SIMPLE RECIPROCAL FUNCTIONS

Question 1 (**+)

The curves C_1 and C_2 have respective equations

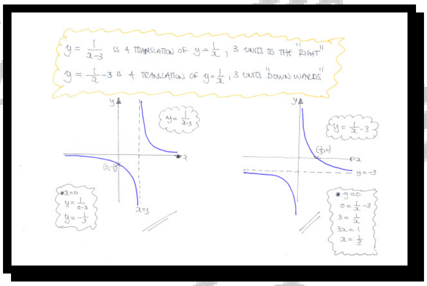
$$C_1: \ y = \frac{1}{x-3}, \ x \neq 3$$

$$C_2: \ y = \frac{1}{x} - 3, \ x \neq 0$$

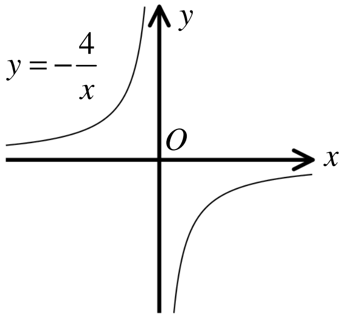
Sketch on separate diagrams the graph of C_1 and the graph of C_2.

Indicate clearly in each graph any asymptotes and the coordinates of any intersections with the coordinate axes.

$\boxed{}$, $\boxed{\text{graph}}$



Question 2 (**+)



The figure above shows the graph of the curve with equation

$$y = -\frac{4}{x}, \quad x \neq 0.$$

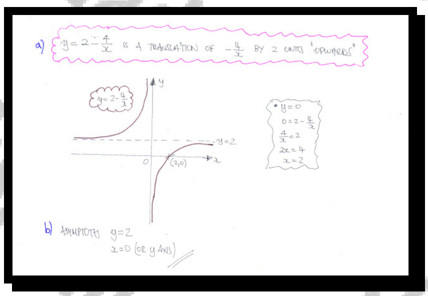
a) Sketch the graph of the curve C with equation

$$y = 2 - \frac{4}{x}, \quad x \neq 0.$$

Indicate clearly the coordinates of any points of intersection between C and the coordinate axes.

b) State the equations of the two asymptotes of C.

$\boxed{}$, $\boxed{x = 0, \ \ y = 2}$



Question 3 (***)

The curves C_1 and C_2 have respective equations

$$C_1: \ y = -\frac{1}{x}, \ x \neq 0$$

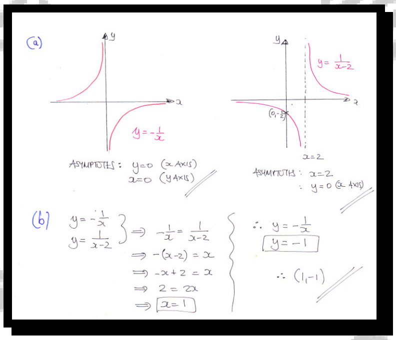
$$C_2: \ y = \frac{1}{x-2}, \ x \neq 2$$

a) Sketch on separate diagrams the graph of C_1 and the graph of C_2.

Indicate clearly in each graph any asymptotes and the coordinates of any intersections with the coordinate axes.

b) Find the coordinates of the point of intersection between C_1 and C_2.

$\boxed{}$, $\boxed{(1,-1)}$



Question 4 (***)

The curves C_1 and C_2 have respective equations

$$C_1: \ y = \frac{1}{x} + 2, \ x \neq 0$$

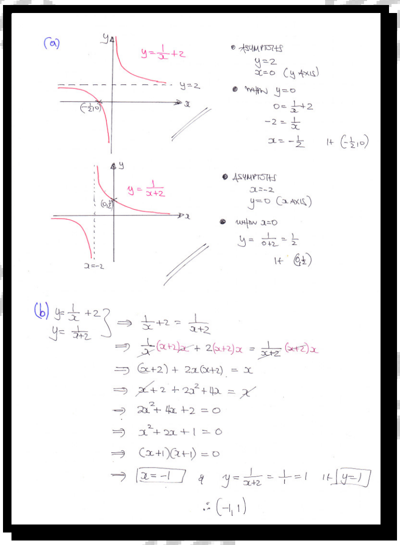
$$C_2: \ y = \frac{1}{x+2}, \ x \neq -2$$

a) Sketch on separate diagrams the graph of C_1 and the graph of C_2.

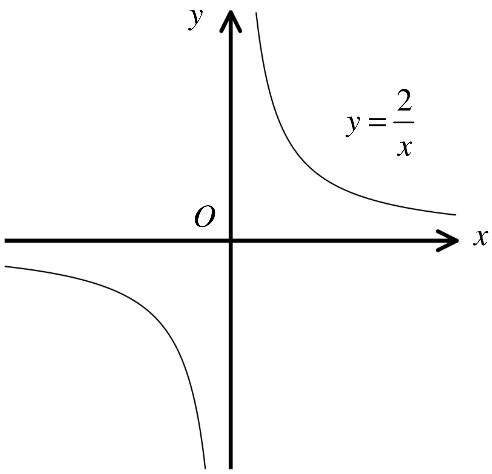
Indicate clearly in each graph any asymptotes and the coordinates of any intersections with the coordinate axes.

b) Find the coordinates of the point of intersection between C_1 and C_2.

$\boxed{}$, $\boxed{(-1,1)}$



Question 5 (***)



The figure above shows the graph of the curve C with equation

$$y = \frac{2}{x}, \ x \neq 0.$$

a) Describe the geometric transformation which maps the graph of C onto the graph with equation

$$y = \frac{2}{x-2}, \ x \neq 0.$$

b) Sketch the graph of the curve with equation

$$y = \frac{2}{x} + 2, \ x \neq 0.$$

Indicate clearly the coordinates of any points of intersections between the curve and the coordinate axes. State the equations of the two asymptotes of the curve.

[continues overleaf]

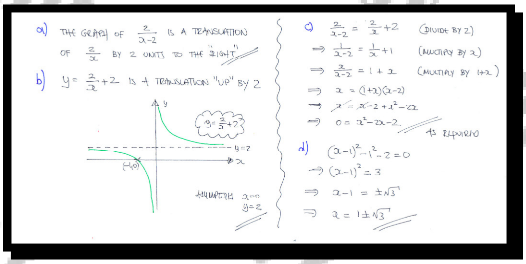
[continued from overleaf]

c) Show that the x coordinates of the points of intersection between the graph of $y=\frac{2}{x-2}$ and the graph of $y=\frac{2}{x}+2$ are the roots of the quadratic equation

$$x^2-2x-2=0.$$

d) Hence find, in exact surd form, the x coordinates of the points of intersection between the graph of $y=\frac{2}{x-2}$ and the graph of $y=\frac{2}{x}+2$.

$\boxed{}$, $\boxed{\text{translation, 2 units to the "right"}}$, $\boxed{x=1\pm\sqrt{3}}$



Question 6 **(***)**

A curve C has equation

$$y = \frac{1}{x^2}, \ x \in \mathbb{R}, \ x \neq 0.$$

a) Sketch the graph of C.

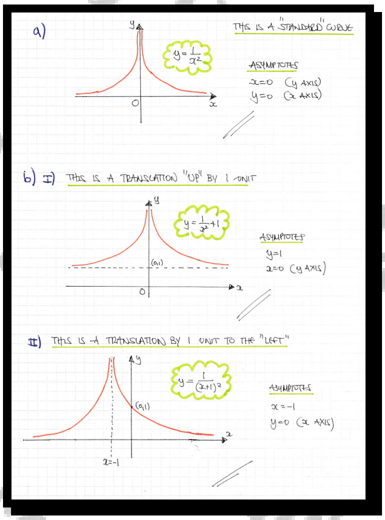
b) Sketch on separate set of axes the graph of ...

i. ... $y = \frac{1}{x^2} + 1, \ x \in \mathbb{R}, \ x \neq 0.$

ii. ... $y = \frac{1}{(x+1)^2}, \ x \in \mathbb{R}, \ x \neq -1.$

Mark clearly in each sketch the equations of any asymptotes to these curves and the coordinates of any intersections with the coordinate axes.

$\boxed{}$, $\boxed{\text{graph}}$



Question 7 (***)

A curve C has equation

$$f(x) = -\frac{1}{x^2}, \ x \in \mathbb{R}, \ x \neq 0.$$

a) Sketch the graph of C.

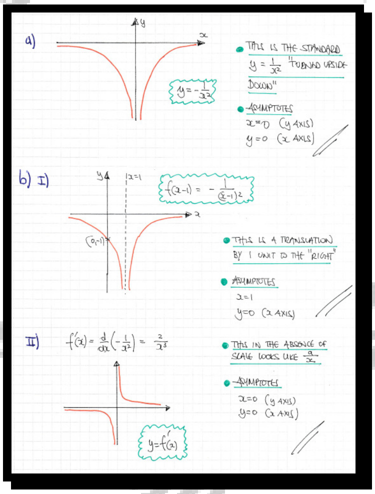
b) Sketch on separate set of axes the graph of …

i. … $f(x-1)$.

ii. … $f'(x)$.

Mark clearly in each sketch the equations of any asymptotes to these curves and the coordinates of any intersections with the coordinate axes.

$\boxed{\quad\quad}$, $\boxed{\text{graph}}$



Question 8 (***+)

$$f(x)=\frac{1}{x},\ x\in\mathbb{R},\ x\neq 0.$$

$$g(x)=\frac{1}{x-1}+1,\ x\in\mathbb{R},\ x\neq 1.$$

a) Describe mathematically the two transformations that map the graph of $f(x)$ onto the graph of $g(x)$.

b) Sketch the graph of $g(x)$.

The sketch must include …

- … the coordinates of any points where $g(x)$ meet the coordinate axes.
- … the equations of any asymptotes of $g(x)$.

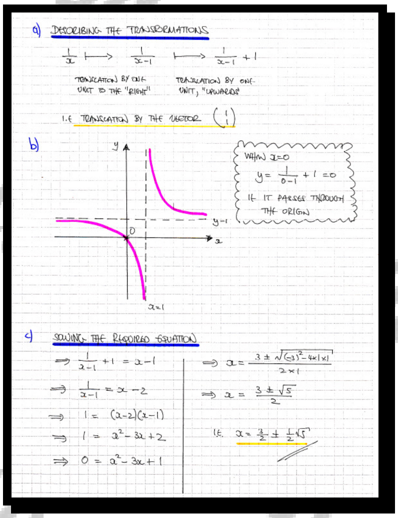
c) Solve the equation

$$g(x)=x-1,$$

giving the answers in the form $a+b\sqrt{5}$, where a and b are constants.

$\boxed{\text{translation "right" by 1 unit, followed by translation "upwards" by 1 unit}}$,

$\boxed{}$, $\boxed{x=\frac{3}{2}\pm\frac{1}{2}\sqrt{5}}$



Question 9 (***+)

A curve has equation $y = f(x)$ given by

$$f(x) = 2 + \frac{1}{2x-1}, \quad x \neq \frac{1}{2}.$$

a) Express $f(x)$ as a single simplified fraction.

Consider the following sequence of transformations T_1, T_2 and T_3.

$$\frac{1}{x} \xrightarrow{T_1} \frac{1}{x-1} \xrightarrow{T_2} \frac{1}{2x-1} \xrightarrow{T_3} 2 + \frac{1}{2x-1}.$$

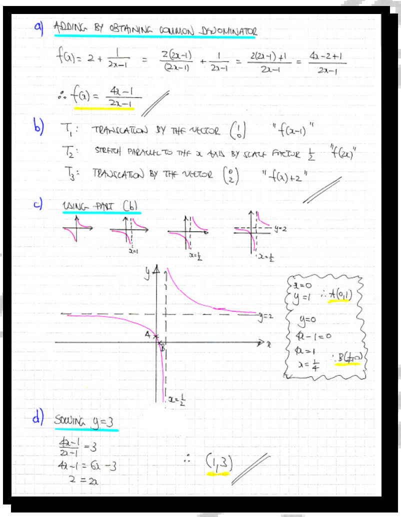
b) Describe geometrically the transformations T_1, T_2 and T_3.

c) Hence sketch the graph of $f(x)$.

Indicate clearly any asymptotes and the coordinates of any intersections with the coordinate axes.

d) Find the coordinates of the point of intersection of $f(x)$ and the line $y = 3$.

$\boxed{}$, $\boxed{T_1 = \text{translation, "right", 1 unit}}$, $\boxed{T_2 = \text{horizontal stretch by scale factor } \frac{1}{2}}$, $\boxed{T_3 = \text{translation, "upwards", 2 units}}$, $\boxed{(1,3)}$



Question 10 (***+)

Consider the following sequence of transformations T_1, T_2 and T_3.

$$\frac{1}{x} \xrightarrow{\quad T_1 \quad} -\frac{1}{x} \xrightarrow{\quad T_2 \quad} -\frac{1}{x+1} \xrightarrow{\quad T_3 \quad} 2-\frac{1}{x+1}$$

a) Describe geometrically the transformations T_1, T_2 and T_3.

b) Hence sketch the graph of

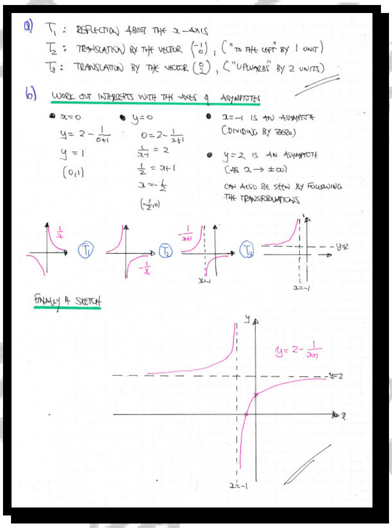
$$y = 2-\frac{1}{x+1}, \quad x \neq -1 .$$

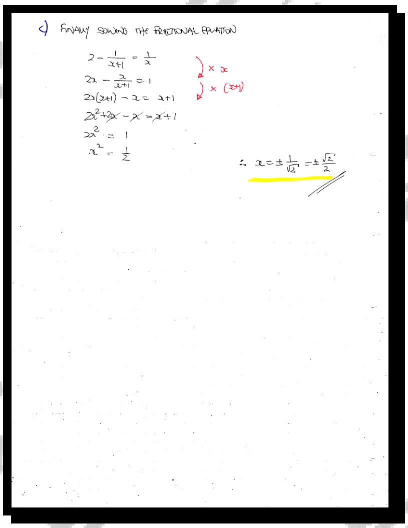
Indicate clearly any asymptotes and the coordinates of any intersections with the coordinate axes.

c) Solve the equation

$$2-\frac{1}{x+1} = \frac{1}{x} .$$

$\boxed{}$, $\boxed{T_1 = \text{reflection in the } x\text{-axis}}$, $\boxed{T_2 = \text{translation, "left", 1 unit}}$, $\boxed{T_3 = \text{translation, "upwards", 2 units}}$, $\boxed{x = \pm\frac{\sqrt{2}}{2}}$





Question 11 (***+)

Consider a sequence of geometric transformations T_1, T_2 and T_3 which map the graph of the curve with equation $y_1 = \frac{1}{x}$ onto the graph of y_2.

T_1 : reflection in the x axis.

T_2 : translation in the negative x direction by 2 units.

T_3 : translation in the positive y direction by 2 units.

a) Show that the equation of y_2 is given by

$$y_2 = \frac{2x+3}{x+2}, \quad x \neq -2 .$$

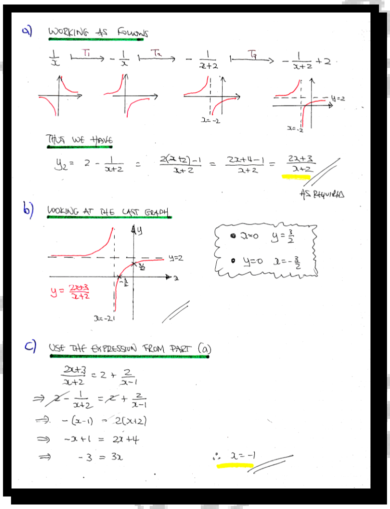
b) Sketch the graph of y_2.

Indicate clearly any asymptotes and coordinates of intersections with the axes.

c) Solve the equation

$$\frac{2x+3}{x+2} = 2 + \frac{2}{x-1} .$$

$\boxed{}$, $\boxed{x=-1}$



Question 12 (***+)

The curve C_1 has equation

$$y = -\frac{2}{x}, \quad x \in \mathbb{R}, \quad x \neq 0.$$

a) Sketch the graph of C_1.

The curve C_2 has equation

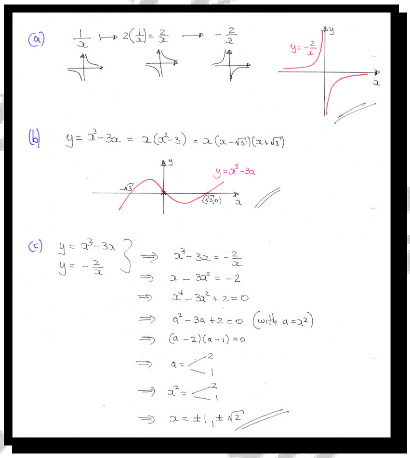
$$y = x^3 - 3x, \quad x \in \mathbb{R}.$$

b) Sketch the graph of C_2.

The sketch must include the coordinates, in exact surd form where appropriate, of all the points where the curve meets the coordinate axes.

c) Find the x coordinates of the points of intersection between C_1 and C_2.

$\boxed{}$, $\boxed{x = \pm 1, \ \pm\sqrt{2}}$



Question 13 (***+)

$$f(x) = \frac{1}{x},\ x \in \mathbb{R},\ x \neq 0.$$

$$g(x) = 2 - \frac{1}{x},\ x \in \mathbb{R},\ x \neq 0.$$

a) Describe mathematically the two transformations that map the graph of $f(x)$ onto the graph of $g(x)$.

$$h(x) = \frac{6}{x+2},\ x \in \mathbb{R},\ x \neq -2.$$

b) Sketch in the same diagram the graphs of $g(x)$ and $h(x)$.

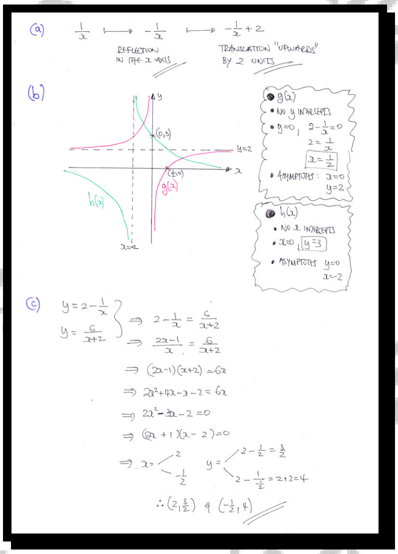
The sketch must include the coordinates of …

- … all the points where the curves meet the coordinate axes.
- … the equations of any asymptotes of the curves.

c) Solve the equation $g(x) = h(x)$.

$\boxed{}$, $\boxed{\text{reflection in the } x \text{ axis, followed by translation "upwards" by 1 unit}}$,

$\boxed{x = -\frac{1}{2}, 2}$



Question 14 (***+)

$$y = \frac{2}{x}, \ x \in \mathbb{R}, \ x \neq 0.$$

a) Describe mathematically the transformation that maps the graph of $y = \frac{1}{x}$ onto the graph of $y = \frac{2}{x}$.

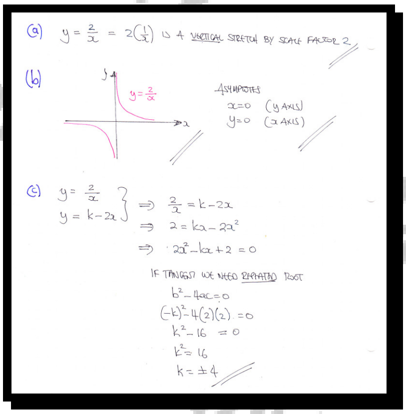
b) Sketch the graph of $y = \frac{2}{x}$.

Write down the equations of the asymptotes of the curve.

The straight line with equation $y = k - 2x$, where k is a constant, is a tangent to the curve with equation $y = \frac{2}{x}$.

c) Determine the possible values of k.

 , $\boxed{\text{stretch, vertically, by scale factor of } 2}$, $\boxed{k = \pm 4}$



(a) $y = \frac{2}{x} = 2\left(\frac{1}{x}\right)$ IS A VERTICAL STRETCH BY SCALE FACTOR 2

(b)

ASYMPTOTES
$x = 0$ (y AXIS)
$y = 0$ (x AXIS)

(c) $\left.\begin{array}{l} y = \frac{2}{x} \\ y = k - 2x \end{array}\right\} \Rightarrow \frac{2}{x} = k - 2x$
$\Rightarrow 2 = kx - 2x^2$
$\Rightarrow 2x^2 - kx + 2 = 0$

IF TANGENT WE NEED REPEATED ROOT
$b^2 - 4ac = 0$
$(-k)^2 - 4(2)(2) = 0$
$k^2 - 16 = 0$
$k^2 = 16$
$k = \pm 4$

Question 15 (***+)

$$f(x)=\frac{1}{x},\ x\in\mathbb{R},\ x\neq 0.$$

$$g(x)=\frac{1}{x+2}+2,\ x\in\mathbb{R},\ x\neq -2.$$

a) Describe mathematically the two transformations that map the graph of $f(x)$ onto the graph of $g(x)$.

b) Sketch the graph of $g(x)$.

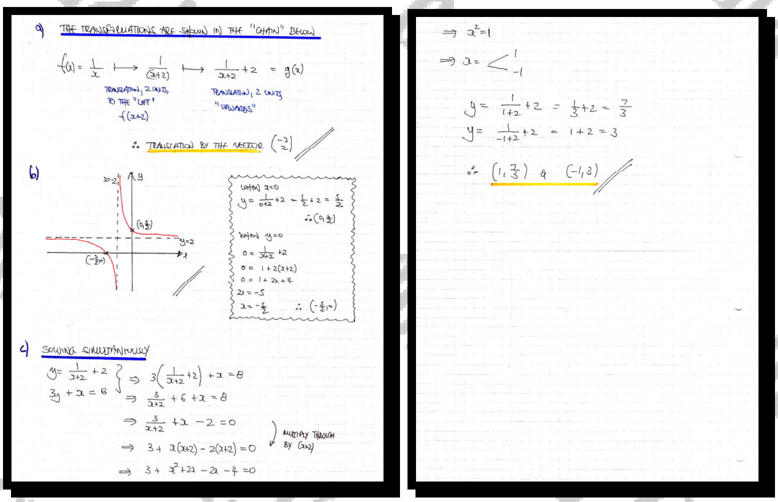
The sketch must include the …

- … coordinates of all the points where the curve meet the coordinate axes.
- … equations of any asymptotes of the curve.

c) Find the coordinates of the points of intersection of $g(x)$ and the line with equation

$$3y+x=8.$$

$\boxed{}$, $\boxed{\text{translation "left" by 2 units, followed by translation "upwards" by 2 units}}$, $\boxed{\left(1,\frac{7}{3}\right),(-1,3)}$



Question 16 (***+)

$$f(x)=\frac{4x-13}{x-3}, \quad x\in\mathbb{R}, \ x\neq 3.$$

a) Show that the equation of $f(x)$ can be written as

$$f(x)=4-\frac{1}{x-3}, \quad x\in\mathbb{R}, \ x\neq 3.$$

b) Sketch the graph of $f(x)$.

The sketch must include …

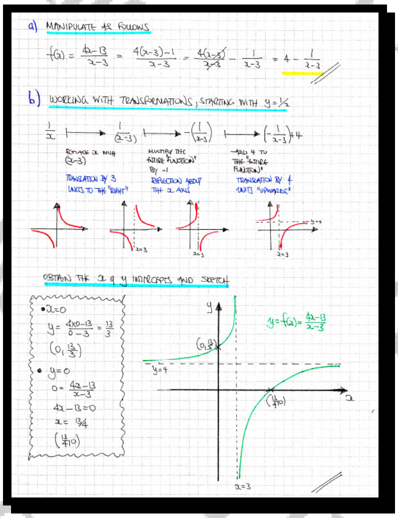
- … the coordinates of the points where $f(x)$ meets the coordinate axes.
- … the equations of any asymptotes of the curve.

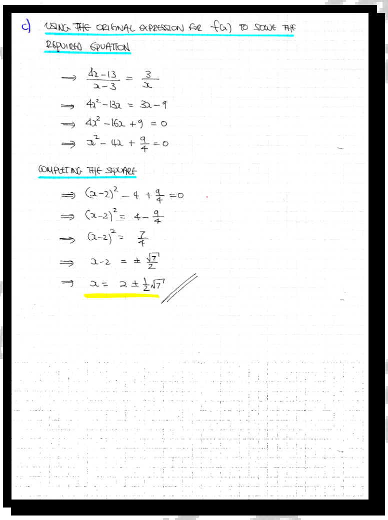
c) Solve the equation

$$f(x)=\frac{3}{x},$$

giving the answers in the form $a+b\sqrt{7}$, where a and b are constants.

$\boxed{}$, $\boxed{x=2\pm\frac{1}{2}\sqrt{7}}$





Question 17 (***+)

$$f(x) = a - \frac{1}{b-x}, \ x \in \mathbb{R}, \ x \neq b,$$

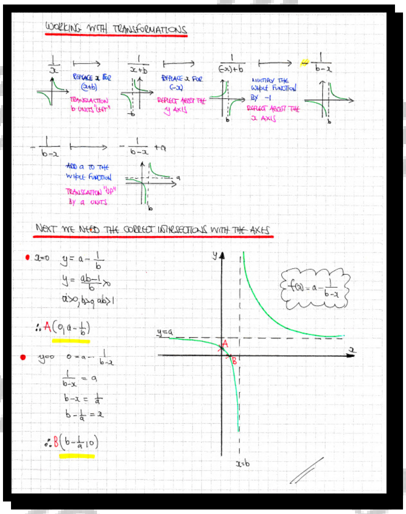
where a and b are positive constants such that $ab > 1$.

Sketch the graph of $f(x)$.

The sketch must include, in terms of a and b, …

- … the coordinates of the points where $f(x)$ meets the coordinate axes.
- … the equations of any asymptotes of the curve.

[ ], [graph]



Question 18 (****)

The curve C_1 has equation

$$y = \frac{a}{x}, \ x \neq 0,$$

where a is a positive constant.

a) Describe geometrically the transformation that maps the graph of C_1 onto the graph of C_2 whose equation is $y = \frac{a}{x} + 1$.

b) Sketch the graph of C_2.

The sketch must include the coordinates of …

- … all the points where the curves meet the coordinate axes.
- … the equations of any asymptotes of the curves.

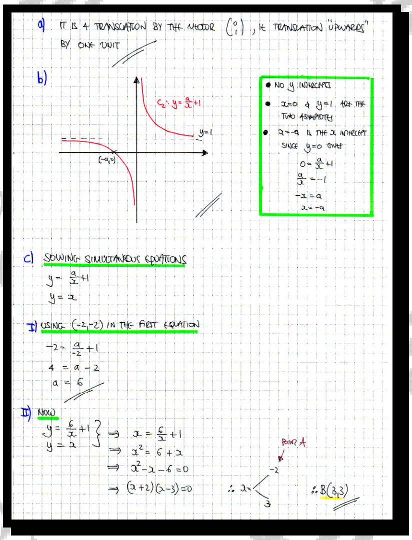
The line with equation $y = x$ intersects C_2 at the point $A(-2,-2)$ and B.

c) Determine …

i. … the value of a.

ii. … the coordinates of B.

$\boxed{}$, $\boxed{\text{translation "upwards" by 1 unit}}$, $\boxed{a = 6}$, $\boxed{B(3,3)}$



Question 19 (****)

The curve C has equation

$$y=\frac{2x+3}{x-2},\ x\in\mathbb{R},\ x\neq 2.$$

a) Show clearly that

$$\frac{2x+3}{x-2}\equiv 2+\frac{7}{x-2}.$$

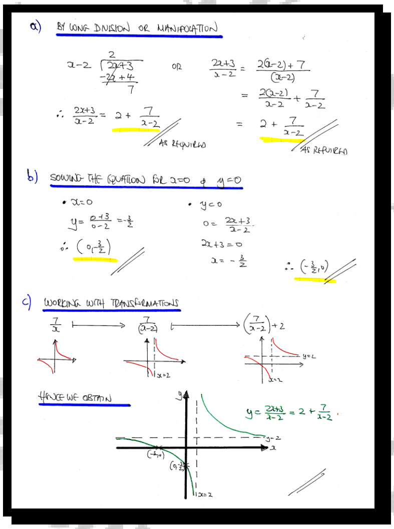
b) Find the coordinates of the points where C meets the coordinate axes.

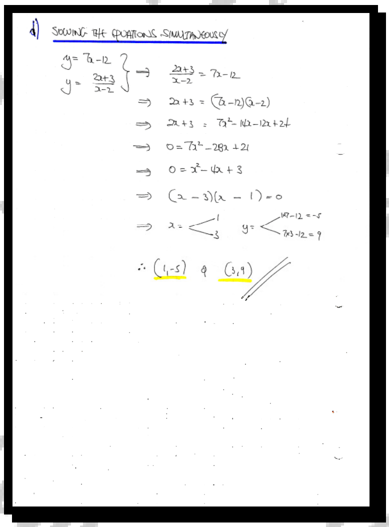
c) Sketch the graph of C showing clearly the equations of any asymptotes.

d) Determine the coordinates of the points of intersection of C and the straight line with equation

$$y=7x-12.$$

$\boxed{}$, $\boxed{\left(0,-\frac{3}{2}\right),\left(-\frac{3}{2},0\right)}$, $\boxed{(1,-5),(3,9)}$





Question 20 (****)

$$f(x) = \frac{1}{x-2}, \; x \in \mathbb{R}, \; x \neq 2.$$

$$g(x) = 1 + \frac{1}{x}, \; x \in \mathbb{R}, \; x \neq 0.$$

a) Describe mathematically the transformation that maps the graph of $y = \frac{1}{x}$ onto the graph of …

i. … $f(x)$.

ii. … $g(x)$.

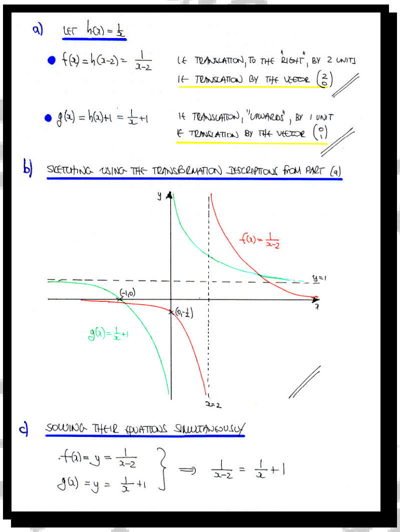
b) Sketch in the same diagram the graphs of $f(x)$ and $g(x)$.

The sketch must include …

- … the coordinates of any the points where the curves meet the coordinate axes.
- … the equations of any asymptotes of the curves.

c) Find as exact surds the coordinates of the points of intersection of the graphs of $f(x)$ and $g(x)$.

$\boxed{}$, $\boxed{\text{translation "right" by 2 units}}$, $\boxed{\text{translation "upwards" by 1 unit}}$, $\boxed{\left(1 \pm \sqrt{3}, \frac{1 \pm \sqrt{3}}{2}\right)}$



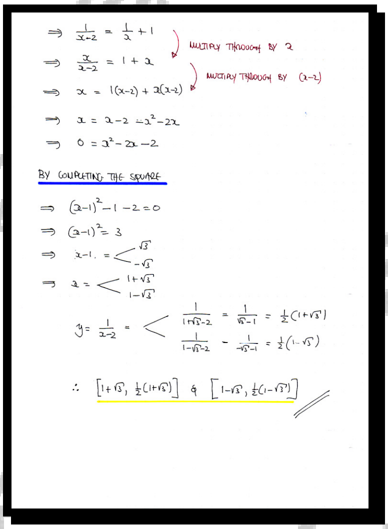
a) LET $h(x) = \frac{1}{x}$

- $f(x) = h(x-2) = \frac{1}{x-2}$ IE TRANSLATION, TO THE "RIGHT", BY 2 UNITS IE TRANSLATION BY THE VECTOR $\binom{2}{0}$
- $g(x) = h(x)+1 = \frac{1}{x}+1$ IE TRANSLATION, "UPWARDS", BY 1 UNIT IE TRANSLATION BY THE VECTOR $\binom{0}{1}$

b) SKETCHING USING THE TRANSFORMATION DESCRIPTIONS FROM PART (a)

c) SOLVING THEIR EQUATIONS SIMULTANEOUSLY

$$\left.\begin{aligned} f(x) = y &= \frac{1}{x-2} \\ g(x) = y &= \frac{1}{x}+1 \end{aligned}\right\} \Rightarrow \frac{1}{x-2} = \frac{1}{x}+1$$



$\Rightarrow \frac{1}{x-2} = \frac{1}{x}+1$ MULTIPLY THROUGH BY x

$\Rightarrow \frac{x}{x-2} = 1 + x$ MULTIPLY THROUGH BY $(x-2)$

$\Rightarrow x = 1(x-2) + x(x-2)$

$\Rightarrow x = x-2+x^2-2x$

$\Rightarrow 0 = x^2-2x-2$

BY COMPLETING THE SQUARE

$\Rightarrow (x-1)^2-1-2=0$

$\Rightarrow (x-1)^2 = 3$

$\Rightarrow x-1 = \sqrt{3}, \; -\sqrt{3}$

$\Rightarrow x = 1+\sqrt{3}, \; 1-\sqrt{3}$

$y = \frac{1}{x-2} = \frac{1}{1+\sqrt{3}-2} = \frac{1}{\sqrt{3}-1} = \frac{1}{2}(1+\sqrt{3})$; $\frac{1}{1-\sqrt{3}-2} = \frac{1}{-\sqrt{3}-1} = \frac{1}{2}(1-\sqrt{3})$

$\therefore \left[1+\sqrt{3}, \frac{1}{2}(1+\sqrt{3})\right]$ & $\left[1-\sqrt{3}, \frac{1}{2}(1-\sqrt{3})\right]$

Question 21 (****)

$$f(x) = \frac{x-2}{x-3}, \; x \in \mathbb{R}, \; x \neq 3.$$

a) Express $f(x)$ in the form

$$f(x) = a + \frac{1}{x+b},$$

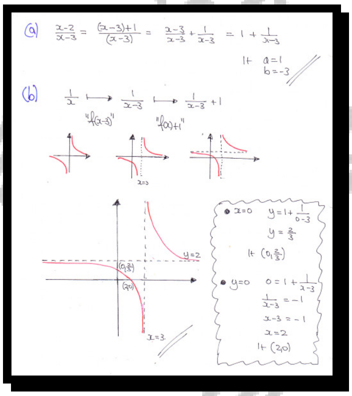
where a and b are integers.

b) By considering a series of transformations which map the graph of $\frac{1}{x}$ onto the graph of $f(x)$, sketch the graph of $f(x)$.

The sketch must include …

- … the coordinates of all the points where the curve meets the coordinate axes.
- … the equations of the two asymptotes of the curve.

$\boxed{a=1}$, $\boxed{b=-3}$



Question 22 (****)

The curves C_1 and C_2 have respective equations

$$C_1: \ y = \frac{1}{x-1}, \ x \neq 1$$

$$C_2: \ y = 1 - \frac{3}{x+3}, \ x \neq -3$$

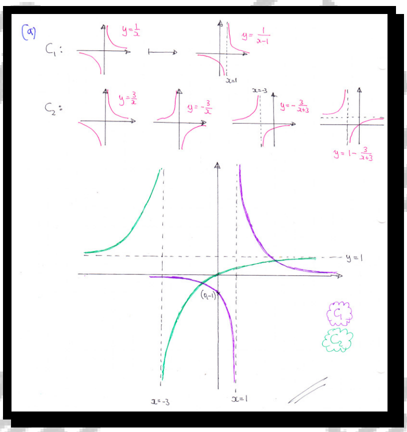
a) Sketch on the same diagram the graphs of C_1 and C_2.

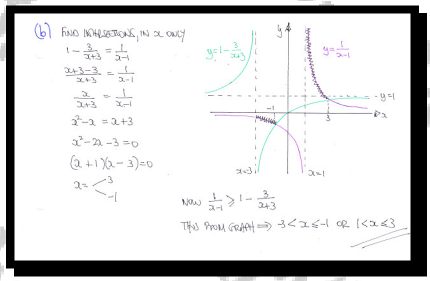
Indicate clearly any asymptotes and coordinates of any intersections with the coordinate axes.

b) By finding the intersections between C_1 and C_2, and considering the graphs sketched in part **(a)**, solve the inequality

$$\frac{1}{x-1} + \frac{3}{x+3} \geq 1.$$

$$\boxed{-3 < x \leq -1 \ \cup \ 1 < x \leq 3}$$





Question 23 (****)

$$f(x) = \frac{3x+3}{x-2}, \ x \in \mathbb{R}, \ x \neq 2.$$

a) Sketch the graph of $f(x)$.

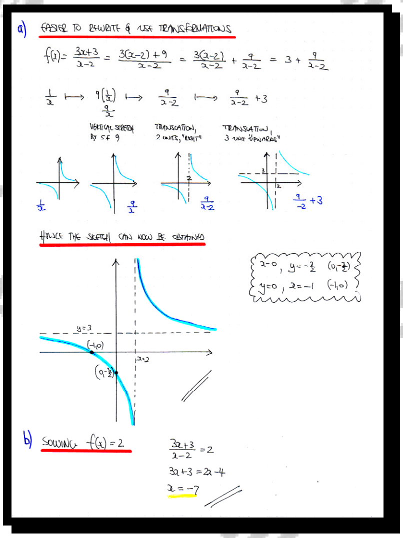
The sketch must include the coordinates of …

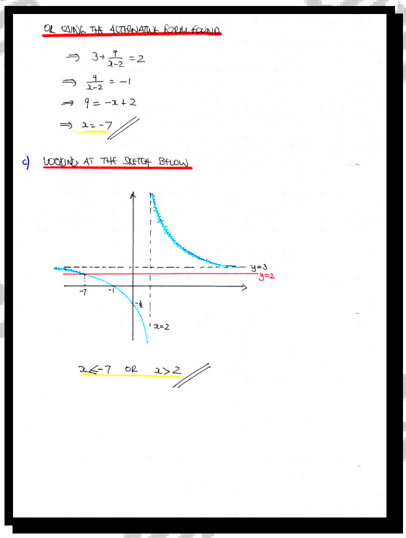
- … all the points where the curve meets the coordinate axes.
- … the equations of the two asymptotes of the curve.

b) Solve the equation $f(x) = 2$.

c) Hence solve the inequality $f(x) \geq 2$.

$\boxed{}$, $\boxed{x=-7}$, $\boxed{x \leq -7 \cup x > 2}$





Question 24 (****+)

A curve has equation $y = f(x)$ given by

$$f(x) = \frac{3x-1}{x+2}, \; x \in \mathbb{R}, \; x \neq 2.$$

a) Sketch the graph of $f(x)$.

The sketch must include the coordinates of …

- … all the points where the curve meets the coordinate axes.
- … the equations of the two asymptotes of the curve.

A different curve has equation $y = g(x)$ given by

$$g(x) = \frac{1}{x} + k, \; x \in \mathbb{R}, \; x \neq 0, \text{ where } k \text{ is a constant.}$$

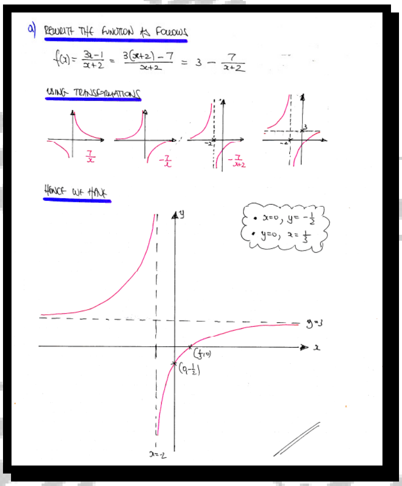
The graph of $f(x)$ meets the graph of $g(x)$ at the points A and B.

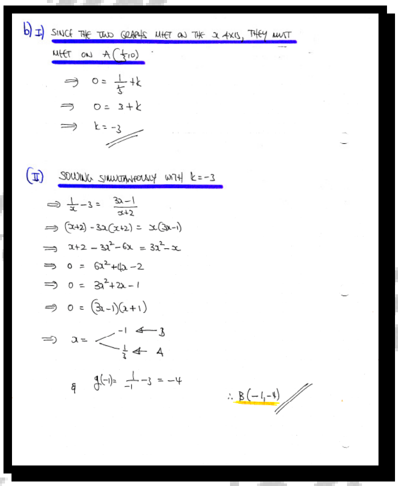
b) Given that A lies on the x axis determine …

i. … the value of k.

ii. … the coordinates of B.

$\boxed{}$, $\boxed{k=-3}$, $\boxed{B(-1,-4)}$




b) I) SINCE THE TWO GRAPHS MEET ON THE x AXIS, THEY MUST MEET ON $A(\frac{1}{3},0)$

$\Rightarrow 0 = \frac{1}{\frac{1}{3}} + k$

$\Rightarrow 0 = 3 + k$

$\Rightarrow k = -3$

(II) SOLVING SIMULTANEOUSLY WITH $k=-3$

$\Rightarrow \frac{1}{x} - 3 = \frac{3x-1}{x+2}$

$\Rightarrow (x+2) - 3x(x+2) = x(3x-1)$

$\Rightarrow x+2-3x^2-6x = 3x^2-x$

$\Rightarrow 0 = 6x^2+4x-2$

$\Rightarrow 0 = 3x^2+2x-1$

$\Rightarrow 0 = (3x-1)(x+1)$

$\Rightarrow x = -1 \leftarrow B$, $\frac{1}{3} \leftarrow A$

& $g(-1) = \frac{1}{-1} - 3 = -4$

$\therefore B(-1,-4)$