# 2$^{nd}$ ORDER O.D.E.s

# 24 EXAM QUESTIONS

# 7 BASIC QUESTIONS

**Question 1** **(**)**

Find a general solution of the following differential equation.

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0 .$$

$\boxed{\phantom{xx}}$ , $\boxed{y = e^{-2x}\left[A\cos 3x + B\sin 3x\right]}$

AUXILIARY EQUATION FOR THE O.D.E IS

$\lambda^2 + 4\lambda + 13 = 0$

$(\lambda+2)^2 - 4 + 13 = 0$

$(\lambda+2)^2 = -9$

$\lambda + 2 = \pm 3i$

$\lambda = -2 \pm 3i$

$\therefore$ GENERAL SOLUTION IS

$y = e^{-2x}(A\cos 3x + B\sin 3x)$

**Question 2** (**)

Find a general solution of the following differential equation.

$$\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=12\left(x+\mathrm{e}^x\right).$$

$$\boxed{y=A\mathrm{e}^{-3x}+B\mathrm{e}^{-2x}+\mathrm{e}^x+2x-\frac{5}{3}}$$

$$\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=12(x+e^x)$$

START WITH THE AUXILIARY EQUATION

$\lambda^2+5\lambda+6=0$

$(\lambda+2)(\lambda+3)=0$

$\lambda = \begin{cases}-2\\-3\end{cases}$

$\therefore$ COMPLEMENTARY FUNCTION: $y=Ae^{-2x}+Be^{-3x}$

FOR PARTICULAR INTEGRAL WE TRY $y=Px+Q+Re^x$

$\frac{dy}{dx}=P+Re^x$

$\frac{d^2y}{dx^2}=Re^x$

SUB INTO THE O.D.E

$(Re^x)+5(P+Re^x)+6(Px+Q+Re^x)\equiv 12x+12e^x$

$6Px+(5P+6Q)+e^x(R+5R+6R)\equiv 12x+12e^x$

$\therefore P=2 \quad R=1 \quad \& \quad 5P+6Q=0$

$10+6Q=0$

$Q=-\frac{5}{3}$

HENCE THE GENERAL SOLUTION IS

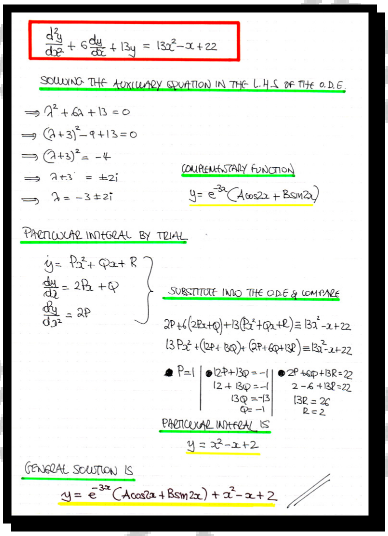
$y=Ae^{-2x}+Be^{-3x}+e^x+2x-\frac{5}{3}$

**Question 3** (**)

Find a general solution of the following differential equation.

$$\frac{d^2y}{dx^2}+6\frac{dy}{dx}+13y=13x^2-x+22.$$

$\boxed{\phantom{xxxx}}$, $\boxed{y=e^{-3x}\left(A\cos 2x+B\sin 2x\right)+x^2-x+2}$



$$\boxed{\frac{d^2y}{dx^2}+6\frac{dy}{dx}+13y=13x^2-x+22}$$

SOLVING THE AUXILIARY EQUATION IN THE L.H.S OF THE O.D.E.

$\Rightarrow \lambda^2+6\lambda+13=0$

$\Rightarrow (\lambda+3)^2-9+13=0$

$\Rightarrow (\lambda+3)^2=-4$

$\Rightarrow \lambda+3=\pm 2i$

$\Rightarrow \lambda=-3\pm 2i$

COMPLEMENTARY FUNCTION

$y=e^{-3x}(A\cos 2x+B\sin 2x)$

PARTICULAR INTEGRAL BY TRIAL

$y=Px^2+Qx+R$

$\frac{dy}{dx}=2Px+Q$

$\frac{d^2y}{dx^2}=2P$

SUBSTITUTE INTO THE O.D.E & COMPARE

$2P+6(2Px+Q)+13(Px^2+Qx+R)\equiv 13x^2-x+22$

$13Px^2+(12P+13Q)x+(2P+6Q+13R)\equiv 13x^2-x+22$

- $P=1$
- $12P+13Q=-1$; $12+13Q=-1$; $13Q=-13$; $Q=-1$
- $2P+6Q+13R=22$; $2-6+13R=22$; $13R=26$; $R=2$

PARTICULAR INTEGRAL IS

$y=x^2-x+2$

GENERAL SOLUTION IS

$y=e^{-3x}(A\cos 2x+B\sin 2x)+x^2-x+2$

**Question 4** (***)

Find a solution of the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 10\sin x,$$

subject to the boundary conditions $y = 6$ and $\frac{dy}{dx} = 5$ at $x = 0$.

$\boxed{\phantom{XX}}$, $\boxed{y = 2e^x + e^{2x} + 3\cos x + \sin x}$

$$\boxed{\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 10\sin x \qquad y(0)=6,\ y'(0)=5}$$

AUXILIARY EQUATION

$\lambda^2 - 3\lambda + 2 = 0$

$(\lambda - 2)(\lambda - 1) = 0$

$\lambda = \begin{cases} 1 \\ 2 \end{cases}$

COMPLEMENTARY FUNCTION

$y = Ae^x + Be^{2x}$

PARTICULAR INTEGRAL BY INSPECTION

$y = P\cos x + Q\sin x$

$y' = -P\sin x + Q\cos x$

$y'' = -P\cos x - Q\sin x$

SUBSTITUTE INTO THE O.D.E

$\Rightarrow (-P\cos x - Q\sin x) - 3(-P\sin x + Q\cos x) + 2(P\cos x + Q\sin x) \equiv 10\sin x$

$\Rightarrow \left\{\begin{array}{l} -P\cos x - Q\sin x \\ -3Q\cos x + 3P\sin x \\ +2P\cos x + 2Q\sin x \end{array}\right\} \equiv 10\sin x$

$\Rightarrow (P - 3Q)\cos x + (3P + Q)\sin x \equiv 10\sin x$

- $P - 3Q = 0$
  $P = 3Q$
- $3P + Q = 10$
  $3(3Q) + Q = 10$
  $10Q = 10$
  $Q = 1$ & $P = 3$

$\therefore$ PARTICULAR INTEGRAL

$y = 3\cos x + \sin x$

$\therefore$ GENERAL SOLUTION

$y = Ae^x + Be^{2x} + 3\cos x + \sin x$

DIFFERENTIATE W.R.T $x$ & APPLY CONDITIONS

$\frac{dy}{dx} = Ae^x + 2Be^{2x} - 3\sin x + \cos x$

- $x=0,\ y=6 \Rightarrow 6 = A + B + 3$
  $\Rightarrow A + B = 3$
- $x=0,\ \frac{dy}{dx}=5 \Rightarrow 5 = A + 2B + 1$
  $\Rightarrow A + 2B = 4$

$\therefore B = 1 \quad A = 2$

FINALLY WE OBTAIN

$y = 2e^x + e^{2x} + 3\cos x + \sin x$

**Question 5** (***)

Find a solution of the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 20\sin 2x,$$

subject to the boundary conditions $y = 1$ and $\frac{dy}{dx} = -5$ at $x = 0$.

$\boxed{\phantom{xxxx}}$, $\boxed{y = 3\cos 2x - \sin 2x - e^{2x} - e^{x}}$

AUXILIARY EQUATION FOR THE L.H.S OF THE O.D.E.

$\lambda^2 - 3\lambda + 2 = 0$

$(\lambda - 2)(\lambda - 1) = 0$

$\lambda = \begin{cases} 1 \\ 2 \end{cases}$

COMPLEMENTARY FUNCTION

$y = Ae^{x} + Be^{2x}$

PARTICULAR INTEGRAL, TRY $y = P\cos 2x + Q\sin 2x$

$\frac{dy}{dx} = -2P\sin 2x + 2Q\cos 2x$

$\frac{d^2y}{dx^2} = -4P\cos 2x - 4Q\sin 2x$

SUB INTO THE O.D.E.

$\frac{d^2y}{dx^2} = -4P\cos 2x - 4Q\sin 2x$

$-3\frac{dy}{dx} = -6Q\cos 2x + 6P\sin 2x$

$+2y = 2P\cos 2x + 2Q\sin 2x$

$(-2P - 6Q)\cos 2x + (6P - 2Q)\sin 2x \equiv 20\sin 2x$

SOLVING SIMULTANEOUS EQUATIONS

- $-2P - 6Q = 0$
  $-6Q = 2P$
  $P = -3Q$
- $6P - 2Q = 20$
  $6(-3Q) - 2Q = 20$
  $-20Q = 20$
  $Q = -1$ & $P = 3$

HENCE THE GENERAL SOLUTION IS

$y = Ae^{x} + Be^{2x} + 3\cos 2x - \sin 2x$

APPLY CONDITIONS

$(0,1) \Rightarrow 1 = A + B + 3$

$A + B = -2$

$\frac{dy}{dx} = Ae^{x} + 2Be^{2x} - 6\sin 2x - 2\cos 2x$

$-5 = A + 2B - 2$

$-3 = A + 2B$

$\therefore \left.\begin{array}{l} A = -2 - B \\ A = -3 - 2B \end{array}\right\} \Rightarrow -2 - B = -3 - 2B$

$B = -1$

& $A = -1$

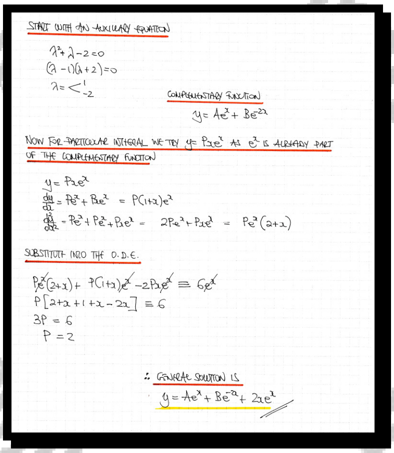
$\therefore y = 3\cos 2x - \sin 2x - e^{x} - e^{2x}$

**Question 6** (***)

Find a general solution of the differential equation

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y=6e^x .$$

$\boxed{\phantom{xxxx}}$ , $\boxed{y=(A+2x)e^x+Be^{-2x}}$



START WITH AN AUXILIARY EQUATION

$\lambda^2+\lambda-2=0$

$(\lambda-1)(\lambda+2)=0$

$\lambda = \begin{cases}1\\-2\end{cases}$

COMPLEMENTARY FUNCTION

$y=Ae^x+Be^{-2x}$

NOW FOR PARTICULAR INTEGRAL WE TRY $y=Pxe^x$ AS $e^x$ IS ALREADY PART OF THE COMPLEMENTARY FUNCTION

$y=Pxe^x$

$\frac{dy}{dx}=Pe^x+Pxe^x = P(1+x)e^x$

$\frac{d^2y}{dx^2}=Pe^x+Pe^x+Pxe^x = 2Pe^x+Pxe^x = Pe^x(2+x)$

SUBSTITUTE INTO THE O.D.E.

$P\cancel{e^x}(2+x)+P(1+x)\cancel{e^x}-2Px\cancel{e^x}\equiv 6\cancel{e^x}$

$P[2+x+1+x-2x]\equiv 6$

$3P=6$

$P=2$

$\therefore$ GENERAL SOLUTION IS

$y=Ae^x+Be^{-2x}+2xe^x$

**Question 7** (***)

Find a general solution of the differential equation

$$\frac{d^2y}{dx^2}-\frac{dy}{dx}-2y=12\left(e^{2x}-e^{-2x}\right).$$

$$\boxed{y=(A+4x)e^{2x}+Be^{-x}-3e^{-2x}}$$

AUXILIARY EQUATION FIRST

$$\lambda^2-\lambda-2=0$$
$$(\lambda+1)(\lambda-2)=0$$
$$\lambda = 2,\ -1$$

COMPLEMENTARY FUNCTION

$$y=Ae^{2x}+Be^{-x}$$

FOR PARTICULAR INTEGRAL TRY

$$y=Pe^{-2x}+Qxe^{2x}$$

BECAUSE $Ae^{2x}$ IS PART OF THE SOLUTION ALREADY

$$\frac{dy}{dx}=-2Pe^{-2x}+Qe^{2x}+2Qxe^{2x}$$
$$\frac{d^2y}{dx^2}=4Pe^{-2x}+2Qe^{2x}+2Qe^{2x}+4Qxe^{2x}$$
$$=4Pe^{-2x}+4Qe^{2x}+4Qxe^{2x}$$

SUBSTITUTE INTO THE O.D.E

$$\underbrace{4Pe^{-2x}+4Qe^{2x}+\cancel{4Qxe^{2x}}}_{\frac{d^2y}{dx^2}}\underbrace{+\cancel{2Pe^{-2x}}-Qe^{2x}-\cancel{2Qxe^{2x}}}_{-\frac{dy}{dx}}\underbrace{-\cancel{2Pe^{-2x}}-\cancel{2Qxe^{2x}}}_{-2y}\equiv 12\left(e^{2x}-e^{-2x}\right)$$

$$4P=-12 \quad \& \quad 3Q=12$$
$$P=-3 \quad\quad Q=4$$

HENCE THE GENERAL SOLUTION IS

$$y=Ae^{2x}+Be^{-x}-3e^{-2x}+4xe^{2x}$$
$$y=(A+4x)e^{2x}+Be^{-x}-3e^{-2x}$$

**Question 8** (***)

$$\frac{d^2y}{dx^2} + y = \sin 2x, \text{ with } y = 0 \text{ , } \frac{dy}{dx} = 0 \text{ at } x = \frac{\pi}{2}.$$

Show that a solution of the above differential equation is

$$y = \frac{2}{3}\cos x(1 - \sin x).$$

[ ] , [proof]

START WITH THE AUXILIARY EQUATION, TO FIND THE COMPLEMENTARY FUNCTION

$$\frac{d^2y}{dx^2} + y = 0$$

$$\lambda^2 + 1 = 0$$

$$\lambda = \pm i$$

$\therefore$ COMPLEMENTARY FUNCTION

$$y = A\cos x + B\sin x$$

FOR PARTICULAR INTEGRAL, AS THE $\frac{dy}{dx}$ IS MISSING, TRY $y = P\sin 2x$

$$\left.\begin{aligned} y &= P\sin 2x \\ \frac{dy}{dx} &= 2P\cos 2x \\ \frac{d^2y}{dx^2} &= -4P\sin 2x \end{aligned}\right\}$$

SUB INTO THE O.D.E

$$-4P\sin 2x + P\sin 2x = \sin 2x$$

$$-3P = 1$$

$$P = -\frac{1}{3}$$

$\therefore$ GENERAL SOLUTION IS

$$y = A\cos x + B\sin x - \frac{1}{3}\sin 2x$$

DIFFERENTIATE & APPLY BOUNDARY CONDITION

$$\frac{dy}{dx} = -A\sin x + B\cos x - \frac{2}{3}\cos 2x$$

- $x = \frac{\pi}{2}$, $y = 0 \implies 0 = 0 + B + 0$

  $B = 0$

- $x = \frac{\pi}{2}$, $\frac{dy}{dx} = 0 \implies 0 = -A + 0 + \frac{2}{3}$

  $\implies A = \frac{2}{3}$

FINALLY WE HAVE

$$y = \frac{2}{3}\cos x - \frac{1}{3}\sin 2x$$

$$y = \frac{2}{3}\cos x - \frac{2}{3}\sin x\cos x$$

$$y = \frac{2}{3}\cos x(1 - \sin x)$$

AS REQUIRED

# 16 STANDARD QUESTIONS

**Question 1** (***+)

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y=6e^{-2x},$$

with $y=3$ and $\frac{dy}{dx}=-2$ at $x=0$.

Show that the solution of the above differential equation is

$$y=2e^{x}+(1-2x)e^{-2x}.$$

$\boxed{\phantom{xx}}$, $\boxed{\text{proof}}$

AUXILIARY EQUATION FOR THE O.D.E IS

$\lambda^2+\lambda-2=0$

$(\lambda-1)(\lambda+2)=0$

$\lambda = 1, -2$

$\therefore$ COMPLEMENTARY FUNCTION

$y=Ae^{x}+Be^{-2x}$

AS THE R.H.S CONTAINS $6e^{-2x}$ WHICH IS PART OF THE SOLUTION FOR THE PARTICULAR INTEGRAL WE TRY

$y=Pxe^{-2x}$

$\frac{dy}{dx}=Pe^{-2x}-2Pxe^{-2x}$

$\frac{d^2y}{dx^2}=-2Pe^{-2x}-2Pe^{-2x}+4Pxe^{-2x}=4Pxe^{-2x}-4Pe^{-2x}$

SUB INTO THE O.D.E

$(\cancel{4Pxe^{-2x}}-4Pe^{-2x})+(Pe^{-2x}-\cancel{2Pxe^{-2x}})-\cancel{2(Pxe^{-2x})}\equiv 6e^{-2x}$

$-3Pe^{-2x}\equiv 6e^{-2x}$

$P=-2$

$\therefore$ PARTICULAR INTEGRAL IS

$y=-2xe^{-2x}$

$\therefore$ GENERAL SOLUTION IS

$y=Ae^{x}+Be^{-2x}-2xe^{-2x}$

DIFFERENTIATE AND APPLY CONDITIONS

$y=Ae^{x}+Be^{-2x}-2xe^{-2x}$

$\frac{dy}{dx}=Ae^{x}-2Be^{-2x}-2e^{-2x}+4xe^{-2x}$

- $x=0$ $y=3$ $\Rightarrow$ $3=A+B$
- $x=0$ $\frac{dy}{dx}=-2$ $\Rightarrow$ $-2=A-2B-2$

$0=A-2B$

$A=2B$

$\therefore$ $3=2B+B$

$B=1$ & $A=2$

FINALLY WE HAVE

$y=2e^{x}+e^{-2x}-2xe^{-2x}$

**Question 2** (***+)

Find a general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2k\frac{dy}{dx} + k^2 y = \frac{1}{4}, \ k > 0 .$$

$\boxed{\phantom{Y}}$ , $\boxed{y = A\mathrm{e}^{kx} + Bx\mathrm{e}^{kx} + \frac{1}{4k^2}}$

OBTAIN THE COMPLEMENTARY FUNCTION — VIA AUXILLARY EQUATION

$\lambda^2 - 2k\lambda + k^2 = 0$

$(\lambda - k)^2 = 0$

$\lambda = k$ (REPEATED)

$\therefore y = Ae^{kx} + Bxe^{kx}$

FOR PARTICULAR INTEGRAL TRY $y = P =$ CONSTANT

$\frac{dy}{dx} = \frac{d^2y}{dx^2} = 0$

SUB INTO THE O.D.E.

$0 + 0 + k^2 P \equiv \frac{1}{4}$

$P = \frac{1}{4k^2}$

$\therefore$ GENERAL SOLUTION

$y = Ae^{kx} + Bxe^{kx} + \frac{1}{4k^2}$

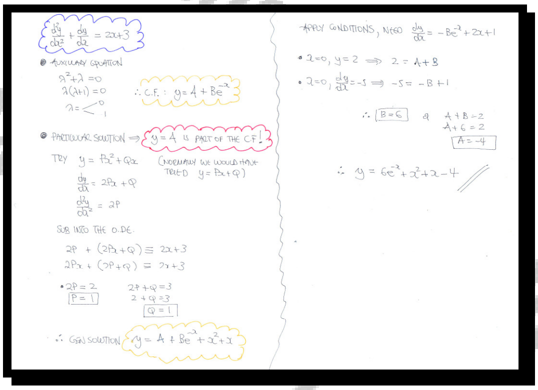
**Question 3** (***+)

Find the solution of the differential equation

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}=2x+3,$$

subject to the conditions $y=2$, $\dfrac{dy}{dx}=-5$ at $x=0$.

$$\boxed{y=x^2+x-4+6e^{-x}}$$



$\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}=2x+3$

- AUXILIARY EQUATION

$\lambda^2+\lambda=0$

$\lambda(\lambda+1)=0$

$\lambda=\begin{cases}0\\-1\end{cases}$

$\therefore$ C.F. : $y=A+Be^{-x}$

- PARTICULAR SOLUTION $\Rightarrow$ $y=A$ IS PART OF THE C.F!

TRY $y=Px^2+Qx$ (NORMALLY WE WOULD HAVE TRIED $y=Px+Q$)

$\frac{dy}{dx}=2Px+Q$

$\frac{d^2y}{dx^2}=2P$

SUB INTO THE O.D.E.

$2P+(2Px+Q)\equiv 2x+3$

$2Px+(2P+Q)\equiv 2x+3$

- $2P=2$, $\boxed{P=1}$    $2P+Q=3$, $2+Q=3$, $\boxed{Q=1}$

$\therefore$ GEN SOLUTION $y=A+Be^{-x}+x^2+x$

APPLY CONDITIONS, NEED $\frac{dy}{dx}=-Be^{-x}+2x+1$

- $x=0,\ y=2 \Rightarrow 2=A+B$
- $x=0,\ \frac{dy}{dx}=-5 \Rightarrow -5=-B+1$

$\therefore \boxed{B=6}$ & $A+B=2$, $A+6=2$, $\boxed{A=-4}$

$\therefore y=6e^{-x}+x^2+x-4$

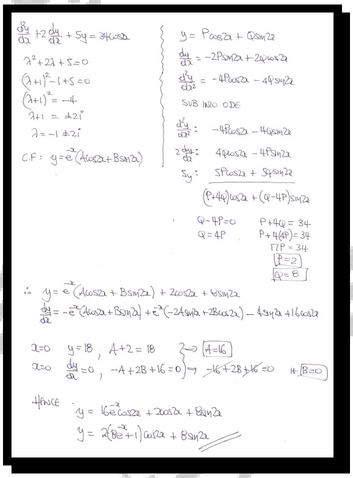
**Question 4 (***+)**

Find a solution of the differential equation

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+5y=34\cos 2x,$$

subject to the boundary conditions $y=18$ and $\frac{dy}{dx}=0$ at $x=0$.

$$\boxed{y=2\left(8e^{-x}+1\right)\cos 2x+8\sin 2x}$$



$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+5y=34\cos 2x$

$\lambda^2+2\lambda+5=0$

$(\lambda+1)^2-1+5=0$

$(\lambda+1)^2=-4$

$\lambda+1=\pm 2i$

$\lambda=-1\pm 2i$

C.F: $y=e^{-x}(A\cos 2x+B\sin 2x)$

$y=P\cos 2x+Q\sin 2x$

$\frac{dy}{dx}=-2P\sin 2x+2Q\cos 2x$

$\frac{d^2y}{dx^2}=-4P\cos 2x-4Q\sin 2x$

SUB INTO ODE

$\frac{d^2y}{dx^2}$: $-4P\cos 2x-4Q\sin 2x$

$2\frac{dy}{dx}$: $4Q\cos 2x-4P\sin 2x$

$5y$: $5P\cos 2x+5Q\sin 2x$

$(P+4Q)\cos 2x+(Q-4P)\sin 2x$

$Q-4P=0$ $\quad P+4Q=34$

$Q=4P$ $\quad P+4(4P)=34$

$17P=34$

$\boxed{P=2}$

$\boxed{Q=8}$

$\therefore\ y=e^{-x}(A\cos 2x+B\sin 2x)+2\cos 2x+8\sin 2x$

$\frac{dy}{dx}=-e^{-x}(A\cos 2x+B\sin 2x)+e^{-x}(-2A\sin 2x+2B\cos 2x)-4\sin 2x+16\cos 2x$

$x=0\ \ y=18,\ A+2=18 \Rightarrow \boxed{A=16}$

$x=0\ \ \frac{dy}{dx}=0,\ -A+2B+16=0 \Rightarrow \cancel{-16}+2B+\cancel{16}=0 \quad \boxed{B=0}$

HENCE

$y=16e^{-x}\cos 2x+2\cos 2x+8\sin 2x$

$y=2(8e^{-x}+1)\cos 2x+8\sin 2x$

**Question 5** (***+)

The curve $C$ has a local minimum at the origin and satisfies the differential equation

$$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+8y=32x^2 .$$

Find an equation for $C$.

$$\boxed{y=\mathrm{e}^{x}\left(\sin 2x+\cos 2x\right)+\left(2x-1\right)^2}$$

$$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+8y=32x^2$$

- AUX EQUATION

$\lambda^2+4\lambda+8=0$
$(\lambda+2)^2-4+8=0$
$(\lambda+2)^2=-4$
$\lambda+2=\pm 2i$
$\lambda=-2\pm 2i$

C.F. : $y=e^{-2x}(A\cos 2x+B\sin 2x)$

- PARTICULAR INTEGRAL, TRY $y=Px^2+Qx+R$

$\frac{dy}{dx}=2Px+Q$
$\frac{d^2y}{dx^2}=2P$

SUB INTO THE O.D.E

$\Rightarrow 2P+4(2Px+Q)+8(Px^2+Qx+R)\equiv 32x^2$
$\Rightarrow 2P+8Px+4Q+8Px^2+8Qx+8R\equiv 32x^2$
$\Rightarrow 8Px^2+(8P+8Q)x+(2P+4Q+8R)\equiv 32x^2$

$\Rightarrow 8P=32$, $\boxed{P=4}$

$8P+8Q=0$, $32+8Q=0$, $\boxed{Q=-4}$

$2P+4Q+8R=0$, $8-16+8R=0$, $8R=8$, $\boxed{R=1}$

$\therefore$ GEN SOLUTION $y=e^{-2x}(A\cos 2x+B\sin 2x)+4x^2-4x+1$

THUS

$\frac{dy}{dx}=-2e^{-2x}(A\cos 2x+B\sin 2x)+e^{-2x}(-2A\sin 2x+2B\cos 2x)+8x-4$

NOW

- $x=0, y=0 \Rightarrow 0=A+1 \Rightarrow \boxed{A=-1}$
- $x=0, \frac{dy}{dx}=0 \Rightarrow 0=-2A+2B-4$
$0=2+2B-4$
$2=2B$
$\boxed{B=1}$

$\therefore\ y=e^{-2x}(\sin 2x-\cos 2x)+4x^2-4x+1$

OR

$y=e^{-2x}(\sin 2x-\cos 2x)+(2x-1)^2$

**Question 6** (***+)

$$\frac{d^2x}{dt^2}+9x+12\sin 3t=0, \ t\geq 0,$$

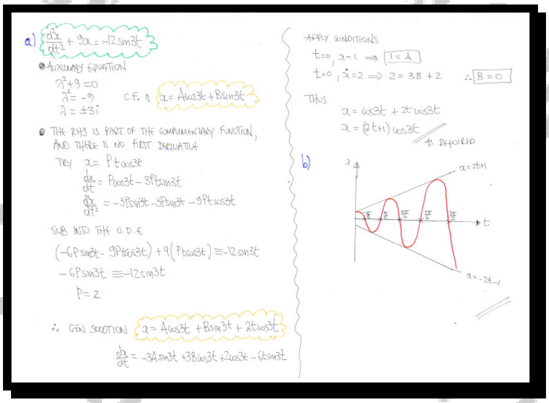
with $x=1$, $\dfrac{dx}{dt}=2$ at $t=0$.

**a)** Show that a solution of the differential equation is

$$x=(2t+1)\cos 3t .$$

**b)** Sketch the graph of $x$.

proof



a) $\frac{d^2x}{dt^2}+9x=-12\sin 3t$

● AUXILIARY EQUATION

$\lambda^2+9=0$

$\lambda^2=-9$

$\lambda=\pm 3i$

C.F. $x=A\cos 3t+B\sin 3t$

● THE RHS IS PART OF THE COMPLEMENTARY FUNCTION, AND THERE IS NO FIRST DERIVATIVE

TRY $x=Pt\cos 3t$

$\frac{dx}{dt}=P\cos 3t-3Pt\sin 3t$

$\frac{d^2x}{dt^2}=-3P\sin 3t-3P\sin 3t-9Pt\cos 3t$

SUB INTO THE O.D.E

$(-6P\sin 3t-9Pt\cos 3t)+9(Pt\cos 3t)\equiv -12\sin 3t$

$-6P\sin 3t\equiv -12\sin 3t$

$P=2$

$\therefore$ GEN SOLUTION $x=A\cos 3t+B\sin 3t+2t\cos 3t$

$\frac{dx}{dt}=-3A\sin 3t+3B\cos 3t+2\cos 3t-6t\sin 3t$

APPLY CONDITIONS

$t=0,\ x=1 \Rightarrow \boxed{1=A}$

$t=0,\ \dot{x}=2 \Rightarrow 2=3B+2 \quad \therefore \boxed{B=0}$

THUS

$x=\cos 3t+2t\cos 3t$

$x=(2t+1)\cos 3t$

AS REQUIRED

b)

**Question 7** **(***+)**

Solve the following differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 2e^{2x},$$

subject to the boundary conditions $y=0$, $\dfrac{dy}{dx}=1$ at $x=0$.

$\boxed{\phantom{xx}}$, $\boxed{y = x(x+1)e^{2x}}$

AUXILIARY EQUATION IS

$\lambda^2 - 4\lambda + 4 = 0$

$(\lambda - 2)^2$

$\lambda = 2$ (REPEATED)

COMPLEMENTARY FUNCTION

$y = Ae^{2x} + Bxe^{2x}$

AS BOTH $e^{2x}$ & $xe^{2x}$ ARE PART OF THE COMPLEMENTARY FUNCTION

WE TRY $y = Px^2e^{2x}$

$y = Px^2e^{2x}$

$\frac{dy}{dx} = 2Pxe^{2x} + 2Px^2e^{2x}$

$\frac{d^2y}{dx^2} = 2Pe^{2x} + 4Pxe^{2x} + 4Pxe^{2x} + 4Px^2e^{2x}$

$= 2Pe^{2x} + 8Pxe^{2x} + 4Px^2e^{2x}$

SUBSTITUTE INTO THE O.D.E.

$2Pe^{2x} + \cancel{8Pxe^{2x}} + \cancel{4Px^2e^{2x}} - \cancel{8Pxe^{2x}} - \cancel{8Px^2e^{2x}} + \cancel{4Px^2e^{2x}} = 2e^{2x}$

$\therefore P = 1$

GENERAL SOLUTION IS

$y = Ae^{2x} + Bxe^{2x} + x^2e^{2x}$

$y = (A + Bx + x^2)e^{2x}$

APPLY BOUNDARY CONDITION $x=0$ $y=0$

$\Rightarrow 0 = A(e^0)$

$\Rightarrow A = 0$

DIFFERENTIATE $y = (Bx + x^2)e^{2x}$ TO APPLY $x=0$ $\frac{dy}{dx} = 1$

$\frac{dy}{dx} = (B + 2x)e^{2x} + 2(Bx + x^2)e^{2x}$

$1 = B + 2\times 0$

$B = 1$

$\therefore y = (x^2 + x)e^{2x}$

$y = x(x+1)e^{2x}$

**Question 8** (***+)

It is given that the functions of $x$, $f$ and $g$, satisfy the following coupled first order differential equations.

$$f'(x) - 5f(x) = 3g(x) \qquad \text{and} \qquad g'(x) + 4g(x) = -6f(x).$$

**a)** Show that

$$f''(x) - f'(x) - 2f(x) = 0.$$

**b)** Given further that $f(0) = 1$ and $g(0) = 3$, solve the differential equation of part **(a)** to obtain simplified expressions for $f(x)$ and $g(x)$.

$\boxed{\phantom{XX}}$, $\boxed{[f(x),\ g(x)] = [5e^{2x} - 4e^{-x},\ 8e^{-x} - 5e^{2x}]}$

a) $\boxed{f'(x) - 5f(x) = 3g(x) \quad \& \quad g'(x) + 4g(x) = -6f(x)}$

DIFFERENTIATE THE FIRST O.D.E. WITH RESPECT TO $x$

$\Rightarrow f''(x) - 5f'(x) = 3g'(x)$

$g'(x) = -4g(x) - 6f(x)$

$\Rightarrow f''(x) - 5f'(x) = 3[-4g(x) - 6f(x)]$

$\Rightarrow f''(x) - 5f'(x) = -12g(x) - 18f(x)$

$\Rightarrow f''(x) - 5f'(x) = -4[3g(x)] - 18f(x)$

$f'(x) - 5f(x) = 3g(x)$

$\Rightarrow f''(x) - 5f'(x) = -4[f'(x) - 5f(x)] - 18f(x)$

$\Rightarrow f''(x) - 5f'(x) = -4f'(x) + 20f(x) - 18f(x)$

$\Rightarrow f''(x) - f'(x) - 2f(x) = 0$ //

b) SOLVING THE 2ND ORDER O.D.E (AUXILIARY EQUATION ETC)

$\lambda^2 - \lambda - 2 = 0$

$(\lambda + 1)(\lambda - 2) = 0$

$\lambda = \begin{cases} -1 \\ 2 \end{cases}$

$\therefore$ GENERAL SOLUTION

$f(x) = Ae^{2x} + Be^{-x}$

DIFFERENTIATING TO OBTAIN $g(x)$

$f'(x) = 2Ae^{2x} - Be^{-x}$

$\Rightarrow 3g(x) = f'(x) - 5f(x)$

$\Rightarrow 3g(x) = 2Ae^{2x} - Be^{-x} - 5[Ae^{2x} + Be^{-x}]$

$\Rightarrow 3g(x) = 2Ae^{2x} - Be^{-x} - 5Ae^{2x} - 5Be^{-x}$

$\Rightarrow 3g(x) = -3Ae^{2x} - 6Be^{-x}$

$\Rightarrow g(x) = -Ae^{2x} - 2Be^{-x}$

APPLY CONDITIONS $f(0) = 1$ $g(0) = 3$

$f(x) = Ae^{2x} + Be^{-x}$

$1 = A + B$

$g(x) = -Ae^{2x} - 2Be^{-x}$

$3 = -A - 2B$

$A + B = -3 - B$

$-3 - B = 1$

$\underline{-4 = B}$

&

$A + B = 1$

$A - 4 = 1$

$\underline{A = 5}$

FINALLY WE HAVE

$f(x) = 5e^{2x} - 4e^{-x}$ & $g(x) = 8e^{-x} - 5e^{2x}$ //

**Question 9** (***+)

The variables $x$ and $y$ satisfy the following coupled first order differential equations.

$$\frac{dx}{dt} = x - 2y \qquad \text{and} \qquad \frac{dy}{dt} = 5x - y .$$

Given further that $x = -1$, $y = 2$ at $t = 0$, solve the differential equations to obtain simplified expressions for $x$ and $y$.

$\boxed{\phantom{xx}}$ , $\boxed{x = -\cos 3t - \frac{5}{3}\sin 3t, \quad y = 2\cos 3t - \frac{7}{3}\sin 3t}$

DIFFERENTIATE THE FIRST EQUATION WITH RESPECT TO $t$

$$\frac{dx}{dt} = x - 2y$$
$$\frac{d^2x}{dt^2} = \frac{dx}{dt} - 2\frac{dy}{dt}$$
$$\frac{d^2x}{dt^2} = \frac{dx}{dt} - 2(5x - y)$$
$$\frac{d^2x}{dt^2} = \frac{dx}{dt} - 10x + 2y$$

BUT $2y = x - \frac{dx}{dt}$

$$\frac{d^2x}{dt^2} = \cancel{\frac{dx}{dt}} - 10x + \left(x - \cancel{\frac{dx}{dt}}\right)$$
$$\frac{d^2x}{dt^2} = -9x$$
$$\frac{d^2x}{dt^2} + 9x = 0$$

AUXILIARY EQUATION

$$\lambda^2 + 9 = 0$$
$$\lambda = \pm 3i$$

$$\therefore \ x(t) = A\cos 3t + B\sin 3t$$

DIFFERENTIATE $x(t)$ WITH RESPECT TO $t$

$$\frac{dx}{dt} = -3A\sin 3t + 3B\cos 3t$$

SUBSTITUTE $x$ & $\dot{x}$ INTO $2y = x - \frac{dx}{dt}$

$$\Rightarrow 2y = x - \frac{dx}{dt}$$
$$\Rightarrow 2y = A\cos 3t + B\sin 3t - (-3A\sin 3t + 3B\cos 3t)$$
$$\Rightarrow 2y = A\cos 3t + B\sin 3t + 3A\sin 3t - 3B\cos 3t$$
$$\Rightarrow y = \frac{A-3B}{2}\cos 3t + \frac{B+3A}{2}\sin 3t$$

APPLY CONDITIONS FIRST INTO $x(t)$

$t=0 \quad x=-1 \Rightarrow -1 = A$

APPLY CONDITION TO $y(t)$

$$t=0 \quad y=2 \Rightarrow 2 = \frac{A-3B}{2}$$
$$4 = A - 3B$$
$$3B = A - 4$$
$$3B = -5$$
$$B = -\frac{5}{3}$$

FINALLY WE HAVE IF $A=-1$ & $B = 5/3$

$$x(t) = -\cos 3t - \frac{5}{3}\sin 3t$$
$$y(t) = 2\cos 3t - \frac{7}{3}\sin 3t$$

**Question 10** (***+)

It is given that the variables $x = f(t)$ and $y = g(t)$ satisfy the following coupled first order differential equations.

$$\frac{dx}{dt} = x + \frac{2}{3}y \qquad \text{and} \qquad \frac{dy}{dt} = 3y - \frac{3}{2}x .$$

Given further that $x = 1$, $y = 3$ at $t = 0$, solve the differential equations to obtain simplified expressions for $f(t)$ and $g(t)$.

$$\boxed{\big[f(t),\ g(x)\big] = \left[e^{2t} + t\,e^{2t},\ 3e^{2t} + \frac{3}{2}t\,e^{2t}\right]}$$

$$\boxed{\frac{dx}{dt} = x + \frac{2}{3}y \qquad \frac{dy}{dt} = 3y - \frac{3}{2}x \qquad \begin{matrix} t=0 \\ x=1 \\ y=3 \end{matrix}}$$

DIFFERENTIATE THE FIRST O.D.E WITH RESPECT TO t

$$\Rightarrow \frac{dx}{dt} = x + \frac{2}{3}y$$

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} = \frac{dx}{dt} + \frac{2}{3}\frac{dy}{dt}} - \mathrm{I}$$

SUBSTITUTE THE SECOND O.D.E. INTO THE ABOVE EXPRESSION

$$\Rightarrow \frac{d^2x}{dt^2} = \frac{dx}{dt} + \frac{2}{3}\left(3y - \frac{3}{2}x\right)$$

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} = \frac{dx}{dt} + 2y - x} - \mathrm{II}$$

REARRANGE THE FIRST O.D.E

$$\Rightarrow \frac{dx}{dt} = x + \frac{2}{3}y$$

$$\Rightarrow 3\frac{dx}{dt} = 3x + 2y$$

$$\Rightarrow \boxed{2y = 3\frac{dx}{dt} - 3x} - \mathrm{III}$$

COMBINING (II) & (III) WE OBTAIN

$$\Rightarrow \frac{d^2x}{dt^2} = \frac{dx}{dt} + \left(3\frac{dx}{dt} - 3x\right) + x$$

$$\Rightarrow \frac{d^2x}{dt^2} = 4\frac{dx}{dt} - 4x$$

$$\Rightarrow \frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = 0$$

AUXILIARY EQUATION FOR THE ABOVE O.D.E

$$\Rightarrow \lambda^2 - 4\lambda + 4 = 0$$

$$\Rightarrow (\lambda - 2)^2 = 0$$

$$\Rightarrow \lambda = 2 \quad \text{(REPEATED)}$$

GENERAL SOLUTION FOR $x = f(t)$

$$\Rightarrow x = f(t) = Ae^{2t} + Bte^{2t}$$

$$\Rightarrow x = f(t) = e^{2t}(A + Bt)$$

APPLY CONDITION $t=0$, $x=1$ YIELDS $A=1$

$$\Rightarrow \boxed{x = f(t) = e^{2t}(1 + Bt)}$$

NOW DIFFERENTIATE $x$ & SUB INTO THE FIRST O.D.E

$$\Rightarrow \frac{dx}{dt} = 2e^{2t}(1+Bt) + Be^{2t} = e^{2t}(2 + B + 2Bt)$$

$$\Rightarrow \underset{\frac{dx}{dt}}{e^{2t}(2+B+2Bt)} = \underset{x}{e^{2t}(1+Bt)} + \underset{\frac{2}{3}y}{\frac{2}{3}y}$$

$$\Rightarrow \frac{2}{3}y = e^{2t}(2+B+2Bt) - e^{2t}(1+Bt)$$

$$\Rightarrow \frac{2}{3}y = e^{2t}(1 + B + Bt)$$

$$\Rightarrow \boxed{y = \frac{3}{2}e^{2t}(1 + B + Bt)}$$

FINALLY APPLY THE CONDITION, $t=0$ $y=3$

$$\Rightarrow 3 = \frac{3}{2}(1+B)$$

$$\Rightarrow 2 = B + 1$$

$$\Rightarrow B = 1$$

$$x = f(t) = e^{2t}(1+t)$$

$$y = g(t) = \frac{3}{2}e^{2t}(2+t)$$

**Question 11** (****)

$$\frac{dx}{dt}+y=\mathrm{e}^{-t} \quad \text{and} \quad \frac{dy}{dt}-x=\mathrm{e}^{t}.$$

Given that $x=0$, $y=0$ at $t=0$, solve the differential equations to obtain simplified expressions for $x=f(t)$ and $y=g(t)$.

$\boxed{\phantom{xxxx}}$, $\boxed{x=-\cosh t+\sin t+\cos t,\quad y=\cosh t+\sin t-\cos t}$

DIFFERENTIATE EITHER OF THE TWO EQUATIONS WITH RESPECT TO $t$, REARRANGE AND SUBSTITUTE INTO THE OTHER

$$\frac{d}{dt}\left(\frac{dx}{dt}+y\right)=\frac{d}{dt}\left(e^{-t}\right)$$
$$\frac{d^2x}{dt^2}+\frac{dy}{dt}=-e^{-t}$$
$$\frac{d^2x}{dt^2}+\left[x+e^{t}\right]=-e^{-t}$$
$$\frac{d^2x}{dt^2}+x=-e^{t}-e^{-t}$$

AUXILIARY EQUATION IS $\lambda^2+1=0$, WHICH YIELDS $\lambda=\pm i$

COMPLEMENTARY FUNCTION IS $x=A\cos t+B\sin t$

FOR PARTICULAR INTEGRAL, LET $x=Pe^{t}+Qe^{-t}$

$$\frac{dx}{dt}=Pe^{t} \quad Qe^{-t}$$
$$\frac{d^2x}{dt^2}=Pe^{t}+Qe^{-t}$$

SUB INTO THE O.D.E

$$\left(Pe^{t}+Qe^{-t}\right)+\left(Pe^{t}+Qe^{-t}\right)=-e^{t}-e^{-t}$$
$$2Pe^{t}+2Qe^{-t}\equiv -e^{t}-e^{-t}$$
$$\therefore P=Q=-\tfrac{1}{2}$$

$\therefore$ GENERAL SOLUTION IS

$$x=A\cos t+B\sin t-\tfrac{1}{2}e^{t}-\tfrac{1}{2}e^{-t}$$
$$x=A\cos t+B\sin t-\cosh t$$

APPLY CONDITIONS $x=0$ $t=0$ YIELDS

$$0=A-1$$
$$A=1$$

$$\therefore x=\cos t+B\sin t-\cosh t$$

DIFFERENTIATE WITH RESPECT TO $t$

$$\frac{dx}{dt}=-\sin t+B\cos t-\sinh t$$
$$-y+e^{-t}=-\sin t+B\cos t-\sinh t$$

APPLY CONDITION $t=0$, $y=0$

$$0+1=0+B-0$$
$$B=1$$

$$\therefore x=\cos t+\sin t-\cosh t$$

FINALLY REARRANGING

$$y=e^{-t}-\frac{dx}{dt} \Rightarrow y=e^{-t}-\frac{d}{dt}\left[\cos t+\sin t-\cosh t\right]$$
$$\Rightarrow y=e^{-t}-\left[-\sin t+\cos t-\sinh t\right]$$
$$\Rightarrow y=e^{-t}+\sin t-\cos t+\sinh t$$
$$\Rightarrow y=e^{-t}+\sin t-\cos t+\tfrac{1}{2}e^{t}-\tfrac{1}{2}e^{-t}$$
$$\Rightarrow y=\sin t-\cos t+\tfrac{1}{2}e^{t}+\tfrac{1}{2}e^{-t}$$
$$\Rightarrow y=\sin t-\cos t+\cosh t$$

**Question 12** (****)

$$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=16+32e^{2x},$$

with $y=8$ and $\frac{dy}{dx}=0$ at $x=0$.

Show that the solution of the above differential equation is

$$y=8\cosh^2 x.$$

$\boxed{\text{proof}}$

$$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=16+32e^{2x}$$

- AUX EQUATION

$\lambda^2+4\lambda+4=0$

$(\lambda-2)^2=0$

$\lambda=-2$ (REPEATED)

$\therefore\ y=Ae^{-2x}+Bxe^{-2x}$

- PARTICULAR INTEGRAL

TRY $y=P+Qe^{2x}$

$\frac{dy}{dx}=2Qe^{2x}$

$\frac{d^2y}{dx^2}=4Qe^{2x}$

SUB INTO THE O.D.E.

$4Qe^{2x}+4(2Qe^{2x})+4(P+Qe^{2x})\equiv 16+32e^{2x}$

$4P+16Qe^{2x}\equiv 16+32e^{2x}$

$\boxed{P=4}$

$\boxed{Q=2}$

$\therefore$ GEN SOLUTION IS $y=Ae^{-2x}+Bxe^{-2x}+2e^{2x}+4$

NOW $\frac{dy}{dx}=-2Ae^{-2x}+Be^{-2x}-2Bxe^{-2x}+4e^{2x}$

APPLY CONDITIONS

$x=0\ y=8 \Rightarrow 8=A+2+4 \Rightarrow \boxed{A=2}$

$x=0\ \frac{dy}{dx}=0 \Rightarrow 0=-2A+B+4$

$2A-4=B$

$\boxed{B=0}$

THUS $y=2e^{-2x}+2e^{2x}+4$

$\Rightarrow y=4\left(\frac{1}{2}e^{2x}+\frac{1}{2}e^{-2x}\right)+4$

$\Rightarrow y=4\cosh 2x+4$

NOW $\cos 2A=2\cos^2 A-1$

$\cosh 2A=2\cosh^2 A-1$

$\Rightarrow y=4\left[2\cosh^2 x-1\right]+4$

$\Rightarrow y=8\cosh^2 x-4+4$

$\Rightarrow y=8\cosh^2 x$

AS REQUIRED

**Question 13** (****)

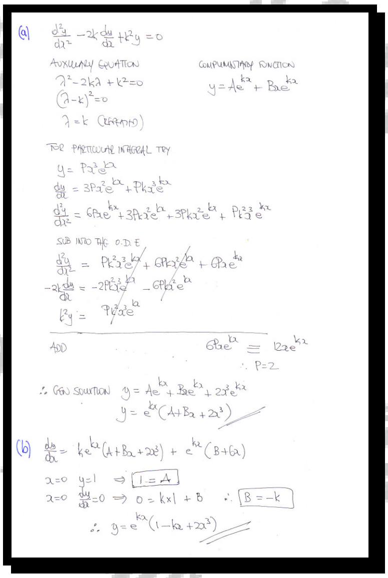
$$\frac{d^2y}{dx^2} - 2k\frac{dy}{dx} + k^2 y = 12x\mathrm{e}^{kx}, \; k > 0$$

**a)** Find a general solution of the differential equation given that $y = Px^3\mathrm{e}^{kx}$, where $P$ is a constant, is part of the solution.

**b)** Given further that $y = 1$, $\frac{dy}{dx} = 0$ at $x = 0$ show that

$$y = \mathrm{e}^{kx}\left(2x^3 - kx + 1\right).$$

$$\boxed{y = \mathrm{e}^{kx}\left(2x^3 + Ax + B\right)}$$


(a) $\frac{d^2y}{dx^2} - 2k\frac{dy}{dx} + k^2y = 0$

AUXILIARY EQUATION

$\lambda^2 - 2k\lambda + k^2 = 0$

$(\lambda - k)^2 = 0$

$\lambda = k$ (REPEATED)

COMPLEMENTARY FUNCTION

$y = Ae^{kx} + Bxe^{kx}$

FOR PARTICULAR INTEGRAL TRY

$y = Px^3e^{kx}$

$\frac{dy}{dx} = 3Px^2e^{kx} + Pkx^3e^{kx}$

$\frac{d^2y}{dx^2} = 6Pxe^{kx} + 3Pkx^2e^{kx} + 3Pkx^2e^{kx} + Pk^2x^3e^{kx}$

SUB INTO THE O.D.E

$\frac{d^2y}{dx^2} = \cancel{Pk^2x^3e^{kx}} + \cancel{6Pkx^2e^{kx}} + 6Pxe^{kx}$

$-2k\frac{dy}{dx} = \cancel{-2Pk^2x^3e^{kx}} \cancel{- 6Pkx^2e^{kx}}$

$k^2y = \cancel{Pk^2x^3e^{kx}}$

ADD $\quad 6Pxe^{kx} \equiv 12xe^{kx}$

$\therefore P = 2$

$\therefore$ GEN SOLUTION $y = Ae^{kx} + Bxe^{kx} + 2x^3e^{kx}$

$y = e^{kx}(A + Bx + 2x^3)$

(b) $\frac{dy}{dx} = ke^{kx}(A + Bx + 2x^3) + e^{kx}(B + 6x)$

$x=0 \; y=1 \Rightarrow \boxed{1 = A}$

$x=0 \; \frac{dy}{dx}=0 \Rightarrow 0 = k\times 1 + B \quad \therefore \boxed{B = -k}$

$\therefore y = e^{kx}(1 - kx + 2x^3)$

**Question 14** (****)

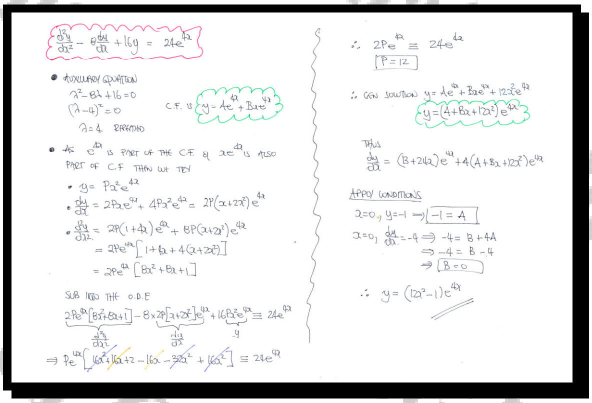
Show that the solution of the differential equation

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 24e^{4x},$$

subject to the boundary conditions $y = -1$, $\frac{dy}{dx} = -4$ at $x = 0$, can be written as

$$y = \left(12x^2 - 1\right)e^{4x}.$$

proof



$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 24e^{4x}$$

- AUXILIARY EQUATION

$\lambda^2 - 8\lambda + 16 = 0$

$(\lambda - 4)^2 = 0$

$\lambda = 4$ REPEATED

C.F. IS $y = Ae^{4x} + Bxe^{4x}$

- AS $e^{4x}$ IS PART OF THE C.F. & $xe^{4x}$ IS ALSO PART OF C.F THEN WE TRY

  - $y = Px^2e^{4x}$
  - $\frac{dy}{dx} = 2Pxe^{4x} + 4Px^2e^{4x} = 2P(x+2x^2)e^{4x}$
  - $\frac{d^2y}{dx^2} = 2P(1+4x)e^{4x} + 8P(x+2x^2)e^{4x}$

$= 2Pe^{4x}\left[1 + 4x + 4(x+2x^2)\right]$

$= 2Pe^{4x}\left[8x^2 + 8x + 1\right]$

SUB INTO THE O.D.E

$$\underbrace{2Pe^{4x}[8x^2+8x+1]}_{\frac{d^2y}{dx^2}} - 8 \times \underbrace{2P[x+2x^2]e^{4x}}_{\frac{dy}{dx}} + 16\underbrace{Px^2e^{4x}}_{y} \equiv 24e^{4x}$$

$$\Rightarrow Pe^{4x}\left[\cancel{16x^2} + \cancel{16x} + 2 - \cancel{16x} - \cancel{32x^2} + \cancel{16x^2}\right] \equiv 24e^{4x}$$

$\therefore 2Pe^{4x} \equiv 24e^{4x}$

$\boxed{P = 12}$

$\therefore$ GEN SOLUTION $y = Ae^{4x} + Bxe^{4x} + 12x^2e^{4x}$

$y = (A + Bx + 12x^2)e^{4x}$

THUS

$\frac{dy}{dx} = (B + 24x)e^{4x} + 4(A + Bx + 12x^2)e^{4x}$

APPLY CONDITIONS

$x = 0,\ y = -1 \Rightarrow \boxed{-1 = A}$

$x = 0,\ \frac{dy}{dx} = -4 \Rightarrow -4 = B + 4A$

$\Rightarrow -4 = B - 4$

$\Rightarrow \boxed{B = 0}$

$\therefore y = (12x^2 - 1)e^{4x}$

**Question 15** (****)

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 4e^{3x}.$$

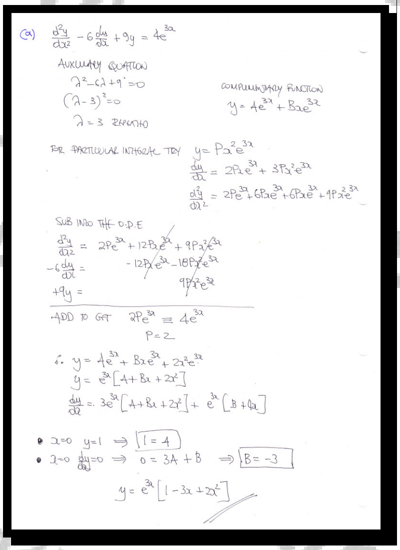
**a)** Find a solution of the differential equation given that $y = 1$, $\frac{dy}{dx} = 0$ at $x = 0$.

**b)** Sketch the graph of $y$.

The sketch must include …

- the coordinates of any points where the graph meets the coordinate axes.
- the coordinates of any stationary points of the curve.
- clear indications of how the graph looks for large positive or negative values of $x$.

$$\boxed{y = e^{3x}\left(2x^2 - 3x + 1\right)}$$



(a) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 4e^{3x}$

AUXILIARY EQUATION

$\lambda^2 - 6\lambda + 9 = 0$

$(\lambda - 3)^2 = 0$

$\lambda = 3$ REPEATED

COMPLEMENTARY FUNCTION

$y = Ae^{3x} + Bxe^{3x}$

FOR PARTICULAR INTEGRAL TRY $y = Px^2e^{3x}$

$\frac{dy}{dx} = 2Pxe^{3x} + 3Px^2e^{3x}$

$\frac{d^2y}{dx^2} = 2Pe^{3x} + 6Pxe^{3x} + 6Pxe^{3x} + 9Px^2e^{3x}$

SUB INTO THE O.D.E

$\frac{d^2y}{dx^2} = 2Pe^{3x} + \cancel{12Pxe^{3x}} + \cancel{9Px^2e^{3x}}$

$-6\frac{dy}{dx} = \cancel{-12Pxe^{3x}} - \cancel{18Px^2e^{3x}}$

$+9y = \cancel{9Px^2e^{3x}}$

ADD TO GET $2Pe^{3x} \equiv 4e^{3x}$

$P = 2$

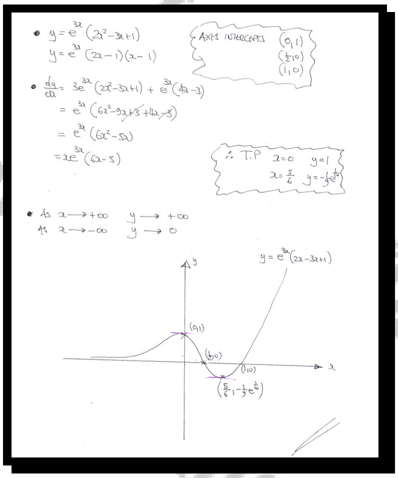
$\therefore y = Ae^{3x} + Bxe^{3x} + 2x^2e^{3x}$

$y = e^{3x}\left[A + Bx + 2x^2\right]$

$\frac{dy}{dx} = 3e^{3x}\left[A + Bx + 2x^2\right] + e^{3x}\left[B + 4x\right]$

- $x=0$ $y=1 \Rightarrow \boxed{1 = A}$
- $x=0$ $\frac{dy}{dx}=0 \Rightarrow 0 = 3A + B \Rightarrow \boxed{B = -3}$

$y = e^{3x}\left[1 - 3x + 2x^2\right]$



- $y = e^{3x}(2x^2 - 3x + 1)$

  $y = e^{3x}(2x - 1)(x - 1)$

AXIS INTERCEPTS $(0,1)$ $(\frac{1}{2},0)$ $(1,0)$

- $\frac{dy}{dx} = 3e^{3x}(2x^2 - 3x + 1) + e^{3x}(4x - 3)$

  $= e^{3x}(6x^2 - 9x \cancel{+3} + 4x \cancel{-3})$

  $= e^{3x}(6x^2 - 5x)$

  $= xe^{3x}(6x - 5)$

$\therefore$ T.P $x = 0$ $y = 1$

$x = \frac{5}{6}$ $y = -\frac{1}{9}e^{\frac{1}{10}}$

- As $x \to +\infty$ $y \to +\infty$

  As $x \to -\infty$ $y \to 0$

**Question 16** (****)

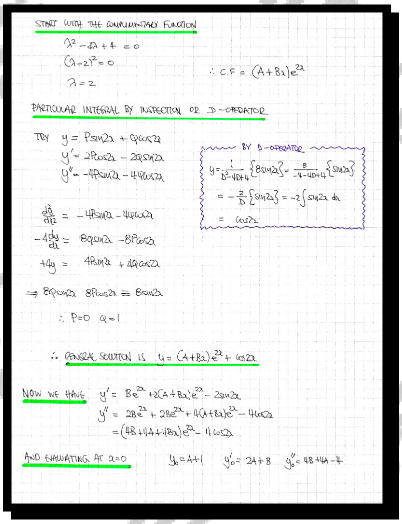
The curve with equation $y = f(x)$ is the solution of the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 8\sin 2x.$$

The first two non zero terms in Maclaurin series expansion of $f(x)$ are $x + kx^2$, where $k$ is a constant.

Determine in any order the value of $k$ and the exact value of $f\left(\frac{1}{4}\pi\right)$.

$\boxed{\phantom{x}}$, $\boxed{k=2}$, $\boxed{f\left(\frac{1}{4}\pi\right) = \frac{1}{4}(3\pi - 4)e^{\frac{1}{2}\pi}}$



START WITH THE COMPLEMENTARY FUNCTION

$\lambda^2 - 4\lambda + 4 = 0$

$(\lambda - 2)^2 = 0$

$\lambda = 2$

$\therefore$ C.F $= (A + Bx)e^{2x}$

PARTICULAR INTEGRAL BY INSPECTION OR D-OPERATOR

TRY $y = P\sin 2x + Q\cos 2x$

$y' = 2P\cos 2x - 2Q\sin 2x$

$y'' = -4P\sin 2x - 4Q\cos 2x$

BY D-OPERATOR

$y = \frac{1}{D^2 - 4D + 4}\{8\sin 2x\} = \frac{8}{-4 - 4D + 4}\{\sin 2x\}$

$= -\frac{2}{D}\{\sin 2x\} = -2\int \sin 2x \, dx$

$= \cos 2x$

$\frac{d^2y}{dx^2} = -4P\sin 2x - 4Q\cos 2x$

$-4\frac{dy}{dx} = 8Q\sin 2x - 8P\cos 2x$

$+4y = 4P\sin 2x + 4Q\cos 2x$

$\Rightarrow 8Q\sin 2x \ 8P\cos 2x \equiv 8\sin 2x$

$\therefore P = 0 \quad Q = 1$

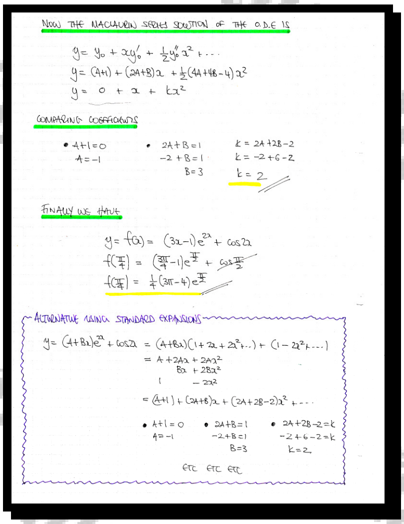
$\therefore$ GENERAL SOLUTION IS $y = (A + Bx)e^{2x} + \cos 2x$

NOW WE HAVE $y' = Be^{2x} + 2(A + Bx)e^{2x} - 2\sin 2x$

$y'' = 2Be^{2x} + 2Be^{2x} + 4(A + Bx)e^{2x} - 4\cos 2x$

$= (4B + 4A + 4Bx)e^{2x} - 4\cos 2x$

AND EVALUATING AT $x = 0$ $\quad y_0 = A + 1 \quad y'_0 = 2A + B \quad y''_0 = 4B + 4A - 4$



NOW THE MACLAURIN SERIES SOLUTION OF THE O.D.E IS

$y = y_0 + xy'_0 + \frac{1}{2}y''_0 x^2 + \dots$

$y = (A + 1) + (2A + B)x + \frac{1}{2}(4A + 4B - 4)x^2$

$y = 0 + x + kx^2$

COMPARING COEFFICIENTS

- $A + 1 = 0$, $A = -1$
- $2A + B = 1$, $-2 + B = 1$, $B = 3$

$k = 2A + 2B - 2$

$k = -2 + 6 - 2$

$k = 2$

FINALLY WE HAVE

$y = f(x) = (3x - 1)e^{2x} + \cos 2x$

$f\left(\frac{\pi}{4}\right) = \left(\frac{3\pi}{4} - 1\right)e^{\frac{\pi}{2}} + \cancel{\cos\frac{\pi}{2}}$

$f\left(\frac{\pi}{4}\right) = \frac{1}{4}(3\pi - 4)e^{\frac{\pi}{2}}$

ALTERNATIVE USING STANDARD EXPANSIONS

$y = (A + Bx)e^{2x} + \cos 2x = (A + Bx)(1 + 2x + 2x^2 + \dots) + (1 - 2x^2 + \dots)$

$= A + 2Ax + 2Ax^2$
$\quad + Bx + 2Bx^2$
$\quad 1 - 2x^2$

$= (A + 1) + (2A + B)x + (2A + 2B - 2)x^2 + \dots$

- $A + 1 = 0$, $A = -1$
- $2A + B = 1$, $-2 + B = 1$, $B = 3$
- $2A + 2B - 2 = k$, $-2 + 6 - 2 = k$, $k = 2$

ETC ETC ETC

# 1 HARD QUESTION

**Question 1** **(****+)**

The function $y = f(x)$ satisfies the following differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 2e^{-x}\left(\sin 2x - 2\cos 2x\right),$$

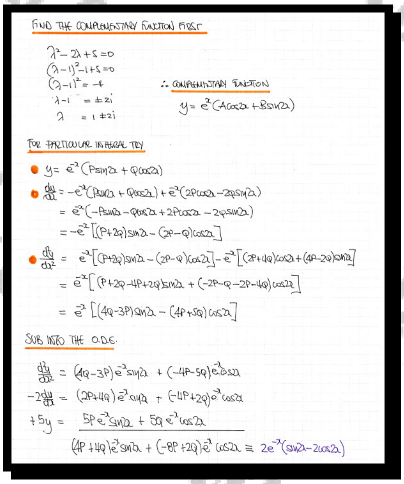
subject to the boundary conditions $y = 0$, $\dfrac{dy}{dx} = 2$ at $x = 0$.

Solve the differential equation to show that

$$y = \cosh x \sin 2x .$$

***No credit will be given for verification methods.***

$\boxed{\text{proof}}$



FIND THE COMPLEMENTARY FUNCTION FIRST

$\lambda^2 - 2\lambda + 5 = 0$
$(\lambda-1)^2 - 1 + 5 = 0$
$(\lambda-1)^2 = -4$
$\lambda - 1 = \pm 2i$
$\lambda = 1 \pm 2i$

∴ COMPLEMENTARY FUNCTION

$y = e^{x}(A\cos 2x + B\sin 2x)$

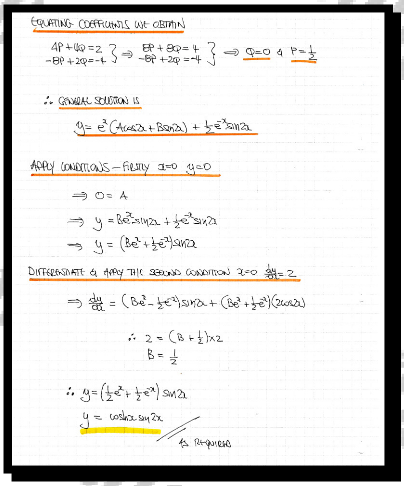
FOR PARTICULAR INTEGRAL TRY

- $y = e^{-x}(P\sin 2x + Q\cos 2x)$
- $\frac{dy}{dx} = -e^{-x}(P\sin 2x + Q\cos 2x) + e^{-x}(2P\cos 2x - 2Q\sin 2x)$
$= e^{-x}(-P\sin 2x - Q\cos 2x + 2P\cos 2x - 2Q\sin 2x)$
$= -e^{-x}\left[(P+2Q)\sin 2x - (2P-Q)\cos 2x\right]$
- $\frac{d^2y}{dx^2} = e^{-x}\left[(P+2Q)\sin 2x - (2P-Q)\cos 2x\right] - e^{-x}\left[(2P+4Q)\cos 2x + (4P-2Q)\sin 2x\right]$
$= e^{-x}\left[(P+2Q-4P+2Q)\sin 2x + (-2P-Q-2P-4Q)\cos 2x\right]$
$= e^{-x}\left[(4Q-3P)\sin 2x - (4P+5Q)\cos 2x\right]$

SUB INTO THE O.D.E.

$\frac{d^2y}{dx^2} = (4Q-3P)e^{-x}\sin 2x + (-4P-5Q)e^{-x}\cos 2x$
$-2\frac{dy}{dx} = (2P+4Q)e^{-x}\sin 2x + (-4P+2Q)e^{-x}\cos 2x$
$+5y = 5Pe^{-x}\sin 2x + 5Qe^{-x}\cos 2x$

$(4P+4Q)e^{-x}\sin 2x + (-8P+2Q)e^{-x}\cos 2x \equiv 2e^{-x}(\sin 2x - 2\cos 2x)$



EQUATING COEFFICIENTS WE OBTAIN

$\left.\begin{array}{l}4P + 4Q = 2\\ -8P + 2Q = -4\end{array}\right\} \Rightarrow \left.\begin{array}{l}8P + 8Q = 4\\ -8P + 2Q = -4\end{array}\right\} \Rightarrow Q = 0 \text{ \& } P = \frac{1}{2}$

∴ GENERAL SOLUTION IS

$y = e^{x}(A\cos 2x + B\sin 2x) + \frac{1}{2}e^{-x}\sin 2x$

APPLY CONDITIONS – FIRSTLY $x=0$ $y=0$

$\Rightarrow 0 = A$
$\Rightarrow y = Be^{x}\sin 2x + \frac{1}{2}e^{-x}\sin 2x$
$\Rightarrow y = \left(Be^{x} + \frac{1}{2}e^{-x}\right)\sin 2x$

DIFFERENTIATE & APPLY THE SECOND CONDITION $x=0$ $\frac{dy}{dx}=2$

$\Rightarrow \frac{dy}{dx} = \left(Be^{x} - \frac{1}{2}e^{-x}\right)\sin 2x + \left(Be^{x} + \frac{1}{2}e^{-x}\right)(2\cos 2x)$

∴ $2 = \left(B + \frac{1}{2}\right) \times 2$
$B = \frac{1}{2}$

∴ $y = \left(\frac{1}{2}e^{x} + \frac{1}{2}e^{-x}\right)\sin 2x$
$y = \cosh x \sin 2x$

AS REQUIRED