POLYNOMIAL EXAM QUESTIONS

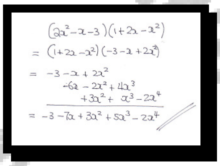
Question 1 **(**)**

Multiply out and simplify

$$\left(2x^2-x-3\right)\left(1+2x-x^2\right),$$

writing the answer in ascending powers of x.

$$\boxed{-3-7x+3x^2+5x^3-2x^4}$$



$(2x^2-x-3)(1+2x-x^2)$

$= (1+2x-x^2)(-3-x+2x^2)$

$= -3-x+2x^2$
$\quad -6x-2x^2+4x^3$
$\quad +3x^2+x^3-2x^4$

$= -3-7x+3x^2+5x^3-2x^4$

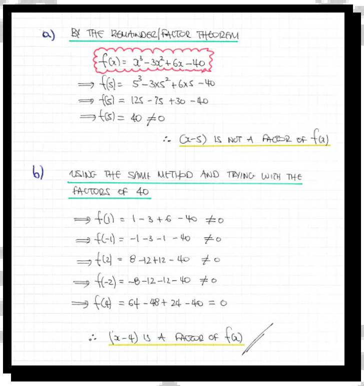
Question 2 **(**)**

$$f(x) \equiv x^3-3x^2+6x-40.$$

a) Show that $(x-5)$ is **not** a factor of $f(x)$.

b) Find a linear factor of $f(x)$.

$\boxed{\quad\quad}$, $\boxed{(x-4)}$



a) BY THE REMAINDER/FACTOR THEOREM

$f(x) = x^3-3x^2+6x-40$

$\Rightarrow f(5) = 5^3-3\times5^2+6\times5-40$

$\Rightarrow f(5) = 125-75+30-40$

$\Rightarrow f(5) = 40 \neq 0$

$\therefore$ $(x-5)$ IS NOT A FACTOR OF $f(x)$

b) USING THE SAME METHOD AND TRYING WITH THE FACTORS OF 40

$\Rightarrow f(1) = 1-3+6-40 \neq 0$

$\Rightarrow f(-1) = -1-3-1-40 \neq 0$

$\Rightarrow f(2) = 8-12+12-40 \neq 0$

$\Rightarrow f(-2) = -8-12-12-40 \neq 0$

$\Rightarrow f(4) = 64-48+24-40 = 0$

$\therefore$ $(x-4)$ IS A FACTOR OF $f(x)$

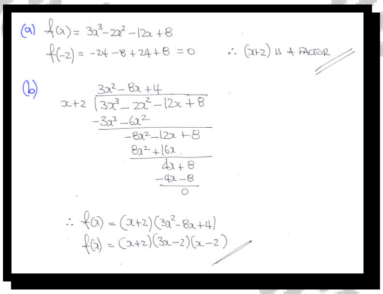
Question 3 (**)

The polynomial $3x^3 - 2x^2 - 12x + 8$ is denoted by $f(x)$.

a) Use the factor theorem to show that $(x+2)$ is a factor of $f(x)$.

b) Factorize $f(x)$ fully.

☐ , $\boxed{f(x) = (3x-2)(x-2)(x+2)}$



(a) $f(x) = 3x^3 - 2x^2 - 12x + 8$

$f(-2) = -24 - 8 + 24 + 8 = 0$ $\quad \therefore (x+2)$ IS A FACTOR //

(b)

$$\begin{array}{r|l} & 3x^2 - 8x + 4 \\ x+2 & 3x^3 - 2x^2 - 12x + 8 \\ & -3x^3 - 6x^2 \\ \hline & \quad -8x^2 - 12x + 8 \\ & \quad 8x^2 + 16x \\ \hline & \qquad 4x + 8 \\ & \qquad -4x - 8 \\ \hline & \qquad\quad 0 \end{array}$$

$\therefore f(x) = (x+2)(3x^2 - 8x + 4)$

$f(x) = (x+2)(3x-2)(x-2)$ //

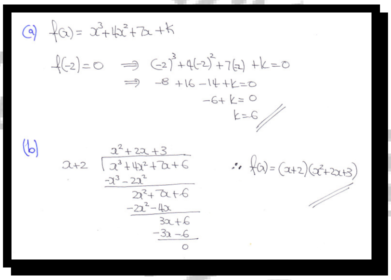
Question 4 (**)

The polynomial $x^3 + 4x^2 + 7x + k$, where k is a constant, is denoted by $f(x)$.

a) Given that $(x+2)$ is a factor of $f(x)$, show that $k = 6$.

b) Express $f(x)$ as a product of a linear factor and a quadratic factor.

☐ , $\boxed{f(x) = (x+2)(x^2 + 2x + 3)}$



(a) $f(x) = x^3 + 4x^2 + 7x + k$

$f(-2) = 0 \Rightarrow (-2)^3 + 4(-2)^2 + 7(-2) + k = 0$

$\Rightarrow -8 + 16 - 14 + k = 0$

$-6 + k = 0$

$k = 6$ //

(b)

$$\begin{array}{r|l} & x^2 + 2x + 3 \\ x+2 & x^3 + 4x^2 + 7x + 6 \\ & -x^3 - 2x^2 \\ \hline & \quad 2x^2 + 7x + 6 \\ & \quad -2x^2 - 4x \\ \hline & \qquad 3x + 6 \\ & \qquad -3x - 6 \\ \hline & \qquad\quad 0 \end{array}$$

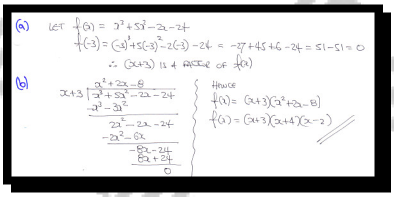
$\therefore f(x) = (x+2)(x^2 + 2x + 3)$ //

Question 5 **(**)**

a) Use the factor theorem to show that $(x+3)$ is a factor of $x^3+5x^2-2x-24$.

b) Factorize $x^3+5x^2-2x-24$ fully.

$$\boxed{(x+3)(x-2)(x+4)}$$



(a) LET $f(x) = x^3+5x^2-2x-24$

$f(-3) = (-3)^3+5(-3)^2-2(-3)-24 = -27+45+6-24 = 51-51 = 0$

$\therefore$ $(x+3)$ IS A FACTOR OF $f(x)$

(b)

$$\begin{array}{r|l} & x^2+2x-8 \\ \hline x+3 & x^3+5x^2-2x-24 \\ & -x^3-3x^2 \\ \hline & 2x^2-2x-24 \\ & -2x^2-6x \\ \hline & -8x-24 \\ & 8x+24 \\ \hline & 0 \end{array}$$

HENCE

$f(x) = (x+3)(x^2+2x-8)$

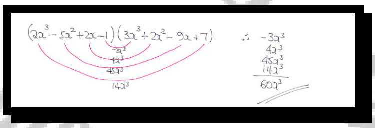
$f(x) = (x+3)(x+4)(x-2)$ //

Question 6 **(**+)**

Find the coefficient of x^3 in the expansion of

$$\left(2x^3-5x^2+2x-1\right)\left(3x^3+2x^2-9x+7\right).$$

$$\boxed{...+60x^3...}$$



$(2x^3-5x^2+2x-1)(3x^3+2x^2-9x+7)$

$-3x^3$

$4x^3$

$45x^3$

$14x^3$

$\therefore$ $-3x^3$
$4x^3$
$45x^3$
$14x^3$
$60x^3$ //

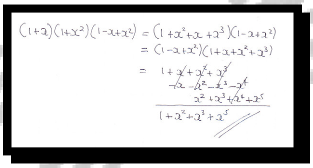
Question 7 **(**+)**

Multiply out and simplify

$$(1+x)\left(1+x^2\right)\left(1-x+x^2\right),$$

writing the answer in ascending powers of x.

$$\boxed{1+x^2+x^3+x^5}$$



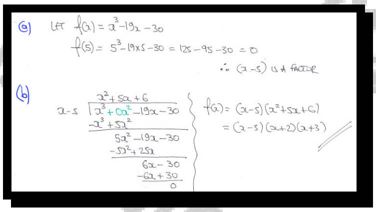
$$\begin{aligned}(1+x)(1+x^2)(1-x+x^2) &= (1+x^2+x+x^3)(1-x+x^2)\\ &= (1-x+x^2)(1+x+x^2+x^3)\\ &= 1+\cancel{x}+\cancel{x^2}+\cancel{x^3}\\ &\quad\;\; -\cancel{x}-\cancel{x^2}-\cancel{x^3}-\cancel{x^4}\\ &\quad\;\; x^2+x^3+\cancel{x^4}+x^5\\ \hline &\quad 1+x^2+x^3+x^5\end{aligned}$$

Question 8 **(**+)**

a) Use the factor theorem to show that $(x-5)$ is a factor of $x^3-19x-30$.

b) Factorize $x^3-19x-30$ into three linear factors.

$$\boxed{(x+3)(x+2)(x-5)}$$



(a) LET $f(x)=x^3-19x-30$

$f(5)=5^3-19\times5-30=125-95-30=0$

$\therefore (x-5)$ IS A FACTOR

(b)

$$\begin{array}{r|l} & x^2+5x+6 \\ \hline x-5 & x^3+0x^2-19x-30 \\ & -x^3+5x^2 \\ \hline & \quad 5x^2-19x-30 \\ & \quad -5x^2+25x \\ \hline & \qquad 6x-30 \\ & \qquad -6x+30 \\ \hline & \qquad\quad 0 \end{array}$$

$f(x)=(x-5)(x^2+5x+6)$

$=(x-5)(x+2)(x+3)$

Question 9 **(**+)**

$$f(x) \equiv ax^3 - x^2 - 5x + b,$$

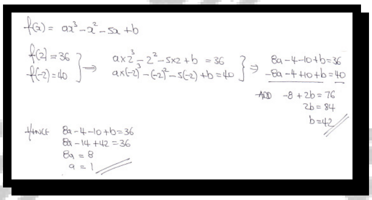
where a and b are constants.

When $f(x)$ is divided by $(x-2)$ the remainder is 36.

When $f(x)$ is divided by $(x+2)$ the remainder is 40.

Find the value of a and the value of b.

$\boxed{\text{C2B}}$, $\boxed{a=1}$, $\boxed{b=42}$



$f(x) = ax^3 - x^2 - 5x + b$

$f(2) = 36$, $f(-2) = 40$ $\Big\}\Rightarrow$ $a \times 2^3 - 2^2 - 5\times 2 + b = 36$, $a\times(-2)^3 - (-2)^2 - 5(-2) + b = 40$ $\Big\}\Rightarrow$ $8a - 4 - 10 + b = 36$, $-8a - 4 + 10 + b = 40$

ADD $-8 + 2b = 76$

$2b = 84$

$b = 42$ //

HENCE $8a - 4 - 10 + b = 36$

$8a - 14 + 42 = 36$

$8a = 8$

$a = 1$ //

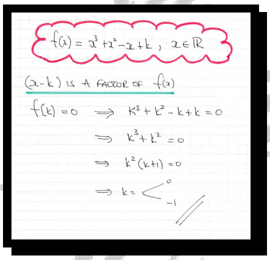
Question 10 **(**+)**

A cubic function is defined in terms of the constant k as

$$f(x) \equiv x^3 + x^2 - x + k, \; x \in \mathbb{R}.$$

Given that $(x-k)$ is a factor of $f(x)$ determine the possible values of k.

$\boxed{\text{N-A}}$, $\boxed{k = -1,\ 0}$



$f(x) = x^3 + x^2 - x + k, \; x \in \mathbb{R}$

$(x-k)$ IS A FACTOR OF $f(x)$

$f(k) = 0 \Rightarrow k^3 + k^2 - k + k = 0$

$\Rightarrow k^3 + k^2 = 0$

$\Rightarrow k^2(k+1) = 0$

$\Rightarrow k = \begin{cases} 0 \\ -1 \end{cases}$ //

Question 11 (**+)

$$f(x) \equiv x^3 - 2x^2 + kx + 6,$$

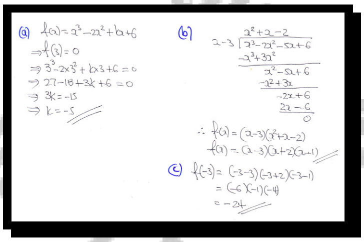
where k is a constant.

a) Given that $(x-3)$ is a factor of $f(x)$, show that $k = -5$.

b) Factorize $f(x)$ into three linear factors.

c) Find the remainder when $f(x)$ is divided by $(x+3)$.

$\boxed{}$, $\boxed{(x-1)(x+2)(x-3)}$, $\boxed{R = -24}$



(a) $f(x) = x^3 - 2x^2 + kx + 6$

$\Rightarrow f(3) = 0$

$\Rightarrow 3^3 - 2 \times 3^2 + k \times 3 + 6 = 0$

$\Rightarrow 27 - 18 + 3k + 6 = 0$

$\Rightarrow 3k = -15$

$\Rightarrow k = -5$

(b)

$$\begin{array}{r|l} & x^2 + x - 2 \\ x-3 & x^3 - 2x^2 - 5x + 6 \\ & -x^3 + 3x^2 \\ \hline & x^2 - 5x + 6 \\ & -x^2 + 3x \\ \hline & -2x + 6 \\ & 2x - 6 \\ \hline & 0 \end{array}$$

$\therefore f(x) = (x-3)(x^2+x-2)$

$f(x) = (x-3)(x+2)(x-1)$

(c) $f(-3) = (-3-3)(-3+2)(-3-1)$

$= (-6)(-1)(-4)$

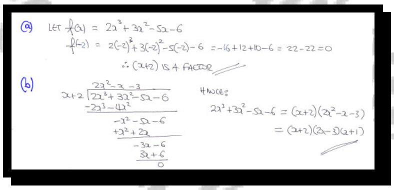
$= -24$

Question 12 (**+)

a) Use the factor theorem to show that $(x+2)$ is a factor of $2x^3 + 3x^2 - 5x - 6$.

b) Factorize $2x^3 + 3x^2 - 5x - 6$ into three linear factors.

$\boxed{(x+1)(x+2)(2x-3)}$



(a) LET $f(x) = 2x^3 + 3x^2 - 5x - 6$

$f(-2) = 2(-2)^3 + 3(-2)^2 - 5(-2) - 6 = -16 + 12 + 10 - 6 = 22 - 22 = 0$

$\therefore (x+2)$ IS A FACTOR

(b)

$$\begin{array}{r|l} & 2x^2 - x - 3 \\ x+2 & 2x^3 + 3x^2 - 5x - 6 \\ & -2x^3 - 4x^2 \\ \hline & -x^2 - 5x - 6 \\ & +x^2 + 2x \\ \hline & -3x - 6 \\ & 3x + 6 \\ \hline & 0 \end{array}$$

HENCE:

$2x^3 + 3x^2 - 5x - 6 = (x+2)(2x^2 - x - 3)$

$= (x+2)(2x-3)(x+1)$

Question 13 **(**+)**

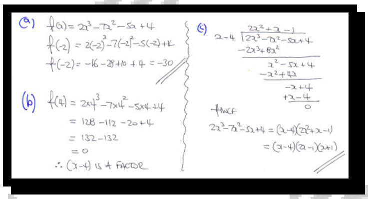
$$f(x) \equiv 2x^3 - 7x^2 - 5x + 4$$

a) Find the remainder when $f(x)$ is divided by $(x+2)$.

b) Use the factor theorem to show that $(x-4)$ is a factor of $f(x)$.

c) Factorize $f(x)$ completely.

$\boxed{}$, $\boxed{R=-30}$, $\boxed{(2x-1)(x+1)(x-4)}$



(a) $f(x) = 2x^3 - 7x^2 - 5x + 4$

$f(-2) = 2(-2)^3 - 7(-2)^2 - 5(-2) + 4$

$f(-2) = -16 - 28 + 10 + 4 = -30$

(b) $f(4) = 2\times4^3 - 7\times4^2 - 5\times4 + 4$

$= 128 - 112 - 20 + 4$

$= 132 - 132$

$= 0$

$\therefore (x-4)$ IS A FACTOR

(c)

$$\begin{array}{r|l} & 2x^2 + x - 1 \\ x-4 & 2x^3 - 7x^2 - 5x + 4 \\ & -2x^3 + 8x^2 \\ \hline & x^2 - 5x + 4 \\ & -x^2 + 4x \\ \hline & -x + 4 \\ & +x - 4 \\ \hline & 0 \end{array}$$

HENCE

$2x^3 - 7x^2 - 5x + 4 = (x-4)(2x^2 + x - 1)$

$= (x-4)(2x-1)(x+1)$

Question 14 (**+)

$$f(x) \equiv x^3 + x^2 + ax + b,$$

where a and b are constants

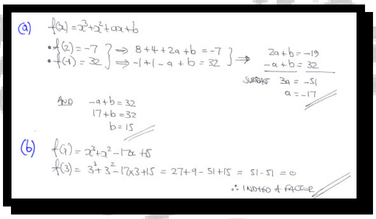
When $f(x)$ is divided by $(x-2)$ the remainder is -7

When $f(x)$ is divided by $(x+1)$ the remainder is 32

a) Find the value of a and the value of b .

b) Show that $(x-3)$ is a factor of $f(x)$.

$\boxed{a=-17}$, $\boxed{b=15}$



(a) $f(x) = x^3+x^2+ax+b$

$\left.\begin{array}{l} \bullet f(2) = -7 \\ \bullet f(-1) = 32 \end{array}\right\} \begin{array}{l} \Rightarrow 8+4+2a+b = -7 \\ \Rightarrow -1+1-a+b = 32 \end{array}\Bigg\} \Rightarrow \begin{array}{r} 2a+b=-19 \\ \underline{-a+b=32} \end{array}$

SUBTRACT $3a = -51$

$a = -17$

AND $-a+b=32$

$17+b=32$

$b=15$

(b) $f(x) = x^3+x^2-17x+15$

$f(3) = 3^3+3^2-17\times3+15 = 27+9-51+15 = 51-51 = 0$

$\therefore$ INDEED A FACTOR

Question 15 **(**+)**

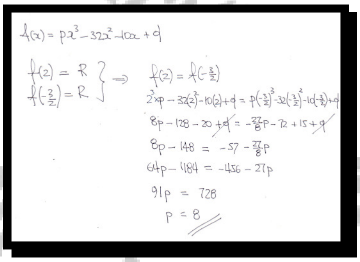
$$f(x) \equiv px^3 - 32x^2 - 10x + q\,,$$

where p and q are constants.

When $f(x)$ is divided by $(x-2)$ the remainder is exactly the same as when $f(x)$ is divided by $(2x+3)$.

Show clearly that $p = 8$.

$\boxed{}$, $\boxed{\text{proof}}$



$f(x) = px^3 - 32x^2 - 10x + q$

$\left.\begin{array}{l} f(2) = R \\ f\left(-\frac{3}{2}\right) = R \end{array}\right\} \Rightarrow f(2) = f\left(-\frac{3}{2}\right)$

$2^3 \times p - 32(2)^2 - 10(2) + q = p\left(-\frac{3}{2}\right)^3 - 32\left(-\frac{3}{2}\right)^2 - 10\left(-\frac{3}{2}\right) + q$

$8p - 128 - 20 \cancel{+q} = -\frac{27}{8}p - 72 + 15 \cancel{+q}$

$8p - 148 = -57 - \frac{27}{8}p$

$64p - 1184 = -456 - 27p$

$91p = 728$

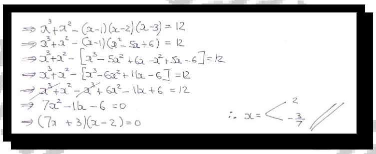
$p = 8$

Question 16 **(***)**

Solve the equation

$$x^3 + x^2 - (x-1)(x-2)(x-3) = 12\,.$$

$\boxed{x = -\frac{3}{7}, 2}$



$\Rightarrow x^3 + x^2 - (x-1)(x-2)(x-3) = 12$

$\Rightarrow x^3 + x^2 - (x-1)(x^2 - 5x + 6) = 12$

$\Rightarrow x^3 + x^2 - \left[x^3 - 5x^2 + 6x - x^2 + 5x - 6\right] = 12$

$\Rightarrow x^3 + x^2 - \left[x^3 - 6x^2 + 11x - 6\right] = 12$

$\Rightarrow \cancel{x^3} + x^2 - \cancel{x^3} + 6x^2 - 11x + 6 = 12$

$\Rightarrow 7x^2 - 11x - 6 = 0$

$\Rightarrow (7x + 3)(x - 2) = 0$

$\therefore x = \begin{cases} 2 \\ -\frac{3}{7} \end{cases}$

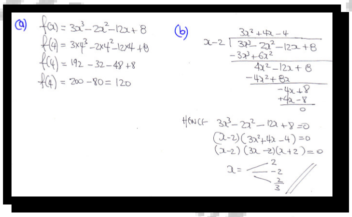
Question 17 **(***)**

$$f(x) \equiv 3x^3 - 2x^2 - 12x + 8.$$

a) Find the remainder when $f(x)$ is divided by $(x-4)$.

b) Given that $(x-2)$ is a factor of $f(x)$ solve the equation $f(x)=0$.

, $\boxed{R=120}$, $\boxed{x=-2,\frac{2}{3},2}$



(a) $f(x) = 3x^3-2x^2-12x+8$

$f(4) = 3\times4^3-2\times4^2-12\times4+8$

$f(4) = 192-32-48+8$

$f(4) = 200-80=120$

(b)

$$\begin{array}{r} 3x^2+4x-4 \\ x-2\,\overline{)\,3x^3-2x^2-12x+8} \\ \underline{-3x^3+6x^2} \\ 4x^2-12x+8 \\ \underline{-4x^2+8x} \\ -4x+8 \\ \underline{+4x-8} \\ 0 \end{array}$$

HENCE $3x^3-2x^2-12x+8=0$

$(x-2)(3x^2+4x-4)=0$

$(x-2)(3x-2)(x+2)=0$

$x = \begin{cases} 2 \\ -2 \\ \frac{2}{3} \end{cases}$ //

Question 18 (***)

It is given that

$$f(x) \equiv 2x^3 + 3x^2 - 8x + c\,,$$

where c is a non zero constant

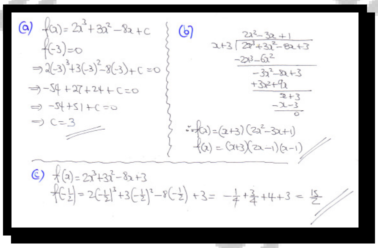
It is further given that $f(-3) = 0$

a) Show that $c = 3$.

b) Factorize $f(x)$ fully.

c) Find the remainder when $f(x)$ is divided by $(2x+1)$

$\boxed{(2x-1)(x-1)(x+3)}$, $\boxed{R = \frac{15}{2}}$



(a) $f(x) = 2x^3 + 3x^2 - 8x + c$

$f(-3) = 0$

$\Rightarrow 2(-3)^3 + 3(-3)^2 - 8(-3) + c = 0$

$\Rightarrow -54 + 27 + 24 + c = 0$

$\Rightarrow -54 + 51 + c = 0$

$\Rightarrow c = 3$

(b)

$$\begin{array}{r} 2x^2 - 3x + 1 \\ x+3 \,\overline{)\, 2x^3 + 3x^2 - 8x + 3} \\ -2x^3 - 6x^2 \qquad\qquad \\ \hline -3x^2 - 8x + 3 \\ +3x^2 + 9x \qquad \\ \hline x + 3 \\ -x - 3 \\ \hline 0 \end{array}$$

$\therefore f(x) = (x+3)(2x^2 - 3x + 1)$

$f(x) = (x+3)(2x-1)(x-1)$

(c) $f(x) = 2x^3 + 3x^2 - 8x + 3$

$f(-\frac{1}{2}) = 2(-\frac{1}{2})^3 + 3(-\frac{1}{2})^2 - 8(-\frac{1}{2}) + 3 = -\frac{1}{4} + \frac{3}{4} + 4 + 3 = \frac{15}{2}$

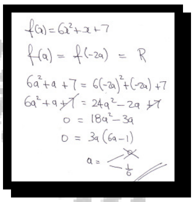
Question 19 **(***)**

$$f(x) \equiv 6x^2 + x + 7, \; x \in \mathbb{R}.$$

The remainder when $f(x)$ is divided by $(x-a)$ is the same as that when $f(x)$ is divided by $(x+2a)$, where a is a non zero constant.

Find the value of a.

$\boxed{}$, $\boxed{a = \frac{1}{6}}$



$f(x) = 6x^2 + x + 7$

$f(a) = f(-2a) = R$

$6a^2 + a + 7 = 6(-2a)^2 + (-2a) + 7$

$6a^2 + a \cancel{+7} = 24a^2 - 2a \cancel{+7}$

$0 = 18a^2 - 3a$

$0 = 3a(6a-1)$

$a = \cancel{0}$, $\frac{1}{6}$

Question 20 **(***)**

A cubic function is defined in terms of the positive constant k as

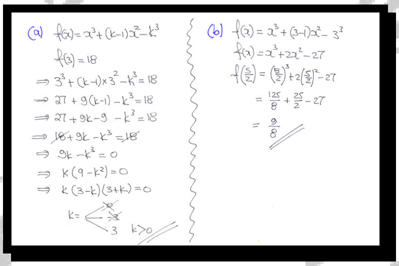
$$f(x) \equiv x^3 + (k-1)x^2 - k^3, \; x \in \mathbb{R}.$$

It is further given that when $f(x)$ is divided by $(x-3)$ the remainder is 18.

a) Determine the value of k.

b) Find the remainder when $f(x)$ is divided by $(2x-5)$.

$\boxed{k = 3}$, $\boxed{\frac{9}{8}}$



(a) $f(x) = x^3 + (k-1)x^2 - k^3$

$f(3) = 18$

$\Rightarrow 3^3 + (k-1) \times 3^2 - k^3 = 18$

$\Rightarrow 27 + 9(k-1) - k^3 = 18$

$\Rightarrow 27 + 9k - 9 - k^3 = 18$

$\Rightarrow \cancel{18} + 9k - k^3 = \cancel{18}$

$\Rightarrow 9k - k^3 = 0$

$\Rightarrow k(9-k^2) = 0$

$\Rightarrow k(3-k)(3+k) = 0$

$k = \cancel{0}$, $\cancel{-3}$, 3 $\quad k > 0$

(b) $f(x) = x^3 + (3-1)x^2 - 3^3$

$f(x) = x^3 + 2x^2 - 27$

$f\left(\frac{5}{2}\right) = \left(\frac{5}{2}\right)^3 + 2\left(\frac{5}{2}\right)^2 - 27$

$= \frac{125}{8} + \frac{25}{2} - 27$

$= \frac{9}{8}$

Question 21 (***)

A cubic graph is defined as

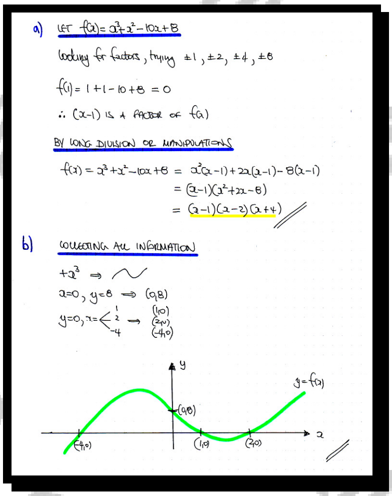
$$f(x) \equiv x^3 + x^2 - 10x + 8, \; x \in \mathbb{R}.$$

a) By considering the integer factors of 8, or otherwise, express $f(x)$ as the product of three linear factors.

b) Sketch the graph of $f(x)$.

The sketch must include the coordinates of any points where the graph of $f(x)$ meets the coordinate axes.

$\boxed{}$, $\boxed{f(x) = (x-2)(x-1)(x+4)}$



a) LET $f(x) = x^3 + x^2 - 10x + 8$

looking for factors, trying $\pm 1, \pm 2, \pm 4, \pm 8$

$f(1) = 1 + 1 - 10 + 8 = 0$

$\therefore (x-1)$ IS A FACTOR OF $f(x)$

BY LONG DIVISION OR MANIPULATIONS

$$\begin{aligned} f(x) = x^3 + x^2 - 10x + 8 &= x^2(x-1) + 2x(x-1) - 8(x-1) \\ &= (x-1)(x^2 + 2x - 8) \\ &= (x-1)(x-2)(x+4) \; // \end{aligned}$$

b) COLLECTING ALL INFORMATION

$+x^3 \Rightarrow \sim$

$x=0, \; y=8 \Rightarrow (0,8)$

$y=0, \; x = \begin{cases} 1 \\ 2 \\ -4 \end{cases} \Rightarrow \begin{matrix} (1,0) \\ (2,0) \\ (-4,0) \end{matrix}$

Question 22 (***)

$$f(x) \equiv 4x^3 + 9x^2 + 3x + 2$$

a) Use the factor theorem to show that $(x+2)$ is a factor of $f(x)$.

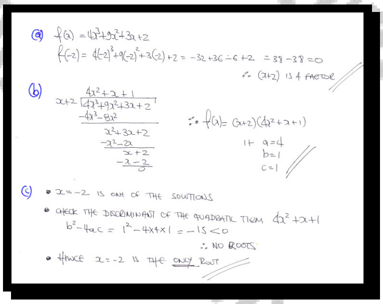
b) Given further that

$$f(x) \equiv (x+2)\left(ax^2 + bx + c\right),$$

find the value of each of the constants a, b and c.

c) Show that the equation $f(x) = 0$ has only one real root.

$\boxed{a=4}$, $\boxed{b=1}$, $\boxed{c=1}$



(a) $f(x) = 4x^3+9x^2+3x+2$

$f(-2) = 4(-2)^3+9(-2)^2+3(-2)+2 = -32+36-6+2 = 38-38 = 0$

$\therefore$ $(x+2)$ IS A FACTOR

(b)

$$\begin{array}{r} 4x^2+x+1 \\ x+2 \overline{)\, 4x^3+9x^2+3x+2} \\ \underline{-4x^3-8x^2} \\ x^2+3x+2 \\ \underline{-x^2-2x} \\ x+2 \\ \underline{-x-2} \\ 0 \end{array}$$

$\therefore f(x) = (x+2)(4x^2+x+1)$

IF $a=4$

$b=1$

$c=1$

(c)

- $x=-2$ IS ONE OF THE SOLUTIONS
- CHECK THE DISCRIMINANT OF THE QUADRATIC TERM $4x^2+x+1$

  $b^2-4ac = 1^2-4\times4\times1 = -15 < 0$

  $\therefore$ NO ROOTS
- HENCE $x=-2$ IS THE ONLY ROOT

Question 23 (***)

$$f(x) \equiv x^3 + px^2 + qx + 6$$

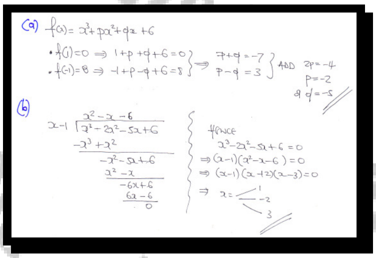
a) Find the value of each of the constants p and q, given that …

… $(x-1)$ is a factor of $f(x)$

… when $f(x)$ is divided by $(x+1)$ the remainder is 8.

b) Hence solve the equation $f(x)=0$.

$\boxed{}$, $\boxed{p=-2}$, $\boxed{q=-5}$, $\boxed{x=1,\ -2,\ 3}$



(a) $f(x) = x^3 + px^2 + qx + 6$

- $f(1)=0 \Rightarrow 1+p+q+6=0$
- $f(-1)=8 \Rightarrow -1+p-q+6=8$

$\Rightarrow p+q=-7$, $p-q=3$ ADD $2p=-4$, $p=-2$, $q=-5$

(b)

$x-1 \,\big)\, x^3-2x^2-5x+6$ quotient x^2-x-6

$-x^3+x^2$

$-x^2-5x+6$

x^2-x

$-6x+6$

$6x-6$

0

HENCE

$x^3-2x^2-5x+6=0$

$\Rightarrow (x-1)(x^2-x-6)=0$

$\Rightarrow (x-1)(x+2)(x-3)=0$

$\Rightarrow x = 1, -2, 3$

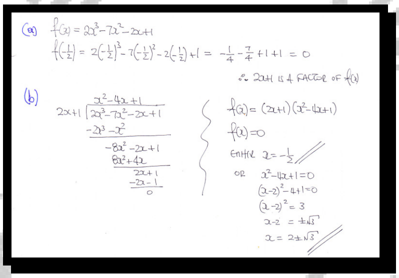
Question 24 (***)

$$f(x) \equiv 2x^3 - 7x^2 - 2x + 1$$

a) Use the factor theorem to show that $(2x+1)$ is a factor of $f(x)$.

b) Find the **exact** solutions of the equation $f(x)=0$.

$\boxed{x=-\frac{1}{2}, 2\pm\sqrt{3}}$



(a) $f(x)=2x^3-7x^2-2x+1$

$f\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right)^3 - 7\left(-\frac{1}{2}\right)^2 - 2\left(-\frac{1}{2}\right)+1 = -\frac{1}{4}-\frac{7}{4}+1+1 = 0$

$\therefore$ $2x+1$ IS A FACTOR OF $f(x)$

(b)

$2x+1 \,\big)\, 2x^3-7x^2-2x+1$ quotient x^2-4x+1

$-2x^3-x^2$

$-8x^2-2x+1$

$8x^2+4x$

$2x+1$

$-2x-1$

0

$f(x)=(2x+1)(x^2-4x+1)$

$f(x)=0$

EITHER $x=-\frac{1}{2}$

OR $x^2-4x+1=0$

$(x-2)^2-4+1=0$

$(x-2)^2=3$

$x-2=\pm\sqrt{3}$

$x=2\pm\sqrt{3}$

Question 25 (***)

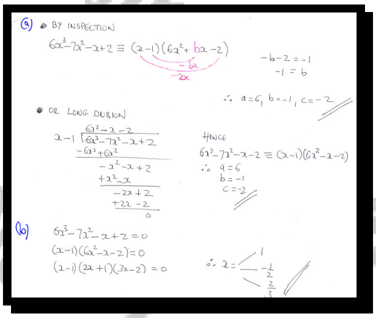
a) Find the value of each of the constants a, b and c so that

$$6x^3-7x^2-x+2 \equiv (x-1)\left(ax^2+bx+c\right).$$

b) Hence solve the equation

$$6x^3-7x^2-x+2=0.$$

$\boxed{}$, $\boxed{a=6,\ b=-1,\ c=-2}$, $\boxed{x=-\frac{1}{2},\frac{2}{3},1}$



(a) • BY INSPECTION

$6x^3-7x^2-x+2 \equiv (x-1)(6x^2+bx-2)$

$-bx$

$-2x$

$-b-2=-1$

$-1=b$

$\therefore a=6,\ b=-1,\ c=-2$

• OR LONG DIVISION

$$\begin{array}{r} 6x^2-x-2 \\ x-1\ \overline{)\,6x^3-7x^2-x+2} \\ -6x^3+6x^2 \\ \hline -x^2-x+2 \\ +x^2-x \\ \hline -2x+2 \\ +2x-2 \\ \hline 0 \end{array}$$

HENCE

$6x^3-7x^2-x-2 \equiv (x-1)(6x^2-x-2)$

$\therefore a=6$

$b=-1$

$c=-2$

(b) $6x^3-7x^2-x+2=0$

$(x-1)(6x^2-x-2)=0$

$(x-1)(2x+1)(3x-2)=0$

$\therefore x = 1,\ -\frac{1}{2},\ \frac{2}{3}$

Question 26 (***)

A cubic polynomial is defined as

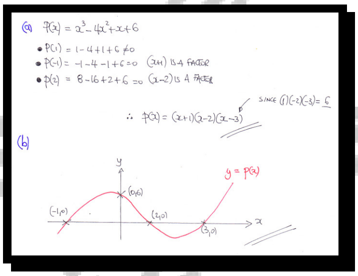
$$p(x) \equiv x^3 - 4x^2 + x + 6, \quad x \in \mathbb{R}.$$

a) By considering the factors of 6, or otherwise, express $p(x)$ as the product of three linear factors.

b) Sketch the graph of $p(x)$.

The sketch must include the coordinates of any points where the graph of $p(x)$ meets the coordinate axes.

$$\boxed{p(x) = (x-3)(x-2)(x+1)}$$



(a) $p(x) = x^3 - 4x^2 + x + 6$

- $p(1) = 1 - 4 + 1 + 6 \neq 0$
- $p(-1) = -1 - 4 - 1 + 6 = 0$ $(x+1)$ IS A FACTOR
- $p(2) = 8 - 16 + 2 + 6 = 0$ $(x-2)$ IS A FACTOR

SINCE $(1)(-2)(-3) = \underline{6}$

$\therefore p(x) = (x+1)(x-2)(x-3)$

(b)

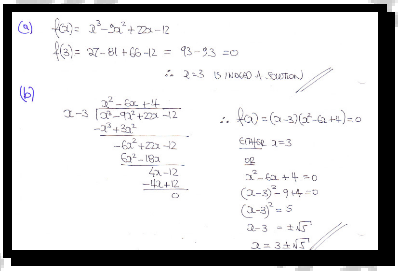
Question 27 **(***)**

$$f(x) \equiv x^3 - 9x^2 + 22x - 12 .$$

a) Show that $x = 3$ is a solution of the equation of the equation $f(x) = 0$.

b) Find, in exact surd form, the other two solutions of the equation $f(x) = 0$.

$\boxed{x = 3 \pm \sqrt{5}}$



(a) $f(x) = x^3 - 9x^2 + 22x - 12$

$f(3) = 27 - 81 + 66 - 12 = 93 - 93 = 0$

$\therefore$ $x = 3$ IS INDEED A SOLUTION

(b)

$$\begin{array}{r|l} & x^2 - 6x + 4 \\ x - 3 & x^3 - 9x^2 + 22x - 12 \\ & -x^3 + 3x^2 \\ \hline & -6x^2 + 22x - 12 \\ & 6x^2 - 18x \\ \hline & 4x - 12 \\ & -4x + 12 \\ \hline & 0 \end{array}$$

$\therefore f(x) = (x-3)(x^2-6x+4) = 0$

EITHER $x = 3$

OR

$x^2 - 6x + 4 = 0$

$(x-3)^2 - 9 + 4 = 0$

$(x-3)^2 = 5$

$x - 3 = \pm\sqrt{5}$

$x = 3 \pm \sqrt{5}$

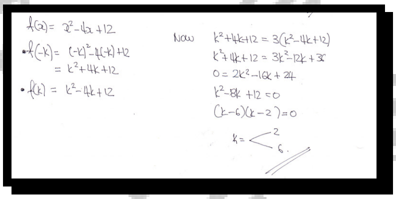
Question 28 **(***)**

$$f(x) \equiv x^2 - 4x + 12 .$$

The remainder when $f(x)$ is divided by $(x+k)$ is three times as large as when $f(x)$ is divided by $(x-k)$.

Determine the possible values of k.

$\boxed{C2P}$, $\boxed{k = 6,\ 2}$



$f(x) = x^2 - 4x + 12$

- $f(-k) = (-k)^2 - 4(-k) + 12$
  $= k^2 + 4k + 12$
- $f(k) = k^2 - 4k + 12$

Now $k^2 + 4k + 12 = 3(k^2 - 4k + 12)$

$k^2 + 4k + 12 = 3k^2 - 12k + 36$

$0 = 2k^2 - 16k + 24$

$k^2 - 8k + 12 = 0$

$(k-6)(k-2) = 0$

$k = \begin{cases} 2 \\ 6. \end{cases}$

Question 29 (***)

$$f(x) \equiv 2x^3 + kx^2 - x - 6,$$

where k is a constant

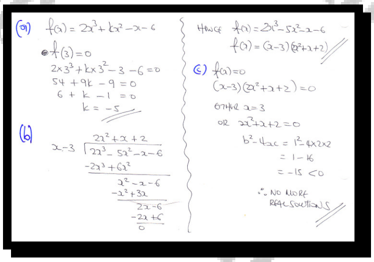
Given that $f(3) = 0$, …

a) … show that $k = -5$

b) … factorize $f(x)$ as a product of one linear and one quadratic factor.

c) … show further that, apart from $x = 3$, the equation $f(x) = 0$ has no other real solutions.

$$\boxed{f(x) = (x-3)\left(2x^2 + x + 2\right)}$$



(a) $f(x) = 2x^3 + kx^2 - x - 6$

$f(3) = 0$

$2 \times 3^3 + k \times 3^2 - 3 - 6 = 0$

$54 + 9k - 9 = 0$

$6 + k - 1 = 0$

$k = -5$

(b)

$$\begin{array}{r l} & 2x^2 + x + 2 \\ x-3 \,\big) & 2x^3 - 5x^2 - x - 6 \\ & \underline{-2x^3 + 6x^2} \\ & x^2 - x - 6 \\ & \underline{-x^2 + 3x} \\ & 2x - 6 \\ & \underline{-2x + 6} \\ & 0 \end{array}$$

HENCE $f(x) = 2x^3 - 5x^2 - x - 6$

$f(x) = (x-3)(2x^2+x+2)$

(c) $f(x) = 0$

$(x-3)(2x^2+x+2) = 0$

EITHER $x = 3$

OR $2x^2 + x + 2 = 0$

$b^2 - 4ac = 1^2 - 4 \times 2 \times 2$

$= 1 - 16$

$= -15 < 0$

∴ NO MORE REAL SOLUTIONS

Question 30 (***)

The polynomial function f is given below

$$f(x) \equiv (2x-1)(x+4) - 4(x-3)^2, \ x \in \mathbb{R}.$$

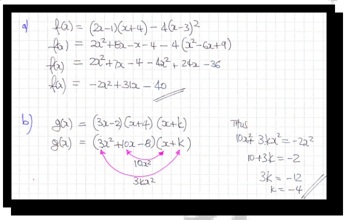
a) Simplify $f(x)$ fully.

The polynomial function g is defined, in terms of the constant k, by

$$g(x) \equiv (3x-2)(x+4)(x+k), \ x \in \mathbb{R}.$$

b) Determine the value of k, given that the coefficient of x^2 in the simplified expansion of $f(x)$ is equal to the coefficient of x^2 in the simplified expansion of $g(x)$.

$\boxed{}$, $\boxed{f(x) \equiv -2x^2 + 31x - 40}$, $\boxed{k = -4}$



a) $f(x) = (2x-1)(x+4) - 4(x-3)^2$

$f(x) = 2x^2 + 8x - x - 4 - 4(x^2 - 6x + 9)$

$f(x) = 2x^2 + 7x - 4 - 4x^2 + 24x - 36$

$f(x) = -2x^2 + 31x - 40$

b) $g(x) = (3x-2)(x+4)(x+k)$

$g(x) = (3x^2 + 10x - 8)(x+k)$

$10x^2$

$3kx^2$

Thus

$10x^2 + 3kx^2 = -2x^2$

$10 + 3k = -2$

$3k = -12$

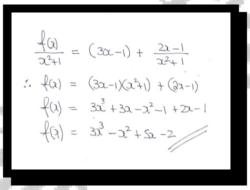
$k = -4$

Question 31 **(***)**

When the polynomial $f(x)$ is divided by $\left(x^2+1\right)$ the quotient is $(3x-1)$ and the remainder is $(2x-1)$.

Determine a fully simplified expression for $f(x)$.

$$\boxed{f(x)=3x^3-x^2+5x-2}$$



$$\frac{f(x)}{x^2+1} = (3x-1) + \frac{2x-1}{x^2+1}$$

$$\therefore f(x) = (3x-1)(x^2+1) + (2x-1)$$

$$f(x) = 3x^3+3x-x^2-1+2x-1$$

$$f(x) = 3x^3-x^2+5x-2$$

Question 32 **(***)**

$$f(x) \equiv (x+p)\left(2x^2+5x-4\right)-4,$$

where p is a non zero constant.

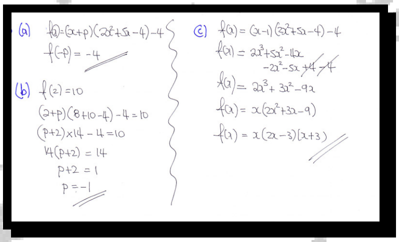
a) State the value of the remainder when $f(x)$ is divided by $(x+p)$.

When $f(x)$ is divided by $(x-2)$ the remainder is 10.

b) Determine the value of p.

c) Factorize $f(x)$ into three linear factors.

$\boxed{}$, $\boxed{-4}$, $\boxed{p=-1}$, $\boxed{f(x)=x(x+3)(2x-3)}$



(a) $f(x)=(x+p)(2x^2+5x-4)-4$

$f(-p) = -4$

(b) $f(2)=10$

$(2+p)(8+10-4)-4=10$

$(p+2)\times 14-4=10$

$14(p+2)=14$

$p+2=1$

$p=-1$

(c) $f(x)=(x-1)(2x^2+5x-4)-4$

$f(x)=2x^3+5x^2-4x$
$-2x^2-5x+\cancel{4}-\cancel{4}$

$f(x)=2x^3+3x^2-9x$

$f(x)=x(2x^2+3x-9)$

$f(x)=x(2x-3)(x+3)$

Question 33 (***)

The polynomial $4x^3 + Ax^2 + Bx + 9$, where A and B are constants, is denoted by $f(x)$.

When $f(x)$ is divided by $(x-2)$ the remainder is R.

When $f(x)$ is divided by $(x-3)$ the remainder is $6R$.

a) Show clearly that

$$B - A = 14.$$

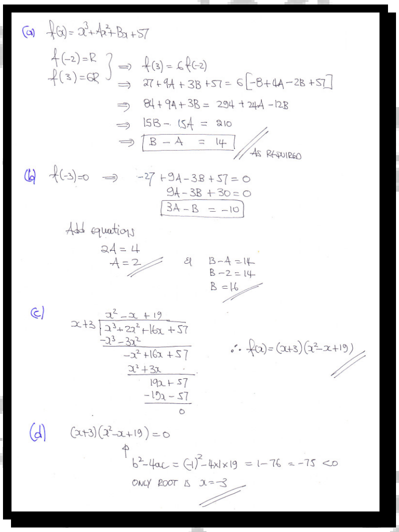
It is further given that $(x+3)$ is factor of $f(x)$.

b) Find the value of A and the value B.

a) Express $f(x)$ as a product of a linear factor and a quadratic factor.

b) Show that the equation $f(x) = 0$ has only one real root.

$\boxed{A=2}$, $\boxed{B=16}$, $\boxed{f(x)=(x+3)\left(x^2-x+19\right)}$



(a) $f(x) = x^3 + Ax^2 + Bx + 57$

$\left.\begin{matrix} f(-2)=R \\ f(3)=6R \end{matrix}\right\} \Rightarrow f(3) = 6f(-2)$

$\Rightarrow 27 + 9A + 3B + 57 = 6\left[-8 + 4A - 2B + 57\right]$

$\Rightarrow 84 + 9A + 3B = 294 + 24A - 12B$

$\Rightarrow 15B - 15A = 210$

$\Rightarrow \boxed{B - A = 14}$ // AS REQUIRED

(b) $f(-3)=0 \Rightarrow -27 + 9A - 3B + 57 = 0$

$9A - 3B + 30 = 0$

$\boxed{3A - B = -10}$

Add equations

$2A = 4$

$A = 2$ // & $B - A = 14$

$B - 2 = 14$

$B = 16$ //

(c)

$x^2 - x + 19$

$x+3 \overline{\smash{)}\, x^3 + 2x^2 + 16x + 57}$

$-x^3 - 3x^2$

$-x^2 + 16x + 57$

$x^2 + 3x$

$19x + 57$

$-19x - 57$

0

$\therefore f(x) = (x+3)(x^2 - x + 19)$ //

(d) $(x+3)(x^2 - x + 19) = 0$

$b^2 - 4ac = (-1)^2 - 4\times1\times19 = 1 - 76 = -75 < 0$

ONLY ROOT IS $x = -3$ //

Question 34 (***)

The polynomial $p(x)$ is defined as

$$p(x) = 2x^3 - 11x^2 + 20x - 12.$$

a) Use the factor theorem to show that $(x-2)$ is a factor of $p(x)$.

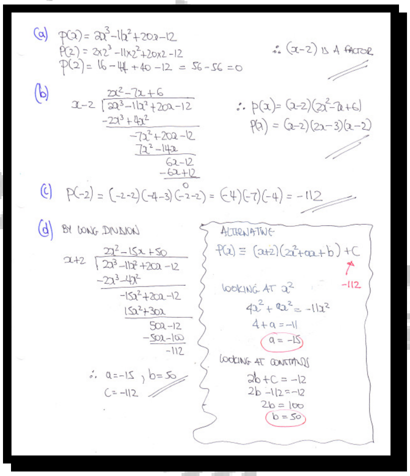
b) Express $p(x)$ as the product of three linear factors.

c) Find the remainder when $p(x)$ is divided by $(x+2)$.

d) Determine the value of each of the constants a, b and c so that

$$p(x) = (x+2)\left(2x^2 + ax + b\right) + c.$$

$\boxed{}$, $\boxed{p(x) = (2x-3)(x-2)^2}$, $\boxed{-112}$, $\boxed{a = -15,\ b = 50,\ c = -112}$



(a) $p(x) = 2x^3 - 11x^2 + 20x - 12$

$p(2) = 2\times2^3 - 11\times2^2 + 20\times2 - 12$

$p(2) = 16 - 44 + 40 - 12 = 56 - 56 = 0$

$\therefore (x-2)$ IS A FACTOR

(b)

$$\begin{array}{r} 2x^2 - 7x + 6 \\ x-2\ \overline{\big)\,2x^3 - 11x^2 + 20x - 12} \\ \underline{-2x^3 + 4x^2} \\ -7x^2 + 20x - 12 \\ \underline{7x^2 - 14x} \\ 6x - 12 \\ \underline{-6x + 12} \\ 0 \end{array}$$

$\therefore p(x) = (x-2)(2x^2 - 7x + 6)$

$p(x) = (x-2)(2x-3)(x-2)$

(c) $p(-2) = (-2-2)(-4-3)(-2-2) = (-4)(-7)(-4) = -112$

(d) BY LONG DIVISION

$$\begin{array}{r} 2x^2 - 15x + 50 \\ x+2\ \overline{\big)\,2x^3 - 11x^2 + 20x - 12} \\ \underline{-2x^3 - 4x^2} \\ -15x^2 + 20x - 12 \\ \underline{15x^2 + 30x} \\ 50x - 12 \\ \underline{-50x - 100} \\ -112 \end{array}$$

$\therefore a = -15,\ b = 50$

$c = -112$

ALTERNATIVE

$p(x) \equiv (x+2)(2x^2 + ax + b) + c$

$\uparrow$ -112

LOOKING AT x^2

$4x^2 + ax^2 = -11x^2$

$4 + a = -11$

$\boxed{a = -15}$

LOOKING AT CONSTANTS

$2b + c = -12$

$2b - 112 = -12$

$2b = 100$

$\boxed{b = 50}$

Question 35 (***)

$$f(x) \equiv x^4 + 2x^3 + x^2 - 4, \ x \in \mathbb{R}.$$

a) Use the factor theorem to show that $(x+2)$ is a factor of $f(x)$.

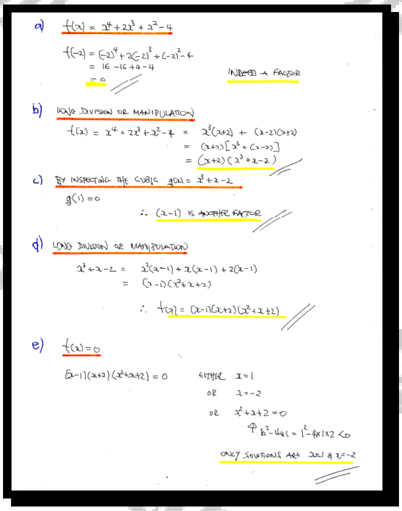
b) Express $f(x)$ as the product of a linear factor and a cubic factor.

c) Find another linear factor of $f(x)$.

d) Express $f(x)$ as the product of two linear factors and a quadratic factor.

e) Show that the equation $f(x)=0$ has exactly two solutions.

$\boxed{}$, $\boxed{f(x) \equiv (x+2)\left(x^3+x-2\right)}$, $\boxed{(x-1)}$, $\boxed{f(x) \equiv (x+2)(x-1)\left(x^2+x+2\right)}$



a) $f(x) = x^4 + 2x^3 + x^2 - 4$

$f(-2) = (-2)^4 + 2(-2)^3 + (-2)^2 - 4$
$= 16 - 16 + 4 - 4$
$= 0$

INDEED A FACTOR

b) LONG DIVISION OR MANIPULATION

$f(x) = x^4 + 2x^3 + x^2 - 4 = x^3(x+2) + (x-2)(x+2)$
$= (x+2)[x^3 + (x-2)]$
$= (x+2)(x^3+x-2)$

c) BY INSPECTING THE CUBIC $g(x) = x^3 + x - 2$

$g(1) = 0$

$\therefore$ $(x-1)$ IS ANOTHER FACTOR

d) LONG DIVISION OR MANIPULATION

$x^3 + x - 2 = x^2(x-1) + x(x-1) + 2(x-1)$
$= (x-1)(x^2+x+2)$

$\therefore$ $f(x) = (x-1)(x+2)(x^2+x+2)$

e) $f(x) = 0$

$(x-1)(x+2)(x^2+x+2) = 0$

EITHER $x = 1$
OR $x = -2$
OR $x^2 + x + 2 = 0$
Φ $b^2 - 4ac = 1^2 - 4 \times 1 \times 2 < 0$

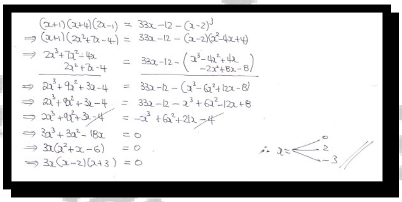
ONLY SOLUTIONS ARE $x=1$ & $x=-2$

Question 36 **(***+)**

Solve the equation

$$(x+1)(x+4)(2x-1) = 33x - 12 - (x-2)^3 .$$

$$\boxed{x = -3, 0, 2}$$



$(x+1)(x+4)(2x-1) = 33x-12-(x-2)^3$

$\Rightarrow (x+1)(2x^2+7x-4) = 33x-12-(x-2)(x^2-4x+4)$

$\Rightarrow \begin{array}{r} 2x^3+7x^2-4x \\ 2x^2+7x-4 \end{array} = 33x-12-\left(\begin{array}{r} x^3-4x^2+4x \\ -2x^2+8x-8 \end{array}\right)$

$\Rightarrow 2x^3+9x^2+3x-4 = 33x-12-(x^3-6x^2+12x-8)$

$\Rightarrow 2x^3+9x^2+3x-4 = 33x-12-x^3+6x^2-12x+8$

$\Rightarrow 2x^3+9x^2+3x\cancel{-4} = -x^3+6x^2+21x\cancel{-4}$

$\Rightarrow 3x^3+3x^2-18x = 0$

$\Rightarrow 3x(x^2+x-6) = 0$

$\Rightarrow 3x(x-2)(x+3) = 0$

$\therefore x = \begin{cases} 0 \\ 2 \\ -3 \end{cases}$ //

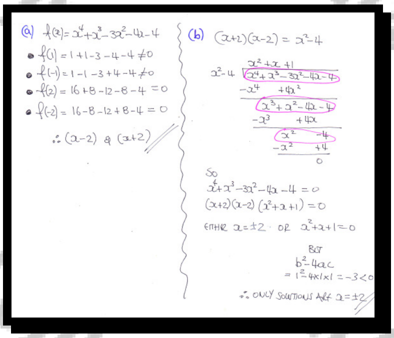
Question 37 **(***+)**

$$f(x) = x^4 + x^3 - 3x^2 - 4x - 4 .$$

a) Use the factor theorem to find two linear factors of $f(x)$.

b) Hence show that the equation $f(x) = 0$ has exactly two real roots.

$$\boxed{(x-2), (x+2)}, \ \boxed{x = \pm 2}$$



(a) $f(x) = x^4+x^3-3x^2-4x-4$

- $f(1) = 1+1-3-4-4 \neq 0$
- $f(-1) = 1-1-3+4-4 \neq 0$
- $f(2) = 16+8-12-8-4 = 0$
- $f(-2) = 16-8-12+8-4 = 0$

$\therefore (x-2)$ & $(x+2)$ //

(b) $(x+2)(x-2) = x^2-4$

$$\begin{array}{r|l} & x^2+x+1 \\ \hline x^2-4 & x^4+x^3-3x^2-4x-4 \\ & -x^4 \qquad +4x^2 \\ \hline & x^3+x^2-4x-4 \\ & -x^3 \qquad +4x \\ \hline & x^2 \qquad -4 \\ & -x^2 \qquad +4 \\ \hline & 0 \end{array}$$

So

$x^4+x^3-3x^2-4x-4 = 0$

$(x+2)(x-2)(x^2+x+1) = 0$

EITHER $x = \pm 2$ OR $x^2+x+1 = 0$

BUT

b^2-4ac

$= 1^2-4\times1\times1 = -3 < 0$

$\therefore$ ONLY SOLUTIONS ARE $x = \pm 2$ //

Question 38 (***+)

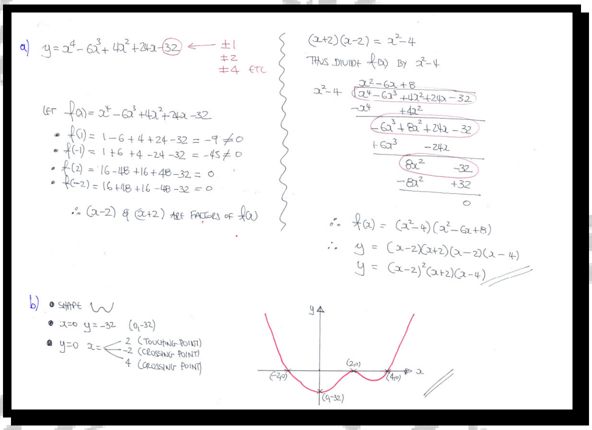
The curve C has equation

$$y = x^4 - 6x^3 + 4x^2 + 24x - 32 .$$

a) Express y as the product of four linear factors.

b) Hence the graph of C, showing clearly the coordinates of any points where the graph of C meets the coordinate axes.

$$\boxed{y = (x+2)(x-4)(x-2)^2}$$



a) $y = x^4 - 6x^3 + 4x^2 + 24x - \boxed{32} \longleftarrow \begin{matrix} \pm 1 \\ \pm 2 \\ \pm 4 \text{ ETC} \end{matrix}$

LET $f(x) = x^4 - 6x^3 + 4x^2 + 24x - 32$

- $f(1) = 1 - 6 + 4 + 24 - 32 = -9 \neq 0$
- $f(-1) = 1 + 6 + 4 - 24 - 32 = -45 \neq 0$
- $f(2) = 16 - 48 + 16 + 48 - 32 = 0$
- $f(-2) = 16 + 48 + 16 - 48 - 32 = 0$

$\therefore (x-2)$ & $(x+2)$ ARE FACTORS OF $f(x)$

$(x+2)(x-2) = x^2 - 4$

THUS DIVIDE $f(x)$ BY $x^2 - 4$

$$\begin{array}{r|l} & x^2 - 6x + 8 \\ x^2 - 4 & x^4 - 6x^3 + 4x^2 + 24x - 32 \\ & -x^4 \qquad\quad + 4x^2 \\ \hline & -6x^3 + 8x^2 + 24x - 32 \\ & +6x^3 \qquad\quad - 24x \\ \hline & 8x^2 \qquad\quad - 32 \\ & -8x^2 \qquad\quad + 32 \\ \hline & 0 \end{array}$$

$\therefore f(x) = (x^2 - 4)(x^2 - 6x + 8)$

$\therefore y = (x-2)(x+2)(x-2)(x-4)$

$y = (x-2)^2(x+2)(x-4)$

b)
- SHAPE W
- $x=0$ $y=-32$ $(0,-32)$
- $y=0$ $x = \begin{cases} 2 & \text{(TOUCHING POINT)} \\ -2 & \text{(CROSSING POINT)} \\ 4 & \text{(CROSSING POINT)} \end{cases}$

Question 39 (***+)

The polynomials $f(x)$ and $g(x)$ are defined in terms of the constants a and b

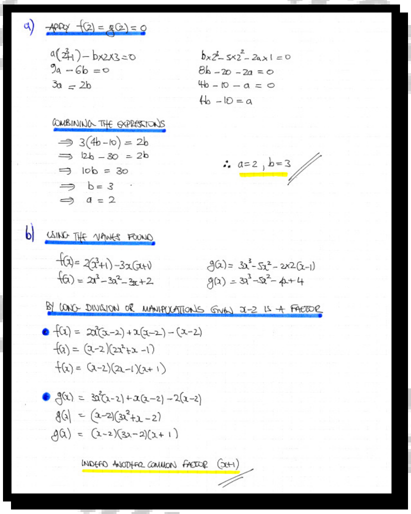
$$f(x)=a\left(x^3+1\right)-bx(x+1)$$

$$g(x)=bx^3-5x^2-2a(x-1).$$

a) Given that $(x-2)$ is a factor of **both** $f(x)$ and $g(x)$, determine the value of a and the value of b.

b) Factorize both $f(x)$ and $g(x)$, and hence show that $f(x)$ and $g(x)$, have another linear common factor.

$\boxed{}$, $\boxed{a=2}$, $\boxed{b=3}$, $\boxed{f(x)=(x-2)(x+1)(2x-1)}$, $\boxed{g(x)=(x-2)(x+1)(3x-2)}$



a) APPLY $f(2)=g(2)=0$

$a(2^3+1)-b\times2\times3=0$
$9a-6b=0$
$3a=2b$

$b\times2^3-5\times2^2-2a\times1=0$
$8b-20-2a=0$
$4b-10-a=0$
$4b-10=a$

COMBINING THE EXPRESSIONS

$\Rightarrow 3(4b-10)=2b$
$\Rightarrow 12b-30=2b$
$\Rightarrow 10b=30$
$\Rightarrow b=3$
$\Rightarrow a=2$

$\therefore a=2,\ b=3$

b) USING THE VALUES FOUND

$f(x)=2(x^3+1)-3x(x+1)$
$f(x)=2x^3-3x^2-3x+2$

$g(x)=3x^3-5x^2-2\times2(x-1)$
$g(x)=3x^3-5x^2-4x+4$

BY LONG DIVISION OR MANIPULATIONS GIVEN $x-2$ IS A FACTOR

- $f(x)=2x^2(x-2)+x(x-2)-(x-2)$
  $f(x)=(x-2)(2x^2+x-1)$
  $f(x)=(x-2)(2x-1)(x+1)$

- $g(x)=3x^2(x-2)+x(x-2)-2(x-2)$
  $g(x)=(x-2)(3x^2+x-2)$
  $g(x)=(x-2)(3x-2)(x+1)$

INDEED ANOTHER COMMON FACTOR $(x+1)$

Question 40 **(***+)**

A polynomial $p(x)$ is defined, in terms of a constant a, by

$$p(x) = x^3 - 16x^2 + 72x + a.$$

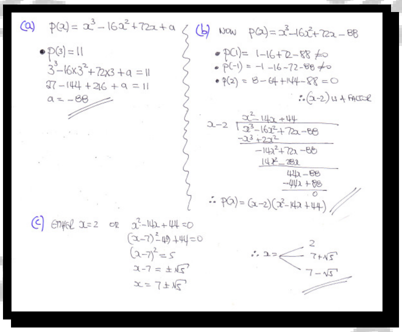
When $p(x)$ is divided by $(x-3)$ the remainder is 11.

a) Determine the value of a.

b) Express $p(x)$ as a product of a linear and one quadratic factor.

c) Hence find, in exact surd form where appropriate, the three solutions of the equation $p(x) = 0$.

$\boxed{a = -88}$, $\boxed{(x-2)(x^2 - 14x + 44)}$, $\boxed{x = 2,\ 7 \pm \sqrt{5}}$



(a) $p(x) = x^3 - 16x^2 + 72x + a$

• $p(3) = 11$

$3^3 - 16 \times 3^2 + 72 \times 3 + a = 11$

$27 - 144 + 216 + a = 11$

$a = -88$

(b) NOW $p(x) = x^3 - 16x^2 + 72x - 88$

• $p(1) = 1 - 16 + 72 - 88 \neq 0$

• $p(-1) = -1 - 16 - 72 - 88 \neq 0$

• $p(2) = 8 - 64 + 144 - 88 = 0$

$\therefore (x-2)$ IS A FACTOR

$$\begin{array}{r|l} & x^2 - 14x + 44 \\ x-2 & x^3 - 16x^2 + 72x - 88 \\ & -x^3 + 2x^2 \\ & -14x^2 + 72x - 88 \\ & 14x^2 - 28x \\ & 44x - 88 \\ & -44x + 88 \\ & 0 \end{array}$$

$\therefore p(x) = (x-2)(x^2 - 14x + 44)$

(c) EITHER $x = 2$ OR $x^2 - 14x + 44 = 0$

$(x-7)^2 - 49 + 44 = 0$

$(x-7)^2 = 5$

$x - 7 = \pm\sqrt{5}$

$x = 7 \pm \sqrt{5}$

$\therefore x = 2,\ 7 + \sqrt{5},\ 7 - \sqrt{5}$

Question 41 **(***+)**

A polynomial $p(x)$ is defined, in terms of a constant k, by

$$p(x) = x^3 + kx^2 - x + 12 .$$

When $p(x)$ is divided by $(x-1)$ the remainder is r.

When $p(x)$ is divided by $(x-4)$ the remainder is $8r$.

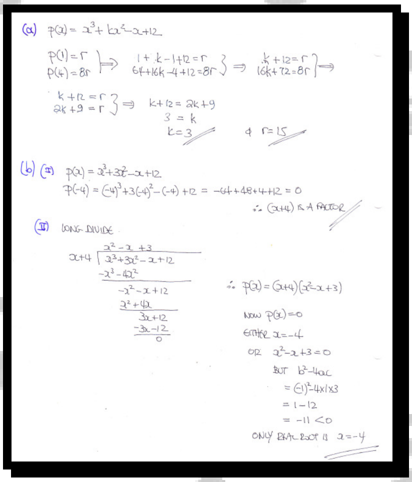
a) Determine in any order …

- **i.** … the value of k.
- **ii.** … the value of r.

b) Show clearly that …

- **i.** … $(x+4)$ is a factor of $p(x)$.
- **ii.** … the equation $p(x)=0$ has only one real root.

$\boxed{\quad}$, $\boxed{k=3}$, $\boxed{r=15}$



(a) $p(x) = x^3 + kx^2 - x + 12$

$\left.\begin{array}{l} p(1) = r \\ p(4) = 8r \end{array}\right\} \Rightarrow \left.\begin{array}{l} 1 + k - 1 + 12 = r \\ 64 + 16k - 4 + 12 = 8r \end{array}\right\} \Rightarrow \left.\begin{array}{l} k + 12 = r \\ 16k + 72 = 8r \end{array}\right\} \Rightarrow$

$\left.\begin{array}{l} k + 12 = r \\ 2k + 9 = r \end{array}\right\} \Rightarrow k + 12 = 2k + 9$

$3 = k$

$k = 3$ & $r = 15$

(b) (I) $p(x) = x^3 + 3x^2 - x + 12$

$p(-4) = (-4)^3 + 3(-4)^2 - (-4) + 12 = -64 + 48 + 4 + 12 = 0$

$\therefore$ $(x+4)$ IS A FACTOR

(II) LONG DIVIDE

$$\begin{array}{r|l} & x^2 - x + 3 \\ \hline x+4 & x^3 + 3x^2 - x + 12 \\ & \underline{-x^3 - 4x^2} \\ & \quad -x^2 - x + 12 \\ & \quad \underline{x^2 + 4x} \\ & \qquad 3x + 12 \\ & \qquad \underline{-3x - 12} \\ & \qquad\quad 0 \end{array}$$

$\therefore p(x) = (x+4)(x^2 - x + 3)$

NOW $p(x) = 0$

EITHER $x = -4$

OR $x^2 - x + 3 = 0$

BUT $b^2 - 4ac$

$= (-1)^2 - 4 \times 1 \times 3$

$= 1 - 12$

$= -11 < 0$

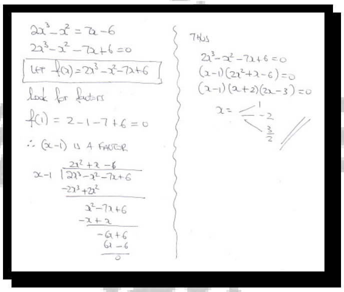
ONLY REAL ROOT IS $x = -4$

Question 42 (***+)

Find the three solutions of the cubic equation

$$2x^3 - x^2 = 7x - 6 \, .$$

$$\boxed{x = -2,\ 1,\ \frac{3}{2}}$$



$2x^3 - x^2 = 7x - 6$

$2x^3 - x^2 - 7x + 6 = 0$

$\boxed{\text{LET } f(x) = 2x^3 - x^2 - 7x + 6}$

look for factors

$f(1) = 2 - 1 - 7 + 6 = 0$

$\therefore (x-1)$ IS A FACTOR

$$\begin{array}{r|l} & 2x^2 + x - 6 \\ \hline x-1 & 2x^3 - x^2 - 7x + 6 \\ & -2x^3 + 2x^2 \\ \hline & x^2 - 7x + 6 \\ & -x + x \\ \hline & -6x + 6 \\ & 6x - 6 \\ \hline & 0 \end{array}$$

Thus

$2x^3 - x^2 - 7x + 6 = 0$

$(x-1)(2x^2 + x - 6) = 0$

$(x-1)(x+2)(2x-3) = 0$

$x = \begin{cases} 1 \\ -2 \\ \frac{3}{2} \end{cases}$ //

Question 43 (***+)

$$f(x) \equiv 2x^3 - 9x^2 - 11x + 30.$$

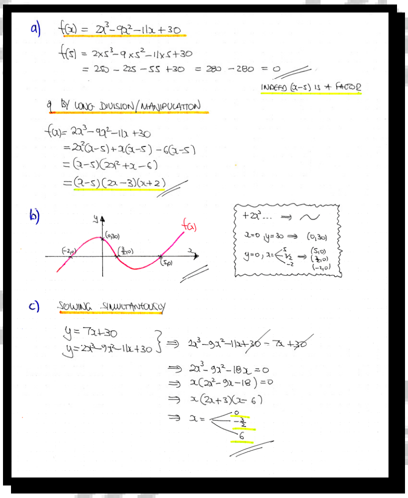
a) Show, by using the factor theorem, that $(x-5)$ is a factor of $f(x)$ and hence factorize $f(x)$ into product of three linear factors.

b) Sketch the graph of $f(x)$.

The sketch must include the coordinates of all the points where the graph meets the coordinate axes.

c) Find the x coordinates of the points where the line with equation $y = 7x + 30$ meets the graph of $f(x)$.

$\boxed{}$, $\boxed{f(x) \equiv (x-5)(2x-3)(x+2)}$, $\boxed{x = -\frac{3}{2} \cup x = 0 \cup x = 6}$



a) $f(x) = 2x^3 - 9x^2 - 11x + 30$

$f(5) = 2 \times 5^3 - 9 \times 5^2 - 11 \times 5 + 30$
$= 250 - 225 - 55 + 30 = 280 - 280 = 0$

INDEED $(x-5)$ IS A FACTOR

BY LONG DIVISION/MANIPULATION

$f(x) = 2x^3 - 9x^2 - 11x + 30$
$= 2x^2(x-5) + x(x-5) - 6(x-5)$
$= (x-5)(2x^2 + x - 6)$
$= (x-5)(2x-3)(x+2)$

b)

$+2x^3 \dots \Rightarrow \sim$

$x=0, y=30 \Rightarrow (0,30)$

$y=0, x = \begin{cases} 5 \\ 3/2 \\ -2 \end{cases} \Rightarrow \begin{matrix} (5,0) \\ (\frac{3}{2},0) \\ (-2,0) \end{matrix}$

c) SOLVING SIMULTANEOUSLY

$\left.\begin{matrix} y = 7x + 30 \\ y = 2x^3 - 9x^2 - 11x + 30 \end{matrix}\right\} \Rightarrow 2x^3 - 9x^2 - 11x + \cancel{30} = 7x + \cancel{30}$

$\Rightarrow 2x^3 - 9x^2 - 18x = 0$
$\Rightarrow x(2x^2 - 9x - 18) = 0$
$\Rightarrow x(2x+3)(x-6)$
$\Rightarrow x = \begin{cases} 0 \\ -\frac{3}{2} \\ 6 \end{cases}$

Question 44 (***+)

A cubic graph is defined by

$$f(x) \equiv x^3 - 3x^2 - 4x + 12, \; x \in \mathbb{R}.$$

a) Show, by using the factor theorem, that $(x-3)$ is a factor of $f(x)$ and hence factorize $f(x)$ into product of three linear factors.

b) Sketch the graph of $f(x)$.
The sketch must include the coordinates of any points where the graph of $f(x)$ meets the coordinate axes.

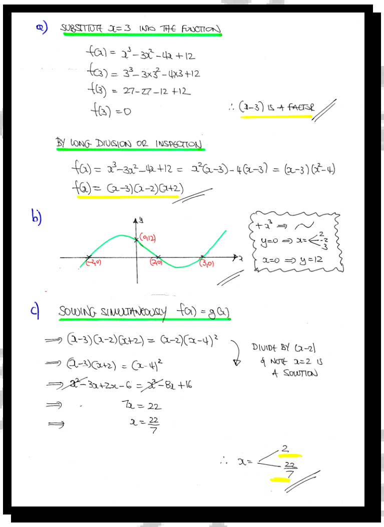
Another cubic graph is defined as

$$g(x) \equiv (x-2)(x-4)^2, \; x \in \mathbb{R}.$$

The two graphs meet at the points P and Q.

c) Determine the x coordinates of P and the x coordinates of Q.

$\boxed{\quad\quad}$, $\boxed{f(x) = (x-2)(x+2)(x-3)}$, $\boxed{x = 2, \frac{22}{7}}$



a) SUBSTITUTE $x=3$ INTO THE FUNCTION

$f(x) = x^3 - 3x^2 - 4x + 12$

$f(3) = 3^3 - 3\times3^2 - 4\times3 + 12$

$f(3) = 27 - 27 - 12 + 12$

$f(3) = 0$ $\quad\therefore (x-3)$ IS A FACTOR

BY LONG DIVISION OR INSPECTION

$f(x) = x^3 - 3x^2 - 4x + 12 = x^2(x-3) - 4(x-3) = (x-3)(x^2-4)$

$f(x) = (x-3)(x-2)(x+2)$

b)

$+x^3 \Rightarrow$ ~

$y=0 \Rightarrow x = 2, -2, 3$

$x=0 \Rightarrow y=12$

c) SOLVING SIMULTANEOUSLY $f(x) = g(x)$

$\Rightarrow (x-3)(x-2)(x+2) = (x-2)(x-4)^2$ DIVIDE BY $(x-2)$ & NOTE $x=2$ IS A SOLUTION

$\Rightarrow (x-3)(x+2) = (x-4)^2$

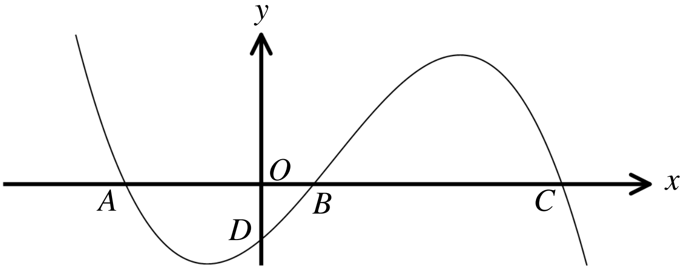
$\Rightarrow \cancel{x^2} - 3x + 2x - 6 = \cancel{x^2} - 8x + 16$

$\Rightarrow 7x = 22$

$\Rightarrow x = \frac{22}{7}$

$\therefore x = 2, \frac{22}{7}$

Question 45 (***+)



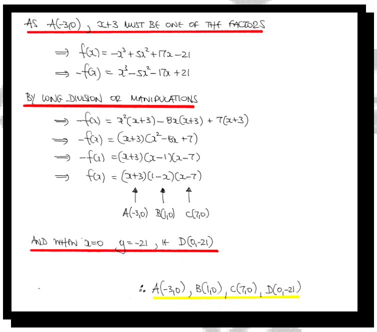
The figure above shows the graph of a cubic polynomial $f(x)$ given by

$$f(x) = -x^3 + 5x^2 + 17x - 21,\ x \in \mathbb{R}.$$

The graph meets the coordinate axes at four distinct points, labelled A, B, C and D.

Given that the coordinates of the point A are $(-3,0)$, determine the coordinates of the points B, C and D.

, $\boxed{B(1,0)}$, $\boxed{C(7,0)}$, $\boxed{D(0,-21)}$



AS $A(-3,0)$, $x+3$ MUST BE ONE OF THE FACTORS

$\Rightarrow f(x) = -x^3 + 5x^2 + 17x - 21$

$\Rightarrow -f(x) = x^3 - 5x^2 - 17x + 21$

BY LONG DIVISION OR MANIPULATIONS

$\Rightarrow -f(x) = x^2(x+3) - 8x(x+3) + 7(x+3)$

$\Rightarrow -f(x) = (x+3)(x^2 - 8x + 7)$

$\Rightarrow -f(x) = (x+3)(x-1)(x-7)$

$\Rightarrow f(x) = (x+3)(1-x)(x-7)$

$A(-3,0)$ $B(1,0)$ $C(7,0)$

AND WHEN $x=0$ $y=-21$, $\therefore D(0,-21)$

$\therefore A(-3,0)$, $B(1,0)$, $C(7,0)$, $D(0,-21)$

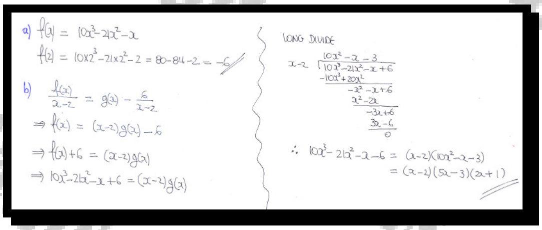
Question 46 **(***+)**

$$f(x) \equiv 10x^3 - 21x^2 - x.$$

a) Find the remainder when $f(x)$ is divided by $(x-2)$.

b) Hence express $10x^3 - 21x^2 - x + 6$ as a product of three linear factors.

$\boxed{}$, $\boxed{R=-6}$, $\boxed{(x-2)(5x-3)(2x+1)}$



a) $f(x) = 10x^3 - 21x^2 - x$

$f(2) = 10 \times 2^3 - 21 \times 2^2 - 2 = 80 - 84 - 2 = -6$

b) $\frac{f(x)}{x-2} = g(x) - \frac{6}{x-2}$

$\Rightarrow f(x) = (x-2)g(x) - 6$

$\Rightarrow f(x) + 6 = (x-2)g(x)$

$\Rightarrow 10x^3 - 21x^2 - x + 6 = (x-2)g(x)$

LONG DIVIDE

$$\begin{array}{r|l} & 10x^2 - x - 3 \\ x-2 & 10x^3 - 21x^2 - x + 6 \\ & -10x^3 + 20x^2 \\ & \quad -x^2 - x + 6 \\ & \quad x^2 - 2x \\ & \qquad -3x + 6 \\ & \qquad 3x - 6 \\ & \qquad\quad 0 \end{array}$$

$\therefore 10x^3 - 21x^2 - x - 6 = (x-2)(10x^2 - x - 3)$

$= (x-2)(5x-3)(2x+1)$

Question 47 (***+)

$$f(x) \equiv x^3 - 3x^2 - 6x + 8, \ x \in \mathbb{R}.$$

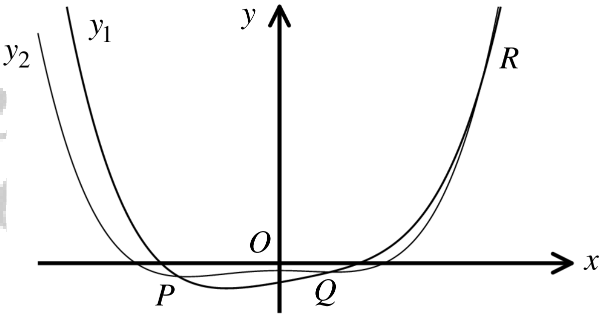
a) Show that $(x-1)$ is a factor of $f(x)$.

b) Hence factorize $f(x)$ into three linear factors.

c) Sketch the graph of $f(x)$.
The sketch must include the coordinates of any points where the graph of $f(x)$ meets the coordinate axes.

The figure below shows the graphs of the curves with equations

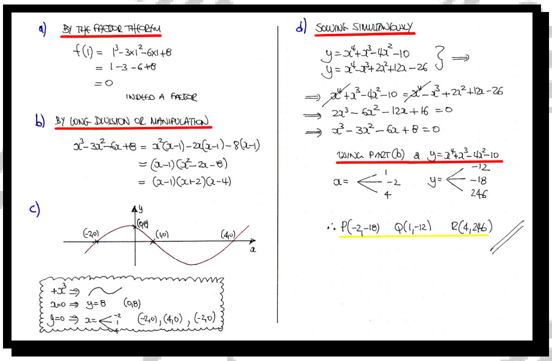
$$y_1 = x^4 + x^3 - 4x^2 - 10 \qquad \text{and} \qquad y_2 = x^4 - x^3 + 2x^2 + 12x - 26.$$



The two graphs meet at the points P, Q and R.

d) Determine the coordinates of P, Q and R.

$\boxed{}$, $\boxed{f(x) = (x-1)(x+2)(x-4)}$, $\boxed{P(-2,-18)}$, $\boxed{Q(1,-12)}$, $\boxed{R(4,246)}$



a) BY THE FACTOR THEOREM

$f(1) = 1^3 - 3\times1^2 - 6\times1 + 8$
$= 1 - 3 - 6 + 8$
$= 0$

INDEED A FACTOR

b) BY LONG DIVISION OR MANIPULATION

$x^3 - 3x^2 - 6x + 8 = x^2(x-1) - 2x(x-1) - 8(x-1)$
$= (x-1)(x^2 - 2x - 8)$
$= (x-1)(x+2)(x-4)$

c)

$+x^3 \Rightarrow$ ~
$x=0 \Rightarrow y=8 \quad (0,8)$
$y=0 \Rightarrow x = \begin{cases} -2 \\ 1 \\ 4 \end{cases} \quad (-2,0), (4,0), (-2,0)$

d) SOLVING SIMULTANEOUSLY

$\left.\begin{array}{l} y = x^4 + x^3 - 4x^2 - 10 \\ y = x^4 - x^3 + 2x^2 + 12x - 26 \end{array}\right\} \Longrightarrow$

$\Longrightarrow \cancel{x^4} + x^3 - 4x^2 - 10 = \cancel{x^4} - x^3 + 2x^2 + 12x - 26$
$\Longrightarrow 2x^3 - 6x^2 - 12x + 16 = 0$
$\Longrightarrow x^3 - 3x^2 - 6x + 8 = 0$

USING PART (b) & $y = x^4 + x^3 - 4x^2 - 10$

$x = \begin{cases} 1 \\ -2 \\ 4 \end{cases} \qquad y = \begin{cases} -12 \\ -18 \\ 246 \end{cases}$

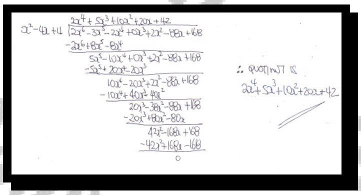
$\therefore P(-2,-18) \quad Q(1,-12) \quad R(4,246)$ //

Question 48 **(***+)**

Find the quotient of the division of

$$2x^6-3x^5-2x^4+2x^2-88x+168 \quad \text{by} \quad x^2-4x+4.$$

$$\boxed{2x^4+5x^3+10x^2+20x+42}$$



$x^2-4x+4 \,\big)\, 2x^6-3x^5-2x^4+0x^3+2x^2-88x+168$ (quotient $2x^4+5x^3+10x^2+20x+42$)

$-2x^6+8x^5-8x^4$

$5x^5-10x^4+0x^3+2x^2-88x+168$

$-5x^5+20x^4-20x^3$

$10x^4-20x^3+2x^2-88x+168$

$-10x^4+40x^3-40x^2$

$20x^3-38x^2-88x+168$

$-20x^3+80x^2-80x$

$42x^2-168x+168$

$-42x^2+168x-168$

0

∴ QUOTIENT IS

$2x^4+5x^3+10x^2+20x+42$

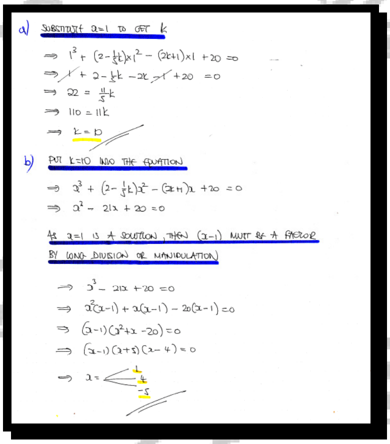
Question 49 **(***+)**

$$x^3+\left(2-\tfrac{1}{5}k\right)x^2-(2k+1)x+20=0.$$

a) Determine the value of the real constant k, if the above equation is to have $x=1$ as one of its roots.

b) Solve the equation for the value of k, found in part **(a)**.

$\boxed{}$, $\boxed{k=10}$, $\boxed{x=-5,4,1}$



a) SUBSTITUTE $x=1$ TO GET k

$\Rightarrow 1^3+\left(2-\frac{1}{5}k\right)\times1^2-(2k+1)\times1+20=0$

$\Rightarrow \cancel{1}+2-\frac{1}{5}k-2k-\cancel{1}+20=0$

$\Rightarrow 22=\frac{11}{5}k$

$\Rightarrow 110=11k$

$\Rightarrow k=10$

b) PUT $k=10$ INTO THE EQUATION

$\Rightarrow x^3+\left(2-\frac{1}{5}k\right)x^2-(2k+1)x+20=0$

$\Rightarrow x^3-21x+20=0$

AS $x=1$ IS A SOLUTION, THEN $(x-1)$ MUST BE A FACTOR

BY LONG DIVISION OR MANIPULATION

$\Rightarrow x^3-21x+20=0$

$\Rightarrow x^2(x-1)+x(x-1)-20(x-1)=0$

$\Rightarrow (x-1)(x^2+x-20)=0$

$\Rightarrow (x-1)(x+5)(x-4)=0$

$\Rightarrow x=1,\ 4,\ -5$

Question 50 **(***+)**

A cubic curve C has equation

$$y = 6x^3 + Ax^2 - 6x + B\,,\ x \in \mathbb{R}\,,$$

where A and B are constants.

The graph of C meets the x axis at $(5,0)$.

When the equation of C is divided by $(x-1)$ the remainder is -24.

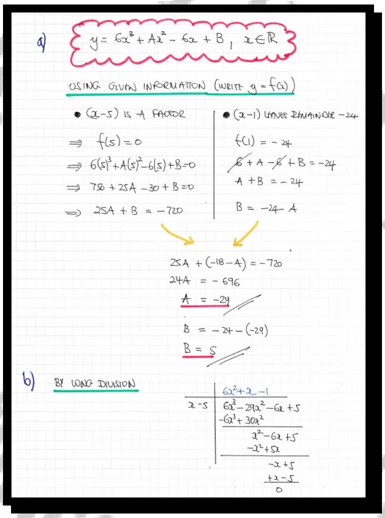
a) Determine the value of A and the value of B.

b) Factorize fully the equation of C.

c) Sketch the graph of C.

The sketch must show clearly the coordinates of any points where the graph of C meets the coordinate axes.

$\boxed{\text{[illegible]}}$, $\boxed{A = -29, \quad B = 5}$, $\boxed{y = (x-5)(3x-1)(2x+1)}$



a) $y = 6x^3 + Ax^2 - 6x + B\,,\ x \in \mathbb{R}$

USING GIVEN INFORMATION (WRITE $y = f(x)$)

- $(x-5)$ IS A FACTOR

$\Rightarrow f(5) = 0$

$\Rightarrow 6(5)^3 + A(5)^2 - 6(5) + B = 0$

$\Rightarrow 750 + 25A - 30 + B = 0$

$\Rightarrow 25A + B = -720$

- $(x-1)$ LEAVES REMAINDER -24

$f(1) = -24$

$\cancel{6} + A - \cancel{6} + B = -24$

$A + B = -24$

$B = -24 - A$

$25A + (-18 - A) = -720$

$24A = -696$

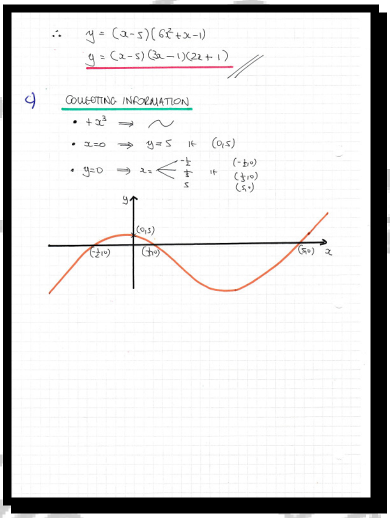
$A = -29$ //

$B = -24 - (-29)$

$B = 5$ //

b) BY LONG DIVISION

$$\begin{array}{r|l} & 6x^2 + x - 1 \\ \hline x-5 & 6x^3 - 29x^2 - 6x + 5 \\ & -6x^3 + 30x^2 \\ \hline & \qquad x^2 - 6x + 5 \\ & \qquad -x^2 + 5x \\ \hline & \qquad\qquad -x + 5 \\ & \qquad\qquad +x - 5 \\ \hline & \qquad\qquad\quad 0 \end{array}$$



$\therefore\ y = (x-5)(6x^2 + x - 1)$

$y = (x-5)(3x-1)(2x+1)$ //

c) COLLECTING INFORMATION

- $+x^3 \Rightarrow \sim$
- $x = 0 \Rightarrow y = 5 \quad \therefore \quad (0,5)$
- $y = 0 \Rightarrow x = \begin{cases} -\frac{1}{2} \\ \frac{1}{3} \\ 5 \end{cases} \quad \therefore \quad \begin{matrix} (-\frac{1}{2},0) \\ (\frac{1}{3},0) \\ (5,0) \end{matrix}$

Question 51 (***+)

The following information is given for a polynomial $f(x)$.

- When $f(x)$ is divided by $(x-2)$ the remainder is 5.
- When $f(x)$ is divided by $(x+2)$ the remainder is -11.
- When $f(x)$ is divided by $(x+2)(x+2)$ the remainder is $ax+b$, and the quotient is $g(x)$, where a and b are constants, so that

$$f(x)=(x-2)(x+2)g(x)+ax+b$$

a) Determine the value of a and the value of b.

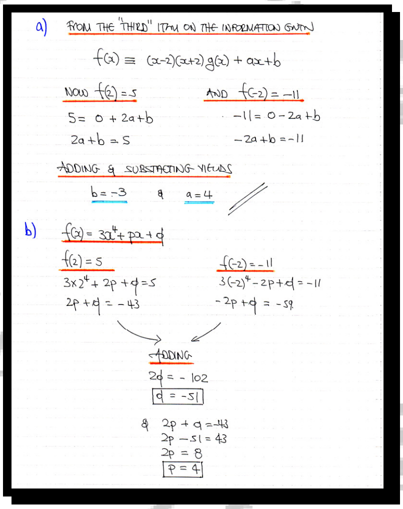
It is further given that

$$f(x)=3x^4+px+q,$$

where p and q are constants.

b) Find the value of p and the value of q.

$\boxed{}$, $\boxed{a=-4}$, $\boxed{b=3}$, $\boxed{p=4}$, $\boxed{q=-51}$



a) FROM THE "THIRD" ITEM ON THE INFORMATION GIVEN

$$f(x) \equiv (x-2)(x+2)g(x)+ax+b$$

NOW $f(2)=5$

$5=0+2a+b$

$2a+b=5$

AND $f(-2)=-11$

$-11=0-2a+b$

$-2a+b=-11$

ADDING & SUBSTRACTING YIELDS

$b=-3$ & $a=4$ //

b) $f(x)=3x^4+px+q$

$f(2)=5$

$3\times 2^4+2p+q=5$

$2p+q=-43$

$f(-2)=-11$

$3(-2)^4-2p+q=-11$

$-2p+q=-59$

ADDING

$2q=-102$

$\boxed{q=-51}$

& $2p+q=-43$

$2p-51=43$

$2p=8$

$\boxed{p=4}$

Question 52 (***+)

A cubic curve and a quartic curve, are both defined for all real numbers, and have respective equations

$$y = x^3 - 3x^2 \quad \text{and} \quad y = x(x-2)^3 .$$

a) Sketch both curves in the same set of axes, indicating the coordinates of any points where each curve meets the coordinate axes.

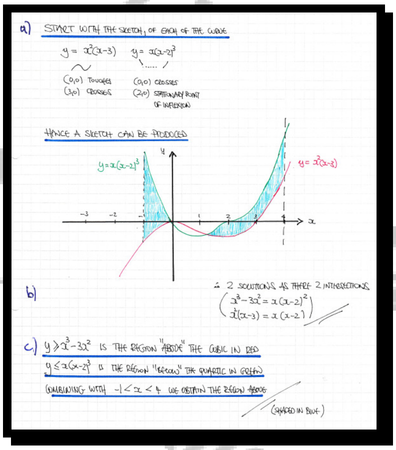
b) State the number of solutions of the equation

$$x^3 - 3x^2 = x(x-2)^3 , \ x \in \mathbb{R} .$$

c) Indicate by shading in the set of axes of part **(a)** the region satisfied by the following inequality.

$$x^3 - 3x^2 \le y \le x(x-2)^3 \quad \cap \quad -1 < x < 4 .$$

$\boxed{}$, $\boxed{\text{2 solutions}}$



a) START WITH THE SKETCH, OF EACH OF THE CURVE

$y = x^2(x-3)$ $y = x(x-2)^3$

(0,0) TOUCHES (0,0) CROSSES

(3,0) CROSSES (2,0) STATIONARY POINT OF INFLEXION

HENCE A SKETCH CAN BE PRODUCED

b) ∴ 2 SOLUTIONS AS THERE 2 INTERSECTIONS

$\left(\begin{array}{l} x^3-3x^2 = x(x-2)^2 \\ x^2(x-3) = x(x-2) \end{array}\right)$

c) $y \geqslant x^3-3x^2$ IS THE REGION "ABOVE" THE CUBIC IN RED

$y \leqslant x(x-2)^3$ IS THE REGION "BELOW" THE QUARTIC IN GREEN

COMBINING WITH $-1 < x < 4$ WE OBTAIN THE REGION ABOVE

(SHADED IN BLUE)

Question 53 (****)

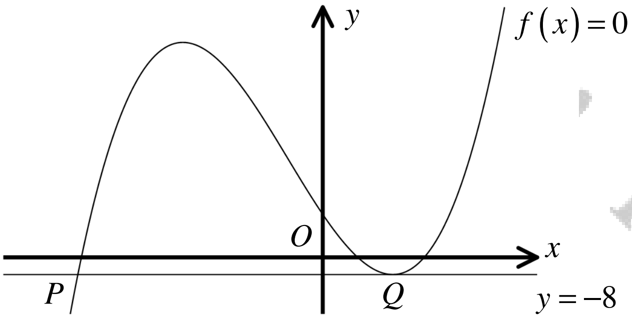
$$f(x) \equiv x^3 + 3x^2 - 24x + 20, \ x \in \mathbb{R}.$$

a) Show that $(x-1)$ is a factor of $f(x)$.

b) Hence factorize $f(x)$ as the product of a linear and a quadratic factor.

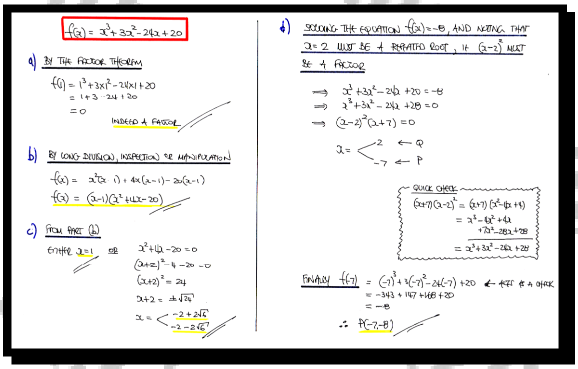
c) Find, in exact form where appropriate, the solutions of the equation $f(x)=0$.

The line with equation $y=-8$ **touches** the graph of $f(x)$ at the point $Q(2,-8)$ and crosses the graph of $f(x)$ at the point P, as shown in the figure below.



d) Determine the coordinates of P.

$\boxed{\ }$, $\boxed{f(x)=(x-1)(x^2+4x-20)}$, $\boxed{x=1, \ -2\pm 2\sqrt{6}}$, $\boxed{P(-7,-8)}$



$\boxed{f(x) = x^3+3x^2-24x+20}$

a) BY THE FACTOR THEOREM

$f(1) = 1^3+3\times 1^2-24\times 1+20$
$= 1+3-24+20$
$=0$
INDEED A FACTOR

b) BY LONG DIVISION, INSPECTION OR MANIPULATION

$f(x) = x^2(x-1)+4x(x-1)-20(x-1)$
$f(x) = (x-1)(x^2+4x-20)$

c) FROM PART (b)

EITHER $x=1$ OR

$x^2+4x-20=0$
$(x+2)^2-4-20=0$
$(x+2)^2=24$
$x+2=\pm\sqrt{24}$
$x = \begin{cases} -2+2\sqrt{6} \\ -2-2\sqrt{6} \end{cases}$

d) SOLVING THE EQUATION $f(x)=-8$, AND NOTING THAT $x=2$ MUST BE A REPEATED ROOT, IE $(x-2)^2$ MUST BE A FACTOR

$\Rightarrow x^3+3x^2-24x+20=-8$
$\Rightarrow x^3+3x^2-24x+28=0$
$\Rightarrow (x-2)^2(x+7)=0$

$x = \begin{cases} 2 \leftarrow Q \\ -7 \leftarrow P \end{cases}$

QUICK CHECK
$(x+7)(x-2)^2 = (x+7)(x^2-4x+4)$
$= x^3-4x^2+4x$
$+7x^2-28x+28$
$= x^3+3x^2-24x+28$

FINALLY $f(-7) = (-7)^3+3(-7)^2-24(-7)+20$ ← JUST AS A CHECK
$= -343+147+168+20$
$= -8$

$\therefore P(-7,-8)$

Question 54 (****)

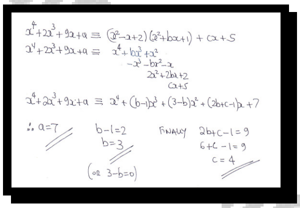
A polynomial $p(x)$ is defined, in terms of a constant a, by

$$p(x) = x^4 + 2x^3 + 9x + a.$$

When $p(x)$ is divided by $x^2 - x + 2$ the quotient is $x^2 + bx + 1$ and the remainder is $cx + 5$, where b and c are constants.

Find the value of a, b and c.

$\boxed{\text{SYN-Q}}$, $\boxed{a = 7,\ b = 3,\ c = 4}$



$x^4 + 2x^3 + 9x + a \equiv (x^2 - x + 2)(x^2 + bx + 1) + cx + 5$

$x^4 + 2x^3 + 9x + a \equiv x^4 + bx^3 + x^2$

$-x^3 - bx^2 - x$

$2x^2 + 2bx + 2$

$cx + 5$

$x^4 + 2x^3 + 9x + a \equiv x^4 + (b-1)x^3 + (3-b)x^2 + (2b+c-1)x + 7$

$\therefore a = 7$ $b - 1 = 2$ FINALLY $2b + c - 1 = 9$

$b = 3$ $6 + c - 1 = 9$

$c = 4$

(OR $3 - b = 0$)

Question 55 (****)

The quadratic function f is given, in terms of three non zero constants a, b and c, by

$$f(x) \equiv ax^2 + bx + c,\ x \in \mathbb{R}.$$

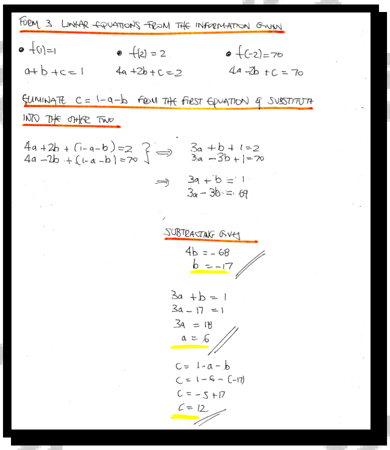
When $f(x)$ is divided by $(x-1)$ the remainder is 1.

When $f(x)$ is divided by $(x-2)$ the remainder is 2.

When $f(x)$ is divided by $(x+2)$ the remainder is 70.

Determine the value of each of the constants a, b and c.

$\boxed{\quad}$, $\boxed{a=6}$, $\boxed{b=-17}$, $\boxed{c=12}$



FORM 3 LINEAR EQUATIONS FROM THE INFORMATION GIVEN

- $f(1)=1$ $\quad$ $a+b+c=1$
- $f(2)=2$ $\quad$ $4a+2b+c=2$
- $f(-2)=70$ $\quad$ $4a-2b+c=70$

ELIMINATE $c=1-a-b$ FROM THE FIRST EQUATION & SUBSTITUTE INTO THE OTHER TWO

$$\left.\begin{array}{l}4a+2b+(1-a-b)=2\\ 4a-2b+(1-a-b)=70\end{array}\right\} \Rightarrow \begin{array}{l}3a+b+1=2\\ 3a-3b+1=70\end{array}$$

$$\Rightarrow \begin{array}{l}3a+b=1\\ 3a-3b=69\end{array}$$

SUBTRACTING GIVES

$$4b=-68$$
$$b=-17 \;//$$

$$3a+b=1$$
$$3a-17=1$$
$$3a=18$$
$$a=6 \;//$$

$$c=1-a-b$$
$$c=1-6-(-17)$$
$$c=-5+17$$
$$c=12 \;//$$

Question 56 (****)

$$f(x) \equiv x^3 - 3x + 2, \; x \in \mathbb{R}.$$

a) Express $f(x)$ as the product of three linear factors.

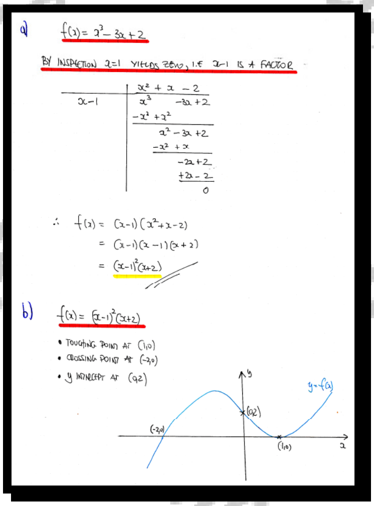
b) Sketch the graph of $f(x)$.

The sketch must include the coordinates of any points where the graph of $f(x)$ meets the coordinate axes.

c) Solve the equation

$$f(x) = (x-1)^2.$$

[illegible], $\boxed{f(x) = (x+2)(x-1)^2}$, $\boxed{x = \pm 1}$



a) $f(x) = x^3 - 3x + 2$

BY INSPECTION $x=1$ YIELDS ZERO, I.E $x-1$ IS A FACTOR

$$\begin{array}{r|l} & x^2 + x - 2 \\ \hline x-1 & x^3 \qquad -3x+2 \\ & \underline{-x^3 + x^2} \\ & \quad x^2 - 3x + 2 \\ & \quad \underline{-x^2 + x} \\ & \qquad -2x + 2 \\ & \qquad \underline{+2x - 2} \\ & \qquad\qquad 0 \end{array}$$

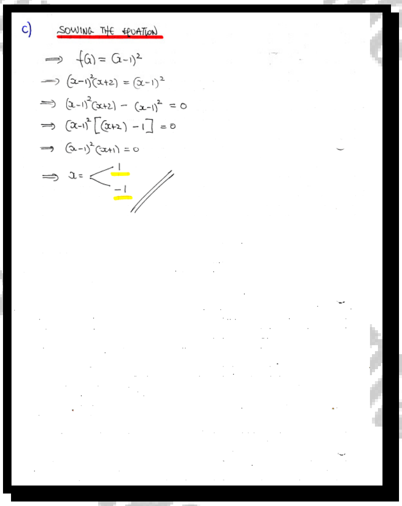
$\therefore \; f(x) = (x-1)(x^2+x-2)$

$= (x-1)(x-1)(x+2)$

$= (x-1)^2(x+2)$ //

b) $f(x) = (x-1)^2(x+2)$

- TOUCHING POINT AT $(1,0)$
- CROSSING POINT AT $(-2,0)$
- y INTERCEPT AT $(0,2)$



c) SOLVING THE EQUATION

$\Longrightarrow f(x) = (x-1)^2$

$\Longrightarrow (x-1)^2(x+2) = (x-1)^2$

$\Longrightarrow (x-1)^2(x+2) - (x-1)^2 = 0$

$\Longrightarrow (x-1)^2\left[(x+2) - 1\right] = 0$

$\Longrightarrow (x-1)^2(x+1) = 0$

$\Longrightarrow x = \begin{cases} 1 \\ -1 \end{cases}$ //

Question 57 **(****)**

A quintic polynomial is defined, in terms of the constants a and b, by

$$f(x) = x^5 + ax^4 + bx^3 - x^2 + 4x - 3.$$

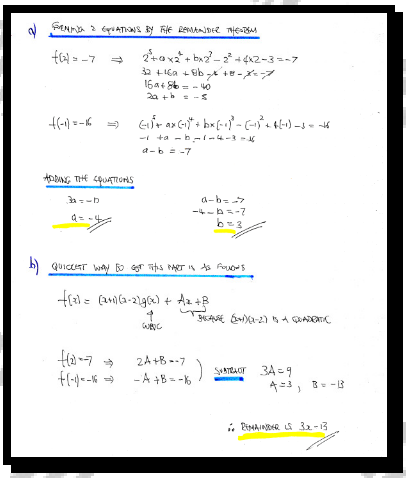
When $f(x)$ is divided by $(x-2)$ the remainder is -7.

When $f(x)$ is divided by $(x+1)$ the remainder is -16.

a) Determine in any order the values of a and b.

b) Find the remainder when $f(x)$ is divided by $(x-2)(x+1)$.

$\boxed{}$, $\boxed{a=-4}$, $\boxed{b=3}$, $\boxed{3x-13}$



a) FORMING 2 EQUATIONS BY THE REMAINDER THEOREM

$$f(2) = -7 \Rightarrow 2^5 + a \times 2^4 + b \times 2^3 - 2^2 + 4 \times 2 - 3 = -7$$
$$32 + 16a + 8b - \cancel{4} + 8 - \cancel{3} = -\cancel{7}$$
$$16a + 8b = -40$$
$$2a + b = -5$$

$$f(-1) = -16 \Rightarrow (-1)^5 + a \times (-1)^4 + b \times (-1)^3 - (-1)^2 + 4(-1) - 3 = -16$$
$$-1 + a - b - 1 - 4 - 3 = -16$$
$$a - b = -7$$

ADDING THE EQUATIONS

$$3a = -12$$
$$a = -4$$

$$a - b = -7$$
$$-4 - b = -7$$
$$b = 3$$

b) QUICKEST WAY TO GET THIS PART IS AS FOLLOWS

$$f(x) = \underset{\substack{\uparrow \\ \text{CUBIC}}}{(x+1)(x-2)g(x)} + \underbrace{Ax+B}_{\text{BECAUSE } (x+1)(x-2) \text{ IS A QUADRATIC}}$$

$$f(2) = -7 \Rightarrow 2A + B = -7$$
$$f(-1) = -16 \Rightarrow -A + B = -16$$

SUBTRACT $3A = 9$

$A = 3$, $B = -13$

∴ REMAINDER IS $3x - 13$

Question 58 (****)

A polynomial $f(x)$ is defined in terms of the constants a, b and c as

$$f(x) = 2x^3 + ax^2 + bx + c, \quad x \in \mathbb{R}.$$

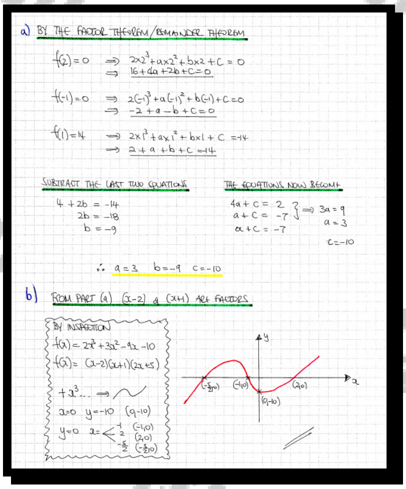
It is further given that

$$f(2) = f(-1) = 0 \quad \text{and} \quad f(1) = -14.$$

- **a)** Find the values of a, b and c.
- **b)** Sketch the graph of $f(x)$.

  The sketch must include the coordinates of any points where the graph of $f(x)$ *meets the coordinate axes.*

$\boxed{}$, $\boxed{a = 3,\ b = -9,\ c = -10}$



a) BY THE FACTOR THEOREM/REMAINDER THEOREM

$f(2)=0 \Rightarrow 2\times2^3+a\times2^2+b\times2+c=0$
$\Rightarrow \underline{16+4a+2b+c=0}$

$f(-1)=0 \Rightarrow 2(-1)^3+a(-1)^2+b(-1)+c=0$
$\Rightarrow \underline{-2+a-b+c=0}$

$f(1)=14 \Rightarrow 2\times1^3+a\times1^2+b\times1+c=-14$
$\Rightarrow \underline{2+a+b+c=-14}$

SUBTRACT THE LAST TWO EQUATIONS

$4+2b=-14$
$2b=-18$
$b=-9$

THE EQUATIONS NOW BECOME

$\left.\begin{array}{l}4a+c=2\\ a+c=-7\end{array}\right\} \Rightarrow 3a=9$
$a+c=-7$
$a=3$
$c=-10$

$\therefore\ a=3 \quad b=-9 \quad c=-10$

b) FROM PART (a) $(x-2)$ & $(x+1)$ ARE FACTORS

BY INSPECTION
$f(x)=2x^3+3x^2-9x-10$
$f(x)=(x-2)(x+1)(2x+5)$

$+x^3 \dots \Rightarrow$

$x=0 \quad y=-10 \quad (0,-10)$

$y=0 \quad x= \begin{cases}-1 & (-1,0)\\ 2 & (2,0)\\ -\frac{5}{2} & (-\frac{5}{2},0)\end{cases}$

Question 59 **(****)**

A polynomial $p(x)$ is given by

$$p(x) = 4x^3 - 2x^2 + x + 5 .$$

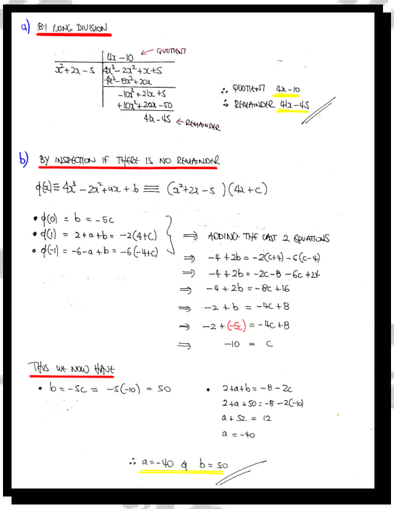
a) Find the remainder and the quotient when $p(x)$ is divided by $x^2 + 2x - 5$.

A different polynomial $q(x)$ is defined as

$$q(x) = 4x^3 - 2x^2 + ax + b .$$

b) Find the value of the constants a and b so that when $q(x)$ is divided by $x^2 + 2x - 5$ there is no remainder.

$\boxed{}$, $\boxed{R = 41x - 45}$, $\boxed{Q = 4x - 10}$, $\boxed{a = -40, \ b = 50}$



a) BY LONG DIVISION

$4x - 10$ ← QUOTIENT

$x^2+2x-5 \,\big|\, 4x^3-2x^2+x+5$

$\cancel{4x^3}-8x^2+20x$

$-10x^2+21x+5$

$+10x^2+20x-50$

$41x-45$ ← REMAINDER

∴ QUOTIENT $4x-10$

∴ REMAINDER $41x-45$

b) BY INSPECTION IF THERE IS NO REMAINDER

$q(x) \equiv 4x^3 - 2x^2 + ax + b \equiv (x^2+2x-5)(4x+c)$

- $q(0) = b = -5c$
- $q(1) = 2+a+b = -2(4+c)$
- $q(-1) = -6-a+b = -6(-4+c)$

$\Rightarrow$ ADDING THE LAST 2 EQUATIONS

$\Rightarrow -4+2b = -2(c+4) - 6(c-4)$

$\Rightarrow -4+2b = -2c-8-6c+24$

$\Rightarrow -4+2b = -8c+16$

$\Rightarrow -2+b = -4c+8$

$\Rightarrow -2+(-5c) = -4c+8$

$\Rightarrow -10 = c$

THUS WE NOW HAVE

- $b = -5c = -5(-10) = 50$
- $2+a+b = -8-2c$

$2+a+50 = -8-2(-10)$

$a+52 = 12$

$a = -40$

$\therefore a = -40$ & $b = 50$

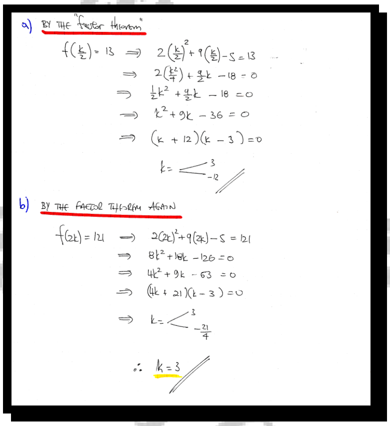
Question 60 (****)

$$f(x) = 2x^2 + 9x - 5$$

a) Given that when $f(x)$ is divided by $(2x-k)$ the remainder is 13, find the possible values of k.

b) Given further that when $f(x)$ is divided by $(x-2k)$ the remainder is 121, find the value of k.

$\boxed{\text{SYN-U}}$, $\boxed{k=-12,3}$, $\boxed{k=3}$



a) BY THE "factor theorem"

$f\left(\frac{k}{2}\right) = 13 \Rightarrow 2\left(\frac{k}{2}\right)^2 + 9\left(\frac{k}{2}\right) - 5 = 13$

$\Rightarrow 2\left(\frac{k^2}{4}\right) + \frac{9}{2}k - 18 = 0$

$\Rightarrow \frac{1}{2}k^2 + \frac{9}{2}k - 18 = 0$

$\Rightarrow k^2 + 9k - 36 = 0$

$\Rightarrow (k+12)(k-3) = 0$

$k = \begin{cases} 3 \\ -12 \end{cases}$ //

b) BY THE FACTOR THEOREM AGAIN

$f(2k) = 121 \Rightarrow 2(2k)^2 + 9(2k) - 5 = 121$

$\Rightarrow 8k^2 + 18k - 126 = 0$

$\Rightarrow 4k^2 + 9k - 63 = 0$

$\Rightarrow (4k+21)(k-3) = 0$

$\Rightarrow k = \begin{cases} 3 \\ -\frac{21}{4} \end{cases}$

$\therefore k = 3$ //

Question 61 (****)

A cubic function is defined in terms of the constants a, b and c as

$$f(x) = x^3 + ax^2 + bx + c, \ x \in \mathbb{R}.$$

a) Given that $(x-1)$ is a factor of $f(x)$ show that

$$a + b + c = -1.$$

It is further given that when $f(x)$ is divided by $(x-2)$ the remainder is -4 and when $f(x)$ is divided by $(x-3)$ the remainder is -12.

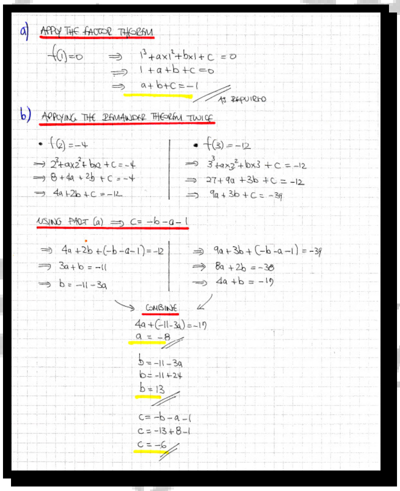
b) Find the values of a, b and c.

c) Hence express $f(x)$ as the product of three linear factors.

d) Sketch the graph of $f(x)$.

The sketch must include the coordinates of any points where the graph of $f(x)$ meets the coordinate axes.

$\boxed{\quad\quad}$, $\boxed{a=-8, \ b=13, \ c=-6}$, $\boxed{(x-1)^2(x-6)}$



a) APPLY THE FACTOR THEOREM

$f(1)=0 \Rightarrow 1^3+a\times1^2+b\times1+c=0$

$\Rightarrow 1+a+b+c=0$

$\Rightarrow a+b+c=-1$ // AS REQUIRED

b) APPLYING THE REMAINDER THEOREM TWICE

- $f(2)=-4$

$\Rightarrow 2^3+a\times2^2+b\times2+c=-4$

$\Rightarrow 8+4a+2b+c=-4$

$\Rightarrow 4a+2b+c=-12$

- $f(3)=-12$

$\Rightarrow 3^3+a\times3^2+b\times3+c=-12$

$\Rightarrow 27+9a+3b+c=-12$

$\Rightarrow 9a+3b+c=-39$

USING PART (a) $\Rightarrow c=-b-a-1$

$\Rightarrow 4a+2b+(-b-a-1)=-12$

$\Rightarrow 3a+b=-11$

$\Rightarrow b=-11-3a$

$\Rightarrow 9a+3b+(-b-a-1)=-39$

$\Rightarrow 8a+2b=-38$

$\Rightarrow 4a+b=-19$

COMBINE

$4a+(-11-3a)=-19$

$a=-8$ //

$b=-11-3a$

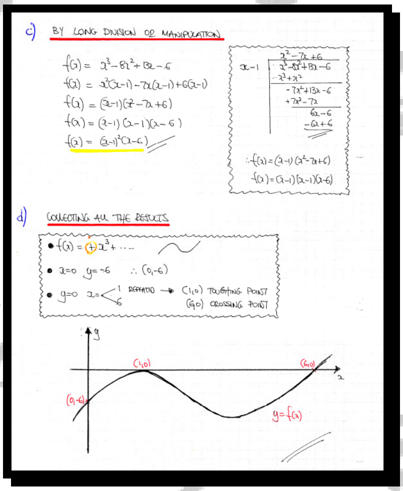
$b=-11+24$

$b=13$ //

$c=-b-a-1$

$c=-13+8-1$

$c=-6$ //



c) BY LONG DIVISION OR MANIPULATION

$f(x)= x^3-8x^2+13x-6$

$f(x) = x^2(x-1)-7x(x-1)+6(x-1)$

$f(x) = (x-1)(x^2-7x+6)$

$f(x) = (x-1)(x-1)(x-6)$

$f(x) = (x-1)^2(x-6)$ //

$$\begin{array}{r|l} & x^2-7x+6 \\ x-1 & x^3-8x^2+13x-6 \\ & -x^3+x^2 \\ \hline & -7x^2+13x-6 \\ & +7x^2-7x \\ \hline & 6x-6 \\ & -6x+6 \end{array}$$

$\therefore f(x)=(x-1)(x^2-7x+6)$

$f(x)=(x-1)(x-1)(x-6)$

d) COLLECTING ALL THE RESULTS

- $f(x) = \textcircled{+}x^3 + \dots$
- $x=0 \ \ y=-6 \ \ \therefore (0,-6)$
- $y=0 \ \ x=<\begin{smallmatrix}1 \text{ REPEATED} \\ 6\end{smallmatrix} \rightarrow$ (1,0) TOUCHING POINT, (6,0) CROSSING POINT

Question 62 (****)

$$f(x) \equiv x^3 + (a+2)x^2 - 2x + b,$$

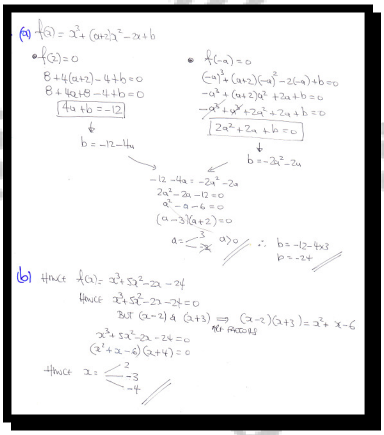
where a and b are non zero constants.

It is given that $(x-2)$ and $(x+a)$ are factors of $f(x)$, $a > 0$.

a) By forming two equations show that $a = 3$ and find the value of b.

b) Solve the equation $f(x) = 0$.

$\boxed{\quad\quad}$, $\boxed{b=-24}$, $\boxed{x=-4,-3,2}$



(a) $f(x) = x^3 + (a+2)x^2 - 2x + b$

- $f(2) = 0$

$8 + 4(a+2) - 4 + b = 0$

$8 + 4a + 8 - 4 + b = 0$

$\boxed{4a + b = -12}$

$\downarrow$

$b = -12 - 4a$

- $f(-a) = 0$

$(-a)^3 + (a+2)(-a)^2 - 2(-a) + b = 0$

$-a^3 + (a+2)a^2 + 2a + b = 0$

$\cancel{-a^3} + \cancel{a^3} + 2a^2 + 2a + b = 0$

$\boxed{2a^2 + 2a + b = 0}$

$\downarrow$

$b = -2a^2 - 2a$

$-12 - 4a = -2a^2 - 2a$

$2a^2 - 2a - 12 = 0$

$a^2 - a - 6 = 0$

$(a-3)(a+2) = 0$

$a = \begin{cases} 3 \\ \cancel{-2} \end{cases}$ $a > 0$ // $\therefore$ $b = -12 - 4 \times 3$

$b = -24$ //

(b) HENCE $f(x) = x^3 + 5x^2 - 2x - 24$

HENCE $x^3 + 5x^2 - 2x - 24 = 0$

BUT $(x-2)$ & $(x+3)$ $\underset{\text{ARE FACTORS}}{\Longrightarrow}$ $(x-2)(x+3) = x^2 + x - 6$

$x^3 + 5x^2 - 2x - 24 = 0$

$(x^2 + x - 6)(x+4) = 0$

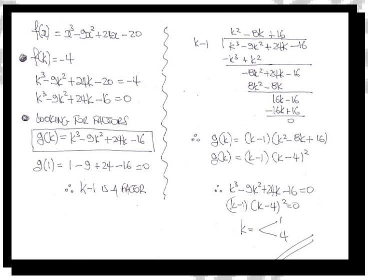
HENCE $x = \begin{cases} 2 \\ -3 \\ -4 \end{cases}$ //

Question 63 **(****)**

$$f(x) = x^3 - 9x^2 + 24x - 20$$

Given that when $f(x)$ is divided by $(x-k)$ the remainder is -4, find the possible values of k.

$\boxed{}$, $\boxed{k = 1,\ 4}$



$f(x) = x^3 - 9x^2 + 24x - 20$

- $f(k) = -4$

$k^3 - 9k^2 + 24k - 20 = -4$

$k^3 - 9k^2 + 24k - 16 = 0$

- LOOKING FOR FACTORS

$\boxed{g(k) = k^3 - 9k^2 + 24k - 16}$

$g(1) = 1 - 9 + 24 - 16 = 0$

$\therefore$ $k-1$ IS A FACTOR

$$\begin{array}{r|l} & k^2 - 8k + 16 \\ k-1 & k^3 - 9k^2 + 24k - 16 \\ & \underline{-k^3 + k^2} \\ & \quad -8k^2 + 24k - 16 \\ & \quad \underline{8k^2 - 8k} \\ & \qquad 16k - 16 \\ & \qquad \underline{-16k + 16} \\ & \qquad\quad 0 \end{array}$$

$\therefore$ $g(k) = (k-1)(k^2 - 8k + 16)$

$g(k) = (k-1)(k-4)^2$

$\therefore$ $k^3 - 9k^2 + 24k - 16 = 0$

$(k-1)(k-4)^2 = 0$

$k = \begin{cases} 1 \\ 4 \end{cases}$

Question 64 (****+)

The quadratic functions f and g are defined by

$$f(x) \equiv 4x^2 + a, \; x \in \mathbb{R}$$

$$g(x) \equiv x^2 + bx + a, \; x \in \mathbb{R},$$

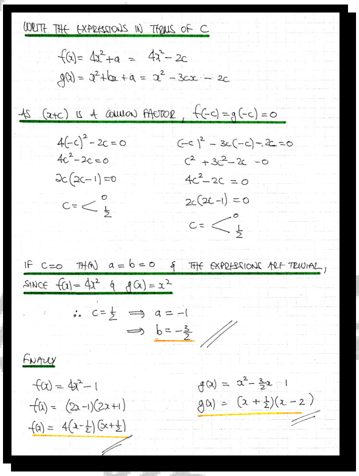
where a, b and c are non zero constants, such that $a = -2c$ and $b = -3c$.

It is further given that $(x+c)$ is a common factor f and g.

Determine the value of a, b and c, and hence factorize f and g, showing clearly the common factor in these factorizations.

$\boxed{a=-1}$, $\boxed{b=-\frac{3}{2}}$, $\boxed{c=\frac{1}{2}}$, $\boxed{f(x)\equiv 4\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)}$,

$\boxed{g(x)\equiv\left(x+\frac{1}{2}\right)(x-2)}$



WRITE THE EXPRESSIONS IN TERMS OF C

$f(x) = 4x^2 + a = 4x^2 - 2c$

$g(x) = x^2 + bx + a = x^2 - 3cx - 2c$

AS $(x+c)$ IS A COMMON FACTOR, $f(-c) = g(-c) = 0$

$4(-c)^2 - 2c = 0$

$4c^2 - 2c = 0$

$2c(2c-1) = 0$

$c = \begin{cases} 0 \\ \frac{1}{2} \end{cases}$

$(-c)^2 - 3c(-c) - 2c = 0$

$c^2 + 3c^2 - 2c = 0$

$4c^2 - 2c = 0$

$2c(2c-1) = 0$

$c = \begin{cases} 0 \\ \frac{1}{2} \end{cases}$

IF $c=0$ THEN $a = b = 0$ & THE EXPRESSIONS ARE TRIVIAL, SINCE $f(x) = 4x^2$ & $g(x) = x^2$

$\therefore c = \frac{1}{2} \Rightarrow a = -1$

$\Rightarrow b = -\frac{3}{2}$

FINALLY

$f(x) = 4x^2 - 1$

$f(x) = (2x-1)(2x+1)$

$f(x) = 4(x-\frac{1}{2})(x+\frac{1}{2})$

$g(x) = x^2 - \frac{3}{2}x - 1$

$g(x) = (x+\frac{1}{2})(x-2)$

Question 65 (****+)

A cubic curve has the following equation.

$$f(x) \equiv x^3 - 6x^2 + 12x + B, \quad x \in \mathbb{R},$$

where B is a non zero constant.

a) If $f(x)$ can be written in the form $(x-A)^3 - 4$, where A is also a non zero constant, find the value of A and the value of B.

A quadratic curve has the following equation.

$$g(x) \equiv x^2 - 4x + 5, \quad x \in \mathbb{R}.$$

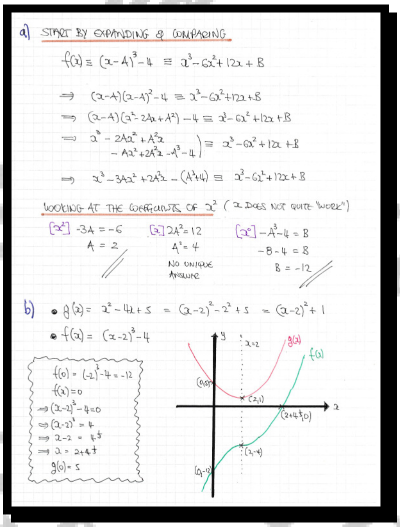
b) Sketch the graph of $f(x)$ and the graph of $g(x)$ in the same set of axes.

The sketch must include the coordinates of any points where each of the graphs meets the coordinate axes, the coordinates of the point of inflexion of $f(x)$ and the coordinates of the minimum point of $g(x)$.

c) Hence, state with full justification the number of real roots of the equation

$$x^3 - 7x^2 + 16x + B = 5.$$

$\boxed{\quad\quad}$, $\boxed{A = 2, \quad B = -12}$, $\boxed{\text{one real root}}$



a) START BY EXPANDING & COMPARING

$f(x) \equiv (x-A)^3 - 4 \equiv x^3 - 6x^2 + 12x + B$

$\Rightarrow (x-A)(x-A)^2 - 4 \equiv x^3 - 6x^2 + 12x + B$

$\Rightarrow (x-A)(x^2 - 2Ax + A^2) - 4 \equiv x^3 - 6x^2 + 12x + B$

$\Rightarrow \left.\begin{matrix} x^3 - 2Ax^2 + A^2x \\ - Ax^2 + 2A^2x - A^3 - 4 \end{matrix}\right\} \equiv x^3 - 6x^2 + 12x + B$

$\Rightarrow x^3 - 3Ax^2 + 2A^2x - (A^3+4) \equiv x^3 - 6x^2 + 12x + B$

LOOKING AT THE COEFFICIENTS OF x^2 (x DOES NOT QUITE "WORK")

$[x^2]$ $-3A = -6$, $A = 2$ //

$[x]$ $2A^2 = 12$, $A^2 = 4$ NO UNIQUE ANSWER

$[x^0]$ $-A^3 - 4 = B$, $-8 - 4 = B$, $B = -12$ //

b) • $g(x) = x^2 - 4x + 5 = (x-2)^2 - 2^2 + 5 = (x-2)^2 + 1$

• $f(x) = (x-2)^3 - 4$

$f(0) = (-2)^3 - 4 = -12$

$f(x) = 0$

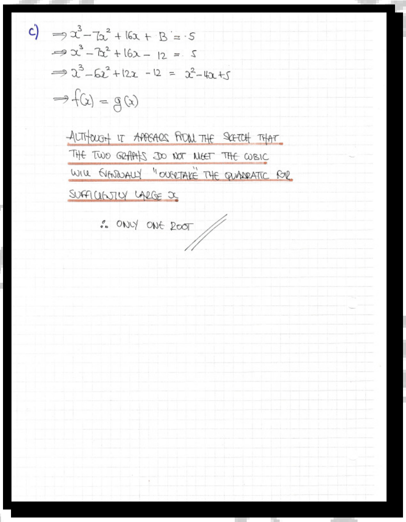
$\Rightarrow (x-2)^3 - 4 = 0$

$\Rightarrow (x-2)^3 = 4$

$\Rightarrow x - 2 = 4^{\frac{1}{3}}$

$\Rightarrow x = 2 + 4^{\frac{1}{3}}$

$g(0) = 5$



c) $\Rightarrow x^3 - 7x^2 + 16x + B = 5$

$\Rightarrow x^3 - 7x^2 + 16x - 12 = 5$

$\Rightarrow x^3 - 6x^2 + 12x - 12 = x^2 - 4x + 5$

$\Rightarrow f(x) = g(x)$

ALTHOUGH IT APPEARS FROM THE SKETCH THAT THE TWO GRAPHS DO NOT MEET THE CUBIC WILL EVENTUALLY "OVERTAKE" THE QUADRATIC FOR SUFFICIENTLY LARGE x

∴ ONLY ONE ROOT //

Question 66 (****+)

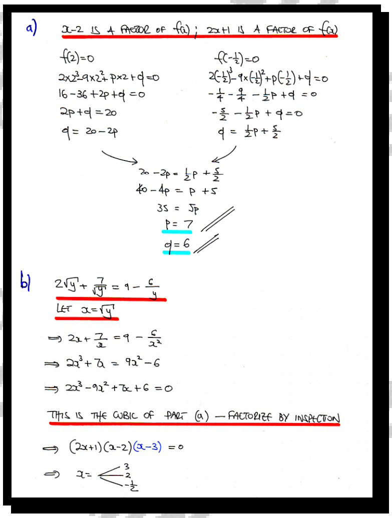
$$f(x) = 2x^3 - 9x^2 + px + q$$

a) Find the values of the constants p and q, given that $(x-2)$ and $(2x+1)$ are factors of $f(x)$.

b) Hence solve the equation

$$2\sqrt{y} + \frac{7}{\sqrt{y}} = 9 - \frac{6}{y}.$$

$\boxed{}$, $\boxed{p=7}$, $\boxed{q=6}$, $\boxed{y=4 \ \cup \ x=9}$



a) $x-2$ IS A FACTOR OF $f(x)$; $2x+1$ IS A FACTOR OF $f(x)$

$f(2)=0$

$2\times2^3-9\times2^2+p\times2+q=0$

$16-36+2p+q=0$

$2p+q=20$

$q=20-2p$

$f(-\frac{1}{2})=0$

$2(-\frac{1}{2})^3-9\times(-\frac{1}{2})^2+p(-\frac{1}{2})+q=0$

$-\frac{1}{4}-\frac{9}{4}-\frac{1}{2}p+q=0$

$-\frac{5}{2}-\frac{1}{2}p+q=0$

$q=\frac{1}{2}p+\frac{5}{2}$

$20-2p=\frac{1}{2}p+\frac{5}{2}$

$40-4p=p+5$

$35=5p$

$p=7$ //

$q=6$ //

b) $2\sqrt{y}+\frac{7}{\sqrt{y}}=9-\frac{6}{y}$

LET $x=\sqrt{y}$

$\Rightarrow 2x+\frac{7}{x}=9-\frac{6}{x^2}$

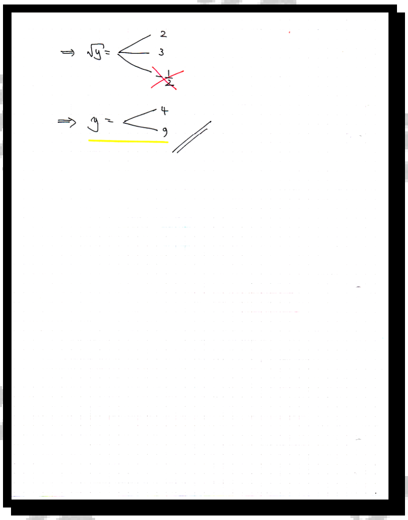
$\Rightarrow 2x^3+7x=9x^2-6$

$\Rightarrow 2x^3-9x^2+7x+6=0$

THIS IS THE CUBIC OF PART (a) — FACTORIZE BY INSPECTION

$\Rightarrow (2x+1)(x-2)(x-3)=0$

$\Rightarrow x = \left\langle \begin{array}{l} 3 \\ 2 \\ -\frac{1}{2} \end{array} \right.$



$\Rightarrow \sqrt{y} = \left\langle \begin{array}{l} 2 \\ 3 \\ \cancel{-\frac{1}{2}} \end{array} \right.$

$\Rightarrow y = \left\langle \begin{array}{l} 4 \\ 9 \end{array} \right.$ //

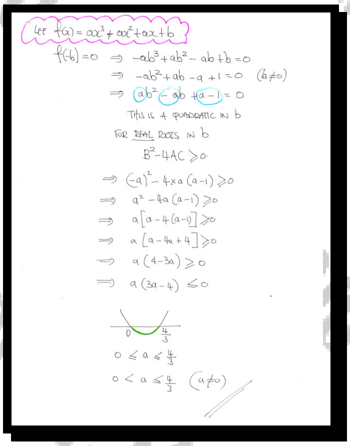
Question 67 (*****)

$$ax^3 + ax^2 + ax + b = 0,$$

where a and b are non zero real constants.

Given that $x = -b$ is a root of the above cubic equation, determine the range of the possible values of a.

$\boxed{}$, $\boxed{0 < a \le \frac{4}{3}}$



LET $f(x) = ax^3 + ax^2 + ax + b$

$f(-b) = 0 \Rightarrow -ab^3 + ab^2 - ab + b = 0$

$\Rightarrow -ab^2 + ab - a + 1 = 0 \quad (b \neq 0)$

$\Rightarrow ab^2 - ab + a - 1 = 0$

THIS IS A QUADRATIC IN b

FOR REAL ROOTS IN b

$B^2 - 4AC \geqslant 0$

$\Rightarrow (-a)^2 - 4 \times a(a-1) \geqslant 0$

$\Rightarrow a^2 - 4a(a-1) \geqslant 0$

$\Rightarrow a[a - 4(a-1)] \geqslant 0$

$\Rightarrow a[a - 4a + 4] \geqslant 0$

$\Rightarrow a(4 - 3a) \geqslant 0$

$\Rightarrow a(3a - 4) \leqslant 0$

$0 \leqslant a \leqslant \frac{4}{3}$

$0 < a \leqslant \frac{4}{3} \quad (a \neq 0)$