MIXED SURD QUESTIONS

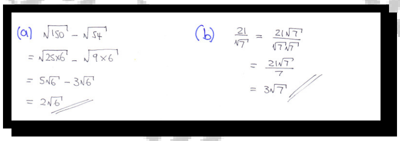
Question 1 **(**)**

Write each of the following expressions a single simplified surd.

a) $\sqrt{150}-\sqrt{54}$.

b) $\dfrac{21}{\sqrt{7}}$.

$\boxed{}$, $\boxed{2\sqrt{6}}$, $\boxed{3\sqrt{7}}$



(a) $\sqrt{150}-\sqrt{54}$

$=\sqrt{25\times6}-\sqrt{9\times6}$

$=5\sqrt{6}-3\sqrt{6}$

$=2\sqrt{6}$

(b) $\frac{21}{\sqrt{7}}=\frac{21\sqrt{7}}{\sqrt{7}\sqrt{7}}$

$=\frac{21\sqrt{7}}{7}$

$=3\sqrt{7}$

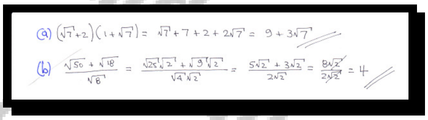
Question 2 **(**)**

Write each of the following surd expressions as simple as possible.

a) $\left(\sqrt{7}+2\right)\left(1+\sqrt{7}\right)$.

b) $\dfrac{\sqrt{50}+\sqrt{18}}{\sqrt{8}}$.

$\boxed{}$, $\boxed{9+3\sqrt{7}}$, $\boxed{4}$



(a) $(\sqrt{7}+2)(1+\sqrt{7})=\sqrt{7}+7+2+2\sqrt{7}=9+3\sqrt{7}$

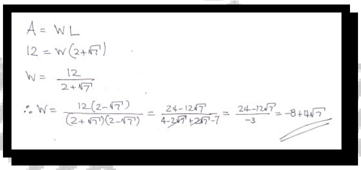
(b) $\frac{\sqrt{50}+\sqrt{18}}{\sqrt{8}}=\frac{\sqrt{25}\sqrt{2}+\sqrt{9}\sqrt{2}}{\sqrt{4}\sqrt{2}}=\frac{5\sqrt{2}+3\sqrt{2}}{2\sqrt{2}}=\frac{8\cancel{\sqrt{2}}}{2\cancel{\sqrt{2}}}=4$

Question 3 (**)

A rectangle has area $12\,\text{cm}^2$ and length $2+\sqrt{7}$ cm.

Find its width in the form $a+b\sqrt{7}$, where a and b are integers.

$\boxed{-8+4\sqrt{7}}$



$A = WL$

$12 = W(2+\sqrt{7})$

$W = \dfrac{12}{2+\sqrt{7}}$

$\therefore W = \dfrac{12(2-\sqrt{7})}{(2+\sqrt{7})(2-\sqrt{7})} = \dfrac{24-12\sqrt{7}}{4-\cancel{2\sqrt{7}}+\cancel{2\sqrt{7}}-7} = \dfrac{24-12\sqrt{7}}{-3} = -8+4\sqrt{7}$

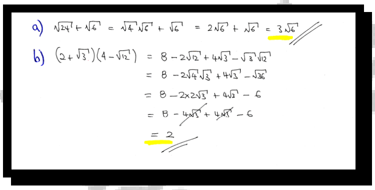
Question 4 (**)

Write each of the following surd expressions as simple as possible.

a) $\sqrt{24}+\sqrt{6}$.

b) $\left(2+\sqrt{3}\right)\left(4-\sqrt{12}\right)$.

$\boxed{}$, $\boxed{3\sqrt{6}}$, $\boxed{2}$



a) $\sqrt{24}+\sqrt{6} = \sqrt{4}\sqrt{6}+\sqrt{6} = 2\sqrt{6}+\sqrt{6} = 3\sqrt{6}$

b) $(2+\sqrt{3})(4-\sqrt{12}) = 8-2\sqrt{12}+4\sqrt{3}-\sqrt{3}\sqrt{12}$

$= 8-2\sqrt{4}\sqrt{3}+4\sqrt{3}-\sqrt{36}$

$= 8-2\times2\sqrt{3}+4\sqrt{3}-6$

$= 8-\cancel{4\sqrt{3}}+\cancel{4\sqrt{3}}-6$

$= 2$

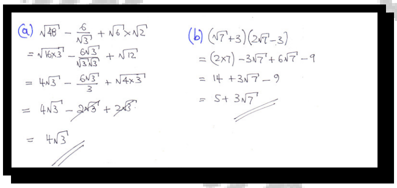
Question 5 **(**)**

Write each of the following surd expressions as simple as possible.

a) $\sqrt{48}-\dfrac{6}{\sqrt{3}}+\sqrt{6}\times\sqrt{2}$.

b) $\left(\sqrt{7}+3\right)\left(2\sqrt{7}-3\right)$.

$\boxed{4\sqrt{3}}$, $\boxed{5+3\sqrt{7}}$



(a) $\sqrt{48}-\dfrac{6}{\sqrt{3}}+\sqrt{6}\times\sqrt{2}$

$=\sqrt{16\times3}-\dfrac{6\sqrt{3}}{\sqrt{3}\sqrt{3}}+\sqrt{12}$

$=4\sqrt{3}-\dfrac{6\sqrt{3}}{3}+\sqrt{4\times3}$

$=4\sqrt{3}-\cancel{2\sqrt{3}}+\cancel{2\sqrt{3}}$

$=4\sqrt{3}$

(b) $(\sqrt{7}+3)(2\sqrt{7}-3)$

$=(2\times7)-3\sqrt{7}+6\sqrt{7}-9$

$=14+3\sqrt{7}-9$

$=5+3\sqrt{7}$

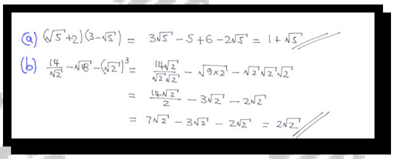
Question 6 **(**)**

Write each of the following surd expressions as simple as possible.

a) $\left(\sqrt{5}+2\right)\left(3-\sqrt{5}\right)$.

b) $\dfrac{14}{\sqrt{2}}-\sqrt{18}-\left(\sqrt{2}\right)^{3}$.

$\boxed{1+\sqrt{5}}$, $\boxed{2\sqrt{2}}$



(a) $(\sqrt{5}+2)(3-\sqrt{5})=3\sqrt{5}-5+6-2\sqrt{5}=1+\sqrt{5}$

(b) $\dfrac{14}{\sqrt{2}}-\sqrt{18}-(\sqrt{2})^{3}=\dfrac{14\sqrt{2}}{\sqrt{2}\sqrt{2}}-\sqrt{9\times2}-\sqrt{2}\sqrt{2}\sqrt{2}$

$=\dfrac{14\sqrt{2}}{2}-3\sqrt{2}-2\sqrt{2}$

$=7\sqrt{2}-3\sqrt{2}-2\sqrt{2}=2\sqrt{2}$

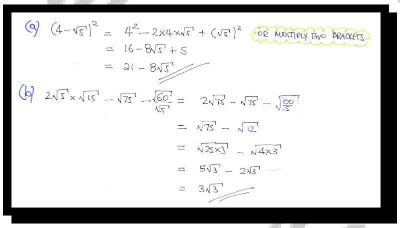
Question 7 **(**+)**

Write each of the following surd expressions as simple as possible.

a) $\left(4-\sqrt{5}\right)^2$.

b) $2\sqrt{5}\times\sqrt{15}-\sqrt{75}-\dfrac{\sqrt{60}}{\sqrt{5}}$.

$\boxed{}$, $\boxed{21-8\sqrt{5}}$, $\boxed{3\sqrt{3}}$



(a) $(4-\sqrt{5})^2 = 4^2-2\times4\times\sqrt{5}+(\sqrt{5})^2$ OR MULTIPLY TWO BRACKETS

$= 16-8\sqrt{5}+5$

$= 21-8\sqrt{5}$

(b) $2\sqrt{5}\times\sqrt{15}-\sqrt{75}-\dfrac{\sqrt{60}}{\sqrt{5}} = 2\sqrt{75}-\sqrt{75}-\sqrt{\dfrac{60}{5}}$

$= \sqrt{75}-\sqrt{12}$

$= \sqrt{25\times3}-\sqrt{4\times3}$

$= 5\sqrt{3}-2\sqrt{3}$

$= 3\sqrt{3}$

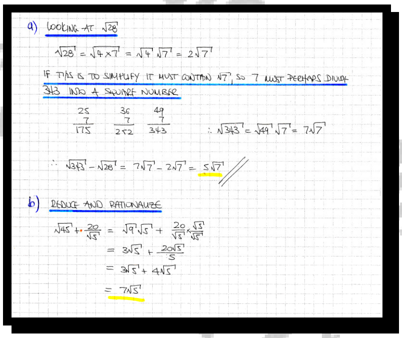
Question 8 **(**+)**

Write each of the following expressions a single simplified surd.

a) $\sqrt{343}-\sqrt{28}$.

b) $\sqrt{45}+\dfrac{20}{\sqrt{5}}$.

$\boxed{\text{SYN-Q}}$, $\boxed{5\sqrt{7}}$, $\boxed{7\sqrt{5}}$



a) LOOKING AT $\sqrt{28}$

$\sqrt{28} = \sqrt{4\times7} = \sqrt{4}\sqrt{7} = 2\sqrt{7}$

IF THIS IS TO SIMPLIFY IT MUST CONTAIN $\sqrt{7}$, SO 7 MUST PERHAPS DIVIDE 343 INTO A SQUARE NUMBER

$\begin{array}{r}25\\ 7\\ \hline 175\end{array}\quad\begin{array}{r}36\\ 7\\ \hline 252\end{array}\quad\begin{array}{r}49\\ 7\\ \hline 343\end{array}$ $\therefore \sqrt{343} = \sqrt{49}\sqrt{7} = 7\sqrt{7}$

$\therefore \sqrt{343}-\sqrt{28} = 7\sqrt{7}-2\sqrt{7} = 5\sqrt{7}$

b) REDUCE AND RATIONALIZE

$\sqrt{45}+\dfrac{20}{\sqrt{5}} = \sqrt{9}\sqrt{5}+\dfrac{20}{\sqrt{5}}\times\dfrac{\sqrt{5}}{\sqrt{5}}$

$= 3\sqrt{5}+\dfrac{20\sqrt{5}}{5}$

$= 3\sqrt{5}+4\sqrt{5}$

$= 7\sqrt{5}$

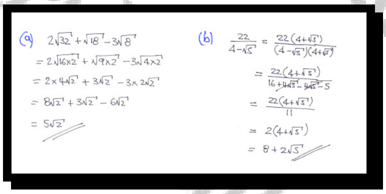
Question 9 **(**+)**

Write each of the following surd expressions as simple as possible.

a) $2\sqrt{32}+\sqrt{18}-3\sqrt{8}$.

b) $\dfrac{22}{4-\sqrt{5}}$.

$\boxed{5\sqrt{2}}$, $\boxed{8+2\sqrt{5}}$



(a) $2\sqrt{32}+\sqrt{18}-3\sqrt{8}$

$= 2\sqrt{16\times2}+\sqrt{9\times2}-3\sqrt{4\times2}$

$= 2\times4\sqrt{2}+3\sqrt{2}-3\times2\sqrt{2}$

$= 8\sqrt{2}+3\sqrt{2}-6\sqrt{2}$

$= 5\sqrt{2}$

(b) $\dfrac{22}{4-\sqrt{5}} = \dfrac{22(4+\sqrt{5})}{(4-\sqrt{5})(4+\sqrt{5})}$

$= \dfrac{22(4+\sqrt{5})}{16\cancel{+4\sqrt{5}}\cancel{-4\sqrt{5}}-5}$

$= \dfrac{22(4+\sqrt{5})}{11}$

$= 2(4+\sqrt{5})$

$= 8+2\sqrt{5}$

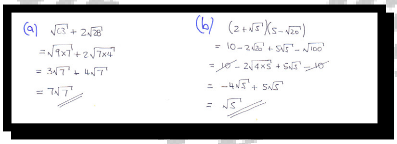
Question 10 **(**+)**

Write each of the following expressions a single simplified surd.

a) $\sqrt{63}+2\sqrt{28}$.

b) $\left(2+\sqrt{5}\right)\left(5-\sqrt{20}\right)$.

$\boxed{7\sqrt{7}}$, $\boxed{\sqrt{5}}$



(a) $\sqrt{63}+2\sqrt{28}$

$=\sqrt{9\times7}+2\sqrt{7\times4}$

$= 3\sqrt{7}+4\sqrt{7}$

$= 7\sqrt{7}$

(b) $(2+\sqrt{5})(5-\sqrt{20})$

$= 10-2\sqrt{20}+5\sqrt{5}-\sqrt{100}$

$= \cancel{10}-2\sqrt{4\times5}+5\sqrt{5}\cancel{-10}$

$= -4\sqrt{5}+5\sqrt{5}$

$= \sqrt{5}$

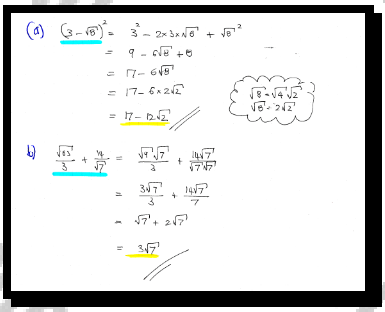
Question 11 **(**+)**

Write each of the following surd expressions as simple as possible.

a) $\left(3-\sqrt{8}\right)^2$.

b) $\dfrac{\sqrt{63}}{3}+\dfrac{14}{\sqrt{7}}$.

$\boxed{}$, $\boxed{17-12\sqrt{2}}$, $\boxed{3\sqrt{7}}$



(a) $\left(3-\sqrt{8}\right)^2 = 3^2 - 2\times3\times\sqrt{8} + \sqrt{8}^2$

$= 9 - 6\sqrt{8} + 8$

$= 17 - 6\sqrt{8}$

$= 17 - 6\times2\sqrt{2}$

$= 17 - 12\sqrt{2}$ //

$\sqrt{8} = \sqrt{4}\sqrt{2}$
$\sqrt{8} = 2\sqrt{2}$

b) $\dfrac{\sqrt{63}}{3} + \dfrac{14}{\sqrt{7}} = \dfrac{\sqrt{9}\sqrt{7}}{3} + \dfrac{14\sqrt{7}}{\sqrt{7}\sqrt{7}}$

$= \dfrac{3\sqrt{7}}{3} + \dfrac{14\sqrt{7}}{7}$

$= \sqrt{7} + 2\sqrt{7}$

$= 3\sqrt{7}$ //

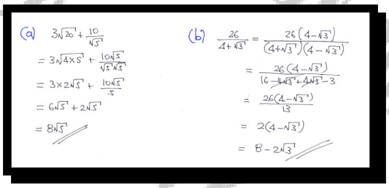
Question 12 **(**+)**

Write each of the following surd expressions as simple as possible.

a) $3\sqrt{20}+\dfrac{10}{\sqrt{5}}$.

b) $\dfrac{26}{4+\sqrt{3}}$.

$\boxed{8\sqrt{5}}$, $\boxed{8-2\sqrt{3}}$



(a) $3\sqrt{20} + \dfrac{10}{\sqrt{5}}$

$= 3\sqrt{4\times5} + \dfrac{10\sqrt{5}}{\sqrt{5}\sqrt{5}}$

$= 3\times2\sqrt{5} + \dfrac{10\sqrt{5}}{5}$

$= 6\sqrt{5} + 2\sqrt{5}$

$= 8\sqrt{5}$ //

(b) $\dfrac{26}{4+\sqrt{3}} = \dfrac{26\left(4-\sqrt{3}\right)}{\left(4+\sqrt{3}\right)\left(4-\sqrt{3}\right)}$

$= \dfrac{26\left(4-\sqrt{3}\right)}{16-\cancel{4\sqrt{3}}+\cancel{4\sqrt{3}}-3}$

$= \dfrac{26\left(4-\sqrt{3}\right)}{13}$

$= 2\left(4-\sqrt{3}\right)$

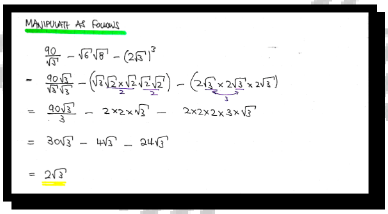
$= 8-2\sqrt{3}$ //

Question 13 (**+)

Write the following expression in the form $k\sqrt{3}$, where k is an integer.

$$\frac{90}{\sqrt{3}}-\sqrt{6}\times\sqrt{8}-\left(2\sqrt{3}\right)^3$$

$\boxed{2\sqrt{3}}$



MANIPULATE AS FOLLOWS

$$\frac{90}{\sqrt{3}}-\sqrt{6}\sqrt{8}-(2\sqrt{3})^3$$

$$=\frac{90\sqrt{3}}{\sqrt{3}\sqrt{3}}-\left(\sqrt{3}\underbrace{\sqrt{2}\times\sqrt{2}}_{2}\underbrace{\sqrt{2}\sqrt{2}}_{2}\right)-\left(2\sqrt{3}\times2\sqrt{3}\times2\sqrt{3}\right)$$

$$=\frac{90\sqrt{3}}{3}-2\times2\times\sqrt{3}-2\times2\times2\times3\times\sqrt{3}$$

$$=30\sqrt{3}-4\sqrt{3}-24\sqrt{3}$$

$$=2\sqrt{3}$$

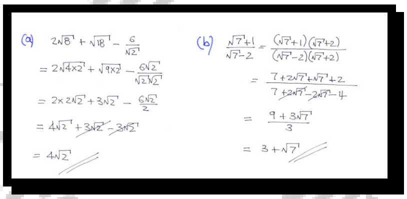
Question 14 (**+)

Write each of the following surd expressions as simple as possible.

a) $2\sqrt{8}+\sqrt{18}-\dfrac{6}{\sqrt{2}}$.

b) $\dfrac{\sqrt{7}+1}{\sqrt{7}-2}$.

$\boxed{4\sqrt{2}}$, $\boxed{3+\sqrt{7}}$



(a) $2\sqrt{8}+\sqrt{18}-\frac{6}{\sqrt{2}}$

$=2\sqrt{4\times2}+\sqrt{9\times2}-\frac{6\sqrt{2}}{\sqrt{2}\sqrt{2}}$

$=2\times2\sqrt{2}+3\sqrt{2}-\frac{6\sqrt{2}}{2}$

$=4\sqrt{2}+\cancel{3\sqrt{2}}-\cancel{3\sqrt{2}}$

$=4\sqrt{2}$

(b) $\frac{\sqrt{7}+1}{\sqrt{7}-2}=\frac{(\sqrt{7}+1)(\sqrt{7}+2)}{(\sqrt{7}-2)(\sqrt{7}+2)}$

$=\frac{7+2\sqrt{7}+\sqrt{7}+2}{7+\cancel{2\sqrt{7}}-\cancel{2\sqrt{7}}-4}$

$=\frac{9+3\sqrt{7}}{3}$

$=3+\sqrt{7}$

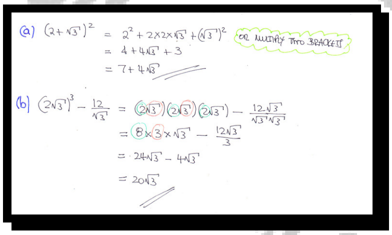
Question 15 **(**+)**

Write each of the following surd expressions as simple as possible.

a) $\left(2+\sqrt{3}\right)^2$.

b) $\left(2\sqrt{3}\right)^3 - \dfrac{12}{\sqrt{3}}$.

$\boxed{7+4\sqrt{3}}$, $\boxed{20\sqrt{3}}$



(a) $(2+\sqrt{3})^2 = 2^2 + 2\times2\times\sqrt{3} + (\sqrt{3})^2$ OR MULTIPLY TWO BRACKETS

$= 4 + 4\sqrt{3} + 3$

$= 7 + 4\sqrt{3}$

(b) $(2\sqrt{3})^3 - \frac{12}{\sqrt{3}} = (2\sqrt{3})(2\sqrt{3})(2\sqrt{3}) - \frac{12\sqrt{3}}{\sqrt{3}\sqrt{3}}$

$= 8\times3\times\sqrt{3} - \frac{12\sqrt{3}}{3}$

$= 24\sqrt{3} - 4\sqrt{3}$

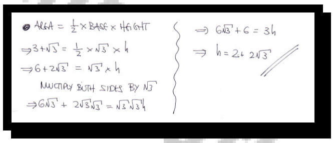
$= 20\sqrt{3}$

Question 16 **(**+)**

The area of a triangle is $\left(3+\sqrt{3}\right)$ cm^2.

Given the base of the triangle is $\sqrt{3}$ cm, find in exact simplified surd form the height of the triangle.

$\boxed{h = 2+2\sqrt{3}}$



- AREA = $\frac{1}{2}\times$ BASE $\times$ HEIGHT

$\Rightarrow 3+\sqrt{3} = \frac{1}{2}\times\sqrt{3}\times h$

$\Rightarrow 6+2\sqrt{3} = \sqrt{3}\times h$

MULTIPLY BOTH SIDES BY $\sqrt{3}$

$\Rightarrow 6\sqrt{3} + 2\sqrt{3}\sqrt{3} = \sqrt{3}\sqrt{3}h$

$\Rightarrow 6\sqrt{3} + 6 = 3h$

$\Rightarrow h = 2 + 2\sqrt{3}$

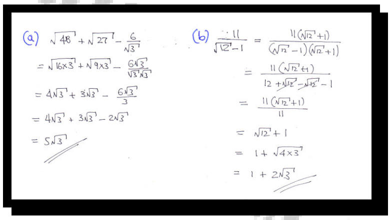
Question 17 **(**+)**

Write each of the following surd expressions as simple as possible.

a) $\sqrt{48}+\sqrt{27}-\dfrac{6}{\sqrt{3}}$.

b) $\dfrac{11}{\sqrt{12}-1}$.

$\boxed{5\sqrt{3}}$, $\boxed{1+2\sqrt{3}}$



(a) $\sqrt{48}+\sqrt{27}-\frac{6}{\sqrt{3}}$

$=\sqrt{16\times3}+\sqrt{9\times3}-\frac{6\sqrt{3}}{\sqrt{3}\sqrt{3}}$

$=4\sqrt{3}+3\sqrt{3}-\frac{6\sqrt{3}}{3}$

$=4\sqrt{3}+3\sqrt{3}-2\sqrt{3}$

$=5\sqrt{3}$

(b) $\frac{11}{\sqrt{12}-1}=\frac{11(\sqrt{12}+1)}{(\sqrt{12}-1)(\sqrt{12}+1)}$

$=\frac{11(\sqrt{12}+1)}{12+\cancel{\sqrt{12}}-\cancel{\sqrt{12}}-1}$

$=\frac{11(\sqrt{12}+1)}{11}$

$=\sqrt{12}+1$

$=1+\sqrt{4\times3}$

$=1+2\sqrt{3}$

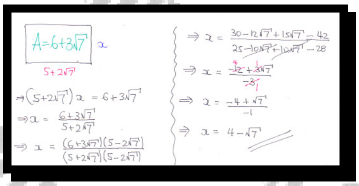
Question 18 **(**+)**

A rectangular room has an area of $6+3\sqrt{7}$ m^2.

The length of the room is $5+2\sqrt{7}$ m.

Find the width of the room, giving the answer as an exact surd in its simplest form.

$\boxed{}$, $\boxed{4-\sqrt{7}}$



$A=6+3\sqrt{7}$

x

$5+2\sqrt{7}$

$\Rightarrow (5+2\sqrt{7})x=6+3\sqrt{7}$

$\Rightarrow x=\frac{6+3\sqrt{7}}{5+2\sqrt{7}}$

$\Rightarrow x=\frac{(6+3\sqrt{7})(5-2\sqrt{7})}{(5+2\sqrt{7})(5-2\sqrt{7})}$

$\Rightarrow x=\frac{30-12\sqrt{7}+15\sqrt{7}-42}{25-\cancel{10\sqrt{7}}+\cancel{10\sqrt{7}}-28}$

$\Rightarrow x=\frac{\overset{4}{\cancel{-12}}+\overset{1}{\cancel{3}}\sqrt{7}}{\underset{1}{\cancel{-3}}}$

$\Rightarrow x=\frac{-4+\sqrt{7}}{-1}$

$\Rightarrow x=4-\sqrt{7}$

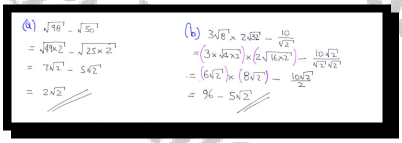
Question 19 **(**+)**

Write each of the following surd expressions as simple as possible.

a) $\sqrt{98}-\sqrt{50}$.

b) $3\sqrt{8}\times 2\sqrt{32}-\dfrac{10}{\sqrt{2}}$.

$\boxed{2\sqrt{2}}$, $\boxed{96-5\sqrt{2}}$



(a) $\sqrt{98}-\sqrt{50}$

$= \sqrt{49\times 2}-\sqrt{25\times 2}$

$= 7\sqrt{2}-5\sqrt{2}$

$= 2\sqrt{2}$

(b) $3\sqrt{8}\times 2\sqrt{32}-\frac{10}{\sqrt{2}}$

$=\left(3\times\sqrt{4\times 2}\right)\times\left(2\sqrt{16\times 2}\right)-\frac{10\sqrt{2}}{\sqrt{2}\sqrt{2}}$

$=\left(6\sqrt{2}\right)\times\left(8\sqrt{2}\right)-\frac{10\sqrt{2}}{2}$

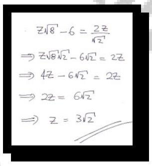
$= 96-5\sqrt{2}$

Question 20 **(**+)**

$$z\sqrt{8}-6=\frac{2z}{\sqrt{2}}.$$

Solve the above equation giving your answer in the form $k\sqrt{2}$, where k is an integer.

$\boxed{}$, $\boxed{z=3\sqrt{2}}$



$z\sqrt{8}-6=\frac{2z}{\sqrt{2}}$

$\Rightarrow z\sqrt{8}\sqrt{2}-6\sqrt{2}=2z$

$\Rightarrow 4z-6\sqrt{2}=2z$

$\Rightarrow 2z=6\sqrt{2}$

$\Rightarrow z=3\sqrt{2}$

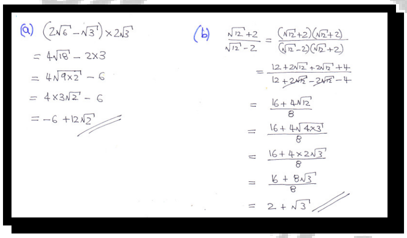
Question 21 **(**+)**

Write each of the following surd expressions as simple as possible.

a) $\left(2\sqrt{6}-\sqrt{3}\right)\times 2\sqrt{3}$.

b) $\dfrac{\sqrt{12}+2}{\sqrt{12}-2}$.

$\boxed{-6+12\sqrt{2}}$, $\boxed{2+\sqrt{3}}$



(a) $\left(2\sqrt{6}-\sqrt{3}\right)\times 2\sqrt{3}$

$= 4\sqrt{18} - 2\times 3$

$= 4\sqrt{9\times 2} - 6$

$= 4\times 3\sqrt{2} - 6$

$= -6 + 12\sqrt{2}$

(b) $\dfrac{\sqrt{12}+2}{\sqrt{12}-2} = \dfrac{(\sqrt{12}+2)(\sqrt{12}+2)}{(\sqrt{12}-2)(\sqrt{12}+2)}$

$= \dfrac{12+2\sqrt{12}+2\sqrt{12}+4}{12+\cancel{2\sqrt{12}}-\cancel{2\sqrt{12}}-4}$

$= \dfrac{16+4\sqrt{12}}{8}$

$= \dfrac{16+4\sqrt{4\times 3}}{8}$

$= \dfrac{16+4\times 2\sqrt{3}}{8}$

$= \dfrac{16+8\sqrt{3}}{8}$

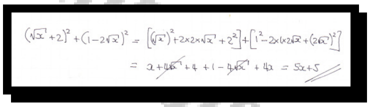
$= 2+\sqrt{3}$

Question 22 **(**+)**

$$f(x) \equiv \left(\sqrt{x}+2\right)^2 + \left(1-2\sqrt{x}\right)^2 .$$

Express $f(x)$, $x \geq 0$ in the form $ax+b$.

$\boxed{5x+5}$



$\left(\sqrt{x}+2\right)^2+\left(1-2\sqrt{x}\right)^2 = \left[\left(\sqrt{x}\right)^2+2\times 2\times\sqrt{x}+2^2\right]+\left[1^2-2\times 1\times 2\sqrt{x}+\left(2\sqrt{x}\right)^2\right]$

$= x+\cancel{4\sqrt{x}}+4+1-\cancel{4\sqrt{x}}+4x = 5x+5$

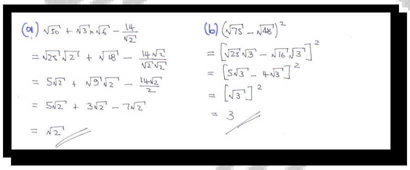
Question 23 (***)

Write each of the following surd expressions as simple as possible.

a) $\sqrt{50}+\sqrt{3}\times\sqrt{6}-\dfrac{14}{\sqrt{2}}$.

b) $\left(\sqrt{75}-\sqrt{48}\right)^2$.

$\boxed{\sqrt{2}}$, $\boxed{3}$



(a) $\sqrt{50}+\sqrt{3}\times\sqrt{6}-\frac{14}{\sqrt{2}}$

$=\sqrt{25}\sqrt{2}+\sqrt{18}-\frac{14\sqrt{2}}{\sqrt{2}\sqrt{2}}$

$=5\sqrt{2}+\sqrt{9}\sqrt{2}-\frac{14\sqrt{2}}{2}$

$=5\sqrt{2}+3\sqrt{2}-7\sqrt{2}$

$=\sqrt{2}$ //

(b) $\left(\sqrt{75}-\sqrt{48}\right)^2$

$=\left[\sqrt{25}\sqrt{3}-\sqrt{16}\sqrt{3}\right]^2$

$=\left[5\sqrt{3}-4\sqrt{3}\right]^2$

$=\left[\sqrt{3}\right]^2$

$=3$ //

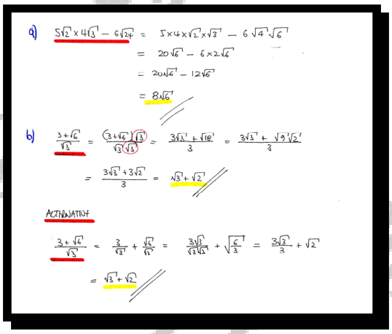
Question 24 (***)

Write each of the following surd expressions as simple as possible.

a) $5\sqrt{2}\times4\sqrt{3}-6\sqrt{24}$.

b) $\dfrac{3+\sqrt{6}}{\sqrt{3}}$.

$\boxed{\text{MP}}$, $\boxed{8\sqrt{6}}$, $\boxed{\sqrt{3}+\sqrt{2}}$



a) $5\sqrt{2}\times4\sqrt{3}-6\sqrt{24}=5\times4\times\sqrt{2}\times\sqrt{3}-6\sqrt{4}\sqrt{6}$

$=20\sqrt{6}-6\times2\sqrt{6}$

$=20\sqrt{6}-12\sqrt{6}$

$=8\sqrt{6}$ //

b) $\frac{3+\sqrt{6}}{\sqrt{3}}=\frac{(3+\sqrt{6})\sqrt{3}}{\sqrt{3}\sqrt{3}}=\frac{3\sqrt{3}+\sqrt{18}}{3}=\frac{3\sqrt{3}+\sqrt{9}\sqrt{2}}{3}$

$=\frac{3\sqrt{3}+3\sqrt{2}}{3}=\sqrt{3}+\sqrt{2}$ //

ALTERNATIVE

$\frac{3+\sqrt{6}}{\sqrt{3}}=\frac{3}{\sqrt{3}}+\frac{\sqrt{6}}{\sqrt{3}}=\frac{3\sqrt{3}}{\sqrt{3}\sqrt{3}}+\sqrt{\frac{6}{3}}=\frac{3\sqrt{3}}{3}+\sqrt{2}$

$=\sqrt{3}+\sqrt{2}$ //

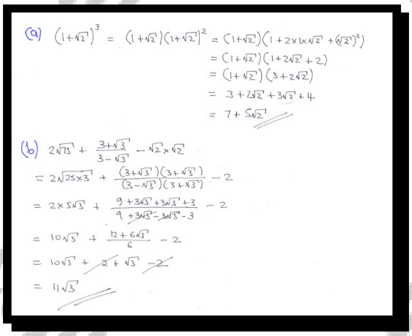
Question 25 (***)

Write each of the following surd expressions as simple as possible.

a) $\left(1+\sqrt{2}\right)^3$.

b) $2\sqrt{75}+\dfrac{3+\sqrt{3}}{3-\sqrt{3}}-\sqrt{2}\times\sqrt{2}$.

$\boxed{}$, $\boxed{7+5\sqrt{2}}$, $\boxed{11\sqrt{3}}$



(a) $(1+\sqrt{2})^3 = (1+\sqrt{2})(1+\sqrt{2})^2 = (1+\sqrt{2})(1+2\times1\times\sqrt{2}+(\sqrt{2})^2)$

$= (1+\sqrt{2})(1+2\sqrt{2}+2)$

$= (1+\sqrt{2})(3+2\sqrt{2})$

$= 3+2\sqrt{2}+3\sqrt{2}+4$

$= 7+5\sqrt{2}$

(b) $2\sqrt{75}+\dfrac{3+\sqrt{3}}{3-\sqrt{3}}-\sqrt{2}\times\sqrt{2}$

$= 2\sqrt{25\times3}+\dfrac{(3+\sqrt{3})(3+\sqrt{3})}{(3-\sqrt{3})(3+\sqrt{3})}-2$

$= 2\times5\sqrt{3}+\dfrac{9+3\sqrt{3}+3\sqrt{3}+3}{9\cancel{+3\sqrt{3}-3\sqrt{3}}-3}-2$

$= 10\sqrt{3}+\dfrac{12+6\sqrt{3}}{6}-2$

$= 10\sqrt{3}+\cancel{2}+\sqrt{3}\cancel{-2}$

$= 11\sqrt{3}$

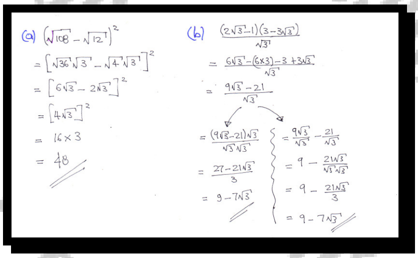
Question 26 (***)

Write each of the following surd expressions as simple as possible.

a) $\left(\sqrt{108}-\sqrt{12}\right)^2$.

b) $\dfrac{\left(2\sqrt{3}-1\right)\left(3-3\sqrt{3}\right)}{\sqrt{3}}$.

$\boxed{48}$, $\boxed{9-7\sqrt{3}}$



(a) $(\sqrt{108}-\sqrt{12})^2$

$= [\sqrt{36}\sqrt{3}-\sqrt{4}\sqrt{3}]^2$

$= [6\sqrt{3}-2\sqrt{3}]^2$

$= [4\sqrt{3}]^2$

$= 16\times3$

$= 48$

(b) $\dfrac{(2\sqrt{3}-1)(3-3\sqrt{3})}{\sqrt{3}}$

$= \dfrac{6\sqrt{3}-(6\times3)-3+3\sqrt{3}}{\sqrt{3}}$

$= \dfrac{9\sqrt{3}-21}{\sqrt{3}}$

$= \dfrac{(9\sqrt{3}-21)\sqrt{3}}{\sqrt{3}\sqrt{3}}$

$= \dfrac{27-21\sqrt{3}}{3}$

$= 9-7\sqrt{3}$

$= \dfrac{9\sqrt{3}}{\sqrt{3}}-\dfrac{21}{\sqrt{3}}$

$= 9-\dfrac{21\sqrt{3}}{\sqrt{3}\sqrt{3}}$

$= 9-\dfrac{21\sqrt{3}}{3}$

$= 9-7\sqrt{3}$

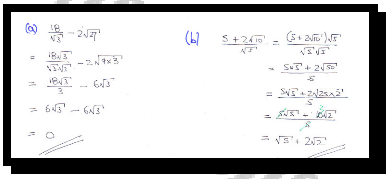
Question 27 (***)

Write each of the following surd expressions as simple as possible.

a) $\frac{18}{\sqrt{3}} - 2\sqrt{27}$.

b) $\frac{5+2\sqrt{10}}{\sqrt{5}}$.

$\boxed{0}$, $\boxed{\sqrt{5}+2\sqrt{2}}$



(a) $\frac{18}{\sqrt{3}} - 2\sqrt{27}$

$= \frac{18\sqrt{3}}{\sqrt{3}\sqrt{3}} - 2\sqrt{9\times3}$

$= \frac{18\sqrt{3}}{3} - 6\sqrt{3}$

$= 6\sqrt{3} - 6\sqrt{3}$

$= 0$

(b) $\frac{5+2\sqrt{10}}{\sqrt{5}} = \frac{(5+2\sqrt{10})\sqrt{5}}{\sqrt{5}\sqrt{5}}$

$= \frac{5\sqrt{5}+2\sqrt{50}}{5}$

$= \frac{5\sqrt{5}+2\sqrt{25\times2}}{5}$

$= \frac{\cancel{5}\sqrt{5}+\overset{2}{\cancel{10}}\sqrt{2}}{\cancel{5}}$

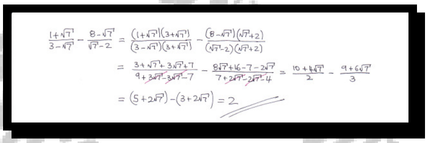
$= \sqrt{5}+2\sqrt{2}$

Question 28 (***)

$$\frac{1+\sqrt{7}}{3-\sqrt{7}} - \frac{8-\sqrt{7}}{\sqrt{7}-2}.$$

Show that the above surd expression simplifies to a positive integer, stating its value.

$\boxed{}$, $\boxed{2}$



$\frac{1+\sqrt{7}}{3-\sqrt{7}} - \frac{8-\sqrt{7}}{\sqrt{7}-2} = \frac{(1+\sqrt{7})(3+\sqrt{7})}{(3-\sqrt{7})(3+\sqrt{7})} - \frac{(8-\sqrt{7})(\sqrt{7}+2)}{(\sqrt{7}-2)(\sqrt{7}+2)}$

$= \frac{3+\sqrt{7}+3\sqrt{7}+7}{9+\cancel{3\sqrt{7}}-\cancel{3\sqrt{7}}-7} - \frac{8\sqrt{7}+16-7-2\sqrt{7}}{7+\cancel{2\sqrt{7}}-\cancel{2\sqrt{7}}-4} = \frac{10+4\sqrt{7}}{2} - \frac{9+6\sqrt{7}}{3}$

$= (5+2\sqrt{7})-(3+2\sqrt{7}) = 2$

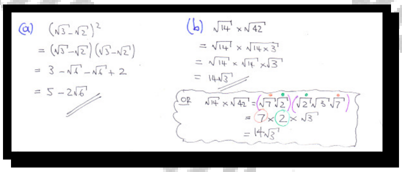
Question 29 **(***)**

Write each of the following surd expressions as simple as possible.

a) $\left(\sqrt{3}-\sqrt{2}\right)^2$.

b) $\sqrt{14}\times\sqrt{42}$.

$\boxed{}$, $\boxed{5-2\sqrt{6}}$, $\boxed{14\sqrt{3}}$



(a) $\left(\sqrt{3}-\sqrt{2}\right)^2$

$= \left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right)$

$= 3-\sqrt{6}-\sqrt{6}+2$

$= 5-2\sqrt{6}$

(b) $\sqrt{14}\times\sqrt{42}$

$= \sqrt{14}\times\sqrt{14\times3}$

$= \sqrt{14}\times\sqrt{14}\times\sqrt{3}$

$= 14\sqrt{3}$

OR $\sqrt{14}\times\sqrt{42} = \left(\sqrt{7}\sqrt{2}\right)\left(\sqrt{2}\sqrt{3}\sqrt{7}\right)$

$= 7\times2\times\sqrt{3}$

$= 14\sqrt{3}$

Question 30 **(***)**

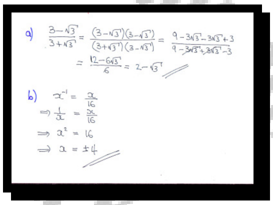
a) Simplify the following expression, writing the final answer in the form $a+b\sqrt{3}$, where a and b are integers

$$\frac{3-\sqrt{3}}{3+\sqrt{3}}.$$

b) Solve the equation

$$x^{-1}=\frac{x}{16}, \quad x\neq 0.$$

$\boxed{}$, $\boxed{2-\sqrt{3}}$, $\boxed{x=\pm4}$



a) $\frac{3-\sqrt{3}}{3+\sqrt{3}} = \frac{(3-\sqrt{3})(3-\sqrt{3})}{(3+\sqrt{3})(3-\sqrt{3})} = \frac{9-3\sqrt{3}-3\sqrt{3}+3}{9-\cancel{3\sqrt{3}}+\cancel{3\sqrt{3}}-3}$

$= \frac{12-6\sqrt{3}}{6} = 2-\sqrt{3}$

b) $x^{-1} = \frac{x}{16}$

$\Rightarrow \frac{1}{x} = \frac{x}{16}$

$\Rightarrow x^2 = 16$

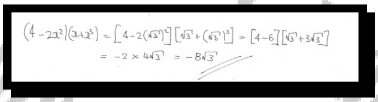
$\Rightarrow x = \pm4$

Question 31 **(***)**

If $x=\sqrt{3}$ show that

$$\left(4-2x^2\right)\left(x+x^3\right)=-8\sqrt{3}.$$

$\boxed{\text{proof}}$



$$\left(4-2x^2\right)\left(x+x^3\right)=\left[4-2\left(\sqrt{3}\right)^2\right]\left[\sqrt{3}+\left(\sqrt{3}\right)^3\right]=\left[4-6\right]\left[\sqrt{3}+3\sqrt{3}\right]$$

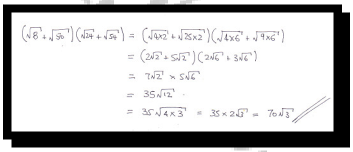
$$=-2\times4\sqrt{3}=-8\sqrt{3}$$

Question 32 **(***)**

Simplify fully

$$\left(\sqrt{8}+\sqrt{50}\right)\left(\sqrt{24}+\sqrt{54}\right).$$

$\boxed{}$, $\boxed{70\sqrt{3}}$



$$\left(\sqrt{8}+\sqrt{50}\right)\left(\sqrt{24}+\sqrt{54}\right)=\left(\sqrt{4\times2}+\sqrt{25\times2}\right)\left(\sqrt{4\times6}+\sqrt{9\times6}\right)$$

$$=\left(2\sqrt{2}+5\sqrt{2}\right)\left(2\sqrt{6}+3\sqrt{6}\right)$$

$$=7\sqrt{2}\times5\sqrt{6}$$

$$=35\sqrt{12}$$

$$=35\sqrt{4\times3}=35\times2\sqrt{3}=70\sqrt{3}$$

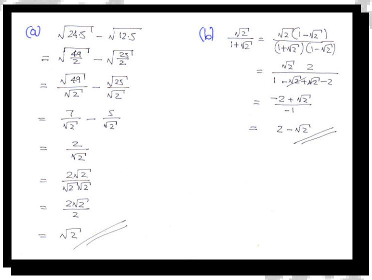
Question 33 (***)

Write each of the following surd expressions as simple as possible.

a) $\sqrt{24.5}-\sqrt{12.5}$.

b) $\dfrac{\sqrt{2}}{1+\sqrt{2}}$.

$\boxed{}$, $\boxed{\sqrt{2}}$, $\boxed{2-\sqrt{2}}$



(a) $\sqrt{24.5}-\sqrt{12.5}$

$=\sqrt{\frac{49}{2}}-\sqrt{\frac{25}{2}}$

$=\frac{\sqrt{49}}{\sqrt{2}}-\frac{\sqrt{25}}{\sqrt{2}}$

$=\frac{7}{\sqrt{2}}-\frac{5}{\sqrt{2}}$

$=\frac{2}{\sqrt{2}}$

$=\frac{2\sqrt{2}}{\sqrt{2}\sqrt{2}}$

$=\frac{2\sqrt{2}}{2}$

$=\sqrt{2}$

(b) $\frac{\sqrt{2}}{1+\sqrt{2}}=\frac{\sqrt{2}(1-\sqrt{2})}{(1+\sqrt{2})(1-\sqrt{2})}$

$=\frac{\sqrt{2}\ \ 2}{1-\cancel{\sqrt{2}}+\cancel{\sqrt{2}}-2}$

$=\frac{-2+\sqrt{2}}{-1}$

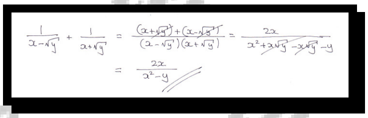
$=2-\sqrt{2}$

Question 34 (***)

$$\frac{1}{x-\sqrt{y}}+\frac{1}{x+\sqrt{y}}\ .$$

Write the above expression as a single fraction in its simplest form.

$$\boxed{\frac{2x}{x^2-y}}$$



$\frac{1}{x-\sqrt{y}}+\frac{1}{x+\sqrt{y}}=\frac{(x+\cancel{\sqrt{y}})+(x-\cancel{\sqrt{y}})}{(x-\sqrt{y})(x+\sqrt{y})}=\frac{2x}{x^2+\cancel{x\sqrt{y}}-\cancel{x\sqrt{y}}-y}$

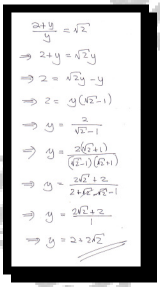
$=\frac{2x}{x^2-y}$

Question 35 (***)

$$\frac{2+y}{y}=\sqrt{2}.$$

Solve the above equation giving the answer in the form $a+b\sqrt{2}$, where a and b are integers.

☐ , $\boxed{y=2+2\sqrt{2}}$



$$\frac{2+y}{y}=\sqrt{2}$$
$$\Rightarrow 2+y=\sqrt{2}y$$
$$\Rightarrow 2=\sqrt{2}y-y$$
$$\Rightarrow 2=y(\sqrt{2}-1)$$
$$\Rightarrow y=\frac{2}{\sqrt{2}-1}$$
$$\Rightarrow y=\frac{2(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)}$$
$$\Rightarrow y=\frac{2\sqrt{2}+2}{2+\cancel{\sqrt{2}}-\cancel{\sqrt{2}}-1}$$
$$\Rightarrow y=\frac{2\sqrt{2}+2}{1}$$
$$\Rightarrow y=2+2\sqrt{2}$$

Question 36 (***)

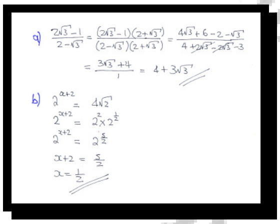
a) Simplify the following expression, writing the final answer in the form $a+b\sqrt{3}$, where a and b are integers

$$\frac{2\sqrt{3}-1}{2-\sqrt{3}}.$$

b) Solve the equation

$$2^{x+2}=4\sqrt{2}.$$

☐ , $\boxed{4+3\sqrt{3}}$, $\boxed{x=\frac{1}{2}}$



a) $\frac{2\sqrt{3}-1}{2-\sqrt{3}}=\frac{(2\sqrt{3}-1)(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})}=\frac{4\sqrt{3}+6-2-\sqrt{3}}{4+\cancel{2\sqrt{3}}-\cancel{2\sqrt{3}}-3}$

$=\frac{3\sqrt{3}+4}{1}=4+3\sqrt{3}$

b) $2^{x+2}=4\sqrt{2}$

$2^{x+2}=2^2\times2^{\frac{1}{2}}$

$2^{x+2}=2^{\frac{5}{2}}$

$x+2=\frac{5}{2}$

$x=\frac{1}{2}$

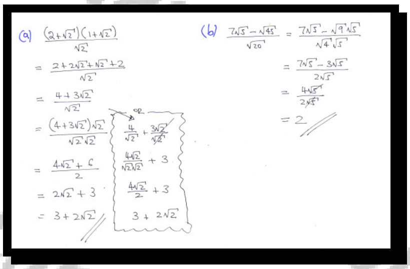
Question 37 (***)

Write each of the following surd expressions as simple as possible.

a) $\dfrac{\left(2+\sqrt{2}\right)\left(1+\sqrt{2}\right)}{\sqrt{2}}$.

b) $\dfrac{5\sqrt{5}-\sqrt{45}}{\sqrt{20}}$.

$\boxed{3+2\sqrt{2}}$, $\boxed{2}$



(a) $\frac{(2+\sqrt{2})(1+\sqrt{2})}{\sqrt{2}}$

$= \frac{2+2\sqrt{2}+\sqrt{2}+2}{\sqrt{2}}$

$= \frac{4+3\sqrt{2}}{\sqrt{2}}$

$= \frac{(4+3\sqrt{2})\sqrt{2}}{\sqrt{2}\sqrt{2}}$

$= \frac{4\sqrt{2}+6}{2}$

$= 2\sqrt{2}+3$

$= 3+2\sqrt{2}$ //

OR

$\frac{4}{\sqrt{2}} + \frac{3\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}}$

$\frac{4\sqrt{2}}{\sqrt{2}\sqrt{2}} + 3$

$\frac{4\sqrt{2}}{2} + 3$

$3 + 2\sqrt{2}$

(b) $\frac{7\sqrt{5}-\sqrt{45}}{\sqrt{20}} = \frac{7\sqrt{5}-\sqrt{9}\sqrt{5}}{\sqrt{4}\sqrt{5}}$

$= \frac{7\sqrt{5}-3\sqrt{5}}{2\sqrt{5}}$

$= \frac{4\cancel{\sqrt{5}}}{2\cancel{\sqrt{5}}}$

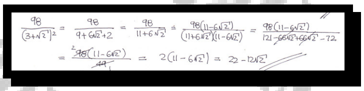
$= 2$ //

Question 38 (***)

$$\frac{98}{\left(3+\sqrt{2}\right)^2}.$$

Write the above surd expression in the form $a+b\sqrt{2}$, where a and b are integers.

$\boxed{\text{C1G}}$, $\boxed{22-12\sqrt{2}}$



$\frac{98}{(3+\sqrt{2})^2} = \frac{98}{9+6\sqrt{2}+2} = \frac{98}{11+6\sqrt{2}} = \frac{98(11-6\sqrt{2})}{(11+6\sqrt{2})(11-6\sqrt{2})} = \frac{98(11-6\sqrt{2})}{121-\cancel{66\sqrt{2}}+\cancel{66\sqrt{2}}-72}$ //

$= \frac{\overset{2}{\cancel{98}}(11-6\sqrt{2})}{\underset{1}{\cancel{49}}} = 2(11-6\sqrt{2}) = 22-12\sqrt{2}$ //

Question 39 (***+)

a) Simplify fully each of the following expressions, writing the final answer in terms of $\sqrt{3}$.

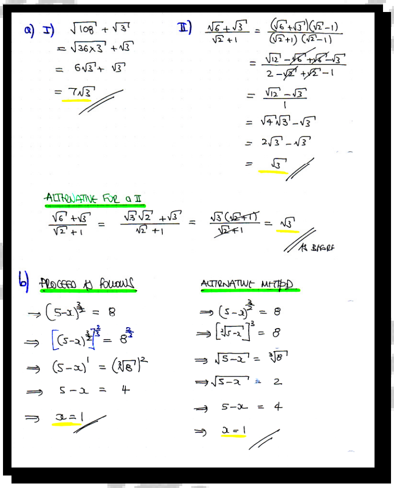
- **i.** $\sqrt{108}+\sqrt{3}$.
- **ii.** $\dfrac{\sqrt{6}+\sqrt{3}}{\sqrt{2}+1}$.

b) Solve the equation

$$(5-x)^{\frac{3}{2}}=8.$$

Detailed workings must be shown in this question.

$\boxed{}$, $\boxed{7\sqrt{3}}$, $\boxed{\sqrt{3}}$, $\boxed{x=1}$



a) I) $\sqrt{108}+\sqrt{3}$

$=\sqrt{36\times 3}+\sqrt{3}$

$=6\sqrt{3}+\sqrt{3}$

$=7\sqrt{3}$ //

II) $\dfrac{\sqrt{6}+\sqrt{3}}{\sqrt{2}+1}=\dfrac{(\sqrt{6}+\sqrt{3})(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)}$

$=\dfrac{\sqrt{12}-\cancel{\sqrt{6}}+\cancel{\sqrt{6}}-\sqrt{3}}{2-\cancel{\sqrt{2}}+\cancel{\sqrt{2}}-1}$

$=\dfrac{\sqrt{12}-\sqrt{3}}{1}$

$=\sqrt{4}\sqrt{3}-\sqrt{3}$

$=2\sqrt{3}-\sqrt{3}$

$=\sqrt{3}$ //

ALTERNATIVE FOR a II

$\dfrac{\sqrt{6}+\sqrt{3}}{\sqrt{2}+1}=\dfrac{\sqrt{3}\sqrt{2}+\sqrt{3}}{\sqrt{2}+1}=\dfrac{\sqrt{3}\cancel{(\sqrt{2}+1)}}{\cancel{\sqrt{2}+1}}=\sqrt{3}$ // AS BEFORE

b) PROCEED AS FOLLOWS

$\Rightarrow (5-x)^{\frac{3}{2}}=8$

$\Rightarrow \left[(5-x)^{\frac{3}{2}}\right]^{\frac{2}{3}}=8^{\frac{2}{3}}$

$\Rightarrow (5-x)^{1}=(\sqrt[3]{8})^{2}$

$\Rightarrow 5-x=4$

$\Rightarrow x=1$ //

ALTERNATIVE METHOD

$\Rightarrow (5-x)^{\frac{3}{2}}=8$

$\Rightarrow \left[\sqrt[2]{5-x}\right]^{3}=8$

$\Rightarrow \sqrt{5-x}=\sqrt[3]{8}$

$\Rightarrow \sqrt{5-x}=2$

$\Rightarrow 5-x=4$

$\Rightarrow x=1$ //

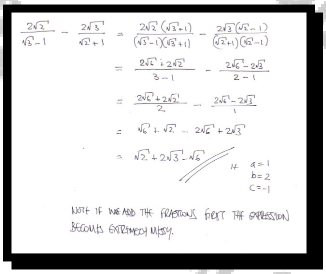
Question 40 (***+)

It is given that that for some constants a, b and c

$$\frac{2\sqrt{2}}{\sqrt{3}-1}-\frac{2\sqrt{3}}{\sqrt{2}+1} \equiv a\sqrt{2}+b\sqrt{3}+c\sqrt{6} .$$

Find the value of a, the value of b and the value of c.

$\boxed{}$, $\boxed{a=1}$, $\boxed{b=2}$, $\boxed{c=-1}$



$$\frac{2\sqrt{2}}{\sqrt{3}-1}-\frac{2\sqrt{3}}{\sqrt{2}+1} = \frac{2\sqrt{2}(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)} - \frac{2\sqrt{3}(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)}$$

$$= \frac{2\sqrt{6}+2\sqrt{2}}{3-1} - \frac{2\sqrt{6}-2\sqrt{3}}{2-1}$$

$$= \frac{2\sqrt{6}+2\sqrt{2}}{2} - \frac{2\sqrt{6}-2\sqrt{3}}{1}$$

$$= \sqrt{6}+\sqrt{2}-2\sqrt{6}+2\sqrt{3}$$

$$= \sqrt{2}+2\sqrt{3}-\sqrt{6}$$

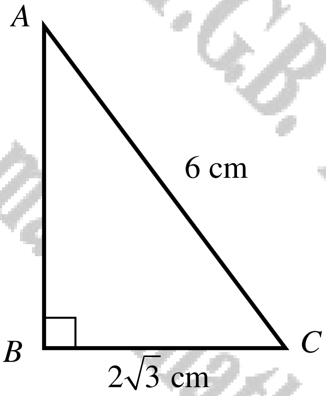
$\therefore$ $a=1$
$b=2$
$c=-1$

NOTE IF WE ADD THE FRACTIONS FIRST THE EXPRESSION BECOMES EXTREMELY MESSY.

Question 41 (*+)**

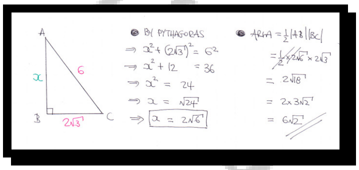
A right angled triangle ABC is shown in the figure below.

The lengths of AC and BC are 6 cm and $2\sqrt{3}$ cm, respectively.



Find the area of the triangle ABC in the form $k\sqrt{2}$, where k is an integer.

$\boxed{k = 6}$



● BY PYTHAGORAS

$\Rightarrow x^2 + (2\sqrt{3})^2 = 6^2$

$\Rightarrow x^2 + 12 = 36$

$\Rightarrow x^2 = 24$

$\Rightarrow x = \sqrt{24}$

$\Rightarrow \boxed{x = 2\sqrt{6}}$

● AREA $= \frac{1}{2}|AB||BC|$

$= \cancel{\frac{1}{2}} \times \cancel{2}\sqrt{6} \times 2\sqrt{3}$

$= 2\sqrt{18}$

$= 2 \times 3\sqrt{2}$

$= 6\sqrt{2}$

Question 42 (***+)

a) Simplify fully each of the following expressions, writing the final answer in terms of $\sqrt{2}$.

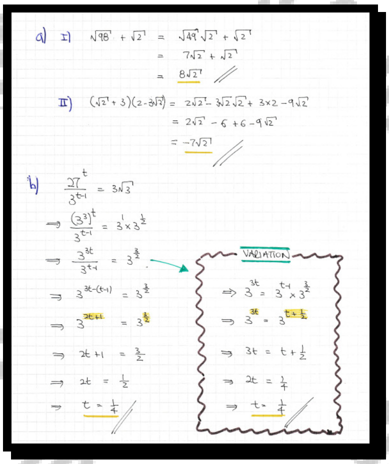
i. $\sqrt{98}+\sqrt{2}$.

ii. $\left(\sqrt{2}+3\right)\left(2-3\sqrt{2}\right)$.

b) Solve the equation

$$\frac{27^t}{3^{t-1}}=3\sqrt{3}.$$

$\boxed{}$, $\boxed{8\sqrt{2}}$, $\boxed{-7\sqrt{2}}$, $\boxed{t=\frac{1}{4}}$



a) I) $\sqrt{98}+\sqrt{2} = \sqrt{49}\sqrt{2}+\sqrt{2}$

$= 7\sqrt{2}+\sqrt{2}$

$= 8\sqrt{2}$ //

II) $(\sqrt{2}+3)(2-3\sqrt{2}) = 2\sqrt{2}-3\sqrt{2}\sqrt{2}+3\times2-9\sqrt{2}$

$= 2\sqrt{2}-6+6-9\sqrt{2}$

$= -7\sqrt{2}$ //

b) $\frac{27^t}{3^{t-1}} = 3\sqrt{3}$

$\Rightarrow \frac{(3^3)^t}{3^{t-1}} = 3^1\times3^{\frac{1}{2}}$

$\Rightarrow \frac{3^{3t}}{3^{t-1}} = 3^{\frac{3}{2}}$

$\Rightarrow 3^{3t-(t-1)} = 3^{\frac{3}{2}}$

$\Rightarrow 3^{2t+1} = 3^{\frac{3}{2}}$

$\Rightarrow 2t+1 = \frac{3}{2}$

$\Rightarrow 2t = \frac{1}{2}$

$\Rightarrow t = \frac{1}{4}$ //

VARIATION

$\Rightarrow 3^{3t} = 3^{t-1}\times3^{\frac{3}{2}}$

$\Rightarrow 3^{3t} = 3^{t+\frac{1}{2}}$

$\Rightarrow 3t = t+\frac{1}{2}$

$\Rightarrow 2t = \frac{1}{4}$

$\Rightarrow t = \frac{1}{4}$ //

Question 43 (***+)

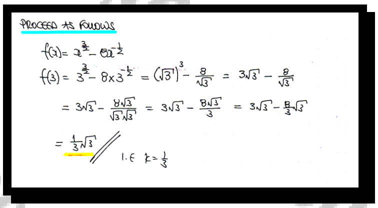
$$f(x) \equiv x^{\frac{3}{2}} - 8x^{-\frac{1}{2}},\ x > 0$$

Show clearly that

$$f(3) = k\sqrt{3},$$

where k is a constant.

$\boxed{\phantom{k = \frac{1}{3}}}$, $\boxed{\text{proof}}$



PROCEED AS FOLLOWS

$f(x) = x^{\frac{3}{2}} - 8x^{-\frac{1}{2}}$

$f(3) = 3^{\frac{3}{2}} - 8 \times 3^{-\frac{1}{2}} = (\sqrt{3})^3 - \frac{8}{\sqrt{3}} = 3\sqrt{3} - \frac{8}{\sqrt{3}}$

$= 3\sqrt{3} - \frac{8\sqrt{3}}{\sqrt{3}\sqrt{3}} = 3\sqrt{3} - \frac{8\sqrt{3}}{3} = 3\sqrt{3} - \frac{8}{3}\sqrt{3}$

$= \frac{1}{3}\sqrt{3}$ // I.E $k = \frac{1}{3}$

Question 44 **(***+)**

Write each of the following surd expressions as simple as possible.

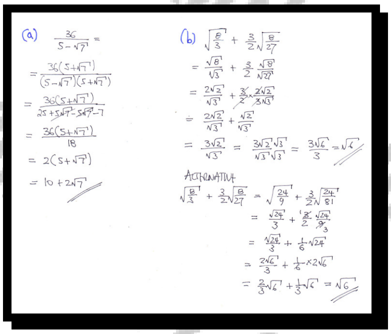
a) $\dfrac{36}{5-\sqrt{7}}$

Give the answer in the form $a+b\sqrt{7}$, where a and b are integers.

b) $\sqrt{\dfrac{8}{3}}+\dfrac{3}{2}\sqrt{\dfrac{8}{27}}$.

Give the answer in the form $\sqrt{k}$, where k is an integer.

$\boxed{\ }$, $\boxed{10+2\sqrt{7}}$, $\boxed{\sqrt{6}}$



(a) $\dfrac{36}{5-\sqrt{7}}=$

$=\dfrac{36(5+\sqrt{7})}{(5-\sqrt{7})(5+\sqrt{7})}$

$=\dfrac{36(5+\sqrt{7})}{25+\cancel{5\sqrt{7}}-\cancel{5\sqrt{7}}-7}$

$=\dfrac{36(5+\sqrt{7})}{18}$

$=2(5+\sqrt{7})$

$=10+2\sqrt{7}$

(b) $\sqrt{\dfrac{8}{3}}+\dfrac{3}{2}\sqrt{\dfrac{8}{27}}$

$=\dfrac{\sqrt{8}}{\sqrt{3}}+\dfrac{3}{2}\dfrac{\sqrt{8}}{\sqrt{27}}$

$=\dfrac{2\sqrt{2}}{\sqrt{3}}+\dfrac{\cancel{3}}{\cancel{2}}\times\dfrac{\cancel{2}\sqrt{2}}{\cancel{3}\sqrt{3}}$

$=\dfrac{2\sqrt{2}}{\sqrt{3}}+\dfrac{\sqrt{2}}{\sqrt{3}}$

$=\dfrac{3\sqrt{2}}{\sqrt{3}}=\dfrac{3\sqrt{2}\sqrt{3}}{\sqrt{3}\sqrt{3}}=\dfrac{3\sqrt{6}}{3}=\sqrt{6}$

ALTERNATIVE

$\sqrt{\dfrac{8}{3}}+\dfrac{3}{2}\sqrt{\dfrac{8}{27}}=\sqrt{\dfrac{24}{9}}+\dfrac{3}{2}\sqrt{\dfrac{24}{81}}$

$=\dfrac{\sqrt{24}}{3}+\dfrac{\overset{1}{\cancel{3}}}{2}\dfrac{\sqrt{24}}{\underset{3}{\cancel{9}}}$

$=\dfrac{\sqrt{24}}{3}+\dfrac{1}{6}\sqrt{24}$

$=\dfrac{2\sqrt{6}}{3}+\dfrac{1}{6}\times2\sqrt{6}$

$=\dfrac{2}{3}\sqrt{6}+\dfrac{1}{3}\sqrt{6}=\sqrt{6}$

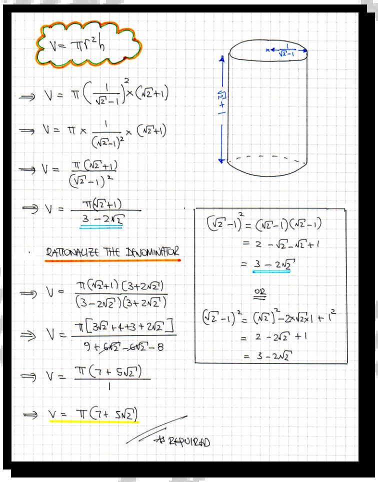
Question 45 **(***+)**

A cylinder has a radius of $\left(\dfrac{1}{\sqrt{2}-1}\right)$ cm and a height of $\left(\sqrt{2}+1\right)$ cm.

Show, by detailed working, that the volume of this cylinder is exactly

$$\pi\left(7+5\sqrt{2}\right)\text{cm}^3 .$$

$\boxed{\quad\quad}$, $\boxed{\text{proof}}$



$$\boxed{V=\pi r^2 h}$$

$$\Longrightarrow V=\pi\left(\frac{1}{\sqrt{2}-1}\right)^2\times\left(\sqrt{2}+1\right)$$

$$\Longrightarrow V=\pi\times\frac{1}{\left(\sqrt{2}-1\right)^2}\times\left(\sqrt{2}+1\right)$$

$$\Longrightarrow V=\frac{\pi\left(\sqrt{2}+1\right)}{\left(\sqrt{2}-1\right)^2}$$

$$\Longrightarrow V=\frac{\pi\left(\sqrt{2}+1\right)}{3-2\sqrt{2}}$$

$$\left(\sqrt{2}-1\right)^2=\left(\sqrt{2}-1\right)\left(\sqrt{2}-1\right)$$
$$=2-\sqrt{2}-\sqrt{2}+1$$
$$=3-2\sqrt{2}$$

OR

$$\left(\sqrt{2}-1\right)^2=\left(\sqrt{2}\right)^2-2\times\sqrt{2}\times1+1^2$$
$$=2-2\sqrt{2}+1$$
$$=3-2\sqrt{2}$$

RATIONALIZE THE DENOMINATOR

$$\Longrightarrow V=\frac{\pi\left(\sqrt{2}+1\right)\left(3+2\sqrt{2}\right)}{\left(3-2\sqrt{2}\right)\left(3+2\sqrt{2}\right)}$$

$$\Longrightarrow V=\frac{\pi\left[3\sqrt{2}+4+3+2\sqrt{2}\right]}{9+\cancel{6\sqrt{2}}-\cancel{6\sqrt{2}}-8}$$

$$\Longrightarrow V=\frac{\pi\left(7+5\sqrt{2}\right)}{1}$$

$$\Longrightarrow V=\pi\left(7+5\sqrt{2}\right)$$

// AS REQUIRED

Question 46 (***+)

a) Solve the equation

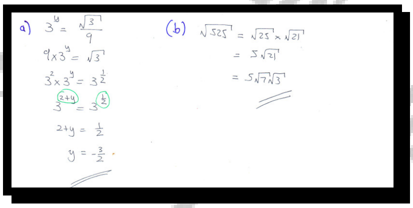
$$3^y = \frac{\sqrt{3}}{9}.$$

b) Express

$$\sqrt{525},$$

in the form $a\sqrt{b}\sqrt{c}$, where a, b and c are prime numbers.

$\boxed{}$, $\boxed{y = -\frac{3}{2}}$, $\boxed{5\sqrt{3}\sqrt{7}}$



a) $3^y = \frac{\sqrt{3}}{9}$

$9 \times 3^y = \sqrt{3}$

$3^2 \times 3^y = 3^{\frac{1}{2}}$

$3^{2+y} = 3^{\frac{1}{2}}$

$2+y = \frac{1}{2}$

$y = -\frac{3}{2}$

(b) $\sqrt{525} = \sqrt{25} \times \sqrt{21}$

$= 5\sqrt{21}$

$= 5\sqrt{7}\sqrt{3}$

Question 47 **(***+)**

a) If x is a real number solve the following indicial equation

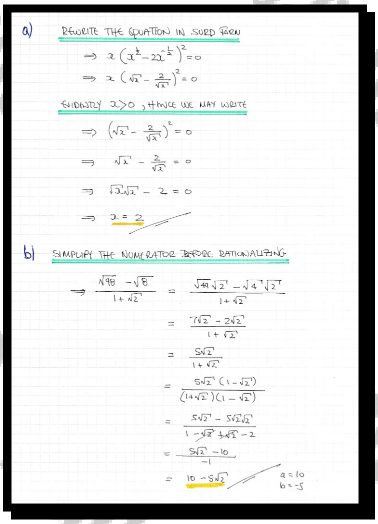
$$x\left(x^{\frac{1}{2}}-2x^{-\frac{1}{2}}\right)^2=0\,.$$

b) Express

$$\frac{\sqrt{98}-\sqrt{8}}{1+\sqrt{2}},$$

in the form $a+b\sqrt{2}$, where a and b are integers.

$\boxed{}$, $\boxed{x=2}$, $\boxed{10-5\sqrt{2}}$



a) REWRITE THE EQUATION IN SURD FORM

$$\Rightarrow x\left(x^{\frac{1}{2}}-2x^{-\frac{1}{2}}\right)^2=0$$

$$\Rightarrow x\left(\sqrt{x}-\frac{2}{\sqrt{x}}\right)^2=0$$

EVIDENTLY $x>0$, HENCE WE MAY WRITE

$$\Rightarrow \left(\sqrt{x}-\frac{2}{\sqrt{x}}\right)^2=0$$

$$\Rightarrow \sqrt{x}-\frac{2}{\sqrt{x}}=0$$

$$\Rightarrow \sqrt{x}\sqrt{x}-2=0$$

$$\Rightarrow x=2$$

b) SIMPLIFY THE NUMERATOR BEFORE RATIONALIZING

$$\Rightarrow \frac{\sqrt{98}-\sqrt{8}}{1+\sqrt{2}}=\frac{\sqrt{49}\sqrt{2}-\sqrt{4}\sqrt{2}}{1+\sqrt{2}}$$

$$=\frac{7\sqrt{2}-2\sqrt{2}}{1+\sqrt{2}}$$

$$=\frac{5\sqrt{2}}{1+\sqrt{2}}$$

$$=\frac{5\sqrt{2}(1-\sqrt{2})}{(1+\sqrt{2})(1-\sqrt{2})}$$

$$=\frac{5\sqrt{2}-5\sqrt{2}\sqrt{2}}{1-\cancel{\sqrt{2}}+\cancel{\sqrt{2}}-2}$$

$$=\frac{5\sqrt{2}-10}{-1}$$

$$=10-5\sqrt{2}$$

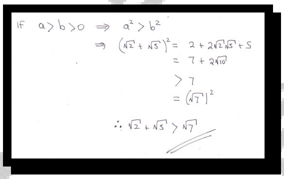
$a=10$
$b=-5$

Question 48 (***+)

Show clearly, without approximating and without using any calculating aid that

$$\sqrt{2}+\sqrt{5}>\sqrt{7}\,.$$

$\boxed{\text{proof}}$



If $a>b>0 \Rightarrow a^2>b^2$

$\Rightarrow (\sqrt{2}+\sqrt{5})^2 = 2+2\sqrt{2}\sqrt{5}+5$

$= 7+2\sqrt{10}$

>7

$=(\sqrt{7})^2$

$\therefore \sqrt{2}+\sqrt{5}>\sqrt{7}$

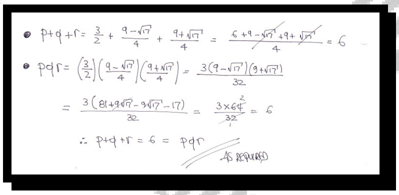
Question 49 (***+)

$$p=\frac{3}{2},\ q=\frac{9-\sqrt{17}}{4} \quad \text{and} \quad r=\frac{9+\sqrt{17}}{4}\,.$$

Prove that

$$p+q+r=pqr\,.$$

$\boxed{\quad}$, $\boxed{\text{proof}}$



- $p+q+r=\frac{3}{2}+\frac{9-\sqrt{17}}{4}+\frac{9+\sqrt{17}}{4}=\frac{6+9\cancel{-\sqrt{17}}+9\cancel{+\sqrt{17}}}{4}=6$
- $pqr=\left(\frac{3}{2}\right)\left(\frac{9-\sqrt{17}}{4}\right)\left(\frac{9+\sqrt{17}}{4}\right)=\frac{3(9-\sqrt{17})(9+\sqrt{17})}{32}$

$=\frac{3(81\cancel{+9\sqrt{17}}-9\sqrt{17}-17)}{32}=\frac{3\times\overset{2}{\cancel{64}}}{\underset{1}{\cancel{32}}}=6$

$\therefore p+q+r=6=pqr$

AS REQUIRED

Question 50 (***+)

a) Simplify fully each of the following expressions, writing the final answer as a single simplified surd.

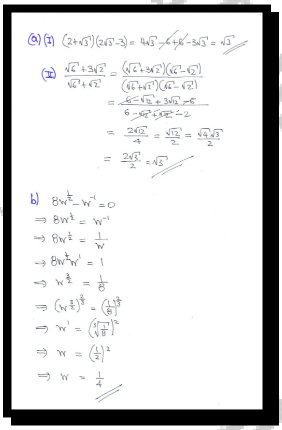
i. $\left(2+\sqrt{3}\right)\left(2\sqrt{3}-3\right)$.

ii. $\dfrac{\sqrt{6}+3\sqrt{2}}{\sqrt{6}+\sqrt{2}}$.

b) Solve the equation

$$8w^{\frac{1}{2}}-w^{-1}=0, \quad w \neq 0 .$$

$\boxed{}$, $\boxed{\sqrt{3}}$, $\boxed{\sqrt{3}}$, $\boxed{w=\frac{1}{4}}$



(a) (I) $(2+\sqrt{3})(2\sqrt{3}-3) = 4\sqrt{3} \cancel{-6} \cancel{+6} - 3\sqrt{3} = \sqrt{3}$

(II) $\dfrac{\sqrt{6}+3\sqrt{2}}{\sqrt{6}+\sqrt{2}} = \dfrac{(\sqrt{6}+3\sqrt{2})(\sqrt{6}-\sqrt{2})}{(\sqrt{6}+\sqrt{2})(\sqrt{6}-\sqrt{2})}$

$= \dfrac{\cancel{6}-\sqrt{12}+3\sqrt{12}\cancel{-6}}{6\cancel{-\sqrt{12}}\cancel{+\sqrt{12}}-2}$

$= \dfrac{2\sqrt{12}}{4} = \dfrac{\sqrt{12}}{2} = \dfrac{\sqrt{4}\sqrt{3}}{2}$

$= \dfrac{2\sqrt{3}}{2} = \sqrt{3}$

b) $8w^{\frac{1}{2}} - w^{-1} = 0$

$\Rightarrow 8w^{\frac{1}{2}} = w^{-1}$

$\Rightarrow 8w^{\frac{1}{2}} = \frac{1}{w}$

$\Rightarrow 8w^{\frac{1}{2}}w^{1} = 1$

$\Rightarrow w^{\frac{3}{2}} = \frac{1}{8}$

$\Rightarrow \left(w^{\frac{3}{2}}\right)^{\frac{2}{3}} = \left(\frac{1}{8}\right)^{\frac{2}{3}}$

$\Rightarrow w^{1} = \left(\sqrt[3]{\frac{1}{8}}\right)^{2}$

$\Rightarrow w = \left(\frac{1}{2}\right)^{2}$

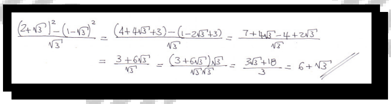
$\Rightarrow w = \frac{1}{4}$

Question 51 **(***+)**

$$\frac{\left(2+\sqrt{3}\right)^2-\left(1-\sqrt{3}\right)^2}{\sqrt{3}}.$$

Write the above surd expression in the form $a+b\sqrt{3}$, where a and b are integers.

$\boxed{}$, $\boxed{6+\sqrt{3}}$



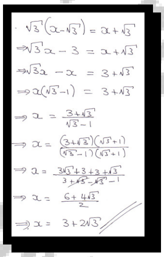
$$\frac{(2+\sqrt{3})^2-(1-\sqrt{3})^2}{\sqrt{3}}=\frac{(4+4\sqrt{3}+3)-(1-2\sqrt{3}+3)}{\sqrt{3}}=\frac{7+4\sqrt{3}-4+2\sqrt{3}}{\sqrt{3}}$$
$$=\frac{3+6\sqrt{3}}{\sqrt{3}}=\frac{(3+6\sqrt{3})\sqrt{3}}{\sqrt{3}\sqrt{3}}=\frac{3\sqrt{3}+18}{3}=6+\sqrt{3}$$

Question 52 **(***+)**

$$\sqrt{3}\left(x-\sqrt{3}\right)=x+\sqrt{3}.$$

Solve the above equation giving the answer in the form $a+b\sqrt{3}$, where a and b are integers.

$\boxed{}$, $\boxed{x=3+2\sqrt{3}}$



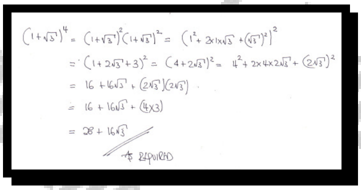
$$\sqrt{3}(x-\sqrt{3})=x+\sqrt{3}$$
$$\Rightarrow \sqrt{3}x-3=x+\sqrt{3}$$
$$\Rightarrow \sqrt{3}x-x=3+\sqrt{3}$$
$$\Rightarrow x(\sqrt{3}-1)=3+\sqrt{3}$$
$$\Rightarrow x=\frac{3+\sqrt{3}}{\sqrt{3}-1}$$
$$\Rightarrow x=\frac{(3+\sqrt{3})(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}$$
$$\Rightarrow x=\frac{3\sqrt{3}+3+3+\sqrt{3}}{3+\cancel{\sqrt{3}}-\cancel{\sqrt{3}}-1}$$
$$\Rightarrow x=\frac{6+4\sqrt{3}}{2}$$
$$\Rightarrow x=3+2\sqrt{3}$$

Question 53 **(****)**

Show clearly that

$$\left(1+\sqrt{3}\right)^4 = 28+16\sqrt{3}\,.$$

$\boxed{}$, $\boxed{\text{proof}}$



$$\left(1+\sqrt{3}\right)^4 = \left(1+\sqrt{3}\right)^2\left(1+\sqrt{3}\right)^2 = \left(1^2+2\times1\times\sqrt{3}+\left(\sqrt{3}\right)^2\right)^2$$

$$= \left(1+2\sqrt{3}+3\right)^2 = \left(4+2\sqrt{3}\right)^2 = 4^2+2\times4\times2\sqrt{3}+\left(2\sqrt{3}\right)^2$$

$$= 16+16\sqrt{3}+\left(2\sqrt{3}\right)\left(2\sqrt{3}\right)$$

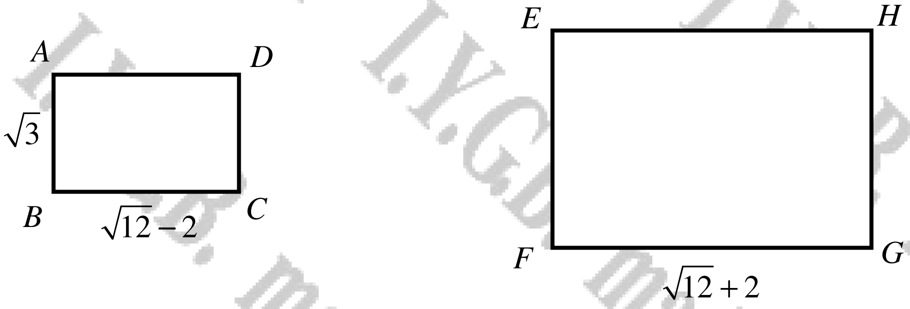
$$= 16+16\sqrt{3}+\left(4\times3\right)$$

$$= 28+16\sqrt{3}$$

AS REQUIRED

Question 54 (****)

The two rectangles shown in the figure below are similar.

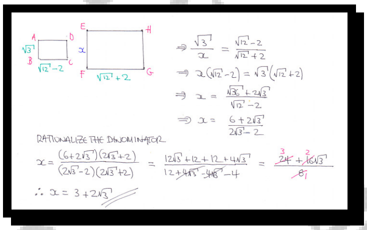


It is further given that in suitable units

$$|AB| = \sqrt{3}, \quad |BC| = \sqrt{12}-2 \quad \text{and} \quad |FG| = \sqrt{12}+2 .$$

Find the exact length of EF .

$\boxed{}$, $\boxed{|EF| = 3+2\sqrt{3}}$



$\Rightarrow \frac{\sqrt{3}}{x} = \frac{\sqrt{12}-2}{\sqrt{12}+2}$

$\Rightarrow x(\sqrt{12}-2) = \sqrt{3}(\sqrt{12}+2)$

$\Rightarrow x = \frac{\sqrt{36}+2\sqrt{3}}{\sqrt{12}-2}$

$\Rightarrow x = \frac{6+2\sqrt{3}}{2\sqrt{3}-2}$

RATIONALIZE THE DENOMINATOR

$x = \frac{(6+2\sqrt{3})(2\sqrt{3}+2)}{(2\sqrt{3}-2)(2\sqrt{3}+2)} = \frac{12\sqrt{3}+12+12+4\sqrt{3}}{12+\cancel{4\sqrt{3}}-\cancel{4\sqrt{3}}-4} = \frac{\overset{3}{\cancel{24}}+\overset{2}{\cancel{16}}\sqrt{3}}{\cancel{8}_{1}}$

$\therefore x = 3+2\sqrt{3}$

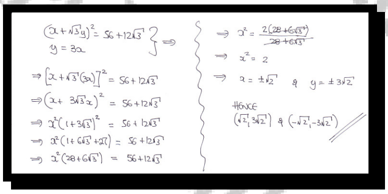
Question 55 **(****)**

Solve the following system of simultaneous equations

$$\left(x+y\sqrt{3}\right)^2=56+12\sqrt{3}$$

$$y=3x\,.$$

$\boxed{}$, $\boxed{\left(\sqrt{2},3\sqrt{2}\right),\left(-\sqrt{2},-3\sqrt{2}\right)}$



$$\left.\begin{aligned}\left(x+\sqrt{3}y\right)^2&=56+12\sqrt{3}\\ y&=3x\end{aligned}\right\}\Rightarrow$$

$$\Rightarrow\left[x+\sqrt{3}(3x)\right]^2=56+12\sqrt{3}$$

$$\Rightarrow\left(x+3\sqrt{3}x\right)^2=56+12\sqrt{3}$$

$$\Rightarrow x^2\left(1+3\sqrt{3}\right)^2=56+12\sqrt{3}$$

$$\Rightarrow x^2\left(1+6\sqrt{3}+27\right)=56+12\sqrt{3}$$

$$\Rightarrow x^2\left(28+6\sqrt{3}\right)=56+12\sqrt{3}$$

$$\Rightarrow x^2=\frac{2\cancel{(28+6\sqrt{3})}}{\cancel{28+6\sqrt{3}}}$$

$$\Rightarrow x^2=2$$

$$\Rightarrow x=\pm\sqrt{2}\quad \& \quad y=\pm3\sqrt{2}$$

HENCE

$\left(\sqrt{2},3\sqrt{2}\right)$ & $\left(-\sqrt{2},-3\sqrt{2}\right)$ //

Question 56 (****+)

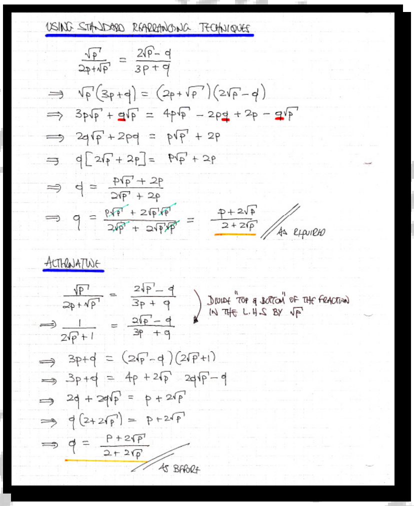
The positive constants p and q satisfy the following equation

$$\frac{\sqrt{p}}{2p+\sqrt{p}}=\frac{2\sqrt{p}-q}{3p+q}.$$

Show by a detailed method that

$$q=\frac{p+2\sqrt{p}}{2+2\sqrt{p}}.$$

$\boxed{}$, $\boxed{\text{proof}}$



USING STANDARD REARRANGING TECHNIQUES

$$\frac{\sqrt{p}}{2p+\sqrt{p}}=\frac{2\sqrt{p}-q}{3p+q}$$

$$\Rightarrow \sqrt{p}(3p+q)=(2p+\sqrt{p})(2\sqrt{p}-q)$$

$$\Rightarrow 3p\sqrt{p}+q\sqrt{p}=4p\sqrt{p}-2pq+2p-q\sqrt{p}$$

$$\Rightarrow 2q\sqrt{p}+2pq=p\sqrt{p}+2p$$

$$\Rightarrow q[2\sqrt{p}+2p]=p\sqrt{p}+2p$$

$$\Rightarrow q=\frac{p\sqrt{p}+2p}{2\sqrt{p}+2p}$$

$$\Rightarrow q=\frac{p\cancel{\sqrt{p}}+2\sqrt{p}\cancel{\sqrt{p}}}{2\cancel{\sqrt{p}}+2\sqrt{p}\cancel{\sqrt{p}}}=\frac{p+2\sqrt{p}}{2+2\sqrt{p}}$$

// AS REQUIRED

ALTERNATIVE

$$\frac{\sqrt{p}}{2p+\sqrt{p}}=\frac{2\sqrt{p}-q}{3p+q}$$

DIVIDE "TOP & BOTTOM" OF THE FRACTION IN THE L.H.S BY $\sqrt{p}$

$$\Rightarrow \frac{1}{2\sqrt{p}+1}=\frac{2\sqrt{p}-q}{3p+q}$$

$$\Rightarrow 3p+q=(2\sqrt{p}-q)(2\sqrt{p}+1)$$

$$\Rightarrow 3p+q=4p+2\sqrt{p}\quad 2q\sqrt{p}-q$$

$$\Rightarrow 2q+2q\sqrt{p}=p+2\sqrt{p}$$

$$\Rightarrow q(2+2\sqrt{p})=p+2\sqrt{p}$$

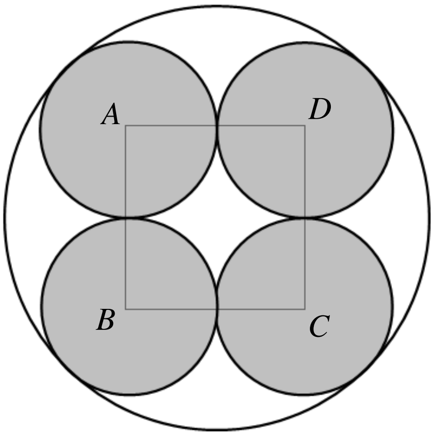
$$\Rightarrow q=\frac{p+2\sqrt{p}}{2+2\sqrt{p}}$$

// AS BEFORE

Question 57 (****+)

Four circles are touching in such a way so that their centres form the corners of a square $ABCD$. These four circles are circumscribed by a larger circle.

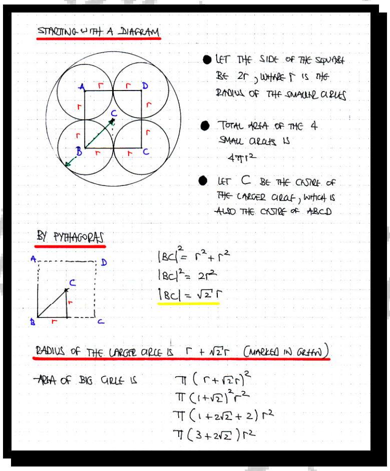
This is shown in the figure below.



Show that the ratio of the total area of the four smaller circles to the area of the larger circle is given by

$$12-8\sqrt{2}:1$$

☐ , proof



STARTING WITH A DIAGRAM

- LET THE SIDE OF THE SQUARE BE $2r$, WHERE r IS THE RADIUS OF THE SMALLER CIRCLES
- TOTAL AREA OF THE 4 SMALL CIRCLES IS $4\pi r^2$
- LET C BE THE CENTRE OF THE LARGER CIRCLE, WHICH IS ALSO THE CENTRE OF ABCD

BY PYTHAGORAS

$|BC|^2 = r^2+r^2$

$|BC|^2 = 2r^2$

$|BC| = \sqrt{2}\,r$

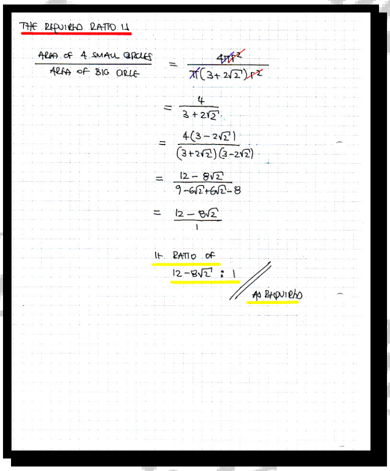
RADIUS OF THE LARGER CIRCLE IS $r + \sqrt{2}r$ (MARKED IN GREEN)

AREA OF BIG CIRCLE IS $\pi(r+\sqrt{2}r)^2$

$\pi(1+\sqrt{2})^2r^2$

$\pi(1+2\sqrt{2}+2)r^2$

$\pi(3+2\sqrt{2})r^2$



THE REQUIRED RATIO IS

$$\frac{\text{AREA OF 4 SMALL CIRCLES}}{\text{AREA OF BIG CIRCLE}} = \frac{4\cancel{\pi}\cancel{r^2}}{\cancel{\pi}(3+2\sqrt{2})\cancel{r^2}}$$

$$= \frac{4}{3+2\sqrt{2}}$$

$$= \frac{4(3-2\sqrt{2})}{(3+2\sqrt{2})(3-2\sqrt{2})}$$

$$= \frac{12-8\sqrt{2}}{9-6\sqrt{2}+6\sqrt{2}-8}$$

$$= \frac{12-8\sqrt{2}}{1}$$

IE RATIO OF

$12-8\sqrt{2} : 1$ //

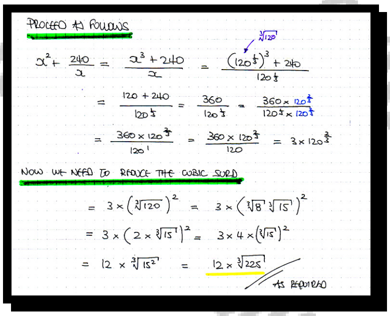
AS REQUIRED

Question 58 **(****+)**

If $x=\sqrt[3]{120}$, show clearly that

$$x^2+\frac{240}{x}=12\sqrt[3]{225}\,.$$

MP1-V , proof



PROCEED AS FOLLOWS

$$x^2+\frac{240}{x}=\frac{x^3+240}{x}=\frac{\left(\overset{\sqrt[3]{120}}{120^{\frac{1}{3}}}\right)^3+240}{120^{\frac{1}{3}}}$$

$$=\frac{120+240}{120^{\frac{1}{3}}}=\frac{360}{120^{\frac{1}{3}}}=\frac{360\times 120^{\frac{2}{3}}}{120^{\frac{1}{3}}\times 120^{\frac{2}{3}}}$$

$$=\frac{360\times 120^{\frac{2}{3}}}{120^{1}}=\frac{360\times 120^{\frac{2}{3}}}{120}=3\times 120^{\frac{2}{3}}$$

NOW WE NEED TO REDUCE THE CUBIC SURD

$$=3\times\left(\sqrt[3]{120}\right)^2=3\times\left(\sqrt[3]{8}\sqrt[3]{15}\right)^2$$

$$=3\times\left(2\times\sqrt[3]{15}\right)^2=3\times 4\times\left(\sqrt[3]{15}\right)^2$$

$$=12\times\sqrt[3]{15^2}=12\times\sqrt[3]{225}$$

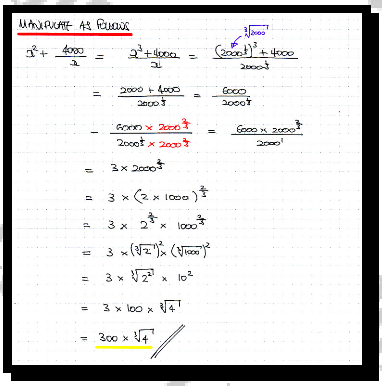
AS REQUIRED

Question 59 **(****+)**

If $x=\sqrt[3]{2000}$, show clearly that

$$x^2+\frac{4000}{x}=300\times\sqrt[3]{4}\,.$$

, proof



MANIPULATE AS FOLLOWS

$$x^2+\frac{4000}{x}=\frac{x^3+4000}{x}=\frac{\left(\overset{\sqrt[3]{2000}}{2000^{\frac{1}{3}}}\right)^3+4000}{2000^{\frac{1}{3}}}$$

$$=\frac{2000+4000}{2000^{\frac{1}{3}}}=\frac{6000}{2000^{\frac{1}{3}}}$$

$$=\frac{6000\times 2000^{\frac{2}{3}}}{2000^{\frac{1}{3}}\times 2000^{\frac{2}{3}}}=\frac{6000\times 2000^{\frac{2}{3}}}{2000^{1}}$$

$$=3\times 2000^{\frac{2}{3}}$$

$$=3\times(2\times 1000)^{\frac{2}{3}}$$

$$=3\times 2^{\frac{2}{3}}\times 1000^{\frac{2}{3}}$$

$$=3\times\left(\sqrt[3]{2}\right)^2\times\left(\sqrt[3]{1000}\right)^2$$

$$=3\times\sqrt[3]{2^2}\times 10^2$$

$$=3\times 100\times\sqrt[3]{4}$$

$$=300\times\sqrt[3]{4}$$

Question 60 (****+)

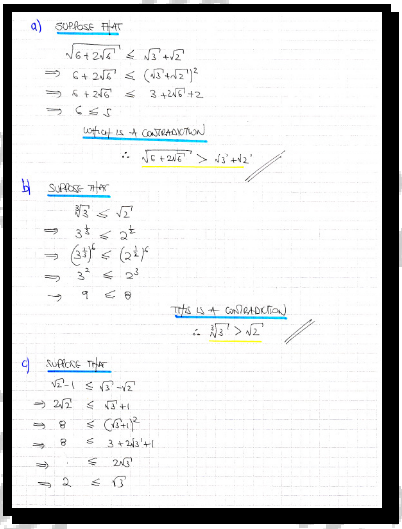
Show clearly, without approximating and without using any calculating aid, that

a) $\sqrt{6+2\sqrt{6}} > \sqrt{3}+\sqrt{2}$.

b) $\sqrt[3]{3} > \sqrt{2}$.

c) $\sqrt{2}-1 > \sqrt{3}-\sqrt{2}$.

$\boxed{}$, $\boxed{\text{proof}}$



a) SUPPOSE THAT

$$\sqrt{6+2\sqrt{6}} \leq \sqrt{3}+\sqrt{2}$$
$$\Rightarrow 6+2\sqrt{6} \leq (\sqrt{3}+\sqrt{2})^2$$
$$\Rightarrow 6+2\sqrt{6} \leq 3+2\sqrt{6}+2$$
$$\Rightarrow 6 \leq 5$$

WHICH IS A CONTRADICTION

$$\therefore \sqrt{6+2\sqrt{6}} > \sqrt{3}+\sqrt{2} \quad //$$

b) SUPPOSE THAT

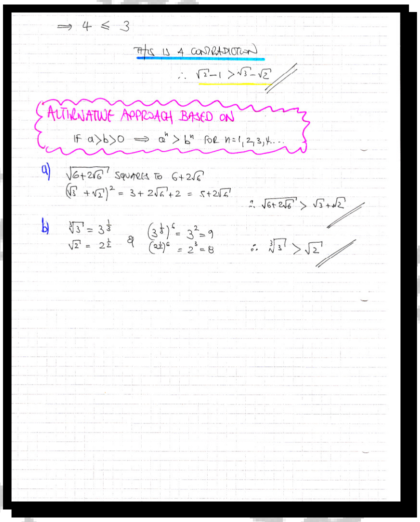
$$\sqrt[3]{3} \leq \sqrt{2}$$
$$\Rightarrow 3^{\frac{1}{3}} \leq 2^{\frac{1}{2}}$$
$$\Rightarrow \left(3^{\frac{1}{3}}\right)^6 \leq \left(2^{\frac{1}{2}}\right)^6$$
$$\Rightarrow 3^2 \leq 2^3$$
$$\Rightarrow 9 \leq 8$$

THIS IS A CONTRADICTION

$$\therefore \sqrt[3]{3} > \sqrt{2} \quad //$$

c) SUPPOSE THAT

$$\sqrt{2}-1 \leq \sqrt{3}-\sqrt{2}$$
$$\Rightarrow 2\sqrt{2} \leq \sqrt{3}+1$$
$$\Rightarrow 8 \leq (\sqrt{3}+1)^2$$
$$\Rightarrow 8 \leq 3+2\sqrt{3}+1$$
$$\Rightarrow \quad \leq 2\sqrt{3}$$
$$\Rightarrow 2 \leq \sqrt{3}$$



$$\Rightarrow 4 \leq 3$$

THIS IS A CONTRADICTION

$$\therefore \sqrt{2}-1 > \sqrt{3}-\sqrt{2} \quad //$$

ALTERNATIVE APPROACH BASED ON

IF $a>b>0 \Rightarrow a^n > b^n$ FOR $n=1,2,3,4\ldots$

a) $\sqrt{6+2\sqrt{6}}$ SQUARES TO $6+2\sqrt{6}$

$(\sqrt{3}+\sqrt{2})^2 = 3+2\sqrt{6}+2 = 5+2\sqrt{6}$

$$\therefore \sqrt{6+2\sqrt{6}} > \sqrt{3}+\sqrt{2} \quad //$$

b) $\sqrt[3]{3} = 3^{\frac{1}{3}}$ & $\left(3^{\frac{1}{3}}\right)^6 = 3^2 = 9$

$\sqrt{2} = 2^{\frac{1}{2}}$ $\left(2^{\frac{1}{2}}\right)^6 = 2^3 = 8$

$$\therefore \sqrt[3]{3} > \sqrt{2} \quad //$$

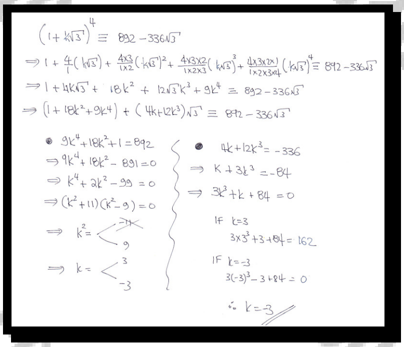
Question 61 (****+)

It is given that if k is a non zero constant then

$$\left(1+k\sqrt{3}\right)^4 \equiv 892-336\sqrt{3} .$$

Determine the value of k .

$$\boxed{k=-3}$$



$$\left(1+k\sqrt{3}\right)^4 \equiv 892-336\sqrt{3}$$

$$\Rightarrow 1+\frac{4}{1}\left(k\sqrt{3}\right)+\frac{4\times3}{1\times2}\left(k\sqrt{3}\right)^2+\frac{4\times3\times2}{1\times2\times3}\left(k\sqrt{3}\right)^3+\frac{4\times3\times2\times1}{1\times2\times3\times4}\left(k\sqrt{3}\right)^4 \equiv 892-336\sqrt{3}$$

$$\Rightarrow 1+4k\sqrt{3}+18k^2+12\sqrt{3}k^3+9k^4 \equiv 892-336\sqrt{3}$$

$$\Rightarrow \left(1+18k^2+9k^4\right)+\left(4k+12k^3\right)\sqrt{3} \equiv 892-336\sqrt{3}$$

- $9k^4+18k^2+1=892$

$\Rightarrow 9k^4+18k^2-891=0$

$\Rightarrow k^4+2k^2-99=0$

$\Rightarrow (k^2+11)(k^2-9)=0$

$\Rightarrow k^2 = \begin{cases} \cancel{-11} \\ 9 \end{cases}$

$\Rightarrow k = \begin{cases} 3 \\ -3 \end{cases}$

- $4k+12k^3=-336$

$\Rightarrow k+3k^3=-84$

$\Rightarrow 3k^3+k+84=0$

IF $k=3$

$3\times3^3+3+84=162$

IF $k=-3$

$3(-3)^3-3+84=0$

$\therefore k=-3$

Question 62 (****+) **non calculator**

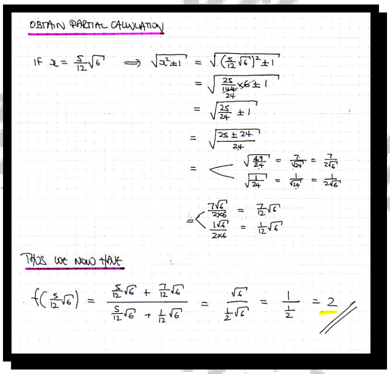
$$f(x) \equiv \frac{x+\sqrt{x^2+1}}{x+\sqrt{x^2-1}}, \ x \in \mathbb{R}, \ |x| \geq 1.$$

Find without the use of a calculator the value of

$$f\left(\tfrac{5}{12}\sqrt{6}\right).$$

Detailed workings must be shown in this question.

$\boxed{\text{YN H}}$, $\boxed{f\left(\tfrac{5}{12}\sqrt{6}\right)=2}$



OBTAIN PARTIAL CALCULATION

$$\text{IF } x = \tfrac{5}{12}\sqrt{6} \implies \sqrt{x^2 \pm 1} = \sqrt{\left(\tfrac{5}{12}\sqrt{6}\right)^2 \pm 1}$$

$$= \sqrt{\tfrac{25}{\cancel{144}_{24}} \times \cancel{6} \pm 1}$$

$$= \sqrt{\tfrac{25}{24} \pm 1}$$

$$= \sqrt{\tfrac{25 \pm 24}{24}}$$

$$= \begin{cases} \sqrt{\tfrac{49}{24}} = \tfrac{7}{\sqrt{24}} = \tfrac{7}{2\sqrt{6}} \\ \sqrt{\tfrac{1}{24}} = \tfrac{1}{\sqrt{24}} = \tfrac{1}{2\sqrt{6}} \end{cases}$$

$$= \begin{cases} \tfrac{7\sqrt{6}}{2\times 6} = \tfrac{7}{12}\sqrt{6} \\ \tfrac{1\sqrt{6}}{2\times 6} = \tfrac{1}{12}\sqrt{6} \end{cases}$$

THUS WE NOW HAVE

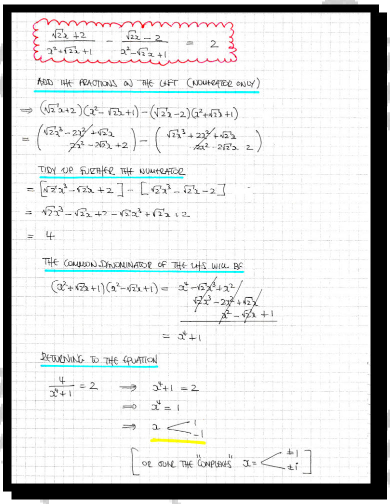
$$f\left(\tfrac{5}{12}\sqrt{6}\right) = \frac{\tfrac{5}{12}\sqrt{6} + \tfrac{7}{12}\sqrt{6}}{\tfrac{5}{12}\sqrt{6} + \tfrac{1}{12}\sqrt{6}} = \frac{\sqrt{6}}{\tfrac{1}{2}\sqrt{6}} = \frac{1}{\tfrac{1}{2}} = 2 \ //$$

Question 63 (****+)

Solve the following equation.

$$\frac{2+\sqrt{2}\,x}{x^2+\sqrt{2}\,x+1}+\frac{2-\sqrt{2}\,x}{x^2-\sqrt{2}\,x+1}=2,\quad x\in\mathbb{R}.$$

$\boxed{\text{YN}\quad\text{I}}$, $\boxed{x=\pm 1}$



$$\frac{\sqrt{2}x+2}{x^2+\sqrt{2}x+1}-\frac{\sqrt{2}x-2}{x^2-\sqrt{2}x+1}=2$$

ADD THE FRACTIONS ON THE LEFT (NUMERATOR ONLY)

$$\Rightarrow (\sqrt{2}x+2)(x^2-\sqrt{2}x+1)-(\sqrt{2}x-2)(x^2+\sqrt{2}x+1)$$

$$=\left(\begin{matrix}\sqrt{2}x^3-\cancel{2x^2}+\sqrt{2}x\\ \cancel{2x^2}-2\sqrt{2}x+2\end{matrix}\right)-\left(\begin{matrix}\sqrt{2}x^3+\cancel{2x^2}+\sqrt{2}x\\ -\cancel{2x^2}-2\sqrt{2}x-2\end{matrix}\right)$$

TIDY UP FURTHER THE NUMERATOR

$$=\left[\sqrt{2}x^3-\sqrt{2}x+2\right]-\left[\sqrt{2}x^3-\sqrt{2}x-2\right]$$

$$=\sqrt{2}x^3-\sqrt{2}x+2-\sqrt{2}x^3+\sqrt{2}x+2$$

$$=4$$

THE COMMON DENOMINATOR OF THE LHS WILL BE

$$(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)=\begin{matrix}x^4-\cancel{\sqrt{2}x^3}+\cancel{x^2}\\ \cancel{\sqrt{2}x^3}-2\cancel{x^2}+\cancel{\sqrt{2}x}\\ \cancel{x^2}-\cancel{\sqrt{2}x}+1\end{matrix}$$

$$=x^4+1$$

RETURNING TO THE EQUATION

$$\frac{4}{x^4+1}=2 \implies x^4+1=2$$

$$\implies x^4=1$$

$$\implies x \begin{matrix}\nearrow 1\\ \searrow -1\end{matrix}$$

$$\left[\text{OR USING THE "COMPLEXES" } x=\begin{matrix}\nearrow \pm 1\\ \searrow \pm i\end{matrix}\right]$$

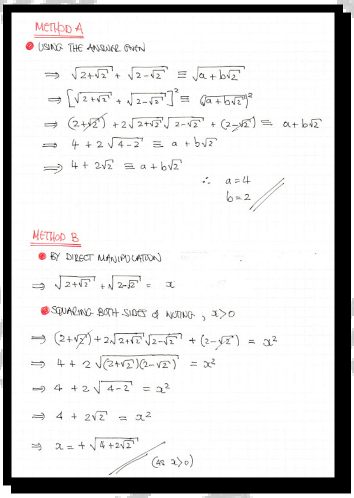
Question 64 (****+)

Show that $\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}}$ can be expressed in the form

$$\sqrt{a+b\sqrt{2}}\ ,$$

where a and b are integers to be found.

$\boxed{}$, $\boxed{\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}}=\sqrt{4+2\sqrt{2}}}$



METHOD A

- USING THE ANSWER GIVEN

$\Rightarrow \sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}} \equiv \sqrt{a+b\sqrt{2}}$

$\Rightarrow \left[\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}}\right]^2 \equiv \left(\sqrt{a+b\sqrt{2}}\right)^2$

$\Rightarrow (2+\cancel{\sqrt{2}})+2\sqrt{2+\sqrt{2}}\sqrt{2-\sqrt{2}}+(2-\cancel{\sqrt{2}}) \equiv a+b\sqrt{2}$

$\Rightarrow 4+2\sqrt{4-2} \equiv a+b\sqrt{2}$

$\Rightarrow 4+2\sqrt{2} \equiv a+b\sqrt{2}$

$\therefore\ a=4$
$b=2$ //

METHOD B

- BY DIRECT MANIPULATION

$\Rightarrow \sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}} = x$

- SQUARING BOTH SIDES & NOTING, $x>0$

$\Rightarrow (2+\cancel{\sqrt{2}})+2\sqrt{2+\sqrt{2}}\sqrt{2-\sqrt{2}}+(2-\cancel{\sqrt{2}}) = x^2$

$\Rightarrow 4+2\sqrt{(2+\sqrt{2})(2-\sqrt{2})} = x^2$

$\Rightarrow 4+2\sqrt{4-2} = x^2$

$\Rightarrow 4+2\sqrt{2} = x^2$

$\Rightarrow x = +\sqrt{4+2\sqrt{2}}$ //

(AS $x>0$)

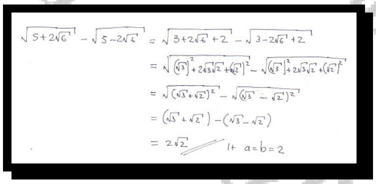
Question 65 (*****)

Show clearly without the use of any calculating aid that

$$\sqrt{5+2\sqrt{6}}-\sqrt{5-2\sqrt{6}}=a\sqrt{b}\ ,$$

where a and b are integers to be found.

$\boxed{a=b=2}$



$\sqrt{5+2\sqrt{6}}-\sqrt{5-2\sqrt{6}} = \sqrt{3+2\sqrt{6}+2}-\sqrt{3-2\sqrt{6}+2}$

$= \sqrt{\left(\sqrt{3}\right)^2+2\sqrt{3}\sqrt{2}+\left(\sqrt{2}\right)^2}-\sqrt{\left(\sqrt{3}\right)^2+2\sqrt{3}\sqrt{2}+\left(\sqrt{2}\right)^2}$

$= \sqrt{(\sqrt{3}+\sqrt{2})^2}-\sqrt{(\sqrt{3}-\sqrt{2})^2}$

$= (\sqrt{3}+\sqrt{2})-(\sqrt{3}-\sqrt{2})$

$= 2\sqrt{2}$ // i.e. $a=b=2$

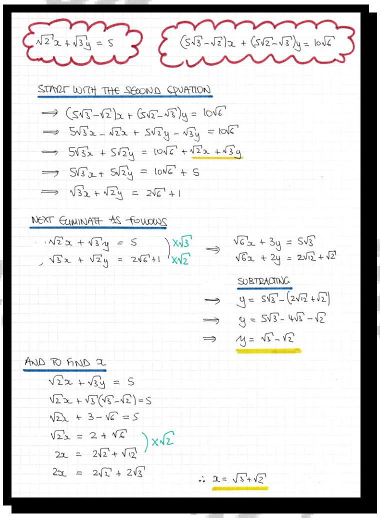
Question 66 (*****)

Determine, in exact simplified surd form, the solution pair (a,b) of the following simultaneous equations.

$$\sqrt{2}\,x+\sqrt{3}\,y=5 \qquad \text{and} \qquad \left(5\sqrt{3}-\sqrt{2}\right)x+\left(5\sqrt{2}-\sqrt{3}\right)y=10\sqrt{6}\,.$$

Detailed workings must be shown in this question.

$\boxed{}$, $\boxed{x=\sqrt{3}+\sqrt{2},\quad y=\sqrt{3}-\sqrt{2}}$



$\sqrt{2}x+\sqrt{3}y=5$ $\qquad$ $(5\sqrt{3}-\sqrt{2})x+(5\sqrt{2}-\sqrt{3})y=10\sqrt{6}$

START WITH THE SECOND EQUATION

$\Longrightarrow (5\sqrt{3}-\sqrt{2})x+(5\sqrt{2}-\sqrt{3})y=10\sqrt{6}$

$\Longrightarrow 5\sqrt{3}x-\sqrt{2}x+5\sqrt{2}y-\sqrt{3}y=10\sqrt{6}$

$\Longrightarrow 5\sqrt{3}x+5\sqrt{2}y=10\sqrt{6}+\underline{\sqrt{2}x+\sqrt{3}y}$

$\Longrightarrow 5\sqrt{3}x+5\sqrt{2}y=10\sqrt{6}+5$

$\Longrightarrow \sqrt{3}x+\sqrt{2}y=2\sqrt{6}+1$

NEXT ELIMINATE AS FOLLOWS

$$\left.\begin{aligned}\sqrt{2}x+\sqrt{3}y&=5\\ \sqrt{3}x+\sqrt{2}y&=2\sqrt{6}+1\end{aligned}\right)\begin{matrix}\times\sqrt{3}\\ \times\sqrt{2}\end{matrix} \Longrightarrow \begin{aligned}\sqrt{6}x+3y&=5\sqrt{3}\\ \sqrt{6}x+2y&=2\sqrt{12}+\sqrt{2}\end{aligned}$$

SUBTRACTING

$\Longrightarrow y=5\sqrt{3}-(2\sqrt{12}+\sqrt{2})$

$\Longrightarrow y=5\sqrt{3}-4\sqrt{3}-\sqrt{2}$

$\Longrightarrow \underline{y=\sqrt{3}-\sqrt{2}}$

AND TO FIND x

$\sqrt{2}x+\sqrt{3}y=5$

$\sqrt{2}x+\sqrt{3}(\sqrt{3}-\sqrt{2})=5$

$\sqrt{2}x+3-\sqrt{6}=5$

$$\left.\begin{aligned}\sqrt{2}x&=2+\sqrt{6}\\ 2x&=2\sqrt{2}+\sqrt{12}\end{aligned}\right)\times\sqrt{2}$$

$2x=2\sqrt{2}+2\sqrt{3}$ $\qquad$ $\therefore \underline{x=\sqrt{3}+\sqrt{2}}$

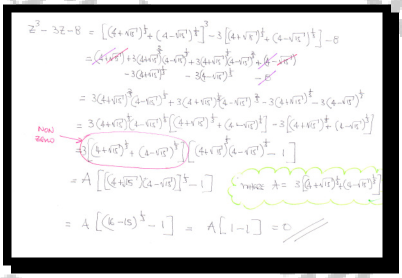
Question 67 **(*****)**

$$z=\sqrt[3]{4+\sqrt{15}}+\sqrt[3]{4-\sqrt{15}}.$$

Verify that z is a solution of the equation

$$z^3-3z-8=0.$$

$\boxed{\text{proof}}$



$$z^3-3z-8=\left[(4+\sqrt{15})^{\frac{1}{3}}+(4-\sqrt{15})^{\frac{1}{3}}\right]^3-3\left[(4+\sqrt{15})^{\frac{1}{3}}+(4-\sqrt{15})^{\frac{1}{3}}\right]-8$$

$$=\cancel{(4+\sqrt{15})}+3(4+\sqrt{15})^{\frac{2}{3}}(4-\sqrt{15})^{\frac{1}{3}}+3(4+\sqrt{15})^{\frac{1}{3}}(4-\sqrt{15})^{\frac{2}{3}}+\cancel{(4-\sqrt{15})}$$
$$-3(4+\sqrt{15})^{\frac{1}{3}}\quad-3(4-\sqrt{15})^{\frac{1}{3}}\quad\cancel{-8}$$

$$=3(4+\sqrt{15})^{\frac{2}{3}}(4-\sqrt{15})^{\frac{1}{3}}+3(4+\sqrt{15})^{\frac{1}{3}}(4-\sqrt{15})^{\frac{2}{3}}-3(4+\sqrt{15})^{\frac{1}{3}}-3(4-\sqrt{15})^{\frac{1}{3}}$$

$$=3(4+\sqrt{15})^{\frac{1}{3}}(4-\sqrt{15})^{\frac{1}{3}}\left[(4+\sqrt{15})^{\frac{1}{3}}+(4-\sqrt{15})^{\frac{1}{3}}\right]-3\left[(4+\sqrt{15})^{\frac{1}{3}}+(4-\sqrt{15})^{\frac{1}{3}}\right]$$

NON ZERO

$$=3\left[(4+\sqrt{15})^{\frac{1}{3}}+(4-\sqrt{15})^{\frac{1}{3}}\right]\left[(4+\sqrt{15})^{\frac{1}{3}}(4-\sqrt{15})^{\frac{1}{3}}-1\right]$$

$$=A\left[\left[(4+\sqrt{15})(4-\sqrt{15})\right]^{\frac{1}{3}}-1\right]\qquad \text{WHERE } A=3\left[(4+\sqrt{15})^{\frac{1}{3}}+(4-\sqrt{15})^{\frac{1}{3}}\right]$$

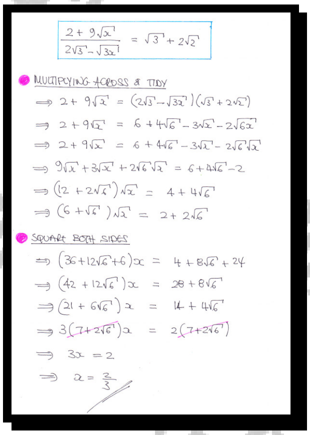
$$=A\left[(16-15)^{\frac{1}{3}}-1\right]=A[1-1]=0$$

Question 68 **(*****)**

Find the rational solution of the following equation

$$\frac{2+9\sqrt{x}}{2\sqrt{3}-\sqrt{3x}}=\sqrt{3}+2\sqrt{2},\ x\in\mathbb{Q}.$$

$\boxed{}$, $\boxed{x=\frac{2}{3}}$



$$\boxed{\frac{2+9\sqrt{x}}{2\sqrt{3}-\sqrt{3x}}=\sqrt{3}+2\sqrt{2}}$$

- MULTIPLYING ACROSS & TIDY

$$\Rightarrow 2+9\sqrt{x}=(2\sqrt{3}-\sqrt{3x})(\sqrt{3}+2\sqrt{2})$$
$$\Rightarrow 2+9\sqrt{x}=6+4\sqrt{6}-3\sqrt{x}-2\sqrt{6x}$$
$$\Rightarrow 2+9\sqrt{x}=6+4\sqrt{6}-3\sqrt{x}-2\sqrt{6}\sqrt{x}$$
$$\Rightarrow 9\sqrt{x}+3\sqrt{x}+2\sqrt{6}\sqrt{x}=6+4\sqrt{6}-2$$
$$\Rightarrow (12+2\sqrt{6})\sqrt{x}=4+4\sqrt{6}$$
$$\Rightarrow (6+\sqrt{6})\sqrt{x}=2+2\sqrt{6}$$

- SQUARE BOTH SIDES

$$\Rightarrow (36+12\sqrt{6}+6)x=4+8\sqrt{6}+24$$
$$\Rightarrow (42+12\sqrt{6})x=28+8\sqrt{6}$$
$$\Rightarrow (21+6\sqrt{6})x=14+4\sqrt{6}$$
$$\Rightarrow 3\cancel{(7+2\sqrt{6})}x=2\cancel{(7+2\sqrt{6})}$$
$$\Rightarrow 3x=2$$
$$\Rightarrow x=\frac{2}{3}$$

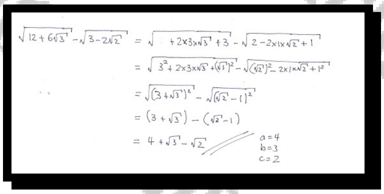
Question 69 **(*****)**

Show clearly without the use of any calculating aid that

$$\sqrt{12+6\sqrt{3}}-\sqrt{3-2\sqrt{2}}=a+\sqrt{b}-\sqrt{c}\ ,$$

where a, b and c are integers to be found.

$\boxed{a=4,\ \ b=3,\ \ c=2}$



$\sqrt{12+6\sqrt{3}}-\sqrt{3-2\sqrt{2}} = \sqrt{\quad +2\times3\times\sqrt{3}+3}-\sqrt{2-2\times1\times\sqrt{2}+1}$

$= \sqrt{3^2+2\times3\times\sqrt{3}+(\sqrt{3})^2}-\sqrt{(\sqrt{2})^2-2\times1\times\sqrt{2}+1^2}$

$= \sqrt{(3+\sqrt{3})^2}-\sqrt{(\sqrt{2}-1)^2}$

$= (3+\sqrt{3})-(\sqrt{2}-1)$

$= 4+\sqrt{3}-\sqrt{2}$ //

$a=4$
$b=3$
$c=2$

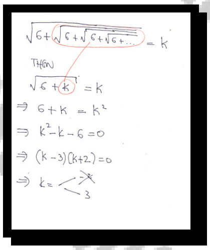
Question 70 **(*****)**

Show clearly without the use of any calculating aid that

$$\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\ldots}}}}=k\ ,$$

where k is an integer to be found.

$\boxed{k=3}$



$\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\ldots}}}}=k$

THEN

$\sqrt{6+k}=k$

$\Rightarrow 6+k=k^2$

$\Rightarrow k^2-k-6=0$

$\Rightarrow (k-3)(k+2)=0$

$\Rightarrow k = \cancel{-2},\ 3$

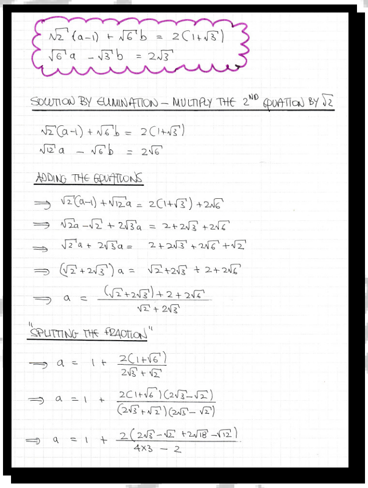
Question 71 (*****)

Determine, in exact simplified surd form, the solution pair (a,b) of the following simultaneous equations.

$$\sqrt{2}(a-1)+\sqrt{6}\,b=2\left(1+\sqrt{3}\right) \quad \text{and} \quad \sqrt{6}\,a-\sqrt{3}\,b=2\sqrt{3}.$$

Detailed workings must be shown in this question.

$\boxed{a=1+\sqrt{2}, \quad b=\sqrt{2}}$



$\sqrt{2}(a-1)+\sqrt{6}\,b = 2(1+\sqrt{3})$

$\sqrt{6}\,a - \sqrt{3}\,b = 2\sqrt{3}$

SOLUTION BY ELIMINATION – MULTIPLY THE 2ND EQUATION BY $\sqrt{2}$

$\sqrt{2}(a-1)+\sqrt{6}\,b = 2(1+\sqrt{3})$

$\sqrt{12}\,a - \sqrt{6}\,b = 2\sqrt{6}$

ADDING THE EQUATIONS

$\Rightarrow \sqrt{2}(a-1)+\sqrt{12}\,a = 2(1+\sqrt{3})+2\sqrt{6}$

$\Rightarrow \sqrt{2}a-\sqrt{2}+2\sqrt{3}a = 2+2\sqrt{3}+2\sqrt{6}$

$\Rightarrow \sqrt{2}a+2\sqrt{3}a = 2+2\sqrt{3}+2\sqrt{6}+\sqrt{2}$

$\Rightarrow (\sqrt{2}+2\sqrt{3})a = \sqrt{2}+2\sqrt{3}+2+2\sqrt{6}$

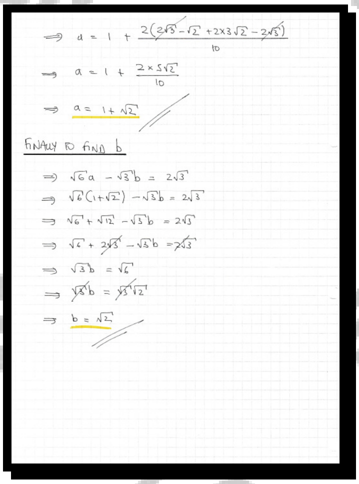
$\Rightarrow a = \dfrac{(\sqrt{2}+2\sqrt{3})+2+2\sqrt{6}}{\sqrt{2}+2\sqrt{3}}$

"SPLITTING THE FRACTION"

$\Rightarrow a = 1+\dfrac{2(1+\sqrt{6})}{2\sqrt{3}+\sqrt{2}}$

$\Rightarrow a = 1+\dfrac{2(1+\sqrt{6})(2\sqrt{3}-\sqrt{2})}{(2\sqrt{3}+\sqrt{2})(2\sqrt{3}-\sqrt{2})}$

$\Rightarrow a = 1+\dfrac{2(2\sqrt{3}-\sqrt{2}+2\sqrt{18}-\sqrt{12})}{4\times3-2}$



$\Rightarrow a = 1+\dfrac{2(\cancel{2\sqrt{3}}-\sqrt{2}+2\times3\sqrt{2}-\cancel{2\sqrt{3}})}{10}$

$\Rightarrow a = 1+\dfrac{2\times5\sqrt{2}}{10}$

$\Rightarrow a = 1+\sqrt{2}$

FINALLY TO FIND b

$\Rightarrow \sqrt{6}\,a-\sqrt{3}\,b = 2\sqrt{3}$

$\Rightarrow \sqrt{6}(1+\sqrt{2})-\sqrt{3}\,b = 2\sqrt{3}$

$\Rightarrow \sqrt{6}+\sqrt{12}-\sqrt{3}\,b = 2\sqrt{3}$

$\Rightarrow \sqrt{6}+\cancel{2\sqrt{3}}-\sqrt{3}\,b = \cancel{2\sqrt{3}}$

$\Rightarrow \sqrt{3}\,b = \sqrt{6}$

$\Rightarrow \cancel{\sqrt{3}}\,b = \cancel{\sqrt{3}}\sqrt{2}$

$\Rightarrow b = \sqrt{2}$

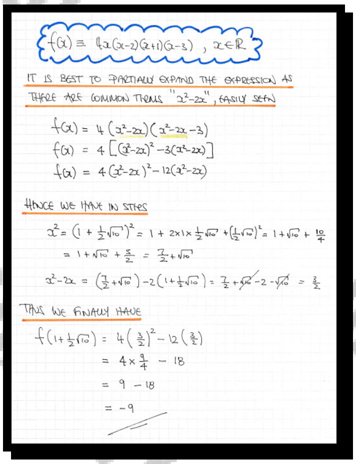
Question 72 (**+) non calculator**

$$f(x) \equiv 4x(x-2)(x+1)(x-3), \quad x \in \mathbb{R}.$$

Evaluate $f\left(1+\frac{1}{2}\sqrt{10}\right)$.

You must show detailed workings in this question.

$\boxed{S_1}$, $\boxed{f\left(1+\frac{1}{2}\sqrt{10}\right)=-9}$



$f(x) \equiv 4x(x-2)(x+1)(x-3)$, $x \in \mathbb{R}$

IT IS BEST TO PARTIALLY EXPAND THE EXPRESSION AS THERE ARE COMMON TERMS "x^2-2x", EASILY SEEN

$$f(x) = 4(x^2-2x)(x^2-2x-3)$$
$$f(x) = 4\left[(x^2-2x)^2-3(x^2-2x)\right]$$
$$f(x) = 4(x^2-2x)^2-12(x^2-2x)$$

HENCE WE HAVE IN STEPS

$$x^2 = \left(1+\tfrac{1}{2}\sqrt{10}\right)^2 = 1+2\times1\times\tfrac{1}{2}\sqrt{10}+\left(\tfrac{1}{2}\sqrt{10}\right)^2 = 1+\sqrt{10}+\tfrac{10}{4}$$
$$= 1+\sqrt{10}+\tfrac{5}{2} = \tfrac{7}{2}+\sqrt{10}$$
$$x^2-2x = \left(\tfrac{7}{2}+\sqrt{10}\right)-2\left(1+\tfrac{1}{2}\sqrt{10}\right) = \tfrac{7}{2}+\cancel{\sqrt{10}}-2-\cancel{\sqrt{10}} = \tfrac{3}{2}$$

THUS WE FINALLY HAVE

$$f\left(1+\tfrac{1}{2}\sqrt{10}\right) = 4\left(\tfrac{3}{2}\right)^2-12\left(\tfrac{3}{2}\right)$$
$$= 4\times\tfrac{9}{4}-18$$
$$= 9-18$$
$$= -9$$

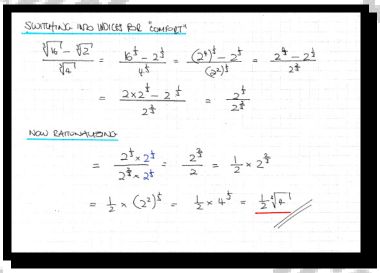
Question 73 **(****+)**

Show with a detailed method that

$$\frac{\sqrt[3]{16}-\sqrt[3]{2}}{\sqrt[3]{4}} = k\,\sqrt[3]{4}$$

where k is a constant to be found.

$\boxed{}$, $\boxed{k=\frac{1}{2}}$



SWITCHING INTO INDICES FOR "COMFORT"

$$\frac{\sqrt[3]{16}-\sqrt[3]{2}}{\sqrt[3]{4}} = \frac{16^{\frac{1}{3}}-2^{\frac{1}{3}}}{4^{\frac{1}{3}}} = \frac{(2^4)^{\frac{1}{3}}-2^{\frac{1}{3}}}{(2^2)^{\frac{1}{3}}} = \frac{2^{\frac{4}{3}}-2^{\frac{1}{3}}}{2^{\frac{2}{3}}}$$

$$= \frac{2\times 2^{\frac{1}{3}}-2^{\frac{1}{3}}}{2^{\frac{2}{3}}} = \frac{2^{\frac{1}{3}}}{2^{\frac{2}{3}}}$$

NOW RATIONALIZING

$$= \frac{2^{\frac{1}{3}}\times 2^{\frac{1}{3}}}{2^{\frac{2}{3}}\times 2^{\frac{1}{3}}} = \frac{2^{\frac{2}{3}}}{2} = \frac{1}{2}\times 2^{\frac{2}{3}}$$

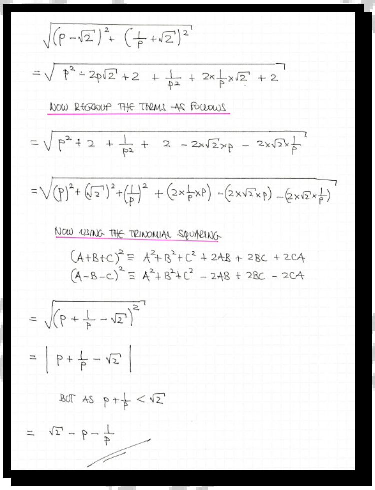
$$= \frac{1}{2}\times(2^2)^{\frac{1}{3}} = \frac{1}{2}\times 4^{\frac{1}{3}} = \underline{\frac{1}{2}\sqrt[3]{4}}$$

Question 74 (*****)

$$f(p) \equiv \left(p-\sqrt{2}\right)^2 + \left(\frac{1}{p}-\sqrt{2}\right)^2, \ p \in \mathbb{R}, \ p \neq 0.$$

Given that $p+\frac{1}{p}<\sqrt{2}$, find $\sqrt{f(p)}$ in its simplest form.

$\boxed{}$, $\boxed{\text{proof}}$



$\sqrt{\left(p-\sqrt{2}\right)^2+\left(\frac{1}{p}+\sqrt{2}\right)^2}$

$=\sqrt{p^2-2p\sqrt{2}+2+\frac{1}{p^2}+2\times\frac{1}{p}\times\sqrt{2}+2}$

NOW REGROUP THE TERMS AS FOLLOWS

$=\sqrt{p^2+2+\frac{1}{p^2}+2-2\times\sqrt{2}\times p-2\times\sqrt{2}\times\frac{1}{p}}$

$=\sqrt{(p)^2+(\sqrt{2})^2+\left(\frac{1}{p}\right)^2+\left(2\times\frac{1}{p}\times p\right)-\left(2\times\sqrt{2}\times p\right)-\left(2\times\sqrt{2}\times\frac{1}{p}\right)}$

NOW USING THE TRINOMIAL SQUARING

$(A+B+C)^2 \equiv A^2+B^2+C^2+2AB+2BC+2CA$

$(A-B-C)^2 \equiv A^2+B^2+C^2-2AB+2BC-2CA$

$=\sqrt{\left(p+\frac{1}{p}-\sqrt{2}\right)^2}$

$=\left|p+\frac{1}{p}-\sqrt{2}\right|$

BUT AS $p+\frac{1}{p}<\sqrt{2}$

$=\sqrt{2}-p-\frac{1}{p}$

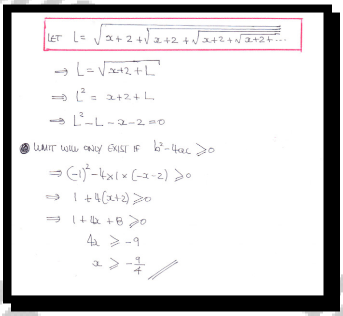
Question 75 (*****)

$$\sqrt{x+2+\sqrt{x+2+\sqrt{x+2+\sqrt{x+2+\sqrt{x+2+\ldots}}}}}\ ,$$

It is given that the above nested radical converges to a limit L, $L \in \mathbb{R}$.

Determine the range of possible values of x.

$\boxed{}$, $\boxed{x \geq -\frac{9}{4}}$



LET $L = \sqrt{x+2+\sqrt{x+2+\sqrt{x+2+\sqrt{x+2+\ldots}}}}$

$\Rightarrow L = \sqrt{x+2+L}$

$\Rightarrow L^2 = x+2+L$

$\Rightarrow L^2 - L - x - 2 = 0$

● LIMIT WILL ONLY EXIST IF $b^2 - 4ac \geqslant 0$

$\Rightarrow (-1)^2 - 4 \times 1 \times (-x-2) \geqslant 0$

$\Rightarrow 1 + 4(x+2) \geqslant 0$

$\Rightarrow 1 + 4x + 8 \geqslant 0$

$4x \geqslant -9$

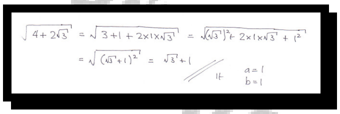
$x \geqslant -\frac{9}{4}$ //

Question 76 (*****)

Show clearly that

$$\sqrt{4+2\sqrt{3}} = 1+\sqrt{3}\,.$$

$\boxed{\text{proof}}$



$\sqrt{4+2\sqrt{3}} = \sqrt{3+1+2\times1\times\sqrt{3}} = \sqrt{(\sqrt{3})^2 + 2\times1\times\sqrt{3} + 1^2}$

$= \sqrt{(\sqrt{3}+1)^2} = \sqrt{3}+1$ //

If $a=1$
$b=1$

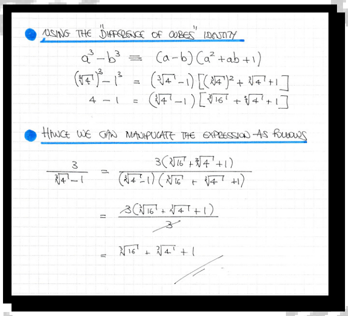
Question 77 (*****)

Show that

$$\frac{3}{\sqrt[3]{4}-1}$$

can be written in the form $\sqrt[3]{a}+\sqrt[3]{b}+1$, where a and b are integers to be found.

$\boxed{\quad}$, $\boxed{\sqrt[3]{16}+\sqrt[3]{4}+1}$



- USING THE "DIFFERENCE OF CUBES" IDENTITY

$$a^3-b^3 \equiv (a-b)(a^2+ab+1)$$
$$(\sqrt[3]{4})^3-1^3 = (\sqrt[3]{4}-1)\left[(\sqrt[3]{4})^2+\sqrt[3]{4}+1\right]$$
$$4-1 = (\sqrt[3]{4}-1)\left[\sqrt[3]{16}+\sqrt[3]{4}+1\right]$$

- HENCE WE CAN MANIPULATE THE EXPRESSION AS FOLLOWS

$$\frac{3}{\sqrt[3]{4}-1} = \frac{3(\sqrt[3]{16}+\sqrt[3]{4}+1)}{(\sqrt[3]{4}-1)(\sqrt[3]{16}+\sqrt[3]{4}+1)}$$
$$= \frac{\cancel{3}(\sqrt[3]{16}+\sqrt[3]{4}+1)}{\cancel{3}}$$
$$= \sqrt[3]{16}+\sqrt[3]{4}+1$$

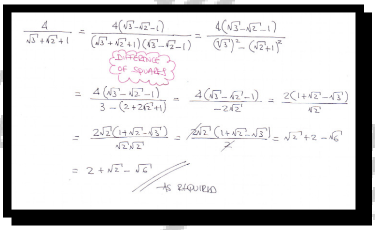
Question 78 (*****)

Show clearly that

$$\frac{4}{\sqrt{3}+\sqrt{2}+1} = 2+\sqrt{2}-\sqrt{6}.$$

You may not use verification in this question

$\boxed{\quad}$, $\boxed{\text{proof}}$



$$\frac{4}{\sqrt{3}+\sqrt{2}+1} = \frac{4(\sqrt{3}-\sqrt{2}-1)}{(\sqrt{3}+\sqrt{2}+1)(\sqrt{3}-\sqrt{2}-1)} = \frac{4(\sqrt{3}-\sqrt{2}-1)}{(\sqrt{3})^2-(\sqrt{2}+1)^2}$$

DIFFERENCE OF SQUARES

$$= \frac{4(\sqrt{3}-\sqrt{2}-1)}{3-(2+2\sqrt{2}+1)} = \frac{4(\sqrt{3}-\sqrt{2}-1)}{-2\sqrt{2}} = \frac{2(1+\sqrt{2}-\sqrt{3})}{\sqrt{2}}$$
$$= \frac{2\sqrt{2}(1+\sqrt{2}-\sqrt{3})}{\sqrt{2}\sqrt{2}} = \frac{\cancel{2}\sqrt{2}(1+\sqrt{2}-\sqrt{3})}{\cancel{2}} = \sqrt{2}+2-\sqrt{6}$$
$$= 2+\sqrt{2}-\sqrt{6}$$

AS REQUIRED

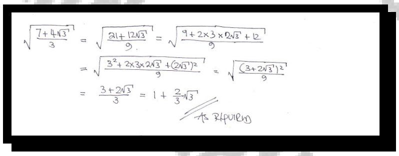
Question 79 **(*****)**

Show clearly that

$$\sqrt{\frac{1+4\sqrt{3}}{3}} = a + b\sqrt{3},$$

where a and b are constants to be found.

$\boxed{a=1}$, $\boxed{b=\frac{2}{3}}$



$$\sqrt{\frac{7+4\sqrt{3}}{3}} = \sqrt{\frac{21+12\sqrt{3}}{9}} = \sqrt{\frac{9+2\times3\times2\sqrt{3}+12}{9}}$$

$$= \sqrt{\frac{3^2+2\times3\times2\sqrt{3}+(2\sqrt{3})^2}{9}} = \sqrt{\frac{(3+2\sqrt{3})^2}{9}}$$

$$= \frac{3+2\sqrt{3}}{3} = 1+\frac{2}{3}\sqrt{3}$$

// AS REQUIRED

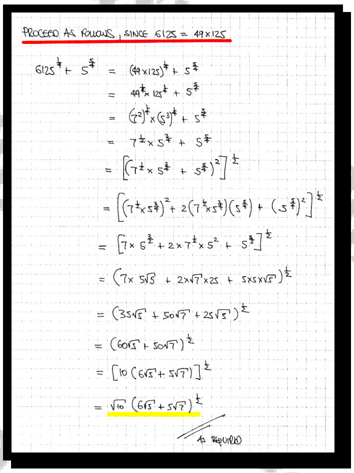
Question 80 **(*****)**

Show that

$$6125^{\frac{1}{4}} + 5^{\frac{5}{4}}$$

can be written in the form $\sqrt{10}\left(a\sqrt{5}+b\sqrt{7}\right)^{\frac{1}{2}}$, where a and b are positive integers to be found.

$\boxed{\sqrt{10}\left(6\sqrt{5}+5\sqrt{7}\right)^{\frac{1}{2}}}$



PROCEED AS FOLLOWS, SINCE 6125 = 49×125

$$6125^{\frac{1}{4}} + 5^{\frac{5}{4}} = (49\times125)^{\frac{1}{4}} + 5^{\frac{5}{4}}$$

$$= 49^{\frac{1}{4}}\times125^{\frac{1}{4}} + 5^{\frac{5}{4}}$$

$$= (7^2)^{\frac{1}{4}}\times(5^3)^{\frac{1}{4}} + 5^{\frac{5}{4}}$$

$$= 7^{\frac{1}{2}}\times5^{\frac{3}{4}} + 5^{\frac{5}{4}}$$

$$= \left[\left(7^{\frac{1}{2}}\times5^{\frac{3}{4}} + 5^{\frac{5}{4}}\right)^2\right]^{\frac{1}{2}}$$

$$= \left[\left(7^{\frac{1}{2}}\times5^{\frac{3}{4}}\right)^2 + 2\left(7^{\frac{1}{2}}\times5^{\frac{3}{4}}\right)\left(5^{\frac{5}{4}}\right) + \left(5^{\frac{5}{4}}\right)^2\right]^{\frac{1}{2}}$$

$$= \left[7\times5^{\frac{3}{2}} + 2\times7^{\frac{1}{2}}\times5^2 + 5^{\frac{5}{2}}\right]^{\frac{1}{2}}$$

$$= \left(7\times5\sqrt{5} + 2\times\sqrt{7}\times25 + 5\times5\times\sqrt{5}\right)^{\frac{1}{2}}$$

$$= \left(35\sqrt{5} + 50\sqrt{7} + 25\sqrt{5}\right)^{\frac{1}{2}}$$

$$= \left(60\sqrt{5} + 50\sqrt{7}\right)^{\frac{1}{2}}$$

$$= \left[10\left(6\sqrt{5}+5\sqrt{7}\right)\right]^{\frac{1}{2}}$$

$$= \sqrt{10}\left(6\sqrt{5}+5\sqrt{7}\right)^{\frac{1}{2}}$$

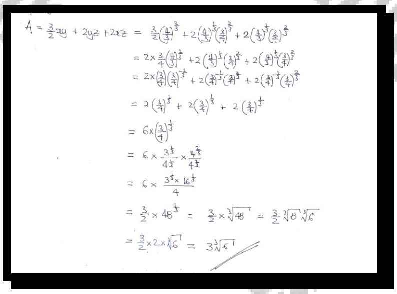
// AS REQUIRED

Question 81 (*****)

$$A=\frac{3}{2}xy+2yz+2xz\,.$$

Given that $x=\left(\frac{4}{3}\right)^{\frac{1}{3}}$, $y=\left(\frac{4}{3}\right)^{\frac{1}{3}}$ and $z=\left(\frac{3}{4}\right)^{\frac{2}{3}}$, show clearly that $A=3\sqrt[3]{6}$

proof



$A=\frac{3}{2}xy+2yz+2xz=\frac{3}{2}\left(\frac{4}{3}\right)^{\frac{2}{3}}+2\left(\frac{4}{3}\right)^{\frac{1}{3}}\left(\frac{3}{4}\right)^{\frac{2}{3}}+2\left(\frac{4}{3}\right)^{\frac{1}{3}}\left(\frac{3}{4}\right)^{\frac{2}{3}}$

$=2\times\frac{3}{4}\left(\frac{4}{3}\right)^{\frac{2}{3}}+2\left(\frac{4}{3}\right)^{\frac{1}{3}}\left(\frac{3}{4}\right)^{\frac{2}{3}}+2\left(\frac{4}{3}\right)^{\frac{1}{3}}\left(\frac{3}{4}\right)^{\frac{2}{3}}$

$=2\times\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)^{-\frac{2}{3}}+2\left(\frac{3}{4}\right)^{-\frac{1}{3}}\left(\frac{3}{4}\right)^{\frac{2}{3}}+2\left(\frac{3}{4}\right)^{-\frac{1}{3}}\left(\frac{3}{4}\right)^{\frac{2}{3}}$

$=2\left(\frac{3}{4}\right)^{\frac{1}{3}}+2\left(\frac{3}{4}\right)^{\frac{1}{3}}+2\left(\frac{3}{4}\right)^{\frac{1}{3}}$

$=6\times\left(\frac{3}{4}\right)^{\frac{1}{3}}$

$=6\times\frac{3^{\frac{1}{3}}}{4^{\frac{1}{3}}}\times\frac{4^{\frac{2}{3}}}{4^{\frac{2}{3}}}$

$=6\times\frac{3^{\frac{1}{3}}\times16^{\frac{1}{3}}}{4}$

$=\frac{3}{2}\times48^{\frac{1}{3}}=\frac{3}{2}\times\sqrt[3]{48}=\frac{3}{2}\sqrt[3]{8}\sqrt[3]{6}$

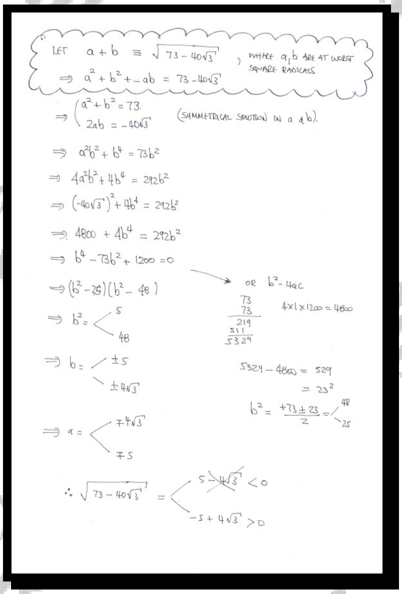
$=\frac{3}{2}\times2\times\sqrt[3]{6}=3\sqrt[3]{6}$

Question 82 (*****)

Find an exact simplified value for

$$\sqrt{73-40\sqrt{3}}\,.$$

$\boxed{-5+4\sqrt{3}}$



LET $a+b\equiv\sqrt{73-40\sqrt{3}}$, WHERE a, b ARE AT WORST SQUARE RADICALS

$\Rightarrow a^2+b^2+2ab=73-40\sqrt{3}$

$\Rightarrow\begin{cases}a^2+b^2=73\\2ab=-40\sqrt{3}\end{cases}$ (SYMMETRICAL SOLUTION IN a & b).

$\Rightarrow a^2b^2+b^4=73b^2$

$\Rightarrow 4a^2b^2+4b^4=292b^2$

$\Rightarrow (-40\sqrt{3})^2+4b^4=292b^2$

$\Rightarrow 4800+4b^4=292b^2$

$\Rightarrow b^4-73b^2+1200=0$

$\Rightarrow (b^2-25)(b^2-48)$

$\Rightarrow b^2=\begin{cases}5\\48\end{cases}$

$\Rightarrow b=\begin{cases}\pm5\\\pm4\sqrt{3}\end{cases}$

$\Rightarrow a=\begin{cases}\mp4\sqrt{3}\\\mp5\end{cases}$

$\therefore\sqrt{73-40\sqrt{3}}=\begin{cases}~~5-4\sqrt{3}~~<0\\-5+4\sqrt{3}>0\end{cases}$

OR b^2-4ac

$$\begin{array}{r}73\\73\\\hline219\\511\\\hline5329\end{array}$$

$4\times1\times1200=4800$

$5329-4800=529$

$=23^2$

$b^2=\frac{+73\pm23}{2}=\begin{cases}48\\25\end{cases}$

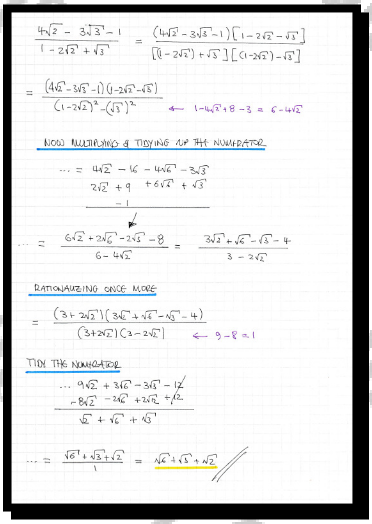
Question 83 (*****)

Rationalize the denominator of the following surd.

$$\frac{4}{\sqrt{3}+\sqrt{2}+1} = 2+\sqrt{2}-\sqrt{6}\,.$$

Show detailed workings in this question.

$\boxed{}$, $\boxed{\text{proof}}$



$$\frac{4\sqrt{2}-3\sqrt{3}-1}{1-2\sqrt{2}+\sqrt{3}} = \frac{(4\sqrt{2}-3\sqrt{3}-1)[1-2\sqrt{2}-\sqrt{3}]}{[(1-2\sqrt{2})+\sqrt{3}][(1-2\sqrt{2})-\sqrt{3}]}$$

$$= \frac{(4\sqrt{2}-3\sqrt{3}-1)(1-2\sqrt{2}-\sqrt{3})}{(1-2\sqrt{2})^2-(\sqrt{3})^2} \quad \leftarrow 1-4\sqrt{2}+8-3 = 6-4\sqrt{2}$$

NOW MULTIPLYING & TIDYING UP THE NUMERATOR

$$\dots = \begin{array}{l} 4\sqrt{2}-16-4\sqrt{6}-3\sqrt{3} \\ 2\sqrt{2}+9+6\sqrt{6}+\sqrt{3} \\ -1 \\ \hline \end{array}$$

$$\dots = \frac{6\sqrt{2}+2\sqrt{6}-2\sqrt{3}-8}{6-4\sqrt{2}} = \frac{3\sqrt{2}+\sqrt{6}-\sqrt{3}-4}{3-2\sqrt{2}}$$

RATIONALIZING ONCE MORE

$$= \frac{(3+2\sqrt{2})(3\sqrt{2}+\sqrt{6}-\sqrt{3}-4)}{(3+2\sqrt{2})(3-2\sqrt{2})} \quad \leftarrow 9-8=1$$

TIDY THE NUMERATOR

$$\dots \begin{array}{l} 9\sqrt{2}+3\sqrt{6}-3\sqrt{3}-\cancel{12} \\ -8\sqrt{2}-2\sqrt{6}+2\sqrt{12}+\cancel{12} \\ \hline \sqrt{2}+\sqrt{6}+\sqrt{3} \end{array}$$

$$\dots = \frac{\sqrt{6}+\sqrt{3}+\sqrt{2}}{1} = \underline{\sqrt{6}+\sqrt{3}+\sqrt{2}} \;//$$

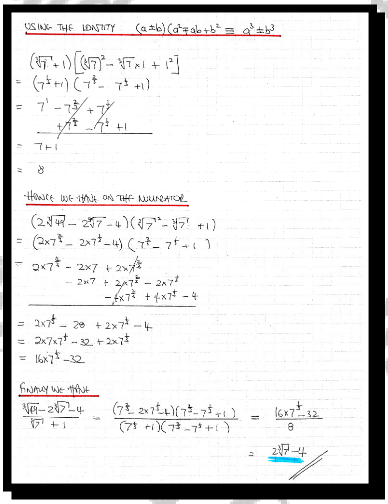
Question 84 (*****)

Show that

$$\frac{\sqrt[3]{49}-2\sqrt[3]{7}-4}{\sqrt[3]{7}+1}$$

can be written in the form $a\sqrt[3]{7}+b$, where a and b are integers to be found.

$\boxed{}$, $\boxed{2\sqrt[3]{7}-4}$



USING THE IDENTITY $(a\pm b)(a^2\mp ab+b^2 \equiv a^3\pm b^3$

$(\sqrt[3]{7}+1)\left[(\sqrt[3]{7})^2-\sqrt[3]{7}\times 1+1^2\right]$

$=(7^{\frac{1}{3}}+1)(7^{\frac{2}{3}}-7^{\frac{1}{3}}+1)$

$=7^1-\cancel{7^{\frac{2}{3}}}+\cancel{7^{\frac{1}{3}}}$

$+\cancel{7^{\frac{2}{3}}}-\cancel{7^{\frac{1}{3}}}+1$

$=7+1$

$=8$

HENCE WE HAVE ON THE NUMERATOR

$(2\sqrt[3]{49}-2\sqrt[3]{7}-4)(\sqrt[3]{7}^2-\sqrt[3]{7}+1)$

$=(2\times7^{\frac{2}{3}}-2\times7^{\frac{1}{3}}-4)(7^{\frac{2}{3}}-7^{\frac{1}{3}}+1)$

$=2\times7^{\frac{4}{3}}-2\times7+\cancel{2\times7^{\frac{2}{3}}}$

$-2\times7+\cancel{2\times7^{\frac{2}{3}}}-2\times7^{\frac{1}{3}}$

$-\cancel{4\times7^{\frac{2}{3}}}+4\times7^{\frac{1}{3}}-4$

$=2\times7^{\frac{4}{3}}-28+2\times7^{\frac{1}{3}}-4$

$=2\times7\times7^{\frac{1}{3}}-32+2\times7^{\frac{1}{3}}$

$=16\times7^{\frac{1}{3}}-32$

FINALLY WE HAVE

$$\frac{\sqrt[3]{49}-2\sqrt[3]{7}-4}{\sqrt[3]{7}+1}=\frac{(7^{\frac{2}{3}}-2\times7^{\frac{1}{3}}-4)(7^{\frac{2}{3}}-7^{\frac{1}{3}}+1)}{(7^{\frac{1}{3}}+1)(7^{\frac{2}{3}}-7^{\frac{1}{3}}+1)}=\frac{16\times7^{\frac{1}{3}}-32}{8}$$

$=2\sqrt[3]{7}-4$

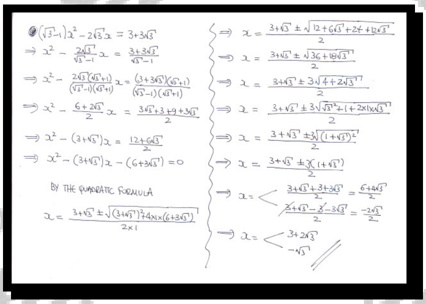
Question 85 **(*****)**

Solve the following quadratic equation

$$\left(\sqrt{3}-1\right)x^2-2\sqrt{3}x=3+3\sqrt{3}.$$

Give one of the roots in the form $p+q\sqrt{3}$ and the other root in the form $r\sqrt{3}$, where p, q and r are integers.

$$\boxed{x=-\sqrt{3},\quad x=3+2\sqrt{3}}$$



$$\left(\sqrt{3}-1\right)x^2-2\sqrt{3}x=3+3\sqrt{3}$$

$$\Rightarrow x^2-\frac{2\sqrt{3}}{\sqrt{3}-1}x=\frac{3+3\sqrt{3}}{\sqrt{3}-1}$$

$$\Rightarrow x^2-\frac{2\sqrt{3}(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}x=\frac{(3+3\sqrt{3})(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}$$

$$\Rightarrow x^2-\frac{6+2\sqrt{3}}{2}x=\frac{3\sqrt{3}+3+9+3\sqrt{3}}{2}$$

$$\Rightarrow x^2-(3+\sqrt{3})x=\frac{12+6\sqrt{3}}{2}$$

$$\Rightarrow x^2-(3+\sqrt{3})x-(6+3\sqrt{3})=0$$

BY THE QUADRATIC FORMULA

$$x=\frac{3+\sqrt{3}\pm\sqrt{(3+\sqrt{3})^2+4\times1\times(6+3\sqrt{3})}}{2\times1}$$

$$\Rightarrow x=\frac{3+\sqrt{3}\pm\sqrt{12+6\sqrt{3}+24+12\sqrt{3}}}{2}$$

$$\Rightarrow x=\frac{3+\sqrt{3}\pm\sqrt{36+18\sqrt{3}}}{2}$$

$$\Rightarrow x=\frac{3+\sqrt{3}\pm3\sqrt{4+2\sqrt{3}}}{2}$$

$$\Rightarrow x=\frac{3+\sqrt{3}\pm3\sqrt{\sqrt{3}^2+1+2\times1\times\sqrt{3}}}{2}$$

$$\Rightarrow x=\frac{3+\sqrt{3}\pm3\sqrt{(1+\sqrt{3})^2}}{2}$$

$$\Rightarrow x=\frac{3+\sqrt{3}\pm3(1+\sqrt{3})}{2}$$

$$\Rightarrow x=\begin{cases}\frac{3+\sqrt{3}+3+3\sqrt{3}}{2}=\frac{6+4\sqrt{3}}{2}\\ \frac{\cancel{3}+\sqrt{3}-\cancel{3}-3\sqrt{3}}{2}=\frac{-2\sqrt{3}}{2}\end{cases}$$

$$\Rightarrow x=\begin{cases}3+2\sqrt{3}\\ -\sqrt{3}\end{cases}$$

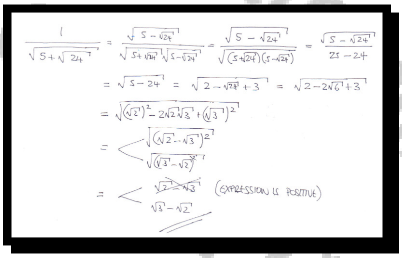
Question 86 **(*****)**

Express

$$\frac{1}{\sqrt{5+\sqrt{24}}},$$

in the form $\sqrt{p}-\sqrt{q}$, where p and q are integers.

$$\boxed{\sqrt{3}-\sqrt{2}}$$



$$\frac{1}{\sqrt{5+\sqrt{24}}}=\frac{\sqrt{5-\sqrt{24}}}{\sqrt{5+\sqrt{24}}\sqrt{5-\sqrt{24}}}=\frac{\sqrt{5-\sqrt{24}}}{\sqrt{(5+\sqrt{24})(5-\sqrt{24})}}=\frac{\sqrt{5-\sqrt{24}}}{25-24}$$

$$=\sqrt{5-24}=\sqrt{2-\sqrt{24}+3}=\sqrt{2-2\sqrt{6}+3}$$

$$=\sqrt{(\sqrt{2})^2-2\sqrt{2}\sqrt{3}+(\sqrt{3})^2}$$

$$=\begin{cases}\sqrt{(\sqrt{2}-\sqrt{3})^2}\\ \sqrt{(\sqrt{3}-\sqrt{2})^2}\end{cases}$$

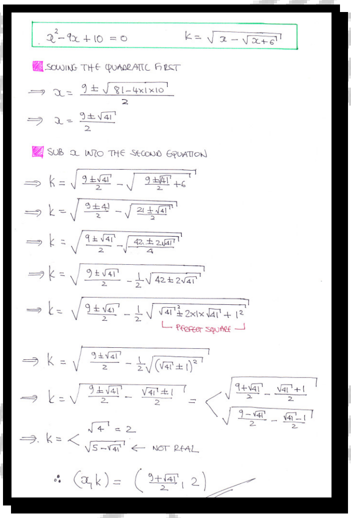
$$=\begin{cases}\cancel{\sqrt{2}-\sqrt{3}}\quad \text{(EXPRESSION IS POSITIVE)}\\ \sqrt{3}-\sqrt{2}\end{cases}$$

Question 87 (*****)

Solve the following simultaneous equations, to find in exact form where appropriate, the value or values of x and k.

$$x^2-9x+10=0 \quad \text{and} \quad k=\sqrt{x-\sqrt{x+6}}$$

$\boxed{\quad}$, $\boxed{x=\frac{1}{2}\left(9+\sqrt{41}\right), \quad k=2}$



$\boxed{x^2-9x+10=0 \qquad k=\sqrt{x-\sqrt{x+6}}}$

▨ SOLVING THE QUADRATIC FIRST

$\Rightarrow x=\frac{9\pm\sqrt{81-4\times1\times10}}{2}$

$\Rightarrow x=\frac{9\pm\sqrt{41}}{2}$

▨ SUB x INTO THE SECOND EQUATION

$\Rightarrow k=\sqrt{\frac{9\pm\sqrt{41}}{2}-\sqrt{\frac{9\pm\sqrt{41}}{2}+6}}$

$\Rightarrow k=\sqrt{\frac{9\pm41}{2}-\sqrt{\frac{21\pm\sqrt{41}}{2}}}$

$\Rightarrow k=\sqrt{\frac{9\pm\sqrt{41}}{2}-\sqrt{\frac{42\pm2\sqrt{41}}{4}}}$

$\Rightarrow k=\sqrt{\frac{9\pm\sqrt{41}}{2}-\frac{1}{2}\sqrt{42\pm2\sqrt{41}}}$

$\Rightarrow k=\sqrt{\frac{9\pm\sqrt{41}}{2}-\frac{1}{2}\sqrt{\underbrace{\sqrt{41}^2\pm2\times1\times\sqrt{41}+1^2}_{\text{PERFECT SQUARE}}}}$

$\Rightarrow k=\sqrt{\frac{9\pm\sqrt{41}}{2}-\frac{1}{2}\sqrt{\left(\sqrt{41}\pm1\right)^2}}$

$\Rightarrow k=\sqrt{\frac{9\pm\sqrt{41}}{2}-\frac{\sqrt{41}\pm1}{2}}=\left\langle\begin{array}{l}\sqrt{\frac{9+\sqrt{41}}{2}-\frac{\sqrt{41}+1}{2}}\\ \sqrt{\frac{9-\sqrt{41}}{2}-\frac{\sqrt{41}-1}{2}}\end{array}\right.$

$\Rightarrow k=\left\langle\begin{array}{l}\sqrt{4}=2\\ \sqrt{5-\sqrt{41}} \leftarrow \text{NOT REAL}\end{array}\right.$

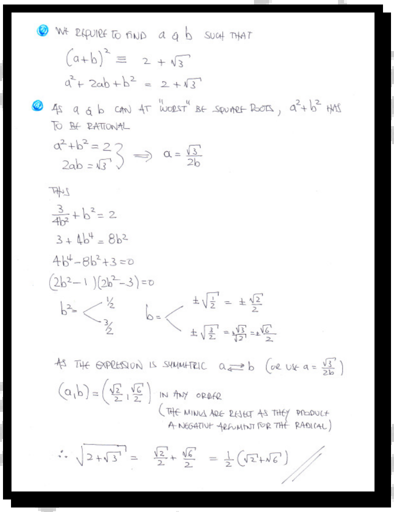
$\therefore (x,k)=\left(\frac{9+\sqrt{41}}{2},2\right)$

Question 88 **(*****)**

Find in exact simplified form

$$\sqrt{2+\sqrt{3}}\ .$$

$$\boxed{\frac{1}{2}\left(\sqrt{2}+\sqrt{6}\right)}$$



- WE REQUIRE TO FIND a & b SUCH THAT

$(a+b)^2 \equiv 2+\sqrt{3}$

$a^2+2ab+b^2 = 2+\sqrt{3}$

- AS a & b CAN AT "WORST" BE SQUARE ROOTS, a^2+b^2 HAS TO BE RATIONAL

$\left.\begin{array}{r} a^2+b^2=2 \\ 2ab=\sqrt{3} \end{array}\right\} \Longrightarrow a=\frac{\sqrt{3}}{2b}$

THUS

$\frac{3}{4b^2}+b^2=2$

$3+4b^4=8b^2$

$4b^4-8b^2+3=0$

$(2b^2-1)(2b^2-3)=0$

$b^2 = \begin{cases} \frac{1}{2} \\ \frac{3}{2} \end{cases} \qquad b = \begin{cases} \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2} \\ \pm\sqrt{\frac{3}{2}} = \pm\frac{\sqrt{3}}{\sqrt{2}} = \pm\frac{\sqrt{6}}{2} \end{cases}$

AS THE EXPRESSION IS SYMMETRIC $a \rightleftarrows b$ (OR USE $a=\frac{\sqrt{3}}{2b}$)

$(a,b)=\left(\frac{\sqrt{2}}{2},\frac{\sqrt{6}}{2}\right)$ IN ANY ORDER

(THE MINUS ARE REJECT AS THEY PRODUCE A NEGATIVE ARGUMENT FOR THE RADICAL)

$\therefore \sqrt{2+\sqrt{3}} = \frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{2} = \frac{1}{2}\left(\sqrt{2}+\sqrt{6}\right)$ //

Question 89 (*****)

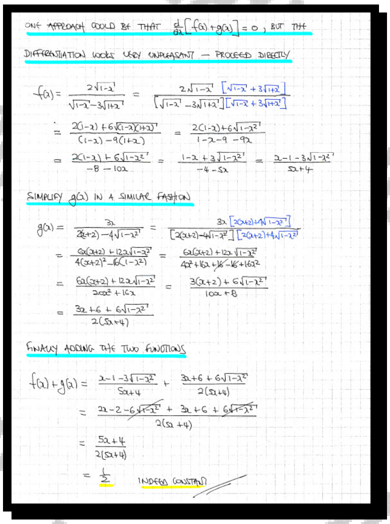
The functions f and f are defined as

$$f(x) \equiv \frac{2\sqrt{1-x}}{\sqrt{1-x}-3\sqrt{1+x}}, \quad -1 \le x \le 1.$$

$$g(x) \equiv \frac{3x}{2(x+2)-4\sqrt{1-x^2}}, \quad -1 \le x \le 1.$$

Show that $f(x)+g(x)$ is a **constant** function.

$\boxed{}$, $\boxed{\text{proof}}$



ONE APPROACH COULD BE THAT $\frac{d}{dx}[f(x)+g(x)]=0$, BUT THE DIFFERENTIATION LOOKS VERY UNPLEASANT — PROCEED DIRECTLY

$$f(x) = \frac{2\sqrt{1-x}}{\sqrt{1-x}-3\sqrt{1+x}} = \frac{2\sqrt{1-x}\,[\sqrt{1-x}+3\sqrt{1+x}]}{[\sqrt{1-x}-3\sqrt{1+x}][\sqrt{1-x}+3\sqrt{1+x}]}$$

$$= \frac{2(1-x)+6\sqrt{(1-x)(1+x)}}{(1-x)-9(1+x)} = \frac{2(1-x)+6\sqrt{1-x^2}}{1-x-9-9x}$$

$$= \frac{2(1-x)+6\sqrt{1-x^2}}{-8-10x} = \frac{1-x+3\sqrt{1-x^2}}{-4-5x} = \frac{x-1-3\sqrt{1-x^2}}{5x+4}$$

SIMPLIFY $g(x)$ IN A SIMILAR FASHION

$$g(x) = \frac{3x}{2(x+2)-4\sqrt{1-x^2}} = \frac{3x\,[2(x+2)+4\sqrt{1-x^2}]}{[2(x+2)-4\sqrt{1-x^2}][2(x+2)+4\sqrt{1-x^2}]}$$

$$= \frac{6x(x+2)+12x\sqrt{1-x^2}}{4(x+2)^2-16(1-x^2)} = \frac{6x(x+2)+12x\sqrt{1-x^2}}{4x^2+16x+\cancel{16}-\cancel{16}+16x^2}$$

$$= \frac{6x(x+2)+12x\sqrt{1-x^2}}{20x^2+16x} = \frac{3(x+2)+6\sqrt{1-x^2}}{10x+8}$$

$$= \frac{3x+6+6\sqrt{1-x^2}}{2(5x+4)}$$

FINALLY ADDING THE TWO FUNCTIONS

$$f(x)+g(x) = \frac{x-1-3\sqrt{1-x^2}}{5x+4} + \frac{3x+6+6\sqrt{1-x^2}}{2(5x+4)}$$

$$= \frac{2x-2-\cancel{6\sqrt{1-x^2}}+3x+6+\cancel{6\sqrt{1-x^2}}}{2(5x+4)}$$

$$= \frac{5x+4}{2(5x+4)}$$

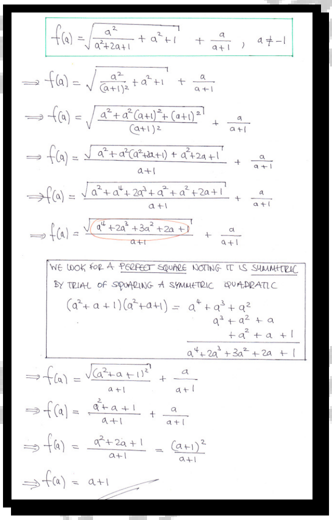
$$= \frac{1}{2}$$ INDEED CONSTANT

Question 90 (*****)

$$f(a) \equiv \frac{a}{a+1} + \sqrt{1 + a^2 + \frac{a^2}{a^2 + 2a + 1}}, \ a \in \mathbb{R}, \ a \neq -1.$$

Show that $f(a)$ can be simplified to a linear polynomial in a.

$\boxed{}$, $\boxed{f(a) = a + 1}$



$$f(a) = \sqrt{\frac{a^2}{a^2+2a+1} + a^2 + 1} + \frac{a}{a+1}, \quad a \neq -1$$

$$\Rightarrow f(a) = \sqrt{\frac{a^2}{(a+1)^2} + a^2 + 1} + \frac{a}{a+1}$$

$$\Rightarrow f(a) = \sqrt{\frac{a^2 + a^2(a+1)^2 + (a+1)^2}{(a+1)^2}} + \frac{a}{a+1}$$

$$\Rightarrow f(a) = \frac{\sqrt{a^2 + a^2(a^2+2a+1) + a^2+2a+1}}{a+1} + \frac{a}{a+1}$$

$$\Rightarrow f(a) = \frac{\sqrt{a^2 + a^4 + 2a^3 + a^2 + a^2 + 2a + 1}}{a+1} + \frac{a}{a+1}$$

$$\Rightarrow f(a) = \frac{\sqrt{a^4 + 2a^3 + 3a^2 + 2a + 1}}{a+1} + \frac{a}{a+1}$$

WE LOOK FOR A PERFECT SQUARE NOTING IT IS SYMMETRIC
BY TRIAL OF SQUARING A SYMMETRIC QUADRATIC

$$\begin{aligned}(a^2 + a + 1)(a^2 + a + 1) = a^4 + a^3 + a^2 & \\ a^3 + a^2 + a & \\ + a^2 + a + 1 & \\ \hline a^4 + 2a^3 + 3a^2 + 2a + 1 & \end{aligned}$$

$$\Rightarrow f(a) = \frac{\sqrt{(a^2+a+1)^2}}{a+1} + \frac{a}{a+1}$$

$$\Rightarrow f(a) = \frac{a^2+a+1}{a+1} + \frac{a}{a+1}$$

$$\Rightarrow f(a) = \frac{a^2+2a+1}{a+1} = \frac{(a+1)^2}{a+1}$$

$$\Rightarrow f(a) = a+1$$

Question 91 **(*****)**

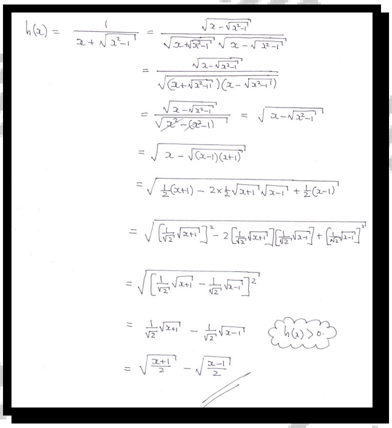
$$h(x) \equiv \frac{1}{\sqrt{x+\sqrt{x^2-1}}}, \ x \in \mathbb{R}, \ x \geq 1.$$

Show that $h(x)$ can be expressed in the form

$$\sqrt{f(x)} - \sqrt{g(x)},$$

where $f(x)$ and $g(x)$ are linear functions to be found.

$\boxed{}$, $\boxed{h(x) = \sqrt{\frac{x+1}{2}} - \sqrt{\frac{x-1}{2}}}$



$$h(x) = \frac{1}{x+\sqrt{x^2-1}} = \frac{\sqrt{x-\sqrt{x^2-1}}}{\sqrt{x+\sqrt{x^2-1}}\sqrt{x-\sqrt{x^2-1}}}$$

$$= \frac{\sqrt{x-\sqrt{x^2-1}}}{\sqrt{(x+\sqrt{x^2-1})(x-\sqrt{x^2-1})}}$$

$$= \frac{\sqrt{x-\sqrt{x^2-1}}}{\sqrt{\cancel{x^2}-(\cancel{x^2}-1)}} = \sqrt{x-\sqrt{x^2-1}}$$

$$= \sqrt{x-\sqrt{(x-1)(x+1)}}$$

$$= \sqrt{\tfrac{1}{2}(x+1) - 2\times\tfrac{1}{2}\sqrt{x+1}\sqrt{x-1} + \tfrac{1}{2}(x-1)}$$

$$= \sqrt{\left[\tfrac{1}{\sqrt{2}}\sqrt{x+1}\right]^2 - 2\left[\tfrac{1}{\sqrt{2}}\sqrt{x+1}\right]\left[\tfrac{1}{\sqrt{2}}\sqrt{x-1}\right] + \left[\tfrac{1}{\sqrt{2}}\sqrt{x-1}\right]^2}$$

$$= \sqrt{\left[\tfrac{1}{\sqrt{2}}\sqrt{x+1} - \tfrac{1}{\sqrt{2}}\sqrt{x-1}\right]^2}$$

$$= \tfrac{1}{\sqrt{2}}\sqrt{x+1} - \tfrac{1}{\sqrt{2}}\sqrt{x-1} \qquad h(x)>0.$$

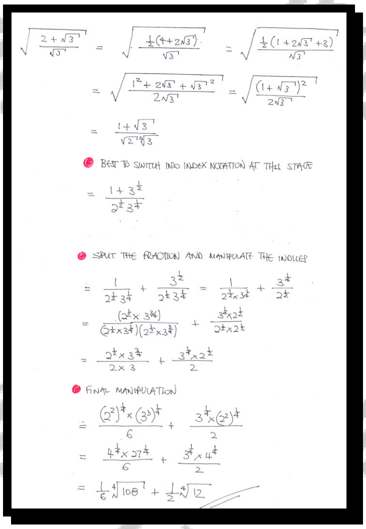
$$= \sqrt{\frac{x+1}{2}} - \sqrt{\frac{x-1}{2}}$$

Question 92 (*****)

Using a detailed method show that

$$\sqrt{\frac{2+\sqrt{3}}{\sqrt{3}}} = \frac{\sqrt[4]{108}}{6} + \frac{\sqrt[4]{12}}{2}.$$

$\boxed{\text{S-S}}$, $\boxed{\text{proof}}$



$$\sqrt{\frac{2+\sqrt{3}}{\sqrt{3}}} = \sqrt{\frac{\frac{1}{2}(4+2\sqrt{3})}{\sqrt{3}}} = \sqrt{\frac{\frac{1}{2}(1+2\sqrt{3}+3)}{\sqrt{3}}}$$

$$= \sqrt{\frac{1^2+2\sqrt{3}+\sqrt{3}^2}{2\sqrt{3}}} = \sqrt{\frac{(1+\sqrt{3})^2}{2\sqrt{3}}}$$

$$= \frac{1+\sqrt{3}}{\sqrt{2}\sqrt[4]{3}}$$

- BEST TO SWITCH INTO INDEX NOTATION AT THIS STAGE

$$= \frac{1+3^{\frac{1}{2}}}{2^{\frac{1}{2}}3^{\frac{1}{4}}}$$

- SPLIT THE FRACTION AND MANIPULATE THE INDICES

$$= \frac{1}{2^{\frac{1}{2}}3^{\frac{1}{4}}} + \frac{3^{\frac{1}{2}}}{2^{\frac{1}{2}}3^{\frac{1}{4}}} = \frac{1}{2^{\frac{1}{2}}\times 3^{\frac{1}{4}}} + \frac{3^{\frac{1}{4}}}{2^{\frac{1}{2}}}$$

$$= \frac{(2^{\frac{1}{2}}\times 3^{\frac{3}{4}})}{(2^{\frac{1}{2}}\times 3^{\frac{1}{4}})(2^{\frac{1}{2}}\times 3^{\frac{3}{4}})} + \frac{3^{\frac{1}{4}}\times 2^{\frac{1}{2}}}{2^{\frac{1}{2}}\times 2^{\frac{1}{2}}}$$

$$= \frac{2^{\frac{1}{2}}\times 3^{\frac{3}{4}}}{2\times 3} + \frac{3^{\frac{1}{4}}\times 2^{\frac{1}{2}}}{2}$$

- FINAL MANIPULATION

$$= \frac{(2^2)^{\frac{1}{4}}\times(3^3)^{\frac{1}{4}}}{6} + \frac{3^{\frac{1}{4}}\times(2^2)^{\frac{1}{4}}}{2}$$

$$= \frac{4^{\frac{1}{4}}\times 27^{\frac{1}{4}}}{6} + \frac{3^{\frac{1}{4}}\times 4^{\frac{1}{4}}}{2}$$

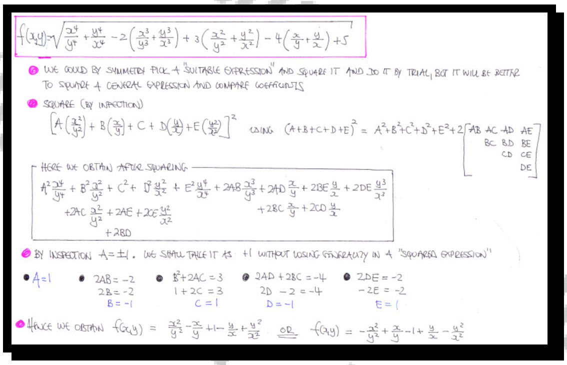
$$= \frac{1}{6}\sqrt[4]{108} + \frac{1}{2}\sqrt[4]{12}$$

Question 93 (*****)

$$f(x,y) \equiv \sqrt{\frac{x^4}{y^4}+\frac{y^4}{x^4}-2\left(\frac{x^3}{y^3}+\frac{y^3}{x^3}\right)+3\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}\right)-4\left(\frac{x}{y}+\frac{y}{x}\right)+5} .$$

Simplify $f(x,y)$ in a form not involving square roots.

$\boxed{}$, $\boxed{f(x,y) \equiv \pm\left(\frac{x^2}{y^2}-\frac{x}{y}+1-\frac{y}{x}+\frac{y^2}{x^2}\right)}$



$\boxed{f(x,y) = \sqrt{\frac{x^4}{y^4}+\frac{y^4}{x^4}-2\left(\frac{x^3}{y^3}+\frac{y^3}{x^3}\right)+3\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}\right)-4\left(\frac{x}{y}+\frac{y}{x}\right)+5}}$

- WE COULD BY SYMMETRY PICK A "SUITABLE EXPRESSION" AND SQUARE IT AND DO IT BY TRIAL, BUT IT WILL BE BETTER TO SQUARE A GENERAL EXPRESSION AND COMPARE COEFFICIENTS
- SQUARE (BY INSPECTION)

$\left[A\left(\frac{x^2}{y^2}\right)+B\left(\frac{x}{y}\right)+C+D\left(\frac{y}{x}\right)+E\left(\frac{y^2}{x^2}\right)\right]^2$ USING $(A+B+C+D+E)^2 = A^2+B^2+C^2+D^2+E^2+2\begin{bmatrix} AB & AC & AD & AE \\ & BC & BD & BE \\ & & CD & CE \\ & & & DE \end{bmatrix}$

HERE WE OBTAIN AFTER SQUARING

$$\boxed{\begin{aligned} &A^2\frac{x^4}{y^4}+B^2\frac{x^2}{y^2}+C^2+D^2\frac{y^2}{x^2}+E^2\frac{y^4}{x^4}+2AB\frac{x^3}{y^3}+2AD\frac{x}{y}+2BE\frac{y}{x}+2DE\frac{y^3}{x^3}\\ &\quad +2AC\frac{x^2}{y^2}+2AE+2CE\frac{y^2}{x^2} \qquad +2BC\frac{x}{y}+2CD\frac{y}{x}\\ &\qquad +2BD \end{aligned}}$$

- BY INSPECTION $A=\pm 1$. WE SHALL TAKE IT AS $+1$ WITHOUT LOSING GENERALITY IN A "SQUARED EXPRESSION"

- $A=1$
- $2AB=-2$
  $2B=-2$
  $B=-1$
- $B^2+2AC=3$
  $1+2C=3$
  $C=1$
- $2AD+2BC=-4$
  $2D-2=-4$
  $D=-1$
- $2DE=-2$
  $-2E=-2$
  $E=1$

- HENCE WE OBTAIN $f(x,y) = \frac{x^2}{y^2}-\frac{x}{y}+1-\frac{y}{x}+\frac{y^2}{x^2}$ OR $f(x,y) = -\frac{x^2}{y^2}+\frac{x}{y}-1+\frac{y}{x}-\frac{y^2}{x^2}$

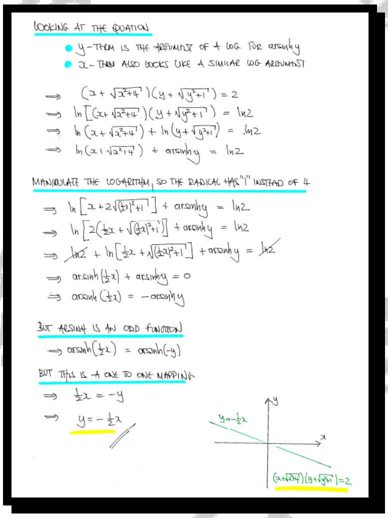
Question 94 (*****)

Sketch the graph of

$$\left[x+\sqrt{x^2+4}\right]\left[y+\sqrt{y^2+1}\right]=2\,,\quad x\in(-\infty,\infty)\,,\ \ y\in(-\infty,\infty)$$

You must show a detailed method in this question

$\boxed{}$, $\boxed{\text{proof}}$



LOOKING AT THE EQUATION

- y–TERM IS THE ARGUMENT OF A LOG FOR arsinh y
- x–TERM ALSO LOOKS LIKE A SIMILAR LOG ARGUMENT

$\Rightarrow \left(x+\sqrt{x^2+4}\right)\left(y+\sqrt{y^2+1}\right)=2$

$\Rightarrow \ln\left[\left(x+\sqrt{x^2+4}\right)\left(y+\sqrt{y^2+1}\right)\right)=\ln 2$

$\Rightarrow \ln\left(x+\sqrt{x^2+4}\right)+\ln\left(y+\sqrt{y^2+1}\right)=\ln 2$

$\Rightarrow \ln\left(x+\sqrt{x^2+4}\right)+\operatorname{arsinh}y=\ln 2$

MANIPULATE THE LOGARITHM, SO THE RADICAL HAS "1" INSTEAD OF 4

$\Rightarrow \ln\left[x+2\sqrt{(\tfrac{1}{2}x)^2+1}\right]+\operatorname{arsinh}y=\ln 2$

$\Rightarrow \ln\left[2\left(\tfrac{1}{2}x+\sqrt{(\tfrac{1}{2}x)^2+1}\right)\right]+\operatorname{arsinh}y=\ln 2$

$\Rightarrow \cancel{\ln 2}+\ln\left[\tfrac{1}{2}x+\sqrt{(\tfrac{1}{2}x)^2+1}\right]+\operatorname{arsinh}y=\cancel{\ln 2}$

$\Rightarrow \operatorname{arsinh}(\tfrac{1}{2}x)+\operatorname{arsinh}y=0$

$\Rightarrow \operatorname{arsinh}(\tfrac{1}{2}x)=-\operatorname{arsinh}y$

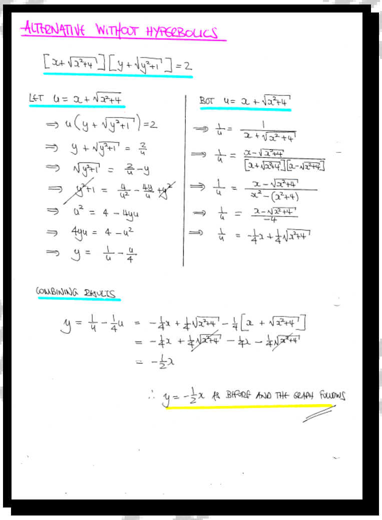
BUT ARSINH IS AN ODD FUNCTION

$\Rightarrow \operatorname{arsinh}(\tfrac{1}{2}x)=\operatorname{arsinh}(-y)$

BUT THIS IS A ONE TO ONE MAPPING

$\Rightarrow \tfrac{1}{2}x=-y$

$\Rightarrow y=-\tfrac{1}{2}x$ //



ALTERNATIVE WITHOUT HYPERBOLICS

$\left[x+\sqrt{x^2+4}\right]\left[y+\sqrt{y^2+1}\right]=2$

LET $u=x+\sqrt{x^2+4}$

$\Rightarrow u\left(y+\sqrt{y^2+1}\right)=2$

$\Rightarrow y+\sqrt{y^2+1}=\frac{2}{u}$

$\Rightarrow \sqrt{y^2+1}=\frac{2}{u}-y$

$\Rightarrow \cancel{y^2}+1=\frac{4}{u^2}-\frac{4y}{u}+\cancel{y^2}$

$\Rightarrow u^2=4-4yu$

$\Rightarrow 4yu=4-u^2$

$\Rightarrow y=\frac{1}{u}-\frac{u}{4}$

BUT $u=x+\sqrt{x^2+4}$

$\Rightarrow \frac{1}{u}=\frac{1}{x+\sqrt{x^2+4}}$

$\Rightarrow \frac{1}{u}=\frac{x-\sqrt{x^2+4}}{\left[x+\sqrt{x^2+4}\right]\left[x-\sqrt{x^2+4}\right]}$

$\Rightarrow \frac{1}{u}=\frac{x-\sqrt{x^2+4}}{x^2-(x^2+4)}$

$\Rightarrow \frac{1}{u}=\frac{x-\sqrt{x^2+4}}{-4}$

$\Rightarrow \frac{1}{u}=-\frac{1}{4}x+\frac{1}{4}\sqrt{x^2+4}$

COMBINING RESULTS

$y=\frac{1}{u}-\frac{1}{4}u=-\frac{1}{4}x+\frac{1}{4}\sqrt{x^2+4}-\frac{1}{4}\left[x+\sqrt{x^2+4}\right]$

$=-\frac{1}{4}x+\cancel{\frac{1}{4}\sqrt{x^2+4}}-\frac{1}{4}x-\cancel{\frac{1}{4}\sqrt{x^2+4}}$

$=-\frac{1}{2}x$

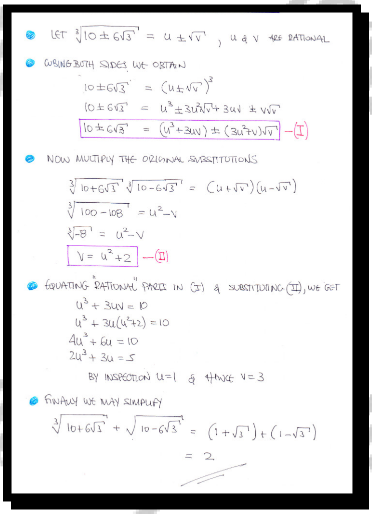
$\therefore y=-\frac{1}{2}x$ AS BEFORE AND THE GRAPH FOLLOWS //

Question 95 (*****)

By using the substitution $\sqrt[3]{10 \pm 6\sqrt{3}} = u \pm \sqrt{v}$, where $u \in \mathbb{Q}$, $v \in \mathbb{Q}$, simplify fully the following cubic radical expression.

$$\sqrt[3]{10+6\sqrt{3}} + \sqrt[3]{10-6\sqrt{3}} .$$

$\boxed{\text{SP}}$, $\boxed{2}$



- LET $\sqrt[3]{10 \pm 6\sqrt{3}} = u \pm \sqrt{v}$, u & v ARE RATIONAL
- CUBING BOTH SIDES WE OBTAIN

$$10 \pm 6\sqrt{3} = (u \pm \sqrt{v})^3$$
$$10 \pm 6\sqrt{3} = u^3 \pm 3u^2\sqrt{v} + 3uv \pm v\sqrt{v}$$
$$\boxed{10 \pm 6\sqrt{3} = (u^3+3uv) \pm (3u^2+v)\sqrt{v}} \text{ —(I)}$$

- NOW MULTIPLY THE ORIGINAL SUBSTITUTIONS

$$\sqrt[3]{10+6\sqrt{3}}\,\sqrt[3]{10-6\sqrt{3}} = (u+\sqrt{v})(u-\sqrt{v})$$
$$\sqrt[3]{100-108} = u^2-v$$
$$\sqrt[3]{-8} = u^2-v$$
$$\boxed{v = u^2+2} \text{ —(II)}$$

- EQUATING "RATIONAL" PARTS IN (I) & SUBSTITUTING (II), WE GET

$$u^3 + 3uv = 10$$
$$u^3 + 3u(u^2+2) = 10$$
$$4u^3 + 6u = 10$$
$$2u^3 + 3u = 5$$

BY INSPECTION $u=1$ & HENCE $v=3$

- FINALLY WE MAY SIMPLIFY

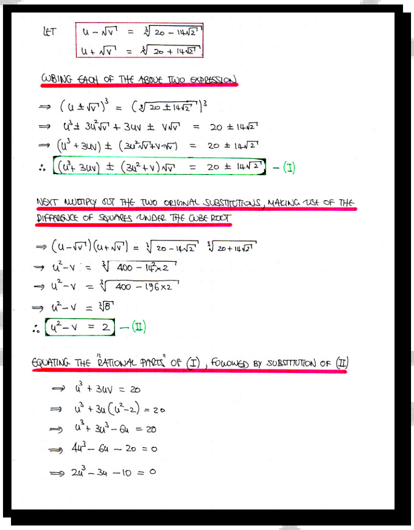
$$\sqrt[3]{10+6\sqrt{3}} + \sqrt{10-6\sqrt{3}} = (1+\sqrt{3}) + (1-\sqrt{3})$$
$$= 2$$

Question 96 (*****)

By using the substitution $\sqrt[3]{20 \pm 14\sqrt{2}} = u \pm \sqrt{v}$, where $u \in \mathbb{Q}$, $v \in \mathbb{Q}$, simplify fully the following cubic radical expression.

$$\sqrt[3]{20+14\sqrt{2}}\ .$$

[illegible], $\boxed{2-\sqrt{2}}$



LET $\boxed{\begin{aligned} u-\sqrt{v} &= \sqrt[3]{20-14\sqrt{2}} \\ u+\sqrt{v} &= \sqrt[3]{20+14\sqrt{2}} \end{aligned}}$

CUBING EACH OF THE ABOVE TWO EXPRESSION

$\Rightarrow (u \pm \sqrt{v})^3 = \left(\sqrt[3]{20 \pm 14\sqrt{2}}\right)^3$

$\Rightarrow u^3 \pm 3u^2\sqrt{v} + 3uv \pm v\sqrt{v} = 20 \pm 14\sqrt{2}$

$\Rightarrow (u^3 + 3uv) \pm (3u^2\sqrt{v} + v\sqrt{v}) = 20 \pm 14\sqrt{2}$

$\therefore \boxed{(u^3 + 3uv) \pm (3u^2 + v)\sqrt{v} = 20 \pm 14\sqrt{2}}$ — (I)

NEXT MULTIPLY OUT THE TWO ORIGINAL SUBSTITUTIONS, MAKING USE OF THE DIFFERENCE OF SQUARES UNDER THE CUBE ROOT

$\Rightarrow (u-\sqrt{v})(u+\sqrt{v}) = \sqrt[3]{20-14\sqrt{2}}\ \sqrt[3]{20+14\sqrt{2}}$

$\Rightarrow u^2 - v = \sqrt[3]{400 - 14^2 \times 2}$

$\Rightarrow u^2 - v = \sqrt[3]{400 - 196 \times 2}$

$\Rightarrow u^2 - v = \sqrt[3]{8}$

$\therefore \boxed{u^2 - v = 2}$ — (II)

EQUATING THE "RATIONAL PARTS" OF (I), FOLLOWED BY SUBSTITUTION OF (II)

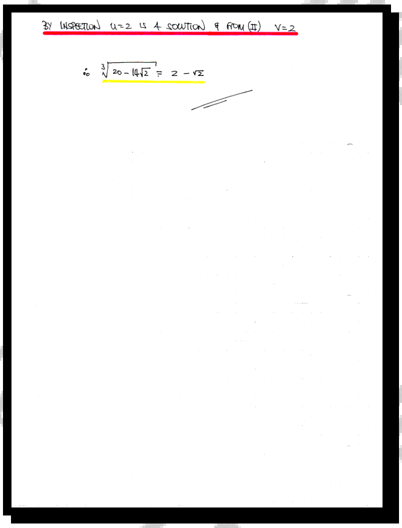
$\Rightarrow u^3 + 3uv = 20$

$\Rightarrow u^3 + 3u(u^2-2) = 20$

$\Rightarrow u^3 + 3u^3 - 6u = 20$

$\Rightarrow 4u^3 - 6u - 20 = 0$

$\Rightarrow 2u^3 - 3u - 10 = 0$



BY INSPECTION $u=2$ IS A SOLUTION & FROM (II) $v=2$

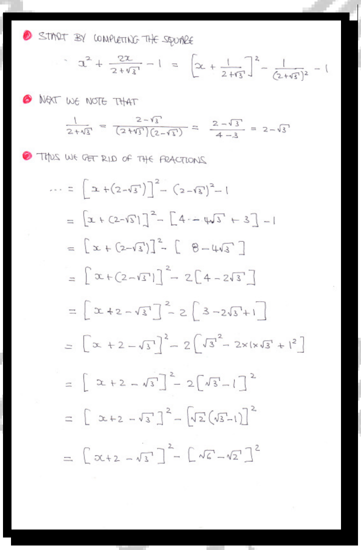
$\therefore \sqrt[3]{20-14\sqrt{2}} = 2 - \sqrt{2}$

Question 97 (*****)

$$f(x) \equiv x^2 + \frac{2x}{2+\sqrt{3}} - 1, \; x \in \mathbb{R}.$$

Factorize $f(x)$ into a product of 2 simple linear factors.

$\boxed{}$, $\boxed{\left(x+\sqrt{6}-\sqrt{3}-\sqrt{2}+2\right)\left(x-\sqrt{6}-\sqrt{3}+\sqrt{2}+2\right)}$



● START BY COMPLETING THE SQUARE

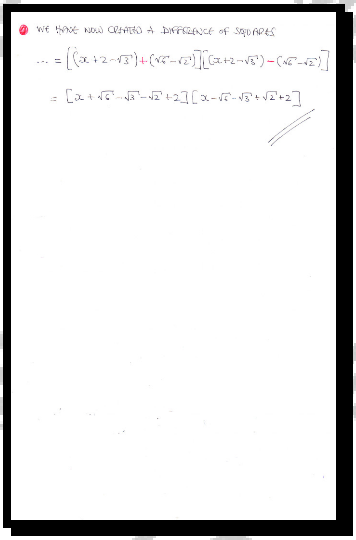
$$x^2 + \frac{2x}{2+\sqrt{3}} - 1 = \left[x + \frac{1}{2+\sqrt{3}}\right]^2 - \frac{1}{(2+\sqrt{3})^2} - 1$$

● NEXT WE NOTE THAT

$$\frac{1}{2+\sqrt{3}} = \frac{2-\sqrt{3}}{(2+\sqrt{3})(2-\sqrt{3})} = \frac{2-\sqrt{3}}{4-3} = 2-\sqrt{3}$$

● THUS WE GET RID OF THE FRACTIONS

$$\begin{aligned}
\ldots &= \left[x+(2-\sqrt{3})\right]^2 - (2-\sqrt{3})^2 - 1 \\
&= \left[x+(2-\sqrt{3})\right]^2 - \left[4-4\sqrt{3}+3\right] - 1 \\
&= \left[x+(2-\sqrt{3})\right]^2 - \left[8-4\sqrt{3}\right] \\
&= \left[x+(2-\sqrt{3})\right]^2 - 2\left[4-2\sqrt{3}\right] \\
&= \left[x+2-\sqrt{3}\right]^2 - 2\left[3-2\sqrt{3}+1\right] \\
&= \left[x+2-\sqrt{3}\right]^2 - 2\left[\sqrt{3}^2 - 2\times1\times\sqrt{3} + 1^2\right] \\
&= \left[x+2-\sqrt{3}\right]^2 - 2\left[\sqrt{3}-1\right]^2 \\
&= \left[x+2-\sqrt{3}\right]^2 - \left[\sqrt{2}(\sqrt{3}-1)\right]^2 \\
&= \left[x+2-\sqrt{3}\right]^2 - \left[\sqrt{6}-\sqrt{2}\right]^2
\end{aligned}$$



● WE HAVE NOW CREATED A DIFFERENCE OF SQUARES

$$\begin{aligned}
\ldots &= \left[(x+2-\sqrt{3})+(\sqrt{6}-\sqrt{2})\right]\left[(x+2-\sqrt{3})-(\sqrt{6}-\sqrt{2})\right] \\
&= \left[x+\sqrt{6}-\sqrt{3}-\sqrt{2}+2\right]\left[x-\sqrt{6}-\sqrt{3}+\sqrt{2}+2\right]
\end{aligned}$$