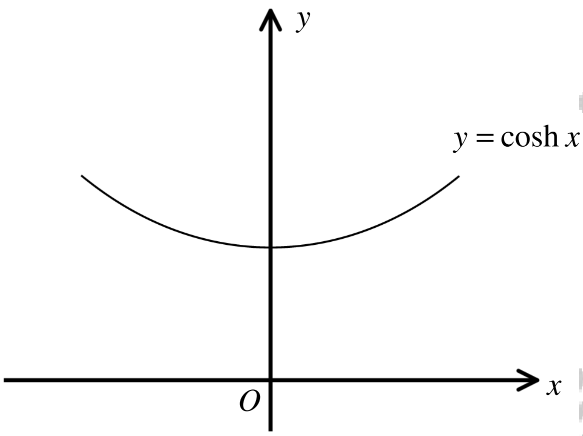
INTEGRATION
ARCLENGTHS
&
SURFACES

ARCLENGTH

Question 1 (**)



The figure above shows the graph of the curve with equation

$$y = \cosh x, \text{ for } -1 \le x \le 1.$$

Find the length of the curve, in terms of e.

$$\boxed{\mathrm{e} - \frac{1}{\mathrm{e}}}$$

$s = \int_{x_1}^{x_2} \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{1}{2}} dx$

$y = \cosh x$
$\frac{dy}{dx} = \sinh x$

HERE

$s = \int_{-1}^{1} \left(1 + \sinh^2 x\right)^{\frac{1}{2}} dx$

$s = \int_{-1}^{1} \cosh x \, dx$ (BUT $\cosh x$ IS EVEN)

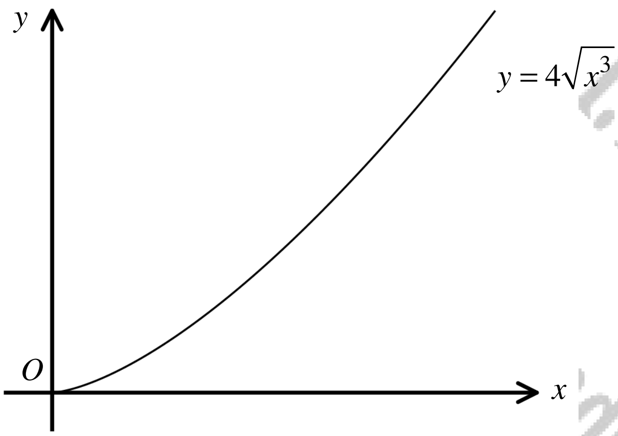
$s = 2\left[\sinh x\right]_0^1$

$s = 2(\sinh 1 - \sinh 0)$

$s = 2 \times \frac{1}{2}(e - e^{-1})$

$s = e - \frac{1}{e}$

Question 2 (**)



The figure above shows the graph of the curve with equation

$$y = 4\sqrt{x^3}\ , \ x \geq 0 .$$

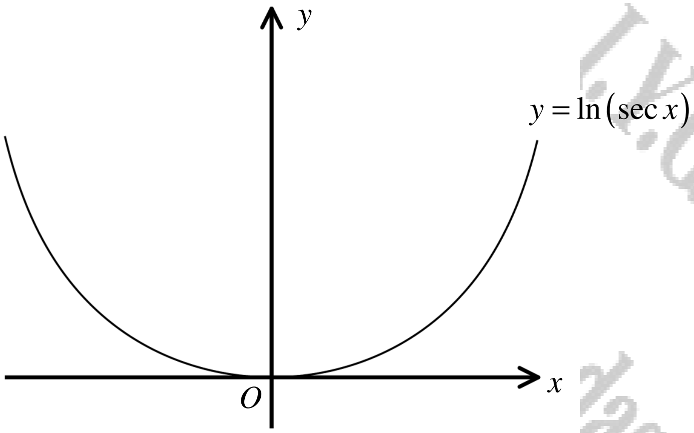
Find the length of the arc of the curve for $0 \leq x \leq 10$.

$\boxed{127}$

$$y = 4\sqrt{x^3} = 4x^{\frac{3}{2}}$$
$$\frac{dy}{dx} = 6x^{\frac{1}{2}}$$
$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + 36x$$

$$\therefore \ s = \int_{x=0}^{x=10} \sqrt{1+36x}\, dx$$
$$\Rightarrow s = \int_0^{36} (1+36x)^{\frac{1}{2}}\, dx$$
$$\Rightarrow s = \left[\frac{1}{54}(1+36x)^{\frac{3}{2}}\right]_0^{10}$$
$$\Rightarrow s = \frac{1}{54}\left[19^3 - 1\right] = 127$$

Question 3 (**+)



The figure above shows the graph of the curve with equation

$$y = \ln(\sec x), \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

Show that the length of the curve for $\frac{\pi}{6} \le x \le \frac{\pi}{3}$ is

$$\ln\left(1 + \frac{2}{3}\sqrt{3}\right).$$

proof

$y = \ln(\sec x)$

$\frac{dy}{dx} = \frac{1}{\cancel{\sec x}} \times \cancel{\sec x} \tan x$

$\frac{dy}{dx} = \tan x$

$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \tan^2 x = \sec^2 x$

$\therefore \ S = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sqrt{\sec^2 x}\ dx$

$\Rightarrow S = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec x \ dx$

$\Rightarrow S = \Big[\ln|\sec x + \tan x|\Big]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$\Rightarrow S = \ln(2+\sqrt{3}) - \ln\left(\frac{2}{\sqrt{3}} + \frac{\sqrt{3}}{3}\right)$

$\Rightarrow S = \ln(2+\sqrt{3}) - \ln(\sqrt{3})$

$\Rightarrow S = \ln\left(\frac{2+\sqrt{3}}{\sqrt{3}}\right)$

$\Rightarrow S = \ln\left(1 + \frac{2}{\sqrt{3}}\right)$

$\Rightarrow S = \ln\left(1 + \frac{2}{3}\sqrt{3}\right)$ // As required

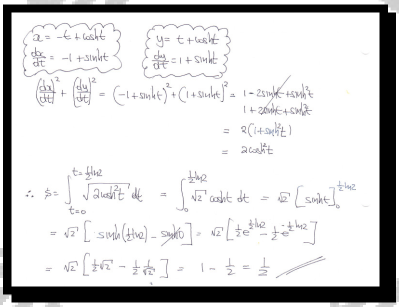
Question 4 **(***)**

A curve is given parametrically by

$$x = -t + \cosh t\,,\ \ y = t + \cosh t\,,\ \ 0 \le t \le \frac{1}{2}\ln 2\,.$$

Show that the length the curve is $\dfrac{1}{2}$.

proof



$x = -t + \cosh t$, $\quad \dfrac{dx}{dt} = -1 + \sinh t$

$y = t + \cosh t$, $\quad \dfrac{dy}{dt} = 1 + \sinh t$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (-1+\sinh t)^2 + (1+\sinh t)^2 = 1 - \cancel{2\sinh t} + \sinh^2 t + 1 + \cancel{2\sinh t} + \sinh^2 t$$

$$= 2(1+\sinh^2 t)$$

$$= 2\cosh^2 t$$

$$\therefore\ S = \int_{t=0}^{t=\frac{1}{2}\ln 2} \sqrt{2\cosh^2 t}\,dt = \int_0^{\frac{1}{2}\ln 2} \sqrt{2}\cosh t\,dt = \sqrt{2}\left[\sinh t\right]_0^{\frac{1}{2}\ln 2}$$

$$= \sqrt{2}\left[\sinh\left(\tfrac{1}{2}\ln 2\right) - \cancel{\sinh 0}\right] = \sqrt{2}\left[\tfrac{1}{2}e^{\frac{1}{2}\ln 2} - \tfrac{1}{2}e^{-\frac{1}{2}\ln 2}\right]$$

$$= \sqrt{2}\left[\tfrac{1}{2}\sqrt{2} - \tfrac{1}{2}\tfrac{1}{\sqrt{2}}\right] = 1 - \frac{1}{2} = \frac{1}{2}$$

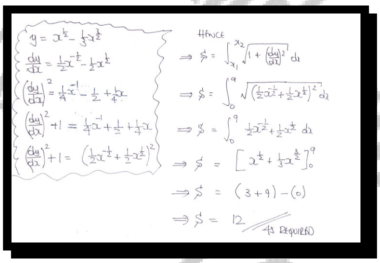
Question 5 **(***+)**

A curve C has equation

$$y = x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}}\,,\quad x \ge 0$$

Show that the length of the arc of C from $A(0,0)$ to $B(9,-6)$ is 12 units.

proof



$y = x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}}$

$\dfrac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{\frac{1}{2}}$

$\left(\dfrac{dy}{dx}\right)^2 = \frac{1}{4}x^{-1} - \frac{1}{2} + \frac{1}{4}x$

$\left(\dfrac{dy}{dx}\right)^2 + 1 = \frac{1}{4}x^{-1} + \frac{1}{2} + \frac{1}{4}x$

$\left(\dfrac{dy}{dx}\right)^2 + 1 = \left(\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{\frac{1}{2}}\right)^2$

HENCE

$$\Rightarrow S = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$\Rightarrow S = \int_0^9 \sqrt{\left(\tfrac{1}{2}x^{-\frac{1}{2}} + \tfrac{1}{2}x^{\frac{1}{2}}\right)^2}\,dx$$

$$\Rightarrow S = \int_0^9 \tfrac{1}{2}x^{-\frac{1}{2}} + \tfrac{1}{2}x^{\frac{1}{2}}\,dx$$

$$\Rightarrow S = \left[x^{\frac{1}{2}} + \tfrac{1}{3}x^{\frac{3}{2}}\right]_0^9$$

$$\Rightarrow S = (3+9) - (0)$$

$$\Rightarrow S = 12$$

AS REQUIRED

Question 6 (***+)

A curve is given parametrically by

$$x = 2\sinh t\,,\ \ y = \cosh^2 t\,,\ \ 0 \le t \le \ln 3\,.$$

Show that the length the curve is exactly

$$\frac{20}{9} + \ln 3\,.$$

$\boxed{\text{proof}}$

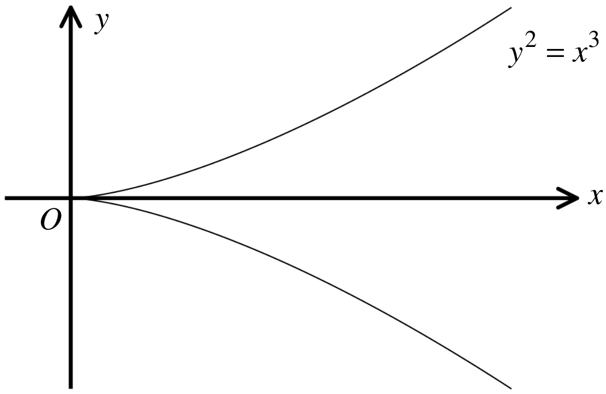
$$\dot{x}^2 + \dot{y}^2 = (2\cosh t)^2 + (2\cosh t \sinh t)^2 = 4\cosh^2 t + 4\cosh^2 t \sinh^2 t$$
$$= 4\cosh^2 t(1 + \sinh^2 t) = 4\cosh^2 t \cosh^2 t = 4\cosh^4 t$$

$\therefore$ IN PARAMETRIC $\quad s = \displaystyle\int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2}\,dt$

$$s = \int_0^{\ln 3} \sqrt{4\cosh^4 t}\,dt = \int_0^{\ln 3} 2\cosh^2 t\,dt$$
$$= \int_0^{\ln 3} 2\left(\tfrac{1}{2} + \tfrac{1}{2}\cosh 2t\right)dt = \int_0^{\ln 3} 1 + \cosh 2t\,dt$$
$$= \left[t + \tfrac{1}{2}\sinh 2t\right]_0^{\ln 3} = \left[\ln 3 + \tfrac{1}{2}\sinh(2\ln 3)\right] - [0]$$
$$= \ln 3 + \tfrac{1}{2}\sinh(\ln 9) = \ln 3 + \tfrac{1}{4}\left[e^{\ln 9} - e^{-\ln 9}\right]$$
$$= \ln 3 + \tfrac{1}{4}\left(9 - \tfrac{1}{9}\right) = \ln 3 + \tfrac{1}{4} \times \tfrac{80}{9}$$
$$= \ln 3 + \tfrac{20}{9}$$

AS REQUIRED

Question 7 **(***+)**



A curve C has equation by $y^2 = x^3$ and its graph is shown in the figure above.

Show that the length of the arc of C from $A\left(5,-5\sqrt{5}\right)$ to $B\left(5,5\sqrt{5}\right)$ is exactly $\dfrac{670}{27}$.

$\boxed{\text{proof}}$

$y^2 = x^3$

$2y\dfrac{dy}{dx} = 3x^2$

$4y^2\left(\dfrac{dy}{dx}\right)^2 = 9x^4$

$4x^3\left(\dfrac{dy}{dx}\right)^2 = 9x^4$

$\left(\dfrac{dy}{dx}\right)^2 = \dfrac{9}{4}x$

$1+\left(\dfrac{dy}{dx}\right)^2 = 1+\dfrac{9}{4}x$

$s = 2\int_0^5 \sqrt{1+\frac{9}{4}x}\,dx$

BY SYMMETRY SINCE THERE ARE TWO CURVES
$y = +\sqrt{x^3}$
$y = -\sqrt{x^3}$

$s = 2\int_0^5 \left(1+\frac{9}{4}x\right)^{\frac{1}{2}}dx$

$s = \left[2\times\frac{8}{27}\left(1+\frac{9}{4}x\right)^{\frac{3}{2}}\right]_0^5$

$s = \frac{16}{27}\left[\left(1+\frac{9}{4}x\right)^{\frac{3}{2}}\right]_0^5$

$s = \frac{16}{27}\left[\frac{343}{8}-1\right] = \frac{670}{27}$

Question 8 (****)

A curve C has equation

$$y = \frac{1}{2}\ln\left(\tanh x\right), \quad x \in \mathbb{R}, \ x > 0 .$$

Show that the length of C from the point where $x = \ln 2$ to the point where $x = \ln 4$ is exactly

$$\ln\left(\frac{\sqrt{17}}{4}\right).$$

$\boxed{\text{proof}}$

- $y = \frac{1}{2}\ln(\tanh x)$

$$\frac{dy}{dx} = \frac{1}{2} \times \frac{1}{\tanh x} \times \text{sech}^2 x = \frac{1}{2\tanh x \cosh^2 x} = \frac{1}{2\frac{\sinh x}{\cosh x} \times \cosh^2 x} = \frac{1}{2\sinh x \cosh x}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{\sinh 2x}}$$

- $$s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{\ln 2}^{\ln 4} \sqrt{1 + \frac{1}{\sinh^2 2x}}\, dx = \int_{\ln 2}^{\ln 4} \sqrt{\frac{\sinh^2 2x + 1}{\sinh^2 2x}}\, dx$$

$$= \int_{\ln 2}^{\ln 4} \sqrt{\frac{\cosh^2 2x}{\sinh^2 2x}}\, dx = \int_{\ln 2}^{\ln 4} \coth 2x \, dx = \left[\frac{1}{2}\ln|\sinh 2x|\right]_{\ln 2}^{\ln 4}$$

$$= \frac{1}{2}\left\{\ln(\sinh(\ln 16)) - \ln(\sinh(\ln 4))\right\} =$$

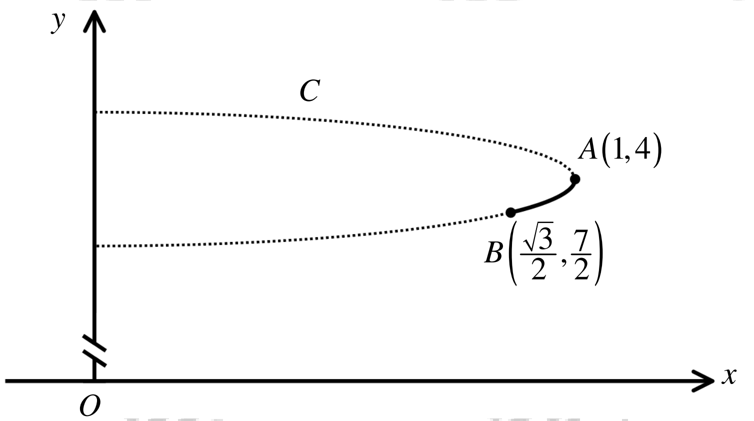
$$= \frac{1}{2}\left\{\ln\left[\frac{1}{2}e^{\ln 16} - \frac{1}{2}e^{-\ln 16}\right] - \ln\left[\frac{1}{2}e^{\ln 4} - \frac{1}{2}e^{-\ln 4}\right]\right\}$$

$$= \frac{1}{2}\left\{\ln\left(8 - \frac{1}{32}\right) - \ln\left(2 - \frac{1}{8}\right)\right\} = \frac{1}{2}\left[\ln\frac{255}{32} - \ln\frac{15}{8}\right]$$

$$= \frac{1}{2}\left[\ln\frac{255}{32} + \ln\frac{8}{15}\right] = \frac{1}{2}\ln\frac{17}{4} = \ln\left(\frac{\sqrt{17}}{2}\right)$$

// AS REQUIRED

Question 9 (****)



The curve C, shown above, is given parametrically by the equations

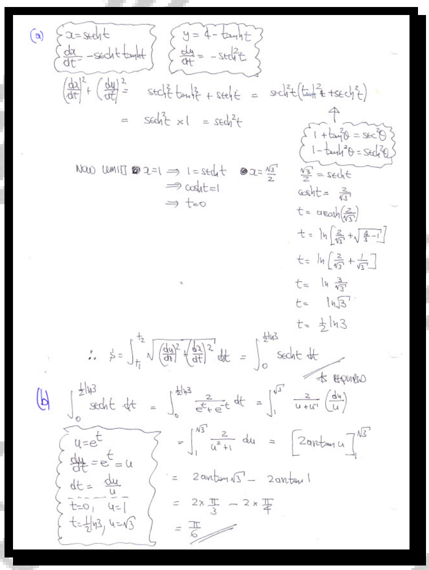
$$x = \operatorname{sech} t, \quad y = 4 - \tanh t, \quad t \in \mathbb{R}$$

a) Show that the length of the arc of C from $A(1,4)$ to $B\left(\frac{\sqrt{3}}{2}, \frac{7}{2}\right)$ is given by

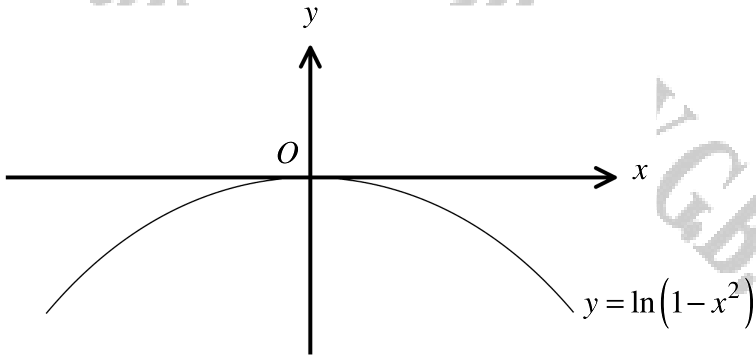
$$s = \int_0^{\frac{1}{2}\ln 3} \operatorname{sech} t \; dx .$$

b) Use the substitution $u = e^t$, to find the exact value of s.

$$\boxed{\frac{\pi}{6}}$$



Question 10 (****)



The figure above shows the graph of the curve with equation

$$y=\ln\left(1-x^2\right),\ \ \frac{1}{2}\le x\le\frac{1}{2}.$$

a) Show that the length s of the curve is given by

$$s=\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1+x^2}{1-x^2}\,dx.$$

b) Hence find the exact length of the curve.

$\boxed{}$, $\boxed{-1+2\ln 3}$

USING THE STANDARD ARCLENGTH FORMULA IN CARTESIAN

$$s=\int_{x_1}^{x_2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$\boxed{\begin{array}{l} y=\ln(1-x^2) \\ \frac{dy}{dx}=\frac{1}{1-x^2}\times(-2x)\end{array}}$$

$$s=\int_{-\frac{1}{2}}^{\frac{1}{2}}\sqrt{1+\left(\frac{-2x}{1-x^2}\right)^2}\,dx$$

$$s=\int_{-\frac{1}{2}}^{\frac{1}{2}}\sqrt{1+\frac{4x^2}{1-2x^2+x^4}}\,dx$$

$$s=\int_{-\frac{1}{2}}^{\frac{1}{2}}\sqrt{\frac{1-2x^2+x^4+4x^2}{1-2x^2+x^4}}\,dx$$

$$s=\int_{-\frac{1}{2}}^{\frac{1}{2}}\sqrt{\frac{x^4+2x^2+1}{x^4-2x^2+1}}\,dx$$

$$s=\int_{-\frac{1}{2}}^{\frac{1}{2}}\sqrt{\frac{(x^2+1)^2}{(x^2-1)^2}}\,dx$$

$$s=\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{x^2+1}{x^2-1}\right|\,dx$$

$$s=\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1+x^2}{1-x^2}\,dx \quad // \text{ AS REQUIRED}$$

TO INTEGRATE THE EXPRESSION, NOTE THAT THE INTEGRAND IS EVEN AND THE DOMAIN IS SYMMETRICAL

$$s=2\int_{0}^{\frac{1}{2}}\frac{1+x^2}{1-x^2}\,dx$$

MANIPULATE THE IMPROPER FRACTION IN THE INTEGRAND

$$s=2\int_{0}^{\frac{1}{2}}\frac{2-(1-x^2)}{(1-x^2)}\,dx=2\int_{0}^{\frac{1}{2}}\frac{2}{1-x^2}-1\,dx.$$

$$s=\int_{0}^{\frac{1}{2}}\frac{4}{(1-x)(1+x)}-2\,dx$$

PARTIAL FRACTIONS BY INSPECTION

$$s=\int_{0}^{\frac{1}{2}}\frac{2}{1-x}+\frac{2}{1+x}-2\,dx$$

$$s=\Big[-2\ln|1-x|+2\ln|1+x|-2x\Big]_{0}^{\frac{1}{2}}$$

$$s=\left(-2\ln\tfrac{1}{2}+2\ln\tfrac{3}{2}-1\right)-\left(\cancel{-2\ln 1}+\cancel{2\ln 1}-0\right)$$

$$s=2\ln\tfrac{3}{2}-\ln\tfrac{1}{2}-1$$

$$s=2\left[\ln\tfrac{3}{2}-\ln\tfrac{1}{2}\right]-1$$

$$s=2\left[\ln\tfrac{3}{2}+\ln 2\right]-1$$

$$s=2\ln 3-1 \quad //$$

Question 11 (****)

$$I(a,x) \equiv \int \sqrt{x^2+a^2}\,dx, \quad x \in \mathbb{R}, \quad a \in \mathbb{R}, \quad x>0.$$

a) Use a suitable hyperbolic substitution to show that

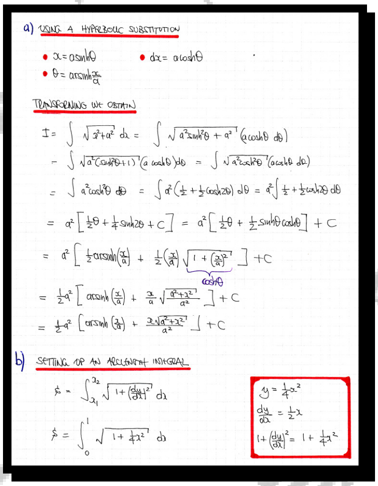
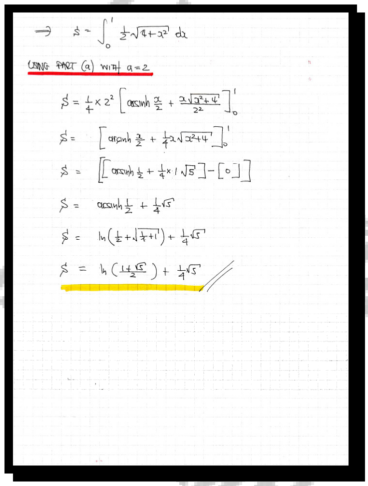
$$I(a,x) = \frac{1}{2}a^2\left[\operatorname{arsinh}\left(\frac{x}{a}\right) + \frac{x\sqrt{x^2+a^2}}{a^2}\right] + \text{constant}.$$

b) Hence find in exact form the length of the curve with equation

$$y = \frac{1}{4}x^2,$$

from the origin O to the point with coordinates $\left(1,\frac{1}{4}\right)$.

$\boxed{s = \frac{1}{4}\sqrt{5} + \ln\left[\frac{1}{2}\left(1+\sqrt{5}\right)\right]}$

Question 12 (****)

A curve has equation

$$y = \ln(1+\cos x), \quad x \in \left[-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi\right]$$

Show that the length this curve is $\ln\left(17+12\sqrt{2}\right)$ units.

$\boxed{}$, $\boxed{\text{proof}}$

PRELIMINARIES FIRST

$$\Rightarrow y = \ln(1+\cos x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\sin x}{\cos x + 1}$$

$$\Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{-\sin x}{\cos x+1}\right)^2 = 1 + \frac{\sin^2 x}{(1+\cos x)^2} = \frac{(1+\cos x)^2 + \sin^2 x}{(1+\cos x)^2}$$

$$= \frac{1 + 2\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)} = \frac{2+2\cos x}{(1+\cos x)^2} = \frac{2(1+\cos x)}{(1+\cos x)^2}$$

$$\Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = \frac{2}{1+\cos x} \quad (\cos x \neq -1)$$

SETTING AN ARCLENGTH INTEGRAL

$$s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\frac{2}{1+\cos x}}\, dx = 2\int_0^{\frac{\pi}{2}} \sqrt{\frac{2}{1+\cos x}}\, dx$$

USING DOUBLE ANGLE IDENTITIES

$$s = 2\int_0^{\frac{\pi}{2}} \sqrt{\frac{2}{1+\left(2\cos^2\frac{x}{2}-1\right)}}\, dx = 2\int_0^{\frac{\pi}{2}} \sqrt{\frac{2}{2\cos^2 \frac{x}{2}}}\, dx$$

$$s = 2\int_0^{\frac{\pi}{2}} \frac{1}{\cos\frac{x}{2}}\, dx = 2\int_0^{\frac{\pi}{2}} \sec\frac{x}{2}\, dx = 2\left[2\ln\left|\sec\frac{x}{2} + \tan\frac{x}{2}\right|\right]_0^{\frac{\pi}{2}}$$

STANDARD RESULT

$$s = 4\left[\ln\left(\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right) - \ln(\sec 0 + \tan 0)\right] = 4\left[\ln(\sqrt{2}+1) - \cancel{\ln(1+0)}\right]$$

$$s = 4\ln(\sqrt{2}+1) = 2\ln(\sqrt{2}+1)^2 = 2\ln(2+2\sqrt{2}+1) = 2\ln(3+2\sqrt{2})$$

REPEATING ONCE MORE

$$s = \ln(3+2\sqrt{2})^2 = \ln(9+12\sqrt{2}+8) = \ln(17+12\sqrt{2})$$

AS REQUIRED

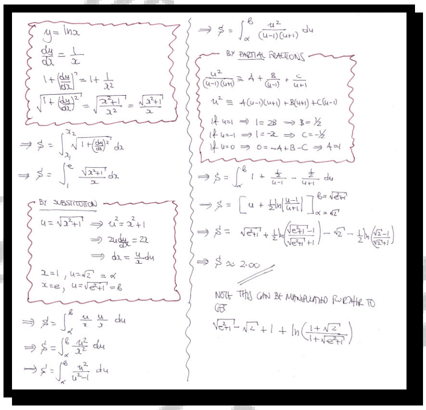
Question 13 (**+)**

Find an exact value for the length of the curve with equation

$$y = \ln x, \quad 1 \le x \le \mathrm{e}.$$

$$\boxed{\sqrt{\mathrm{e}^2+1} - \sqrt{2} + \frac{1}{2}\ln\left(\frac{\sqrt{\mathrm{e}^2+1}-1}{\sqrt{\mathrm{e}^2+1}+1}\right) - \frac{1}{2}\ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right) \approx 2.00}$$

$$\boxed{\sqrt{\mathrm{e}^2+1} - \sqrt{2} + 1 + \ln\left(\frac{1+\sqrt{2}}{1+\sqrt{\mathrm{e}^2+1}}\right) \approx 2.00}$$



$y = \ln x$

$\frac{dy}{dx} = \frac{1}{x}$

$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1}{x^2}$

$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{\frac{x^2+1}{x^2}} = \frac{\sqrt{x^2+1}}{x}$

$\Rightarrow s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$\Rightarrow s = \int_1^{e} \frac{\sqrt{x^2+1}}{x}\, dx$

BY SUBSTITUTION

$u = \sqrt{x^2+1} \Rightarrow u^2 = x^2+1$

$\Rightarrow 2u\frac{du}{dx} = 2x$

$\Rightarrow dx = \frac{u}{x}\, du$

$x = 1,\ u = \sqrt{2} = \alpha$

$x = e,\ u = \sqrt{e^2+1} = \beta$

$\Rightarrow s = \int_\alpha^\beta \frac{u}{x}\frac{u}{x}\, du$

$\Rightarrow s = \int_\alpha^\beta \frac{u^2}{x^2}\, du$

$\Rightarrow s = \int_\alpha^\beta \frac{u^2}{u^2-1}\, du$

$\Rightarrow s = \int_\alpha^\beta \frac{u^2}{(u-1)(u+1)}\, du$

BY PARTIAL FRACTIONS

$\frac{u^2}{(u-1)(u+1)} \equiv A + \frac{B}{(u-1)} + \frac{C}{u+1}$

$u^2 \equiv A(u-1)(u+1) + B(u+1) + C(u-1)$

If $u=1 \Rightarrow 1 = 2B \Rightarrow B = \frac{1}{2}$

If $u=-1 \Rightarrow 1 = -2C \Rightarrow C = -\frac{1}{2}$

If $u=0 \Rightarrow 0 = -A+B-C \Rightarrow A = 1$

$\Rightarrow s = \int_\alpha^\beta 1 + \frac{\frac{1}{2}}{u-1} - \frac{\frac{1}{2}}{u+1}\, du$

$\Rightarrow s = \left[u + \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right|\right]_{\alpha=\sqrt{2}}^{\beta=\sqrt{e^2+1}}$

$\Rightarrow s = \sqrt{e^2+1} + \frac{1}{2}\ln\left(\frac{\sqrt{e^2+1}-1}{\sqrt{e^2+1}+1}\right) - \sqrt{2} - \frac{1}{2}\ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)$

$\Rightarrow s \approx 2.00$

NOTE THIS CAN BE MANIPULATED FURTHER TO GET

$\sqrt{e^2+1} - \sqrt{2} + 1 + \ln\left(\frac{1+\sqrt{2}}{1+\sqrt{e^2+1}}\right)$

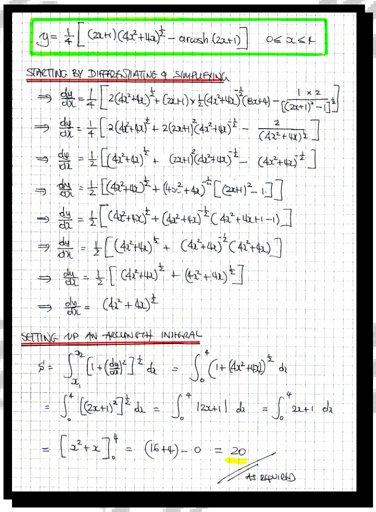
Question 14 **(****+)**

A curve has equation

$$y=\frac{1}{4}\left[(2x+1)\sqrt{4x^2+4x}-\operatorname{arcosh}(2x+1)\right],\quad 0\le x\le 4.$$

Show that the length of the curve is 20 units.

$\boxed{}$, $\boxed{\text{proof}}$



$\boxed{y=\frac{1}{4}\left[(2x+1)(4x^2+4x)^{\frac{1}{2}}-\operatorname{arcosh}(2x+1)\right]\quad 0\le x\le 4}$

STARTING BY DIFFERENTIATING & SIMPLIFYING

$\Rightarrow \frac{dy}{dx}=\frac{1}{4}\left[2(4x^2+4x)^{\frac{1}{2}}+(2x+1)\times\frac{1}{2}(4x^2+4x)^{-\frac{1}{2}}(8x+4)-\frac{1\times 2}{[(2x+1)^2-1]^{\frac{1}{2}}}\right]$

$\Rightarrow \frac{dy}{dx}=\frac{1}{4}\left[2(4x^2+4x)^{\frac{1}{2}}+2(2x+1)^2(4x^2+4x)^{-\frac{1}{2}}-\frac{2}{(4x^2+4x)^{\frac{1}{2}}}\right]$

$\Rightarrow \frac{dy}{dx}=\frac{1}{2}\left[(4x^2+4x)^{\frac{1}{2}}+(2x+1)^2(4x^2+4x)^{-\frac{1}{2}}-(4x^2+4x)^{-\frac{1}{2}}\right]$

$\Rightarrow \frac{dy}{dx}=\frac{1}{2}\left[(4x^2+4x)^{\frac{1}{2}}+(4x^2+4x)^{-\frac{1}{2}}\left[(2x+1)^2-1\right]\right]$

$\Rightarrow \frac{dy}{dx}=\frac{1}{2}\left[(4x^2+4x)^{\frac{1}{2}}+(4x^2+4x)^{-\frac{1}{2}}(4x^2+4x+1-1)\right]$

$\Rightarrow \frac{dy}{dx}=\frac{1}{2}\left[(4x^2+4x)^{\frac{1}{2}}+(4x^2+4x)^{-\frac{1}{2}}(4x^2+4x)\right]$

$\Rightarrow \frac{dy}{dx}=\frac{1}{2}\left[(4x^2+4x)^{\frac{1}{2}}+(4x^2+4x)^{\frac{1}{2}}\right]$

$\Rightarrow \frac{dy}{dx}=(4x^2+4x)^{\frac{1}{2}}$

SETTING UP AN ARCLENGTH INTEGRAL

$s=\int_{x_1}^{x_2}\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}dx=\int_0^4\left(1+(4x^2+4x)\right)^{\frac{1}{2}}dx$

$=\int_0^4\left[(2x+1)^2\right]^{\frac{1}{2}}dx=\int_0^4|2x+1|\,dx=\int_0^4 2x+1\,dx$

$=\left[x^2+x\right]_0^4=(16+4)-0=20$

AS REQUIRED

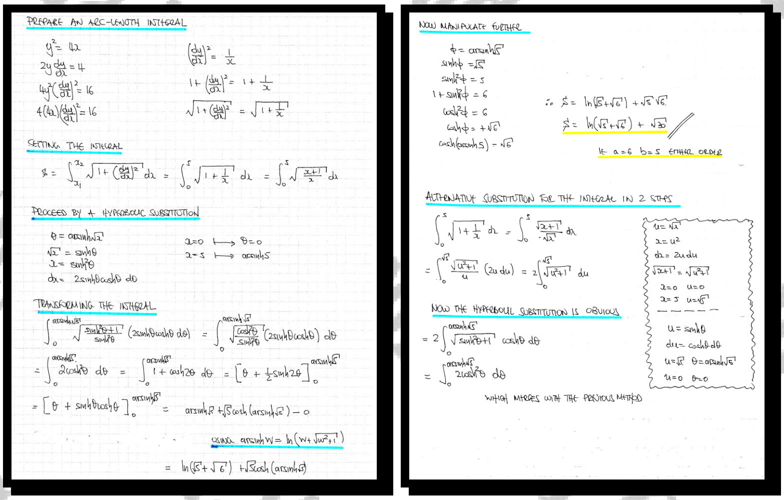
Question 15 (*****)

A parabola has equation

$$y^2 = 4x\,, \quad 0 \le x \le 5$$

Show that the length this parabola is exactly $\ln\left(\sqrt{a}+\sqrt{b}\right)+\sqrt{ab}$ where a and b are positive integers.

$\boxed{}\,, \quad \boxed{(a,b)=(6,5)=(5,6)}$



SURFACES OF REVOLUTION

Question 1 (***)

The part of the curve with equation

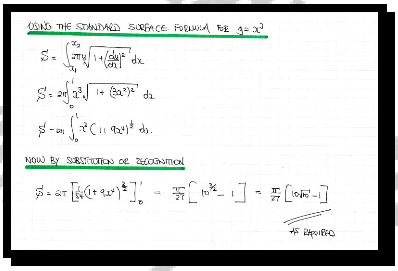
$$y = x^3, \quad 0 \le x \le 1$$

is rotated through 2π radians about the x axis.

Show that the area surface generated is

$$\frac{\pi}{27}\left[10\sqrt{10} - 1\right].$$

$\boxed{}$, $\boxed{\text{proof}}$



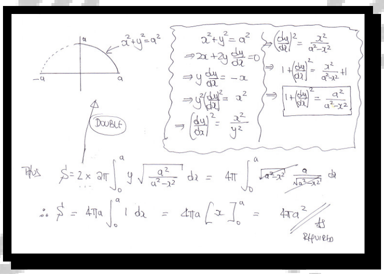
Question 2 (***)

By considering the top half of the circle with equation

$$x^2 + y^2 = a^2, \quad y \ge 0$$

show that the surface generated when the circle's top half is rotated through 2π radians about the x axis has an area of $4\pi a^2$ square units.

$\boxed{\text{proof}}$



Question 3 (***)

A parabola has equation

$$y^2 = 12x \, , \;\; x \geq 0 \, .$$

The arc of the parabola from the point $A(0,0)$ to the point $B(3,6)$ is rotated through 2π radians about the x axis, to form a solid of revolution.

Show clearly that the area of the curved surface of the solid produced is exactly

$$24\pi\left(2\sqrt{2}-1\right).$$

$\boxed{\text{proof}}$

$y^2 = 12x$

$2y\frac{dy}{dx} = 12$

$\frac{dy}{dx} = \frac{6}{y}$

$\left(\frac{dy}{dx}\right)^2 = \frac{36}{y^2}$

$1+\left(\frac{dy}{dx}\right)^2 = 1 + \frac{36}{y^2}$

$1+\left(\frac{dy}{dx}\right)^2 = \frac{y^2+36}{y^2}$

$\boxed{1+\left(\frac{dy}{dx}\right)^2 = \frac{12x+36}{y^2}}$

$S = \int_{x_1}^{x_2} 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$\Rightarrow S = 2\pi \int_0^3 y \sqrt{\frac{12x+36}{y^2}}\, dx$

$\Rightarrow S = 2\pi \int_0^3 \cancel{y} \frac{\sqrt{12x+36}}{\cancel{y}}\, dx$

$\Rightarrow S = 2\pi \int_0^3 2(3x+9)^{\frac{1}{2}}\, dx$

$\Rightarrow S = 2\pi \left[2 \times \frac{2}{9}(3x+9)^{\frac{3}{2}}\right]_0^3$

$\Rightarrow S = 2\pi \times \frac{4}{9}\left[(3x+9)^{\frac{3}{2}}\right]_0^3$

$\Rightarrow S = \frac{8}{9}\pi\left[18^{\frac{3}{2}} - 9^{\frac{3}{2}}\right]$

$\Rightarrow S = \frac{8}{9}\pi\left[\left(\sqrt{18}\right)^3 - 27\right]$

$\Rightarrow S = \frac{8}{9}\pi\left[18\sqrt{18} - 27\right]$

$\Rightarrow S = \frac{8}{9}\pi\left[54\sqrt{3} - 27\right]$

$\Rightarrow S = 24\pi\left(2\sqrt{3}-1\right)$ // As Required

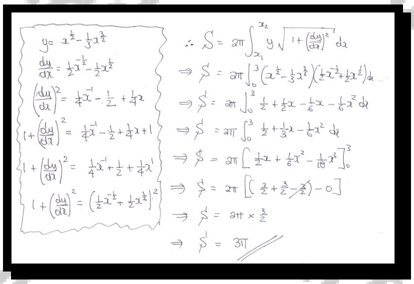
Question 4 (***+)

A curve C has equation given by

$$y = x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}}, \quad x \ge 0 .$$

Show that the area of the surface generated when the arc of C for which $0 \le x \le 3$ is rotated through 2π radians about the x axis is 3π square units.

proof



$y = x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}}$

$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{\frac{1}{2}}$

$\left(\frac{dy}{dx}\right)^2 = \frac{1}{4}x^{-1} - \frac{1}{2} + \frac{1}{4}x$

$1 + \left(\frac{dy}{dx}\right)^2 = \frac{1}{4}x^{-1} - \frac{1}{2} + \frac{1}{4}x + 1$

$1 + \left(\frac{dy}{dx}\right)^2 = \frac{1}{4}x^{-1} + \frac{1}{2} + \frac{1}{4}x^{1}$

$1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{\frac{1}{2}}\right)^2$

$\therefore S = 2\pi \int_{x_1}^{x_2} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$\Rightarrow S = 2\pi \int_0^3 \left(x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}}\right)\left(\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{\frac{1}{2}}\right) dx$

$\Rightarrow S = 2\pi \int_0^3 \frac{1}{2} + \frac{1}{2}x - \frac{1}{6}x - \frac{1}{6}x^2\, dx$

$\Rightarrow S = 2\pi \int_0^3 \frac{1}{2} + \frac{1}{3}x - \frac{1}{6}x^2\, dx$

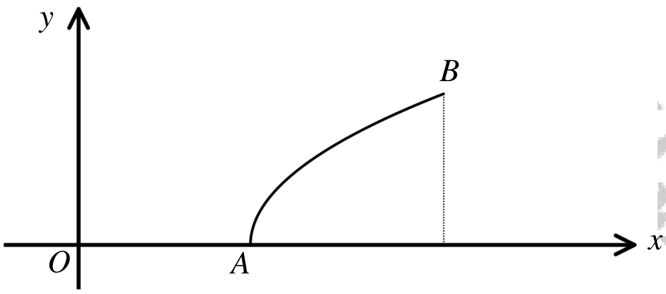
$\Rightarrow S = 2\pi \left[\frac{1}{2}x + \frac{1}{6}x^2 - \frac{1}{18}x^3\right]_0^3$

$\Rightarrow S = 2\pi \left[\left(\frac{3}{2} + \cancel{\frac{3}{2}} - \cancel{\frac{3}{2}}\right) - 0\right]$

$\Rightarrow S = 2\pi \times \frac{3}{2}$

$\Rightarrow S = 3\pi$

Question 5 **(****)**



The figure above shows the curve C, given parametrically by the equations

$$x = \frac{1}{2}\cosh 2t\,, \quad y = 2\sinh t\,, \ t \in \mathbb{R}\,.$$

a) Show that

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 2\cosh^2 t$$

The arc of C from the point $A\left(\frac{1}{2},0\right)$ to the point $B\left(\frac{17}{16},\frac{3}{2}\right)$ is rotated through 2π radians about the x axis.

b) Show that the area of the surface generated is $\frac{61}{24}\pi$ square units.

proof

(a) $\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} = \sqrt{(\sinh 2t)^2+(2\cosh t)^2} = \sqrt{\sinh^2 2t + 4\cosh^2 t}$

$= \sqrt{(2\sinh t\cosh t)^2 + 4\cosh^2 t} = \sqrt{4\cosh^2 t\sinh^2 t + 4\cosh^2 t}$

$= \sqrt{4\cosh^2 t(\sinh^2 t+1)} = \sqrt{4\cosh^2 t\cosh^2 t} = \sqrt{4\cosh^4 t}$

$= 2\cosh^2 t$ // AS REQUIRED

(b) • $y=0$: $t=0$
• $y=\frac{3}{2}$: $\frac{3}{2} = 2\sinh t$, $\sinh t = \frac{3}{4}$, $t = \text{arsinh}\frac{3}{4} = T$

so $\cosh^2 t - \sinh^2 t = 1$, $\cosh^2 t - \frac{9}{16} = 1$, $\cosh^2 t = \frac{25}{16}$, $\cosh t = +\frac{5}{4}$

$S = \int_{t_1}^{t_2} 2\pi y(t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

$S = 2\pi\int_0^T 2\sinh t\,(2\cosh^2 t)\,dt$

$S = 8\pi\int_0^T \sinh t\cosh^2 t\,dt$

$S = \frac{8\pi}{3}\left[\cosh^3 t\right]_0^T$

$\Rightarrow S = \frac{8\pi}{3}\left[\frac{125}{64}-1\right]$

$\Rightarrow S = \frac{8\pi}{3}\times\frac{61}{64}$

$\Rightarrow S = \frac{61}{24}\pi$ //

Question 6 (****)

The curve C has parametric equations

$$x = \cos\theta, \quad y = \ln(\sec\theta + \tan\theta) - \sin\theta, \; 0 \le \theta \le \frac{\pi}{3}$$

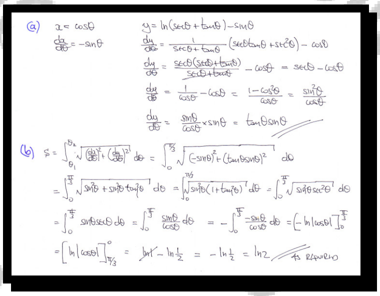
a) Show that

$$\frac{dy}{d\theta} = f(\theta)\,g(\theta),$$

where $f(\theta)$ and $g(\theta)$ are simple trigonometric functions.

b) Hence show that the length of C is $\ln 2$.

$$\boxed{\frac{dy}{d\theta} = \sin\theta\tan\theta}$$



(a) $x = \cos\theta$

$\frac{dx}{d\theta} = -\sin\theta$

$y = \ln(\sec\theta + \tan\theta) - \sin\theta$

$\frac{dy}{d\theta} = \frac{1}{\sec\theta + \tan\theta}(\sec\theta\tan\theta + \sec^2\theta) - \cos\theta$

$\frac{dy}{d\theta} = \frac{\sec\theta(\cancel{\sec\theta + \tan\theta})}{\cancel{\sec\theta + \tan\theta}} - \cos\theta = \sec\theta - \cos\theta$

$\frac{dy}{d\theta} = \frac{1}{\cos\theta} - \cos\theta = \frac{1 - \cos^2\theta}{\cos\theta} = \frac{\sin^2\theta}{\cos\theta}$

$\frac{dy}{d\theta} = \frac{\sin\theta}{\cos\theta} \times \sin\theta = \tan\theta\sin\theta$

(b) $s = \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_0^{\pi/3} \sqrt{(-\sin\theta)^2 + (\tan\theta\sin\theta)^2}\, d\theta$

$= \int_0^{\frac{\pi}{3}} \sqrt{\sin^2\theta + \sin^2\theta\tan^2\theta}\, d\theta = \int_0^{\pi/3} \sqrt{\sin^2\theta(1 + \tan^2\theta)}\, d\theta = \int_0^{\frac{\pi}{3}} \sqrt{\sin^2\theta\sec^2\theta}\, d\theta$

$= \int_0^{\frac{\pi}{3}} \sin\theta\sec\theta\, d\theta = \int_0^{\frac{\pi}{3}} \frac{\sin\theta}{\cos\theta}\, d\theta = -\int_0^{\frac{\pi}{3}} \frac{-\sin\theta}{\cos\theta}\, d\theta = \left[-\ln|\cos\theta|\right]_0^{\frac{\pi}{3}}$

$= \left[\ln|\cos\theta|\right]_{\pi/3}^{0} = \cancel{\ln 1} - \ln\frac{1}{2} = -\ln\frac{1}{2} = \ln 2$ AS REQUIRED

Question 7 (****)

A curve has parametric equations

$$x = t - \tanh t\,, \qquad y = \operatorname{sech} t\,, \qquad 0 \le t < \ln 2\,.$$

Determine, in exact simplified form, the area of the surface of a complete revolution of the curve, about the x axis.

V, $\boxed{\frac{2}{5}\pi}$

USING THE STANDARD FORMULA OF SURFACE OF REVOLUTION ABOUT THE x-AXIS IN PARAMETRIC

$$S = \int_{t_1}^{t_2} 2\pi y(t)\,ds = 2\pi \int_{t_1}^{t_2} 2\pi y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$S = 2\pi \int_0^{\ln 2} \operatorname{sech} t \sqrt{(1-\operatorname{sech}^2 t)^2 + (-\operatorname{sech} t \tanh t)^2}\,dt$$

$$S = 2\pi \int_0^{\ln 2} \operatorname{sech} t \sqrt{1 - 2\operatorname{sech}^2 t + \operatorname{sech}^4 t + \operatorname{sech}^2 t \tanh^2 t}\,dt$$

NOW USING IDENTITIES: $1 + \tan^2\theta \equiv \sec^2\theta \Rightarrow 1 - \tanh^2\theta \equiv \operatorname{sech}^2\theta$

$$S = 2\pi \int_0^{\ln 2} \operatorname{sech} t \sqrt{1 - 2\operatorname{sech}^2 t + \operatorname{sech}^4 t + \operatorname{sech}^2 t (1 - \operatorname{sech}^2 t)}\,dt$$

$$S = 2\pi \int_0^{\ln 2} \operatorname{sech} t \sqrt{1 - 2\operatorname{sech}^2 t \cancel{+ \operatorname{sech}^4 t} + \operatorname{sech}^2 t \cancel{- \operatorname{sech}^4 t}}\,dt$$

$$S = 2\pi \int_0^{\ln 2} \operatorname{sech} t \sqrt{1 - \operatorname{sech}^2 t}\,dt$$

$$S = 2\pi \int_0^{\ln 2} \operatorname{sech} t \sqrt{\tanh^2 t}\,dt$$

$$S = 2\pi \int \operatorname{sech} t \tanh t\,dt$$

INTEGRATING

$$S = 2\pi \Big[-\operatorname{sech} t\Big]_0^{\ln 2} = -2\pi \left[\frac{1}{\cosh t}\right]_0^{\ln 2} = -2\pi \left[\frac{1}{\frac{1}{2}e^{t} + \frac{1}{2}e^{-t}}\right]_0^{\ln 2}$$

$$= -4\pi \left[\frac{1}{e^{\ln 2} + e^{-\ln 2}} - \frac{1}{1+1}\right] = -4\pi \left[\frac{1}{2 + \frac{1}{2}} - \frac{1}{2}\right]$$

$$= -4\pi \left[\frac{2}{5} - \frac{1}{2}\right] = \frac{2}{5}\pi$$

Question 8 (****+)

The curve C has equation given by

$$y^2 = x^2 + 32, \ x \in \mathbb{R}, \ 0 \le x \le 4$$

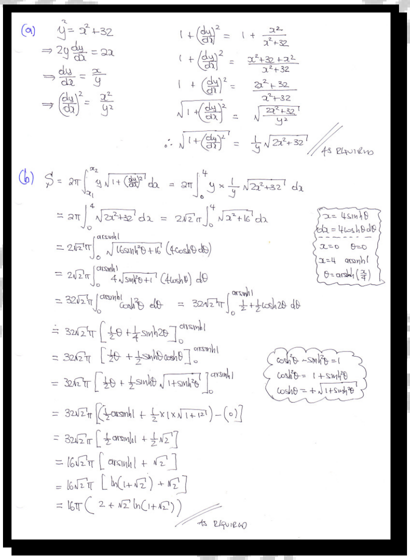
a) Show that

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{\sqrt{2x^2+32}}{y}.$$

b) Hence show further that the area of the surface generated when C is rotated by 2π radians in the x axis is given by

$$16\pi\left[2+\sqrt{2}\ln\left(1+\sqrt{2}\right)\right].$$

$\boxed{\text{proof}}$



(a) $y^2 = x^2 + 32$

$\Rightarrow 2y\frac{dy}{dx} = 2x$

$\Rightarrow \frac{dy}{dx} = \frac{x}{y}$

$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{x^2}{y^2}$

$1+\left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{x^2+32}$

$1+\left(\frac{dy}{dx}\right)^2 = \frac{x^2+32+x^2}{x^2+32}$

$1+\left(\frac{dy}{dx}\right)^2 = \frac{2x^2+32}{x^2+32}$

$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{\frac{2x^2+32}{y^2}}$

$\therefore \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{1}{y}\sqrt{2x^2+32}$ // AS REQUIRED

(b) $S = 2\pi\int_{x_1}^{x_2} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 2\pi\int_0^4 y \times \frac{1}{y}\sqrt{2x^2+32}\,dx$

$= 2\pi\int_0^4 \sqrt{2x^2+32}\,dx = 2\sqrt{2}\pi\int_0^4 \sqrt{x^2+16}\,dx$

$x = 4\sinh\theta$
$dx = 4\cosh\theta\,d\theta$
$x=0 \quad \theta=0$
$x=4 \quad \text{arsinh}1$
$\theta = \text{arsinh}\left(\frac{x}{4}\right)$

$= 2\sqrt{2}\pi\int_0^{\text{arsinh}1} \sqrt{16\sinh^2\theta+16}\,(4\cosh\theta\,d\theta)$

$= 2\sqrt{2}\pi\int_0^{\text{arsinh}1} 4\sqrt{\sinh^2\theta+1}\,(4\cosh\theta)\,d\theta$

$= 32\sqrt{2}\pi\int_0^{\text{arsinh}1} \cosh^2\theta\,d\theta = 32\sqrt{2}\pi\int_0^{\text{arsinh}1} \frac{1}{2}+\frac{1}{2}\cosh2\theta\,d\theta$

$= 32\sqrt{2}\pi\left[\frac{1}{2}\theta+\frac{1}{4}\sinh2\theta\right]_0^{\text{arsinh}1}$

$= 32\sqrt{2}\pi\left[\frac{1}{2}\theta+\frac{1}{2}\sinh\theta\cosh\theta\right]_0^{\text{arsinh}1}$

$\cosh^2\theta - \sinh^2\theta = 1$
$\cosh^2\theta = 1+\sinh^2\theta$
$\cosh\theta = +\sqrt{1+\sinh^2\theta}$

$= 32\sqrt{2}\pi\left[\frac{1}{2}\theta+\frac{1}{2}\sinh\theta\sqrt{1+\sinh^2\theta}\right]_0^{\text{arsinh}1}$

$= 32\sqrt{2}\pi\left[\left(\frac{1}{2}\text{arsinh}1+\frac{1}{2}\times1\times\sqrt{1+1^2}\right)-(0)\right]$

$= 32\sqrt{2}\pi\left[\frac{1}{2}\text{arsinh}1+\frac{1}{2}\sqrt{2}\right]$

$= 16\sqrt{2}\pi\left[\text{arsinh}1+\sqrt{2}\right]$

$= 16\sqrt{2}\pi\left[\ln(1+\sqrt{2})+\sqrt{2}\right]$

$= 16\pi\left(2+\sqrt{2}\ln(1+\sqrt{2})\right)$ // AS REQUIRED

Question 9 (****+)

A curve is defined parametrically by the following equations.

$$x = 2\ln t \ , \quad y = t + \frac{1}{t}, \quad t \in \mathbb{R}, \quad 1 \le t \le 4.$$

The curve is fully revolved about the y axis forming a surface of revolution.

Show that the area of this surface is

$$k\pi\left[-3 + 10\ln 2\right],$$

where k is a positive integer to be found.

$\boxed{\text{FP[illegible]-X}}$, $\boxed{k = 3}$

$$\boxed{x = 2\ln t \qquad y = t + \frac{1}{t} \qquad 1 \leqslant t \leqslant 4}$$

START BY OBTAINING A SIMPLIFIED EXPRESSION FOR THE ARC LENGTH ELEMENT IN PARAMETRIC

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\left(\frac{2}{t}\right)^2 + \left(1 - \frac{1}{t^2}\right)^2}$$

$$= \sqrt{\frac{4}{t^2} + 1 - \frac{2}{t^2} + \frac{1}{t^4}} = \sqrt{1 + \frac{2}{t^2} + \frac{1}{t^4}} = \sqrt{\left(1 + \frac{1}{t^2}\right)^2}$$

$$= 1 + \frac{1}{t^2}$$

SETTING UP A SURFACE OF REVOLUTION INTEGRAL ABOUT THE y AXIS

$$\Rightarrow S = \int_{y_1}^{y_2} 2\pi x \, ds = \int_{t_1}^{t_2} 2\pi x(t)\left[1 + \frac{1}{t^2}\right] dt$$

$$= \int_1^4 2\pi(2\ln t)\left(1 + \frac{1}{t^2}\right) dt = 4\pi \int_1^4 \left(1 + \frac{1}{t^2}\right)\ln t \, dt$$

PROCEED BY INTEGRATION BY PARTS

$\ln t$	$\frac{1}{t}$
$t - \frac{1}{t}$	$1 + \frac{1}{t^2}$

$$\Rightarrow S = 4\pi\left\{\left[\left(t - \frac{1}{t}\right)\ln t\right]_1^4 - \int_1^4 \frac{1}{t}\left(t - \frac{1}{t}\right) dt\right\}$$

$$\Rightarrow S = 4\pi\left\{\left(\frac{15}{4}\ln 4\right) - (0) - \int_1^4 1 - \frac{1}{t^2} \, dt\right\}$$

$$\Rightarrow S = 4\pi\left\{\frac{15}{4}\ln 4 - \left[t + \frac{1}{t}\right]_1^4\right\}$$

$$\Rightarrow S = 4\pi\left\{\frac{15}{4} + \left[t + \frac{1}{t}\right]_4^1\right\}$$

$$\Rightarrow S = 4\pi\left\{\frac{15}{4}\ln 4 + \left(2 - \frac{17}{4}\right)\right\}$$

$$\Rightarrow S = \pi\left[15\ln 4 + 8 - 17\right]$$

$$\Rightarrow S = \pi\left[30\ln 2 - 9\right]$$

$$\Rightarrow S = 3\pi\left[-3 + 10\ln 2\right]$$

Question 10 (****+)

A curve has parametric equations

$$x = \cosh t + t, \quad y = \cosh t - t, \quad t \in \mathbb{R}.$$

The part of the curve, for which $0 \le t \le \ln 2$, is rotated through 2π radians about the x axis.

Show that the exact area of the surface generated is

$$\frac{\pi\sqrt{2}}{16}\left(23 - 8\ln 2\right).$$

[ ], [proof]

$$x = \cosh t + t \qquad y = \cosh t - t \qquad t \in \mathbb{R}$$

START BY FORMING A SIMPLIFIED EXPRESSION FOR THE ARCLENGTH ELEMENT IN PARAMETRIC

$$\frac{dx}{dt} = \sinh t + 1 \qquad \frac{dy}{dt} = \sinh t - 1$$

$$\left(\frac{dx}{dt}\right)^2 = \sinh^2 t + 2\sinh t + 1 \qquad \left(\frac{dy}{dt}\right)^2 = \sinh^2 t - 2\sinh t + 1$$

$$\Rightarrow \frac{ds}{dt} = \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} = \sqrt{2\sinh^2 t + 2} = \sqrt{2(1+\sinh^2 t)}$$

$$\Rightarrow \frac{ds}{dt} = \sqrt{2\cosh^2 t} = \sqrt{2}\cosh t$$

NOW THE SURFACE OF REVOLUTION

$$\Rightarrow S = \int_{t_1}^{t_2} 2\pi y(t)\, ds = 2\pi \int_{t_1}^{t_2} y(t) \frac{ds}{dt}\, dt$$

$$\Rightarrow S = 2\pi \int_0^{\ln 2} (\cosh t - t)\sqrt{2}\cosh t \, dt$$

$$\Rightarrow S = \sqrt{2}\pi \int_0^{\ln 2} 2\cosh^2 t - 2t\cosh t \, dt$$

$\cos 2t \equiv 2\cos^2 t - 1$
$\cosh 2t \equiv 2\cosh^2 t - 1$
$2\cosh^2 t \equiv 1 + \cosh 2t$

BY PARTS

$-2t$	-2
$\sinh t$	$\cosh t$

$$\Rightarrow S = \sqrt{2}\pi\left[\int_0^{\ln 2} 1 + \cosh 2t \, dt - \left[2t\sinh t\right]_0^{\ln 2} + \int_0^{\ln 2} 2\sinh t \, dt\right]$$

$$\Rightarrow S = \sqrt{2}\pi\left[t + \tfrac{1}{2}\sinh 2t - 2t\sinh t + 2\cosh t\right]_0^{\ln 2}$$

$$\Rightarrow S = \sqrt{2}\pi\left[t + \tfrac{1}{2}(2\sinh t\cosh t) - 2t\sinh t + 2\cosh t\right]_0^{\ln 2}$$

$$\Rightarrow S = \sqrt{2}\pi\left[t + \sinh t\cosh t - 2t\sinh t + 2\cosh t\right]_0^{\ln 2}$$

NOW WE HAVE

- $\sinh(\ln 2) = \frac{1}{2}\left(e^{\ln 2} - e^{-\ln 2}\right) = \frac{1}{2}\left(2 - \frac{1}{2}\right) = \frac{3}{4}$
- $\cosh(\ln 2) = \frac{1}{2}\left(e^{\ln 2} + e^{-\ln 2}\right) = \frac{1}{2}\left(2 + \frac{1}{2}\right) = \frac{5}{4}$

FINALLY WE OBTAIN

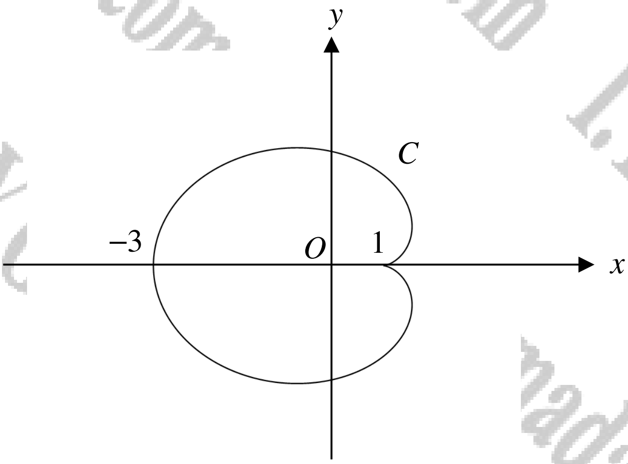
$$\Rightarrow S = \sqrt{2}\pi\left[\left(\ln 2 + \tfrac{3}{4}\times\tfrac{5}{4} - 2\times\tfrac{3}{4}\ln 2 + 2\times\tfrac{5}{4}\right) - (2)\right]$$

$$\Rightarrow S = \sqrt{2}\pi\left[\ln 2 + \tfrac{15}{16} - \tfrac{3}{2}\ln 2 + \tfrac{5}{2} - 2\right]$$

$$\Rightarrow S = \sqrt{2}\pi\left[\tfrac{23}{16} - \tfrac{1}{2}\ln 2\right]$$

$$\Rightarrow S = \frac{\sqrt{2}\pi}{16}\left(23 - 8\ln 2\right)$$

Question 11 (****+)



The figure above shows the cardioid C with parametric equations

$$x = 2\cos\theta - \cos 2\theta\,, \quad y = 2\sin\theta - \sin 2\theta\,, \quad 0 \le \theta < 2\pi\,.$$

The curve is revolved by a full turn in the x axis, forming a surface of revolution.

Find in exact simplified form the area of this surface.

$\boxed{}$, $\boxed{\dfrac{128\pi}{5}}$

START BY FORMING A SIMPLIFIED EXPRESSION FOR THE ARCLENGTH ELEMENT ds

$$\Rightarrow ds = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta = \sqrt{(-2\sin\theta + 2\sin 2\theta)^2 + (2\cos\theta - 2\cos 2\theta)^2}\,d\theta$$

$$\Rightarrow ds = \sqrt{\begin{matrix} 4\sin^2\theta - 8\sin\theta\sin 2\theta + 4\sin^2 2\theta \\ 4\cos^2\theta - 8\cos\theta\cos 2\theta + 4\cos^2 2\theta \end{matrix}}\,d\theta$$

$$\Rightarrow ds = \sqrt{4 - 8(\cos 2\theta\cos\theta + \sin 2\theta\sin\theta) + 4}\,d\theta = \sqrt{8 - 8\cos(2\theta - \theta)}\,d\theta$$

$$\Rightarrow ds = \sqrt{8(1 - \cos\theta)}\,d\theta$$

BY INSPECTION THE PARAMETRIC LIMITS ARE π AT -3 & 0 AT 1 – REVOLVING THE TOP HALF BY A FULL TURN GIVES

$$\Rightarrow S = 2\pi\int_{\theta_1}^{\theta_2} y(\theta)\,ds = 2\pi\int_0^{\pi} (2\sin\theta - \sin 2\theta)\sqrt{8(1 - \cos\theta)}\,d\theta$$

$$\Rightarrow S = 2\pi\int_0^{\pi} (2\sin\theta - 2\sin\theta\cos\theta)\sqrt{8}\,\sqrt{1 - \cos\theta}\,d\theta$$

$$\Rightarrow S = 2\pi\int_0^{\pi} 2\sin\theta(1 - \cos\theta) \times 2\sqrt{2} \times (1 - \cos\theta)^{\frac{1}{2}}\,d\theta$$

$$\Rightarrow S = 2\pi\int_0^{\pi} 4\sqrt{2}\sin\theta(1 - \cos\theta)^{\frac{3}{2}}\,d\theta$$

$$\Rightarrow S = 8\pi\sqrt{2}\int_0^{\pi} \sin\theta(1 - \cos\theta)^{\frac{3}{2}}\,d\theta$$

BY INSPECTION OR SUBSTITUTION

$$\Rightarrow S = 8\pi\sqrt{2}\left[\tfrac{2}{5}(1 - \cos\theta)^{\frac{5}{2}}\right]_0^{\pi}$$

$$\Rightarrow S = \frac{16\pi\sqrt{2}}{5}\left[(1 - \cos\theta)^{\frac{5}{2}}\right]_0^{\pi}$$

$$\Rightarrow S = \frac{16\pi\sqrt{2}}{5}\left[[1 - (-1)]^{\frac{5}{2}} - \cancel{(1 - 1)^{\frac{5}{2}}}\right]$$

$$\Rightarrow S = \frac{16\pi\sqrt{2}}{5} \times 2^{\frac{5}{2}}$$

$$\Rightarrow S = \frac{16\pi}{5} \times 2^{\frac{1}{2}} \times 2^{\frac{5}{2}}$$

$$\Rightarrow S = \frac{16\pi}{5} \times 2^3$$

$$\Rightarrow S = \frac{128\pi}{5}$$

Question 12 (*****)

A curve has parametric equations

$$x = t - \sin t, \quad y = 1 - \cos t, \quad 0 \le t < 2\pi .$$

Determine, in exact simplified form, the area of the surface of a complete revolution of the curve, about the x axis.

V, $\boxed{}$, $\boxed{\dfrac{64\pi}{3}}$

USING THE STANDARD FORMULA FOR SURFACE OF REVOLUTION ABOUT x

$$S = \int_{t_1}^{t_2} 2\pi y(t)\, ds = 2\pi \int_{t_1}^{t_2} y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\Rightarrow S = 2\pi \int_0^{2\pi} (1-\cos t)\sqrt{(1-\cos t)^2 + \sin^2 t}\, dt$$

$$\Rightarrow S = 2\pi \int_0^{2\pi} (1-\cos t)\sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\, dt$$

$$\Rightarrow S = 2\pi \int_0^{2\pi} (1-\cos t)\sqrt{2 - 2\cos t}\, dt$$

$$\Rightarrow S = 2\pi\sqrt{2} \int_0^{2\pi} (1-\cos t)(1-\cos t)^{\frac{1}{2}}\, dt$$

$$\Rightarrow S = 2\pi\sqrt{2} \int_0^{2\pi} (1-\cos t)^{\frac{3}{2}}\, dt$$

NOW USING TRIG IDENTITIES $\cos 2t \equiv 1 - 2\sin^2 t \Rightarrow \cos t \equiv 1 - 2\sin^2\frac{t}{2}$

$$\Rightarrow S = 2\pi\sqrt{2} \int_0^{2\pi} \left[1 - \left(1 - 2\sin^2\tfrac{t}{2}\right)\right]^{\frac{3}{2}} dt$$

$$\Rightarrow S = 2\pi\sqrt{2} \int_0^{2\pi} \left(2\sin^2\tfrac{t}{2}\right)^{\frac{3}{2}} dt$$

$$\Rightarrow S = 2\pi\sqrt{2} \int_0^{2\pi} 2^{\frac{3}{2}} \sin^3\tfrac{t}{2}\, dt$$

$$\Rightarrow S = 8\pi \int_0^{2\pi} \sin^3\tfrac{t}{2}\, dt$$

BY SUBSTITUTION $u = \cos\frac{t}{2}$ OR MANIPULATIONS

$$\Rightarrow S = 8\pi \int_0^{2\pi} \sin\tfrac{t}{2}\sin^2\tfrac{t}{2}\, dt$$

$$\Rightarrow S = 8\pi \int_0^{2\pi} \sin\tfrac{t}{2}\left(1 - \cos^2\tfrac{t}{2}\right) dt$$

$$\Rightarrow S = 8\pi \int_0^{2\pi} \sin\tfrac{t}{2} - \sin\tfrac{t}{2}\cos^2\tfrac{t}{2}\, dt$$

$$\Rightarrow S = 8\pi \left[-2\cos\tfrac{t}{2} + \tfrac{2}{3}\cos^3\tfrac{t}{2}\right]_0^{2\pi}$$

$$\Rightarrow S = 8\pi \left[\tfrac{2}{3}\cos^3\tfrac{t}{2} - 2\cos\tfrac{t}{2}\right]_0^{2\pi}$$

$$\Rightarrow S = 8\pi \left[\left(-\tfrac{2}{3} + 2\right) - \left(\tfrac{2}{3} - 2\right)\right]$$

$$\Rightarrow S = 8\pi\left(4 - \tfrac{4}{3}\right)$$

$$\Rightarrow S = \frac{64\pi}{3}$$

Question 13 (*****)

Gabriel's horn is the geometric figure which is formed by revolving the graph of

$$y = \frac{1}{x}, \quad x \in [1, \infty),$$

by 2π radians about the x axis.

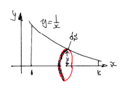
Gabriel's horn gives rise to the "Painter's Paradox", that the "horn" could be filled with a finite quantity of paint and yet that paint would not be sufficient to coat its inner or outer surface.

> Use calculus to verify the validity of the apparent paradox, however you need **not** resolve the flaw in the paradox.

You must show any limiting processes and further advised NOT to find $\displaystyle\int \frac{\sqrt{1+x^2}}{x^2}\,dx$.

$\boxed{}$, $\boxed{\text{proof}}$

BEST TO FIND EXPRESSIONS FOR THE VOLUME & SURFACE AREA BETWEEN $x=1$ & $x=k>1$



$$V = \pi\int_{x_1}^{x_2} [y(x)]^2\,dx$$

$$V = \pi\int_1^k \frac{1}{x^2}\,dx = \pi\left[-\frac{1}{x}\right]_1^k$$

$$V = \pi\left[\frac{1}{x}\right]_k^1 = \pi\left[1-\frac{1}{k}\right]$$

$$\therefore \boxed{V = \pi\left(1-\frac{1}{k}\right)}$$

NEXT THE SURFACE AREA (INSIDE THE HORN OR OUTSIDE)

$$A = 2\pi\int_{x_1}^{x_2} y\,ds = 2\pi\int_{x_1}^{x_2} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 2\pi\int_1^k \left(\frac{1}{x}\right)\sqrt{1+\left(-\frac{1}{x^2}\right)^2}\,dx$$

$$A = 2\pi\int_1^k \frac{1}{x}\sqrt{1+\frac{1}{x^2}}\,dx = 2\pi\int_1^k \frac{1}{x}\sqrt{\frac{x^2+1}{x^2}}\,dx = 2\pi\int_1^k \frac{\sqrt{x^2+1}}{x^2}\,dx$$

NOW THIS INTEGRAL IS VERY STRAIGHT FORWARD BUT LIMITING AS $k\to\infty$ IS VERY HARD — SO WE CAN ARGUE AS FOLLOWS

$$\Longrightarrow (x^2+1)^{\frac{1}{2}} > x \qquad [\text{IF } x \geqslant 1]$$

$$\Longrightarrow \frac{(x^2+1)^{\frac{1}{2}}}{x^2} > \frac{x}{x^2}$$

$$\Longrightarrow \frac{(x^2+1)^{\frac{1}{2}}}{x^2} > \frac{1}{x}$$

$$\therefore \int_1^k \frac{x^2+1}{x^2}\,dx > \int_1^k \frac{1}{x}\,dx$$

THUS WE NOW HAVE

$$2\pi\int_1^k \frac{\sqrt{1+x^2}}{x^2}\,dx > 2\pi\int_1^k \frac{1}{x}\,dx$$

$$A > 2\pi\left[\ln x\right]_1^k$$

$$A > 2\pi\left[\ln k - \cancel{\ln 1}\right]$$

$$A > 2\pi\ln k$$

COLLECT ALL THE RESULTS FOR VOLUME & SURFACE AREA

$$V = \pi\left(1-\frac{1}{k}\right) \qquad A > 2\pi\ln k$$

AS $k\to\infty$, $\frac{1}{k}\to 0$, BUT $\ln k$ DIVERGES

$\therefore$ VOLUME OF THE "HORN" IS π

$\therefore$ AREA IS GREATER THAN $2\pi\ln k$, WHICH DIVERGES

$\therefore$ FINITE VOLUME BUT INFINITE AREA

Question 14 (*****)

The part of the graph of the exponential curve

$$y = e^x \ , \ \ln\left(\frac{3}{4}\right) \le x \le \ln\left(\frac{4}{3}\right),$$

is rotated by 2π radians in the x axis, forming a surface of revolution S.

Show that area of S is

$$\pi\left[\frac{185}{144} + \ln\left(\frac{3}{2}\right)\right].$$

$\boxed{\text{proof}}$

USING THE STANDARD "SURFACE OF REVOLUTION FORMULA"

$$S = 2\pi \int_{x_1}^{x} y \, ds = 2\pi \int_{x_1}^{x_2} y \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx$$

$$= 2\pi \int_{\ln\frac{3}{4}}^{\ln\frac{4}{3}} (e^x)\left[1 + (e^x)^2\right]^{\frac{1}{2}} dx = 2\pi \int_{\ln\frac{3}{4}}^{\ln\frac{4}{3}} e^x(1+e^{2x})^{\frac{1}{2}} dx$$

BY SUBSTITUTION NEXT EITHER $e^x = \sinh\theta$ (OR $u = e^x$)

- $e^x = \sinh\theta$
  $e^x dx = \cosh\theta \, d\theta$
  $\sinh\theta \, dx = \cosh\theta \, d\theta$
  $dx = \frac{\cosh\theta}{\sinh\theta} d\theta$
- $x = \ln\frac{4}{3}$
  $\sinh\theta = e^{\ln\frac{4}{3}}$
  $\sinh\theta = \frac{4}{3}$
  $\theta = \text{arsinh}\frac{4}{3}$
- $x = \ln\frac{3}{4}$
  $\sinh\theta = e^{\ln\frac{3}{4}}$
  $\sinh\theta = \frac{3}{4}$
  $\theta = \text{arsinh}\frac{3}{4}$

TRANSFORMING THE INTEGRAL

$$= 2\pi \int_{\text{arsinh}\frac{3}{4}}^{\text{arsinh}\frac{4}{3}} (\cancel{\sinh\theta})\left[1 + \sinh^2\theta\right]^{\frac{1}{2}}\left(\frac{\cosh\theta}{\cancel{\sinh\theta}} d\theta\right)$$

$$= 2\pi \int_{\text{arsinh}\frac{3}{4}}^{\text{arsinh}\frac{4}{3}} \cosh^2\theta \, d\theta$$

$$= 2\pi \int_{\text{arsinh}\frac{3}{4}}^{\text{arsinh}\frac{4}{3}} \frac{1}{2} + \frac{1}{2}\cosh 2\theta \, d\theta$$

$$= 2\pi \left[\frac{1}{2}\theta + \frac{1}{4}\sinh 2\theta\right]_{\text{arsinh}\frac{3}{4}}^{\text{arsinh}\frac{4}{3}}$$

$$= \pi \left[\theta + \sinh\theta\cosh\theta\right]_{\text{arsinh}\frac{3}{4}}^{\text{arsinh}\frac{4}{3}}$$

NOW IF $\sinh\theta = \frac{3}{4}$

$\sinh^2\theta = \frac{9}{16}$

$1 + \sinh^2\theta = 1 + \frac{9}{16} = \frac{25}{16}$

$\cosh^2\theta = \frac{25}{16}$

$\cosh\theta = +\frac{5}{4}$ & SIMILARLY $\sinh\theta = \frac{4}{3}$

$\cosh\theta = +\frac{5}{3}$

$$\ldots = \pi\left[\theta + \sinh\theta\cosh\theta\right]_{\text{arsinh}\frac{3}{4},\ \cosh\theta = \frac{5}{4}}^{\text{arsinh}\frac{4}{3},\ \cosh\theta = \frac{5}{3}}$$

$$= \pi\left[\left(\text{arsinh}\frac{4}{3} + \frac{4}{3}\times\frac{5}{3}\right) - \left(\text{arsinh}\frac{3}{4} + \frac{3}{4}\times\frac{5}{4}\right)\right]$$

$$= \pi\left[\text{arsinh}\frac{4}{3} + \frac{20}{9} - \text{arsinh}\frac{3}{4} - \frac{15}{16}\right]$$

$$= \pi\left[\text{arsinh}\frac{4}{3} - \text{arsinh}\frac{3}{4} + \frac{20}{9} - \frac{15}{16}\right]$$

$$= \pi\left[\ln\left(\frac{4}{3} + \sqrt{\frac{16}{9}+1}\right) - \ln\left(\frac{3}{4} + \sqrt{\frac{9}{16}+1}\right) + \frac{20\times16 - 15\times9}{9\times16}\right]$$

$$= \pi\left[\ln\left(\frac{4}{3}+\frac{5}{3}\right) - \ln\left(\frac{3}{4}+\frac{5}{4}\right) + \frac{185}{144}\right]$$

$$= \pi\left[\ln 3 - \ln 2 + \frac{185}{144}\right]$$

$$= \pi\left[\frac{185}{144} + \ln\frac{3}{2}\right]$$

Question 15 (*****)

The part of the curve with equation

$$y = \sin 2x\,, \quad 0 \le x \le \frac{\pi}{2}$$

is rotated by $360°$ about the x axis.

Show that the area of the surface generated is

$$\pi\left[\frac{1}{2}\ln\left(2+\sqrt{5}\right)+\sqrt{5}\right].$$

$\boxed{}$, $\boxed{\text{proof}}$

OBTAIN ALL THE AUXILIARIES FOR THE PROBLEM

$$y = \sin 2x$$

$$\frac{dy}{dx} = 2\cos 2x$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+4\cos^2 2x}$$

SETTING UP AN EXPRESSION FOR THE SURFACE

$$S = \int_{x_1}^{x_2} 2\pi y(x)\, ds = 2\pi \int_{x_1}^{x_2} y(x)\frac{ds}{dx}\, dx$$

$$S = 2\pi \int_0^{\frac{\pi}{2}} (\sin 2x)\sqrt{1+4\cos^2 2x}\, dx$$

PROCEED BY A HYPERBOLIC SUBSTITUTION

$$2\cos 2x = \sinh\theta \implies \theta = \text{arsinh}(2\cos 2x)$$

$$-4\sin 2x\, dx = \cosh\theta\, d\theta$$

$$dx = \frac{\cosh\theta}{-4\sin 2x}$$

$$x = \frac{\pi}{2} \longmapsto \theta = \text{arsinh}(-2) = -\text{arsinh}\,2$$

$$x = 0 \longmapsto \theta = \text{arsinh}\,2$$

TRANSFORMING THE INTEGRAL

$$\implies S = 2\pi \int_{\text{arsinh}2}^{-\text{arsinh}2} \cancel{(\sin 2x)}\sqrt{1+\sinh^2\theta}\,\frac{\cosh\theta}{-4\cancel{\sin 2x}}\, d\theta$$

$$\implies S = 2\pi \int_{\text{arsinh}(-2)}^{\text{arsinh}2} +\frac{1}{4}\cosh\theta\cosh\theta\, d\theta$$

EVEN INTEGRAND IN A SYMMETRICAL DOMAIN

$$\implies S = \cancel{4}\pi \int_0^{\text{arsinh}2} \frac{1}{\cancel{4}}\cosh^2\theta\, d\theta$$

$$\implies S = \pi \int_0^{\text{arsinh}2} \frac{1}{2}+\frac{1}{2}\cosh 2\theta\, d\theta$$

$$\implies S = \pi\left[\frac{1}{2}\theta + \frac{1}{4}\sinh 2\theta\right]_0^{\text{arsinh}2}$$

$$\implies S = \pi\left[\frac{1}{2}\theta + \frac{1}{2}\sinh\theta\cosh\theta\right]_0^{\text{arsinh}2}$$

$$\implies S = \pi\left[\frac{1}{2}\theta + \frac{1}{2}\sinh\theta\sqrt{1+\sinh^2\theta}\right]_0^{\text{arsinh}2}$$

$$\implies S = \pi\left[\left(\frac{1}{2}\text{arsinh}\,2 + \cancel{\frac{1}{2}\times 2}\times\sqrt{1+2^2}\right)-0\right]$$

$$\implies S = \pi\left[\frac{1}{2}\ln\left(2+\sqrt{1+2^2}\right)+\sqrt{5}\right]$$

$$\implies S = \pi\left[\frac{1}{2}\ln\left(2+\sqrt{5}\right)+\sqrt{5}\right]$$

Question 16 (*****)

A curve has parametric equations

$$x = 2 + \tanh t\,, \quad y = \operatorname{sech} t\,, \quad t \in \mathbb{R}$$

The part of the curve for which

$$0 \le t \le \ln\left[\frac{\sqrt{2}+\sqrt{6}}{2}\right],$$

is rotated through 2π radians in the x axis.

Show that the exact area of the surface generated is

$$\tfrac{1}{6}\pi\left[4+3\sqrt{3}\right].$$

$\boxed{}$, $\boxed{\text{proof}}$

$$x = 2 + \tanh t \quad \bullet \quad y = \operatorname{sech} t \quad \bullet \quad 0 \le t \le \ln\left(\frac{\sqrt{6}+\sqrt{2}}{2}\right)$$

OBTAINING THE ARCLENGTH DIFFERENTIAL IN PARAMETRIC

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(\operatorname{sech}^2 t) + (\operatorname{sech} t \tanh t)^2}$$

$$= \sqrt{\operatorname{sech}^4 t + \operatorname{sech}^2 t \tanh^2 t} = |\operatorname{sech} t|\sqrt{1+\tanh^2 t}$$

$$= \operatorname{sech} t\sqrt{1+\tanh^2 t}$$

SETTING UP THE INTEGRAL FOR THE SURFACE OF REVOLUTION

$$S = \int_{t_1}^{t_2} 2\pi\, y(t)\, \frac{ds}{dt}\, dt$$

$$S = 2\pi\int_0^{\ln\left(\frac{\sqrt{2}+\sqrt{6}}{2}\right)} (\operatorname{sech} t)\operatorname{sech} t\sqrt{1+\tanh^2 t}\; dt$$

$$S = 2\pi\int_0^{\ln\left(\frac{\sqrt{6}+\sqrt{2}}{2}\right)} \operatorname{sech}^2 t\sqrt{1+\tanh^2 t}\; dt$$

USING A SUBSTITUTION

$$\tanh t = \tan\theta$$
$$\operatorname{sech}^2 t\, dt = \sec^2\theta\, d\theta$$
$$dt = \frac{\sec^2\theta\, d\theta}{\operatorname{sech}^2 t}$$

$$@\ t = 0 \longmapsto \theta = 0$$
$$t = \ln\left(\frac{\sqrt{2}+\sqrt{6}}{2}\right) \longmapsto \theta = \alpha$$

$$\Rightarrow S = 2\pi\int_0^{\alpha} \cancel{\operatorname{sech}^2 t}\sqrt{1+\tan^2\theta}\;\frac{\sec^2\theta}{\cancel{\operatorname{sech}^2 t}}\, d\theta$$

$$\Rightarrow S = 2\pi\int_0^{\alpha} \sec^3\theta\, d\theta$$

WE NEED THE INTEGRAL OF $\sec^3\theta$ — IGNORING LIMITS

$$\int \sec^3\theta\, d\theta = \int \sec\theta\sec^2\theta\, d\theta$$

BY PARTS

$\sec\theta$	$\sec\theta\tan\theta$
$\tan\theta$	$\sec^2\theta$

$$\int \sec^3\theta\, d\theta = \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\, d\theta$$
$$\int \sec^3\theta\, d\theta = \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)\, d\theta$$
$$\int \sec^3\theta\, d\theta = \sec\theta\tan\theta - \int \sec^3\theta - \sec\theta\, d\theta$$
$$\int \sec^3\theta\, d\theta = \sec\theta\tan\theta - \int \sec^3\theta\, d\theta + \int \sec\theta\, d\theta$$
$$2\int \sec^3\theta\, d\theta = \sec\theta\tan\theta + \int \sec\theta\, d\theta$$
$$2\int \sec^3\theta\, d\theta = \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| + C$$
$$\int \sec^3\theta\, d\theta = \tfrac{1}{2}\sec\theta\tan\theta + \tfrac{1}{2}\ln|\sec\theta + \tan\theta| + C.$$

NOW THE LIMIT α

- $$e^{2\ln\left(\frac{\sqrt{2}+\sqrt{6}}{2}\right)} = \frac{(\sqrt{2}+\sqrt{6})^2}{4} = \frac{2+2\sqrt{12}+6}{4} = \frac{8+4\sqrt{3}}{4}$$
  $$= 2+\sqrt{3}$$
- $$\tanh t = \frac{e^{2t}-1}{e^{2t}+1} = \frac{(2+\sqrt{3})-1}{(2+\sqrt{3})+1} = \frac{1+\sqrt{3}}{3+\sqrt{3}}$$
  $$= \frac{(1+\sqrt{3})(3-\sqrt{3})}{(3+\sqrt{3})(3-\sqrt{3})} = \frac{\cancel{3}-\sqrt{3}+3\sqrt{3}-\cancel{3}}{9-3} = \frac{2\sqrt{3}}{6} = \frac{\sqrt{3}}{3}$$
- $$\tan\theta = \tanh t$$
  $$\tan\theta = \frac{\sqrt{3}}{3}$$
  $$\theta = \frac{\pi}{6}$$

$$\Rightarrow S = 2\pi\int_0^{\frac{\pi}{6}} \sec^3\theta\, d\theta$$
$$\Rightarrow S = 2\pi\left[\tfrac{1}{2}\sec\theta\tan\theta + \tfrac{1}{2}\ln|\sec\theta+\tan\theta|\right]_0^{\frac{\pi}{6}}$$
$$\Rightarrow S = \pi\left[\left[\sec\tfrac{\pi}{6}\tan\tfrac{\pi}{6} + \ln\left(\sec\tfrac{\pi}{6}+\tan\tfrac{\pi}{6}\right)\right] - \left[0 + \cancel{\ln 1}\right]\right]$$
$$\Rightarrow S = \pi\left[\frac{2}{\sqrt{3}}\times\frac{1}{\sqrt{3}} + \ln\left(\frac{2}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right)\right]$$
$$\Rightarrow S = \pi\left[\frac{2}{3} + \ln\frac{3}{\sqrt{3}}\right] = \pi\left[\frac{2}{3} + \ln\sqrt{3}\right]$$
$$\Rightarrow S = \pi\left[\frac{2}{3} + \frac{1}{2}\ln 3\right] = \frac{1}{6}\pi\left[4 + 3\ln 3\right]$$

Question 17 (*****)

A curve is defined parametrically by the following equations.

$$x = 2\ln t \ , \quad y = t + \frac{1}{t} \ , \quad t \in \mathbb{R} \ , \quad t \geq 1 .$$

The curve is fully revolved about the y axis forming a surface of revolution.

The surface is modelling the casing of a rocket

The vertex of the surface is held just above a container full of paint, with its line of symmetry vertical.

Its line of symmetry is vertically lowered into the paint, at a rate of $\dfrac{1}{\pi \ln t}$, $t > 1$.

Show that the outer section of the surface is covered in paint at the rate $4\coth\left(\frac{1}{2}x\right)$.

$\boxed{}$, $\boxed{\text{proof}}$

FIRSTLY TO SKETCH THE CURVE THIS IS A COSH CURVE

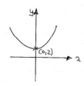
$\frac{x}{2} = \ln t$

$t = e^{\frac{1}{2}x}$

$y = t + \frac{1}{t}$

$y = e^{\frac{1}{2}x} + e^{-\frac{1}{2}x}$

$y = 2\cosh\left(\frac{1}{2}x\right)$



WE ARE UNFORTUNATELY "FORCED" TO WORK IN PARAMETRIC

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\left(\frac{2}{t}\right)^2 + \left(1 - \frac{1}{t^2}\right)^2} = \sqrt{\frac{4}{t^2} + 1 - \frac{2}{t^2} + \frac{1}{t^4}}$$
$$= \sqrt{1 + \frac{2}{t^2} + \frac{1}{t^4}} = \sqrt{\left(1 + \frac{1}{t^2}\right)^2} = \left|1 + \frac{1}{t^2}\right| = 1 + \frac{1}{t^2}$$

SETTING A SURFACE OF REVOLUTION ABOUT THE y AXIS

$$S = \int_{y_1}^{y_2} 2\pi x \, ds = \int_{t_1}^{t_2} 2\pi \, x(t) \, ds(t) = \int_1^T 2\pi(2\ln t)\left(1 + \frac{1}{t^2}\right) dt$$
$$= 4\pi \int_1^T \left(1 + \frac{1}{t^2}\right) \ln t \, dt$$

PROCEED BY INTEGRATION BY PARTS

$\ln t$	$\frac{1}{t}$
$t - \frac{1}{t}$	$1 + \frac{1}{t^2}$

$$S = 4\pi \left\{ \left[\left(t - \frac{1}{t}\right)\ln t\right]_1^T - \int_1^T \frac{1}{t}\left(t - \frac{1}{t}\right) dt \right\}$$
$$S = 4\pi \left\{ \left[\left(t - \frac{1}{t}\right)\ln t\right]_1^T - \int_1^T 1 - \frac{1}{t^2} \, dt \right\}$$
$$S = 4\pi \left\{ \left[\left(t - \frac{1}{t}\right)\ln t - t - \frac{1}{t}\right]_1^T \right\}$$
$$S = 4\pi \left\{ \left(T - \frac{1}{T}\right)\ln T - T - \frac{1}{T} - (0 - 1 - 1) \right\}$$
$$S = 4\pi \left[\left(T - \frac{1}{T}\right)\ln T + 2 - T - \frac{1}{T} \right]$$

NOW WE NEED TO BE CAREFUL WITH t, T AND TIME

$$\frac{dS}{d(\text{TIME})} = \frac{dS}{dT} \times \frac{dT}{d(\text{TIME})} = \frac{dS}{dT} \times \frac{dT}{dy} \times \frac{dy}{d(\text{TIME})}$$

- $\frac{dS}{dT} = 4\pi\left[\left(1 + \frac{1}{T^2}\right)\ln T + \left(T - \frac{1}{T}\right)\frac{1}{T} - 1 + \frac{1}{T^2}\right]$

  $\frac{dS}{dT} = 4\pi\left[\left(1 + \frac{1}{T^2}\right)\ln T + \cancel{1 - \frac{1}{T^2}} \cancel{- 1 + \frac{1}{T^2}}\right]$

  $\frac{dS}{dT} = 4\pi\left(\frac{T^2 + 1}{T^2}\right)\ln T$

- $\frac{dy}{dt} = 1 - \frac{1}{t^2} = \frac{t^2 - 1}{t^2}$

  $\left.\frac{dy}{dt}\right|_{t=T} = \frac{T^2 - 1}{T^2}$

  $\frac{dy}{dT} = \frac{T^2 - 1}{T^2}$

  $\frac{dT}{dy} = \frac{T^2}{T^2 - 1}$

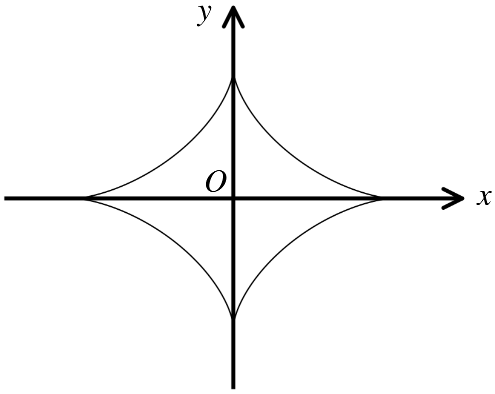
THUS WE FINALLY HAVE, IGNORING "DIRECTIONAL MINUSES"

$$\frac{dS}{d(\text{time})} = 4\pi\left(\frac{T^2 + 1}{T^2}\right)\ln T \times \frac{T^2}{T^2 - 1} \times \frac{1}{\pi \ln T}$$

$$\frac{dS}{d(\text{time})} = \frac{4(T^2 + 1)}{T^2 - 1} = \frac{4(t^2 + 1)}{t^2 - 1} = \frac{4\left(t + \frac{1}{t}\right)}{t - \frac{1}{t}}$$
$$= \frac{4\left(2\cosh\frac{1}{2}x\right)}{\left(2\sinh\frac{1}{2}x\right)} \leftarrow \text{FOUND EARLIER} \quad = 4\coth\frac{1}{2}x$$

MIXED QUESTIONS

Question 1 (****)



The parametric equations of an astroid are

$$x = a\cos^3\theta, \quad y = a\sin^3\theta, \quad 0 \le \theta < 2\pi$$

a) Show that the total length of the curve is $6a$ units.

The curve is rotated by $360°$ about the x axis forming a solid of revolution.

b) Show further that the surface area of the solid is $\frac{12}{5}\pi a^2$.

proof

(a) $x = a\cos^3\theta$
$y = a\sin^3\theta$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(-3a\cos^2\theta\sin\theta\right)^2 + \left(3a\sin^2\theta\cos\theta\right)^2$$
$$= 9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta$$
$$= 9a^2\cos^2\theta\sin^2\theta\left[\cos^2\theta + \sin^2\theta\right]$$
$$= 9a^2\cos^2\theta\sin^2\theta$$

Thus

BY SYMMETRY

$$s = \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = 4\int_0^{\frac{\pi}{2}} \sqrt{9a^2\cos^2\theta\sin^2\theta}\, d\theta$$
$$= 4\int_0^{\frac{\pi}{2}} 3a\cos\theta\sin\theta\, d\theta = 4\left[\frac{3a}{2}\sin^2\theta\right]_0^{\frac{\pi}{2}}$$
$$= 4\left[\frac{3a}{2}\sin^2\frac{\pi}{2} - \cancel{\frac{3a}{2}\sin^2 0}\right] = 6a$$

AS REQUIRED

(b)
$$S = 2\pi\int_{\theta_1}^{\theta_2} y\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

BY SYMMETRY

FROM part a

$$S = 2 \times 2\pi\int_0^{\frac{\pi}{2}} a\sin^3\theta\left(3a\cos\theta\sin\theta\right)\, d\theta$$
$$S = \pi a^2\int_0^{\frac{\pi}{2}} 12\sin^4\theta\cos\theta\, d\theta$$
$$S = \pi a^2\left[\frac{12}{5}\sin^5\theta\right]_0^{\frac{\pi}{2}}$$
$$S = \frac{12\pi a^2}{5}\left[1^5 - 0^5\right]$$
$$S = \frac{12\pi}{5}a^2$$

AS REQUIRED

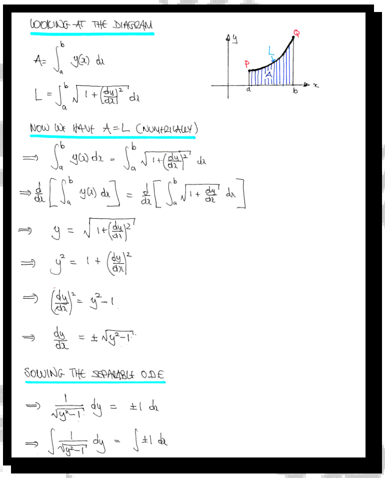
Question 2 (****)

The curve with equation $y = f(x)$ satisfies $y > 0$, for $x \in [a, b]$.

- The area of the region bounded by the curve with equation $y = f(x)$ and the x axis, for $a \le x \le b$, is denoted by A.
- The length along the curve from the point $P[a, f(a)]$ to the point $Q[b, f(b)]$, is denoted by L.

If A is **numerically equal** to L, determine the equation of the curve.

$\boxed{y = \cosh(x + C)}$



LOOKING AT THE DIAGRAM

$A = \int_a^b y(x)\,dx$

$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$

NOW WE HAVE $A = L$ (NUMERICALLY)

$\Rightarrow \int_a^b y(x)\,dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$

$\Rightarrow \frac{d}{dx}\left[\int_a^b y(x)\,dx\right] = \frac{d}{dx}\left[\int_a^b \sqrt{1 + \frac{dy}{dx}}\,dx\right]$

$\Rightarrow y = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

$\Rightarrow y^2 = 1 + \left(\frac{dy}{dx}\right)^2$

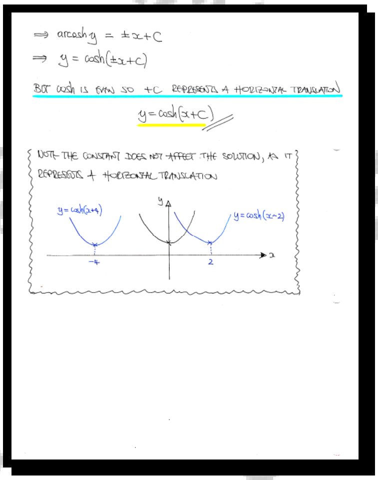
$\Rightarrow \left(\frac{dy}{dx}\right)^2 = y^2 - 1$

$\Rightarrow \frac{dy}{dx} = \pm\sqrt{y^2 - 1}$

SOLVING THE SEPARABLE O.D.E

$\Rightarrow \frac{1}{\sqrt{y^2 - 1}}\,dy = \pm 1\,dx$

$\Rightarrow \int \frac{1}{\sqrt{y^2 - 1}}\,dy = \int \pm 1\,dx$



$\Rightarrow \text{arcosh}\,y = \pm x + C$

$\Rightarrow y = \cosh(\pm x + C)$

BUT cosh IS EVEN SO $+C$ REPRESENTS A HORIZONTAL TRANSLATION

$y = \cosh(x + C)$

NOTE THE CONSTANT DOES NOT AFFECT THE SOLUTION, AS IT REPRESENTS A HORIZONTAL TRANSLATION

Question 3 (****+)

A cycloid has parametric equations

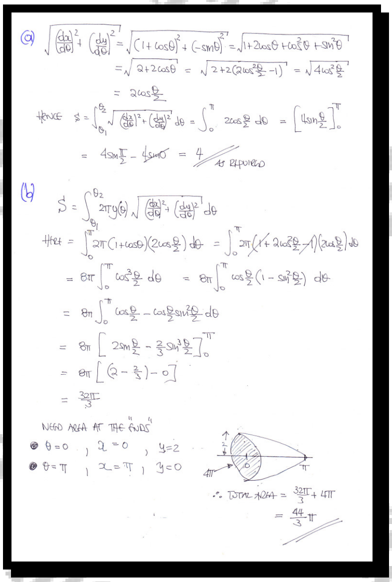
$$x = \theta + \sin\theta, \quad y = 1 + \cos\theta, \quad 0 \le \theta \le \pi$$

a) Show that the total length of the curve is 4 units.

The cycloid is rotated by $360°$ about the x axis, forming a solid of revolution.

b) Show further that the **total** surface area of the solid is $\dfrac{44\pi}{3}$.

$\boxed{\text{proof}}$



(a) $\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2} = \sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}$

$= \sqrt{2 + 2\cos\theta} = \sqrt{2 + 2\left(2\cos^2\frac{\theta}{2} - 1\right)} = \sqrt{4\cos^2\frac{\theta}{2}}$

$= 2\cos\frac{\theta}{2}$

HENCE $s = \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_0^{\pi} 2\cos\frac{\theta}{2}\, d\theta = \left[4\sin\frac{\theta}{2}\right]_0^{\pi}$

$= 4\sin\frac{\pi}{2} - \cancel{4\sin 0} = 4$ AS REQUIRED

(b) $S = \int_{\theta_1}^{\theta_2} 2\pi y(\theta) \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$

THAT $= \int_0^{\pi} 2\pi(1+\cos\theta)\left(2\cos\frac{\theta}{2}\right) d\theta = \int_0^{\pi} 2\pi\left(\cancel{1} + 2\cos^2\frac{\theta}{2} - \cancel{1}\right)\left(2\cos\frac{\theta}{2}\right) d\theta$

$= 8\pi \int_0^{\pi} \cos^3\frac{\theta}{2}\, d\theta = 8\pi \int_0^{\pi} \cos\frac{\theta}{2}\left(1 - \sin^2\frac{\theta}{2}\right) d\theta$

$= 8\pi \int_0^{\pi} \cos\frac{\theta}{2} - \cos\frac{\theta}{2}\sin^2\frac{\theta}{2}\, d\theta$

$= 8\pi \left[2\sin\frac{\theta}{2} - \frac{2}{3}\sin^3\frac{\theta}{2}\right]_0^{\pi}$

$= 8\pi\left[\left(2 - \frac{2}{3}\right) - 0\right]$

$= \frac{32\pi}{3}$

NEED AREA AT THE "ENDS"

- $\theta = 0$, $x = 0$, $y = 2$
- $\theta = \pi$, $x = \pi$, $y = 0$

2, 0, π, 4π

$\therefore$ TOTAL AREA $= \frac{32\pi}{3} + 4\pi$

$= \frac{44}{3}\pi$