6$

If $\left(\frac{9}{2}, 6\right)$ OR $\left(6, \frac{9}{2}\right)$

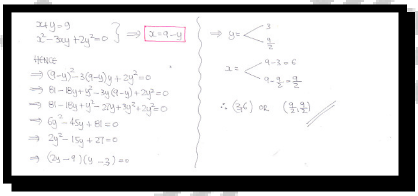
$\therefore$ 4.5m by 6m

Question 24 **(***+)**

Solve the following simultaneous equations

$$x + y = 9$$
$$x^2 - 3xy + 2y^2 = 0$$

$\boxed{}$, $\boxed{(6,3),\left(\frac{9}{2},\frac{9}{2}\right)}$



$\left.\begin{array}{l} x+y=9 \\ x^2-3xy+2y^2=0 \end{array}\right\} \Rightarrow \boxed{x=9-y}$

HENCE

$\Rightarrow (9-y)^2-3(9-y)y+2y^2=0$

$\Rightarrow 81-18y+y^2-3y(9-y)+2y^2=0$

$\Rightarrow 81-18y+y^2-27y+3y^2+2y^2=0$

$\Rightarrow 6y^2-45y+81=0$

$\Rightarrow 2y^2-15y+27=0$

$\Rightarrow (2y-9)(y-3)=0$

$\Rightarrow y = \begin{cases} 3 \\ \frac{9}{2} \end{cases}$

$x = \begin{cases} 9-3=6 \\ 9-\frac{9}{2}=\frac{9}{2} \end{cases}$

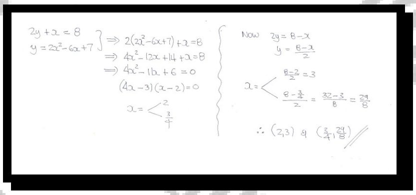
$\therefore (3,6)$ OR $\left(\frac{9}{2},\frac{9}{2}\right)$ //

Question 25 **(***+)**

Solve the following simultaneous equations

$$2y + x = 8$$
$$y = 2x^2 - 6x + 7$$

$\boxed{}$, $\boxed{(2,3)\ \&\ \left(\frac{3}{4},\frac{29}{8}\right)}$



$\left.\begin{array}{l} 2y+x=8 \\ y=2x^2-6x+7 \end{array}\right\} \Rightarrow 2(2x^2-6x+7)+x=8$

$\Rightarrow 4x^2-12x+14+x=8$

$\Rightarrow 4x^2-11x+6=0$

$(4x-3)(x-2)=0$

$x = \begin{cases} 2 \\ \frac{3}{4} \end{cases}$

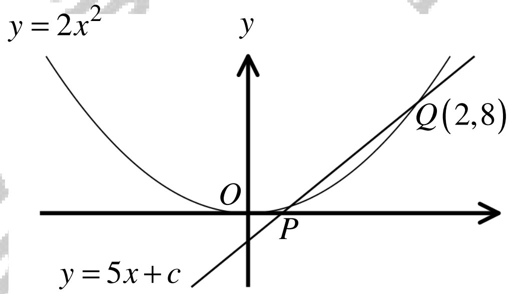
Now $2y=8-x$

$y=\frac{8-x}{2}$

$x = \begin{cases} \frac{8-2}{2}=3 \\ \frac{8-\frac{3}{4}}{2}=\frac{32-3}{8}=\frac{29}{8} \end{cases}$

$\therefore (2,3)$ & $\left(\frac{3}{4},\frac{29}{8}\right)$ //

Question 26 (***+)

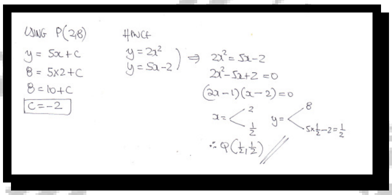


The figure above shows the graph of the curve with equation $y = 2x^2$ and the line with equation $y = 5x + c$, where c is a constant.

The line meets the curve at the point P and at the point $Q(2,8)$.

Determine the coordinates of P.

$$\boxed{P\left(\frac{1}{2},\frac{1}{2}\right)}$$



USING $P(2,8)$

$y = 5x + c$

$8 = 5 \times 2 + c$

$8 = 10 + c$

$\boxed{c = -2}$

HENCE

$\left.\begin{matrix} y = 2x^2 \\ y = 5x - 2 \end{matrix}\right) \Rightarrow 2x^2 = 5x - 2$

$2x^2 - 5x + 2 = 0$

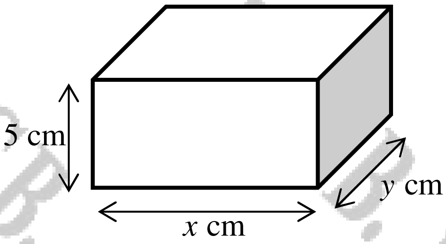
$(2x-1)(x-2) = 0$

$x = \begin{cases} 2 \\ \frac{1}{2} \end{cases} \quad y = \begin{cases} 8 \\ 5 \times \frac{1}{2} - 2 = \frac{1}{2} \end{cases}$

$\therefore P\left(\frac{1}{2},\frac{1}{2}\right)$ //

Question 27 **(***+)**

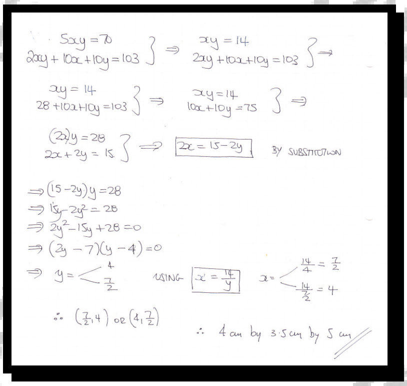
The figure below shows a cuboid of length x cm, width y cm and height 5 cm.



The cuboid has a volume of 70 cm^3 and a surface area of 103 cm^2.

Determine the measurements of the cuboid.

$\boxed{4 \text{ by } 3.5 \text{ by } 5}$



$$\left.\begin{array}{l} 5xy = 70 \\ 2xy + 10x + 10y = 103 \end{array}\right\} \Rightarrow \left.\begin{array}{l} xy = 14 \\ 2xy + 10x + 10y = 103 \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l} xy = 14 \\ 28 + 10x + 10y = 103 \end{array}\right\} \Rightarrow \left.\begin{array}{l} xy = 14 \\ 10x + 10y = 75 \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l} (2x)y = 28 \\ 2x + 2y = 15 \end{array}\right\} \Longrightarrow \boxed{2x = 15 - 2y}$$ BY SUBSTITUTION

$$\Rightarrow (15 - 2y)y = 28$$
$$\Rightarrow 15y - 2y^2 = 28$$
$$\Rightarrow 2y^2 - 15y + 28 = 0$$
$$\Rightarrow (2y - 7)(y - 4) = 0$$
$$\Rightarrow y = \begin{cases} 4 \\ \frac{7}{2} \end{cases}$$ USING $\boxed{x = \frac{14}{y}}$ $x = \begin{cases} \frac{14}{4} = \frac{7}{2} \\ \frac{14}{\frac{7}{2}} = 4 \end{cases}$

$\therefore \left(\frac{7}{2}, 4\right)$ or $\left(4, \frac{7}{2}\right)$

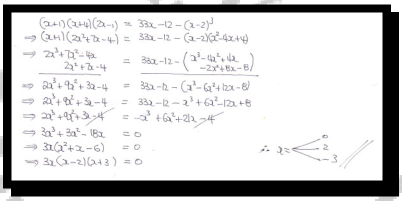
$\therefore$ 4 cm by 3.5 cm by 5 cm //

Question 28 **(***+)**

Solve the following equation

$$(x+1)(x+4)(2x-1) = 33x - 12 - (x-2)^3 .$$

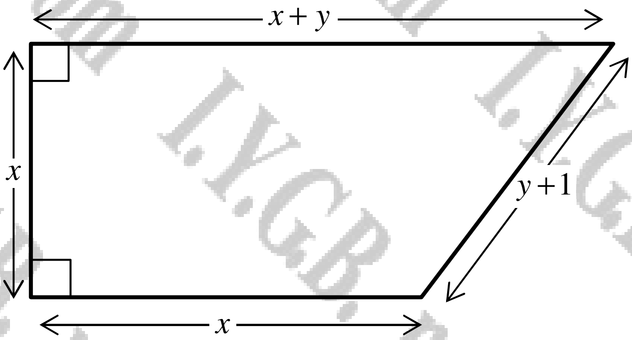
$\boxed{x = -3, 0, 2}$



$$(x+1)(x+4)(2x-1) = 33x - 12 - (x-2)^3$$
$$\Rightarrow (x+1)(2x^2+7x-4) = 33x - 12 - (x-2)(x^2-4x+4)$$
$$\Rightarrow \begin{array}{r} 2x^3 + 7x^2 - 4x \\ 2x^2 + 7x - 4 \end{array} = 33x - 12 - \left(\begin{array}{r} x^3 - 4x^2 + 4x \\ -2x^2 + 8x - 8 \end{array}\right)$$
$$\Rightarrow 2x^3 + 9x^2 + 3x - 4 = 33x - 12 - (x^3 - 6x^2 + 12x - 8)$$
$$\Rightarrow 2x^3 + 9x^2 + 3x - 4 = 33x - 12 - x^3 + 6x^2 - 12x + 8$$
$$\Rightarrow 2x^3 + 9x^2 + 3x \cancel{-4} = -x^3 + 6x^2 + 21x \cancel{-4}$$
$$\Rightarrow 3x^3 + 3x^2 - 18x = 0$$
$$\Rightarrow 3x(x^2 + x - 6) = 0$$
$$\Rightarrow 3x(x-2)(x+3) = 0$$

$$\therefore x = \begin{cases} 0 \\ 2 \\ -3 \end{cases}$$ //

Question **(***+)**

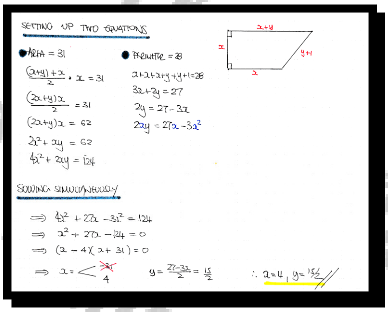


The figure above shows a right angled trapezium whose measurements are given in terms of x and y.

The trapezium has and a perimeter of 28 and an area of 31.

Determine the value x and the value of y.

$\boxed{x = 4, \ \ y = 7.5}$



SETTING UP TWO EQUATIONS

- AREA = 31

$\frac{(x+y)+x}{2} \cdot x = 31$

$\frac{(2x+y)x}{2} = 31$

$(2x+y)x = 62$

$2x^2 + xy = 62$

$4x^2 + 2xy = 124$

- PERIMETER = 28

$x+x+x+y+y+1=28$

$3x+2y=27$

$2y=27-3x$

$2xy=27x-3x^2$

SOLVING SIMULTANEOUSLY

$\Rightarrow 4x^2 + 27x - 3x^2 = 124$

$\Rightarrow x^2 + 27x - 124 = 0$

$\Rightarrow (x-4)(x+31) = 0$

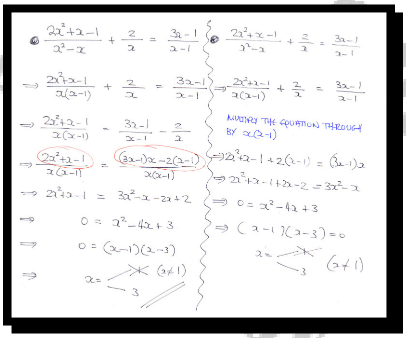
$\Rightarrow x = \begin{cases} \cancel{-31} \\ 4 \end{cases}$ $\qquad y = \frac{27-3x}{2} = \frac{15}{2}$ $\qquad \therefore x=4, \ y=15/2$

Question 29 **(***+)**

Find the solution of the following equation

$$\frac{2x^2+x-1}{x^2-x}+\frac{2}{x}=\frac{3x-1}{x-1}.$$

$\boxed{x=3, x\neq 1}$



$$\frac{2x^2+x-1}{x^2-x}+\frac{2}{x}=\frac{3x-1}{x-1}$$

$$\Rightarrow \frac{2x^2+x-1}{x(x-1)}+\frac{2}{x}=\frac{3x-1}{x-1}$$

$$\Rightarrow \frac{2x^2+x-1}{x(x-1)}=\frac{3x-1}{x-1}-\frac{2}{x}$$

$$\Rightarrow \frac{2x^2+x-1}{x(x-1)}=\frac{(3x-1)x-2(x-1)}{x(x-1)}$$

$$\Rightarrow 2x^2+x-1=3x^2-x-2x+2$$

$$\Rightarrow 0=x^2-4x+3$$

$$\Rightarrow 0=(x-1)(x-3)$$

$$\Rightarrow x=\begin{cases}\times & (x\neq 1)\\ 3\end{cases}$$

$$\frac{2x^2+x-1}{x^2-x}+\frac{2}{x}=\frac{3x-1}{x-1}$$

$$\Rightarrow \frac{2x^2+x-1}{x(x-1)}+\frac{2}{x}=\frac{3x-1}{x-1}$$

MULTIPLY THE EQUATION THROUGH BY $x(x-1)$

$$\Rightarrow 2x^2+x-1+2(x-1)=(3x-1)x$$

$$\Rightarrow 2x^2+x-1+2x-2=3x^2-x$$

$$\Rightarrow 0=x^2-4x+3$$

$$\Rightarrow (x-1)(x-3)=0$$

$$x=\begin{cases}\times\\ 3\end{cases}\quad(x\neq 1)$$

Question 30 **(***+)**

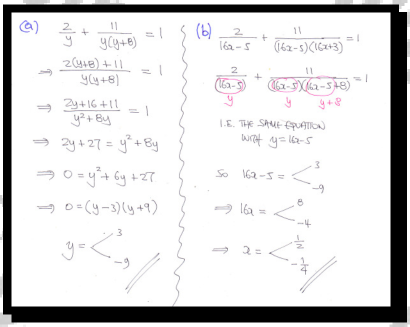
a) Find the solutions of the following equation

$$\frac{2}{y}+\frac{11}{y(y+8)}=1,\ y\neq -8,\ y\neq 0.$$

b) Hence, or otherwise, solve the equation

$$\frac{2}{16x-5}+\frac{11}{(16x-5)(16x+3)}=1,\ x\neq -\frac{3}{16},\ x\neq \frac{5}{16}.$$

$\boxed{x=-9,3}$, $\boxed{x=-\frac{1}{4},\frac{1}{2}}$



(a) $$\frac{2}{y}+\frac{11}{y(y+8)}=1$$

$$\Rightarrow \frac{2(y+8)+11}{y(y+8)}=1$$

$$\Rightarrow \frac{2y+16+11}{y^2+8y}=1$$

$$\Rightarrow 2y+27=y^2+8y$$

$$\Rightarrow 0=y^2+6y+27$$

$$\Rightarrow 0=(y-3)(y+9)$$

$$y=\begin{cases}3\\ -9\end{cases}$$

(b) $$\frac{2}{16x-5}+\frac{11}{(16x-5)(16x+3)}=1$$

$$\frac{2}{\underbrace{(16x-5)}_{y}}+\frac{11}{\underbrace{(16x-5)}_{y}\underbrace{(16x-5+8)}_{y+8}}=1$$

I.E. THE SAME EQUATION WITH $y=16x-5$

$$\text{So}\quad 16x-5=\begin{cases}3\\ -9\end{cases}$$

$$\Rightarrow 16x=\begin{cases}8\\ -4\end{cases}$$

$$\Rightarrow x=\begin{cases}\frac{1}{2}\\ -\frac{1}{4}\end{cases}$$

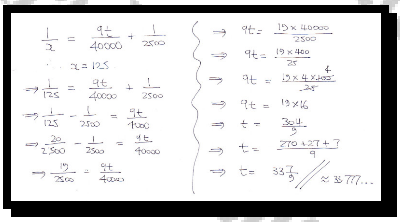
Question 31 (*+)**

A relationship between two variables is given below

$$\frac{1}{x}=\frac{9t}{40000}+\frac{1}{2500}.$$

Find the value of t when $x=125$.

$$\boxed{t=33\frac{7}{9}\approx 33.8}$$



$\frac{1}{x}=\frac{9t}{40000}+\frac{1}{2500}$

$x=125$

$\Rightarrow \frac{1}{125}=\frac{9t}{40000}+\frac{1}{2500}$

$\Rightarrow \frac{1}{125}-\frac{1}{2500}=\frac{9t}{4000}$

$\Rightarrow \frac{20}{2500}-\frac{1}{2500}=\frac{9t}{40000}$

$\Rightarrow \frac{19}{2500}=\frac{9t}{40000}$

$\Rightarrow 9t=\frac{19\times 40000}{2500}$

$\Rightarrow 9t=\frac{19\times 400}{25}$

$\Rightarrow 9t=\frac{19\times 4\times \cancel{100}^{4}}{\cancel{25}}$

$\Rightarrow 9t=19\times 16$

$\Rightarrow t=\frac{304}{9}$

$\Rightarrow t=\frac{270+27+7}{9}$

$\Rightarrow t=33\frac{7}{9}$ // $\approx 33.777\ldots$

Question 32 (*+)**

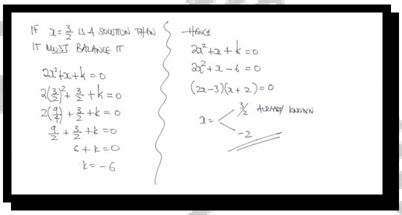
The quadratic equation given below

$$2x^2+x+k=0,$$

where k is a constant, has solutions $x=\frac{3}{2}$ and $x=x_0$.

Find the value of x_0.

$$\boxed{}, \quad \boxed{x_0=-2}$$



IF $x=\frac{3}{2}$ IS A SOLUTION THEN IT MUST BALANCE IT

$2x^2+x+k=0$

$2\left(\frac{3}{2}\right)^2+\frac{3}{2}+k=0$

$2\left(\frac{9}{4}\right)+\frac{3}{2}+k=0$

$\frac{9}{2}+\frac{3}{2}+k=0$

$6+k=0$

$k=-6$

HENCE

$2x^2+x+k=0$

$2x^2+x-6=0$

$(2x-3)(x+2)=0$

$x=\left\langle\begin{array}{l}3/2 \text{ ALREADY KNOWN}\\ -2\end{array}\right.$ //

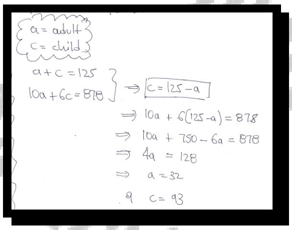
Question 33 **(***+)**

In a cinema adult tickets cost £10 each while child tickets cost £6.

For a certain film there were 125 people in the cinema, having paid in total £878.

Find how many adults and how many children were watching this film?

$\boxed{\text{93 children and 32 adults}}$



a = adult
c = child

$$\left.\begin{array}{l} a+c=125 \\ 10a+6c=878 \end{array}\right\} \Rightarrow \boxed{c=125-a}$$

$$\Rightarrow 10a+6(125-a)=878$$

$$\Rightarrow 10a+750-6a=878$$

$$\Rightarrow 4a=128$$

$$\Rightarrow a=32$$

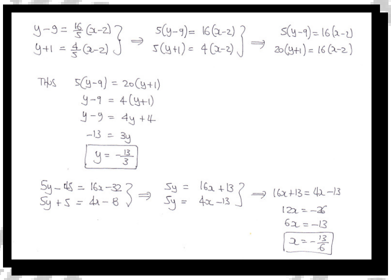
$$\therefore c=93$$

Question 34 **(***+)** **non calculator**

Find the exact solution of the following simultaneous equations

$$y-9=\frac{16}{5}(x-2)$$
$$y+1=\frac{4}{5}(x-2)$$

$$\boxed{\left(-\frac{13}{6},-\frac{13}{3}\right)}$$



$$\left.\begin{array}{l} y-9=\frac{16}{5}(x-2) \\ y+1=\frac{4}{5}(x-2) \end{array}\right\} \Rightarrow \left.\begin{array}{l} 5(y-9)=16(x-2) \\ 5(y+1)=4(x-2) \end{array}\right\} \Rightarrow \begin{array}{l} 5(y-9)=16(x-2) \\ 20(y+1)=16(x-2) \end{array}$$

THUS $5(y-9)=20(y+1)$

$$y-9=4(y+1)$$
$$y-9=4y+4$$
$$-13=3y$$
$$\boxed{y=-\frac{13}{3}}$$

$$\left.\begin{array}{l} 5y-45=16x-32 \\ 5y+5=4x-8 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 5y=16x+13 \\ 5y=4x-13 \end{array}\right\} \Rightarrow 16x+13=4x-13$$

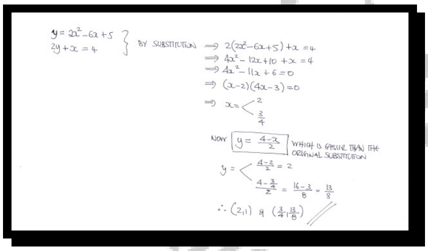
$$12x=-26$$
$$6x=-13$$
$$\boxed{x=-\frac{13}{6}}$$

Question 35 (*+) non calculator**

Find the coordinates of the points of intersection between the graphs of

$$y = 2x^2 - 6x + 5 \quad \text{and} \quad 2y + x = 4 \, .$$

$$\boxed{(2,1), \quad \left(\frac{3}{4}, \frac{13}{8}\right)}$$



$$\left.\begin{array}{l} y = 2x^2 - 6x + 5 \\ 2y + x = 4 \end{array}\right\} \text{ BY SUBSTITUTION } \Rightarrow 2(2x^2 - 6x + 5) + x = 4$$

$$\Rightarrow 4x^2 - 12x + 10 + x = 4$$

$$\Rightarrow 4x^2 - 11x + 6 = 0$$

$$\Rightarrow (x-2)(4x-3) = 0$$

$$\Rightarrow x = \begin{cases} 2 \\ \frac{3}{4} \end{cases}$$

NOW $\boxed{y = \frac{4-x}{2}}$ WHICH IS EASIER THAN THE ORIGINAL SUBSTITUTION

$$y = \begin{cases} \frac{4-2}{2} = 2 \\ \frac{4-\frac{3}{4}}{2} = \frac{16-3}{8} = \frac{13}{8} \end{cases}$$

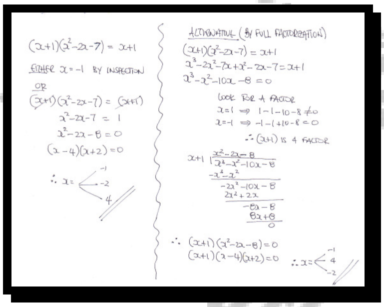
$$\therefore (2,1) \;\&\; \left(\tfrac{3}{4}, \tfrac{13}{8}\right) \; //$$

Question 36 (*+)**

Solve the following equation

$$(x+1)\left(x^2 - 2x - 7\right) = x + 1$$

$$\boxed{x = -2, -1, 4}$$



$$(x+1)(x^2-2x-7) = x+1$$

EITHER $x = -1$ BY INSPECTION

OR

$$\cancel{(x+1)}(x^2-2x-7) = \cancel{(x+1)}$$

$$x^2-2x-7 = 1$$

$$x^2-2x-8 = 0$$

$$(x-4)(x+2) = 0$$

$$\therefore x = \begin{cases} -1 \\ -2 \\ 4 \end{cases} //$$

ALTERNATIVE (BY FULL FACTORIZATION)

$$(x+1)(x^2-2x-7) = x+1$$

$$x^3-2x^2-7x+x^2-2x-7 = x+1$$

$$x^3-x^2-10x-8 = 0$$

LOOK FOR A FACTOR

$x = 1 \Rightarrow 1-1-10-8 \neq 0$

$x = -1 \Rightarrow -1-1+10-8 = 0$

$\therefore (x+1)$ IS A FACTOR

$$\begin{array}{r} x^2-2x-8 \\ x+1 \;\big)\; x^3-x^2-10x-8 \\ -x^3-x^2 \\ \hline -2x^2-10x-8 \\ 2x^2+2x \\ \hline -8x-8 \\ 8x+8 \\ \hline 0 \end{array}$$

$$\therefore (x+1)(x^2-2x-8) = 0$$

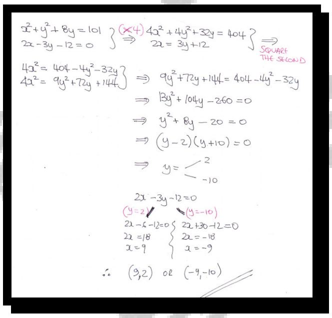
$$(x+1)(x-4)(x+2) = 0 \qquad \therefore x = \begin{cases} -1 \\ 4 \\ -2 \end{cases} //$$

Question 37 **(***+)**

Find the coordinates of the points of intersection between

$$x^2 + y^2 + 8y = 101 \quad \text{and} \quad 2x - 3y - 12 = 0 .$$

$$\boxed{\quad}, \boxed{(9,2),(-9,10)}$$



$$\left.\begin{array}{l} x^2+y^2+8y=101 \\ 2x-3y-12=0 \end{array}\right\} \overset{(\times 4)}{\Rightarrow} \left.\begin{array}{l} 4x^2+4y^2+32y=404 \\ 2x=3y+12 \end{array}\right\} \underset{\text{SQUARE THE SECOND}}{\Longrightarrow}$$

$$\left.\begin{array}{l} 4x^2=404-4y^2-32y \\ 4x^2=9y^2+72y+144 \end{array}\right\} \Rightarrow 9y^2+72y+144=404-4y^2-32y$$

$$\Rightarrow 13y^2+104y-260=0$$

$$\Rightarrow y^2+8y-20=0$$

$$\Rightarrow (y-2)(y+10)=0$$

$$\Rightarrow y = \begin{cases} 2 \\ -10 \end{cases}$$

$$2x-3y-12=0$$

$$(y=2) \qquad (y=-10)$$

$$\left.\begin{array}{l} 2x-6-12=0 \\ 2x=18 \\ x=9 \end{array}\right\} \begin{array}{l} 2x+30-12=0 \\ 2x=-18 \\ x=-9 \end{array}$$

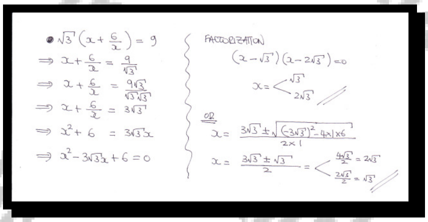
$$\therefore \ (9,2) \text{ or } (-9,-10)$$

Question 38 **(***+)**

Find in exact surd form the roots of the following equation

$$\sqrt{3}\left(x + \frac{6}{x}\right) = 9, \; x \neq 0 .$$

$$\boxed{x = \sqrt{3}, \; x = 2\sqrt{3}}$$



$$\bullet \sqrt{3}\left(x+\frac{6}{x}\right)=9$$

$$\Rightarrow x+\frac{6}{x}=\frac{9}{\sqrt{3}}$$

$$\Rightarrow x+\frac{6}{x}=\frac{9\sqrt{3}}{\sqrt{3}\sqrt{3}}$$

$$\Rightarrow x+\frac{6}{x}=3\sqrt{3}$$

$$\Rightarrow x^2+6=3\sqrt{3}x$$

$$\Rightarrow x^2-3\sqrt{3}x+6=0$$

FACTORIZATION

$$(x-\sqrt{3})(x-2\sqrt{3})=0$$

$$x = \begin{cases} \sqrt{3} \\ 2\sqrt{3} \end{cases}$$

OR

$$x=\frac{3\sqrt{3}\pm\sqrt{(-3\sqrt{3})^2-4\times1\times6}}{2\times1}$$

$$x=\frac{3\sqrt{3}\pm\sqrt{3}}{2}= \begin{cases} \frac{4\sqrt{3}}{2}=2\sqrt{3} \\ \frac{2\sqrt{3}}{2}=\sqrt{3} \end{cases}$$

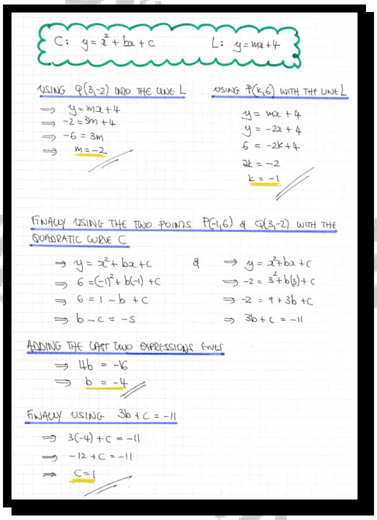
Question 39 (****)

$$C: \quad y = x^2 + bx + c$$
$$L: \quad y = mx + 4$$

The quadratic curve C intersects the straight line L at the points with coordinates $(k, 6)$ and $(3, -2)$, where k, m, b and c are constants.

Find the value of k, m, b and c.

$\boxed{}$, $\boxed{m = -2, \; k = -1, \; b = -4, \; c = 1}$



$C: \; y = x^2 + bx + c \qquad L: \; y = mx + 4$

USING $Q(3,-2)$ INTO THE LINE L

$\Rightarrow y = mx + 4$
$\Rightarrow -2 = 3m + 4$
$\Rightarrow -6 = 3m$
$\Rightarrow m = -2$

USING $P(k,6)$ WITH THE LINE L

$y = mx + 4$
$y = -2x + 4$
$6 = -2k + 4$
$2k = -2$
$k = -1$

FINALLY USING THE TWO POINTS $P(-1,6)$ & $Q(3,-2)$ WITH THE QUADRATIC CURVE C

$\Rightarrow y = x^2 + bx + c$ & $\Rightarrow y = x^2 + bx + c$
$\Rightarrow 6 = (-1)^2 + b(-1) + c$ $\qquad \Rightarrow -2 = 3^2 + b(3) + c$
$\Rightarrow 6 = 1 - b + c$ $\qquad \Rightarrow -2 = 9 + 3b + c$
$\Rightarrow b - c = -5$ $\qquad \Rightarrow 3b + c = -11$

ADDING THE LAST TWO EXPRESSIONS GIVES

$\Rightarrow 4b = -16$
$\Rightarrow b = -4$

FINALLY USING $3b + c = -11$

$\Rightarrow 3(-4) + c = -11$
$\Rightarrow -12 + c = -11$
$\Rightarrow c = 1$

Question 40 **(****)**

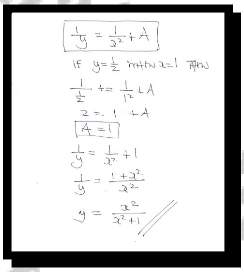
A relationship between three variables is given below

$$\frac{1}{y} = \frac{1}{x^2} + A\,.$$

Given further that when $x = 1$, $y = \frac{1}{2}$, show clearly that

$$y = \frac{x^2}{1 + x^2}\,.$$

$\boxed{\text{proof}}$



$\boxed{\frac{1}{y} = \frac{1}{x^2} + A}$

IF $y = \frac{1}{2}$ WHEN $x = 1$ THEN

$\frac{1}{\frac{1}{2}} += \frac{1}{1^2} + A$

$2 = 1 + A$

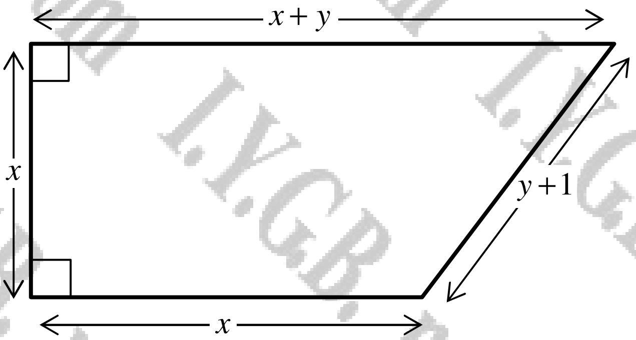
$\boxed{A = 1}$

$\frac{1}{y} = \frac{1}{x^2} + 1$

$\frac{1}{y} = \frac{1 + x^2}{x^2}$

$y = \frac{x^2}{x^2 + 1}$ //

Question 41 (****)

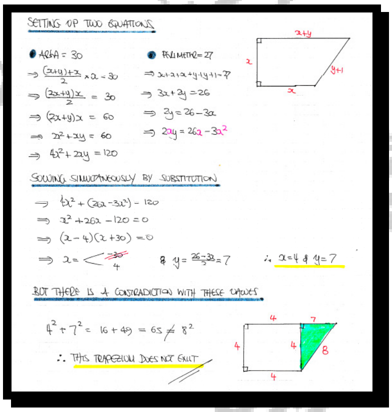


The figure above shows a right angled trapezium whose measurements are given in terms of x and y.

The trapezium has and a perimeter of 27 and an area of 30.

Determine the value x and the value of y, and hence show that the above trapezium does **not** exist.

$\boxed{}$, $\boxed{x=4,\ \ y=7}$



SETTING UP TWO EQUATIONS

- AREA = 30

$\Rightarrow \frac{(x+y)+x}{2} \times x = 30$

$\Rightarrow \frac{(2x+y)x}{2} = 30$

$\Rightarrow (2x+y)x = 60$

$\Rightarrow 2x^2+xy = 60$

$\Rightarrow 4x^2+2xy = 120$

- PERIMETER = 27

$\Rightarrow x+x+x+y+y+1=27$

$\Rightarrow 3x+2y=26$

$\Rightarrow 2y=26-3x$

$\Rightarrow 2xy=26x-3x^2$

SOLVING SIMULTANEOUSLY BY SUBSTITUTION

$\Rightarrow 4x^2+(26x-3x^2)=120$

$\Rightarrow x^2+26x-120=0$

$\Rightarrow (x-4)(x+30)=0$

$\Rightarrow x = \begin{cases} \cancel{-30} \\ 4 \end{cases}$ & $y=\frac{26-3x}{2}=7$ $\therefore x=4$ & $y=7$

BUT THERE IS A CONTRADICTION WITH THESE VALUES

$4^2+7^2=16+49=65 \neq 8^2$

$\therefore$ THIS TRAPEZIUM DOES NOT EXIST

Question 42 (****)

$$f(x) = x^2(x-4), \ x \in \mathbb{R}.$$

$$g(x) = x(10-x), \ x \in \mathbb{R}.$$

a) Determine the coordinates of the points of intersection between the graphs of $f(x)$ and $g(x)$.

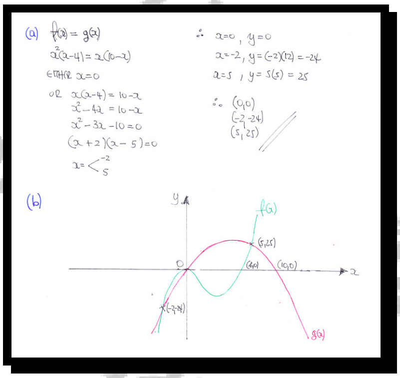
b) Sketch the graph of $f(x)$ and the graph of $g(x)$ in the same diagram.

The sketch must include …

… the coordinates of any points where the graph of $f(x)$ and the graph of $g(x)$ meet the coordinate axes.

… the coordinates of the points of intersection between the graph of $f(x)$ and the graph of $g(x)$.

$$\boxed{(0,0),(-2,-24),(5,25)}$$



(a) $f(x) = g(x)$

$x^2(x-4) = x(10-x)$

EITHER $x=0$

OR $x(x-4) = 10-x$

$x^2 - 4x = 10 - x$

$x^2 - 3x - 10 = 0$

$(x+2)(x-5)=0$

$x = \left\langle\begin{matrix}-2\\5\end{matrix}\right.$

$\therefore$ $x=0, \ y=0$

$x=-2, \ y=(-2)(12)=-24$

$x=5, \ y=5(5)=25$

$\therefore$ $(0,0)$

$(-2,-24)$

$(5,25)$ //

(b)

Question 43 (****)

$$f(x) = x^3 - 9x^2 + 13x + 2, \ x \in \mathbb{R}.$$

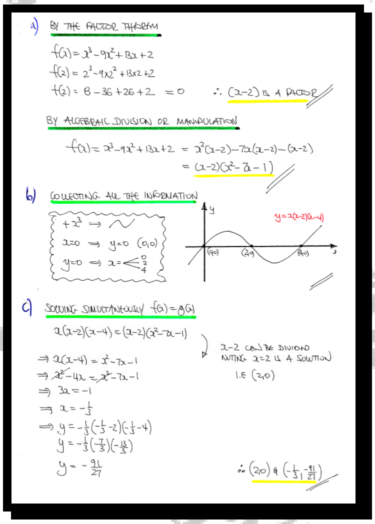
a) Show, by using the factor theorem, that $(x-2)$ is a factor of $f(x)$ and hence express $f(x)$ as a product of a linear and a quadratic factor.

$$g(x) = x(x-2)(x-4), \ x \in \mathbb{R}.$$

b) Sketch the graph of $g(x)$, indicating clearly the coordinates of any points where the graph of $g(x)$ meets the coordinate axes.

c) Determine the exact coordinates, where appropriate, of the points of intersection between the graph of $f(x)$ and the graph of $g(x)$.

$\boxed{}$, $\boxed{f(x) = (x-2)(x^2-7x-1)}$, $\boxed{(2,0), \left(-\frac{1}{3}, -\frac{91}{27}\right)}$



a) BY THE FACTOR THEOREM

$f(x) = x^3 - 9x^2 + 13x + 2$

$f(2) = 2^3 - 9 \times 2^2 + 13 \times 2 + 2$

$f(2) = 8 - 36 + 26 + 2 = 0$ $\therefore$ $(x-2)$ IS A FACTOR

BY ALGEBRAIC DIVISION OR MANIPULATION

$f(x) = x^3 - 9x^2 + 13x + 2 = x^2(x-2) - 7x(x-2) - (x-2)$

$= (x-2)(x^2 - 7x - 1)$

b) COLLECTING ALL THE INFORMATION

$+x^3 \Rightarrow$ ∿

$x=0 \Rightarrow y=0 \ (0,0)$

$y=0 \Rightarrow x = \begin{cases} 0 \\ 2 \\ 4 \end{cases}$

c) SOLVING SIMULTANEOUSLY $f(x) = g(x)$

$x(x-2)(x-4) = (x-2)(x^2-x-1)$

$x-2$ CAN BE DIVIDED NOTING $x=2$ IS A SOLUTION I.E $(2,0)$

$\Rightarrow x(x-4) = x^2 - 7x - 1$

$\Rightarrow \cancel{x^2} - 4x = \cancel{x^2} - 7x - 1$

$\Rightarrow 3x = -1$

$\Rightarrow x = -\frac{1}{3}$

$\Rightarrow y = -\frac{1}{3}\left(-\frac{1}{3} - 2\right)\left(-\frac{1}{3} - 4\right)$

$y = -\frac{1}{3}\left(-\frac{7}{3}\right)\left(-\frac{13}{3}\right)$

$y = -\frac{91}{27}$

$\therefore (2,0)$ & $\left(-\frac{1}{3}, -\frac{91}{27}\right)$

Question 44 (****)

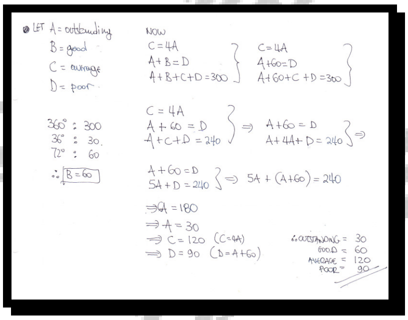
The 300 Year 11 pupils of a certain school are classed as “outstanding”, “good”, “average” or “poor”.

- The following information is also available about these pupils.
- In a standard pie chart the sector that represents the “good” pupils is $72°$.
- The “poor” pupils are as many as the “good” and “outstanding” pupils added together.

There are four times as many “average” pupils as “outstanding” ones.

Determine the number of students in each class.

$$\boxed{O = 30,\ G = 60,\ A = 120,\ P = 90}$$



- LET A = outstanding
  B = good
  C = average
  D = poor

$360° : 300$
$36° : 30$
$72° : 60$
$\therefore \boxed{B = 60}$

NOW

$$\left.\begin{array}{l} C = 4A \\ A + B = D \\ A + B + C + D = 300 \end{array}\right\} \quad \left.\begin{array}{l} C = 4A \\ A + 60 = D \\ A + 60 + C + D = 300 \end{array}\right\}$$

$$\left.\begin{array}{l} C = 4A \\ A + 60 = D \\ A + C + D = 240 \end{array}\right\} \Rightarrow \left.\begin{array}{l} A + 60 = D \\ A + 4A + D = 240 \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l} A + 60 = D \\ 5A + D = 240 \end{array}\right\} \Rightarrow 5A + (A + 60) = 240$$

$\Rightarrow 6A = 180$
$\Rightarrow A = 30$
$\Rightarrow C = 120 \quad (C = 4A)$
$\Rightarrow D = 90 \quad (D = A + 60)$

$\therefore$ OUTSTANDING = 30
GOOD = 60
AVERAGE = 120
POOR = 90

Question 45 (****)

Andrew and Bethany are preparing for a Mathematics exam by doing the same set of practice papers.

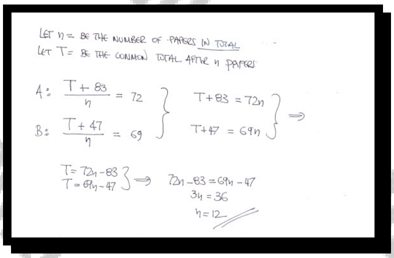
They both have one practice paper left to do and their mean scores are identical.

Andrew scores 83% on his last paper and his mean score is rises to 72%.

Bethany scores 47% on her last paper and her mean score is drops to 69%.

Determine the number of practice papers in the set.

$\boxed{n = 12}$



LET n = BE THE NUMBER OF PAPERS IN TOTAL
LET T = BE THE COMMON TOTAL AFTER n PAPERS

$$\left.\begin{array}{l} A: \dfrac{T+83}{n} = 72 \\ B: \dfrac{T+47}{n} = 69 \end{array}\right\} \quad \left.\begin{array}{l} T+83 = 72n \\ T+47 = 69n \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l} T = 72n - 83 \\ T = 69n - 47 \end{array}\right\} \Rightarrow \quad \begin{array}{c} 72n - 83 = 69n - 47 \\ 3n = 36 \\ n = 12 \end{array}$$

Question 46 **(****)**

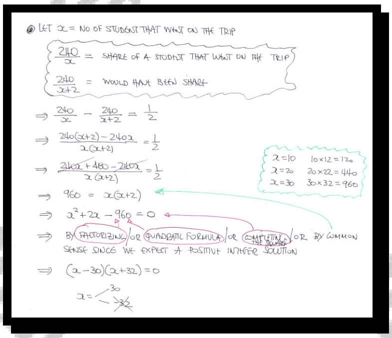
The students in a class hired a coach for a day trip, at a cost of £240.

They agreed to share **equally** the cost of the coach hire among them.

On the day of the trip 2 students fell ill so the share of the remaining students increased by £0.50.

How many students went on the school trip.

$\boxed{30}$



LET x = NO OF STUDENT THAT WENT ON THE TRIP

$\frac{240}{x}$ = SHARE OF A STUDENT THAT WENT ON THE TRIP

$\frac{240}{x+2}$ = WOULD HAVE BEEN SHARE

$\Rightarrow \frac{240}{x} - \frac{240}{x+2} = \frac{1}{2}$

$\Rightarrow \frac{240(x+2) - 240x}{x(x+2)} = \frac{1}{2}$

$\Rightarrow \frac{\cancel{240x} + 480 - \cancel{240x}}{x(x+2)} = \frac{1}{2}$

$\Rightarrow 960 = x(x+2)$

$\Rightarrow x^2 + 2x - 960 = 0$

$\Rightarrow$ BY FACTORIZING / OR QUADRATIC FORMULA / OR COMPLETING THE SQUARE / OR BY COMMON SENSE SINCE WE EXPECT A POSITIVE INTEGER SOLUTION

$\Rightarrow (x-30)(x+32) = 0$

$x = \begin{cases} 30 \\ \cancel{-32} \end{cases}$

$x = 10 \quad 10 \times 12 = 120$

$x = 20 \quad 20 \times 22 = 440$

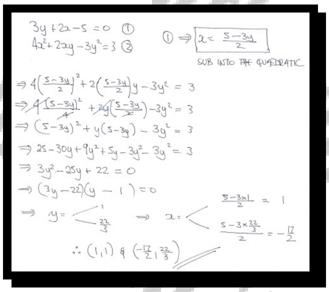
$x = 30 \quad 30 \times 32 = 960$

Question 47 **(****)**

Solve the following simultaneous equations

$$3y + 2x - 5 = 0$$
$$4x^2 + 2xy - 3y^2 = 3$$

$\boxed{(1,1) \ \& \ \left(-\frac{17}{2}, \frac{22}{3}\right)}$



$3y + 2x - 5 = 0$ ①

$4x^2 + 2xy - 3y^2 = 3$ ②

① $\Rightarrow \boxed{x = \frac{5-3y}{2}}$

SUB INTO THE QUADRATIC

$\Rightarrow 4\left(\frac{5-3y}{2}\right)^2 + 2\left(\frac{5-3y}{2}\right)y - 3y^2 = 3$

$\Rightarrow \cancel{4}\frac{(5-3y)^2}{\cancel{4}} + \cancel{2}y\left(\frac{5-3y}{\cancel{2}}\right) - 3y^2 = 3$

$\Rightarrow (5-3y)^2 + y(5-3y) - 3y^2 = 3$

$\Rightarrow 25 - 30y + 9y^2 + 5y - 3y^2 - 3y^2 = 3$

$\Rightarrow 3y^2 - 25y + 22 = 0$

$\Rightarrow (3y-22)(y-1) = 0$

$\Rightarrow y = \begin{cases} 1 \\ \frac{22}{3} \end{cases} \Rightarrow x = \begin{cases} \frac{5-3\times1}{2} = 1 \\ \frac{5-3\times\frac{22}{3}}{2} = -\frac{17}{2} \end{cases}$

$\therefore (1,1)$ & $\left(-\frac{17}{2}, \frac{22}{3}\right)$

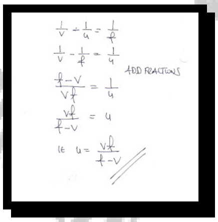
Question 48 **(****)**

Make u the subject of the equation

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f} .$$

Give the answer as a single simplified fraction.

$$\boxed{u = \frac{vf}{f - v}}$$



$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$

$\frac{1}{v} - \frac{1}{f} = \frac{1}{u}$ ADD FRACTIONS

$\frac{f-v}{vf} = \frac{1}{u}$

$\frac{vf}{f-v} = u$

ie $u = \frac{vf}{f-v}$ //

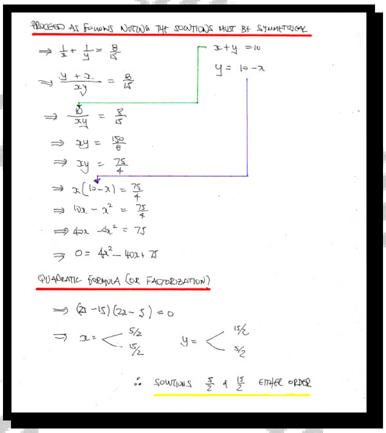
Question 49 **(****)**

Solve the following system of simultaneous equations

$$\frac{1}{x} + \frac{1}{y} = \frac{8}{15}$$

$$x + y = 10 .$$

$\boxed{\text{SYN-V}}$, $\boxed{\left(\frac{5}{2}, \frac{15}{2}\right), \left(\frac{15}{2}, \frac{5}{2}\right)}$



PROCEED AS FOLLOWS NOTING THE SOLUTIONS MUST BE SYMMETRICAL

$\Rightarrow \frac{1}{x} + \frac{1}{y} = \frac{8}{15}$

$x + y = 10$

$y = 10 - x$

$\Rightarrow \frac{y + x}{xy} = \frac{8}{15}$

$\Rightarrow \frac{10}{xy} = \frac{8}{15}$

$\Rightarrow xy = \frac{150}{8}$

$\Rightarrow xy = \frac{75}{4}$

$\Rightarrow x(10 - x) = \frac{75}{4}$

$\Rightarrow 10x - x^2 = \frac{75}{4}$

$\Rightarrow 40x - 4x^2 = 75$

$\Rightarrow 0 = 4x^2 - 40x + 75$

QUADRATIC FORMULA (OR FACTORIZATION)

$\Rightarrow (2x - 15)(2x - 5) = 0$

$\Rightarrow x = \begin{cases} 5/2 \\ 15/2 \end{cases} \quad y = \begin{cases} 15/2 \\ 5/2 \end{cases}$

$\therefore$ SOLUTIONS $\frac{5}{2}$ & $\frac{15}{2}$ EITHER ORDER

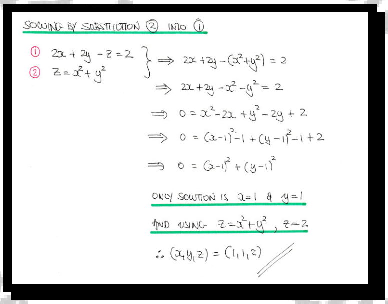
Question 50 **(****)**

Find the solution of the following simultaneous equations

$$2x + 2y - z = 2$$
$$z = x^2 + y^2$$

assuming that x, y, z are all real numbers.

$\boxed{}$, $\boxed{(x, y, z) = (1,1,2)}$



SOLVING BY SUBSTITUTION ② INTO ①

① $2x + 2y - z = 2$
② $z = x^2 + y^2$

$$\left.\begin{aligned} 2x+2y-z&=2 \\ z&=x^2+y^2 \end{aligned}\right\} \Rightarrow 2x + 2y - (x^2+y^2) = 2$$
$$\Rightarrow 2x + 2y - x^2 - y^2 = 2$$
$$\Rightarrow 0 = x^2 - 2x + y^2 - 2y + 2$$
$$\Rightarrow 0 = (x-1)^2 - 1 + (y-1)^2 - 1 + 2$$
$$\Rightarrow 0 = (x-1)^2 + (y-1)^2$$

ONLY SOLUTION IS $x=1$ & $y=1$

AND USING $z = x^2 + y^2$, $z = 2$

$\therefore (x, y, z) = (1, 1, 2)$ //

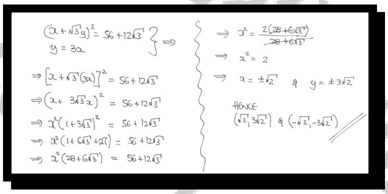
Question 51 **(****)**

Solve the following system of simultaneous equations

$$\left(x + y\sqrt{3}\right)^2 = 56 + 12\sqrt{3}$$

$$y = 3x.$$

$\boxed{}$, $\boxed{\left(\sqrt{2}, 3\sqrt{2}\right), \left(-\sqrt{2}, -3\sqrt{2}\right)}$



$$\left.\begin{aligned} (x+\sqrt{3}y)^2 &= 56+12\sqrt{3} \\ y &= 3x \end{aligned}\right\} \Rightarrow$$
$$\Rightarrow \left[x + \sqrt{3}(3x)\right]^2 = 56 + 12\sqrt{3}$$
$$\Rightarrow \left(x + 3\sqrt{3}x\right)^2 = 56 + 12\sqrt{3}$$
$$\Rightarrow x^2\left(1 + 3\sqrt{3}\right)^2 = 56 + 12\sqrt{3}$$
$$\Rightarrow x^2\left(1 + 6\sqrt{3} + 27\right) = 56 + 12\sqrt{3}$$
$$\Rightarrow x^2\left(28 + 6\sqrt{3}\right) = 56 + 12\sqrt{3}$$
$$\Rightarrow x^2 = \frac{2\cancel{(28+6\sqrt{3})}}{\cancel{28+6\sqrt{3}}}$$
$$\Rightarrow x^2 = 2$$
$$\Rightarrow x = \pm\sqrt{2} \quad \& \quad y = \pm 3\sqrt{2}$$

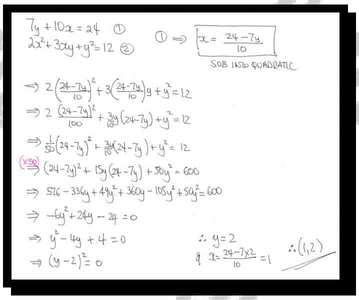
HENCE
$(\sqrt{2}, 3\sqrt{2})$ & $(-\sqrt{2}, -3\sqrt{2})$ //

Question 52 (****+)

Solve the following system of simultaneous equations

$$7y + 10x = 24$$
$$2x^2 + 3xy + y^2 = 12$$

$$\boxed{(1,2)}$$



$7y + 10x = 24$ ①

$2x^2 + 3xy + y^2 = 12$ ②

① $\Rightarrow \boxed{x = \frac{24-7y}{10}}$

SUB INTO QUADRATIC

$\Rightarrow 2\left(\frac{24-7y}{10}\right)^2 + 3\left(\frac{24-7y}{10}\right)y + y^2 = 12$

$\Rightarrow 2\,\frac{(24-7y)^2}{100} + \frac{3y}{10}(24-7y) + y^2 = 12$

$\Rightarrow \frac{1}{50}(24-7y)^2 + \frac{3y}{10}(24-7y) + y^2 = 12$

$(\times 50)$ $\Rightarrow (24-7y)^2 + 15y(24-7y) + 50y^2 = 600$

$\Rightarrow 576 - 336y + 49y^2 + 360y - 105y^2 + 50y^2 = 600$

$\Rightarrow -6y^2 + 24y - 24 = 0$

$\Rightarrow y^2 - 4y + 4 = 0$

$\Rightarrow (y-2)^2 = 0$

$\therefore y = 2$

& $x = \frac{24-7\times 2}{10} = 1$

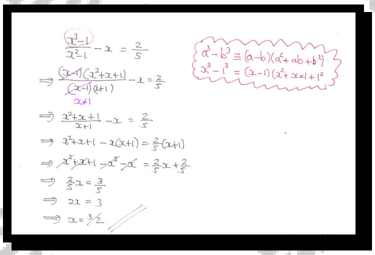
$\therefore (1,2)$

Question 53 (****+)

Solve the following equation

$$\frac{x^3-1}{x^2-1} - x = \frac{2}{5}, \quad x \neq \pm 1.$$

$$\boxed{x = \frac{3}{2}}$$



$\frac{x^3-1}{x^2-1} - x = \frac{2}{5}$

$a^3 - b^3 \equiv (a-b)(a^2+ab+b^2)$

$x^3 - 1^3 = (x-1)(x^2+x\times 1+1^2$

$\Rightarrow \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}(x+1)} - x = \frac{2}{5}$

$x \neq 1$

$\Rightarrow \frac{x^2+x+1}{x+1} - x = \frac{2}{5}$

$\Rightarrow x^2+x+1 - x(x+1) = \frac{2}{5}(x+1)$

$\Rightarrow \cancel{x^2} + \cancel{x} + 1 - \cancel{x^2} - \cancel{x} = \frac{2}{5}x + \frac{2}{5}$

$\Rightarrow \frac{2}{5}x = \frac{3}{5}$

$\Rightarrow 2x = 3$

$\Rightarrow x = 3/2$

Question 54 (****+)

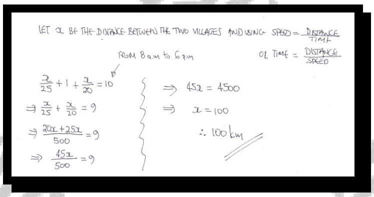
A cyclist leaves village A at 8 a.m. cycling towards village B at constant speed of 25 km h^{-1}.

After arriving at B the cyclist spends exactly 1 hour there before he cycles back to A, following exactly the same route he took on his outward journey.

On his return journey he cycles at a constant speed 20 km h^{-1}.

Given the cyclist returns back to village A at 6 p.m. determine the distance between the two villages.

$\boxed{100 \text{ km}}$



LET x BE THE DISTANCE BETWEEN THE TWO VILLAGES AND USING SPEED = $\frac{\text{DISTANCE}}{\text{TIME}}$

OR TIME = $\frac{\text{DISTANCE}}{\text{SPEED}}$

FROM 8 a.m to 6 p.m

$$\frac{x}{25} + 1 + \frac{x}{20} = 10$$
$$\Rightarrow \frac{x}{25} + \frac{x}{20} = 9$$
$$\Rightarrow \frac{20x + 25x}{500} = 9$$
$$\Rightarrow \frac{45x}{500} = 9$$
$$\Rightarrow 45x = 4500$$
$$\Rightarrow x = 100$$

$\therefore$ 100 km

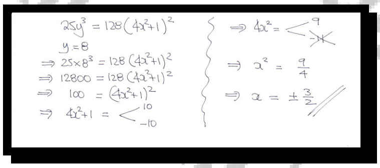
Question 55 (****+)

A relationship between two variables is given below

$$25y^3 = 128\left(4x^2+1\right)^2.$$

Find the possible values of x when $y = 8$.

$\boxed{}$, $\boxed{x = \pm\frac{3}{2}}$



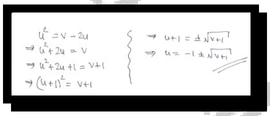
$$25y^3 = 128(4x^2+1)^2$$
$$y = 8$$
$$\Rightarrow 25 \times 8^3 = 128(4x^2+1)^2$$
$$\Rightarrow 12800 = 128(4x^2+1)^2$$
$$\Rightarrow 100 = (4x^2+1)^2$$
$$\Rightarrow 4x^2+1 = \begin{cases} 10 \\ -10 \end{cases}$$
$$\Rightarrow 4x^2 = \begin{cases} 9 \\ \cancel{-11} \end{cases}$$
$$\Rightarrow x^2 = \frac{9}{4}$$
$$\Rightarrow x = \pm\frac{3}{2}$$

Question 56 (****+)

Make u the subject of the equation

$$u^2 = v - 2u\,.$$

$$\boxed{u = -1 \pm \sqrt{v+1}}$$



$u^2 = v - 2u$
$\Rightarrow u^2 + 2u = v$
$\Rightarrow u^2 + 2u + 1 = v + 1$
$\Rightarrow (u+1)^2 = v + 1$

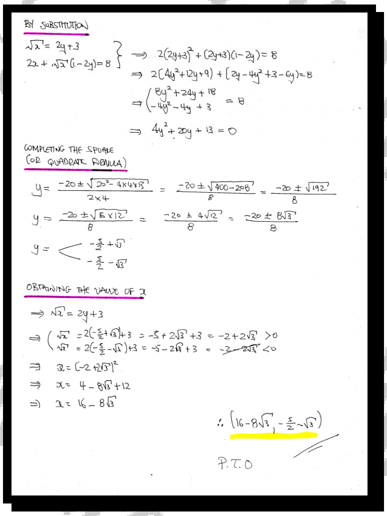
$\Rightarrow u + 1 = \pm\sqrt{v+1}$
$\Rightarrow u = -1 \pm \sqrt{v+1}$

Question 57 (****+)

Find as exact simplified surds the coordinates of the point of intersection between the graphs of

$$\sqrt{x} = 2y + 3 \qquad \text{and} \qquad 2x + \sqrt{x} - 2y\sqrt{x} = 8\,.$$

$$\boxed{}\,, \ \boxed{\left(16 - 8\sqrt{3}, -\frac{5}{2} + \sqrt{3}\right)}$$



BY SUBSTITUTION

$\left.\begin{array}{l}\sqrt{x} = 2y + 3 \\ 2x + \sqrt{x}(1-2y) = 8\end{array}\right\} \Rightarrow 2(2y+3)^2 + (2y+3)(1-2y) = 8$

$\Rightarrow 2(4y^2 + 12y + 9) + (2y - 4y^2 + 3 - 6y) = 8$

$\Rightarrow \left(\begin{array}{l}8y^2 + 24y + 18 \\ -4y^2 - 4y + 3\end{array}\right. = 8$

$\Rightarrow 4y^2 + 20y + 13 = 0$

COMPLETING THE SQUARE
(OR QUADRATIC FORMULA)

$y = \frac{-20 \pm \sqrt{20^2 - 4\times4\times13}}{2\times4} = \frac{-20 \pm \sqrt{400-208}}{8} = \frac{-20 \pm \sqrt{192}}{8}$

$y = \frac{-20 \pm \sqrt{16\times12}}{8} = \frac{-20 \pm 4\sqrt{12}}{8} = \frac{-20 \pm 8\sqrt{3}}{8}$

$y = \begin{cases} -\frac{5}{2} + \sqrt{3} \\ -\frac{5}{2} - \sqrt{3} \end{cases}$

OBTAINING THE VALUE OF x

$\Rightarrow \sqrt{x} = 2y + 3$

$\Rightarrow \left(\begin{array}{l}\sqrt{x} = 2(-\frac{5}{2} + \sqrt{3}) + 3 = -5 + 2\sqrt{3} + 3 = -2 + 2\sqrt{3} > 0 \\ \sqrt{x} = 2(-\frac{5}{2} - \sqrt{3}) + 3 = -5 - 2\sqrt{3} + 3 = ~~-2 - 2\sqrt{3}~~ < 0\end{array}\right.$

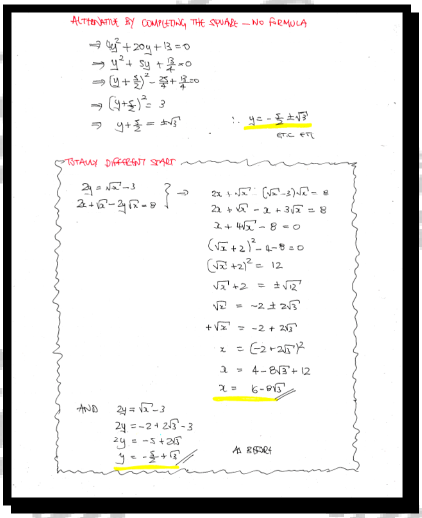
$\Rightarrow x = (-2 + 2\sqrt{3})^2$

$\Rightarrow x = 4 - 8\sqrt{3} + 12$

$\Rightarrow x = 16 - 8\sqrt{3}$

$\therefore \left(16 - 8\sqrt{3}, -\frac{5}{2} - \sqrt{3}\right)$

P.T.O



ALTERNATIVE BY COMPLETING THE SQUARE – NO FORMULA

$\Rightarrow 4y^2 + 20y + 13 = 0$
$\Rightarrow y^2 + 5y + \frac{13}{4} = 0$
$\Rightarrow (y + \frac{5}{2})^2 - \frac{25}{4} + \frac{13}{4} = 0$
$\Rightarrow (y + \frac{5}{2})^2 = 3$
$\Rightarrow y + \frac{5}{2} = \pm\sqrt{3}$

$\therefore y = -\frac{5}{2} \pm \sqrt{3}$

ETC ETC

TOTALLY DIFFERENT START

$\left.\begin{array}{l}2y = \sqrt{x} - 3 \\ 2x + \sqrt{x} - 2y\sqrt{x} = 8\end{array}\right\} \Rightarrow$

$2x + \sqrt{x} - (\sqrt{x} - 3)\sqrt{x} = 8$
$2x + \sqrt{x} - x + 3\sqrt{x} = 8$
$x + 4\sqrt{x} - 8 = 0$
$(\sqrt{x} + 2)^2 - 4 - 8 = 0$
$(\sqrt{x} + 2)^2 = 12$
$\sqrt{x} + 2 = \pm\sqrt{12}$
$\sqrt{x} = -2 \pm 2\sqrt{3}$
$+\sqrt{x} = -2 + 2\sqrt{3}$
$x = (-2 + 2\sqrt{3})^2$
$x = 4 - 8\sqrt{3} + 12$
$x = 6 - 8\sqrt{3}$

AND
$2y = \sqrt{x} - 3$
$2y = -2 + 2\sqrt{3} - 3$
$2y = -5 + 2\sqrt{3}$
$y = -\frac{5}{2} + \sqrt{3}$

AS BEFORE

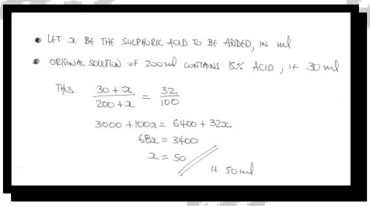
Question 58 (****+)

Sulphuric acid is a colourless liquid which can be diluted with water.

Pure sulphuric acid is to be added to a 200 ml water solution, which also contains sulphuric acid of concentration 15% by volume.

How many ml of pure sulphuric acid must be added so that the resulting solution contains sulphuric acid of concentration 32% by volume.

$\boxed{50 \text{ ml}}$



- LET x BE THE SULPHURIC ACID TO BE ADDED, IN ml
- ORIGINAL SOLUTION OF 200 ml CONTAINS 15% ACID, ie 30 ml

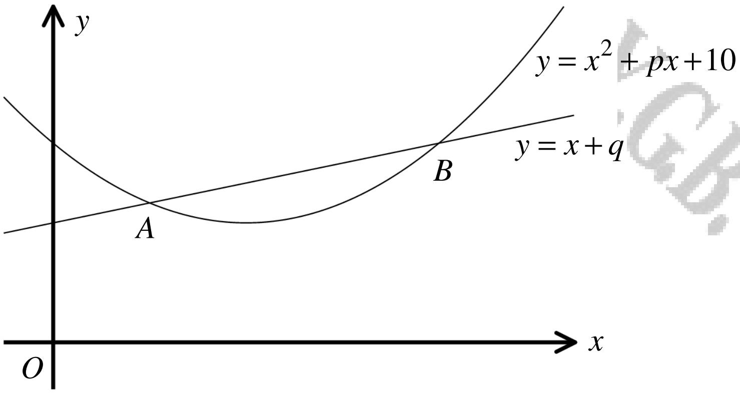
THUS $\dfrac{30+x}{200+x} = \dfrac{32}{100}$

$3000 + 100x = 6400 + 32x$

$68x = 3400$

$x = 50$ // ie 50 ml

Question 59 (****+)



The figure above shows the graph of the curve with equation

$$y = x^2 + px + 10$$

and the straight line with equation

$$y = x + q,$$

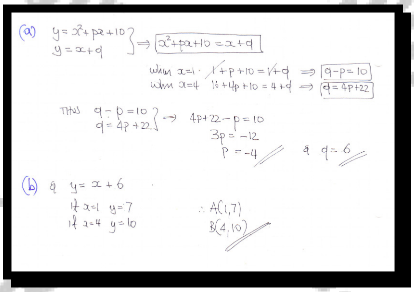
where p and q are constants.

The curve and the straight line intersect at the points A and B whose x coordinates are 1 and 4, respectively.

a) Determine the value of p and the value of q.

b) Find the coordinates of A and B.

$\boxed{}$, $\boxed{p = -4,\ q = 6}$, $\boxed{A(1,7),\ B(4,10)}$



(a) $\left.\begin{array}{l} y = x^2 + px + 10 \\ y = x + q \end{array}\right\} \Rightarrow \boxed{x^2 + px + 10 = x + q}$

when $x=1$ · $\cancel{1} + p + 10 = \cancel{1} + q \Rightarrow \boxed{q - p = 10}$
when $x=4$ $16 + 4p + 10 = 4 + q \Rightarrow \boxed{q = 4p + 22}$

Thus $\left.\begin{array}{l} q - p = 10 \\ q = 4p + 22 \end{array}\right\} \Rightarrow 4p + 22 - p = 10$
$3p = -12$
$p = -4$ // & $q = 6$ //

(b) & $y = x + 6$
if $x=1$ $y=7$
if $x=4$ $y=10$
$\therefore A(1,7)$
$B(4,10)$ //

Question 60 (****+)

A pupil is heard saying to another pupil …

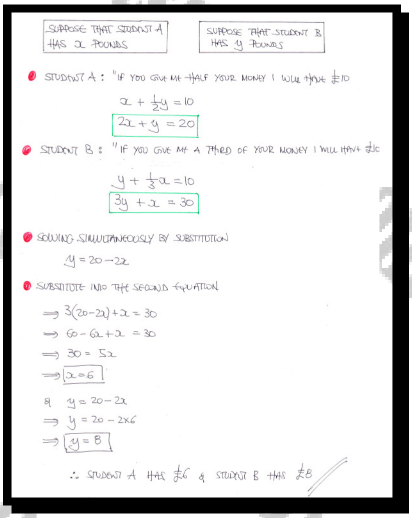
"… if you give me half your pocket money I will have £10."

The other pupil replied …

"…if you give me one third of your pocket money I will have £10."

Determine how much money each pupil has.

$\boxed{\text{£6 and £8}}$



SUPPOSE THAT STUDENT A HAS x POUNDS

SUPPOSE THAT STUDENT B HAS y POUNDS

- STUDENT A : "IF YOU GIVE ME HALF YOUR MONEY I WILL HAVE £10

$$x + \frac{1}{2}y = 10$$
$$\boxed{2x + y = 20}$$

- STUDENT B : "IF YOU GIVE ME A THIRD OF YOUR MONEY I WILL HAVE £10

$$y + \frac{1}{3}x = 10$$
$$\boxed{3y + x = 30}$$

- SOLVING SIMULTANEOUSLY BY SUBSTITUTION

$$y = 20 - 2x$$

- SUBSTITUTE INTO THE SECOND EQUATION

$$\Rightarrow 3(20-2x) + x = 30$$
$$\Rightarrow 60 - 6x + x = 30$$
$$\Rightarrow 30 = 5x$$
$$\Rightarrow \boxed{x = 6}$$

& $y = 20 - 2x$
$$\Rightarrow y = 20 - 2 \times 6$$
$$\Rightarrow \boxed{y = 8}$$

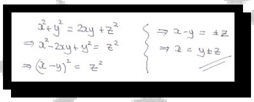
$\therefore$ STUDENT A HAS £6 & STUDENT B HAS £8 //

Question 61 (****+)

Make x the subject of the equation

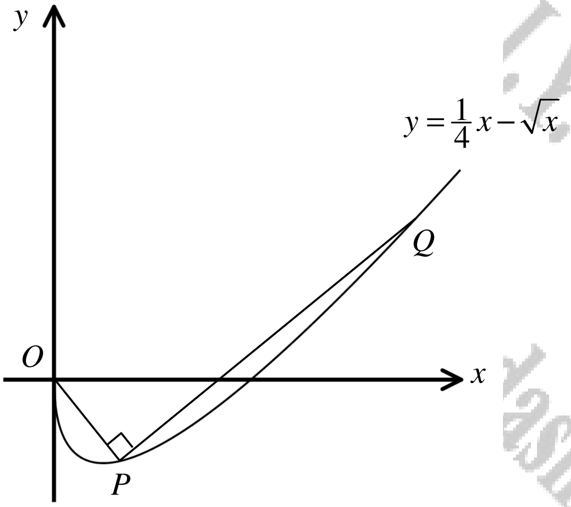
$$x^2 + y^2 = 2xy + z^2.$$

$\boxed{}$, $\boxed{x = y \pm z}$



$$x^2 + y^2 = 2xy + z^2$$
$$\Rightarrow x^2 - 2xy + y^2 = z^2$$
$$\Rightarrow (x-y)^2 = z^2$$
$$\Rightarrow x - y = \pm z$$
$$\Rightarrow x = y \pm z$$ //

Question 62 (****+)



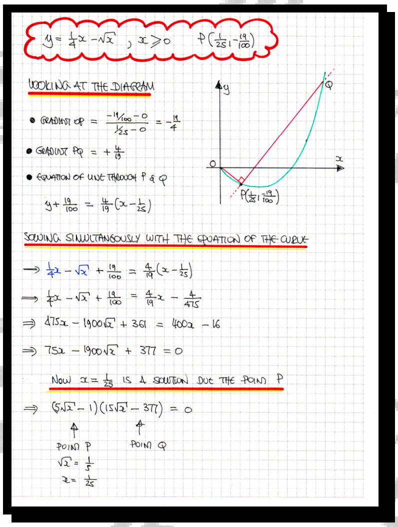
The figure above show the curve with equation

$$y=\frac{1}{4}x-\sqrt{x}\,,\ x\in\mathbb{R}\,,\ x\geq 0\,.$$

The points $P(0.04,-0.19)$ and Q lie on the curve, so that $\measuredangle OPQ = 90°$, where O is the origin.

Show that the y coordinate of Q is $\frac{k}{900}$, where k is a six digit integer.

$\boxed{\quad\quad}$, $\boxed{k=119509}$



$y=\frac{1}{4}x-\sqrt{x}\ ,\ x\geqslant 0 \qquad P\left(\frac{1}{25},-\frac{19}{100}\right)$

LOOKING AT THE DIAGRAM

- GRADIENT OP $= \dfrac{-19/100-0}{1/25-0} = -\frac{19}{4}$
- GRADIENT PQ $= +\frac{4}{19}$
- EQUATION OF LINE THROUGH P & Q

$y+\frac{19}{100} = \frac{4}{19}\left(x-\frac{1}{25}\right)$

SOLVING SIMULTANEOUSLY WITH THE EQUATION OF THE CURVE

$\Longrightarrow \frac{1}{4}x-\sqrt{x}+\frac{19}{100} = \frac{4}{19}\left(x-\frac{1}{25}\right)$

$\Longrightarrow \frac{1}{4}x-\sqrt{x}+\frac{19}{100} = \frac{4}{19}x-\frac{4}{475}$

$\Longrightarrow 475x-1900\sqrt{x}+361 = 400x-16$

$\Longrightarrow 75x-1900\sqrt{x}+377=0$

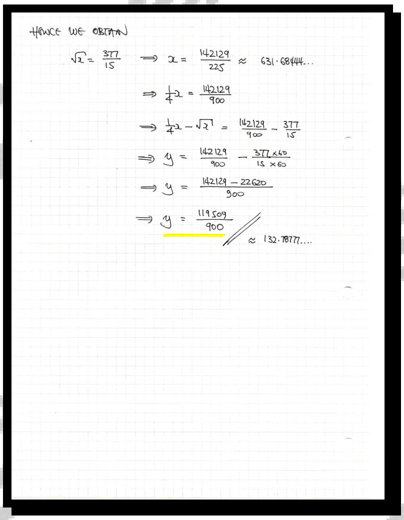
NOW $x=\frac{1}{25}$ IS A SOLUTION DUE THE POINT P

$\Longrightarrow (5\sqrt{x}-1)(15\sqrt{x}-377)=0$

$\uparrow$ POINT P $\quad \uparrow$ POINT Q

$\sqrt{x}=\frac{1}{5}$

$x=\frac{1}{25}$



HENCE WE OBTAIN

$\sqrt{x}=\frac{377}{15} \Longrightarrow x=\frac{142129}{225} \approx 631.68444\ldots$

$\Longrightarrow \frac{1}{4}x=\frac{142129}{900}$

$\Longrightarrow \frac{1}{4}x-\sqrt{x}=\frac{142129}{900}-\frac{377}{15}$

$\Longrightarrow y=\frac{142129}{900}-\frac{377\times 60}{15\times 60}$

$\Longrightarrow y=\frac{142129-22620}{900}$

$\Longrightarrow y=\frac{119509}{900}$ // $\approx 132.78777\ldots$

Question 63 (*****)

Two joggers, A and B ran a standard route of 5 km, which consists of a downhill section to start with, a flat section in the middle of the run and an uphill section all the way to the finish line.

A ran the three sections with respective speeds 2.4 ms^{-1}, 3.2 ms^{-1} and 2 ms^{-1}.

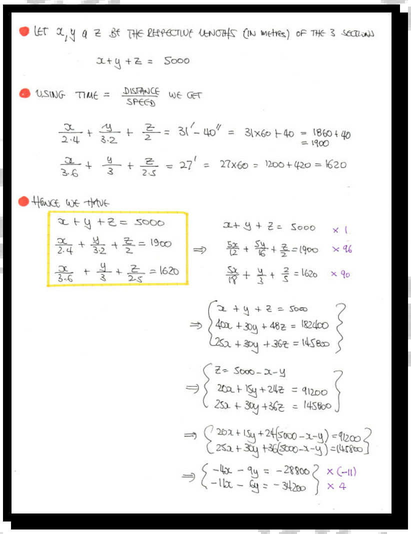
A took 31 minutes and 40 seconds to complete the run.

B ran the three sections with respective speeds 3.6 ms^{-1}, 3 ms^{-1} and 2.5 ms^{-1}.

A took exactly 27 minutes to complete the run.

Assuming that both runners started at the same time, determine the distance between A and B, as B crosses the finish line.

$\boxed{\quad}$, $\boxed{560\text{ m}}$



● LET x, y & z BE THE RESPECTIVE LENGTHS (IN metres) OF THE 3 SECTIONS

$$x+y+z=5000$$

● USING $\text{TIME} = \frac{\text{DISTANCE}}{\text{SPEED}}$ WE GET

$$\frac{x}{2.4}+\frac{y}{3.2}+\frac{z}{2}=31'-40''=31\times60+40=1860+40=1900$$

$$\frac{x}{3.6}+\frac{y}{3}+\frac{z}{2.5}=27'=27\times60=1200+420=1620$$

● HENCE WE HAVE

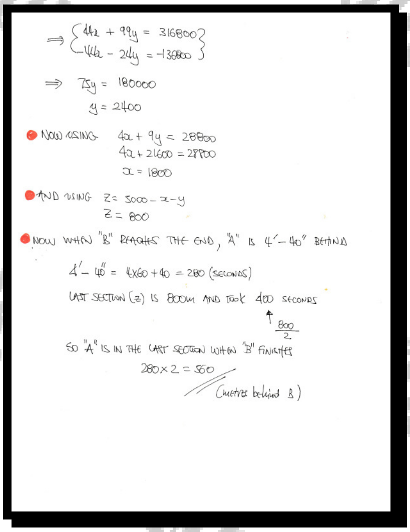
$$\boxed{\begin{array}{l}x+y+z=5000\\ \frac{x}{2.4}+\frac{y}{3.2}+\frac{z}{2}=1900\\ \frac{x}{3.6}+\frac{y}{3}+\frac{z}{2.5}=1620\end{array}} \Rightarrow \begin{array}{ll}x+y+z=5000 & \times 1\\ \frac{5x}{12}+\frac{5y}{16}+\frac{z}{2}=1900 & \times 96\\ \frac{5x}{18}+\frac{y}{3}+\frac{2}{5}=1620 & \times 90\end{array}$$

$$\Rightarrow \left\{\begin{array}{l}x+y+z=5000\\ 40x+30y+48z=182400\\ 25x+30y+36z=145800\end{array}\right\}$$

$$\Rightarrow \left\{\begin{array}{l}z=5000-x-y\\ 20x+15y+24z=91200\\ 25x+30y+36z=145800\end{array}\right\}$$

$$\Rightarrow \left\{\begin{array}{l}20x+15y+24(5000-x-y)=91200\\ 25x+30y+36(5000-x-y)=145800\end{array}\right\}$$

$$\Rightarrow \left\{\begin{array}{l}-4x-9y=-28800\\ -11x-6y=-34200\end{array}\right\} \begin{array}{l}\times(-11)\\ \times 4\end{array}$$



$$\Rightarrow \left\{\begin{array}{l}44x+99y=316800\\ -44x-24y=-136800\end{array}\right\}$$

$$\Rightarrow 75y=180000$$

$$y=2400$$

● NOW USING $4x+9y=28800$

$$4x+21600=28800$$

$$x=1800$$

● AND USING $z=5000-x-y$

$$z=800$$

● NOW WHEN "B" REACHES THE END, "A" IS $4'-40''$ BEHIND

$$4'-40''=4\times60+40=280\text{ (SECONDS)}$$

LAST SECTION (z) IS 800m AND TOOK 400 SECONDS $\uparrow \frac{800}{2}$

SO "A" IS IN THE LAST SECTION WHEN "B" FINISHES

$$280\times2=560$$

(metres behind B)

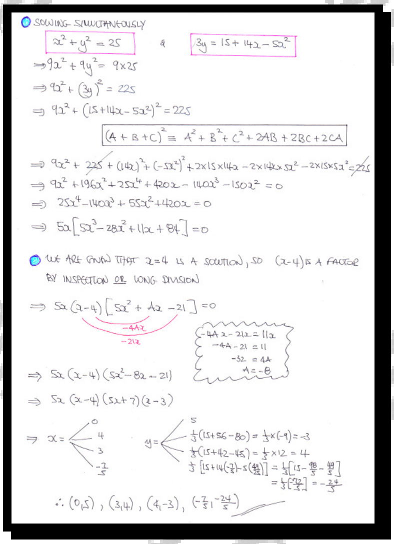
Question 64 (*****)

Find the coordinates of the points of intersections between

$$x^2+y^2=25 \qquad \text{and} \qquad 3y=15+14x-5x^2,$$

given further that the x coordinate of one of these points is 4.

$\boxed{\text{SP-X}}$, $\boxed{(0,5),(3,4),(4,-3),\left(-\frac{2}{5},-\frac{24}{5}\right)}$



① SOLVING SIMULTANEOUSLY

$\boxed{x^2+y^2=25}$ & $\boxed{3y=15+14x-5x^2}$

$\Rightarrow 9x^2+9y^2=9\times25$

$\Rightarrow 9x^2+(3y)^2=225$

$\Rightarrow 9x^2+(15+14x-5x^2)^2=225$

$\boxed{(A+B+C)^2\equiv A^2+B^2+C^2+2AB+2BC+2CA}$

$\Rightarrow 9x^2+\cancel{225}+(14x)^2+(-5x^2)^2+2\times15\times14x-2\times14x\times5x^2-2\times15\times5x^2=\cancel{225}$

$\Rightarrow 9x^2+196x^2+25x^4+420x-140x^3-150x^2=0$

$\Rightarrow 25x^4-140x^3+55x^2+420x=0$

$\Rightarrow 5x\left[5x^3-28x^2+11x+84\right]=0$

② WE ARE GIVEN THAT $x=4$ IS A SOLUTION, SO $(x-4)$ IS A FACTOR BY INSPECTION OR LONG DIVISION

$\Rightarrow 5x(x-4)\left[5x^2+Ax-21\right]=0$

$-4Ax$

$-21x$

$-4Ax-21x=11x$
$-4A-21=11$
$-32=4A$
$A=-8$

$\Rightarrow 5x(x-4)(5x^2-8x-21)$

$\Rightarrow 5x(x-4)(5x+7)(x-3)$

$\Rightarrow x = 0,\ 4,\ 3,\ -\frac{7}{5}$

$y = 5$
$\frac{1}{3}(15+56-80)=\frac{1}{3}\times(-9)=-3$
$\frac{1}{3}(15+42-45)=\frac{1}{3}\times12=4$
$\frac{1}{3}\left[15+14\left(-\frac{7}{5}\right)-5\left(\frac{49}{25}\right)\right]=\frac{1}{3}\left[15-\frac{98}{5}-\frac{49}{5}\right]$
$=\frac{1}{3}\left[-\frac{72}{5}\right]=-\frac{24}{5}$

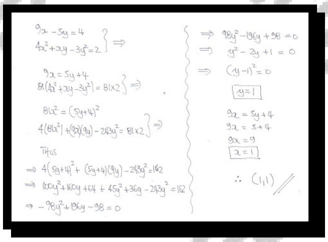
$\therefore (0,5),\ (3,4),\ (4,-3),\ \left(-\frac{7}{5},-\frac{24}{5}\right)$

Question 65 (*****) **non calculator**

Solve the simultaneous equations

$$9x-5y=4$$
$$4x^2+xy-3y^2=2$$

$\boxed{(1,1)}$



$\left.\begin{array}{l}9x-5y=4\\4x^2+xy-3y^2=2\end{array}\right\}\Rightarrow$

$\left.\begin{array}{l}9x=5y+4\\81(4x^2+xy-3y^2)=81\times2\end{array}\right\}\Rightarrow$

$\left.\begin{array}{l}81x^2=(5y+4)^2\\4(81x^2)+(9x)(9y)-243y^2=81\times2\end{array}\right\}\Rightarrow$

Thus

$\Rightarrow 4(5y+4)^2+(5y+4)(9y)-243y^2=162$

$\Rightarrow 100y^2+160y+64+45y^2+36y-243y^2=162$

$\Rightarrow -98y^2+196y-98=0$

$\Rightarrow 98y^2-196y+98=0$

$\Rightarrow y^2-2y+1=0$

$\Rightarrow (y-1)^2=0$

$\boxed{y=1}$

$9x=5y+4$
$9x=5+4$
$9x=9$
$\boxed{x=1}$

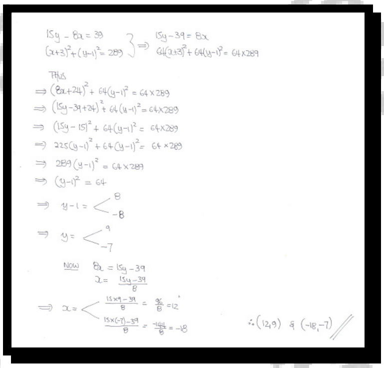
$\therefore (1,1)$

Question 66 **(*****)**

Solve the simultaneous equations

$$15y - 8x = 39$$
$$(x+3)^2 + (y-1)^2 = 289$$

$\boxed{}$, $\boxed{(12,9) \ \& \ (-18,-7)}$



$$\left.\begin{array}{l} 15y - 8x = 39 \\ (x+3)^2+(y-1)^2 = 289 \end{array}\right\} \Rightarrow \begin{array}{l} 15y - 39 = 8x \\ 64(x+3)^2 + 64(y-1)^2 = 64 \times 289 \end{array}$$

Thus

$$\Rightarrow (8x+24)^2 + 64(y-1)^2 = 64 \times 289$$
$$\Rightarrow (15y-39+24)^2 + 64(y-1)^2 = 64 \times 289$$
$$\Rightarrow (15y-15)^2 + 64(y-1)^2 = 64 \times 289$$
$$\Rightarrow 225(y-1)^2 + 64(y-1)^2 = 64 \times 289$$
$$\Rightarrow 289(y-1)^2 = 64 \times 289$$
$$\Rightarrow (y-1)^2 = 64$$
$$\Rightarrow y-1 = \left\langle \begin{array}{l} 8 \\ -8 \end{array} \right.$$
$$\Rightarrow y = \left\langle \begin{array}{l} 9 \\ -7 \end{array} \right.$$

Now $8x = 15y - 39$

$x = \dfrac{15y-39}{8}$

$$\Rightarrow x = \left\langle \begin{array}{l} \frac{15 \times 9 - 39}{8} = \frac{96}{8} = 12 \\ \frac{15 \times (-7) - 39}{8} = \frac{-144}{8} = -18 \end{array} \right.$$

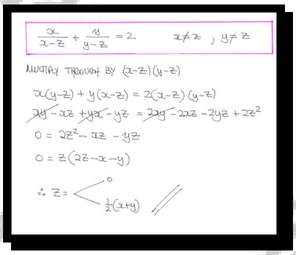
$\therefore (12,9)$ & $(-18,-7)$ //

Question 67 **(*****)**

Solve the following equation for x.

$$\frac{x}{x-z} + \frac{y}{y-z} = 2, \quad x \neq z, \ y \neq z.$$

$\boxed{\text{SP-O}}$, $\boxed{z = 0, \quad z = \frac{1}{2}(x+y)}$



$$\boxed{\frac{x}{x-z} + \frac{y}{y-z} = 2 \qquad x \neq z, \ y \neq z}$$

MULTIPLY THROUGH BY $(x-z)(y-z)$

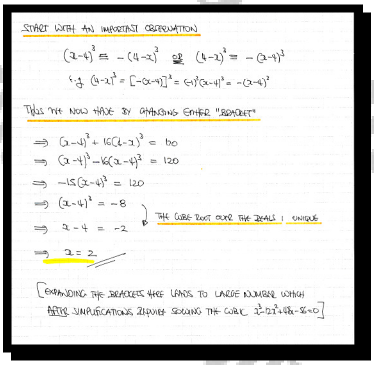
$$x(y-z) + y(x-z) = 2(x-z)(y-z)$$
$$\cancel{xy} - xz + \cancel{yx} - yz = \cancel{2xy} - 2xz - 2yz + 2z^2$$
$$0 = 2z^2 - xz - yz$$
$$0 = z(2z - x - y)$$
$$\therefore z = \left\langle \begin{array}{l} 0 \\ \frac{1}{2}(x+y) \end{array} \right. \quad //$$

Question 68 **(*****)**

Use algebra to solve the equation

$$(x-4)^3+16(4-x)^3=120, \quad x\in\mathbb{R}.$$

V, $\boxed{}$, $\boxed{x=2}$



START WITH AN IMPORTANT OBSERVATION

$$(x-4)^3 \equiv -(4-x)^3 \quad \text{OR} \quad (4-x)^3 \equiv -(x-4)^3$$

$$\text{e.g. } (4-x)^3 = [-(x-4)]^3 = (-1)^3(x-4)^3 = -(x-4)^2$$

THUS WE NOW HAVE BY CHANGING EITHER "BRACKET"

$$\Rightarrow (x-4)^3+16(4-x)^3 = 120$$
$$\Rightarrow (x-4)^3-16(x-4)^3 = 120$$
$$\Rightarrow -15(x-4)^3 = 120$$
$$\Rightarrow (x-4)^3 = -8$$
$$\Rightarrow x-4 = -2$$
$$\Rightarrow x = 2$$

THE CUBE ROOT OVER THE REALS IS UNIQUE

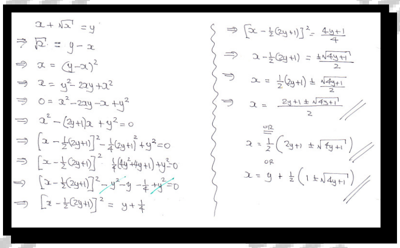
[EXPANDING THE BRACKETS HERE LEADS TO LARGE NUMBER WHICH AFTER SIMPLIFICATIONS REQUIRE SOLVING THE CUBIC $x^3-12x^2+48x-56=0$]

Question 69 **(*****)**

Make x the subject of the equation

$$x+\sqrt{x}=y \ .$$

$$\boxed{x=y+\frac{1}{2}\left[1\pm\sqrt{4y+1}\right]}$$



$$x+\sqrt{x}=y$$
$$\Rightarrow \sqrt{x}=y-x$$
$$\Rightarrow x=(y-x)^2$$
$$\Rightarrow x=y^2-2xy+x^2$$
$$\Rightarrow 0=x^2-2xy-x+y^2$$
$$\Rightarrow x^2-(2y+1)x+y^2=0$$
$$\Rightarrow \left[x-\tfrac{1}{2}(2y+1)\right]^2-\tfrac{1}{4}(2y+1)^2+y^2=0$$
$$\Rightarrow \left[x-\tfrac{1}{2}(2y+1)\right]^2 \ \tfrac{1}{4}(4y^2+4y+1)+y^2=0$$
$$\Rightarrow \left[x-\tfrac{1}{2}(2y+1)\right]^2\cancel{-y^2}-y-\tfrac{1}{4}\cancel{+y^2}=0$$
$$\Rightarrow \left[x-\tfrac{1}{2}(2y+1)\right]^2=y+\tfrac{1}{4}$$
$$\Rightarrow \left[x-\tfrac{1}{2}(2y+1)\right]^2=\frac{4y+1}{4}$$
$$\Rightarrow x-\tfrac{1}{2}(2y+1)=\frac{\pm\sqrt{4y+1}}{2}$$
$$\Rightarrow x=\tfrac{1}{2}(2y+1)\pm\frac{\sqrt{4y+1}}{2}$$
$$\Rightarrow x=\frac{2y+1\pm\sqrt{4y+1}}{2}$$

OR

$$x=\tfrac{1}{2}\left(2y+1\pm\sqrt{4y+1}\right)$$

OR

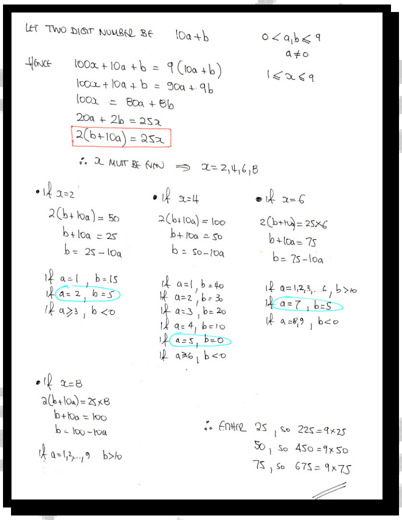
$$x=y+\tfrac{1}{2}\left(1\pm\sqrt{4y+1}\right)$$

Question 70 (*****)

It is required to add a single digit at the front of a two digit number so that the resulting three digit number is nine times as large as the original two digit number.

Determine with full justification the three possible cases that satisfy this requirement.

$\boxed{25, \ 50, \ 75}$



LET TWO DIGIT NUMBER BE $10a+b$ $\quad 0 < a,b \leqslant 9$, $a \neq 0$

HENCE
$$100x + 10a + b = 9(10a+b)$$
$$100x + 10a + b = 90a + 9b$$
$$100x = 80a + 8b$$
$$20a + 2b = 25x$$
$$\boxed{2(b+10a) = 25x}$$
$1 \leqslant x \leqslant 9$

$\therefore$ x MUST BE EVEN $\Rightarrow$ $x = 2,4,6,8$

- If $x=2$

$2(b+10a) = 50$
$b + 10a = 25$
$b = 25 - 10a$

If $a=1$, $b=15$
If $a=2$, $b=5$
If $a \geqslant 3$, $b<0$

- If $x=4$

$2(b+10a) = 100$
$b+10a = 50$
$b = 50 - 10a$

If $a=1$, $b=40$
If $a=2$, $b=30$
If $a=3$, $b=20$
If $a=4$, $b=10$
If $a=5$, $b=0$
If $a \geqslant 6$, $b<0$

- If $x=6$

$2(b+10a) = 25 \times 6$
$b+10a = 75$
$b = 75 - 10a$

If $a=1,2,3,\ldots 6$, $b>10$
If $a=7$, $b=5$
If $a=8,9$, $b<0$

- If $x=8$

$2(b+10a) = 25 \times 8$
$b+10a = 100$
$b = 100 - 10a$

If $a=1,2,\ldots,9$ $b>10$

$\therefore$ EITHER 25, so $225 = 9 \times 25$
50, so $450 = 9 \times 50$
75, so $675 = 9 \times 75$

Question 71 (*****)

When a man is asked how old he is, he replied.

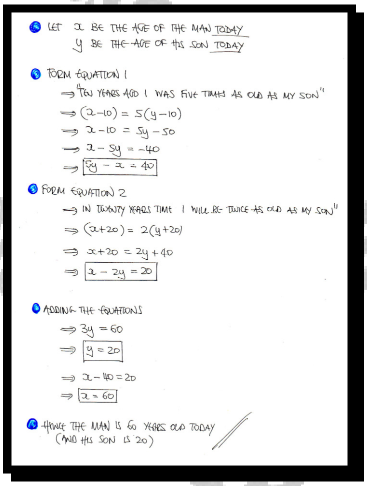
"Ten years ago I was five times as old as my son."

He continued …

"… in twenty years time I will be twice as old as my son."

Determine how old the man is.

$\boxed{\text{SP}}$, $\boxed{60 \text{ years old}}$



● LET x BE THE AGE OF THE MAN TODAY
y BE THE AGE OF HIS SON TODAY

● FORM EQUATION 1

$\Rightarrow$ "TEN YEARS AGO I WAS FIVE TIMES AS OLD AS MY SON"

$\Rightarrow (x-10) = 5(y-10)$

$\Rightarrow x-10 = 5y-50$

$\Rightarrow x-5y = -40$

$\Rightarrow \boxed{5y-x=40}$

● FORM EQUATION 2

$\Rightarrow$ IN TWENTY YEARS TIME I WILL BE TWICE AS OLD AS MY SON"

$\Rightarrow (x+20) = 2(y+20)$

$\Rightarrow x+20 = 2y+40$

$\Rightarrow \boxed{x-2y=20}$

● ADDING THE EQUATIONS

$\Rightarrow 3y = 60$

$\Rightarrow \boxed{y=20}$

$\Rightarrow x-40=20$

$\Rightarrow \boxed{x=60}$

● HENCE THE MAN IS 60 YEARS OLD TODAY
(AND HIS SON IS 20) //

Question 72 (*****)

When a man is asked how old he is, he replied.

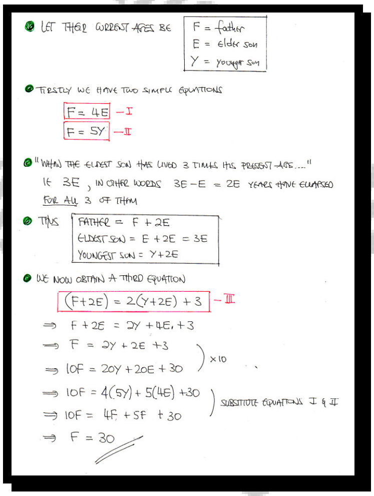
"I am four times as old as my eldest son and five times as old as my youngest son."

He continued …

"… when my eldest son is three times as old as he is now I will be exceeding twice my youngest son's age by three years."

Determine how old the man is.

$\boxed{\text{SP}}$, $\boxed{30 \text{ years old}}$



- LET THEIR CURRENT AGES BE

$$\boxed{\begin{array}{l} F = \text{father} \\ E = \text{elder son} \\ Y = \text{younger son} \end{array}}$$

- FIRSTLY WE HAVE TWO SIMPLE EQUATIONS

$$\boxed{F = 4E} \text{ — I}$$
$$\boxed{F = 5Y} \text{ — II}$$

- "WHEN THE ELDEST SON HAS LIVED 3 TIMES HIS PRESENT AGE…."

IE $3E$, IN OTHER WORDS $3E - E = 2E$ YEARS HAVE ELAPSED <u>FOR ALL</u> 3 OF THEM

- THUS

$$\boxed{\begin{array}{l} \text{FATHER} = F + 2E \\ \text{ELDEST SON} = E + 2E = 3E \\ \text{YOUNGEST SON} = Y + 2E \end{array}}$$

- WE NOW OBTAIN A THIRD EQUATION

$$\boxed{(F+2E) = 2(Y+2E) + 3} \text{ — III}$$

$$\Rightarrow F + 2E = 2Y + 4E + 3$$
$$\Rightarrow F = 2Y + 2E + 3$$
$$\Rightarrow 10F = 20Y + 20E + 30 \quad \Big) \times 10$$
$$\Rightarrow 10F = 4(5Y) + 5(4E) + 30$$
$$\Rightarrow 10F = 4F + 5F + 30 \quad \Big) \text{ SUBSTITUTE EQUATIONS I \& II}$$
$$\Rightarrow F = 30$$

Question 73 (*****)

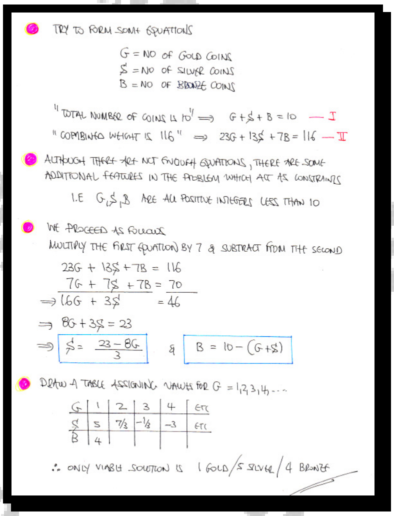
It is known that a box contains 10 coins of which some are gold, some are silver and some are bronze.

The combined weight of the 10 coins is 116 grams

Each gold coin weighs 23 grams, each silver coin weighs 13 grams and each bronze coin weighs 7 grams.

Determine the number of each type of coin.

$\boxed{\text{SP-B}}$, $\boxed{(G,S,B)=(1,5,4)}$



● TRY TO FORM SOME EQUATIONS

G = NO OF GOLD COINS
S = NO OF SILVER COINS
B = NO OF BRONZE COINS

"TOTAL NUMBER OF COINS IS 10" $\Rightarrow$ $G + S + B = 10$ — I
"COMBINED WEIGHT IS 116" $\Rightarrow$ $23G + 13S + 7B = 116$ — II

● ALTHOUGH THERE ARE NOT ENOUGH EQUATIONS, THERE ARE SOME ADDITIONAL FEATURES IN THE PROBLEM WHICH ACT AS CONSTRAINTS

I.E G, S, B ARE ALL POSITIVE INTEGERS LESS THAN 10

● WE PROCEED AS FOLLOWS
MULTIPLY THE FIRST EQUATION BY 7 & SUBTRACT FROM THE SECOND

$$\begin{aligned} 23G + 13S + 7B &= 116 \\ 7G + 7S + 7B &= 70 \\ \Rightarrow 16G + 3S &= 46 \end{aligned}$$

$$\Rightarrow 8G + 3S = 23$$

$$\Rightarrow \boxed{S = \frac{23 - 8G}{3}} \quad \& \quad \boxed{B = 10 - (G + S)}$$

● DRAW A TABLE ASSIGNING VALUES FOR $G = 1, 2, 3, 4, \ldots$

G	1	2	3	4	ETC
S	5	$7/3$	$-1/3$	-3	ETC
B	4				

$\therefore$ ONLY VIABLE SOLUTION IS 1 GOLD / 5 SILVER / 4 BRONZE

Question 74 (*****)

A water tank is full of water.

The tank has 3 outlet pipes, each having a constant drainage rate, when the water is allowed to flow out of the tank.

Let A, B and C be labels for each of the three outlet pipes.

If only A and B are turned on, it takes 12 hours to drain the tank.

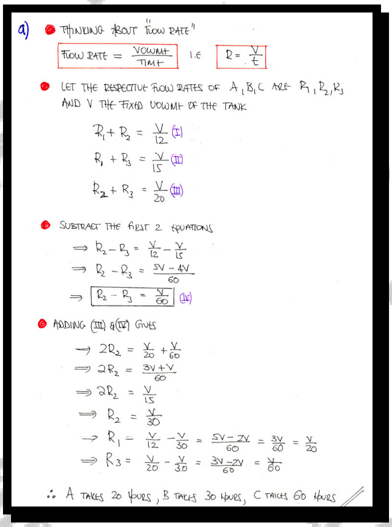
If only A and C are both turned on, it takes 15 hours to drain the tank.

If only B and C are both turned on, it takes 20 hours to drain the tank.

a) Find how long does each outlet pipe on its own take to drain a full tank.

b) Determine the time it takes to drain a full tank, if all three outlet pipes are turned on.

$\boxed{}$, $\boxed{(A,B,C) = (20,30,60) \text{ hours}}$, $\boxed{\text{10 hours}}$



a) ● THINKING ABOUT "FLOW RATE"

$\boxed{\text{FLOW RATE} = \frac{\text{VOLUME}}{\text{TIME}}}$ I.E $\boxed{R = \frac{V}{t}}$

● LET THE RESPECTIVE FLOW RATES OF A, B, C ARE R_1, R_2, R_3 AND V THE FIXED VOLUME OF THE TANK

$$R_1 + R_2 = \frac{V}{12} \quad (\text{I})$$

$$R_1 + R_3 = \frac{V}{15} \quad (\text{II})$$

$$R_2 + R_3 = \frac{V}{20} \quad (\text{III})$$

● SUBTRACT THE FIRST 2 EQUATIONS

$$\Rightarrow R_2 - R_3 = \frac{V}{12} - \frac{V}{15}$$

$$\Rightarrow R_2 - R_3 = \frac{5V - 4V}{60}$$

$$\Rightarrow \boxed{R_2 - R_3 = \frac{V}{60}} \quad (\text{IV})$$

● ADDING (III) & (IV) GIVES

$$\Rightarrow 2R_2 = \frac{V}{20} + \frac{V}{60}$$

$$\Rightarrow 2R_2 = \frac{3V + V}{60}$$

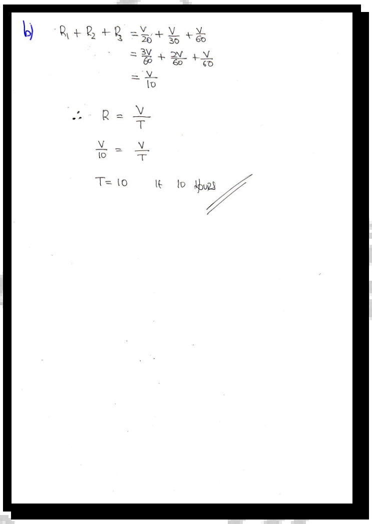
$$\Rightarrow 2R_2 = \frac{V}{15}$$

$$\Rightarrow R_2 = \frac{V}{30}$$

$$\Rightarrow R_1 = \frac{V}{12} - \frac{V}{30} = \frac{5V - 2V}{60} = \frac{3V}{60} = \frac{V}{20}$$

$$\Rightarrow R_3 = \frac{V}{20} - \frac{V}{30} = \frac{3V - 2V}{60} = \frac{V}{60}$$

∴ A TAKES 20 HOURS, B TAKES 30 HOURS, C TAKES 60 HOURS



b) $R_1 + R_2 + R_3 = \frac{V}{20} + \frac{V}{30} + \frac{V}{60}$

$= \frac{3V}{60} + \frac{2V}{60} + \frac{V}{60}$

$= \frac{V}{10}$

$\therefore \quad R = \frac{V}{T}$

$\frac{V}{10} = \frac{V}{T}$

$T = 10$ IE 10 HOURS

Question 75 (*****)

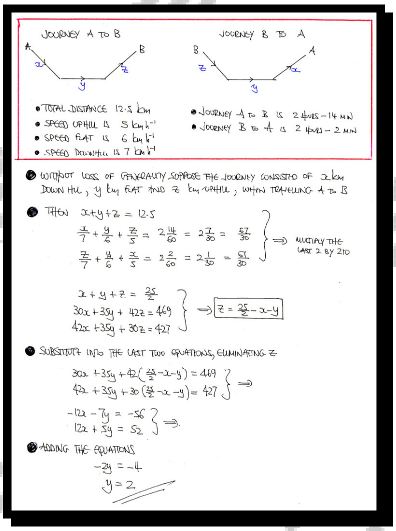
A man walked from his village to the nearby town in 2 hours and 14 minutes.

His return journey over the same route took him 2 hours and 2 minutes.

It is further known that the man always walks at 5 km h^{-1} uphill, at 6 km h^{-1} on flat ground and at 7 km h^{-1} downhill.

Given that the distance between the village and the town is 12.5 km, determine how long the flat distance between the village and the nearby town is.

$\boxed{\quad}$, $\boxed{2 \text{ km}}$



JOURNEY A TO B

JOURNEY B TO A

- TOTAL DISTANCE 12.5 km
- SPEED UPHILL IS 5 km h^{-1}
- SPEED FLAT IS 6 km h^{-1}
- SPEED DOWNHILL IS 7 km h^{-1}

- JOURNEY A TO B IS 2 HOURS – 14 MIN
- JOURNEY B TO A IS 2 HOURS – 2 MIN

- WITHOUT LOSS OF GENERALITY SUPPOSE THE JOURNEY CONSISTED OF x km DOWN HILL, y km FLAT AND z km UPHILL, WHEN TRAVELLING A TO B

- THEN

$$\left.\begin{aligned} x+y+z &= 12.5 \\ \frac{x}{7}+\frac{y}{6}+\frac{z}{5} &= 2\tfrac{14}{60} = 2\tfrac{7}{30} = \frac{67}{30} \\ \frac{z}{7}+\frac{y}{6}+\frac{x}{5} &= 2\tfrac{2}{60} = 2\tfrac{1}{30} = \frac{61}{30} \end{aligned}\right\} \Longrightarrow \text{MULTIPLY THE LAST 2 BY 210}$$

$$\left.\begin{aligned} x+y+z &= \frac{25}{2} \\ 30x+35y+42z &= 469 \\ 42x+35y+30z &= 427 \end{aligned}\right\} \Longrightarrow \boxed{z = \frac{25}{2}-x-y}$$

- SUBSTITUTE INTO THE LAST TWO EQUATIONS, ELIMINATING z

$$\left.\begin{aligned} 30x+35y+42\left(\frac{25}{2}-x-y\right) &= 469 \\ 42x+35y+30\left(\frac{25}{2}-x-y\right) &= 427 \end{aligned}\right\} \Longrightarrow$$

$$\left.\begin{aligned} -12x-7y &= -56 \\ 12x+5y &= 52 \end{aligned}\right\} \Longrightarrow$$

- ADDING THE EQUATIONS

$$-2y = -4$$

$$y = 2$$

Question 76 (*****)

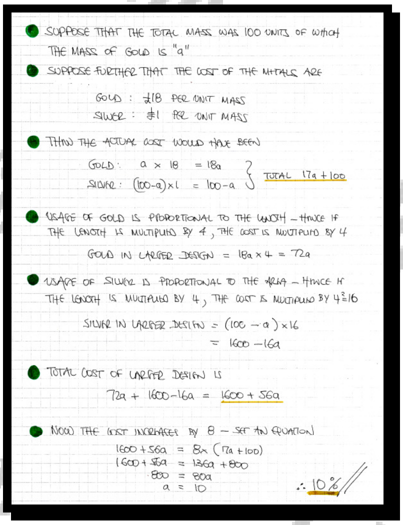
A square jewellery design is made of gold and silver.

The amount of gold used is proportional to the side of the square but the amount of silver used is proportional to the area of the square.

If the side of the square was to be enlarged by a factor of 8, the cost of the jewellery design would increase by a factor of 8.

Given that gold is 18 times more expensive than silver, determine the percentage of gold used in the standard design.

☐ , $\boxed{10\%}$



- SUPPOSE THAT THE TOTAL MASS WAS 100 UNITS OF WHICH THE MASS OF GOLD IS "a"
- SUPPOSE FURTHER THAT THE COST OF THE METALS ARE

GOLD : £18 PER UNIT MASS
SILVER : £1 PER UNIT MASS

- THEN THE ACTUAL COST WOULD HAVE BEEN

$$\left.\begin{array}{ll} \text{GOLD}: & a \times 18 = 18a \\ \text{SILVER}: & (100-a)\times 1 = 100-a \end{array}\right\} \text{TOTAL } 17a+100$$

- USAGE OF GOLD IS PROPORTIONAL TO THE LENGTH – HENCE IF THE LENGTH IS MULTIPLIED BY 4, THE COST IS MULTIPLIED BY 4

$$\text{GOLD IN LARGER DESIGN} = 18a \times 4 = 72a$$

- USAGE OF SILVER IS PROPORTIONAL TO THE AREA – HENCE IF THE LENGTH IS MULTIPLIED BY 4, THE COST IS MULTIPLIED BY $4^2 = 16$

$$\begin{aligned} \text{SILVER IN LARGER DESIGN} &= (100-a)\times 16 \\ &= 1600-16a \end{aligned}$$

- TOTAL COST OF LARGER DESIGN IS

$$72a + 1600 - 16a = 1600 + 56a$$

- NOW THE COST INCREASES BY 8 – SET AN EQUATION

$$\begin{aligned} 1600+56a &= 8\times(17a+100) \\ 1600+56a &= 136a+800 \\ 800 &= 80a \\ a &= 10 \end{aligned}$$

$\therefore 10\%$ //

Question 77 (*****)

Two walkers, A and B, start their walk at the point P, at the same time.

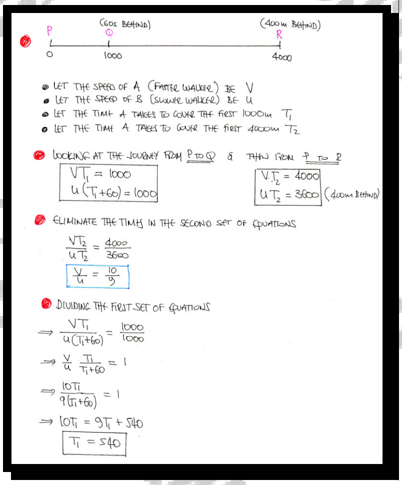
They both walk in the same direction along a straight road, each walker with different constant speed.

The points Q and R lies on that road so that $|PQ| = 1$ km and $|QR| = 3$ km.

- Walker B passes through Q 60 s after walker A passed through Q.
- When walker A passes through R, walker B is 400 m behind A.

Determine the speed of each of the two walkers, in $\text{km}\,\text{h}^{-1}$.

SP-Z, $\boxed{V_A = 6\frac{2}{3}\ \text{km}\,\text{h}^{-1}}$, $\boxed{V_B = 6\ \text{km}\,\text{h}^{-1}}$



- LET THE SPEED OF A (FASTER WALKER) BE V
- LET THE SPEED OF B (SLOWER WALKER) BE u
- LET THE TIME A TAKES TO COVER THE FIRST 1000m T_1
- LET THE TIME A TAKES TO COVER THE FIRST 4000m T_2

LOOKING AT THE JOURNEY FROM P TO Q & THEN FROM P TO R

$$\boxed{\begin{array}{l} VT_1 = 1000 \\ u(T_1+60) = 1000 \end{array}} \qquad \boxed{\begin{array}{l} V.T_2 = 4000 \\ uT_2 = 3600 \end{array}} \text{(400m BEHIND)}$$

ELIMINATE THE TIMES IN THE SECOND SET OF EQUATIONS

$$\frac{VT_2}{uT_2} = \frac{4000}{3600}$$

$$\boxed{\frac{V}{u} = \frac{10}{9}}$$

DIVIDING THE FIRST SET OF EQUATIONS

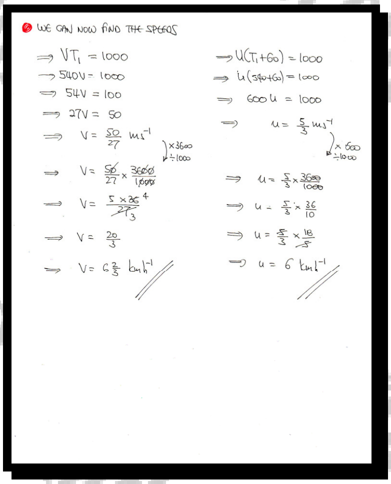
$$\Rightarrow \frac{VT_1}{u(T_1+60)} = \frac{1000}{1000}$$

$$\Rightarrow \frac{V}{u}\,\frac{T_1}{T_1+60} = 1$$

$$\Rightarrow \frac{10T_1}{9(T_1+60)} = 1$$

$$\Rightarrow 10T_1 = 9T_1 + 540$$

$$\boxed{T_1 = 540}$$



WE CAN NOW FIND THE SPEEDS

$$\Rightarrow VT_1 = 1000 \qquad \Rightarrow u(T_1+60) = 1000$$

$$\Rightarrow 540V = 1000 \qquad \Rightarrow u(540+60) = 1000$$

$$\Rightarrow 54V = 100 \qquad \Rightarrow 600u = 1000$$

$$\Rightarrow 27V = 50 \qquad \Rightarrow u = \frac{5}{3}\ \text{ms}^{-1}$$

$$\Rightarrow V = \frac{50}{27}\ \text{ms}^{-1}$$

×3600 ÷1000 ×600 ÷1000

$$\Rightarrow V = \frac{\cancel{50}}{27} \times \frac{36\cancel{00}}{1\cancel{000}} \qquad \Rightarrow u = \frac{5}{3} \times \frac{3600}{1000}$$

$$\Rightarrow V = \frac{5 \times \overset{4}{\cancel{36}}}{\underset{3}{\cancel{27}}} \qquad \Rightarrow u = \frac{5}{3} \times \frac{36}{10}$$

$$\Rightarrow V = \frac{20}{3} \qquad \Rightarrow u = \frac{\cancel{5}}{3} \times \frac{18}{\cancel{5}}$$

$$\Rightarrow V = 6\tfrac{2}{3}\ \text{kmh}^{-1} \qquad \Rightarrow u = 6\ \text{kmh}^{-1}$$

Question 78 (*****)

Two thin rigid vertical poles AB and CD are standing on level horizontal ground.

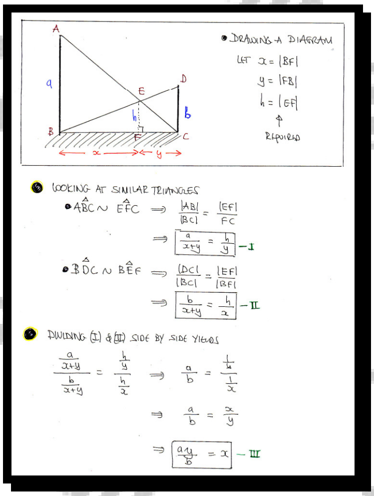
- AB has length a and the point B is level with the ground.
- CD has length b, $b < a$, and the point C is level with the ground.

A taut string is connecting A to C and another taut string is connecting B to D.

The two strings cross each other at the point E.

Find, in terms of a and b, the vertical height of E above the ground.

$\boxed{\text{SP-N}}$, $\boxed{h = \dfrac{ab}{a+b}}$



● DRAWING A DIAGRAM

LET $x = |BF|$

$y = |FB|$

$h = |EF|$

↑

REQUIRED

● LOOKING AT SIMILAR TRIANGLES

● $\hat{ABC} \sim \hat{EFC} \Rightarrow \frac{|AB|}{|BC|} = \frac{|EF|}{FC}$

$\Rightarrow \boxed{\frac{a}{x+y} = \frac{h}{y}}$ — I

● $\hat{BDC} \sim \hat{BEF} \Rightarrow \frac{|DC|}{|BC|} = \frac{|EF|}{|BF|}$

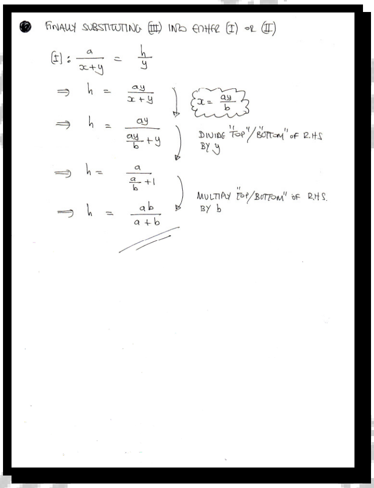
$\Rightarrow \boxed{\frac{b}{x+y} = \frac{h}{x}}$ — II

● DIVIDING (I) & (II) SIDE BY SIDE YIELDS

$$\frac{\frac{a}{x+y}}{\frac{b}{x+y}} = \frac{\frac{h}{y}}{\frac{h}{x}} \Rightarrow \frac{a}{b} = \frac{\frac{1}{y}}{\frac{1}{x}}$$

$$\Rightarrow \frac{a}{b} = \frac{x}{y}$$

$$\Rightarrow \boxed{\frac{ay}{b} = x} \text{ — III}$$



● FINALLY SUBSTITUTING (III) INTO EITHER (I) OR (II)

$$(I): \frac{a}{x+y} = \frac{h}{y}$$

$$\Rightarrow h = \frac{ay}{x+y}$$

$\left\{x = \frac{ay}{b}\right\}$

$$\Rightarrow h = \frac{ay}{\frac{ay}{b}+y}$$

DIVIDE "TOP"/"BOTTOM" OF R.H.S BY y

$$\Rightarrow h = \frac{a}{\frac{a}{b}+1}$$

MULTIPLY "TOP/BOTTOM" OF R.H.S. BY b

$$\Rightarrow h = \frac{ab}{a+b}$$

Question 79 (*****)

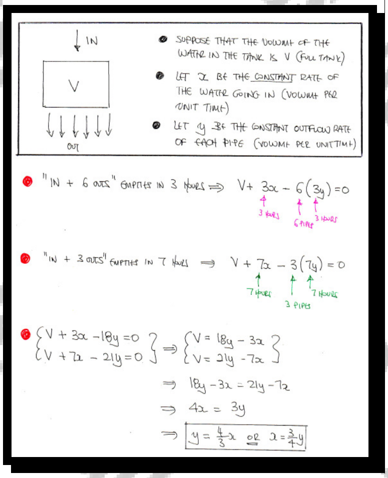
A water tank is fed by one inlet pipe which feeds into the tank at constant rate

The tank has 6 outlet pipes, each having the same constant drainage rate. The drainage rate of one of the outlet pipes is greater than the inflow rate of the inlet pipe.

- When the inlet pipe and all 6 outlet pipes are turned on, it takes 3 hours to empty the full tank.
- When the inlet pipe and 3 outlet pipes are turned on, it takes 7 hours to empty the full tank.

Determine the number of hours it takes to empty a full tank with the inlet pipe and just one of the outlet pipes turned on.

$\boxed{\text{SP-A}}$, $\boxed{63 \text{ hours}}$



- SUPPOSE THAT THE VOLUME OF THE WATER IN THE TANK IS V (FULL TANK)
- LET x BE THE CONSTANT RATE OF THE WATER GOING IN (VOLUME PER UNIT TIME)
- LET y BE THE CONSTANT OUTFLOW RATE OF EACH PIPE (VOLUME PER UNIT TIME)

● "IN + 6 OUTS" EMPTIES IN 3 HOURS $\Rightarrow V + 3x - 6(3y) = 0$

3 HOURS 6 PIPES 3 HOURS

● "IN + 3 OUTS" EMPTIES IN 7 HOURS $\Rightarrow V + 7x - 3(7y) = 0$

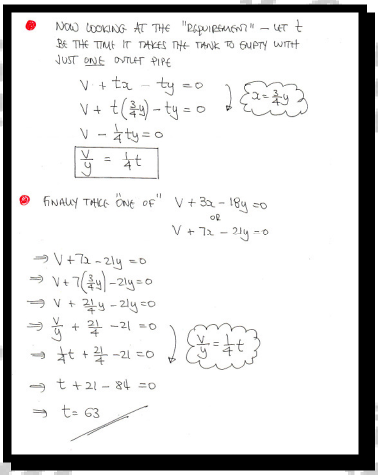
7 HOURS 3 PIPES 7 HOURS

● $\left\{\begin{array}{l} V + 3x - 18y = 0 \\ V + 7x - 21y = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} V = 18y - 3x \\ V = 21y - 7x \end{array}\right\}$

$\Rightarrow 18y - 3x = 21y - 7x$

$\Rightarrow 4x = 3y$

$\Rightarrow \boxed{y = \frac{4}{3}x \text{ or } x = \frac{3}{4}y}$



● NOW LOOKING AT THE "REQUIREMENT" – LET t BE THE TIME IT TAKES THE TANK TO EMPTY WITH JUST ONE OUTLET PIPE

$V + tx - ty = 0$

$V + t\left(\frac{3}{4}y\right) - ty = 0$ $\quad x = \frac{3}{4}y$

$V - \frac{1}{4}ty = 0$

$\boxed{\frac{V}{y} = \frac{1}{4}t}$

● FINALLY TAKE "ONE OF" $V + 3x - 18y = 0$ OR $V + 7x - 21y = 0$

$\Rightarrow V + 7x - 21y = 0$

$\Rightarrow V + 7\left(\frac{3}{4}y\right) - 21y = 0$

$\Rightarrow V + \frac{21}{4}y - 21y = 0$

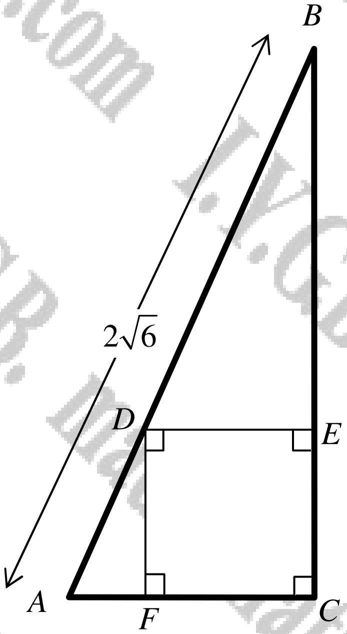
$\Rightarrow \frac{V}{y} + \frac{21}{4} - 21 = 0$

$\Rightarrow \frac{1}{4}t + \frac{21}{4} - 21 = 0$ $\quad \frac{V}{y} = \frac{1}{4}t$

$\Rightarrow t + 21 - 84 = 0$

$\Rightarrow t = 63$

Question 80 (*****)

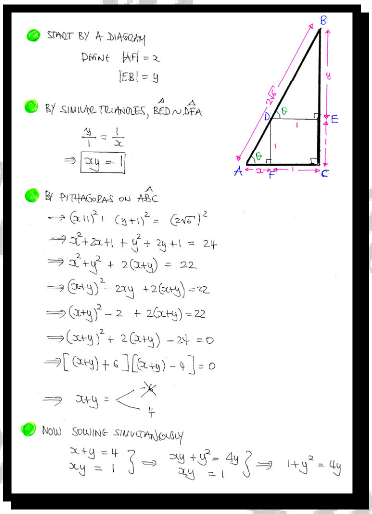


The figure above shows a right angled triangle ABC, where $|AB| = 2\sqrt{6}$.

A square $DECF$, of side length 1, is drawn inside ABC, so that D lies on AB, E lies on BC and F lies on AC.

Determine, in exact simplified surd form, the possible values of the tangent of the angle BAC.

$\boxed{}$, $\boxed{2 \pm \sqrt{3}}$



- START BY A DIAGRAM

DEFINE $|AF| = x$

$|EB| = y$

- BY SIMILAR TRIANGLES, $\overset{\triangle}{BED} \sim \overset{\triangle}{DFA}$

$$\frac{y}{1} = \frac{1}{x}$$

$$\Rightarrow \boxed{xy = 1}$$

- BY PYTHAGORAS ON $\overset{\triangle}{ABC}$

$$\Rightarrow (x+1)^2 + (y+1)^2 = (2\sqrt{6})^2$$

$$\Rightarrow x^2 + 2x + 1 + y^2 + 2y + 1 = 24$$

$$\Rightarrow x^2 + y^2 + 2(x+y) = 22$$

$$\Rightarrow (x+y)^2 - 2xy + 2(x+y) = 22$$

$$\Rightarrow (x+y)^2 - 2 + 2(x+y) = 22$$

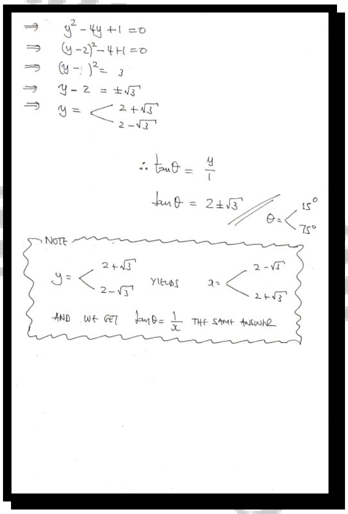
$$\Rightarrow (x+y)^2 + 2(x+y) - 24 = 0$$

$$\Rightarrow [(x+y)+6][(x+y)-4] = 0$$

$$\Rightarrow x+y = \left\langle \begin{array}{l} \cancel{-6} \\ 4 \end{array} \right.$$

- NOW SOLVING SIMULTANEOUSLY

$$\left.\begin{array}{l} x+y = 4 \\ xy = 1 \end{array}\right\} \Rightarrow \left.\begin{array}{l} xy + y^2 = 4y \\ xy = 1 \end{array}\right\} \Rightarrow 1 + y^2 = 4y$$



$$\Rightarrow y^2 - 4y + 1 = 0$$

$$\Rightarrow (y-2)^2 - 4 + 1 = 0$$

$$\Rightarrow (y-2)^2 = 3$$

$$\Rightarrow y - 2 = \pm\sqrt{3}$$

$$\Rightarrow y = \left\langle \begin{array}{l} 2+\sqrt{3} \\ 2-\sqrt{3} \end{array} \right.$$

$$\therefore \tan\theta = \frac{y}{1}$$

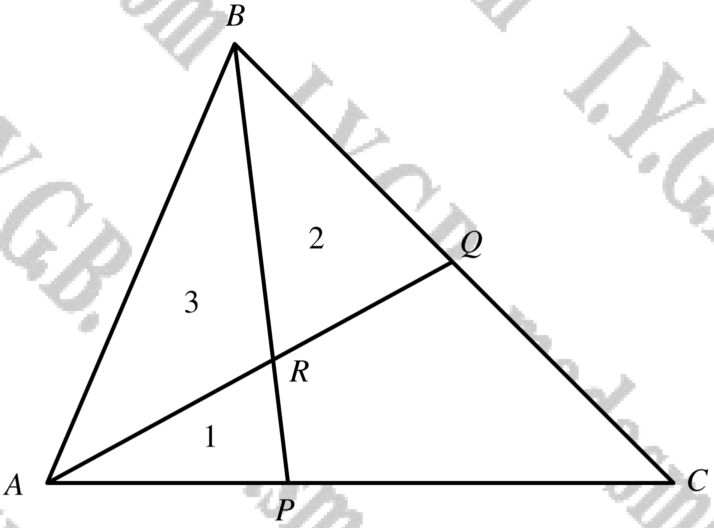
$$\tan\theta = 2 \pm \sqrt{3} \quad \theta = \left\langle \begin{array}{l} 15^\circ \\ 75^\circ \end{array} \right.$$

NOTE

$$y = \left\langle \begin{array}{l} 2+\sqrt{3} \\ 2-\sqrt{3} \end{array} \right. \text{ YIELDS } x = \left\langle \begin{array}{l} 2-\sqrt{3} \\ 2+\sqrt{3} \end{array} \right.$$

AND WE GET $\tan\theta = \frac{1}{x}$ THE SAME ANSWER

Question 81 (*****)



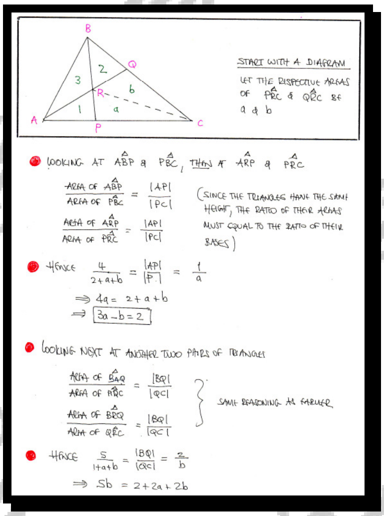
The figure above shows a triangle ABC.

The point P lies on AC and the point Q lies on BC.

The point R is the intersection of BP and AQ.

Given that the respective areas of the triangles APR, BQR and ABR are 1, 2 and 3 square units, determine the exact area of the quadrilateral $CPRQ$.

$\boxed{}$, $\boxed{\frac{18}{7}}$



START WITH A DIAGRAM

LET THE RESPECTIVE AREAS OF $\overset{\triangle}{PRC}$ & $\overset{\triangle}{QRC}$ BE a & b

- LOOKING AT $\overset{\triangle}{ABP}$ & $\overset{\triangle}{PBC}$, THEN AT $\overset{\triangle}{ARP}$ & $\overset{\triangle}{PRC}$

$$\frac{\text{AREA OF } \overset{\triangle}{ABP}}{\text{AREA OF } \overset{\triangle}{PBC}} = \frac{|AP|}{|PC|}$$

$$\frac{\text{AREA OF } \overset{\triangle}{ARP}}{\text{AREA OF } \overset{\triangle}{PRC}} = \frac{|AP|}{|PC|}$$

(SINCE THE TRIANGLES HAVE THE SAME HEIGHT, THE RATIO OF THEIR AREAS MUST EQUAL TO THE RATIO OF THEIR BASES)

- HENCE $\frac{4}{2+a+b} = \frac{|AP|}{|P|} = \frac{1}{a}$

$$\Rightarrow 4a = 2+a+b$$

$$\Rightarrow \boxed{3a-b=2}$$

- LOOKING NEXT AT ANOTHER TWO PAIRS OF TRIANGLES

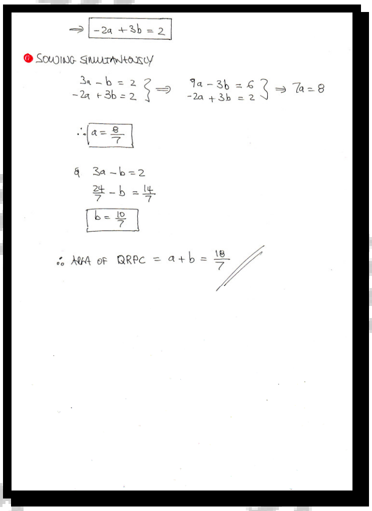
$$\frac{\text{AREA OF } \overset{\triangle}{BAQ}}{\text{AREA OF } \overset{\triangle}{AQC}} = \frac{|BQ|}{|QC|}$$

$$\frac{\text{AREA OF } \overset{\triangle}{BRQ}}{\text{AREA OF } \overset{\triangle}{QRC}} = \frac{|BQ|}{|QC|}$$

SAME REASONING AS EARLIER

- HENCE $\frac{5}{1+a+b} = \frac{|BQ|}{|QC|} = \frac{2}{b}$

$$\Rightarrow 5b = 2+2a+2b$$



$$\Rightarrow \boxed{-2a+3b=2}$$

- SOLVING SIMULTANEOUSLY

$$\left.\begin{array}{r} 3a-b=2 \\ -2a+3b=2 \end{array}\right\} \Rightarrow \left.\begin{array}{r} 9a-3b=6 \\ -2a+3b=2 \end{array}\right\} \Rightarrow 7a=8$$

$$\therefore \boxed{a=\frac{8}{7}}$$

& $3a-b=2$

$$\frac{24}{7}-b=\frac{14}{7}$$

$$\boxed{b=\frac{10}{7}}$$

$\therefore$ AREA OF QRPC $= a+b = \frac{18}{7}$ //