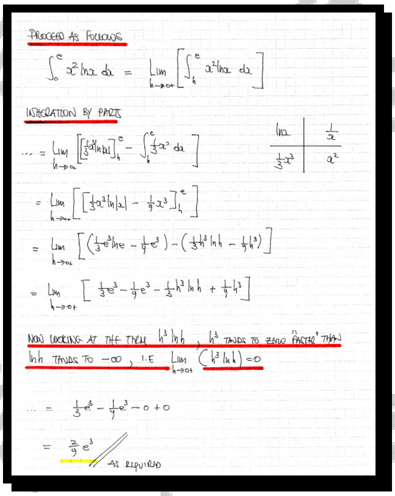
IMPROPER INTEGRALS

Question 1 **(***)**

Indicating clearly the limiting processes used, show that

$$\int_0^{\mathrm{e}} x^2 \ln x \; dx = \frac{2}{9}\mathrm{e}^3 .$$

FP2-Q , proof



PROCEED AS FOLLOWS

$$\int_0^e x^2 \ln x \, dx = \lim_{h \to 0^+}\left[\int_h^e x^2 \ln x \, dx\right]$$

INTEGRATION BY PARTS

$\ln x$	$\frac{1}{x}$
$\frac{1}{3}x^3$	x^2

$$\dots = \lim_{h\to 0^+}\left[\left[\tfrac{1}{3}x^3 \ln|x|\right]_h^e - \int_h^e \tfrac{1}{3}x^2 \, dx\right]$$

$$= \lim_{h\to 0^+}\left[\left[\tfrac{1}{3}x^3 \ln|x| - \tfrac{1}{9}x^3\right]_h^e\right]$$

$$= \lim_{h\to 0^+}\left[\left(\tfrac{1}{3}e^3 \ln e - \tfrac{1}{9}e^3\right) - \left(\tfrac{1}{3}h^3 \ln h - \tfrac{1}{9}h^3\right)\right]$$

$$= \lim_{h\to 0^+}\left[\tfrac{1}{3}e^3 - \tfrac{1}{9}e^3 - \tfrac{1}{3}h^3 \ln h + \tfrac{1}{9}h^3\right]$$

NOW LOOKING AT THE TERM $h^3 \ln h$, h^3 TENDS TO ZERO "FASTER" THAN $\ln h$ TENDS TO $-\infty$, I.E $\lim_{h\to 0^+}\left(h^3 \ln h\right) = 0$

$$\dots = \tfrac{1}{3}e^3 - \tfrac{1}{9}e^3 - 0 + 0$$

$$= \tfrac{2}{9}e^3$$

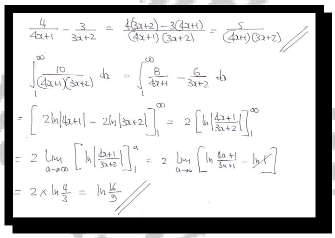
AS REQUIRED

Question 2 **(***)**

Indicating clearly the limiting processes used, show that

$$\int_1^{\infty} \frac{10}{(3x+2)(4x+1)} \; dx = \ln\frac{16}{9} .$$

proof



$$\frac{4}{4x+1} - \frac{3}{3x+2} = \frac{4(3x+2)-3(4x+1)}{(4x+1)(3x+2)} = \frac{5}{(4x+1)(3x+2)}$$

$$\int_1^{\infty} \frac{10}{(4x+1)(3x+2)} \, dx = \int_1^{\infty} \frac{8}{4x+1} - \frac{6}{3x+2} \, dx$$

$$= \Big[2\ln|4x+1| - 2\ln|3x+2|\Big]_1^{\infty} = 2\left[\ln\left|\frac{4x+1}{3x+2}\right|\right]_1^{\infty}$$

$$= 2\lim_{a\to\infty}\left[\ln\left|\frac{4x+1}{3x+2}\right|\right]_1^{a} = 2\lim_{a\to\infty}\left[\ln\frac{4a+1}{3a+1} - \cancel{\ln 1}\right]$$

$$= 2 \times \ln\frac{4}{3} = \ln\frac{16}{9}$$

Question 3 (***)

$$I = \int 16x\mathrm{e}^{-4x}\, dx \, .$$

a) Show that

$$I = -\mathrm{e}^{-4x}\left(4x+1\right) + \text{constant} \, .$$

b) Hence find

$$\int_{1}^{\infty} 16x\mathrm{e}^{-4x}\, dx \, ,$$

showing clearly the limiting process used.

$$\boxed{5\mathrm{e}^{-4}}$$

a) $I = \int 16x e^{-4x}\, dx = \dots$ BY PARTS

$16x$	16
$-\frac{1}{4}e^{-4x}$	e^{-4x}

$I = -4xe^{-4x} - \int -4e^{-4x}\, dx$

$I = -4xe^{-4x} + \int 4e^{-4x}\, dx$

$I = -4xe^{-4x} - e^{-4x} + C$

$I = -e^{-4x}(4x+1) + C$

AS REQUIRED

b) $\int_{1}^{\infty} 16xe^{-4x}\, dx = \lim_{k \to \infty}\left[\int_{1}^{k} 16xe^{-4x}\, dx\right]$

$= \lim_{k \to \infty}\left[\left[-e^{-4x}(4x+1)\right]_{1}^{k}\right] = \lim_{k \to \infty}\left[\left[e^{-4x}(4x+1)\right]_{k}^{1}\right]$

$= \lim_{k \to \infty}\left[e^{-4} \times 5 - e^{-4k}(4k+1)\right]$

$= \lim_{k \to \infty}\left[5e^{-4} - \cancel{4ke^{-4k}} - \cancel{e^{-4k}}\right]$

$= 5e^{-4}$

NOTE THAT $k \to \infty$ "SLOWER" THAN $e^{-4k} \to 0$

Question 4 (***)

Find the exact value of the following integral.

$$\int_{e}^{\infty} \frac{1-\ln x}{x^2}\ dx .$$

$\boxed{\text{F, 2 M}}$, $\boxed{-e^{-1}}$

$$\int_{e}^{\infty} \frac{1-\ln x}{x^2}\,dx = \ldots \text{ INTEGRATION BY PARTS}$$

$1-\ln x$	$-\frac{1}{x}$
$-\frac{1}{x}$	$\frac{1}{x^2}$

$$= \left[-\frac{1}{x}(1-\ln x)\right]_{e}^{\infty} - \int_{e}^{\infty} \frac{1}{x^2}\,dx$$

$$= \left[-\frac{1}{x} + \frac{1}{x}\ln x\right]_{e}^{\infty} - \left[-\frac{1}{x}\right]_{e}^{\infty}$$

$$= \left[\cancel{-\frac{1}{x}} + \frac{1}{x}\ln x + \cancel{\frac{1}{x}}\right]_{e}^{\infty}$$

$$= \left[\frac{\ln x}{x}\right]_{e}^{\infty}$$

$$= \lim_{k\to\infty}\left[\left[\frac{\ln x}{x}\right]_{e}^{k}\right]$$

$$= \lim_{k\to\infty}\left[\frac{\ln k}{k} - \frac{\ln e}{e}\right]$$

$$= 0 - \frac{1}{e}$$

$\frac{\ln k}{k} \to 0$ AS $k \to \infty$

AS $\frac{1}{k} \to 0$ FASTER THAN $\ln k \to \infty$

$$= -\frac{1}{e}$$

Question 5 (***+)

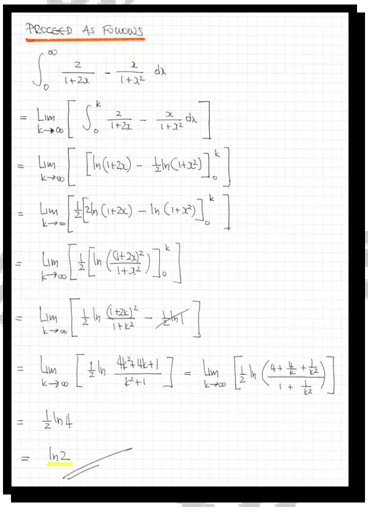
Evaluate the integral

$$\int_0^{\infty} \frac{2}{1+2x} - \frac{x}{1+x^2}\ dx\,,$$

showing clearly the limiting processes used.

Give the answer in the form $\ln N$, where N is a positive integer.

$\boxed{}$, $\boxed{\ln 2}$


PROCEED AS FOLLOWS

$$\int_0^{\infty} \frac{2}{1+2x} - \frac{x}{1+x^2}\ dx$$

$$= \lim_{k\to\infty}\left[\int_0^k \frac{2}{1+2x} - \frac{x}{1+x^2}\ dx\right]$$

$$= \lim_{k\to\infty}\left[\left[\ln(1+2x) - \tfrac{1}{2}\ln(1+x^2)\right]_0^k\right]$$

$$= \lim_{k\to\infty}\left[\tfrac{1}{2}\left[2\ln(1+2x) - \ln(1+x^2)\right]_0^k\right]$$

$$= \lim_{k\to\infty}\left[\tfrac{1}{2}\left[\ln\left(\frac{(1+2x)^2}{1+x^2}\right)\right]_0^k\right]$$

$$= \lim_{k\to\infty}\left[\tfrac{1}{2}\ln\frac{(1+2k)^2}{1+k^2} - \cancel{\tfrac{1}{2}\ln 1}\right]$$

$$= \lim_{k\to\infty}\left[\tfrac{1}{2}\ln\frac{4k^2+4k+1}{k^2+1}\right] = \lim_{k\to\infty}\left[\tfrac{1}{2}\ln\left(\frac{4+\frac{4}{k}+\frac{1}{k^2}}{1+\frac{1}{k^2}}\right)\right]$$

$$= \tfrac{1}{2}\ln 4$$

$$= \ln 2$$

Question 6 **(***+)**

The function f is defined by

$$f(x) \equiv \frac{ax+b}{e^{x}}, \quad x \in \mathbb{R},$$

where a and b are non zero constants.

The mean value of f in the interval $(\ln 2, \ln 4)$ is $\frac{1}{4\ln 2}$.

Given further that

$$\int_{1}^{\infty} f(x)\ dx = \frac{3}{e},$$

determine the value of a and the value of b.

$\boxed{}$, $\boxed{a=2}$, $\boxed{b=-1}$

$$f(x) \equiv \frac{ax+b}{e^{x}} = (ax+b)e^{-x}$$

PROCEED BY INDEFINITE INTEGRATION BY PARTS

$$\int \frac{ax+b}{e^{x}}\,dx = -(ax+b)e^{-x} - \int -ae^{-x}\,dx$$

$ax+b$	a
$-e^{-x}$	e^{-x}

$$= -(ax+b)e^{-x} + \int ae^{-x}\,dx$$

$$= -(ax+b)e^{-x} - ae^{-x} + C$$

$$= -e^{-x}\left[ax+b+a\right] + C$$

NEXT THE MEAN VALUE OF THE FUNCTION IN $(\ln 2, \ln 4)$

$$\Rightarrow \text{MEAN VALUE} = \frac{1}{b-a}\int_{a}^{b} f(x)\,dx$$

$$\Rightarrow \frac{1}{4\ln 2} = \frac{1}{\ln 4 - \ln 2}\int_{\ln 2}^{\ln 4} \frac{ax+b}{e^{x}}\,dx$$

$$\Rightarrow \frac{1}{4\cancel{\ln 2}} = \frac{1}{\cancel{\ln 2}}\Big[-e^{-x}\left[ax+b+a\right]\Big]_{\ln 2}^{\ln 4}$$

$$\Rightarrow \frac{1}{4} = \Big[+e^{-x}(ax+a+b)\Big]_{\ln 4}^{\ln 2}$$

$$\Rightarrow \frac{1}{4} = e^{-\ln 2}(a\ln 2 + a + b) - e^{-\ln 4}(a\ln 4 + a + b)$$

$$\Rightarrow \frac{1}{4} = \frac{1}{2}(a\ln 2 + a + b) - \frac{1}{4}(2a\ln 2 + a + b)$$

$$\Rightarrow 1 = \cancel{2(a\ln 2} + a + b) - (\cancel{2a\ln 2} + a + b)$$

$$\Rightarrow 1 = 2a + 2b - a - b$$

$$\Rightarrow a + b = 1$$

NEXT THE IMPROPER INTEGRAL

$$\Rightarrow \int_{1}^{\infty} \frac{ax+b}{e^{x}}\,dx = \lim_{k \to \infty}\left[\Big[-e^{-x}(ax+b+a)\Big]_{1}^{k}\right]$$

$$\Rightarrow \frac{3}{e} = \lim_{k \to \infty}\left[\Big[+e^{-x}(ax+b+a)\Big]_{k}^{1}\right]$$

$$\Rightarrow \frac{3}{e} = \lim_{k \to \infty}\left[e^{-1}(2a+b) - e^{-k}(ak+b+a)\right]$$

$$\Rightarrow \frac{3}{e} = \lim_{k \to \infty}\left[\frac{2a+b}{e} - \cancel{\frac{ak}{e^{k}}} - \cancel{\frac{a+b}{e^{k}}}\right]$$

TENDS TO ZERO, AS e^{k} GROWS FASTER THAN THE ALGEBRAIC NUMERATOR GROWS, AS $k \to \infty$

TENDS TO ZERO AS $k \to \infty$

$$\Rightarrow \frac{3}{e} = \frac{2a+b}{e}$$

$$\Rightarrow 2a + b = 3$$

SOLVING THE EQUATIONS BY INSPECTION

$$\left.\begin{aligned} a+b &= 1 \\ 2a+b &= 3 \end{aligned}\right\} \Rightarrow a = 2 \ \ \& \ \ b = -1$$

Question 7 (***+)

$$f(y) \equiv \frac{4y}{y^4 - 1}, \quad y \in \mathbb{R}, \; |y| \neq 1.$$

a) Express $f(y)$ into three partial fractions.

b) Hence evaluate the improper integral

$$\int_2^{\infty} f(y)\, dy,$$

showing clearly the limiting processes used.

c) Find, in exact form, the mean value of f, in the interval $\{y \in \mathbb{R} : 2 \leq y \leq 4\}$.

$\boxed{f(y) = \frac{1}{y+1} + \frac{1}{y-1} - \frac{2y}{y^2+1}}$, $\boxed{\ln\left(\frac{5}{3}\right)}$, $\boxed{\frac{1}{2}\ln\left(\frac{25}{17}\right) = \ln\left(\frac{5}{\sqrt{17}}\right)}$

a) FACTORIZE THE DENOMINATOR FULLY

$$f(y) = \frac{4y}{y^4-1} = \frac{4y}{(y^2-1)(y^2+1)} = \frac{4y}{(y-1)(y+1)(y^2+1)}$$

NOW WE HAVE

$$\frac{4y}{(y-1)(y+1)(y^2+1)} \equiv \frac{A}{y-1} + \frac{B}{y+1} + \frac{Cx+D}{y^2+1}$$

$$\Longrightarrow \boxed{4y \equiv A(y+1)(y^2+1) + B(y-1)(y^2+1) + (y-1)(y+1)(Cx+D)}$$

- IF $y=1$: $4 = 4A$, $A=1$
- IF $y=-1$: $-4 = -4B$, $B=1$
- IF $y=0$: $0 = A - B - D$, $D=0$
- IF $y=2$: $8 = 15A + 5B + 6C$, $8 = 15 + 5 + 6C$, $-12 = 6C$, $C=-2$

$$\therefore \; f(y) = \frac{1}{y-1} + \frac{1}{y+1} - \frac{2y}{y^2+1}$$

b)
$$\int_2^{\infty} f(y)\,dy = \int_2^{\infty} \frac{1}{y-1} + \frac{1}{y+1} - \frac{2y}{y^2+1}\,dy$$
$$= \lim_{a \to \infty}\left[\int_2^{a} \frac{1}{y-1} + \frac{1}{y+1} - \frac{2y}{y^2+1}\,dy\right]$$
$$= \lim_{a \to \infty}\left[\Big[\ln(y-1) + \ln(y+1) - \ln(y^2+1)\Big]_2^a\right]$$
$$= \lim_{a \to \infty}\left[\left[\ln\left(\frac{y^2-1}{y^2+1}\right)\right]_2^a\right]$$
$$= \lim_{a \to \infty}\left[\ln\left(\frac{a^2-1}{a^2+1}\right) - \ln\left(\frac{3}{5}\right)\right]$$
$$= \ln 1 - \ln\frac{3}{5}$$
$$= \ln\frac{5}{3}$$

c) USING THE INTEGRATION OF PART (b) WE HAVE

MEAN OF $f(y)$, OVER $[2,4]$ $= \frac{1}{4-2}\int_2^4 f(y)\,dy$

$$= \frac{1}{2}\left[\ln\left(\frac{a^2-1}{a^2+1}\right) - \ln\left(\frac{3}{5}\right)\right]$$

WHERE $a=4$

$$= \frac{1}{2}\left[\ln\frac{15}{17} - \ln\frac{3}{5}\right]$$
$$= \frac{1}{2}\left[\ln\frac{15}{17} + \ln\frac{5}{3}\right]$$
$$= \frac{1}{2}\ln\frac{25}{17}$$

OR $\ln\left(\frac{5}{\sqrt{17}}\right)$

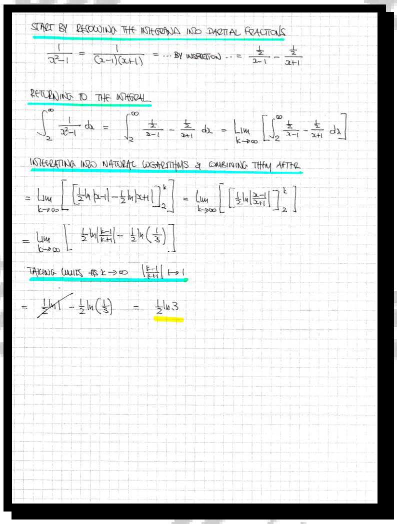
Question 8 (***+)

By showing formally all the limiting processes evaluate the following integral

$$\int_2^{\infty} \frac{1}{x^2-1}\, dx \ .$$

Give the answer in the form $k \ln 3$, where k is a positive constant to be found.

$\boxed{}$, $\boxed{k = \frac{1}{2}}$



START BY RESOLVING THE INTEGRAND INTO PARTIAL FRACTIONS

$$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \dots \text{BY INSPECTION} \dots = \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}$$

RETURNING TO THE INTEGRAL

$$\int_2^{\infty} \frac{1}{x^2-1}\, dx = \int_2^{\infty} \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}\, dx = \lim_{k \to \infty}\left[\int_2^{k} \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}\, dx\right]$$

INTEGRATING INTO NATURAL LOGARITHMS & COMBINING THEM AFTER

$$= \lim_{k \to \infty}\left[\left[\tfrac{1}{2}\ln|x-1| - \tfrac{1}{2}\ln|x+1|\right]_2^k\right] = \lim_{k \to \infty}\left[\left[\tfrac{1}{2}\ln\left|\frac{x-1}{x+1}\right|\right]_2^k\right]$$

$$= \lim_{k \to \infty}\left[\tfrac{1}{2}\ln\left|\frac{k-1}{k+1}\right| - \tfrac{1}{2}\ln\left(\tfrac{1}{3}\right)\right]$$

TAKING LIMITS AS $k \to \infty$ $\left|\frac{k-1}{k+1}\right| \mapsto 1$

$$= \cancel{\tfrac{1}{2}\ln 1} - \tfrac{1}{2}\ln\left(\tfrac{1}{3}\right) = \tfrac{1}{2}\ln 3$$

Question 9 (****)

$$\int \frac{\ln x^2}{x^3}\,dx\,,\ \ x \neq 0\,.$$

a) Show that the substitution $y = \frac{1}{x}$ transforms the above integral into

$$\int 2y\ln y\ dy\,.$$

b) Hence evaluate

$$\int_1^{\infty} \frac{\ln x^2}{x^3}\ dx\,,$$

showing clearly the limiting process used.

$\boxed{}\,,\ \boxed{\frac{1}{2}}$

a) USING THE SUBSTITUTION GIVEN

$$y = \frac{1}{x} \implies \frac{dy}{dx} = -\frac{1}{x^2}$$
$$\implies dx = -x^2\,dy$$
$$\implies dx = -\frac{1}{y^2}\,dy$$

$$\implies \int \frac{\ln x^2}{x^3}\,dx = \int \frac{2\ln x}{x^3}\,dx = \int (2\ln x)\left(\frac{1}{x^3}\right)dx$$
$$= \int 2\ln\left(\frac{1}{y}\right) \times y^3 \times \left(-\frac{1}{y^2}\,dy\right)$$
$$= \int -2y\ln\left(\frac{1}{y}\right)dy = \int -2y\ln y^{-1}\,dy$$
$$= \int 2y\ln y\ dy$$

AS REQUIRED

b) PROCEED BY INTEGRATION BY PARTS

$$\int_1^{\infty} \frac{\ln x^2}{x^3}\,dx = \ldots \text{ SUBSTITUTION FROM PART (a) } \ldots = \int_1^0 2y\ln y\ dy$$

$x = 1 \mapsto y = 1$

$x = \infty \mapsto y = 0$

BY PARTS

$\ln y$	$\frac{1}{y}$
y^2	$2y$

$$= \left[y^2\ln y\right]_1^0 - \int_1^0 y\ dy$$
$$= \left[y^2\ln y - \frac{1}{2}y^2\right]_1^0$$
$$= \left[\frac{1}{2}y^2 - y^2\ln y\right]_0^1$$

APPLY LIMITS

$$\ldots = \lim_{h\to 0}\left[\frac{1}{2}y^2 - y^2\ln y\right]_h^1$$
$$= \lim_{h\to 0}\left[\left(\frac{1}{2} - \cancel{\ln 1}\right) - \left(\frac{1}{2}h^2 - h^2\ln h\right)\right]$$

NOW h^2 TENDS TO ZERO FASTER THAN $\ln h$ TENDS TO $-\infty$

$$= \lim_{h\to 0}\left[\frac{1}{2} - \cancel{\frac{1}{2}h^2} + \cancel{h^2\ln h}\right]$$
$$= \frac{1}{2}$$

Question 10 (****)

By showing formally all the limiting processes evaluate the following integral

$$\int_0^{\frac{1}{4}\pi} \frac{1}{x} - \frac{\sin 2x}{1-\cos 2x} \, dx \ .$$

Give the answer in the form $\ln\left[\frac{\pi\sqrt{2}}{n}\right]$, where n is a positive integer to be found.

$\boxed{}$, $\boxed{n=4}$

START BY REPLACING "0" AT THE BOTTOM LIMIT WITH k, $k>0$

$$\int_k^{\frac{\pi}{4}} \frac{1}{x} - \frac{\sin 2x}{1-\cos 2x} \, dx = \left[\ln x - \frac{1}{2}\ln(1-\cos 2x)\right]_k^{\pi/4}$$

$$= \frac{1}{2}\left[2\ln x - \ln(1-\cos 2x)\right]_k^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left[\ln x^2 - \ln(1-\cos 2x)\right]_k^{\pi/4}$$

$$= \frac{1}{2}\left[\ln\left(\frac{x^2}{1-\cos 2x}\right)\right]_k^{\pi/4}$$

EVALUATING AND TAKE LIMITS

$$= \frac{1}{2}\ln\left(\frac{\pi^2/16}{1-\cancel{\cos\pi/2}}\right) - \frac{1}{2}\ln\left(\frac{k^2}{1-\cos 2k}\right)$$

$$= \frac{1}{2}\ln\frac{\pi^2}{16} - \frac{1}{2}\ln\left[\frac{k^2}{1-\left(1-\frac{4k^2}{2!}+\frac{16k^4}{4!}-\dots\right)}\right]$$

$$= \frac{1}{2}\ln\frac{\pi^2}{16} - \frac{1}{2}\ln\left[\frac{k^2}{2k^2-\frac{2}{3}k^4+O(k^6)}\right]$$

$$= \frac{1}{2}\ln\frac{\pi^2}{16} - \frac{1}{2}\ln\left(\frac{1}{2-\frac{2}{3}k^2+O(k^4)}\right)$$

$$\therefore \int_0^{\pi/4} \frac{1}{x} - \frac{\sin 2x}{1-\cos 2x} \, dx = \frac{1}{2}\ln\frac{\pi^2}{16} - \frac{1}{2}\lim_{k\to 0}\left[\ln\left(\frac{1}{2-\frac{2}{3}k^2+O(k^4)}\right)\right]$$

$$= \frac{1}{2}\ln\frac{\pi^2}{16} - \frac{1}{2}\ln\left(\frac{1}{2}\right)$$

$$= \ln\left(\frac{\pi}{4}\right) - \ln\left(\frac{1}{2}\right)^{\frac{1}{2}}$$

$$= \ln\frac{\pi}{4} - \ln\frac{\sqrt{2}}{2}$$

$$= \ln\frac{\pi}{4} + \ln\frac{2}{\sqrt{2}}$$

$$= \ln\left(\frac{\pi}{2\sqrt{2}}\right)$$

$$= \ln\left(\frac{\pi\sqrt{2}}{4}\right)$$

ie $n=4$

Question 11 (*****)

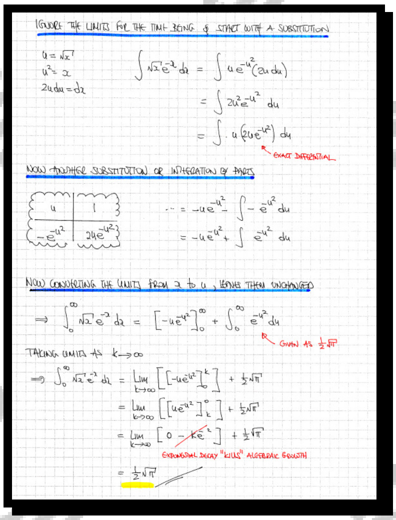
By showing formally all the limiting processes, find an exact simplified value for the following improper integral.

$$\int_0^{\infty} \sqrt{x}\,\mathrm{e}^{-x}\ dx\,.$$

You may assume without proof that

$$\int_0^{\infty} \mathrm{e}^{-x^2}\ dx = \frac{1}{2}\sqrt{\pi}\,.$$

$\boxed{}$, $\boxed{\frac{1}{2}\sqrt{\pi}}$



IGNORE THE LIMITS FOR THE TIME BEING & START WITH A SUBSTITUTION

$u = \sqrt{x}$
$u^2 = x$
$2u\,du = dx$

$$\int \sqrt{x}\,e^{-x}\,dx = \int u e^{-u^2}(2u\,du) = \int 2u^2 e^{-u^2}\,du = \int u\left(2ue^{-u^2}\right)du$$

EXACT DIFFERENTIAL

NOW ANOTHER SUBSTITUTION OR INTEGRATION BY PARTS

u	1
$-e^{-u^2}$	$2ue^{-u^2}$

$$\ldots = -ue^{-u^2} - \int -e^{-u^2}\,du = -ue^{-u^2} + \int e^{-u^2}\,du$$

NOW CONVERTING THE LIMITS FROM x TO u, LEAVES THEM UNCHANGED

$$\Rightarrow \int_0^{\infty} \sqrt{x}\,e^{-x}\,dx = \left[-ue^{-u^2}\right]_0^{\infty} + \int_0^{\infty} e^{-u^2}\,du$$

GIVEN AS $\frac{1}{2}\sqrt{\pi}$

TAKING LIMITS AS $k \to \infty$

$$\Rightarrow \int_0^{\infty} \sqrt{x}\,e^{-x}\,dx = \lim_{k\to\infty}\left[\left[-ue^{-u^2}\right]_0^k\right] + \frac{1}{2}\sqrt{\pi}$$
$$= \lim_{k\to\infty}\left[\left[ue^{-u^2}\right]_k^0\right] + \frac{1}{2}\sqrt{\pi}$$
$$= \lim_{k\to\infty}\left[0 - \cancel{ke^{-k^2}}\right] + \frac{1}{2}\sqrt{\pi}$$

EXPONENTIAL DECAY "KILLS" ALGEBRAIC GROWTH

$$= \frac{1}{2}\sqrt{\pi}$$

Question 12 (*****)

The function f is defined as

$$f(x) = \arctan\left(\frac{1}{2x^2}\right), \quad x \in (-\infty, \infty).$$

a) Find a simplified expression for $f'(x)$.

b) Show that $\lim_{x \to \pm\infty} \left[x f(x) \right] = 0$.

c) Determine the value of $\lim_{x \to \pm\infty} \left[\ln \left[\frac{2x^2 - 2x + 1}{2x^2 + 2x + 1} \right] \right]$.

d) Hence find the value of $\int_{-\infty}^{\infty} f(x) \, dx$.

$\square$, $\boxed{f'(x) = -\frac{4x}{4x^4+1}}$, $\boxed{\lim_{x \to \pm\infty} \left[\frac{2x^2-2x+1}{2x^2+2x+1} \right] = 0}$, $\boxed{\int_{-\infty}^{\infty} f(x) \, dx = \pi}$

a) DIFFERENTIATE AND TIDY

$$\frac{d}{dx}\left[\arctan\left(\frac{1}{2x^2}\right)\right] = \frac{1}{1+\left(\frac{1}{2x^2}\right)^2} \times \frac{d}{dx}\left(\frac{1}{2x^2}\right) = \frac{1}{1+\frac{1}{4x^4}} \times \frac{-1}{x^3}$$

$$= \frac{4x^4}{4x^4+1} \times \frac{-1}{x^3} = -\frac{4x}{4x^4+1}$$

b) NEXT THE FIRST LIMIT

$$\lim_{x \to \pm\infty}\left[x f(x)\right] = \lim_{x \to \pm\infty}\left[x \arctan\left(\frac{1}{2x^2}\right)\right]$$

THIS IS AN INDETERMINATE FORM $(\infty)(0)$ — REWRITE TO USE L'HOSPITAL

$$= \lim_{x \to \pm\infty}\left[\frac{\arctan\left(\frac{1}{2x^2}\right)}{\frac{1}{x}}\right] \quad \leftarrow \text{NOW ZERO/ZERO TYPE, L'HOSPITAL NEXT}$$

$$= \lim_{x \to \pm\infty}\left[\frac{-\frac{4x}{4x^4+1}}{-\frac{1}{x^2}}\right] \leftarrow \text{PART (a)} \quad = \lim_{x \to \pm\infty}\left[\frac{4x^3}{4x^4+1}\right]$$

$$= \lim_{x \to \pm\infty}\left[\frac{\frac{4}{x}}{4+\frac{1}{x^4}}\right] = \frac{0}{4} = 0$$

c) THE NEXT LIMIT IS VERY EASY

$$\lim_{x \to \pm\infty}\left[\ln\left[\frac{2x^2-2x+1}{2x^2+2x+1}\right]\right] = \lim_{x = \pm\infty}\left[\ln\left[\frac{2-\frac{2}{x}+\frac{1}{x^2}}{2+\frac{2}{x}+\frac{1}{x^2}}\right]\right]$$

$$= \ln\left[\frac{2}{2}\right]$$

$$= \ln 1$$

$$= 0$$

d) PROCEED BY INTEGRATION BY PARTS

$$\int_{-\infty}^{\infty} \arctan\left(\frac{1}{2x^2}\right) dx$$

NOTE THAT AT $x=0$, $f(0) = \frac{\pi}{2}$ (FROM BOTH SIDES)

$$\begin{array}{c|c} \arctan\left(\frac{1}{2x^2}\right) & \frac{-4x}{4x^4+1} \\ \hline x & 1 \end{array}$$

$$= \cancel{\left[x \arctan\left(\frac{1}{2x^2}\right)\right]_{-\infty}^{\infty}} - \int_{-\infty}^{\infty} -\frac{4x^2}{4x^4+1} \, dx$$

PART b

$$= \int_{-\infty}^{\infty} \frac{4x^2}{4x^4+1} \, dx$$

NOW BY THE SOPHIE GERMAIN IDENTITY

$$a^4 + 4b^4 \equiv (a^2+2b^2+2ab)(a^2+2b^2-2ab)$$

OR BY COMPLETING THE SQUARE

$$4x^4+1 = (4x^4+4x^2+1) - 4x^2 = (2x^2+1)^2 - (2x)^2$$

$$= (2x^2+1-2x)(2x^2+1+2x)$$

OR BY LOOKING AT THE LOGARITHMIC LIMIT OF PART (c) & SUSPECTING THAT $(2x^2-2x+1)(2x^2+2x+1) = 4x^4+1$

$$\dots = \int_{-\infty}^{\infty} \frac{4x^2}{4x^4+1} \, dx = \int_{-\infty}^{\infty} \frac{4x^2}{(2x^2-2x+1)(2x^2+2x+1)} \, dx$$

BOTH ARE IRREDUCIBLE

PARTIAL FRACTIONS NEXT

$$\frac{4x^2}{(2x^2-2x+1)(2x^2+2x+1)} \equiv \frac{Ax+B}{2x^2-2x+1} + \frac{Cx+D}{2x^2+2x+1}$$

$$4x^2 \equiv (Ax+B)(2x^2+2x+1) + (Cx+D)(2x^2-2x+1)$$

$$\begin{aligned} 4x^2 \equiv\ & 2Ax^3 + 2Ax^2 + Ax \\ & + 2Bx^2 + 2Bx + B \\ & 2Cx^3 - 2Cx^2 + Cx \\ & + 2Dx^2 - 2Dx + D \end{aligned}$$

$$4x^2 \equiv 2(A+C)x^3 + 2(A+B-C+D)x^2 + (A+2B+C-2D)x + (B+D)$$

$\boxed{A+C=0} \implies A+2B+C-2D = 0$

$\implies 2B - 2D = 0$

$\implies B - D = 0$ BUT $\boxed{B+D=0}$

$\boxed{B=D=0}$

THUS $2(A+B-C+D) = 4$

$2(A-C) = 4$

$\boxed{A-C=2}$

$\therefore \boxed{A=1}$ & $\boxed{C=-1}$

RETURNING TO THE INTEGRAL

$$\dots = \int_{-\infty}^{\infty} \frac{x}{2x^2-2x+1} - \frac{x}{2x^2+2x+1} \, dx$$

$$= \int_{-\infty}^{\infty} \frac{x}{2x^2-2x+1} \, dx - \int_{-\infty}^{\infty} \frac{x}{2x^2+2x+1} \, dx$$

MANIPULATE INTO LOGS & ARCTANS AS FOLLOWS

$$= \frac{1}{4}\int_{-\infty}^{\infty} \frac{(4x-2)+2}{2x^2-2x+1} \, dx - \frac{1}{4}\int_{-\infty}^{\infty} \frac{(4x+2)-2}{2x^2+2x+1} \, dx$$

$$= \frac{1}{4}\int_{-\infty}^{\infty} \frac{4x-2}{2x^2-2x+1} \, dx - \frac{1}{4}\int_{-\infty}^{\infty} \frac{4x+2}{2x^2+2x+1} \, dx + \int_{-\infty}^{\infty} \frac{\frac{1}{2}}{2x^2-2x+1} \, dx$$

$$+ \int_{-\infty}^{\infty} \frac{\frac{1}{2}}{2x^2+2x+1} \, dx$$

$$= \left[\frac{1}{4}\ln(2x^2-2x+1) - \frac{1}{4}\ln(2x^2+2x+1)\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{4x^2-x+2} \, dx$$

$$+ \int_{-\infty}^{\infty} \frac{1}{4x^2+4x+2} \, dx$$

$$= \cancel{\left[\frac{1}{4}\ln\left[\frac{2x^2-2x+1}{2x^2+2x+1}\right]\right]_{-\infty}^{\infty}} + \int_{-\infty}^{\infty} \frac{1}{(2x-1)^2+1} \, dx + \int_{-\infty}^{\infty} \frac{1}{(2x+1)^2+1} \, dx$$

PART C

THESE ARE STANDARD ARCTAN INTEGRALS, $\frac{d}{dx}\left[\arctan(2x \pm 1)\right] = \frac{2}{(2x \pm 1)^2+1}$

$$= \left[\frac{1}{2}\arctan(2x-1) + \frac{1}{2}\arctan(2x+1)\right]_{-\infty}^{\infty}$$

$$= \frac{1}{2}\lim_{\substack{a \to \infty \\ b \to -\infty}}\left[\arctan(2a-1) + \arctan(2a+1) - \arctan(2b-1) - \arctan(2b+1)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2}\right]$$

$$= \pi$$