REDUCTION FORMULAS

Question 1 (**)

$$I_n = \int_0^1 x^n \, \mathrm{e}^{-\frac{1}{2}x^2} \, dx, \;\; n \in \mathbb{N}.$$

Show clearly that …

a) … $I_n = (n-1) I_{n-2} - \mathrm{e}^{-\frac{1}{2}}$, $n \geq 2$.

b) … $\displaystyle\int_0^1 x^5 \, \mathrm{e}^{-\frac{1}{2}x^2} \, dx = 8 - 13\mathrm{e}^{-\frac{1}{2}}$.

$\boxed{\text{proof}}$

(a) $\boxed{I_n = \int_0^1 x^n e^{-\frac{1}{2}x^2} \, dx}$

$I_n = \int_0^1 x^{n-1}\left(x e^{-\frac{1}{2}x^2}\right) dx$

BY PARTS

$\begin{array}{c|c} x^{n-1} & (n-1)x^{n-2} \\ \hline -e^{-\frac{1}{2}x^2} & x e^{-\frac{1}{2}x^2} \end{array}$

$I_n = \left[-x^{n-1} e^{-\frac{1}{2}x^2}\right]_0^1 - \int_0^1 -(n-1)x^{n-2} e^{-\frac{1}{2}x^2} \, dx$

$I_n = -e^{-\frac{1}{2}} + (n-1)\int_0^1 x^{n-2} e^{-\frac{1}{2}x^2} \, dx$

$I_n = -e^{-\frac{1}{2}} + (n-1) I_{n-2}$

AS REQUIRED

(b) $\int_0^1 x^5 e^{-\frac{1}{2}x^2} \, dx = I_5$

$= 4I_3 - e^{-\frac{1}{2}}$

$= 4\left[2I_1 - e^{-\frac{1}{2}}\right] - e^{-\frac{1}{2}} = 8I_1 - 5e^{-\frac{1}{2}}$

$= 8\int_0^1 x e^{-\frac{1}{2}x^2} \, dx - 5e^{-\frac{1}{2}}$

$= 8\left[-e^{-\frac{1}{2}x^2}\right]_0^1 - 5e^{-\frac{1}{2}}$

$= 8\left[-e^{-\frac{1}{2}} + 1\right] - 5e^{-\frac{1}{2}}$

$= 8 - 13e^{-\frac{1}{2}}$

AS REQUIRED

Question 2 (**)

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x \ dx \,, \ n \in \mathbb{N} \,.$$

Show clearly that …

a) … $I_n = \dfrac{n-1}{n} I_{n-2} \,, \ n \geq 2 \,.$

b) … $\displaystyle\int_0^{\frac{\pi}{2}} \sin^6 x \left(1+\cos^2 x\right) \ dx = \frac{45\pi}{256} \,.$

$\boxed{\text{proof}}$

(a) $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx = \int_0^{\frac{\pi}{2}} \sin^{n-1} x \sin x \, dx$

BY PARTS

$\sin^{n-1} x$	$(n-1)\sin^{n-2} x \cos x$
$-\cos x$	$\sin x$

$I_n = \cancel{\left[-\cos x \sin^{n-1} x\right]_0^{\frac{\pi}{2}}} + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x \, dx$

$I_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x (1-\sin^2 x) \, dx$

$I_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x - \sin^n x \, dx$

$I_n = (n-1) I_{n-2} - (n-1) I_n$

$[1+(n-1)] I_n = (n-1) I_{n-2}$

$n I_n = (n-1) I_{n-2}$

$I_n = \frac{n-1}{n} I_{n-2}$

AS REQUIRED

(b) $\int_0^{\frac{\pi}{2}} \sin^6 x (1+\cos^2 x) \, dx = \int_0^{\frac{\pi}{2}} \sin^6 x (1+1-\sin^2 x) \, dx$

$= \int_0^{\frac{\pi}{2}} 2\sin^6 x - \sin^8 x \, dx$

$= 2I_6 - I_8 = 2I_6 - \frac{7}{8} I_6$

$= \frac{9}{8} I_6 = \frac{9}{8} \times \frac{5}{6} I_4 = \frac{9}{8} \times \frac{5}{6} \times \frac{3}{4} I_2$

$= \frac{9}{8} \times \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} I_0 = \frac{45}{128} I_0$

$= \frac{45}{128} \int_0^{\frac{\pi}{2}} 1 \, dx$

$= \frac{45}{128} \times \frac{\pi}{2}$

$= \frac{45}{256}\pi$

Question 3 (**)

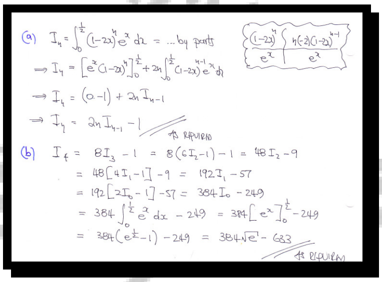
$$I_n = \int_0^{\frac{1}{2}} (1-2x)^n \, \mathrm{e}^x \, dx, \quad n \in \mathbb{N}.$$

Show clearly that …

a) …. $I_n = 2nI_{n-1} - 1, \; n \geq 1$.

b) … $I_4 = 384\sqrt{\mathrm{e}} - 633$.

$\boxed{\text{proof}}$



(a) $I_n = \int_0^{\frac{1}{2}} (1-2x)^n e^x \, dx = \dots$ by parts

$\begin{array}{c|c} (1-2x)^n & n(-2)(1-2x)^{n-1} \\ \hline e^x & e^x \end{array}$

$\Rightarrow I_n = \left[e^x (1-2x)^n \right]_0^{\frac{1}{2}} + 2n \int_0^{\frac{1}{2}} (1-2x)^{n-1} e^x \, dx$

$\Rightarrow I_n = (0-1) + 2n I_{n-1}$

$\Rightarrow I_n = 2n I_{n-1} - 1$ AS REQUIRED

(b) $I_4 = 8I_3 - 1 = 8(6I_2 - 1) - 1 = 48 I_2 - 9$

$= 48[4I_1 - 1] - 9 = 192 I_1 - 57$

$= 192[2I_0 - 1] - 57 = 384 I_0 - 249$

$= 384 \int_0^{\frac{1}{2}} e^x \, dx - 249 = 384 \left[e^x \right]_0^{\frac{1}{2}} - 249$

$= 384(e^{\frac{1}{2}} - 1) - 249 = 384\sqrt{e} - 633$ AS REQUIRED

Question 4 (**)

$$I_n = \int_0^{\frac{\pi}{2}} \cos^n x \;\; dx \,, \;\; n \in \mathbb{N} \,.$$

Show clearly that …

a) … $I_n = \dfrac{n-1}{n} I_{n-2} \,, \; n \geq 2 \,.$

b) … $\displaystyle\int_0^{\frac{\pi}{2}} \cos^5 x \sin^2 x \;\; dx = \frac{8}{105} \,.$

$\boxed{\text{proof}}$

(a) $I_n = \int_0^{\frac{\pi}{2}} \cos x \cos^{n-1} x \, dx = \dots$ BY PARTS

$\cos^{n-1} x$	$(n-1)\cos^{n-1} x(-\sin x)$
$-\sin x$	$\cos x$

$\Rightarrow I_n = \cancel{\left[-\sin x \cos^{n-1} x\right]_0^{\frac{\pi}{2}}} + (n-1)\int_0^{\frac{\pi}{2}} \sin^2 x \cos^{n-1} x \, dx$

$\Rightarrow I_n = (n-1)\int_0^{\frac{\pi}{2}} (1-\cos^2 x)\cos^{n-1} x \, dx$

$\Rightarrow I_n = (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x - \cos^n x \, dx$

$\Rightarrow I_n = (n-1) I_{n-2} - (n-1) I_n$

$\Rightarrow I_n + (n-1) I_n = (n-1) I_{n-2}$

$\Rightarrow n I_n = (n-1) I_{n-2}$

$\Rightarrow I_n = \frac{n}{n-1} I_{n-2}$ // AS REQUIRED

(b) $\int_0^{\frac{\pi}{2}} \cos^5 x \sin^2 x \, dx = \int_0^{\frac{\pi}{2}} \cos^5 x (1-\cos^2 x) \, dx = \int_0^{\frac{\pi}{2}} \cos^5 x - \cos^7 x \, dx$

$= I_5 - I_7 = I_5 - \frac{6}{7} I_5 = \frac{1}{7} I_5$

$= \frac{1}{7} \times \frac{4}{5} I_3 = \frac{1}{7} \times \frac{4}{5} \times \frac{2}{3} I_1 = \frac{8}{105} \int_0^{\frac{\pi}{2}} \cos x \, dx$

$= \frac{8}{105} \left[\sin x\right]_0^{\frac{\pi}{2}} = \frac{8}{105}(1-0) = \frac{8}{105}$ // AS REQUIRED

Question 5 (**+)

$$I_n = \int_1^{e} x(\ln x)^n \, dx, \; n \in \mathbb{N}.$$

Show clearly that …

a) … $2I_n = e^2 - nI_{n-1}$.

b) … $I_4 = \frac{1}{4}\left(e^2 - 3\right)$.

$\boxed{\text{proof}}$

(a) $\boxed{I_n = \int_1^e x(\ln x)^n \, dx}$

$I_n = \int_1^e x(\ln x)^n \, dx$ ... by parts

$(\ln x)^n$	$n(\ln x)^{n-1} \times \frac{1}{x}$
$\frac{1}{2}x^2$	x

$I_n = \left[\frac{1}{2}x^2(\ln x)^n\right]_1^e - \int_1^e \frac{1}{2}nx(\ln x)^{n-1} \, dx$

$I_n = \frac{1}{2}e^2 - 0 - \frac{1}{2}n\int_1^e x(\ln x)^{n-1} \, dx$

$I_n = \frac{1}{2}e^2 - \frac{1}{2}n I_{n-1}$

$2I_n = e^2 - nI_{n-1}$ // AS REQUIRED

(b) $\boxed{I_n = \frac{1}{2}e^2 - \frac{1}{2}nI_{n-1}}$

$I_4 = \frac{1}{2}e^2 - 2I_3 = \frac{1}{2}e^2 - 2\left[\frac{1}{2}e^2 - \frac{3}{2}I_2\right] = -\frac{1}{2}e^2 + 3I_2$

$I_4 = -\frac{1}{2}e^2 + 3\left[\frac{1}{2}e^2 - I_1\right] = -\frac{1}{2}e^2 + \frac{3}{2}e^2 - 3I_1 = e^2 - 3I_1$

$I_4 = e^2 - 3\left[\frac{1}{2}e^2 - \frac{1}{2}I_0\right] = e^2 - \frac{3}{2}e^2 + \frac{3}{2}I_0$

$I_4 = -\frac{1}{2}e^2 + \frac{3}{2}\int_1^e x \, dx = -\frac{1}{2}e^2 + \frac{3}{2}\left[\frac{1}{2}x^2\right]_1^e$

$I_4 = -\frac{1}{2}e^2 + \frac{3}{2}\left[\frac{1}{2}e^2 - \frac{1}{2}\right] = -\frac{1}{2}e^2 + \frac{3}{4}e^2 - \frac{3}{4}$

$I_4 = \frac{1}{4}e^2 - \frac{3}{4}$

$I_4 = \frac{1}{4}\left(e^2 - 3\right)$ //

Question 6 (**+)

$$I_n = \int_1^{e} x^2 (\ln x)^n \, dx, \; n \in \mathbb{N}.$$

a) Show clearly that

$$I_n = \frac{1}{3}\left(e^3 - nI_{n-1}\right).$$

The part of the curve with equation $y = 3x(\ln x)^2$, for $1 \le x \le e$, is rotated by 2π radians about the x axis.

b) Show that the volume of the solid generated is given by

$$\frac{\pi}{9}\left(11e^3 - 8\right).$$

proof

(a) $I_n = \int_1^e x^2 (\ln x)^n \, dx$ ... BY PARTS

$(\ln x)^n$	$n(\ln x)^{n-1} \times \frac{1}{x}$
$\frac{1}{3}x^3$	x^2

$I_n = \left[\frac{1}{3}x^3(\ln x)^n\right]_1^e - \int_1^e \frac{1}{3}nx^2(\ln x)^{n-1}\,dx$

$I_n = \left(\frac{1}{3}e^3 - 0\right) - \frac{1}{3}n\int_1^e x^2(\ln x)^{n-1}\,dx$

$I_n = \frac{1}{3}e^3 - \frac{1}{3}nI_{n-1}$

$I_n = \frac{1}{3}\left[e^3 - nI_{n-1}\right]$ // AS REQUIRED

(b) $V = \pi\int_{x_1}^{x_2} (y(x))^2\,dx = \pi\int_1^e \left[3x(\ln x)^2\right]^2 dx = 9\pi\int_1^e x^2(\ln x)^4\,dx$

$V = 9\pi I_4 = 9\pi \times \frac{1}{3}\left[e^3 - 4I_3\right] = 3\pi\left[e^3 - 4\times\frac{1}{3}(e^3 - 3I_2)\right]$

$V = 3\pi\left[e^3 - \frac{4}{3}e^3 + 4I_2\right] = 3\pi\left[-\frac{1}{3}e^3 + 4I_2\right] = \pi\left[-e^3 + 12I_2\right]$

$V = \pi\left[-e^3 + 12\times\frac{1}{3}(e^3 - 2I_1)\right] = \pi\left[-e^3 + 4e^3 - 8I_1\right]$

$V = \pi\left[3e^3 - 8I_1\right] = \pi\left[3e^3 - 8\times\frac{1}{3}(e^3 - I_0)\right]$

$V = \pi\left[3e^3 - \frac{8}{3}e^3 + \frac{8}{3}I_0\right] = \pi\left[\frac{1}{3}e^3 + \frac{8}{3}\int_1^e x^2\,dx\right]$

$V = \pi\left[\frac{1}{3}e^3 + \frac{8}{3}\left[\frac{1}{3}x^3\right]_1^e\right] = \pi\left[\frac{1}{3}e^3 + \frac{8}{3}\left(\frac{1}{3}e^3 - \frac{1}{3}\right)\right]$

$V = \pi\left[\frac{11}{9}e^3 - \frac{8}{9}\right] = \frac{\pi}{9}\left[11e^3 - 8\right]$ // AS REQUIRED

Question 7 (+)**

$$I_n = \int \sec^n x \, dx \, , \; n \in \mathbb{N} \, .$$

Show clearly that

$$I_n = \frac{1}{n-1}(\tan x)(\sec x)^{n-2} + \frac{n-2}{n-1} I_{n-2} \, , \; n \geq 2 \, .$$

$\boxed{\text{proof}}$

$$I_n = \int \sec^n x \, dx$$

$$I_n = \int \sec^{n-2} x \sec^2 x \, dx = \ldots \text{by parts}$$

$\sec^{n-2} x$	$(n-2)\sec^{n-2} x \tan x$
$\tan x$	$\sec^2 x$

$$I_n = \tan x \sec^{n-2} x - \int (n-2) \sec^{n-2} x \tan^2 x \, dx$$

$$I_n = \tan x \sec^{n-2} x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) \, dx$$

$$I_n = \tan x \sec^{n-2} x - (n-2) \int \sec^n x - \sec^{n-2} x \, dx$$

$$I_n = \tan x \sec^{n-2} x - (n-2) \int \sec^n x \, dx + (n-2) \int \sec^{n-2} x \, dx$$

$$I_n = \tan x \sec^{n-2} x - (n-2) I_n + (n-2) I_{n-2}$$

$$I_n + (n-2) I_n = \tan x \sec^{n-2} x \, dx + (n-2) I_{n-2}$$

$$(n-1) I_n = \tan x \sec^{n-2} x \, dx + (n-2) I_{n-2}$$

$$I_n = \frac{1}{n-1} \tan x \sec^{n-2} + \frac{n-2}{n-1} I_{n-2} \; // \text{ As required}$$

Question 8 (***)

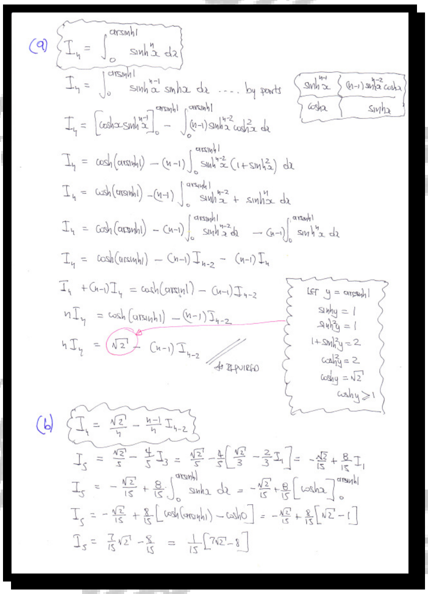
$$I_n = \int_0^{\operatorname{arsinh} 1} \sinh^n x \; dx \, , \; n \in \mathbb{N} .$$

Show clearly that …

a) … $nI_n = \sqrt{2} - (n-1) I_{n-2}$, $n \geq 2$.

b) … $I_5 = \frac{1}{15}\left(7\sqrt{2} - 8\right)$.

$\boxed{\text{proof}}$



(a) $I_n = \int_0^{\operatorname{arsinh}1} \sinh^n x \, dx$

$I_n = \int_0^{\operatorname{arsinh}1} \sinh^{n-1} x \sinh x \, dx$ …. by parts

$\sinh^{n-1} x$	$(n-1)\sinh^{n-2} x \cosh x$
$\cosh x$	$\sinh x$

$I_n = \left[\cosh x \sinh^{n-1} x\right]_0^{\operatorname{arsinh}1} - \int_0^{\operatorname{arsinh}1} (n-1)\sinh^{n-2} x \cosh^2 x \, dx$

$I_n = \cosh(\operatorname{arsinh}1) - (n-1)\int_0^{\operatorname{arsinh}1} \sinh^{n-2} x (1 + \sinh^2 x) \, dx$

$I_n = \cosh(\operatorname{arsinh}1) - (n-1)\int_0^{\operatorname{arsinh}1} \sinh^{n-2} x + \sinh^n x \, dx$

$I_n = \cosh(\operatorname{arsinh}1) - (n-1)\int_0^{\operatorname{arsinh}1} \sinh^{n-2} x \, dx - (n-1)\int_0^{\operatorname{arsinh}1} \sinh^n x \, dx$

$I_n = \cosh(\operatorname{arsinh}1) - (n-1) I_{n-2} - (n-1) I_n$

$I_n + (n-1) I_n = \cosh(\operatorname{arsinh}1) - (n-1) I_{n-2}$

$n I_n = \cosh(\operatorname{arsinh}1) - (n-1) I_{n-2}$

$n I_n = \sqrt{2} - (n-1) I_{n-2}$ // AS REQUIRED

LET $y = \operatorname{arsinh}1$
$\sinh y = 1$
$\sinh^2 y = 1$
$1 + \sinh^2 y = 2$
$\cosh^2 y = 2$
$\cosh y = \sqrt{2}$
$\cosh y \geq 1$

(b) $I_n = \frac{\sqrt{2}}{n} - \frac{n-1}{n} I_{n-2}$

$I_5 = \frac{\sqrt{2}}{5} - \frac{4}{5} I_3 = \frac{\sqrt{2}}{5} - \frac{4}{5}\left[\frac{\sqrt{2}}{3} - \frac{2}{3} I_1\right] = -\frac{\sqrt{2}}{15} + \frac{8}{15} I_1$

$I_5 = -\frac{\sqrt{2}}{15} + \frac{8}{15}\int_0^{\operatorname{arsinh}1} \sinh x \, dx = -\frac{\sqrt{2}}{15} + \frac{8}{15}\left[\cosh x\right]_0^{\operatorname{arsinh}1}$

$I_5 = -\frac{\sqrt{2}}{15} + \frac{8}{15}\left[\cosh(\operatorname{arsinh}1) - \cosh 0\right] = -\frac{\sqrt{2}}{15} + \frac{8}{15}\left[\sqrt{2} - 1\right]$

$I_5 = \frac{7}{15}\sqrt{2} - \frac{8}{15} = \frac{1}{15}\left[7\sqrt{2} - 8\right]$

Question 9 **(***)**

The integral I_n is defined for $n \geq 0$ as

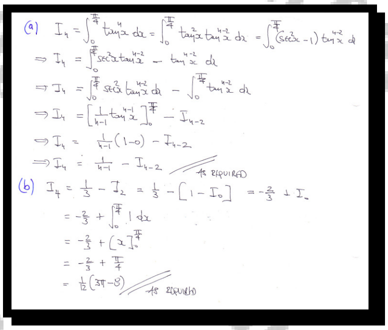
$$I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx \,, \; n \in \mathbb{N} \,.$$

Show clearly that …

a) … $I_n = \dfrac{1}{n-1} - I_{n-2} \,, \; n \geq 1$.

b) … $I_4 = \dfrac{1}{12}(3\pi - 8)$.

$\boxed{\text{proof}}$



(a) $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx = \int_0^{\frac{\pi}{4}} \tan^2 x \tan^{n-2} x \, dx = \int_0^{\frac{\pi}{4}} (\sec^2 x - 1) \tan^{n-2} x \, dx$

$\Rightarrow I_n = \int_0^{\frac{\pi}{4}} \sec^2 x \tan^{n-2} x - \tan^{n-2} x \, dx$

$\Rightarrow I_n = \int_0^{\frac{\pi}{4}} \sec^2 x \tan^{n-2} x \, dx - \int_0^{\frac{\pi}{4}} \tan^{n-2} x \, dx$

$\Rightarrow I_n = \left[\frac{1}{n-1} \tan^{n-1} x\right]_0^{\frac{\pi}{4}} - I_{n-2}$

$\Rightarrow I_n = \frac{1}{n-1}(1-0) - I_{n-2}$

$\Rightarrow I_n = \frac{1}{n-1} - I_{n-2}$ AS REQUIRED

(b) $I_4 = \frac{1}{3} - I_2 = \frac{1}{3} - [1 - I_0] = -\frac{2}{3} + I_0$

$= -\frac{2}{3} + \int_0^{\frac{\pi}{4}} 1 \, dx$

$= -\frac{2}{3} + [x]_0^{\frac{\pi}{4}}$

$= -\frac{2}{3} + \frac{\pi}{4}$

$= \frac{1}{12}(3\pi - 8)$ AS REQUIRED

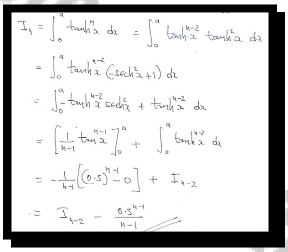
Question 10 (***)

$$I_n = \int_0^a \tanh^n x \ dx\,,\ n \in \mathbb{N}.$$

Given that $a = \text{artanh}\,\frac{1}{2}$, show clearly that

$$I_n = I_{n-2} - \frac{(0.5)^{n-1}}{n-1}\,,\ n \geq 2\,.$$

$\boxed{\text{proof}}$



$$I_n = \int_0^a \tanh^n x \ dx = \int_0^a \tanh^{n-2} x \tanh^2 x \ dx$$

$$= \int_0^a \tanh^{n-2} x \left(-\text{sech}^2 x + 1\right) dx$$

$$= \int_0^a -\tanh^{n-2} x \,\text{sech}^2 x + \tanh^{n-2} x \ dx$$

$$= \left[\frac{1}{n-1} \tan^{n-1} x\right]_0^a + \int_0^a \tanh^{n-2} x \ dx$$

$$= -\frac{1}{n-1}\left[(0.5)^{n-1} - 0\right] + I_{n-2}$$

$$= I_{n-2} - \frac{0.5^{n-1}}{n-1}$$

Question 11 **(***+)**

By forming and using a suitable reduction formula show that

$$\int_0^1 x^5 \, e^{-x^2} \, dx = \frac{2e-5}{2e}.$$

No credit will be given if no reduction formula is not used in this question

$\boxed{}$, $\boxed{\text{proof}}$

SET UP A REDUCTION FORMULA IN TERMS OF n, $n \in \mathbb{N}$

$$I_n = \int_0^1 x^n e^{-x^2} \, dx$$

$$I_n = \int_0^1 x^{n-1} \left(x e^{-x^2} \right) dx$$

PROCEED BY INTEGRATION BY PARTS

x^{n-1}	$(n-1)x^{n-2}$
$-\frac{1}{2}e^{-x^2}$	xe^{-x^2}

$$I_n = \left[-\tfrac{1}{2} x^{n-1} e^{-x^2} \right]_0^1 - \int_0^1 -\tfrac{1}{2}(n-1) x^{n-2} e^{-x^2} \, dx$$

$$I_n = -\tfrac{1}{2}e^{-1} + \tfrac{1}{2}(n-1) \int_0^1 x^{n-2} e^{-x^2} \, dx$$

$$I_n = -\tfrac{1}{2}e^{-1} + \tfrac{1}{2}(n-1) I_{n-2}$$

USING THE FORMULA DERIVED TO OBTAIN I_5

$$\Rightarrow I_5 = -\tfrac{1}{2}e^{-1} + \tfrac{1}{2} \times 4 \times I_3 = -\tfrac{1}{2}e^{-1} + 2I_3$$

$$\Rightarrow I_5 = -\tfrac{1}{2}e^{-1} + 2\left[-\tfrac{1}{2}e^{-1} + \tfrac{1}{2} \times 2 \, I_1 \right] = -\tfrac{3}{2}e^{-1} + 2I_1$$

$$\Rightarrow I_5 = -\tfrac{3}{2}e^{-1} + 2\int_0^1 x e^{-x^2} \, dx$$

BY RECOGNITION

$$\Rightarrow I_5 = -\tfrac{3}{2}e^{-1} + 2\left[-\tfrac{1}{2}e^{-x^2} \right]_0^1$$

$$\Rightarrow I_5 = -\tfrac{3}{2}e^{-1} + 2\left[-\tfrac{1}{2}e^{-1} + \tfrac{1}{2} \right] = -\tfrac{5}{2}e^{-1} + 1$$

$$\Rightarrow I_5 = 1 - \frac{5}{2e}$$

$$\Rightarrow I_5 = \frac{2e-5}{2e}$$

AS REQUIRED

Question 12 (***+)

The integral I_n is defined for $n \geq 0$ as

$$I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, dx .$$

a) Show clearly that …

$$\ldots \ I_n = \left(\frac{\pi}{2}\right)^n - n(n-1) I_{n-2}, \ n \geq 2 .$$

b) Hence find, in terms of π, exact expressions for …

i. … $\displaystyle\int_0^{\frac{\pi}{2}} x^4 \cos x \, dx$.

ii. … $\displaystyle\int_0^{\frac{\pi}{2}} x^5 \sin x \, dx$.

$\boxed{}$, $\boxed{\displaystyle\int_0^{\frac{\pi}{2}} x^4 \cos x \, dx = \frac{\pi^4}{16} - 3\pi^2 + 24}$, $\boxed{\displaystyle\int_0^{\frac{\pi}{2}} x^5 \sin x \, dx = \frac{5\pi^4}{16} - 15\pi^2 + 120}$

a) PROCEED BY INTEGRATION BY PARTS

$$\Rightarrow I_n = \int_0^{\pi/2} x^n \cos x \, dx$$

$$\begin{array}{c|c} x^n & nx^{n-1} \\ \hline \sin x & \cos x \end{array}$$

$$\Rightarrow I_n = \left[x^n \sin x\right]_0^{\pi/2} - n\int_0^{\frac{\pi}{2}} x^{n-1} \sin x \, dx$$

$$\Rightarrow I_n = \left(\frac{\pi}{2}\right)^n \times 1 - 0 - n\int_0^{\frac{\pi}{2}} x^{n-1} \sin x \, dx$$

$$\Rightarrow I_n = \left(\frac{\pi}{2}\right)^n - n\int_0^{\frac{\pi}{2}} x^{n-1} \sin x$$

INTEGRATION BY PARTS AGAIN

$$\begin{array}{c|c} x^{n-1} & (n-1)x^{n-2} \\ \hline -\cos x & \sin x \end{array}$$

$$\Rightarrow I_n = \left(\frac{\pi}{2}\right)^n - n\left[\cancel{\left[-x^{n-1}\cos x\right]_0^{\pi/2}} + (n-1)\int_0^{\pi/2} x^{n-2} \cos x \, dx\right]$$

ZERO AT BOTH ENDS

$$\Rightarrow I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)\int_0^{\pi/2} x^{n-2} \cos x \, dx$$

$$\therefore I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)\int_0^{\frac{\pi}{2}} x^{n-2} \cos x \, dx$$

AS REQUIRED

b) I) REWRITE IN "I" NOTATION

$$\begin{aligned} \int_0^{\pi/2} x^4 \cos x \, dx &= I_4 \\ &= \left(\frac{\pi}{2}\right)^4 - 4\times 3\, I_2 = \frac{\pi^4}{16} - 12 I_2 \\ &= \frac{\pi^4}{16} - 12\left[\left(\frac{\pi}{2}\right)^2 - 2\times 1\times I_0\right] \\ &= \frac{\pi^4}{16} - 12\times\frac{\pi^2}{4} + 24\, I_0 \\ &= \frac{\pi^4}{16} - 3\pi^2 + 24\int_0^{\pi/2} \cos x \, dx \\ &= \frac{\pi^4}{16} - 3\pi^2 + 24\left[\sin x\right]_0^{\pi/2} \\ &= \frac{\pi^4}{16} - 3\pi^2 + 24 \end{aligned}$$

II) START BY INTEGRATION BY PARTS

$$\begin{array}{c|c} x^5 & 5x^4 \\ \hline -\cos x & \sin x \end{array}$$

$$\int_0^{\frac{\pi}{2}} x^5 \sin x = \cancel{\left[-x^5 \cos x\right]_0^{\pi/2}} - \int -5x^4 \cos x \, dx$$

ZERO AT BOTH ENDS

$$\begin{aligned} &= 5\int x^4 \cos x \, dx \\ &= 5 I_4 \\ &= 5\left(\frac{\pi^4}{16} - 3\pi^2 + 24\right) \\ &= \frac{5}{16}\pi^4 - 15\pi^2 + 120 \end{aligned}$$

Question 13 (***+)

The integral I_n is defined for $n \geq 0$ as

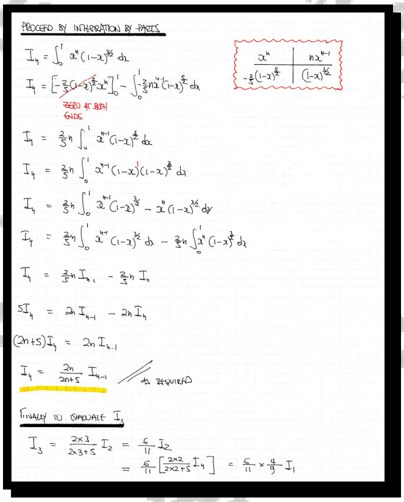
$$I_n = \int_0^1 x^n (1-x)^{\frac{3}{2}} \, dx \, , \; n \in \mathbb{N} .$$

Show that

$$I_n = \left(\frac{2n}{2n+5} \right) I_{n-1} \, , \; n \geq 1,$$

and use it to find as an exact fraction the value of I_3.

$\boxed{}$, $\boxed{I_3 = \frac{32}{1155}}$



PROCEED BY INTEGRATION BY PARTS

x^n	nx^{n-1}
$-\frac{2}{5}(1-x)^{\frac{5}{2}}$	$(1-x)^{\frac{3}{2}}$

$I_n = \int_0^1 x^n (1-x)^{3/2} \, dx$

$I_n = \left[\cancel{-\frac{2}{5}(1-x)^{\frac{5}{2}} x^n} \right]_0^1 - \int_0^1 -\frac{2}{5} n x^{n-1} (1-x)^{\frac{5}{2}} \, dx$

ZERO AT BOTH ENDS

$I_n = \frac{2}{5} n \int_0^1 x^{n-1} (1-x)^{\frac{5}{2}} \, dx$

$I_n = \frac{2}{5} n \int_0^1 x^{n-1} (1-x)^1 (1-x)^{\frac{3}{2}} \, dx$

$I_n = \frac{2}{5} n \int_0^1 x^{n-1} (1-x)^{\frac{3}{2}} - x^n (1-x)^{3/2} \, dx$

$I_n = \frac{2}{5} n \int_0^1 x^{n-1} (1-x)^{3/2} \, dx - \frac{2}{5} n \int_0^1 x^n (1-x)^{\frac{3}{2}} \, dx$

$I_n = \frac{2}{5} n I_{n-1} - \frac{2}{5} n I_n$

$5 I_n = 2n I_{n-1} - 2n I_n$

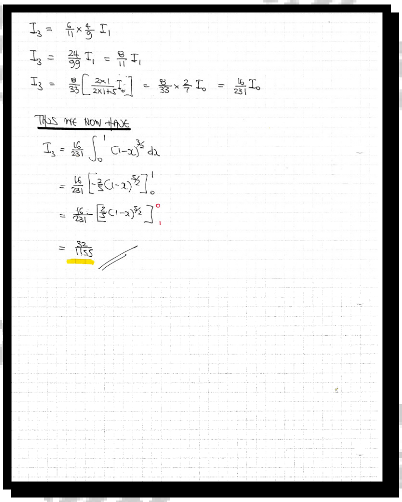
$(2n+5) I_n = 2n I_{n-1}$

$I_n = \frac{2n}{2n+5} I_{n-1}$ // AS REQUIRED

FINALLY TO EVALUATE I_3

$I_3 = \frac{2 \times 3}{2 \times 3 + 5} I_2 = \frac{6}{11} I_2$

$= \frac{6}{11} \left[\frac{2 \times 2}{2 \times 2 + 5} I_1 \right] = \frac{6}{11} \times \frac{4}{9} I_1$



$I_3 = \frac{6}{11} \times \frac{4}{9} I_1$

$I_3 = \frac{24}{99} I_1 = \frac{8}{11} I_1$

$I_3 = \frac{8}{33} \left[\frac{2 \times 1}{2 \times 1 + 5} I_0 \right] = \frac{8}{33} \times \frac{2}{7} I_0 = \frac{16}{231} I_0$

THUS WE NOW HAVE

$I_3 = \frac{16}{231} \int_0^1 (1-x)^{3/2} \, dx$

$= \frac{16}{231} \left[-\frac{2}{5} (1-x)^{5/2} \right]_0^1$

$= \frac{16}{231} \left[\frac{2}{5} (1-x)^{5/2} \right]_1^0$

$= \frac{32}{1155}$ //

Question 14 (***+)

$$I_n = \int_0^{\ln 2} \tanh^n x \ dx, \ n \in \mathbb{N}.$$

Show clearly that …

a) … $I_n = I_{n-2} - \dfrac{1}{n-1}\left(\dfrac{3}{5}\right)^{n-1}, \ n \geq 2$.

b) … $\displaystyle\sum_{r=1}^{\infty} \frac{1}{2r}\left(\frac{3}{5}\right)^{2r} = \ln\left(\frac{5}{4}\right)$.

$\boxed{\text{proof}}$

(a) $I_n = \int_0^{\ln 2} \tanh^n x \, dx = \int_0^{\ln 2} \tanh^{n-2} x \tanh^2 x \, dx = \int_0^{\ln 2} \tanh^{n-2} x (1 - \operatorname{sech}^2 x) \, dx$

$= \int_0^{\ln 2} \tanh^{n-2} x \, dx - \operatorname{sech}^2 x \tanh^{n-2} x \, dx = \int_0^{\ln 2} \tanh^{n-2} x \, dx - \int_0^{\ln 2} \operatorname{sech}^2 x \tanh^{n-2} x \, dx$

$= I_{n-2} - \frac{1}{n-1}\left[\tanh^{n-1} x\right]_0^{\ln 2} = I_{n-2} - \frac{1}{n-1}\left(\tanh(\ln 2)\right)^{n-1}$

Now $\tanh(\ln 2) = \dfrac{e^{2\ln 2} - 1}{e^{2\ln 2} + 1} = \dfrac{4-1}{4+1} = \dfrac{3}{5}$

$\therefore I_n = I_{n-2} - \frac{1}{n-1}\left(\frac{3}{5}\right)^{n-1}$ // AS REQUIRED

(b) $\frac{1}{n-1}\left(\frac{3}{5}\right)^{n-1} \equiv I_{n-2} - I_n$

$n=3$ $\quad \frac{1}{2}\left(\frac{3}{5}\right)^2 = I_1 - \cancel{I_3}$

$n=5$ $\quad \frac{1}{4}\left(\frac{3}{5}\right)^4 = \cancel{I_3} - \cancel{I_5}$

$n=7$ $\quad \frac{1}{6}\left(\frac{3}{5}\right)^6 = \cancel{I_5} - \cancel{I_7}$

$n=9$ $\quad \frac{1}{8}\left(\frac{3}{5}\right)^8 = \cancel{I_7} - \cancel{I_9}$

$\quad \vdots \qquad \vdots \qquad \vdots$

$\therefore \sum_{r=1}^{\infty} \frac{1}{2r}\left(\frac{3}{5}\right)^{2r} = I_1 = \int_0^{\ln 2} \tanh x \, dx = \int_0^{\ln 2} \frac{\sinh x}{\cosh x} \, dx$

$= \left[\ln|\cosh x|\right]_0^{\ln 2} = \ln(\cosh(\ln 2)) - \ln(\cosh 0)$

$= \ln\left[\frac{1}{2}e^{\ln 2} + \frac{1}{2}e^{-\ln 2}\right] - \cancel{\ln 1} = \ln\left[1 + \frac{1}{4}\right]$

$= \ln \frac{5}{4}$ //

Question 15 (***+)

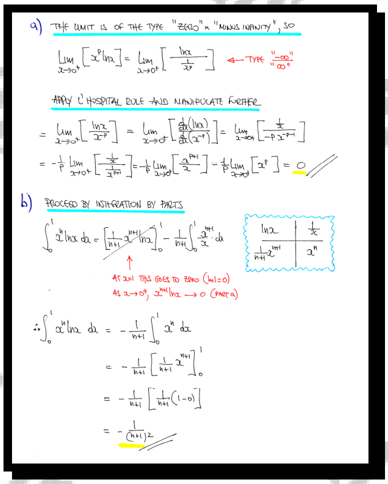
a) If $p \in (0,\infty)$, show that

$$\lim_{x \to 0^+} \left[x^p \ln x \right] = 0, \quad x \in (0,\infty).$$

b) Hence find a simplified expression for

$$\int_0^1 x^n \ln x \ dx, \ n \in \mathbb{N}.$$

$$\boxed{-\frac{1}{(n+1)^2}}$$



a) THE LIMIT IS OF THE TYPE "ZERO" × "MINUS INFINITY", SO

$$\lim_{x \to 0^+}\left[x^p \ln x\right] = \lim_{x \to 0^+}\left[\frac{\ln x}{\frac{1}{x^p}}\right] \quad \longleftarrow \text{TYPE } \frac{\text{"}-\infty\text{"}}{\text{"}\infty\text{"}}$$

APPLY L'HOSPITAL RULE AND MANIPULATE FURTHER

$$= \lim_{x \to 0^+}\left[\frac{\ln x}{x^{-p}}\right] = \lim_{x \to 0^+}\left[\frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x^{-p})}\right] = \lim_{x \to 0^+}\left[\frac{\frac{1}{x}}{-p x^{-p-1}}\right]$$

$$= -\frac{1}{p}\lim_{x \to 0^+}\left[\frac{\frac{1}{x}}{\frac{1}{x^{p+1}}}\right] = -\frac{1}{p}\lim_{x \to 0^+}\left[\frac{x^{p+1}}{x}\right] = -\frac{1}{p}\lim_{x \to 0^+}\left[x^p\right] = 0$$

b) PROCEED BY INTEGRATION BY PARTS

$$\int_0^1 x^n \ln x \, dx = \cancel{\left[\frac{1}{n+1}x^{n+1}\ln x\right]_0^1} - \frac{1}{n+1}\int_0^1 \frac{x^{n+1}}{x} \, dx$$

$\ln x$	$\frac{1}{x}$
$\frac{1}{n+1}x^{n+1}$	x^n

AT $x=1$ THIS GOES TO ZERO ($\ln 1 = 0$)
AS $x \to 0^+$, $x^{n+1}\ln x \to 0$ (PART a)

$$\therefore \int_0^1 x^n \ln x \, dx = -\frac{1}{n+1}\int_0^1 x^n \, dx$$

$$= -\frac{1}{n+1}\left[\frac{1}{n+1}x^{n+1}\right]_0^1$$

$$= -\frac{1}{n+1}\left[\frac{1}{n+1}(1-0)\right]$$

$$= -\frac{1}{(n+1)^2}$$

Question 16 (***+)

$$I_n = \int \operatorname{cosec}^n x \, dx, \quad n \in \mathbb{N}.$$

a) Show clearly that

$$I_n = \frac{n-2}{n-1} I_{n-2} - \frac{1}{n-1}(\cot x)(\operatorname{cosec} x)^{n-2}, \quad n \geq 2.$$

b) Use part **(a)** to evaluate

$$I_n = \int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \operatorname{cosec}^6 x \; dx$$

$\boxed{}$, $\boxed{\dfrac{28}{15}}$

a) PROCEED AS PER USUAL BY INTEGRATION BY PARTS

$$I_n = \int \operatorname{cosec}^n x \, dx = \int \operatorname{cosec}^2 x \operatorname{cosec}^{n-2} x \, dx$$

$\operatorname{cosec}^{n-2} x$	$-(n-2)\operatorname{cosec}^{n-2} x \cot x$
$-\cot x$	$\operatorname{cosec}^2 x$

$$I_n = -\cot x \operatorname{cosec}^{n-2} x - (n-2)\int \operatorname{cosec}^{n-2} x \cot^2 x \, dx$$

$$I_n = -\cot x \operatorname{cosec}^{n-2} x - (n-2)\int \operatorname{cosec}^{n-2} x (\operatorname{cosec}^2 x - 1) \, dx$$

$$I_n = -\cot x \operatorname{cosec}^{n-2} x - (n-2)\int \operatorname{cosec}^{n} x \, dx - \operatorname{cosec}^{n-2} x \, dx$$

$$I_n = -\cot x \operatorname{cosec}^{n-2} x - (n-2)\int \operatorname{cosec}^{n} x \, dx + (n-2)\int \operatorname{cosec}^{n-2} x \, dx$$

$$I_n = -\cot x \operatorname{cosec}^{n-2} x - (n-2) I_n + (n-2) I_{n-2}$$

$$(n-1) I_n = (n-2) I_{n-2} - \cot x \operatorname{cosec}^{n-2} x$$

$$I_n = \frac{n-2}{n-1} I_{n-2} - \frac{1}{n-1} \cot x \operatorname{cosec}^{n-2} x$$

AS REQUIRED

b) USING THE FORMULA OF PART (a) WITH LIMITS

$$I_n = \frac{n-2}{n-1} I_{n-2} - \frac{1}{n-1}\Big[\cot x \operatorname{cosec}^{n-2} x\Big]_{\pi/4}^{\pi/2}$$

$$I_n = \frac{n-2}{n-1} I_{n-2} - \frac{1}{n-1}\Big[0 - 1 \times (\sqrt{2})^{n-2}\Big]$$

$$I_n = \frac{n-2}{n-1} I_{n-2} + \frac{(\sqrt{2})^{n-2}}{n-1}$$

USING THE ABOVE FOR I_6

$$I_6 = \frac{4}{5} I_4 + \frac{(\sqrt{2})^4}{5} = \frac{4}{5} I_4 + \frac{4}{5}$$

$$I_6 = \frac{4}{5}\left[\frac{2}{3} I_2 + \frac{(\sqrt{2})^2}{3}\right] + \frac{4}{5}$$

$$I_6 = \frac{8}{15} I_2 + \frac{8}{15} + \frac{4}{5}$$

$$I_6 = \frac{8}{15} I_2 + \frac{4}{3}$$

REAPPLY ONCE MORE BY DIRECT INTEGRATION OR THE FORMULA WITH $n=2$

$$I_6 = \frac{8}{15}\left[0 + \frac{(\sqrt{2})^0}{1}\right] + \frac{4}{3}$$

$$I_6 = \frac{8}{15} + \frac{4}{3}$$

$$I_6 = \frac{28}{15}$$

Question 17 **(****)**

$$I_n = \int_0^1 \left(1 - x^2\right)^n \ dx \, , n \in \mathbb{N} \, .$$

Show clearly that

$$I_n = \frac{2n}{2n+1} I_{n-1} \, , \ n \geq 1 \, .$$

proof

$$I_n = \int_0^1 (1-x^2)^n \, dx = \int^1 1 \times (1-x^2)^n \, dx$$

BY PARTS

$(1-x^2)^n$	$n(-2x)(1-x^2)^{n-1}$
x	1

$$I_n = \cancel{\left[x(1-x^2)^n\right]_0^1} + 2n\int_0^1 x^2(1-x^2)^{n-1} \, dx.$$

$$I_n = 2n\int_0^1 \left[1-(1-x^2)\right](1-x^2)^{n-1} \, dx$$

$$I_n = 2n\int_0^1 (1-x^2)^{n-1} - (1-x^2)(1-x^2)^{n-1} \, dx$$

$$I_n = 2n\int_0^1 (1-x^2)^{n-1} - (1-x^2)^n \, dx$$

$$I_n = 2n\, I_{n-1} - 2n\, I_n$$

$$(1+2n) I_n = 2n\, I_{n-1}$$

$$I_n = \frac{2n}{2n+1} I_{n-1}$$

AS REQUIRED

Question 18 (****)

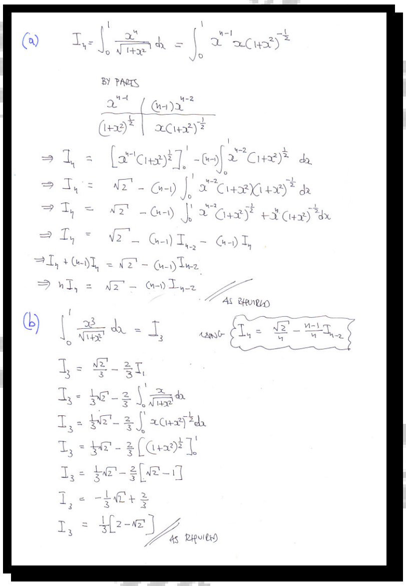
$$I_n = \int_0^1 \frac{x^n}{\sqrt{1+x^2}}\ dx\ ,\ n \in \mathbb{N}\,.$$

Show clearly that…

a) … $nI_n = \sqrt{2} - (n-1)I_{n-2}\,,\ n \geq 2\,.$

b) … $\displaystyle\int_0^1 \frac{x^3}{\sqrt{1+x^2}}\ dx = \frac{1}{3}\left(2-\sqrt{2}\right).$

$\boxed{\text{proof}}$



(a) $I_n = \int_0^1 \frac{x^n}{\sqrt{1+x^2}}\,dx = \int_0^1 x^{n-1}\,x(1+x^2)^{-\frac{1}{2}}$

BY PARTS

$$\begin{array}{c|c} x^{n-1} & (n-1)x^{n-2} \\ \hline (1+x^2)^{\frac{1}{2}} & x(1+x^2)^{-\frac{1}{2}} \end{array}$$

$\Rightarrow I_n = \left[x^{n-1}(1+x^2)^{\frac{1}{2}}\right]_0^1 - (n-1)\int_0^1 x^{n-2}(1+x^2)^{\frac{1}{2}}\,dx$

$\Rightarrow I_n = \sqrt{2} - (n-1)\int_0^1 x^{n-2}(1+x^2)(1+x^2)^{-\frac{1}{2}}\,dx$

$\Rightarrow I_n = \sqrt{2} - (n-1)\int_0^1 x^{n-2}(1+x^2)^{-\frac{1}{2}} + x^n(1+x^2)^{-\frac{1}{2}}\,dx$

$\Rightarrow I_n = \sqrt{2} - (n-1)I_{n-2} - (n-1)I_n$

$\Rightarrow I_n + (n-1)I_n = \sqrt{2} - (n-1)I_{n-2}$

$\Rightarrow nI_n = \sqrt{2} - (n-1)I_{n-2}$ // AS REQUIRED

(b) $\int_0^1 \frac{x^3}{\sqrt{1+x^2}}\,dx = I_3$ USING $\left\{I_n = \frac{\sqrt{2}}{n} - \frac{n-1}{n}I_{n-2}\right\}$

$I_3 = \frac{\sqrt{2}}{3} - \frac{2}{3}I_1$

$I_3 = \frac{1}{3}\sqrt{2} - \frac{2}{3}\int_0^1 \frac{x}{\sqrt{1+x^2}}\,dx$

$I_3 = \frac{1}{3}\sqrt{2} - \frac{2}{3}\int_0^1 x(1+x^2)^{-\frac{1}{2}}\,dx$

$I_3 = \frac{1}{3}\sqrt{2} - \frac{2}{3}\left[(1+x^2)^{\frac{1}{2}}\right]_0^1$

$I_3 = \frac{1}{3}\sqrt{2} - \frac{2}{3}\left[\sqrt{2} - 1\right]$

$I_3 = -\frac{1}{3}\sqrt{2} + \frac{2}{3}$

$I_3 = \frac{1}{3}\left[2 - \sqrt{2}\right]$ // AS REQUIRED

Question 19 (****)

$$I_n = \int_0^{\sqrt{3}} \left(3-x^2\right)^n \, dx \, , \ n \in \mathbb{N} .$$

Show clearly that…

a) … $(2n+1)I_n = 6nI_{n-1}$, $n \geq 1$.

b) … $\displaystyle\int_0^{\sqrt{3}} \left(3-x^2\right)^4 \, dx = \frac{1152}{35}\sqrt{3}$.

proof

a) $I_n = \int_0^{\sqrt{3}} (3-x^2)^n \, dx = \dots$ BY PARTS

$(3-x^2)^n$	$-2xn(3-x^2)^{n-1}$
x	1

$I_n = \cancel{\left[x(3-x^2)^n\right]_0^{\sqrt{3}}} + 2n\int_0^{\sqrt{3}} x^2(3-x^2)^{n-1} \, dx$

$I_n = -2n\int_0^{\sqrt{3}} -x^2(3-x^2)^{n-1} \, dx$

$I_n = -2n\int_0^{\sqrt{3}} 3(3-x^2)^{n-1} - x^2(3-x^2)^{n-1} - 3(3-x^2)^{n-1} \, dx$

$I_n = -2n\int_0^{\sqrt{3}} (3-x^2)(3-x^2)^{n-1} - 3(3-x^2)^{n-1} \, dx$

$I_n = -2n\int_0^{\sqrt{3}} (3-x^2)^n - 3(3-x^2)^{n-1} \, dx$

$I_n = -2nI_n + 6nI_{n-1}$

$(2n+1)I_n = 6nI_{n-1}$

$I_n = \frac{6n}{2n+1}I_{n-1}$

b) $I_4 = \frac{24}{9}I_3 = \frac{8}{3}I_3 = \frac{8}{3} \times \frac{18}{7}I_2 = \frac{48}{7}I_2 = \frac{48}{7} \times \frac{12}{5}I_1$

$= \frac{576}{35}I_1 = \frac{576}{35} \times \frac{6}{3}I_0 = \frac{1152}{35}I_0 = \frac{1152}{35}\int_0^{\sqrt{3}} 1 \, dx$

$= \frac{1152}{35}\sqrt{3}$

Question 20 (****)

$$I_n = \int_0^1 x^n \sqrt{1-x^2}\ dx\,,\ n \in \mathbb{N}\,.$$

Show clearly that…

a) … $(n+2)I_n = (n-1)I_{n-2}\,,\ n \geq 2\,.$

b) … $\displaystyle\int_0^1 x^n \sqrt{1-x^2}\ dx = \frac{16}{315}\,.$

proof

(a) $I_n = \int_0^1 x^n(1-x^2)^{\frac{1}{2}}\,dx = \int_0^1 x^{n-1}\,x(1-x^2)^{\frac{1}{2}}\,dx$

BY PARTS

x^{n-1}	$(n-1)x^{n-2}$
$-\frac{1}{3}(1-x^2)^{\frac{3}{2}}$	$x(1-x^2)^{\frac{1}{2}}$

$I_n = \cancel{\left[-\frac{1}{3}x^{n-1}(1-x^2)^{\frac{3}{2}}\right]_0^1} - \int_0^1 -\frac{1}{3}(n-1)x^{n-2}(1-x^2)^{\frac{3}{2}}\,dx$

$I_n = \frac{1}{3}(n-1)\int_0^1 x^{n-2}(1-x^2)^1(1-x^2)^{\frac{1}{2}}\,dx$

$I_n = \frac{1}{3}(n-1)\int_0^1 x^{n-2}(1-x^2)^{\frac{1}{2}} - x^n(1-x^2)^{\frac{1}{2}}\,dx$

$3I_n = (n-1)\left[I_{n-2} - I_n\right]$

$3I_n = (n-1)I_{n-2} - (n-1)I_n$

$\left[3+(n-1)\right]I_n = (n-1)I_{n-2}$

$(n+2)I_n = (n-1)I_{n-2}$ // AS REQUIRED

(b) $I_n = \frac{n-1}{n+2}I_{n-2}$

$\int_0^1 x^7(1-x^2)^{\frac{1}{2}}\,dx = I_7 = \frac{6}{9}I_5 = \frac{2}{3}\left(\frac{4}{7}I_3\right)$

$= \frac{2}{3}\times\frac{4}{7}\times\left(\frac{2}{5}I_1\right) = \frac{16}{105}I_1$

$= \frac{16}{105}\int_0^1 x(1-x^2)^{\frac{1}{2}}\,dx$

$= \frac{16}{105}\left[-\frac{1}{3}(1-x^2)^{\frac{3}{2}}\right]_0^1$

$= \frac{16}{315}\left[(1-x^2)^{\frac{3}{2}}\right]_1^0$

$= \frac{16}{315}$ //

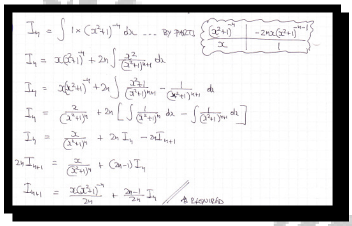
Question 21 (****)

$$I_n = \int \left(1+x^2\right)^{-n} \, dx, \; n \in \mathbb{N}.$$

Show clearly that

$$I_{n+1} = \frac{x\left(x^2+1\right)^{-n}}{2n} + \frac{2n-1}{2n} I_n, \; n \geq 1.$$

proof



$I_n = \int 1 \times (x^2+1)^{-n} \, dx$ ... BY PARTS $\begin{array}{c|c} (x^2+1)^{-n} & -2nx(x^2+1)^{-n-1} \\ \hline x & 1 \end{array}$

$I_n = x(x^2+1)^{-n} + 2n \int \frac{x^2}{(x^2+1)^{n+1}} \, dx$

$I_n = x(x^2+1)^{-n} + 2n \int \frac{x^2+1}{(x^2+1)^{n+1}} - \frac{1}{(x^2+1)^{n+1}} \, dx$

$I_n = \frac{x}{(x^2+1)^n} + 2n \left[\int \frac{1}{(x^2+1)^n} \, dx - \int \frac{1}{(x^2+1)^{n+1}} \, dx \right]$

$I_n = \frac{x}{(x^2+1)^n} + 2n I_n - 2n I_{n+1}$

$2n I_{n+1} = \frac{x}{(x^2+1)^n} + (2n-1) I_n$

$I_{n+1} = \frac{x(x^2+1)^{-n}}{2n} + \frac{2n-1}{2n} I_n$ // AS REQUIRED

Question 22 (****)

The integral I_n is defined for $n \geq 0$ as

$$I_n = \int_0^{\pi} \theta^n \sin\theta \, d\theta .$$

Show clearly that …

a) … $I_n = \pi^n - n(n-1)I_{n-2}$, $n \geq 2$.

b) … $\displaystyle\int_0^{\frac{\pi}{2}} x^4 \sin 2x \, dx = \frac{1}{32}\left(\pi^4 - 12\pi^2 + 48\right)$.

$\boxed{\text{proof}}$

(a) $I_n = \int_0^{\pi} \theta^n \sin\theta \, d\theta = $ by parts

θ^n	$n\theta^{n-1}$
$-\cos\theta$	$\sin\theta$

$I_n = \left[-\theta^n \cos\theta\right]_0^{\pi} - \int_0^{\pi} -n\theta^{n-1}\cos\theta \, d\theta$

$I_n = \left(-\pi^n \cos\pi\right) - (0) + n\int_0^{\pi} \theta^{n-1}\cos\theta \, d\theta$

$I_n = \pi^n + n\int_0^{\pi} \theta^{n-1}\cos\theta \, d\theta = \dots$ by parts again

θ^{n-1}	$(n-1)\theta^{n-2}$
$\sin\theta$	$\cos\theta$

$I_n = \pi^n + n\left\{\cancel{\left[\theta^{n-1}\sin\theta\right]_0^{\pi}} - \int_0^{\pi} (n-1)\theta^{n-2}\sin\theta \, d\theta\right.$

$I_n = \pi^n - n(n-1)\int_0^{\pi} \theta^{n-2}\sin\theta \, d\theta$

$I_n = \pi^n - n(n-1)I_{n-2}$ // AS REQUIRED

(b) $\int_0^{\frac{\pi}{2}} x^4 \sin 2x \, dx = \dots$ by substitution

$\theta = 2x$, $\frac{d\theta}{dx} = 2$, $dx = \frac{d\theta}{2}$

$x = \frac{\pi}{2}, \theta = \pi$; $x = 0, \theta = 0$

$= \int_0^{\pi} \left(\frac{\theta}{2}\right)^4 \sin\theta \, \frac{d\theta}{2} = \frac{1}{32}\int_0^{\pi} \theta^4 \sin\theta \, d\theta$

$= \frac{1}{32} I_4 = \frac{1}{32}\left[\pi^4 - 12I_2\right]$

$= \frac{1}{32}\left[\pi^4 - 12\left(\pi^2 - 2I_0\right)\right]$

$= \frac{1}{32}\left[\pi^4 - 12\pi^2 + 24I_0\right] = \frac{1}{32}\left[\pi^4 - 12\pi^2 + 24\int_0^{\pi} \sin\theta \, d\theta\right]$

$= \frac{1}{32}\left[\pi^4 - 12\pi^2 + 24\left[-\cos\theta\right]_0^{\pi}\right] = \frac{1}{32}\left[\pi^4 - 12\pi^2 + 24\left[\cos\theta\right]_{\pi}^{0}\right]$

$= \frac{1}{32}\left[\pi^4 - 12\pi^2 + 24\left(1 - (-1)\right)\right] = \frac{1}{32}\left[\pi^4 - 12\pi^2 + 48\right]$ // AS REQUIRED

Question 23 (****)

$$I_n = \int \frac{x^n}{\sqrt{1+x^2}}\, dx\, ,\ n \in \mathbb{N}\, .$$

a) Find an expression for

$$\frac{d}{dx}\left[x^{n-1}\left(x^2+1\right)^{\frac{1}{2}} \right].$$

b) Use part (a) to show that

$$nI_n + (n-1)I_{n-2} = x^{n-1}\sqrt{x^2+1}\, ,\ n \geq 2\, .$$

$$\boxed{\frac{d}{dx}\left[x^{n-1}\left(x^2+1\right)^{\frac{1}{2}} \right] = (n-1)x^{n-2}\left(x^2+1\right)^{\frac{1}{2}} + x^n\left(x^2+1\right)^{-\frac{1}{2}}}$$

(a) $\frac{d}{dx}\left(x^{n-1}(x^2+1)^{\frac{1}{2}}\right) = (n-1)x^{n-2}(x^2+1)^{\frac{1}{2}} + x^{n-1}\times\frac{1}{2}(x^2+1)^{-\frac{1}{2}}\times 2x$

$= (n-1)x^{n-2}(x^2+1)^{\frac{1}{2}} + x^n(x^2+1)^{-\frac{1}{2}}$

(b) $\frac{d}{dx}\left[x^{n-1}(x^2+1)^{\frac{1}{2}}\right] = (n-1)x^{n-2}(x^2+1)^{\frac{1}{2}} + x^n(x^2+1)^{-\frac{1}{2}}$

$\Rightarrow \frac{d}{dx}\left[x^{n-1}(x^2+1)^{\frac{1}{2}}\right] = \frac{(n-1)x^{n-2}(x^2+1)}{(x^2+1)^{\frac{1}{2}}} + \frac{x^n}{(x^2+1)^{\frac{1}{2}}}$

$\Rightarrow \frac{d}{dx}\left[x^{n-1}(x^2+1)^{\frac{1}{2}}\right] = \frac{(n-1)x^n + (n-1)x^{n-2} + x^n}{(x^2+1)^{\frac{1}{2}}}$

$\Rightarrow \frac{d}{dx}\left[x^{n-1}(x^2+1)^{\frac{1}{2}}\right] = \frac{nx^n + (n-1)x^{n-2}}{(x^2+1)^{\frac{1}{2}}}$

$\Rightarrow \frac{d}{dx}\left[x^{n-1}(x^2+1)^{\frac{1}{2}}\right] = \frac{nx^n}{(x^2+1)^{\frac{1}{2}}} + \frac{(n-1)x^{n-2}}{(x^2+1)^{\frac{1}{2}}}$

INTEGRATE EQUATION WITH RESPECT TO x

$\Rightarrow \int \frac{d}{dx}\left[x^{n-1}(x^2+1)^{\frac{1}{2}}\right] dx = n\int \frac{x^n}{(x^2+1)^{\frac{1}{2}}}\, dx + (n-1)\int \frac{x^{n-2}}{(x^2+1)^{\frac{1}{2}}}\, dx$

$\Rightarrow x^{n-1}(x^2+1)^{\frac{1}{2}} = nI_n + (n-1)I_{n-2}$

AS REQUIRED

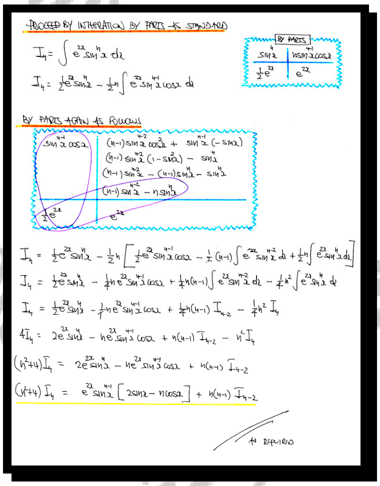
Question 24 (****+)

$$I_n \equiv \int e^{2x} \sin^n x \, dx \, , \; n \in \mathbb{N} \, , \; n \geq 2 \, .$$

Use integration by parts twice to show

$$\left(n^2+4\right) I_n = n(n-1) I_{n-2} + (2\sin x - n\cos x) e^{2x} \sin^{n-1} x \, .$$

$\boxed{}$, $\boxed{\text{proof}}$



PROCEED BY INTEGRATION BY PARTS AS STANDARD

$$I_n = \int e^{2x} \sin^n x \, dx$$

$$I_n = \frac{1}{2} e^{2x} \sin^n x - \frac{1}{2} n \int e^{2x} \sin^{n-1} x \cos x \, dx$$

BY PARTS

$\sin^n x$	$n \sin^{n-1} x \cos x$
$\frac{1}{2} e^{2x}$	e^{2x}

BY PARTS AGAIN AS FOLLOWS

$\sin^{n-1} x \cos x$	$(n-1)\sin^{n-2} x \cos^2 x + \sin^{n-1} x(-\sin x)$ $(n-1)\sin^{n-2} x(1-\sin^2 x) - \sin^n x$ $(n-1)\sin^{n-2} x - (n-1)\sin^n x - \sin^n x$ $(n-1)\sin^{n-2} x - n\sin^n x$
$\frac{1}{2} e^{2x}$	e^{2x}

$$I_n = \frac{1}{2} e^{2x} \sin^n x - \frac{1}{2} n \left[\frac{1}{2} e^{2x} \sin^{n-1} x \cos x - \frac{1}{2}(n-1) \int e^{2x} \sin^{n-2} x \, dx + \frac{1}{2} n \int e^{2x} \sin^n x \, dx \right]$$

$$I_n = \frac{1}{2} e^{2x} \sin^n x - \frac{1}{4} n e^{2x} \sin^{n-1} x \cos x + \frac{1}{4} n(n-1) \int e^{2x} \sin^{n-2} x \, dx - \frac{1}{4} n^2 \int e^{2x} \sin^n x \, dx$$

$$I_n = \frac{1}{2} e^{2x} \sin^n x - \frac{1}{4} n e^{2x} \sin^{n-1} x \cos x + \frac{1}{4} n(n-1) I_{n-2} - \frac{1}{4} n^2 I_n$$

$$4I_n = 2e^{2x} \sin^n x - n e^{2x} \sin^{n-1} x \cos x + n(n-1) I_{n-2} - n^2 I_n$$

$$(n^2+4) I_n = 2e^{2x} \sin^n x - n e^{2x} \sin^{n-1} x \cos x + n(n-1) I_{n-2}$$

$$(n^2+4) I_n = e^{2x} \sin^{n-1} x \left[2\sin x - n\cos x \right] + n(n-1) I_{n-2}$$

AS REQUIRED

Question 25 (****+)

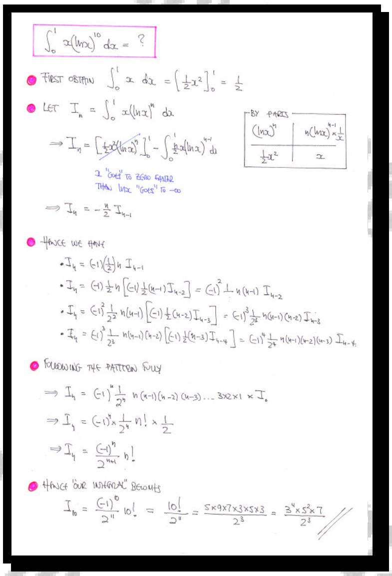
Find a suitable reduction formula and use it to find

$$\int_0^1 x(\ln x)^{10}\ dx .$$

You may assume that the integral converges.

Give the answer as the product of powers of prime factors.

$$\boxed{\int_0^1 x(\ln x)^{10}\ dx = 2^{-3}\times 3^4\times 5^2\times 7}$$



$\boxed{\int_0^1 x(\ln x)^{10}\,dx = ?}$

- FIRST OBTAIN $\int_0^1 x\,dx = \left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{2}$
- LET $I_n = \int_0^1 x(\ln x)^n\,dx$

BY PARTS

$(\ln x)^n$	$n(\ln x)^{n-1}\times\frac{1}{x}$
$\frac{1}{2}x^2$	x

$\Rightarrow I_n = \left[\cancel{\tfrac{1}{2}x^2(\ln x)^n}\right]_0^1 - \int_0^1 \frac{n}{2}x(\ln x)^{n-1}\,dx$

x "GOES" TO ZERO FASTER THAN $\ln x$ "GOES" TO $-\infty$

$\Rightarrow I_n = -\frac{n}{2}I_{n-1}$

- HENCE WE HAVE
  - $I_n = (-1)\left(\frac{1}{2}\right)n\,I_{n-1}$
  - $I_n = (-1)\frac{1}{2}n\left[(-1)\frac{1}{2}(n-1)I_{n-2}\right] = (-1)^2\frac{1}{2^2}n(n-1)\,I_{n-2}$
  - $I_n = (-1)^2\frac{1}{2^2}n(n-1)\left[(-1)\frac{1}{2}(n-2)I_{n-3}\right] = (-1)^3\frac{1}{2^3}n(n-1)(n-2)\,I_{n-3}$
  - $I_n = (-1)^3\frac{1}{2^3}n(n-1)(n-2)\left[(-1)\frac{1}{2}(n-3)I_{n-4}\right] = (-1)^4\frac{1}{2^4}n(n-1)(n-2)(n-3)\,I_{n-4}$
- FOLLOWING THE PATTERN FULLY

$\Rightarrow I_n = (-1)^n\frac{1}{2^n}n(n-1)(n-2)(n-3)\ldots 3\times2\times1\times I_0$

$\Rightarrow I_n = (-1)^n\times\frac{1}{2^n}n!\times\frac{1}{2}$

$\Rightarrow I_n = \frac{(-1)^n}{2^{n+1}}n!$

- HENCE "OUR INTEGRAL" BECOMES

$I_{10} = \frac{(-1)^{10}}{2^{11}}10! = \frac{10!}{2^{11}} = \frac{5\times9\times7\times3\times5\times3}{2^3} = \frac{3^4\times5^2\times7}{2^3}$

Question 26 (****+)

$$I_n = \int \frac{\sin nx}{\sin x}\,dx \ , \ n \in \mathbb{N}.$$

a) Show by considering $I_{n+2} - I_n$ that

$$I_{n+2} = I_n + \frac{2}{n+1}\sin\left[(n+1)x\right] + C \ , \ \ n \geq 0 .$$

b) Show further that

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin 6x}{\sin x}\,dx = \frac{1}{15}\left(12\sqrt{3} - 17\sqrt{2}\right).$$

$\boxed{\text{proof}}$

(a) $I_{n+2} - I_n = \int \frac{\sin[(n+2)x]}{\sin x}\,dx - \int \frac{\sin(nx)}{\sin x}\,dx$

$= \int \frac{\sin[(n+2)x] - \sin nx}{\sin x}\,dx$

$\sin P - \sin Q \equiv 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$

$\Rightarrow I_{n+2} - I_n = \int \frac{2\cos\left[\frac{(n+2)x+nx}{2}\right]\sin\left[\frac{(n+2)x-nx}{2}\right]}{\sin x}\,dx$

$\Rightarrow I_{n+2} - I_n = \int \frac{2\cos[(n+1)x]\cancel{\sin x}}{\cancel{\sin x}}\,dx$

$\Rightarrow I_{n+2} - I_n = \int 2\cos[(n+1)x]\,dx$

$\Rightarrow I_{n+2} - I_n = \frac{2}{n+1}\sin[(n+1)x] + C$

$\Rightarrow I_{n+2} = I_n + \frac{2}{n+1}\sin[(n+1)x] + C$

(b) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin 6x}{\sin x}\,dx = I_6$ WITH LIMITS $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}$

$I_{n+2} = I_n + \frac{2}{n+1}\left[\sin[(n+1)x]\right]_{\pi/4}^{\pi/3}$

$\Rightarrow I_n = I_{n-2} + \frac{2}{n-1}\left[\sin[(n-1)x]\right]_{\pi/4}^{\pi/3}$

$\Rightarrow I_6 = I_4 + \frac{2}{5}\left[\sin 5x\right]_{\pi/4}^{\pi/3}$

$\Rightarrow I_6 = I_4 + \frac{2}{5}\left[-\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\right]$

$\Rightarrow I_6 = I_4 - \frac{1}{5}\sqrt{3} + \frac{1}{5}\sqrt{2}$

$\Rightarrow I_6 = I_2 + \frac{2}{3}\left[\sin 3x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} - \frac{1}{5}\sqrt{3} + \frac{1}{5}\sqrt{2}$

$\Rightarrow I_6 = I_2 + \frac{2}{3}\left[0 - \frac{\sqrt{2}}{2}\right] - \frac{1}{5}\sqrt{3} + \frac{1}{5}\sqrt{2}$

$\Rightarrow I_6 = I_2 - \frac{1}{3}\sqrt{2} - \frac{1}{5}\sqrt{3} + \frac{1}{5}\sqrt{2}$

$\Rightarrow I_6 = I_2 - \frac{1}{5}\sqrt{3} - \frac{2}{15}\sqrt{2}$

$\Rightarrow I_6 = \cancel{I_0} + 2\left[\sin x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} - \frac{1}{5}\sqrt{3} - \frac{2}{15}\sqrt{2}$

$\Rightarrow I_6 = \sqrt{3} - \sqrt{2} - \frac{1}{5}\sqrt{3} - \frac{2}{15}\sqrt{2}$

$\Rightarrow I_6 = \frac{4}{5}\sqrt{3} - \frac{17}{15}\sqrt{2}$

$\Rightarrow I_6 = \frac{1}{15}\left[12\sqrt{3} - 17\sqrt{2}\right]$

AS REQUIRED

Question 27 (****+)

$$I_n = \int_0^{\pi} \frac{\sin(n\theta)}{\sin\theta}\ d\theta .$$

The integral above is defined for positive integer values n.

a) Use trigonometric identities to show that

$$\frac{\sin(n\theta) - \sin\left[(n-2)\theta\right]}{\sin\theta} = 2\cos\left[(n-1)\theta\right].$$

b) Hence show that

$$I_n = I_{n-2},\ n \geq 2 .$$

c) Evaluate I_n in both cases, where n is either odd or even positive integer.

$$\boxed{I_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \pi & \text{if } n \text{ is odd} \end{cases}}$$

(a) $\left\{\sin A - \sin B \equiv 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\right\}$

$$\frac{\sin(\overset{A}{n\theta}) - \sin[\overset{B}{(n-2)\theta}]}{\sin\theta} = \frac{2\cos\left[\frac{n\theta + (n-2)\theta}{2}\right]\sin\left[\frac{n\theta-(n-2)\theta}{2}\right]}{\sin\theta}$$

$$= \frac{2\cos\left(\frac{2n\theta-2\theta}{2}\right)\sin\left(\frac{n\theta-n\theta+2\theta}{2}\right)}{\sin\theta} = \frac{2\cos(n\theta-\theta)\cancel{\sin\theta}}{\cancel{\sin\theta}}$$

$$= 2\cos((n-1)\theta)$$

AS REQUIRED

(b) $$I_n - I_{n-2} = \int_0^{\pi} \frac{\sin n\theta}{\sin\theta}\,d\theta - \int_0^{\pi}\frac{\sin[(n-2)\theta]}{\sin\theta}\,d\theta$$

$$= \int_0^{\pi} \frac{\sin n\theta}{\sin\theta} - \frac{\sin((n-2)\theta)}{\sin\theta}\,d\theta$$

$$= \int_0^{\pi} 2\cos[(n-1)\theta]\,d\theta = \left[\frac{2}{n-1}\sin((n-1)\theta)\right]_0^{\pi}$$

$$= \frac{2}{n-1}[0-0] = 0$$

$$\therefore I_n = I_{n-2}$$

(c) IF $n =$ EVEN THEN $I_n = I_0 = \int_0^{\pi} 0\,d\theta = 0$

IF $n =$ ODD THEN $I_n = I_1 = \int_0^{\pi}\frac{\sin\theta}{\sin\theta}\,d\theta = \int_0^{\pi} 1\,d\theta$

$$= [\theta]_0^{\pi} = \pi$$

$$\therefore I_n = \begin{cases} 0 & \text{IF } n = \text{EVEN} \\ \pi & \text{IF } n = \text{ODD} \end{cases}$$

Question 28 (****+)

$$I_n = \int_0^{\frac{\pi}{3}} e^{3x} \tan^n x \, dx \ , \ n \in \mathbb{N} .$$

a) Show clearly that…

i. … $nI_{n+1} = e^{\pi}\left(\sqrt{3}\right)^n - 3I_n - nI_{n-1}, \ n \geq 1$.

ii. … $I_0 = I_4 + I_3 - 3I_1$.

b) Hence find the exact value of

$$\int_0^{\frac{\pi}{3}} e^{3x} \tan x\left(\tan^3 x + \sec^2 x - 4\right) dx .$$

$\boxed{\text{proof}}$

(a) $I_n = \int_0^{\frac{\pi}{3}} e^{3x} \tan^2 x \tan^{n-2} x \, dx$

$I_n = \int_0^{\frac{\pi}{3}} e^{3x} (\sec^2 x - 1) \tan^{n-2} x \, dx$

$I_n = \int_0^{\frac{\pi}{3}} e^{3x} \sec^2 x \tan^{n-2} x - e^{3x} \tan^{n-2} x \, dx$

$I_n = \int_0^{\frac{\pi}{3}} e^{3x} \sec^2 x \tan^{n-2} x \, dx - I_{n-2}$

BY PARTS

e^{3x}	$3e^{3x}$
$\frac{1}{n-1}\tan^{n-1} x$	$\sec^2 x \tan^{n-2} x$

$I_n = \left[\frac{1}{n-1} e^{3x} \tan^{n-1} x\right]_0^{\frac{\pi}{3}} - \frac{3}{n-1}\int_0^{\frac{\pi}{3}} e^{3x} \tan^{n-1} x \, dx - I_{n-2}$

$\boxed{I_n = \frac{1}{n-1} e^{\pi} (\sqrt{3})^{n-1} - \frac{3}{n-1} I_{n-1} - I_{n-2}}$

MANIPULATE FURTHER $n \mapsto n+1$

$I_{n+1} = \frac{1}{n} e^{\pi} (\sqrt{3})^n - \frac{3}{n} I_n - I_{n-1}$

$n I_{n+1} = e^{\pi} (\sqrt{3})^n - 3I_n - n I_{n-1}$

AS REQUIRED

(b) IF $n=1 \Rightarrow$ $I_2 = e^{\pi}\sqrt{3} - 3I_1 - I_0$
IF $n=3 \Rightarrow$ $3I_4 = 3e^{\pi}\sqrt{3} - 3I_3 - 3I_2$ } BY SUBSTITUTION

HENCE $3I_4 = 3e^{\pi}\sqrt{3} - 3I_3 - 3\left[e^{\pi}\sqrt{3} - 3I_1 - I_0\right]$

$3I_4 = \cancel{3e^{\pi}\sqrt{3}} - 3I_3 - \cancel{3e^{\pi}\sqrt{3}} + 9I_1 + 3I_0$

$3I_4 = 9I_1 + 3I_0 - 3I_3$

$I_4 = 3I_1 + I_0 - I_3$

$I_0 = I_4 + I_3 - 3I_1$

AS REQUIRED

(c) $\int_0^{\frac{\pi}{3}} e^{3x} \tan x (\tan^3 x + \sec^2 x - 4) \, dx$

$= \int_0^{\frac{\pi}{3}} e^{3x} \tan x (\tan^3 x + (1 + \tan^2 x) - 4) \, dx$

$= \int_0^{\frac{\pi}{3}} e^{3x} \tan x (\tan^3 x + \tan^2 x - 3) \, dx$

$= \int_0^{\frac{\pi}{3}} e^{3x} (\tan^4 x + \tan^3 x - 3\tan x) \, dx$

$= I_4 + I_3 - 3I_1$

$= I_0$

$= \int_0^{\frac{\pi}{3}} e^{3x} \, dx$

$= \left[\frac{1}{3} e^{3x}\right]_0^{\frac{\pi}{3}}$

$= \frac{1}{3} e^{\pi} - \frac{1}{3}$

$= \frac{1}{3}(e^{\pi} - 1)$

Question 29 (****+)

$$I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta \; d\theta \;, \; m \in \mathbb{N} \,, \; n \in \mathbb{N} \,.$$

a) Show clearly that

$$I_{m,n} = \frac{m-1}{m+n} I_{m-2,n} \,.$$

b) Hence find an exact value for

$$\int_0^{\frac{\pi}{2}} \sin\theta \sin^2 2\theta \; d\theta \,.$$

$\boxed{\frac{128}{315}}$

a) $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m\theta \cos^n\theta \, d\theta = \int_0^{\frac{\pi}{2}} \sin^{m-1}\theta \left(\sin\theta \cos^n\theta\right) d\theta$

BY PARTS

$\sin^{m-1}\theta$	$(m-1)\sin^{m-2}\theta\cos\theta$
$-\frac{1}{n+1}\cos^{n+1}\theta$	$\sin\theta\cos^n\theta$

$I_{m,n} = \cancel{\left[-\frac{1}{n+1}\cos^{n+1}\theta \sin^{m-1}\theta\right]_0^{\frac{\pi}{2}}} + \frac{m-1}{n+1}\int_0^{\frac{\pi}{2}} \sin^{m-2}\theta \cos^{n+2}\theta \, d\theta$

$I_{m,n} = \frac{m-1}{n+1}\int_0^{\frac{\pi}{2}} \sin^{m-2}\theta \cos^2\theta \cos^n\theta \, d\theta$

$I_{m,n} = \frac{m-1}{n+1}\int_0^{\frac{\pi}{2}} \sin^{m-2}\theta (1-\sin^2\theta)\cos^n\theta \, d\theta = \frac{m-1}{n+1}\int_0^{\frac{\pi}{2}} \sin^{m-2}\theta\cos^n\theta - \sin^m\theta\cos^n\theta \, d\theta$

$I_{m,n} = \frac{m-1}{n+1} I_{m-2,n} - \frac{m-1}{n+1} I_{m,n}$

$\left(1 + \frac{m-1}{n+1}\right) I_{m,n} = \frac{m-1}{n+1} I_{m-2,n}$

$\frac{m+n}{n+1} I_{m,n} = \frac{m-1}{n+1} I_{m-2,n}$

$I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}$

b) THE ABOVE FORMULA IS SYMMETRICAL I.E

$I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}$ $\quad$ $I_{m,n} = \frac{n-1}{m+n} I_{m,n-2}$

THUS $\int_0^{\frac{\pi}{2}} \sin\theta \sin^4 2\theta \, d\theta = \int_0^{\frac{\pi}{2}} \sin\theta (2\sin\theta\cos\theta)^4 \, d\theta = \int_0^{\frac{\pi}{2}} 16\sin^5\theta\cos^4\theta \, d\theta$

$= 16 I_{5,4} = 16 \times \frac{4}{9} I_{3,4} = 16 \times \frac{4}{9} \times \frac{2}{7} I_{1,4}$

NOW CHANGE TO "n"

$\ldots = 16 \times \frac{4}{9} \times \frac{2}{7} \times \frac{3}{5} I_{1,2} = 16 \times \frac{4}{9} \times \frac{2}{7} \times \frac{3}{5} \times \frac{1}{3} I_{1,0} = \frac{128}{315}\int_0^{\frac{\pi}{2}} \sin\theta \, d\theta$

$= \frac{128}{315}\left[-\cos\theta\right]_0^{\frac{\pi}{2}} = \frac{128}{315}$

Question 30 (****+)

It is given that

$$I_n = \int_0^1 x^6\left(x^3+1\right)^n \, dx \,,\ n \in \mathbb{Z},\ n \geq 0 .$$

Show, with detailed workings, that

$$I_n = \frac{2^n}{3n+7} + \frac{3n}{3n+7} I_{n-1}, \quad n \geq 1.$$

$\boxed{}$, $\boxed{\text{proof}}$

PROCEED BY INTEGRATION BY PARTS

$$I_n = \int_0^1 x^6(x^3+1)^n \, dx$$

$(x^3+1)^n$	$3nx^2(x^3+1)^{n-1}$
$\frac{1}{7}x^7$	x^6

THUS WE NOW HAVE

$$\Rightarrow I_n = \left[\tfrac{1}{7}x^7(x^3+1)^n\right]_0^1 - \tfrac{3}{7}n\int_0^1 x^9(x^3+1)^{n-1}\,dx$$

$$\Rightarrow I_n = \tfrac{1}{7}(2^n) - 0 - \tfrac{3}{7}n\int_0^1 x^6x^3(x^3+1)^{n-1}\,dx$$

$$\Rightarrow I_n = \frac{2^n}{7} - \frac{3}{7}n\int_0^1 x^6\left[-1+(x^3+1)\right](x^3+1)^{n-1}\,dx$$

$$\Rightarrow I_n = \frac{2^n}{7} - \frac{3}{7}n\int_0^1 -x^6(x^3+1)^{n-1} + x^6(x^3+1)^n\,dx$$

SPLIT THE INTEGRAL & TIDY

$$\Rightarrow I_n = \frac{2^n}{7} + \frac{3}{7}n\int_0^1 x^6(x^3+1)^{n-1}\,dx - \frac{3}{7}n\int_0^1 x^6(x^3+1)^n\,dx$$

$$\Rightarrow I_n = \frac{2^n}{7} + \frac{3}{7}n\,I_{n-1} - \frac{3}{7}n\,I_n$$

$$\Rightarrow 7I_n = 2^n + 3nI_{n-1} - 3nI_n$$

$$\Rightarrow (7+3n)I_n = 2^n + 3nI_{n-1}$$

$$\therefore I_n = \frac{2^n}{3n+7} + \frac{3n}{3n+7}I_{n-1}$$

AS REQUIRED

Question 31 (****+)

It is given that

$$I_n = \int_0^1 x\left(1-2x^4\right)^n \, dx \,,\; n \in \mathbb{Z} \,,\; n \geq 0 \,.$$

Show, with detailed workings, that

$$I_n = \frac{(-1)^n}{4n+2} + \frac{2n}{2n+1} I_{n-1} \,,\;\; n \geq 1 \,.$$

$\boxed{}$, $\boxed{\text{proof}}$

PROCEED BY STANDARD INTEGRATION BY PARTS

$$I_n = \int_0^1 x(1-2x^4)^n \, dx$$

$(1-2x^4)^n$	$-8nx^3(1-2x^4)^{n-1}$
$\frac{1}{2}x^2$	x

THUS WE NOW HAVE

$$\Rightarrow I_n = \left[\tfrac{1}{2}x^2(1-2x^4)^n\right]_0^1 + 4n\int_0^1 x^5(1-2x^4)^{n-1}\,dx$$

$$\Rightarrow I_n = \tfrac{1}{2}(-1)^n - 0 + 4n\int_0^1 x\,x^4(1-2x^4)^{n-1}\,dx$$

$$\Rightarrow I_n = \tfrac{1}{2}(-1)^n - 2n\int_0^1 x(-2x^4)(1-2x^4)^{n-1}\,dx$$

$$\Rightarrow I_n = \tfrac{1}{2}(-1)^n - 2n\int_0^1 x\left[-1+(1-2x^4)\right](1-2x^4)^{n-1}\,dx$$

$$\Rightarrow I_n = \tfrac{1}{2}(-1)^n - 2n\int_0^1 -x(1-2x^4)^{n-1} + x(1-2x^4)(1-2x^4)^{n-1}\,dx$$

SPLIT THE INTEGRAL & TIDY

$$\Rightarrow I_n = \tfrac{1}{2}(-1)^n + 2n\int_0^1 x(1-2x^4)^{n-1}\,dx - 2n\int_0^1 x(1-2x^4)^n\,dx$$

$$\Rightarrow I_n = \tfrac{1}{2}(-1)^n + 2nI_{n-1} - 2nI_n$$

$$\Rightarrow (2n+1)I_n = \tfrac{1}{2}(-1)^n + 2nI_{n-1}$$

$$\Rightarrow I_n = \frac{(-1)^n}{2(2n+1)} + \frac{2n}{2n+1}I_{n-1}$$

$$\therefore I_n = \frac{(-1)^n}{4n+2} + \frac{2n}{2n+1}I_{n-1}$$

AS REQUIRED

Question 32 (*****)

It is given that $I_n = \int_0^1 \frac{x^{2n+1}}{\sqrt{1-x^2}}\, dx$.

Show with detailed workings that

$$I_n = \frac{2^{2n}(n!)^2}{(2n+1)!}.$$

V, $\boxed{}$, $\boxed{\text{proof}}$

PROCEED BY REGROUPING & INTEGRATION BY PARTS

$$I_n = \int_0^1 \frac{x^{2n+1}}{\sqrt{1-x^2}}\,dx = \int_0^1 x^{2n+1}(1-x^2)^{-\frac{1}{2}}\,dx = \int_0^1 x^{2n}\left[x(1-x^2)^{-\frac{1}{2}}\right]dx$$

$$\boxed{\begin{array}{c|c} x^{2n} & 2nx^{2n-1} \\ \hline -(1-x^2)^{\frac{1}{2}} & (1-x^2)^{-\frac{1}{2}} \end{array}}$$

$$\Rightarrow I_n = \left[\cancel{-x^{2n}(1-x^2)^{\frac{1}{2}}}\right]_0^1 - \int_0^1 -2nx^{2n-1}(1-x^2)^{\frac{1}{2}}\,dx$$

$$\Rightarrow I_n = 2n\int_0^1 x^{2n-1}(1-x^2)^{\frac{1}{2}}\,dx$$

$$\Rightarrow I_n = 2n\int_0^1 \left[x^{2n-1}(1-x^2)^{-\frac{1}{2}}\right](1-x^2)\,dx$$

$$\Rightarrow I_n = 2n\int_0^1 x^{2n-1}(1-x^2)^{-\frac{1}{2}} - x^{2n+1}(1-x^2)^{-\frac{1}{2}}\,dx$$

$$\Rightarrow I_n = 2n\int_0^1 x^{2n-1}(1-x^2)^{-\frac{1}{2}}\,dx - 2n\int_0^1 x^{2n+1}(1-x^2)^{-\frac{1}{2}}\,dx$$

$$\Rightarrow I_n = 2nI_{n-1} - 2nI_n$$

$$\Rightarrow I_n + 2nI_n = 2nI_{n-1}$$

$$\Rightarrow (2n+1)I_n = 2nI_{n-1}$$

$$\Rightarrow I_n = \frac{2n}{2n+1}I_{n-1}$$

CONTINUING AS WE REQUIRE A CLOSED FORM

$$\Rightarrow I_n = \frac{2n}{2n+1}\times\frac{2n-2}{2n-1}I_{n-2}$$

$$\Rightarrow I_n = \frac{2n}{2n+1}\times\frac{2n-2}{2n-1}\times\frac{2n-4}{2n-3}I_{n-3}$$

CONTINUING IN THE SAME FASHION

$$\Rightarrow I_n = \frac{2n}{2n+1}\times\frac{2n-2}{2n-1}\times\frac{2n-4}{2n-3}\times\dots\times\frac{6}{7}\times\frac{4}{5}\times\frac{2}{3}I_0$$

$$\Rightarrow I_n = \frac{2n(2n-2)(2n-4)\dots6\times4\times2}{(2n+1)(2n-1)(2n-3)\dots7\times5\times3}\times\underbrace{\int_0^1 x(1-x^2)^{-\frac{1}{2}}\,dx}_{\text{THIS GIVES 1}}$$

MANIPULATE FURTHER

$$\Rightarrow I_n = \frac{2n}{2n+1}\times\frac{2n}{2n}\times\frac{2n-2}{2n-1}\times\frac{2n-2}{2n-2}\times\frac{2n-4}{2n-3}\times\frac{2n-4}{2n-4}\times\dots\times\frac{6}{7}\,\frac{6}{6}\times\frac{4}{5}\times\frac{4}{4}\times\frac{2}{3}\times\frac{2}{2}$$

$$\Rightarrow I_n = \frac{(2n)^2(2n-2)^2(2n-4)^2\dots\times6^2\times4^2\times2^2}{(2n+1)!}$$

$$\Rightarrow I_n = \frac{2^2n^2\times2^2\times(n-1)^2\times2^2(n-2)^2\times\dots\times(2^2\times3^2)\times(2^2\times2^2)\times(2^2\times1^2)}{(2n+1)!}$$

$$\Rightarrow I_n = \frac{(2^2)^n\left[n^2(n-1)^2(n-2)^2\dots3^2\times2^2\times1^2\right]}{(2n+1)!}$$

$$\Rightarrow I_n = \frac{2^{2n}(n!)^2}{(2n+1)!}$$

AS REQUIRED

Question 33 (*****)

It is given that

$$I_n = \int_0^a x^{n+\frac{1}{2}} \sqrt{a-x} \; dx \,, \; n \in \mathbb{Z} \,, \; n \geq 0$$

where a is a positive constant.

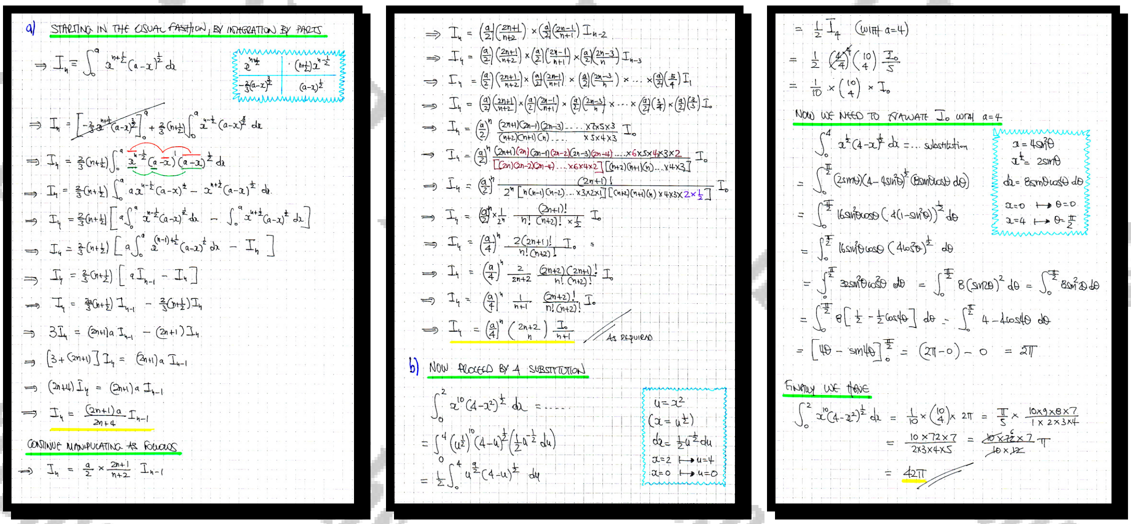
a) Use integration by parts to show

$$I_n = \left(\frac{a}{4}\right)^n \binom{2n+2}{n} \frac{I_0}{n+1} \,, \quad n \geq 1 .$$

b) Determine the value of

$$\int_0^2 x^{10} \sqrt{4-x^2} \; dx \,.$$

$\boxed{}$, $\boxed{42\pi}$



Question 34 (*****)

It is given that

$$I_{n,m} = \int_0^1 (1-x)^n \, x^m \; dx ,$$

where $n, m \in \mathbb{Z}$, with $n, m \geq 0$.

a) Show that …

i. … $I_{n,m} - I_{n-1,m} = -I_{n-1,m+1}$.

ii. … $I_{n,m} = \dfrac{n}{m} I_{n-1,m+1}$

b) Hence derive an expression of $I_{n,m}$ and use it to find

$$\int_0^1 7x^{\frac{1}{2}} (1-x)^3 \; dx .$$

$\boxed{}$, $\boxed{\displaystyle\int_0^1 7x^{\frac{1}{2}} (1-x)^3 \; dx = \frac{32}{45}}$

$$\boxed{I_{n,m} \equiv \int_0^1 (1-x)^n x^m \, dx \quad n, m \geq 0}$$

a) I) BY DIRECT PROOF

$$
\begin{aligned}
I_{n,m} - I_{n-1,m} &= \int_0^1 (1-x)^n x^m \, dx - \int_0^1 (1-x)^{n-1} x^m \, dx \\
&= \int_0^1 (1-x)^n x^m - (1-x)^{n-1} x^m \, dx \\
&= \int_0^1 (1-x)^{n-1} x^m \left[(1-x) - 1\right] dx \\
&= \int_0^1 (1-x)^{n-1} x^m (-x) \, dx \\
&= -\int_0^1 (1-x)^{n-1} x^{m+1} \, dx \\
&= -I_{n-1,m+1} \quad \text{// AS REQUIRED}
\end{aligned}
$$

II) NEXT PROCEED BY INTEGRATION BY PARTS

$$I_{n,m} = \int_0^1 (1-x)^n x^m \, dx$$

$(1-x)^n$	$-n(1-x)^{n-1}$
$\frac{1}{m+1} x^{m+1}$	x^m

$$I_{n,m} = \cancel{\left[\frac{1}{m+1} x^{m+1} (1-x^n)\right]_0^1} - \int_0^1 -\frac{n}{m+1} (1-x)^{n-1} x^{m+1} \, dx$$

$$I_{n,m} = \frac{n}{m+1} \int_0^1 (1-x)^{n-1} x^{m+1} \, dx$$

$$I_{n,m} = \frac{m}{m+1} I_{n-1,m+1} \quad \text{// AS REQUIRED}$$

b) COMBINING THE RESULTS OF PART (a)

$$
\left.
\begin{aligned}
I_{n,m} &= I_{n-1,m} - I_{n-1,m+1} && \times n \\
I_{n,m} &= \frac{n}{m+1} I_{n-1,m+1} && \times (m+1)
\end{aligned}
\right\} \Rightarrow
$$

$$
\left.
\begin{aligned}
n I_{n,m} &= n I_{n-1,m} - n I_{n-1,m+1} \\
(m+1) I_{n,m} &= n I_{n-1,m+1}
\end{aligned}
\right\} \Rightarrow \text{ADDING}
$$

$$(n+m+1) I_{n,m} = n I_{n-1,m}$$

$$\boxed{I_{n,m} = \frac{n}{n+m+1} I_{n-1,m}}$$

USING THIS FORMULA

$$
\begin{aligned}
\int_0^1 7x^{\frac{1}{2}} (1-x)^3 \, dx &= 7 I_{3,\frac{1}{2}} \\
&= \frac{7 \times 3}{3+\frac{1}{2}+1} I_{2,\frac{1}{2}} = \frac{21}{\frac{9}{2}} I_{2,\frac{1}{2}} \\
&= \frac{14}{3} \times \frac{2}{2+\frac{1}{2}+1} I_{1,\frac{1}{2}} \\
&= \frac{14}{3} \times \frac{2}{\frac{7}{2}} I_{1,\frac{1}{2}} = \frac{\overset{2}{\cancel{14}}}{3} \times \frac{4}{\cancel{7}} I_{1,\frac{1}{2}} \\
&= \frac{8}{3} \times \frac{1}{1+\frac{1}{2}+1} I_{0,\frac{1}{2}} \\
&= \frac{8}{3} \times \frac{1}{\frac{5}{2}} I_{0,\frac{1}{2}} = \frac{8}{3} \times \frac{2}{5} I_{0,\frac{1}{2}} \\
&= \frac{16}{15} I_{0,\frac{1}{2}} = \frac{16}{15} \int_0^1 (1-x)^0 x^{\frac{1}{2}} \, dx \\
&= \frac{16}{15} \int_0^1 x^{\frac{1}{2}} \, dx \\
&= \frac{16}{15} \times \frac{2}{3} \left[x^{\frac{3}{2}}\right]_0^1 \\
&= \frac{32}{45} \quad \text{//}
\end{aligned}
$$

FOR THOSE WITH KNOWLEDGE OF BETA & GAMMA FUNCTIONS THE INTEGRAL IS TRIVIAL

$$
\begin{aligned}
\int_0^1 7x^{\frac{1}{2}} (1-x)^3 \, dx &= 7 B\left(\tfrac{3}{2}, 4\right) \\
&= \frac{7 \Gamma\left(\frac{3}{2}\right) \Gamma(4)}{\Gamma\left(\frac{11}{2}\right)} \\
&= \frac{7 \times \cancel{\Gamma\left(\frac{3}{2}\right)} \times 3!}{\frac{9}{2} \times \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \cancel{\Gamma\left(\frac{3}{2}\right)}} \\
&= \frac{\cancel{7} \times 3! \times 8}{9 \times \cancel{7} \times 5 \times 3} \\
&= \frac{\cancel{3} \times 2 \times 8}{9 \times 5 \times \cancel{3}} \\
&= \frac{16}{45}
\end{aligned}
$$

Question 35 (*****)

$$I(m,n)=\int_a^b (b-x)^m (x-a)^n \ dx, \quad m \in \mathbb{N},\ n \in \mathbb{N}.$$

Show that

$$I(m,n)=\frac{m!\,n!}{(m+n+1)!}(b-a)^{m+n+1},$$

where a and b are real constants such that $b>a$

$\boxed{}$, $\boxed{\text{proof}}$

$$I(m,n)=\int_a^b (b-x)^m (x-a)^n \ dx$$

PROCEED BY INTEGRATION BY PARTS

$(b-x)^m$	$-m(b-x)^{m-1}$
$\frac{1}{n+1}(x-a)^{n+1}$	$(x-a)^n$

HENCE WE HAVE

$$\Rightarrow I(m,n)=\cancel{\left[\frac{1}{n+1}(b-x)^m(x-a)\right]_a^b}+\frac{m}{n+1}\int_a^b (x-a)^{n+1}(b-x)^{m-1}\ dx$$

$$\Rightarrow I(m,n)=\frac{m}{n+1}I(m-1,n+1)$$

$$\Rightarrow I(m,n)=\frac{m}{n+1}\times\frac{m-1}{n+2}I(m-2,n+2)$$

$$\Rightarrow I(m,n)=\frac{m}{n+1}\times\frac{m-1}{n+2}\times\frac{m-2}{n+3}I(m-3,n+3)$$

$$\Rightarrow I(m,n)=\frac{m(m-1)(m-2)\ldots\times 3\times 2\times 1}{(n+1)(n+2)(n+3)\ldots(n+m-2)(n+m-1)(n+m)}I(0,n+m)$$

$$\Rightarrow I(m,n)=\frac{m!}{(n+m)(n+m-1)(n+m-2)\ldots(n+2)(n+1)}I(0,n+m)$$

$$\Rightarrow I(m,n)=\frac{m!\times n(n-1)(n-2)(n-3)\ldots 3\times 2\times 1}{(n+m)(n+m-1)(n+m-2)\ldots(n+2)(n+1)\,n(n-1)(n-2)\ldots 3\times 2\times 1}I(0,n+m)$$

$$\Rightarrow I(m,n)=\frac{m!\,n!}{(n+m)!}I(0,m+n)$$

$$\Rightarrow I(m,n)=\frac{m!\,n!}{(n+m)!}\int_a^b (x-a)^{n+m}\ dx$$

$$\Rightarrow I(m,n)=\frac{m!\,n!}{(n+m)!}\times\left[\frac{1}{n+m+1}(x-a)^{n+m+1}\right]_a^b$$

$$\Rightarrow I(m,n)=\frac{m!\,n!}{(n+m)!}\times\frac{1}{n+m+1}\left[(b-a)^{m+n+1}-0\right]$$

$$\Rightarrow I(m,n)=\frac{m!\,n!}{(n+m+1)!}(b-a)^{m+n+1}$$

AS REQUIRED

Question 36 (*****)

$$I_n = \int_0^a \frac{x^n}{\sqrt{a^2-x^2}}\,dx\,,\ n \in \mathbb{N}\,,\ a>0\,.$$

Show clearly that…

a) … $I_n = \dfrac{a^2(n-1)}{n} I_{n-2}\,,\ n \geq 2\,.$

b) … $\displaystyle\int_2^4 \frac{3x^3-18x^2+36x-18}{\sqrt{4x-x^2}}\,dx = 3\pi - 16\,.$

, proof

a) REWRITE AND USE INTEGRATION BY PARTS

$$I_n = \int_0^a x^{n-1}\frac{x}{\sqrt{a^2-x^2}}\,dx$$

x^{n-1}	$(n-1)x^{n-2}$
$-(a^2-x^2)^{\frac{1}{2}}$	$x(a^2-x^2)^{-\frac{1}{2}}$

$$I_n = \left[\cancel{-x^{n-1}(a^2-x^2)^{\frac{1}{2}}}\right]_0^a + (n-1)\int_0^a x^{n-2}(a^2-x^2)^{\frac{1}{2}}\,dx$$

$$I_n = (n-1)\int_0^a \frac{x^{n-2}(a^2-x^2)}{(a^2-x^2)^{\frac{1}{2}}}\,dx$$

MULTIPLIED "TOP & BOTTOM" OF THE FRACTION BY $(a^2-x^2)^{\frac{1}{2}}$

$$I_n = (n-1)\int_0^a \frac{a^2x^{n-2}}{(a^2-x^2)^{\frac{1}{2}}} - \frac{x^n}{(a^2-x^2)^{\frac{1}{2}}}\,dx$$

$$I_n = (n-1)a^2\int_0^a \frac{x^{n-2}}{\sqrt{a^2-x^2}}\,dx - (n-1)\int_0^a \frac{x^n}{\sqrt{a^2-x^2}}\,dx$$

$$I_n = a^2(n-1)I_{n-2} - (n-1)I_n$$

$$nI_n = a^2(n-1)I_{n-2}$$

$$I_n = \frac{a^2(n-1)}{n}I_{n-2}$$

b) START BY COMPLETING THE SQUARE IN THE DENOMINATOR

$$4x-x^2 = -(x^2-4x) = -\left[(x-2)^2-4\right] = 4-(x-2)^2$$

IT APPEARS THAT $a=2$ AND THE SUBSTITUTION $u=x-2$ IS NEEDED

$$\begin{aligned}3x^3-18x^2+36x-18 &= 3\left[x^3-6x^2+12x-6\right]\\ &= 3\left[x^2(x-2)-4x(x-2)+4(x-2)+2\right]\\ &= 3\left[(x-2)(x^2-4x+4)+2\right]\\ &= 3\left[(x-2)(x-2)^2+2\right]\\ &= 3\left[(x-2)^3+2\right]\\ &= 3(x-2)^3+6\end{aligned}$$

RETURNING TO THE INTEGRAL

$$\int_2^4 \frac{3x^3-18x^2+36x-18}{\sqrt{4x-x^2}}\,dx = \int_2^4 \frac{3(x-2)^3+6}{\sqrt{4-(x-2)^2}}\,dx$$

LET $u=x-2$ $\quad du=dx$ $\quad 2 \mapsto 0$, $\quad 4 \mapsto 2$

$$= \int_0^2 \frac{3u^3+6}{\sqrt{4-u^2}}\,du = 3\int_0^2 \frac{u^3}{\sqrt{4-u^2}}\,du + 6\int_0^2 \frac{1}{\sqrt{4-u^2}}\,du$$

$$= 3I_3 + \left[6\arcsin\frac{u}{2}\right]_0^2 = 3I_3 + 6\arcsin 1$$

$$= 3I_3 + \frac{\pi}{2}\times 6 = 3I_3 + 3\pi$$

NOW THE REDUCTION FORMULA HAS $a=2$

$$I_n = \frac{4(n-1)}{n}I_{n-2}$$

$$I_3 = 4\times\frac{2}{3}I_1 = \frac{8}{3}\int_0^2 \frac{u}{\sqrt{4-u^2}}\,du$$

$$I_3 = \frac{8}{3}\int_0^2 u(4-u^2)^{-\frac{1}{2}}\,du$$

$$I_3 = \frac{8}{3}\left[(4-u^2)^{\frac{1}{2}}\times(-1)\right]_0^2$$

$$I_3 = \frac{8}{3}\left[2-0\right)$$

$$I_3 = \frac{16}{3}$$

$$\therefore \int_0^2 \frac{3x^3-18x^2+36x-18}{\sqrt{4x-x^2}}\,dx = 3\times\frac{16}{3} + 3\pi = 3\pi + 16$$