MATRICES PRACTICE

MATRIX INTRODUCTION

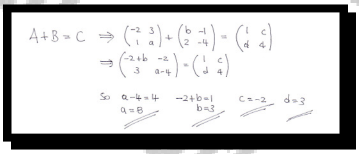
Question 1

The matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are given below in terms of the scalar constants a, b, c and d, by

$$\mathbf{A} = \begin{pmatrix} -2 & 3 \\ 1 & a \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b & -1 \\ 2 & -4 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 & c \\ d & 4 \end{pmatrix}.$$

Given that $\mathbf{A} + \mathbf{B} = \mathbf{C}$, find the value of a, b, c and d.

$$\boxed{a = 8,\ b = 3,\ c = 2,\ d = 3}$$



$$A+B=C \Rightarrow \begin{pmatrix} -2 & 3 \\ 1 & a \end{pmatrix} + \begin{pmatrix} b & -1 \\ 2 & -4 \end{pmatrix} = \begin{pmatrix} 1 & c \\ d & 4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -2+b & -2 \\ 3 & a-4 \end{pmatrix} = \begin{pmatrix} 1 & c \\ d & 4 \end{pmatrix}$$

So $a-4=4$, $a=8$ $\quad -2+b=1$, $b=3$ $\quad c=-2$ $\quad d=3$

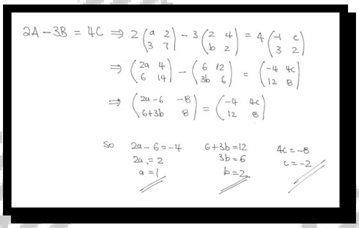
Question 2

The matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are given below in terms of the scalar constants a, b and c, by

$$\mathbf{A} = \begin{pmatrix} a & 2 \\ 3 & 7 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2 & 4 \\ b & 2 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} -1 & c \\ 3 & 2 \end{pmatrix}.$$

Given that $2\mathbf{A} - 3\mathbf{B} = 4\mathbf{C}$, find the value of a, b and c.

$$\boxed{a = 1,\ b = -2,\ c = -2}$$



$$2A-3B=4C \Rightarrow 2\begin{pmatrix} a & 2 \\ 3 & 7 \end{pmatrix} - 3\begin{pmatrix} 2 & 4 \\ b & 2 \end{pmatrix} = 4\begin{pmatrix} -1 & c \\ 3 & 2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2a & 4 \\ 6 & 14 \end{pmatrix} - \begin{pmatrix} 6 & 12 \\ 3b & 6 \end{pmatrix} = \begin{pmatrix} -4 & 4c \\ 12 & 8 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2a-6 & -8 \\ 6+3b & 8 \end{pmatrix} = \begin{pmatrix} -4 & 4c \\ 12 & 8 \end{pmatrix}$$

So $2a-6=-4$, $2a=2$, $a=1$ $\quad 6+3b=12$, $3b=6$, $b=2$ $\quad 4c=-8$, $c=-2$

Question 3

Multiply each of the following matrices.

a) $\begin{pmatrix} 4 & 5 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$

b) $\begin{pmatrix} -4 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & -4 \\ -2 & 7 \end{pmatrix}$

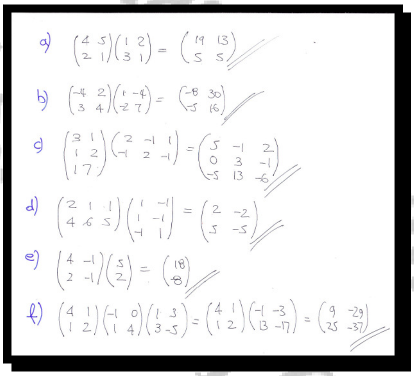
c) $\begin{pmatrix} 3 & 1 \\ 1 & 2 \\ 1 & 7 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \end{pmatrix}$

d) $\begin{pmatrix} 2 & 1 & 1 \\ 4 & 6 & 5 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix}$

e) $\begin{pmatrix} 4 & -1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix}$

f) $\begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 3 & -5 \end{pmatrix}$

$$\boxed{\begin{pmatrix} 19 & 13 \\ 5 & 5 \end{pmatrix}}, \boxed{\begin{pmatrix} -8 & 30 \\ -5 & 16 \end{pmatrix}}, \boxed{\begin{pmatrix} 5 & -1 & 2 \\ 0 & 3 & -1 \\ -5 & 13 & -6 \end{pmatrix}}, \boxed{\begin{pmatrix} 2 & -2 \\ 5 & -5 \end{pmatrix}}, \boxed{\begin{pmatrix} 18 \\ 8 \end{pmatrix}}, \boxed{\begin{pmatrix} 9 & -29 \\ 25 & -37 \end{pmatrix}}$$



a) $\begin{pmatrix} 4 & 5 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 19 & 13 \\ 5 & 5 \end{pmatrix}$

b) $\begin{pmatrix} -4 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & -4 \\ -2 & 7 \end{pmatrix} = \begin{pmatrix} -8 & 30 \\ -5 & 16 \end{pmatrix}$

c) $\begin{pmatrix} 3 & 1 \\ 1 & 2 \\ 1 & 7 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 5 & -1 & 2 \\ 0 & 3 & -1 \\ -5 & 13 & -6 \end{pmatrix}$

d) $\begin{pmatrix} 2 & 1 & 1 \\ 4 & 6 & 5 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 5 & -5 \end{pmatrix}$

e) $\begin{pmatrix} 4 & -1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 18 \\ 8 \end{pmatrix}$

f) $\begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 3 & -5 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -1 & -3 \\ 13 & -17 \end{pmatrix} = \begin{pmatrix} 9 & -29 \\ 25 & -37 \end{pmatrix}$

Question 4

Multiply each of the following matrices.

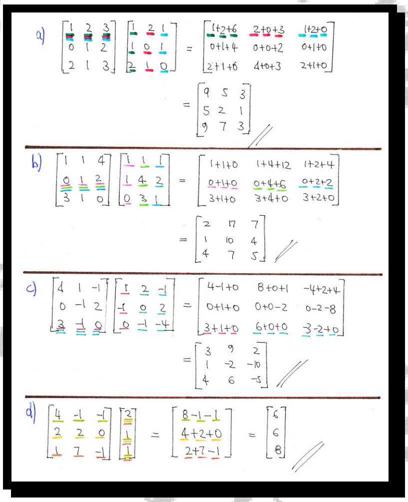
a) $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 1 & 4 \\ 0 & 1 & 2 \\ 3 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & 2 \\ 0 & 3 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 4 & 1 & -1 \\ 0 & -1 & 2 \\ 3 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & -1 \\ -1 & 0 & 2 \\ 0 & -1 & -4 \end{pmatrix}$

d) $\begin{pmatrix} 4 & -1 & -1 \\ 2 & 2 & 0 \\ 1 & 7 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

$$\boxed{\begin{pmatrix} 9 & 5 & 3 \\ 5 & 2 & 1 \\ 9 & 7 & 3 \end{pmatrix}}, \boxed{\begin{pmatrix} 2 & 17 & 7 \\ 1 & 10 & 4 \\ 4 & 7 & 5 \end{pmatrix}}, \boxed{\begin{pmatrix} 3 & 9 & 2 \\ 1 & -2 & -10 \\ 4 & 6 & -5 \end{pmatrix}}, \boxed{\begin{pmatrix} 6 \\ 6 \\ 8 \end{pmatrix}}$$



a) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1+2+6 & 2+0+3 & 1+2+0 \\ 0+1+4 & 0+0+2 & 0+1+0 \\ 2+1+6 & 4+0+3 & 2+1+0 \end{bmatrix}$

$= \begin{bmatrix} 9 & 5 & 3 \\ 5 & 2 & 1 \\ 9 & 7 & 3 \end{bmatrix}$ //

b) $\begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 2 \\ 3 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 2 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1+1+0 & 1+4+12 & 1+2+4 \\ 0+1+0 & 0+4+6 & 0+2+2 \\ 3+1+0 & 3+4+0 & 3+2+0 \end{bmatrix}$

$= \begin{bmatrix} 2 & 17 & 7 \\ 1 & 10 & 4 \\ 4 & 7 & 5 \end{bmatrix}$ //

c) $\begin{bmatrix} 4 & 1 & -1 \\ 0 & -1 & 2 \\ 3 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ -1 & 0 & 2 \\ 0 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 4-1+0 & 8+0+1 & -4+2+4 \\ 0+1+0 & 0+0-2 & 0-2-8 \\ 3+1+0 & 6+0+0 & -3-2+0 \end{bmatrix}$

$= \begin{bmatrix} 3 & 9 & 2 \\ 1 & -2 & -10 \\ 4 & 6 & -5 \end{bmatrix}$ //

d) $\begin{bmatrix} 4 & -1 & -1 \\ 2 & 2 & 0 \\ 1 & 7 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8-1-1 \\ 4+2+0 \\ 2+7-1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 8 \end{bmatrix}$ //

Question 5

A matrix $\mathbf{T}$ represents the linear transformation

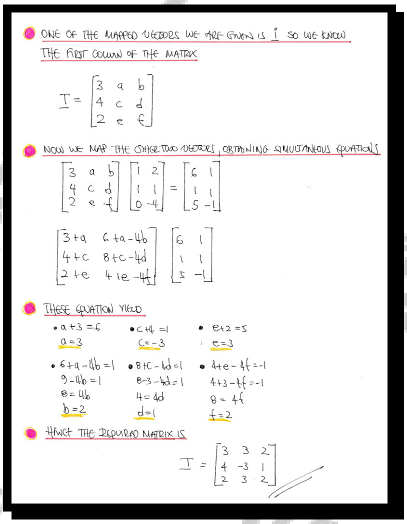
$$\mathbf{T}\begin{pmatrix} x \\ y \\ z \end{pmatrix} :\mapsto \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

so that

$$\mathbf{T}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} :\mapsto \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}, \quad \mathbf{T}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} :\mapsto \begin{pmatrix} 6 \\ 1 \\ 5 \end{pmatrix}, \quad \mathbf{T}\begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} :\mapsto \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

Find the elements of $\mathbf{T}$.

$$\boxed{\mathbf{T} = \begin{pmatrix} 3 & 3 & 2 \\ 4 & -3 & 1 \\ 2 & 3 & 2 \end{pmatrix}}$$



- ONE OF THE MAPPED VECTORS WE ARE GIVEN IS $\underline{i}$ SO WE KNOW THE FIRST COLUMN OF THE MATRIX

$$\underline{T} = \begin{bmatrix} 3 & a & b \\ 4 & c & d \\ 2 & e & f \end{bmatrix}$$

- NOW WE MAP THE OTHER TWO VECTORS, OBTAINING SIMULTANEOUS EQUATIONS

$$\begin{bmatrix} 3 & a & b \\ 4 & c & d \\ 2 & e & f \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 1 & 1 \\ 5 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 3+a & 6+a-4b \\ 4+c & 8+c-4d \\ 2+e & 4+e-4f \end{bmatrix} \begin{bmatrix} 6 & 1 \\ 1 & 1 \\ 5 & -1 \end{bmatrix}$$

- THESE EQUATION YIELD

$a+3=6$, $\underline{a=3}$ · $c+4=1$, $\underline{c=-3}$ · $e+2=5$, $\underline{e=3}$

$6+a-4b=1$, $9-4b=1$, $8=4b$, $\underline{b=2}$

$8+c-4d=1$, $8-3-4d=1$, $4=4d$, $\underline{d=1}$

$4+e-4f=-1$, $4+3-4f=-1$, $8=4f$, $\underline{f=2}$

- HENCE THE REQUIRED MATRIX IS

$$\underline{T} = \begin{bmatrix} 3 & 3 & 2 \\ 4 & -3 & 1 \\ 2 & 3 & 2 \end{bmatrix}$$

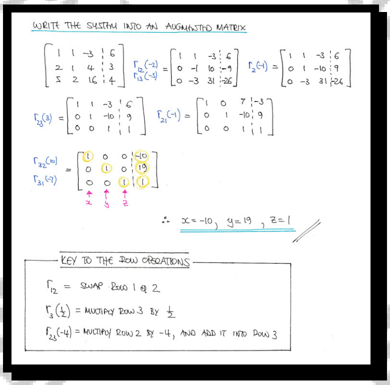
MATRIX
ROW
REDUCTION

Question 1

Solve the following simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

$$\begin{pmatrix} 1 & 1 & -3 \\ 2 & 1 & 4 \\ 5 & 2 & 16 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix}$$

V, $\boxed{x=-10, \quad y=19, \quad z=1}$



WRITE THE SYSTEM INTO AN AUGMENTED MATRIX

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 6 \\ 2 & 1 & 4 & 3 \\ 5 & 2 & 16 & 4 \end{array}\right] \begin{array}{c} r_{12}(-2) \\ r_{13}(-5) \end{array} = \left[\begin{array}{ccc|c} 1 & 1 & -3 & 6 \\ 0 & -1 & 10 & -9 \\ 0 & -3 & 31 & -26 \end{array}\right] r_2(-1) = \left[\begin{array}{ccc|c} 1 & 1 & -3 & 6 \\ 0 & 1 & -10 & 9 \\ 0 & -3 & 31 & -26 \end{array}\right]$$

$$r_{23}(3) = \left[\begin{array}{ccc|c} 1 & 1 & -3 & 6 \\ 0 & 1 & -10 & 9 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad r_{21}(-1) = \left[\begin{array}{ccc|c} 1 & 0 & 7 & -3 \\ 0 & 1 & -10 & 9 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\begin{array}{c} r_{32}(10) \\ r_{31}(-7) \end{array} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -10 \\ 0 & 1 & 0 & 19 \\ 0 & 0 & 1 & 1 \\ \uparrow & \uparrow & \uparrow & \\ x & y & z & \end{array}\right]$$

$\therefore \ x=-10, \ y=19, \ z=1$

KEY TO THE ROW OPERATIONS

r_{12} = SWAP ROW 1 & 2

$r_3\left(\frac{1}{2}\right)$ = MULTIPLY ROW 3 BY $\frac{1}{2}$

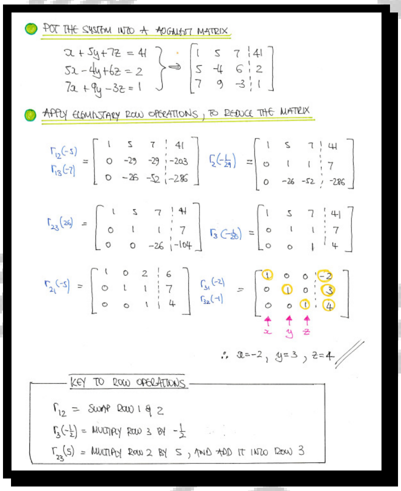
$r_{23}(-4)$ = MULTIPLY ROW 2 BY -4, AND ADD IT INTO ROW 3

Question 2

$$\begin{aligned} x + 5y + 7z &= 41 \\ 5x - 4y + 6z &= 2 \\ 7x + 9y - 3z &= 1 \end{aligned}$$

Solve the system of simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

$$\boxed{x = -2, \quad y = 3, \quad z = 4}$$



PUT THE SYSTEM INTO A AUGMENT MATRIX

$$\left.\begin{aligned} x + 5y + 7z &= 41 \\ 5x - 4y + 6z &= 2 \\ 7x + 9y - 3z &= 1 \end{aligned}\right\} \Rightarrow \left[\begin{array}{ccc|c} 1 & 5 & 7 & 41 \\ 5 & -4 & 6 & 2 \\ 7 & 9 & -3 & 1 \end{array}\right]$$

APPLY ELEMENTARY ROW OPERATIONS, TO REDUCE THE MATRIX

$$\begin{matrix} r_{12}(-5) \\ r_{13}(-7) \end{matrix} = \left[\begin{array}{ccc|c} 1 & 5 & 7 & 41 \\ 0 & -29 & -29 & -203 \\ 0 & -26 & -52 & -286 \end{array}\right] \quad r_2\left(-\tfrac{1}{29}\right) = \left[\begin{array}{ccc|c} 1 & 5 & 7 & 41 \\ 0 & 1 & 1 & 7 \\ 0 & -26 & -52 & -286 \end{array}\right]$$

$$r_{23}(26) = \left[\begin{array}{ccc|c} 1 & 5 & 7 & 41 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & -26 & -104 \end{array}\right] \quad r_3\left(-\tfrac{1}{26}\right) = \left[\begin{array}{ccc|c} 1 & 5 & 7 & 41 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$$r_{21}(-5) = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 6 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 1 & 4 \end{array}\right] \quad \begin{matrix} r_{31}(-2) \\ r_{32}(-1) \end{matrix} = \left[\begin{array}{ccc|c} \textcircled{1} & 0 & 0 & \textcircled{-2} \\ 0 & \textcircled{1} & 0 & \textcircled{3} \\ 0 & 0 & \textcircled{1} & \textcircled{4} \\ \uparrow & \uparrow & \uparrow & \\ x & y & z & \end{array}\right]$$

$$\therefore \ x = -2, \ y = 3, \ z = 4 \ //$$

KEY TO ROW OPERATIONS

r_{12} = SWAP ROW 1 & 2

$r_3(-\tfrac{1}{2})$ = MULTIPLY ROW 3 BY $-\tfrac{1}{2}$

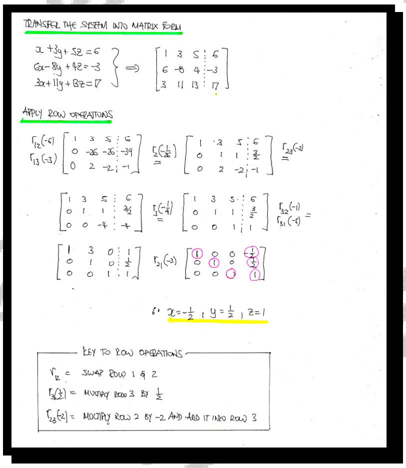
$r_{23}(5)$ = MULTIPLY ROW 2 BY 5, AND ADD IT INTO ROW 3

Question 3

$$\begin{aligned} x + 3y + 5z &= 6 \\ 6x - 8y + 4z &= -3 \\ 3x + 11y + 13z &= 17 \end{aligned}$$

Solve the system of simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

V , M , $\boxed{x = -\frac{1}{2},\ y = \frac{1}{2},\ z = 1}$



TRANSFER THE SYSTEM INTO MATRIX FORM

$$\left.\begin{aligned} x + 3y + 5z &= 6 \\ 6x - 8y + 4z &= -3 \\ 3x + 11y + 13z &= 17 \end{aligned}\right\} \Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 5 & 6 \\ 6 & -8 & 4 & -3 \\ 3 & 11 & 13 & 17 \end{array}\right]$$

APPLY ROW OPERATIONS

$$\begin{matrix} r_{12}(-6) \\ r_{13}(-3) \end{matrix} \left[\begin{array}{ccc|c} 1 & 3 & 5 & 6 \\ 0 & -26 & -26 & -39 \\ 0 & 2 & -2 & -1 \end{array}\right] \underset{=}{r_2\left(-\tfrac{1}{26}\right)} \left[\begin{array}{ccc|c} 1 & 3 & 5 & 6 \\ 0 & 1 & 1 & \frac{3}{2} \\ 0 & 2 & -2 & -1 \end{array}\right] \underset{=}{r_{23}(-2)}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 6 \\ 0 & 1 & 1 & 3/2 \\ 0 & 0 & -4 & -4 \end{array}\right] \underset{=}{r_3\left(-\tfrac{1}{4}\right)} \left[\begin{array}{ccc|c} 1 & 3 & 5 & 6 \\ 0 & 1 & 1 & \frac{3}{2} \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{matrix} r_{32}(-1) \\ r_{31}(-5) \end{matrix} =$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \end{array}\right] r_{21}(-3) \left[\begin{array}{ccc|c} ① & 0 & 0 & -\frac{1}{2} \\ 0 & ① & 0 & \frac{1}{2} \\ 0 & 0 & ① & ① \end{array}\right]$$

$\therefore\ x = -\frac{1}{2},\ y = \frac{1}{2},\ z = 1$

KEY TO ROW OPERATIONS

r_{12} = SWAP ROW 1 & 2

$r_3\left(\frac{1}{2}\right)$ = MULTIPLY ROW 3 BY $\frac{1}{2}$

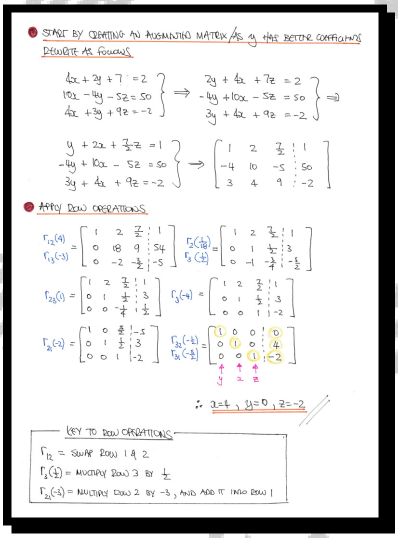
$r_{23}(-2)$ = MULTIPLY ROW 2 BY -2 AND ADD IT INTO ROW 3

Question 4

$$\begin{aligned} 4x \quad +2y + 7z &= \quad 2 \\ 10x - 4y - 5z &= \ 50 \\ 4x \quad +3y + 9z &= -2 \end{aligned}$$

Solve the system of simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

$$\boxed{x = 4, \ \ y = 0, \ \ z = -2}$$



START BY CREATING AN AUGMENTED MATRIX / AS y HAS BETTER COEFFICIENTS REWRITE AS FOLLOWS

$$\left.\begin{aligned} 4x + 2y + 7 &= 2 \\ 10x - 4y - 5z &= 50 \\ 4x + 3y + 9z &= -2 \end{aligned}\right\} \Longrightarrow \left.\begin{aligned} 2y + 4x + 7z &= 2 \\ -4y + 10x - 5z &= 50 \\ 3y + 4x + 9z &= -2 \end{aligned}\right\} \Longrightarrow$$

$$\left.\begin{aligned} y + 2x + \tfrac{7}{2}z &= 1 \\ -4y + 10x - 5z &= 50 \\ 3y + 4x + 9z &= -2 \end{aligned}\right\} \Longrightarrow \left[\begin{array}{ccc:c} 1 & 2 & \frac{7}{2} & 1 \\ -4 & 10 & -5 & 50 \\ 3 & 4 & 9 & -2 \end{array}\right]$$

APPLY ROW OPERATIONS

$$\begin{matrix} r_{12}(4) \\ r_{13}(-3) \end{matrix} = \left[\begin{array}{ccc:c} 1 & 2 & \frac{7}{2} & 1 \\ 0 & 18 & 9 & 54 \\ 0 & -2 & -\frac{3}{2} & -5 \end{array}\right] \quad \begin{matrix} r_2(\frac{1}{18}) \\ r_3(\frac{1}{2}) \end{matrix} = \left[\begin{array}{ccc:c} 1 & 2 & \frac{7}{2} & 1 \\ 0 & 1 & \frac{1}{2} & 3 \\ 0 & -1 & -\frac{3}{4} & -\frac{5}{2} \end{array}\right]$$

$$r_{23}(1) = \left[\begin{array}{ccc:c} 1 & 2 & \frac{7}{2} & 1 \\ 0 & 1 & \frac{1}{2} & 3 \\ 0 & 0 & -\frac{1}{4} & \frac{1}{2} \end{array}\right] \quad r_3(-4) = \left[\begin{array}{ccc:c} 1 & 2 & \frac{7}{2} & 1 \\ 0 & 1 & \frac{1}{2} & 3 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$r_{21}(-2) = \left[\begin{array}{ccc:c} 1 & 0 & \frac{5}{2} & -5 \\ 0 & 1 & \frac{1}{2} & 3 \\ 0 & 0 & 1 & -2 \end{array}\right] \quad \begin{matrix} r_{32}(-\frac{1}{2}) \\ r_{31}(-\frac{5}{2}) \end{matrix} = \left[\begin{array}{ccc:c} ① & 0 & 0 & 0 \\ 0 & ① & 0 & 4 \\ 0 & 0 & ① & -2 \\ \uparrow & \uparrow & \uparrow & \\ y & x & z & \end{array}\right]$$

$$\therefore \ x = 4, \ y = 0, \ z = -2$$

KEY TO ROW OPERATIONS

r_{12} = SWAP ROW 1 & 2

$r_3(\frac{1}{2})$ = MULTIPLY ROW 3 BY $\frac{1}{2}$

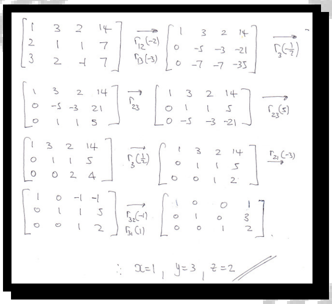
$r_{21}(-3)$ = MULTIPLY ROW 2 BY -3, AND ADD IT INTO ROW 1

Question 5

$$\begin{aligned} x+3y+2z&=14\\ 2x\ \ +y\ \ +z&=\ 7\\ 3x+2y\ \ -z&=\ 7 \end{aligned}$$

Solve the system simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

$$\boxed{x=1,\ y=3,\ z=2}$$



$$\begin{bmatrix} 1 & 3 & 2 & 14\\ 2 & 1 & 1 & 7\\ 3 & 2 & -1 & 7 \end{bmatrix} \xrightarrow[\substack{r_{12}(-2)\\ r_{13}(-3)}]{} \begin{bmatrix} 1 & 3 & 2 & 14\\ 0 & -5 & -3 & -21\\ 0 & -7 & -7 & -35 \end{bmatrix} \xrightarrow[r_3(-\frac{1}{7})]{}$$

$$\begin{bmatrix} 1 & 3 & 2 & 14\\ 0 & -5 & -3 & 21\\ 0 & 1 & 1 & 5 \end{bmatrix} \xrightarrow[r_{23}]{} \begin{bmatrix} 1 & 3 & 2 & 14\\ 0 & 1 & 1 & 5\\ 0 & -5 & -3 & -21 \end{bmatrix} \xrightarrow[r_{23}(5)]{}$$

$$\begin{bmatrix} 1 & 3 & 2 & 14\\ 0 & 1 & 1 & 5\\ 0 & 0 & 2 & 4 \end{bmatrix} \xrightarrow[r_3(\frac{1}{2})]{} \begin{bmatrix} 1 & 3 & 2 & 14\\ 0 & 1 & 1 & 5\\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{r_{21}(-3)}$$

$$\begin{bmatrix} 1 & 0 & -1 & -1\\ 0 & 1 & 1 & 5\\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[\substack{r_{32}(-1)\\ r_{31}(1)}]{} \begin{bmatrix} 1 & 0 & 0 & 1\\ 0 & 1 & 0 & 3\\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

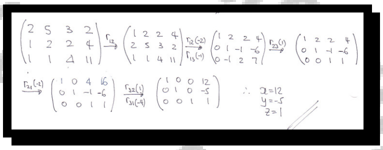
$\therefore\ x=1,\ y=3,\ z=2$ //

Question 6

$$\begin{aligned} 2x+5y+3z&=\ 2\\ x+2y+2z&=\ 4\\ x+\ \ y+4z&=11 \end{aligned}$$

Solve the above simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

$$\boxed{x=12,\ y=-5,\ z=1}$$



$$\begin{pmatrix} 2 & 5 & 3 & 2\\ 1 & 2 & 2 & 4\\ 1 & 1 & 4 & 11 \end{pmatrix} \xrightarrow{r_{12}} \begin{pmatrix} 1 & 2 & 2 & 4\\ 2 & 5 & 3 & 2\\ 1 & 1 & 4 & 11 \end{pmatrix} \xrightarrow[r_{13}(-1)]{r_{12}(-2)} \begin{pmatrix} 1 & 2 & 2 & 4\\ 0 & 1 & -1 & -6\\ 0 & -1 & 2 & 7 \end{pmatrix} \xrightarrow{r_{23}(1)} \begin{pmatrix} 1 & 2 & 2 & 4\\ 0 & 1 & -1 & -6\\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow{r_{21}(-2)} \begin{pmatrix} 1 & 0 & 4 & 16\\ 0 & 1 & -1 & -6\\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow[r_{31}(-4)]{r_{32}(1)} \begin{pmatrix} 1 & 0 & 0 & 12\\ 0 & 1 & 0 & -5\\ 0 & 0 & 1 & 1 \end{pmatrix}$$

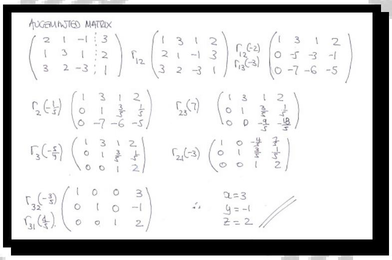
$\therefore$ $x=12$
$y=-5$
$z=1$ //

Question 7

$$\begin{aligned} 2x \ + y \ - z &= 3 \\ x + 3y \ + z &= 2 \\ 3x + 2y - 3z &= 1 \end{aligned}$$

Solve the system of simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

$\boxed{x = 3, y = -1, z = 2}$



AUGMENTED MATRIX

$$\left(\begin{array}{ccc:c} 2 & 1 & -1 & 3 \\ 1 & 3 & 1 & 2 \\ 3 & 2 & -3 & 1 \end{array}\right) r_{12} \begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 1 & -1 & 3 \\ 3 & 2 & -3 & 1 \end{pmatrix} \begin{matrix} r_{12}(-2) \\ r_{13}(-3) \end{matrix} \begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & -5 & -3 & -1 \\ 0 & -7 & -6 & -5 \end{pmatrix}$$

$$r_2\left(-\tfrac{1}{5}\right) \begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & 1 & \frac{3}{5} & \frac{1}{5} \\ 0 & -7 & -6 & -5 \end{pmatrix} \quad r_{23}(7) \begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & 1 & \frac{3}{5} & \frac{1}{5} \\ 0 & 0 & -\frac{9}{5} & -\frac{18}{5} \end{pmatrix}$$

$$r_3\left(-\tfrac{5}{9}\right) \begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & 1 & \frac{3}{5} & \frac{1}{5} \\ 0 & 0 & 1 & 2 \end{pmatrix} \quad r_{21}(-3) \begin{pmatrix} 1 & 0 & -\frac{4}{5} & \frac{7}{5} \\ 0 & 1 & \frac{3}{5} & \frac{1}{5} \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

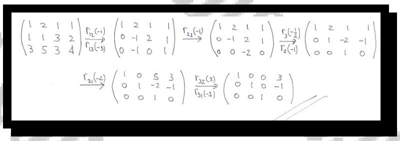
$$\begin{matrix} r_{32}\left(-\frac{3}{5}\right) \\ r_{31}\left(\frac{4}{5}\right) \end{matrix} \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \quad \therefore \quad \begin{aligned} x &= 3 \\ y &= -1 \\ z &= 2 \end{aligned}$$

Question 8

Solve the following simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 3 \\ 3 & 5 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

$\boxed{x = 3, y = -1, z = 0}$



$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 3 & 2 \\ 3 & 5 & 3 & 4 \end{pmatrix} \xrightarrow[r_{13}(-3)]{r_{12}(-1)} \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & 2 & 1 \\ 0 & -1 & 0 & 1 \end{pmatrix} \xrightarrow{r_{23}(-1)} \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & -2 & 0 \end{pmatrix} \xrightarrow[r_2(-1)]{r_3\left(-\frac{1}{2}\right)} \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

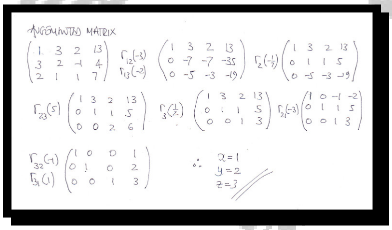
$$\xrightarrow{r_{21}(-2)} \begin{pmatrix} 1 & 0 & 5 & 3 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \xrightarrow[r_{31}(-5)]{r_{32}(2)} \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Question 9

$$\begin{aligned} x+3y+2z &= 13 \\ 3x+2y-z &= 4 \\ 2x+y+z &= 7 \end{aligned}$$

Solve the system of simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

$$\boxed{x=1, y=2, z=3}$$



AUGMENTED MATRIX

$$\begin{pmatrix} 1 & 3 & 2 & 13 \\ 3 & 2 & -1 & 4 \\ 2 & 1 & 1 & 7 \end{pmatrix} \begin{matrix} r_{12}(-3) \\ r_{13}(-2) \end{matrix} \begin{pmatrix} 1 & 3 & 2 & 13 \\ 0 & -7 & -7 & -35 \\ 0 & -5 & -3 & -19 \end{pmatrix} r_2\left(-\tfrac{1}{7}\right) \begin{pmatrix} 1 & 3 & 2 & 13 \\ 0 & 1 & 1 & 5 \\ 0 & -5 & -3 & -19 \end{pmatrix}$$

$$r_{23}(5) \begin{pmatrix} 1 & 3 & 2 & 13 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 2 & 6 \end{pmatrix} r_3\left(\tfrac{1}{2}\right) \begin{pmatrix} 1 & 3 & 2 & 13 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 \end{pmatrix} r_{21}(-3) \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

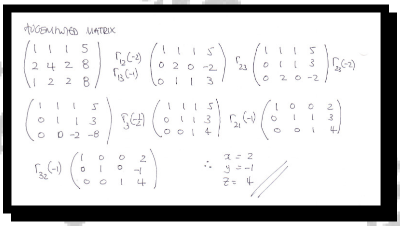
$$\begin{matrix} r_{32}(-1) \\ r_{31}(1) \end{matrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix} \qquad \therefore \begin{matrix} x=1 \\ y=2 \\ z=3 \end{matrix} //$$

Question 10

Solve the following simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ 8 \end{pmatrix}$$

$$\boxed{x=2, y=-1, z=4}$$



AUGMENTED MATRIX

$$\begin{pmatrix} 1 & 1 & 1 & 5 \\ 2 & 4 & 2 & 8 \\ 1 & 2 & 2 & 8 \end{pmatrix} \begin{matrix} r_{12}(-2) \\ r_{13}(-1) \end{matrix} \begin{pmatrix} 1 & 1 & 1 & 5 \\ 0 & 2 & 0 & -2 \\ 0 & 1 & 1 & 3 \end{pmatrix} r_{23} \begin{pmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 0 & -2 \end{pmatrix} r_{23}(-2)$$

$$\begin{pmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -2 & -8 \end{pmatrix} r_3\left(-\tfrac{1}{2}\right) \begin{pmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{pmatrix} r_{21}(-1) \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

$$r_{32}(-1) \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{pmatrix} \qquad \therefore \begin{matrix} x=2 \\ y=-1 \\ z=4 \end{matrix} //$$

Question 11

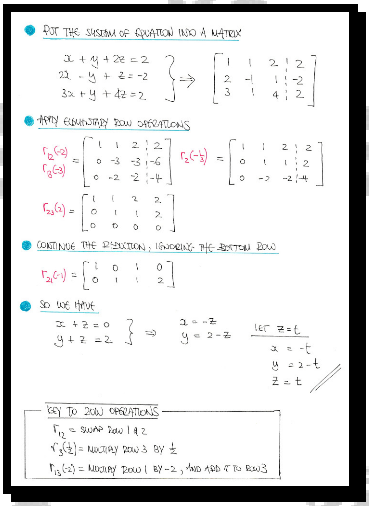
$$\begin{aligned} x+y+2z &= \ \ 2 \\ 2x-y\ \ +z &= -2 \\ 3x+y+4z &= \ \ 2 \end{aligned}$$

Show, by reducing the augmented matrix of the above system of equations into row echelon form, that the solution can be written as

$$x=-t\,,\ \ y=2-t\,,\ \ z=t$$

where t is a scalar parameter.

$\boxed{\text{proof}}$



- PUT THE SYSTEM OF EQUATION INTO A MATRIX

$$\left.\begin{aligned} x+y+2z&=2 \\ 2x-y+z&=-2 \\ 3x+y+4z&=2 \end{aligned}\right\} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 2 & -1 & 1 & -2 \\ 3 & 1 & 4 & 2 \end{array}\right]$$

- APPLY ELEMENTARY ROW OPERATIONS

$$\begin{array}{l} r_{12}(-2) \\ r_{13}(-3) \end{array} = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & -3 & -3 & -6 \\ 0 & -2 & -2 & -4 \end{array}\right] \quad r_2(-\tfrac{1}{3}) = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & -2 & -2 & -4 \end{array}\right]$$

$$r_{23}(2) = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- CONTINUE THE REDUCTION, IGNORING THE BOTTOM ROW

$$r_{21}(-1) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

- SO WE HAVE

$$\left.\begin{aligned} x+z&=0 \\ y+z&=2 \end{aligned}\right\} \Rightarrow \begin{aligned} x&=-z \\ y&=2-z \end{aligned} \qquad \begin{aligned} &\underline{\text{LET } z=t} \\ &x=-t \\ &y=2-t \\ &z=t \end{aligned}$$

KEY TO ROW OPERATIONS

r_{12} = SWAP ROW 1 & 2

$r_3(\frac{1}{2})$ = MULTIPLY ROW 3 BY $\frac{1}{2}$

$r_{13}(-2)$ = MULTIPLY ROW 1 BY -2, AND ADD IT TO ROW 3

Question 12

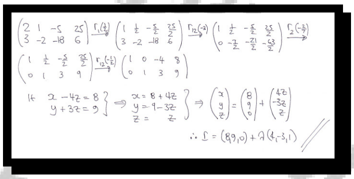
$$\begin{aligned}3x-2y-18z&=6\\2x+y-5z&=25\end{aligned}$$

Show, by reducing the above system of equations into row echelon form, that the solution can be written as

$$\mathbf{r}=8\mathbf{i}+9\mathbf{j}+\lambda(4\mathbf{i}-3\mathbf{j}+\mathbf{k}),$$

where λ is a scalar parameter.

$\boxed{\text{proof}}$



$$\begin{pmatrix}2&1&-5&25\\3&-2&-18&6\end{pmatrix}\xrightarrow{r_1(\frac{1}{2})}\begin{pmatrix}1&\frac{1}{2}&-\frac{5}{2}&\frac{25}{2}\\3&-2&-18&6\end{pmatrix}\xrightarrow{r_{12}(-3)}\begin{pmatrix}1&\frac{1}{2}&-\frac{5}{2}&\frac{25}{2}\\0&-\frac{7}{2}&-\frac{21}{2}&-\frac{63}{2}\end{pmatrix}\xrightarrow{r_2(-\frac{2}{7})}$$

$$\begin{pmatrix}1&\frac{1}{2}&-\frac{5}{2}&\frac{25}{2}\\0&1&3&9\end{pmatrix}\xrightarrow{r_{12}(-\frac{1}{2})}\begin{pmatrix}1&0&-4&8\\0&1&3&9\end{pmatrix}$$

$$\text{If}\quad \left.\begin{aligned}x-4z&=8\\y+3z&=9\end{aligned}\right\}\Rightarrow\left.\begin{aligned}x&=8+4z\\y&=9-3z\\z&=z\end{aligned}\right\}\Rightarrow\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}8\\9\\0\end{pmatrix}+\begin{pmatrix}4z\\-3z\\z\end{pmatrix}$$

$$\therefore \underline{r}=(8,9,0)+\lambda(4,-3,1) \quad //$$

Question 13

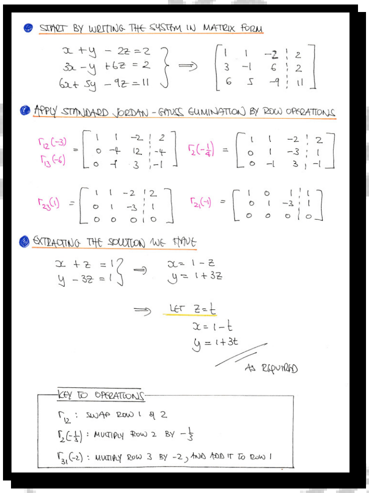
$$\begin{aligned} x + y - 2z &= 2 \\ 3x - y + 6z &= 2 \\ 6x + 5y - 9z &= 11 \end{aligned}$$

Show, by reducing the above equation system into row echelon form, that the consistent solution of the system can be written as

$$x = 1 - t, \qquad y = 3t + 1, \qquad z = t$$

where t is a scalar parameter.

$\boxed{\text{proof}}$



START BY WRITING THE SYSTEM IN MATRIX FORM

$$\left.\begin{aligned} x + y - 2z &= 2 \\ 3x - y + 6z &= 2 \\ 6x + 5y - 9z &= 11 \end{aligned}\right\} \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -2 & 2 \\ 3 & -1 & 6 & 2 \\ 6 & 5 & -9 & 11 \end{array}\right]$$

APPLY STANDARD JORDAN - GAUSS ELIMINATION BY ROW OPERATIONS

$$\begin{matrix} r_{12}(-3) \\ r_{13}(-6) \end{matrix} = \left[\begin{array}{ccc|c} 1 & 1 & -2 & 2 \\ 0 & -4 & 12 & -4 \\ 0 & -1 & 3 & -1 \end{array}\right] \quad r_2(-\tfrac{1}{4}) = \left[\begin{array}{ccc|c} 1 & 1 & -2 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & -1 & 3 & -1 \end{array}\right]$$

$$r_{23}(1) = \left[\begin{array}{ccc|c} 1 & 1 & -2 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad r_{21}(-1) = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

EXTRACTING THE SOLUTION WE HAVE

$$\left.\begin{aligned} x + z &= 1 \\ y - 3z &= 1 \end{aligned}\right\} \Longrightarrow \begin{aligned} x &= 1 - z \\ y &= 1 + 3z \end{aligned}$$

$$\Longrightarrow \text{LET } z = t$$

$$x = 1 - t$$

$$y = 1 + 3t$$

AS REQUIRED

KEY TO OPERATIONS

r_{12} : SWAP ROW 1 & 2

$r_2(-\frac{1}{3})$: MULTIPLY ROW 2 BY $-\frac{1}{3}$

$r_{31}(-2)$: MULTIPLY ROW 3 BY -2, AND ADD IT TO ROW 1

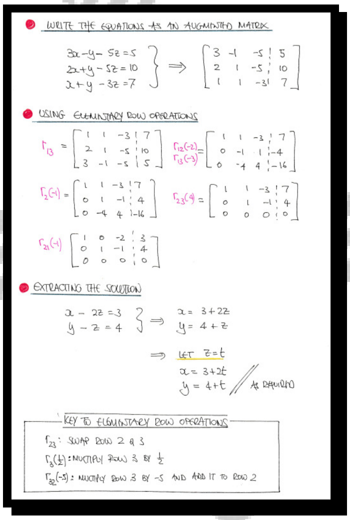
Question 14

$$\begin{aligned}3x - y - 5z &= 5\\ 2x + y - 5z &= 10\\ x + y - 3z &= 7\end{aligned}$$

Show, by reducing the above system into row echelon form, that the consistent solution of the system can be written as

$$x = 2t + 3, \quad y = t + 4, \quad z = t.$$

V, proof



WRITE THE EQUATIONS AS AN AUGMENTED MATRIX

$$\left.\begin{aligned}3x-y-5z&=5\\2x+y-5z&=10\\x+y-3z&=7\end{aligned}\right\} \Longrightarrow \left[\begin{array}{ccc|c}3&-1&-5&5\\2&1&-5&10\\1&1&-3&7\end{array}\right]$$

USING ELEMENTARY ROW OPERATIONS

$$r_{13} = \left[\begin{array}{ccc|c}1&1&-3&7\\2&1&-5&10\\3&-1&-5&5\end{array}\right] \quad \begin{matrix}r_{12}(-2)\\r_{13}(-3)\end{matrix} = \left[\begin{array}{ccc|c}1&1&-3&7\\0&-1&1&-4\\0&-4&4&-16\end{array}\right]$$

$$r_2(-1) = \left[\begin{array}{ccc|c}1&1&-3&7\\0&1&-1&4\\0&-4&4&-16\end{array}\right] \quad r_{23}(4) = \left[\begin{array}{ccc|c}1&1&-3&7\\0&1&-1&4\\0&0&0&0\end{array}\right]$$

$$r_{21}(-1) \left[\begin{array}{ccc|c}1&0&-2&3\\0&1&-1&4\\0&0&0&0\end{array}\right]$$

EXTRACTING THE SOLUTION

$$\left.\begin{aligned}x-2z&=3\\y-z&=4\end{aligned}\right\} \Longrightarrow \begin{aligned}x&=3+2z\\y&=4+z\end{aligned}$$

$$\Longrightarrow \text{LET } z=t$$

$$x = 3+2t$$
$$y = 4+t \quad // \text{ AS REQUIRED}$$

KEY TO ELEMENTARY ROW OPERATIONS

r_{23}: SWAP ROW 2 & 3

$r_3(\frac{1}{2})$: MULTIPLY ROW 3 BY $\frac{1}{2}$

$r_{32}(-5)$: MULTIPLY ROW 3 BY -5 AND ADD IT TO ROW 2

DETERMINANTS

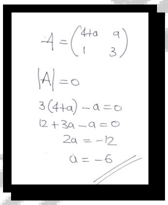
Question 1

The 2×2 matrix $\mathbf{A}$ is defined, in terms of a scalar constant a, by

$$\mathbf{A} = \begin{pmatrix} 4+a & a \\ 1 & 3 \end{pmatrix}.$$

Given that $\mathbf{A}$ is singular, find the value of a.

$$\boxed{a = -6}$$



$A = \begin{pmatrix} 4+a & a \\ 1 & 3 \end{pmatrix}$

$|A| = 0$

$3(4+a) - a = 0$

$12 + 3a - a = 0$

$2a = -12$

$a = -6$

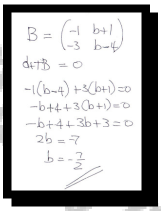
Question 2

The 2×2 matrix $\mathbf{B}$ is defined, in terms of a scalar constant b, by

$$\mathbf{B} = \begin{pmatrix} -1 & b+1 \\ -3 & b-4 \end{pmatrix}.$$

Given that $\mathbf{B}$ is singular, determine the value of b.

$$\boxed{b = -\frac{7}{2}}$$



$B = \begin{pmatrix} -1 & b+1 \\ -3 & b-4 \end{pmatrix}$

$\det B = 0$

$-1(b-4) + 3(b+1) = 0$

$-b + 4 + 3(b+1) = 0$

$-b + 4 + 3b + 3 = 0$

$2b = -7$

$b = -\frac{7}{2}$

Question 3

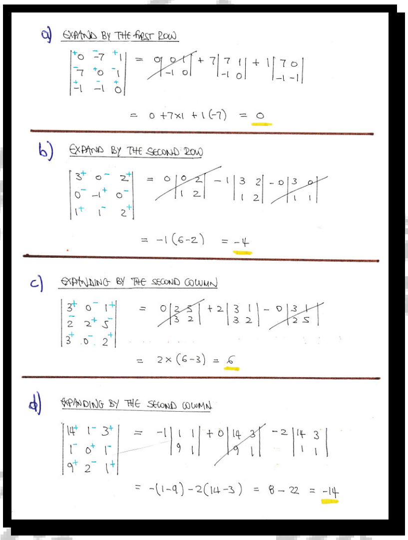
Evaluate each of the following determinants to the answer given.

a) $\begin{vmatrix} 0 & -7 & 1 \\ 7 & 0 & 1 \\ -1 & -1 & 0 \end{vmatrix} = 0$

b) $\begin{vmatrix} 3 & 0 & 2 \\ 0 & -1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = -4$

c) $\begin{vmatrix} 3 & 0 & 1 \\ 2 & 2 & 5 \\ 3 & 0 & 2 \end{vmatrix} = 6$

d) $\begin{vmatrix} 14 & 1 & 3 \\ 1 & 0 & 1 \\ 9 & 2 & 1 \end{vmatrix} = -14$



a) EXPAND BY THE FIRST ROW

$$\begin{vmatrix} {}^{+}0 & {}^{-}-7 & {}^{+}1 \\ {}^{-}7 & {}^{+}0 & {}^{-}1 \\ {}^{+}-1 & {}^{-}-1 & {}^{+}0 \end{vmatrix} = \cancel{0\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}} + 7\begin{vmatrix} 7 & 1 \\ -1 & 0 \end{vmatrix} + 1\begin{vmatrix} 7 & 0 \\ -1 & -1 \end{vmatrix}$$

$$= 0 + 7\times1 + 1(-7) = \underline{0}$$

b) EXPAND BY THE SECOND ROW

$$\begin{vmatrix} 3^{+} & 0^{-} & 2^{+} \\ 0^{-} & -1^{+} & 0^{-} \\ 1^{+} & 1^{-} & 2^{+} \end{vmatrix} = \cancel{0\begin{vmatrix} 0 & 2 \\ 1 & 2 \end{vmatrix}} - 1\begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} - \cancel{0\begin{vmatrix} 3 & 0 \\ 1 & 1 \end{vmatrix}}$$

$$= -1(6-2) = \underline{-4}$$

c) EXPANDING BY THE SECOND COLUMN

$$\begin{vmatrix} 3^{+} & 0^{-} & 1^{+} \\ 2^{-} & 2^{+} & 5^{-} \\ 3^{+} & 0^{-} & 2^{+} \end{vmatrix} = \cancel{0\begin{vmatrix} 2 & 5 \\ 3 & 2 \end{vmatrix}} + 2\begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} - \cancel{0\begin{vmatrix} 3 & 1 \\ 2 & 5 \end{vmatrix}}$$

$$= 2\times(6-3) = \underline{6}$$

d) EXPANDING BY THE SECOND COLUMN

$$\begin{vmatrix} 14^{+} & 1^{-} & 3^{+} \\ 1^{-} & 0^{+} & 1^{-} \\ 9^{+} & 2^{-} & 1^{+} \end{vmatrix} = -1\begin{vmatrix} 1 & 1 \\ 9 & 1 \end{vmatrix} + \cancel{0\begin{vmatrix} 14 & 3 \\ 9 & 1 \end{vmatrix}} - 2\begin{vmatrix} 14 & 3 \\ 1 & 1 \end{vmatrix}$$

$$= -(1-9) - 2(14-3) = 8 - 22 = \underline{-14}$$

Question 4

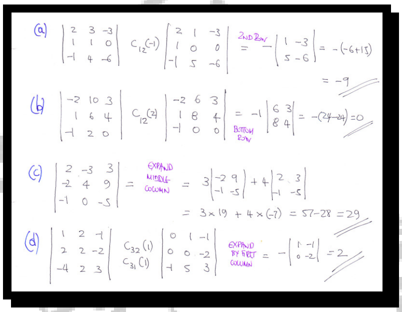
Evaluate each of the following determinants to the answer given.

a) $\begin{vmatrix} 2 & 3 & -3 \\ 1 & 1 & 0 \\ -1 & 4 & -6 \end{vmatrix} = -9$

b) $\begin{vmatrix} -2 & 10 & 3 \\ 1 & 6 & 4 \\ -1 & 2 & 0 \end{vmatrix} = 0$

c) $\begin{vmatrix} 2 & -3 & 3 \\ -2 & 4 & 9 \\ -1 & 0 & -5 \end{vmatrix} = 29$

d) $\begin{vmatrix} 1 & 2 & -1 \\ 2 & 2 & -2 \\ -4 & 2 & 3 \end{vmatrix} = 2$



(a) $\begin{vmatrix} 2 & 3 & -3 \\ 1 & 1 & 0 \\ -1 & 4 & -6 \end{vmatrix} \; C_{12}(-1) \begin{vmatrix} 2 & 1 & -3 \\ 1 & 0 & 0 \\ -1 & 5 & -6 \end{vmatrix} \overset{\text{2ND ROW}}{=} -\begin{vmatrix} 1 & -3 \\ 5 & -6 \end{vmatrix} = -(-6+15)$

$= -9$

(b) $\begin{vmatrix} -2 & 10 & 3 \\ 1 & 6 & 4 \\ -1 & 2 & 0 \end{vmatrix} \; C_{12}(2) \begin{vmatrix} -2 & 6 & 3 \\ 1 & 8 & 4 \\ -1 & 0 & 0 \end{vmatrix} \underset{\text{BOTTOM ROW}}{=} -1 \begin{vmatrix} 6 & 3 \\ 8 & 4 \end{vmatrix} = -(24-24) = 0$

(c) $\begin{vmatrix} 2 & -3 & 3 \\ -2 & 4 & 9 \\ -1 & 0 & -5 \end{vmatrix} =$ EXPAND MIDDLE COLUMN $= 3\begin{vmatrix} -2 & 9 \\ -1 & -5 \end{vmatrix} + 4\begin{vmatrix} 2 & 3 \\ -1 & -5 \end{vmatrix}$

$= 3 \times 19 + 4 \times (-7) = 57 - 28 = 29$

(d) $\begin{vmatrix} 1 & 2 & -1 \\ 2 & 2 & -2 \\ -4 & 2 & 3 \end{vmatrix} \; \begin{matrix} C_{32}(1) \\ C_{31}(1) \end{matrix} \begin{vmatrix} 0 & 1 & -1 \\ 0 & 0 & -2 \\ -1 & 5 & 3 \end{vmatrix}$ EXPAND BY FIRST COLUMN $= -\begin{vmatrix} 1 & -1 \\ 0 & -2 \end{vmatrix} = 2$

Question 5

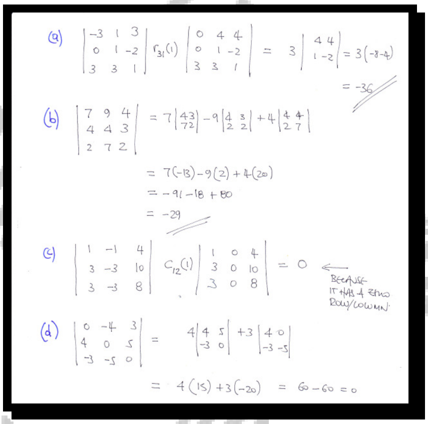
Evaluate the each of the following determinants to the answer given.

a) $\begin{vmatrix} -3 & 1 & 3 \\ 0 & 1 & -2 \\ 3 & 3 & 1 \end{vmatrix} = -36$

b) $\begin{vmatrix} 7 & 9 & 4 \\ 4 & 4 & 3 \\ 2 & 7 & 2 \end{vmatrix} = -29$

c) $\begin{vmatrix} 1 & -1 & 4 \\ 3 & -3 & 10 \\ 3 & -3 & 8 \end{vmatrix} = 0$

d) $\begin{vmatrix} 0 & -4 & 3 \\ 4 & 0 & 5 \\ -3 & -5 & 0 \end{vmatrix} = 0$



(a) $\begin{vmatrix} -3 & 1 & 3 \\ 0 & 1 & -2 \\ 3 & 3 & 1 \end{vmatrix} \; r_{31}(1) \begin{vmatrix} 0 & 4 & 4 \\ 0 & 1 & -2 \\ 3 & 3 & 1 \end{vmatrix} = 3\begin{vmatrix} 4 & 4 \\ 1 & -2 \end{vmatrix} = 3(-8-4)$

$= -36$

(b) $\begin{vmatrix} 7 & 9 & 4 \\ 4 & 4 & 3 \\ 2 & 7 & 2 \end{vmatrix} = 7\begin{vmatrix} 4 & 3 \\ 7 & 2 \end{vmatrix} - 9\begin{vmatrix} 4 & 3 \\ 2 & 2 \end{vmatrix} + 4\begin{vmatrix} 4 & 4 \\ 2 & 7 \end{vmatrix}$

$= 7(-13) - 9(2) + 4(20)$

$= -91 - 18 + 80$

$= -29$

(c) $\begin{vmatrix} 1 & -1 & 4 \\ 3 & -3 & 10 \\ 3 & -3 & 8 \end{vmatrix} \; C_{12}(1) \begin{vmatrix} 1 & 0 & 4 \\ 3 & 0 & 10 \\ 3 & 0 & 8 \end{vmatrix} = 0 \leftarrow$ BECAUSE IT HAS A ZERO ROW/COLUMN.

(d) $\begin{vmatrix} 0 & -4 & 3 \\ 4 & 0 & 5 \\ -3 & -5 & 0 \end{vmatrix} = 4\begin{vmatrix} 4 & 5 \\ -3 & 0 \end{vmatrix} + 3\begin{vmatrix} 4 & 0 \\ -3 & -5 \end{vmatrix}$

$= 4(15) + 3(-20) = 60 - 60 = 0$

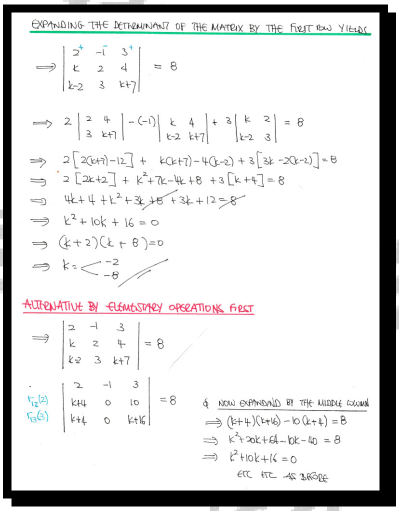
Question 6

The 3×3 matrix $\mathbf{A}$ is defined in terms of the scalar constant k by

$$\mathbf{A}=\begin{pmatrix}2 & -1 & 3\\ k & 2 & 4\\ k-2 & 3 & k+7\end{pmatrix}.$$

Given that $|\mathbf{A}|=8$, find the possible values of k.

$$\boxed{k=-2,\quad k=-8}$$



EXPANDING THE DETERMINANT OF THE MATRIX BY THE FIRST ROW YIELDS

$$\Rightarrow \begin{vmatrix}2^{+} & -1^{-} & 3^{+}\\ k & 2 & 4\\ k-2 & 3 & k+7\end{vmatrix}=8$$

$$\Rightarrow 2\begin{vmatrix}2 & 4\\ 3 & k+7\end{vmatrix}-(-1)\begin{vmatrix}k & 4\\ k-2 & k+7\end{vmatrix}+3\begin{vmatrix}k & 2\\ k-2 & 3\end{vmatrix}=8$$

$$\Rightarrow 2\left[2(k+7)-12\right]+k(k+7)-4(k-2)+3\left[3k-2(k-2)\right]=8$$

$$\Rightarrow 2\left[2k+2\right]+k^2+7k-4k+8+3\left[k+4\right]=8$$

$$\Rightarrow 4k+4+k^2+3k\cancel{+8}+3k+12=\cancel{8}$$

$$\Rightarrow k^2+10k+16=0$$

$$\Rightarrow (k+2)(k+8)=0$$

$$\Rightarrow k=\begin{cases}-2\\-8\end{cases}$$

ALTERNATIVE BY ELEMENTARY OPERATIONS FIRST

$$\Rightarrow \begin{vmatrix}2 & -1 & 3\\ k & 2 & 4\\ k-2 & 3 & k+7\end{vmatrix}=8$$

$$\begin{matrix}\\ r_{12}(2)\\ r_{13}(3)\end{matrix}\begin{vmatrix}2 & -1 & 3\\ k+4 & 0 & 10\\ k+4 & 0 & k+16\end{vmatrix}=8$$

& NOW EXPANDING BY THE MIDDLE COLUMN

$$\Rightarrow (k+4)(k+16)-10(k+4)=8$$

$$\Rightarrow k^2+20k+64-10k-40=8$$

$$\Rightarrow k^2+10k+16=0$$

ETC ETC AS BEFORE

Question 7

A transformation in three dimensional space is defined by the following 3×3 matrix.

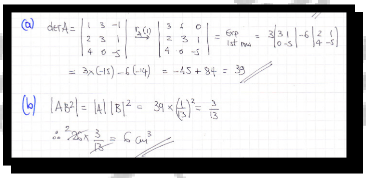
$$\mathbf{A}=\begin{pmatrix}1 & 3 & -1\\ 2 & 3 & 1\\ 4 & 0 & -5\end{pmatrix}.$$

a) Find the value of $\det\mathbf{A}$.

A cone with a volume of 26 cm^3 is transformed by the matrix composition $\mathbf{AB}^2$.

b) Given that $\det\mathbf{B}=\dfrac{1}{13}$, calculate the volume of the transformed cone.

$\boxed{\det\mathbf{A}=39}$, $\boxed{\text{volume}=6}$



(a) $\det A=\begin{vmatrix}1 & 3 & -1\\ 2 & 3 & 1\\ 4 & 0 & -5\end{vmatrix}\xrightarrow{r_2(1)}\begin{vmatrix}3 & 6 & 0\\ 2 & 3 & 1\\ 4 & 0 & -5\end{vmatrix}=\underset{\text{1st row}}{\text{Exp}}=3\begin{vmatrix}3 & 1\\ 0 & -5\end{vmatrix}-6\begin{vmatrix}2 & 1\\ 4 & -5\end{vmatrix}$

$=3\times(-15)-6(-14)=-45+84=39$

(b) $|AB^2|=|A|\,|B|^2=39\times\left(\frac{1}{13}\right)^2=\frac{3}{13}$

$\therefore \overset{2}{\cancel{26}}\times\frac{3}{\cancel{13}}=6\text{ cm}^3$

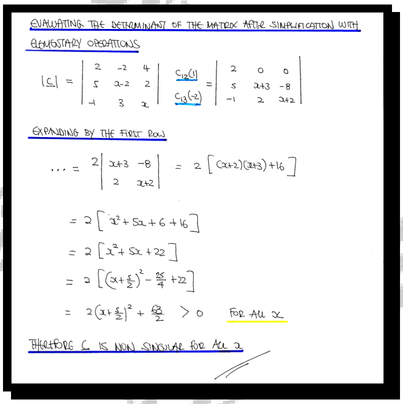
Question 8

A transformation in three dimensional space is defined by the following 3×3 matrix, where x is a scalar constant.

$$\mathbf{C} = \begin{pmatrix} 2 & -2 & 4 \\ 5 & x-2 & 2 \\ -1 & 3 & x \end{pmatrix}.$$

Show that $\mathbf{C}$ is non singular for all values of x.

proof



EVALUATING THE DETERMINANT OF THE MATRIX AFTER SIMPLIFICATION WITH ELEMENTARY OPERATIONS

$$|\underline{C}| = \begin{vmatrix} 2 & -2 & 4 \\ 5 & x-2 & 2 \\ -1 & 3 & x \end{vmatrix} \begin{array}{c} \underline{C_{12}(1)} \\ \underline{C_{13}(-2)} \end{array} = \begin{vmatrix} 2 & 0 & 0 \\ 5 & x+3 & -8 \\ -1 & 2 & x+2 \end{vmatrix}$$

EXPANDING BY THE FIRST ROW

$$\dots = 2\begin{vmatrix} x+3 & -8 \\ 2 & x+2 \end{vmatrix} = 2\left[(x+2)(x+3)+16\right]$$

$$= 2\left[x^2+5x+6+16\right]$$

$$= 2\left[x^2+5x+22\right]$$

$$= 2\left[\left(x+\frac{5}{2}\right)^2-\frac{25}{4}+22\right]$$

$$= 2\left(x+\frac{5}{2}\right)^2+\frac{63}{2} > 0 \quad \text{FOR ALL } x$$

THEREFORE $\underline{C}$ IS NON SINGULAR FOR ALL x

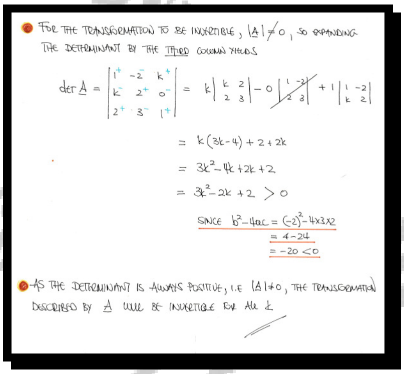
Question 9

A transformation in three dimensional space is defined by the following 3×3 matrix, where k is a scalar constant.

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & k \\ k & 2 & 0 \\ 2 & 3 & 1 \end{pmatrix}.$$

Show that the transformation defined by $\mathbf{A}$ can be inverted for all values of k.

proof



● FOR THE TRANSFORMATION TO BE INVERTIBLE, $|\underline{A}| \neq 0$, SO EXPANDING THE DETERMINANT BY THE THIRD COLUMN YIELDS

$$\det \underline{A} = \begin{vmatrix} 1^{+} & -2^{-} & k^{+} \\ k^{-} & 2^{+} & 0^{-} \\ 2^{+} & 3^{-} & 1^{+} \end{vmatrix} = k\begin{vmatrix} k & 2 \\ 2 & 3 \end{vmatrix} - 0\cancel{\begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix}} + 1\begin{vmatrix} 1 & -2 \\ k & 2 \end{vmatrix}$$

$$= k(3k-4) + 2 + 2k$$

$$= 3k^2 - 4k + 2k + 2$$

$$= 3k^2 - 2k + 2 > 0$$

SINCE $b^2 - 4ac = (-2)^2 - 4 \times 3 \times 2$

$= 4 - 24$

$= -20 < 0$

● AS THE DETERMINANT IS ALWAYS POSITIVE, I.E $|\underline{A}| \neq 0$, THE TRANSFORMATION DESCRIBED BY $\underline{A}$ WILL BE INVERTIBLE FOR ALL k

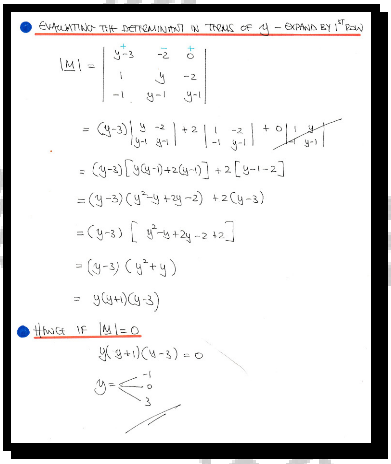
Question 10

A transformation in three dimensional space is defined by the following 3×3 matrix, where y is a scalar constant.

$$\mathbf{M}=\begin{pmatrix} y-3 & -2 & 0 \\ 1 & y & -2 \\ -1 & y-1 & y-1 \end{pmatrix}$$

If $|\mathbf{M}|=0$, find the possible values of y.

$$\boxed{y=-1,\quad y=0,\quad y=3}$$



- EVALUATING THE DETERMINANT IN TERMS OF y – EXPAND BY 1^{ST} ROW

$$|\underline{M}| = \begin{vmatrix} \overset{+}{y-3} & \overset{-}{-2} & \overset{+}{0} \\ 1 & y & -2 \\ -1 & y-1 & y-1 \end{vmatrix}$$

$$= (y-3)\begin{vmatrix} y & -2 \\ y-1 & y-1 \end{vmatrix} + 2\begin{vmatrix} 1 & -2 \\ -1 & y-1 \end{vmatrix} + \cancel{0\begin{vmatrix} 1 & y \\ -1 & y-1 \end{vmatrix}}$$

$$= (y-3)\left[y(y-1)+2(y-1)\right] + 2\left[y-1-2\right]$$

$$= (y-3)(y^2-y+2y-2) + 2(y-3)$$

$$= (y-3)\left[y^2-y+2y-2+2\right]$$

$$= (y-3)(y^2+y)$$

$$= y(y+1)(y-3)$$

- HENCE IF $|\underline{M}|=0$

$$y(y+1)(y-3)=0$$

$$y = \begin{cases} -1 \\ 0 \\ 3 \end{cases}$$

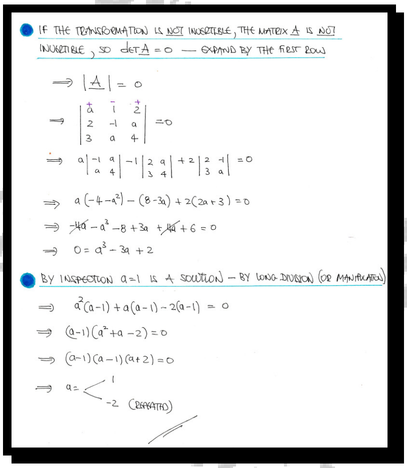
Question 11

A non invertible transformation in three dimensional space is defined by the following 3×3 matrix, where a is a scalar constant.

$$\mathbf{A} = \begin{pmatrix} a & 1 & 2 \\ 2 & -1 & a \\ 3 & a & 4 \end{pmatrix}$$

Determine the possible values of a.

$$\boxed{a = 1,\ \ a = -2}$$



IF THE TRANSFORMATION IS NOT INVERTIBLE, THE MATRIX $\underline{A}$ IS NOT INVERTIBLE, SO $\det \underline{A} = 0$ — EXPAND BY THE FIRST ROW

$$\Longrightarrow |\underline{A}| = 0$$

$$\Longrightarrow \begin{vmatrix} \overset{+}{a} & \overset{-}{1} & \overset{+}{2} \\ 2 & -1 & a \\ 3 & a & 4 \end{vmatrix} = 0$$

$$\Longrightarrow a\begin{vmatrix} -1 & a \\ a & 4 \end{vmatrix} - 1\begin{vmatrix} 2 & a \\ 3 & 4 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 \\ 3 & a \end{vmatrix} = 0$$

$$\Longrightarrow a(-4-a^2) - (8-3a) + 2(2a+3) = 0$$

$$\Longrightarrow \cancel{-4a} - a^3 - 8 + 3a + \cancel{4a} + 6 = 0$$

$$\Longrightarrow 0 = a^3 - 3a + 2$$

BY INSPECTION $a=1$ IS A SOLUTION — BY LONG DIVISION (OR MANIPULATION)

$$\Longrightarrow a^2(a-1) + a(a-1) - 2(a-1) = 0$$

$$\Longrightarrow (a-1)(a^2+a-2) = 0$$

$$\Longrightarrow (a-1)(a-1)(a+2) = 0$$

$$\Longrightarrow a = \begin{cases} 1 \\ -2 \ \text{(REPEATED)} \end{cases}$$

Question 12

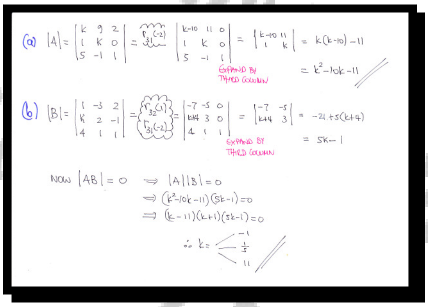
The 3×3 matrices $\mathbf{A}$ and $\mathbf{B}$ are defined in terms of a scalar constant k by

$$\mathbf{A} = \begin{pmatrix} k & 9 & 2 \\ 1 & k & 0 \\ 5 & -1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & -3 & 2 \\ k & 2 & -1 \\ 4 & 1 & 1 \end{pmatrix}.$$

a) Find an expression for $\det \mathbf{A}$, in terms of k.

b) Find the possible values of k given that $\mathbf{AB}$ is singular.

$\boxed{\det \mathbf{A} = k^2 - 10k - 11}$, $\boxed{k = -1,\ 11,\ \frac{1}{5}}$



(a) $|A| = \begin{vmatrix} k & 9 & 2 \\ 1 & k & 0 \\ 5 & -1 & 1 \end{vmatrix} \overset{r_{31}(-2)}{=} \begin{vmatrix} k-10 & 11 & 0 \\ 1 & k & 0 \\ 5 & -1 & 1 \end{vmatrix} = \begin{vmatrix} k-10 & 11 \\ 1 & k \end{vmatrix} = k(k-10)-11$

EXPAND BY THIRD COLUMN

$= k^2-10k-11$

(b) $|B| = \begin{vmatrix} 1 & -3 & 2 \\ k & 2 & -1 \\ 4 & 1 & 1 \end{vmatrix} \overset{\substack{r_{32}(1) \\ r_{31}(-2)}}{=} \begin{vmatrix} -7 & -5 & 0 \\ k+4 & 3 & 0 \\ 4 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -7 & -5 \\ k+4 & 3 \end{vmatrix} = -21+5(k+4)$

EXPAND BY THIRD COLUMN

$= 5k-1$

NOW $|AB| = 0 \Rightarrow |A||B| = 0$

$\Rightarrow (k^2-10k-11)(5k-1)=0$

$\Rightarrow (k-11)(k+1)(5k-1)=0$

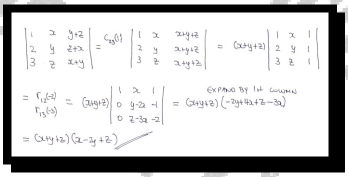
$\therefore k = -1,\ \frac{1}{5},\ 11$

Question 13

Factorize fully the following 3×3 determinant.

$$\begin{vmatrix} 1 & x & y+z \\ 2 & y & z+x \\ 3 & z & x+y \end{vmatrix}.$$

$\boxed{(x+y+z)(x-2y+z)}$



$\begin{vmatrix} 1 & x & y+z \\ 2 & y & z+x \\ 3 & z & x+y \end{vmatrix} \overset{c_{23}(1)}{=} \begin{vmatrix} 1 & x & x+y+z \\ 2 & y & x+y+z \\ 3 & z & x+y+z \end{vmatrix} = (x+y+z)\begin{vmatrix} 1 & x & 1 \\ 2 & y & 1 \\ 3 & z & 1 \end{vmatrix}$

$= \substack{r_{12}(-2) \\ r_{13}(-3)} = (x+y+z)\begin{vmatrix} 1 & x & 1 \\ 0 & y-2x & -1 \\ 0 & z-3x & -2 \end{vmatrix} = (x+y+z)(-2y+4x+z-3x)$

EXPAND BY 1st COLUMN

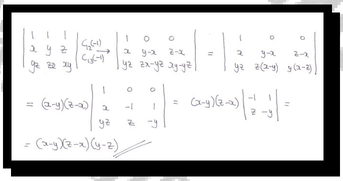
$= (x+y+z)(x-2y+z)$

Question 14

Factorize fully the following 3×3 determinant.

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix}.$$

$$\boxed{(x-y)(y-z)(z-x)}$$



$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix} \xrightarrow[C_{13}(-1)]{C_{12}(-1)} \begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ yz & zx-yz & xy-yz \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ yz & z(x-y) & y(x-z) \end{vmatrix}$$

$$= (x-y)(z-x)\begin{vmatrix} 1 & 0 & 0 \\ x & -1 & 1 \\ yz & z & -y \end{vmatrix} = (x-y)(z-x)\begin{vmatrix} -1 & 1 \\ z & -y \end{vmatrix} =$$

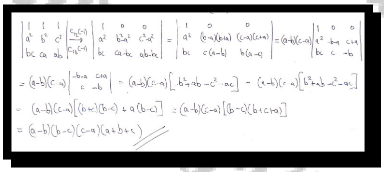
$$= (x-y)(z-x)(y-z)$$

Question 15

Factorize fully the following 3×3 determinant.

$$\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}.$$

$$\boxed{(a-b)(b-c)(c-a)(a+b+c)}$$



$$\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} \xrightarrow[C_{13}(-1)]{C_{12}(-1)} \begin{vmatrix} 1 & 0 & 0 \\ a^2 & b^2-a^2 & c^2-a^2 \\ bc & ca-bc & ab-bc \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ a^2 & (b-a)(b+a) & (c-a)(c+a) \\ bc & c(a-b) & b(a-c) \end{vmatrix} = (a-b)(c-a)\begin{vmatrix} 1 & 0 & 0 \\ a^2 & -b-a & c+a \\ bc & c & -b \end{vmatrix}$$

$$= (a-b)(c-a)\begin{vmatrix} -b-a & c+a \\ c & -b \end{vmatrix} = (a-b)(c-a)\left[b^2+ab-c^2-ac\right] = (a-b)(c-a)\left[b^2+ab-c^2-ac\right]$$

$$= (a-b)(c-a)\left[(b+c)(b-c)+a(b-c)\right] = (a-b)(c-a)\left[(b-c)(b+c+a)\right]$$

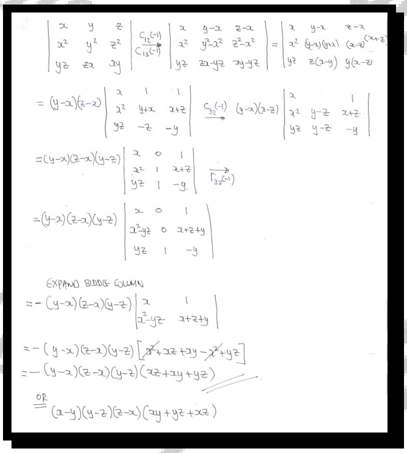
$$= (a-b)(b-c)(c-a)(a+b+c)$$

Question 16

Factorize fully the following 3×3 determinant.

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix}.$$

$$\boxed{(x-y)(y-z)(z-x)(xy+yz+zx)}$$



$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} \xrightarrow[C_{13}(-1)]{C_{12}(-1)} \begin{vmatrix} x & y-x & z-x \\ x^2 & y^2-x^2 & z^2-x^2 \\ yz & zx-yz & xy-yz \end{vmatrix} = \begin{vmatrix} x & y-x & z-x \\ x^2 & (y-x)(y+x) & (z-x)(x+z) \\ yz & z(x-y) & y(x-z) \end{vmatrix}$$

$$= (y-x)(z-x)\begin{vmatrix} x & 1 & 1 \\ x^2 & y+x & x+z \\ yz & -z & -y \end{vmatrix} \xrightarrow{C_{32}(-1)} (y-x)(x-z)\begin{vmatrix} x & & 1 \\ x^2 & y-z & x+z \\ yz & y-z & -y \end{vmatrix}$$

$$=(y-x)(z-x)(y-z)\begin{vmatrix} x & 0 & 1 \\ x^2 & 1 & x+z \\ yz & 1 & -y. \end{vmatrix} \xrightarrow{r_{32}(-1)}$$

$$=(y-x)(z-x)(y-z)\begin{vmatrix} x & 0 & 1 \\ x^2-yz & 0 & x+z+y \\ yz & 1 & -y \end{vmatrix}$$

EXPAND BIDDLE COLUMN

$$=-(y-x)(z-x)(y-z)\begin{vmatrix} x & 1 \\ x^2-yz & x+z+y \end{vmatrix}$$

$$=-(y-x)(z-x)(y-z)\left[\cancel{x^2}+xz+xy-\cancel{x^2}+yz\right]$$

$$=-(y-x)(z-x)(y-z)(xz+xy+yz)$$

$$\overset{\text{OR}}{=} (x-y)(y-z)(z-x)(xy+yz+xz)$$

MATRIX INVERSE

Question 1

Find the inverse for each of the following 2×2 matrices.

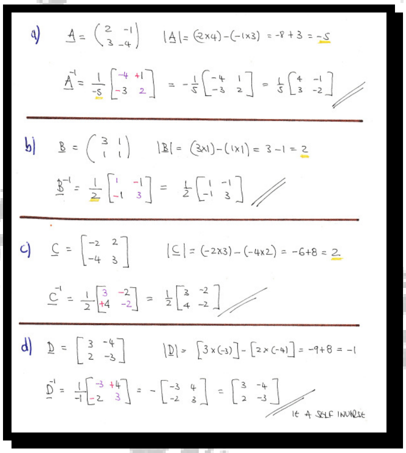
a) $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & -4 \end{pmatrix}$

b) $\mathbf{B} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$

c) $\mathbf{C} = \begin{pmatrix} -2 & 2 \\ -4 & 3 \end{pmatrix}$

d) $\mathbf{D} = \begin{pmatrix} 3 & -4 \\ 2 & -3 \end{pmatrix}$

$\boxed{\mathbf{A}^{-1} = \frac{1}{5}\begin{pmatrix} 4 & -1 \\ 3 & -2 \end{pmatrix}}$, $\boxed{\mathbf{B}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}}$, $\boxed{\mathbf{C}^{-1} = \frac{1}{2}\begin{pmatrix} 3 & -2 \\ 4 & -2 \end{pmatrix}}$, $\boxed{\mathbf{D}^{-1} = \begin{pmatrix} 3 & -4 \\ 2 & -3 \end{pmatrix}}$



a) $\underline{A} = \begin{pmatrix} 2 & -1 \\ 3 & -4 \end{pmatrix}$ $|\underline{A}| = (2\times4) - (-1\times3) = -8+3 = \underline{-5}$

$\underline{A}^{-1} = \frac{1}{-5}\begin{bmatrix} -4 & +1 \\ -3 & 2 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} -4 & 1 \\ -3 & 2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4 & -1 \\ 3 & -2 \end{bmatrix}$ //

b) $\underline{B} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$ $|\underline{B}| = (3\times1) - (1\times1) = 3-1 = \underline{2}$

$\underline{B}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$ //

c) $\underline{C} = \begin{bmatrix} -2 & 2 \\ -4 & 3 \end{bmatrix}$ $|\underline{C}| = (-2\times3) - (-4\times2) = -6+8 = \underline{2}$

$\underline{C}^{-1} = \frac{1}{2}\begin{bmatrix} 3 & -2 \\ +4 & -2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$ //

d) $\underline{D} = \begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix}$ $|\underline{D}| = [3\times(-3)] - [2\times(-4)] = -9+8 = -1$

$\underline{D}^{-1} = \frac{1}{-1}\begin{bmatrix} -3 & +4 \\ -2 & 3 \end{bmatrix} = -\begin{bmatrix} -3 & 4 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix}$ //

It A SELF INVERSE

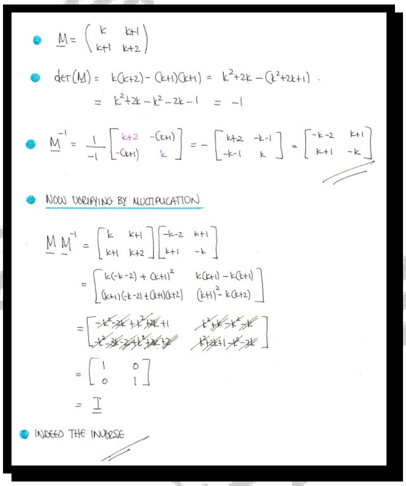
Question 2

Find, in terms of k, the inverse of the following 2×2 matrix.

$$\mathbf{M}=\begin{pmatrix} k & k+1 \\ k+1 & k+2 \end{pmatrix}.$$

Verify your answer by multiplication.

$$\boxed{\mathbf{M}^{-1}=\begin{pmatrix} -k-2 & k+1 \\ k+1 & -k \end{pmatrix}}$$



- $\underline{M}=\begin{pmatrix} k & k+1 \\ k+1 & k+2 \end{pmatrix}$
- $\det(\underline{M}) = k(k+2)-(k+1)(k+1) = k^2+2k-(k^2+2k+1)$.
  $= k^2+2k-k^2-2k-1 = -1$
- $\underline{M}^{-1} = \frac{1}{-1}\begin{bmatrix} k+2 & -(k+1) \\ -(k+1) & k \end{bmatrix} = -\begin{bmatrix} k+2 & -k-1 \\ -k-1 & k \end{bmatrix} = \begin{bmatrix} -k-2 & k+1 \\ k+1 & -k \end{bmatrix}$
- NOW VERIFYING BY MULTIPLICATION

$$\underline{M}\,\underline{M}^{-1} = \begin{bmatrix} k & k+1 \\ k+1 & k+2 \end{bmatrix}\begin{bmatrix} -k-2 & k+1 \\ k+1 & -k \end{bmatrix}$$
$$= \begin{bmatrix} k(-k-2)+(k+1)^2 & k(k+1)-k(k+1) \\ (k+1)(-k-2)+(k+1)(k+2) & (k+1)^2-k(k+2) \end{bmatrix}$$
$$= \begin{bmatrix} \cancel{-k^2}\cancel{-2k}\cancel{+k^2}\cancel{+2k}+1 & \cancel{k^2}\cancel{+k}\cancel{-k^2}\cancel{-k} \\ \cancel{-k^2}\cancel{-3k}\cancel{-2}\cancel{+k^2}\cancel{+3k}\cancel{+2} & \cancel{k^2}\cancel{+2k}+1\cancel{-k^2}\cancel{-2k} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \underline{I}$$

- INDEED THE INVERSE

Question 3

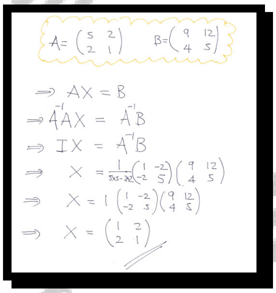
The 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 9 & 12 \\ 4 & 5 \end{pmatrix}.$$

Find the 2×2 matrix $\mathbf{X}$ that satisfy the equation

$$\mathbf{AX} = \mathbf{B}.$$

$$\boxed{\mathbf{X} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}}$$



$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 9 & 12 \\ 4 & 5 \end{pmatrix}$

$\Rightarrow AX = B$

$\Rightarrow A^{-1}AX = A^{-1}B$

$\Rightarrow IX = A^{-1}B$

$\Rightarrow X = \frac{1}{5\times5-2\times2}\begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 9 & 12 \\ 4 & 5 \end{pmatrix}$

$\Rightarrow X = 1\begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 9 & 12 \\ 4 & 5 \end{pmatrix}$

$\Rightarrow X = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

Question 4

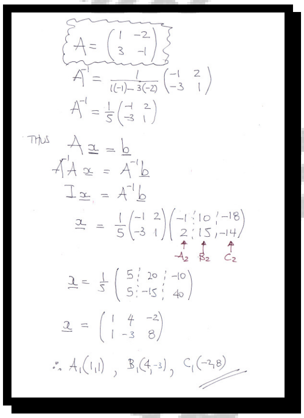
The triangle T_1 is mapped by the 2×2 matrix

$$\mathbf{A}=\begin{pmatrix}1 & -2\\ 3 & -1\end{pmatrix}$$

onto the triangle T_2, whose vertices have coordinates $A_2(-1,2)$, $B_2(10,15)$ and $C_2(-18,-14)$.

Find the coordinates of the vertices of T_1.

$\boxed{A_1(1,1)}$, $\boxed{B_1(4,-3)}$, $\boxed{C_1(-2,8)}$



$$A=\begin{pmatrix}1 & -2\\ 3 & -1\end{pmatrix}$$

$$A^{-1}=\frac{1}{1(-1)-3(-2)}\begin{pmatrix}-1 & 2\\ -3 & 1\end{pmatrix}$$

$$A^{-1}=\frac{1}{5}\begin{pmatrix}-1 & 2\\ -3 & 1\end{pmatrix}$$

THUS $\quad A\underline{x}=\underline{b}$

$$A^{-1}A\underline{x}=A^{-1}\underline{b}$$

$$I\underline{x}=A^{-1}\underline{b}$$

$$\underline{x}=\frac{1}{5}\begin{pmatrix}-1 & 2\\ -3 & 1\end{pmatrix}\begin{pmatrix}-1 & 10 & -18\\ 2 & 15 & -14\end{pmatrix}$$

$$\begin{array}{ccc}\uparrow & \uparrow & \uparrow\\ A_2 & B_2 & C_2\end{array}$$

$$\underline{x}=\frac{1}{5}\begin{pmatrix}5 & 20 & -10\\ 5 & -15 & 40\end{pmatrix}$$

$$\underline{x}=\begin{pmatrix}1 & 4 & -2\\ 1 & -3 & 8\end{pmatrix}$$

$\therefore A_1(1,1)$, $B_1(4,-3)$, $C_1(-2,8)$

Question 5

The triangle T_1 is mapped by the 2×2 matrix

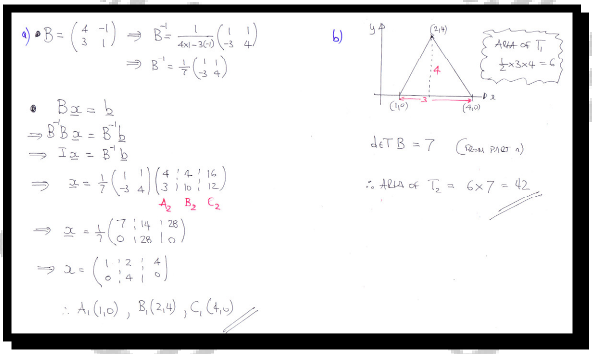
$$\mathbf{B} = \begin{pmatrix} 4 & -1 \\ 3 & 1 \end{pmatrix}$$

onto the triangle T_2, whose vertices have coordinates $A_2(4,3)$, $B_2(4,10)$ and $C_2(16,12)$.

a) Find the coordinates of the vertices of T_1.

b) Determine the area of T_2.

$\boxed{A_1(1,0)}$, $\boxed{B_1(2,4)}$, $\boxed{C_1(4,0)}$, $\boxed{\text{area} = 42}$



a) • $B = \begin{pmatrix} 4 & -1 \\ 3 & 1 \end{pmatrix} \Rightarrow B^{-1} = \frac{1}{4\times1-3(-1)}\begin{pmatrix} 1 & 1 \\ -3 & 4 \end{pmatrix}$

$\Rightarrow B^{-1} = \frac{1}{7}\begin{pmatrix} 1 & 1 \\ -3 & 4 \end{pmatrix}$

• $B\underline{x} = \underline{b}$

$\Rightarrow B^{-1}B\underline{x} = B^{-1}\underline{b}$

$\Rightarrow I\underline{x} = B^{-1}\underline{b}$

$\Rightarrow \underline{x} = \frac{1}{7}\begin{pmatrix} 1 & 1 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 4 & 4 & 16 \\ 3 & 10 & 12 \end{pmatrix}$

A_2 B_2 C_2

$\Rightarrow \underline{x} = \frac{1}{7}\begin{pmatrix} 7 & 14 & 28 \\ 0 & 28 & 0 \end{pmatrix}$

$\Rightarrow x = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 4 & 0 \end{pmatrix}$

$\therefore A_1(1,0)$, $B_1(2,4)$, $C_1(4,0)$

b)

AREA OF T_1
$\frac{1}{2}\times3\times4 = 6$

$\det B = 7$ (FROM PART a)

$\therefore$ AREA OF T_2 $= 6\times7 = 42$

Question 6

The triangle T_1 is mapped by the 2×2 matrix

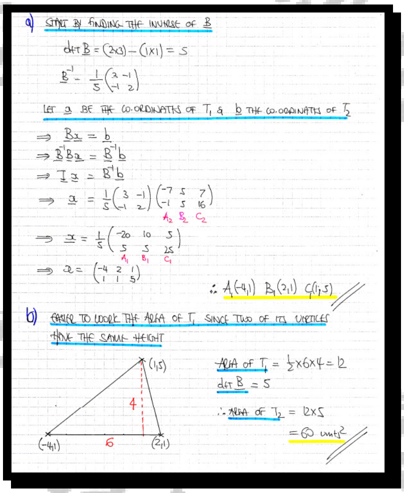
$$\mathbf{B}=\begin{pmatrix}2 & 1\\ 1 & 3\end{pmatrix}$$

onto the triangle T_2, whose vertices have coordinates $A_2(-7,-1)$, $B_2(5,5)$ and $C_2(7,16)$.

a) Find the coordinates of the vertices of T_1.

b) Determine the area of T_2.

$\boxed{A_1(-4,1)}$, $\boxed{B_1(2,1)}$, $\boxed{C_1(1,5)}$, $\boxed{\text{area}=20}$



a) START BY FINDING THE INVERSE OF $\underline{B}$

$\det \underline{B} = (2\times3)-(1\times1) = 5$

$\underline{B}^{-1} = \frac{1}{5}\begin{pmatrix}3 & -1\\ -1 & 2\end{pmatrix}$

LET $\underline{a}$ BE THE CO-ORDINATES OF T_1 & $\underline{b}$ THE CO-ORDINATES OF T_2

$\Rightarrow \underline{B}\underline{x} = \underline{b}$

$\Rightarrow \underline{B}^{-1}\underline{B}\underline{x} = \underline{B}^{-1}\underline{b}$

$\Rightarrow \underline{I}\underline{x} = \underline{B}^{-1}\underline{b}$

$\Rightarrow \underline{x} = \frac{1}{5}\begin{pmatrix}3 & -1\\ -1 & 2\end{pmatrix}\begin{pmatrix}-7 & 5 & 7\\ -1 & 5 & 16\end{pmatrix}$

$\begin{matrix}A_2 & B_2 & C_2\end{matrix}$

$\Rightarrow \underline{x} = \frac{1}{5}\begin{pmatrix}-20 & 10 & 5\\ 5 & 5 & 25\end{pmatrix}$

$\begin{matrix}A_1 & B_1 & C_1\end{matrix}$

$\Rightarrow \underline{x} = \begin{pmatrix}-4 & 2 & 1\\ 1 & 1 & 5\end{pmatrix}$

$\therefore A_1(-4,1)\ \ B_1(2,1)\ \ C_1(1,5)$ //

b) EASIER TO WORK THE AREA OF T_1 SINCE TWO OF ITS VERTICES HAVE THE SAME HEIGHT

AREA OF $T_1 = \frac{1}{2}\times6\times4 = 12$

$\det \underline{B} = 5$

$\therefore$ AREA OF $T_2 = 12\times5$

$= 60$ units^2 //

Question 7

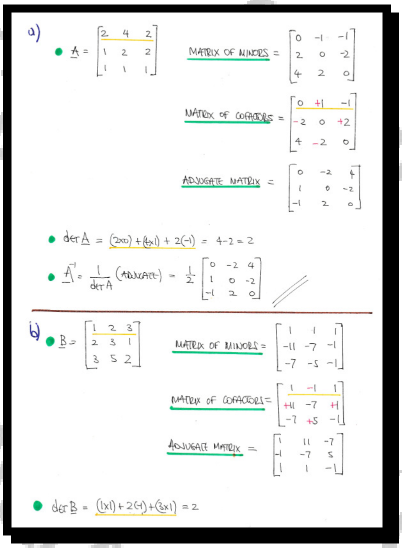
Find the inverse of each the following 3×3 matrices.

a) $\mathbf{A}=\begin{pmatrix}2&4&2\\1&2&2\\1&1&1\end{pmatrix}$

b) $\mathbf{B}=\begin{pmatrix}1&2&3\\2&3&1\\3&5&2\end{pmatrix}$

c) $\mathbf{C}=\begin{pmatrix}1&2&3\\1&1&-2\\2&1&1\end{pmatrix}$

$$\boxed{\mathbf{A}^{-1}=\frac{1}{2}\begin{pmatrix}0&-2&4\\1&0&-2\\-1&2&0\end{pmatrix}}, \boxed{\mathbf{B}^{-1}=\frac{1}{2}\begin{pmatrix}1&11&-7\\-1&-7&5\\1&1&-1\end{pmatrix}}, \boxed{\mathbf{C}^{-1}=\frac{1}{10}\begin{pmatrix}-3&-1&7\\5&5&-5\\1&-3&1\end{pmatrix}}$$



a)

- $\underline{A}=\begin{bmatrix}2&4&2\\1&2&2\\1&1&1\end{bmatrix}$ MATRIX OF MINORS $=\begin{bmatrix}0&-1&-1\\2&0&-2\\4&2&0\end{bmatrix}$

MATRIX OF COFACTORS $=\begin{bmatrix}0&+1&-1\\-2&0&+2\\4&-2&0\end{bmatrix}$

ADJUGATE MATRIX $=\begin{bmatrix}0&-2&4\\1&0&-2\\-1&2&0\end{bmatrix}$

- $\det\underline{A}=(2\times0)+(4\times1)+2(-1)=4-2=2$
- $\underline{A}^{-1}=\frac{1}{\det A}(\text{ADJUGATE})=\frac{1}{2}\begin{bmatrix}0&-2&4\\1&0&-2\\-1&2&0\end{bmatrix}$ //

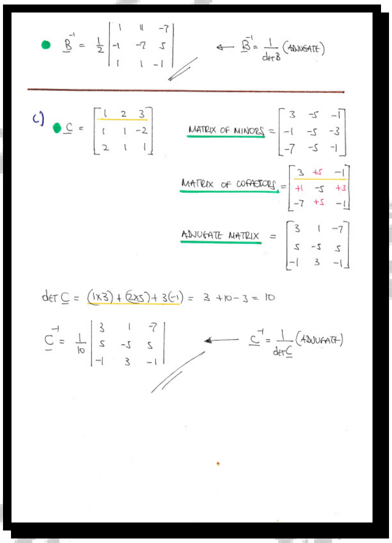
b)

- $\underline{B}=\begin{bmatrix}1&2&3\\2&3&1\\3&5&2\end{bmatrix}$ MATRIX OF MINORS $=\begin{bmatrix}1&1&1\\-11&-7&-1\\-7&-5&-1\end{bmatrix}$

MATRIX OF COFACTORS $=\begin{bmatrix}1&-1&1\\+11&-7&+1\\-7&+5&-1\end{bmatrix}$

ADJUGATE MATRIX $=\begin{bmatrix}1&11&-7\\-1&-7&5\\1&1&-1\end{bmatrix}$

- $\det\underline{B}=(1\times1)+2(-1)+(3\times1)=2$



- $\underline{B}^{-1}=\frac{1}{2}\begin{vmatrix}1&11&-7\\-1&-7&5\\1&1&-1\end{vmatrix}$ // $\leftarrow \underline{B}^{-1}=\frac{1}{\det B}(\text{ADJUGATE})$

c)

- $\underline{C}=\begin{bmatrix}1&2&3\\1&1&-2\\2&1&1\end{bmatrix}$ MATRIX OF MINORS $=\begin{bmatrix}3&-5&-1\\-1&-5&-3\\-7&-5&-1\end{bmatrix}$

MATRIX OF COFACTORS $=\begin{bmatrix}3&+5&-1\\+1&-5&+3\\-7&+5&-1\end{bmatrix}$

ADJUGATE MATRIX $=\begin{bmatrix}3&1&-7\\5&-5&5\\-1&3&-1\end{bmatrix}$

$\det\underline{C}=(1\times3)+(2\times5)+3(-1)=3+10-3=10$

$\underline{C}^{-1}=\frac{1}{10}\begin{vmatrix}3&1&-7\\5&-5&5\\-1&3&-1\end{vmatrix}$ // $\leftarrow \underline{C}^{-1}=\frac{1}{\det\underline{C}}(\text{ADJUGATE})$

Question 8

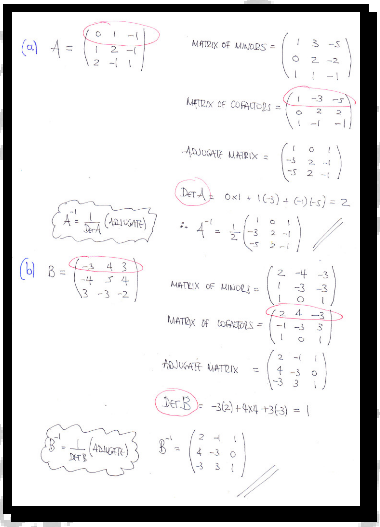
Find the inverse of each the following 3×3 matrices.

a) $\mathbf{A}=\begin{pmatrix}0&1&-1\\1&2&-1\\2&-1&1\end{pmatrix}$

b) $\mathbf{B}=\begin{pmatrix}-3&4&3\\-4&5&4\\3&-3&-2\end{pmatrix}$

c) $\mathbf{C}=\begin{pmatrix}1&2&1\\2&1&1\\1&4&2\end{pmatrix}$

$$\boxed{\mathbf{A}^{-1}=\frac{1}{2}\begin{pmatrix}1&0&1\\-3&2&-1\\-5&2&-1\end{pmatrix}},\ \boxed{\mathbf{B}^{-1}=\begin{pmatrix}2&-1&1\\4&-3&0\\-3&3&1\end{pmatrix}},\ \boxed{\mathbf{C}^{-1}=\begin{pmatrix}2&0&-1\\3&-1&-1\\-7&2&3\end{pmatrix}}$$



(a) $A=\begin{pmatrix}0&1&-1\\1&2&-1\\2&-1&1\end{pmatrix}$

MATRIX OF MINORS $=\begin{pmatrix}1&3&-5\\0&2&-2\\1&1&-1\end{pmatrix}$

MATRIX OF COFACTORS $=\begin{pmatrix}1&-3&-5\\0&2&2\\1&-1&-1\end{pmatrix}$

ADJUGATE MATRIX $=\begin{pmatrix}1&0&1\\-3&2&-1\\-5&2&-1\end{pmatrix}$

DET A $=0\times1+1(-3)+(-1)(-5)=2$

$A^{-1}=\frac{1}{\text{DET}A}(\text{ADJUGATE})$

$\therefore A^{-1}=\frac{1}{2}\begin{pmatrix}1&0&1\\-3&2&-1\\-5&2&-1\end{pmatrix}$ //

(b) $B=\begin{pmatrix}-3&4&3\\-4&5&4\\3&-3&-2\end{pmatrix}$

MATRIX OF MINORS $=\begin{pmatrix}2&-4&-3\\1&-3&-3\\1&0&1\end{pmatrix}$

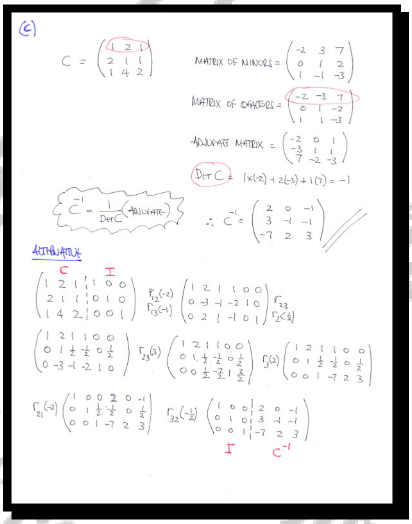
MATRIX OF COFACTORS $=\begin{pmatrix}2&4&-3\\-1&-3&3\\1&0&1\end{pmatrix}$

ADJUGATE MATRIX $=\begin{pmatrix}2&-1&1\\4&-3&0\\-3&3&1\end{pmatrix}$

DET B $=-3(2)+4\times4+3(-3)=1$

$B^{-1}=\frac{1}{\text{DET}B}(\text{ADJUGATE})$

$B^{-1}=\begin{pmatrix}2&-1&1\\4&-3&0\\-3&3&1\end{pmatrix}$ //



(c)

$C=\begin{pmatrix}1&2&1\\2&1&1\\1&4&2\end{pmatrix}$

MATRIX OF MINORS $=\begin{pmatrix}-2&3&7\\0&1&2\\1&-1&-3\end{pmatrix}$

MATRIX OF COFACTORS $=\begin{pmatrix}-2&-3&7\\0&1&-2\\1&1&-3\end{pmatrix}$

ADJUGATE MATRIX $=\begin{pmatrix}-2&0&1\\-3&1&1\\7&-2&-3\end{pmatrix}$

DET C $=1\times(-2)+2(-3)+1(7)=-1$

$C^{-1}=\frac{1}{\text{DET}C}(\text{ADJUGATE})$

$\therefore C^{-1}=\begin{pmatrix}2&0&-1\\3&-1&-1\\-7&2&3\end{pmatrix}$ //

ALTERNATIVE

$$\overset{C\qquad\quad I}{\left(\begin{array}{ccc|ccc}1&2&1&1&0&0\\2&1&1&0&1&0\\1&4&2&0&0&1\end{array}\right)}\ \begin{matrix}r_{12}(-2)\\r_{13}(-1)\end{matrix}\ \begin{pmatrix}1&2&1&1&0&0\\0&-3&-1&-2&1&0\\0&2&1&-1&0&1\end{pmatrix}\ \begin{matrix}r_{23}\\r_2(\frac{1}{2})\end{matrix}$$

$$\begin{pmatrix}1&2&1&1&0&0\\0&1&\frac{1}{2}&-\frac{1}{2}&0&\frac{1}{2}\\0&-3&-1&-2&1&0\end{pmatrix}\ r_{23}(3)\ \begin{pmatrix}1&2&1&1&0&0\\0&1&\frac{1}{2}&-\frac{1}{2}&0&\frac{1}{2}\\0&0&\frac{1}{2}&-\frac{7}{2}&1&\frac{3}{2}\end{pmatrix}\ r_3(2)\ \begin{pmatrix}1&2&1&1&0&0\\0&1&\frac{1}{2}&-\frac{1}{2}&0&\frac{1}{2}\\0&0&1&-7&2&3\end{pmatrix}$$

$$r_{21}(-2)\begin{pmatrix}1&0&0&2&0&-1\\0&1&\frac{1}{2}&-\frac{1}{2}&0&\frac{1}{2}\\0&0&1&-7&2&3\end{pmatrix}\ r_{32}(-\tfrac{1}{2})\ \underset{I\qquad\quad C^{-1}}{\left(\begin{array}{ccc|ccc}1&0&0&2&0&-1\\0&1&0&3&-1&-1\\0&0&1&-7&2&3\end{array}\right)}$$

Question 9

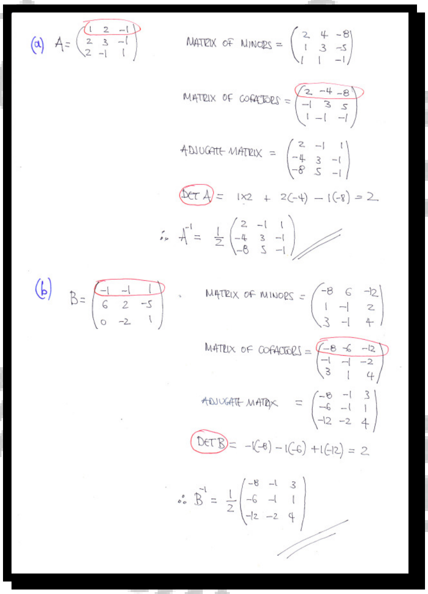
Find the inverse of each the following 3×3 matrices.

a) $\mathbf{A}=\begin{pmatrix}1 & 2 & -1\\ 2 & 3 & -1\\ 2 & -1 & 1\end{pmatrix}$

b) $\mathbf{B}=\begin{pmatrix}-1 & -1 & 1\\ 6 & 2 & -5\\ 0 & -2 & 1\end{pmatrix}$

c) $\mathbf{C}=\begin{pmatrix}5 & -2 & 2\\ 3 & -4 & -5\\ -2 & 3 & 4\end{pmatrix}$

$$\boxed{\mathbf{A}^{-1}=\frac{1}{2}\begin{pmatrix}2 & -1 & 1\\ -4 & 3 & -1\\ -8 & 5 & -1\end{pmatrix}},\ \boxed{\mathbf{B}^{-1}=\frac{1}{2}\begin{pmatrix}-8 & -1 & 3\\ -6 & -1 & 1\\ -12 & -2 & 4\end{pmatrix}},\ \boxed{\mathbf{C}^{-1}=\begin{pmatrix}-1 & 14 & 18\\ -2 & 24 & 31\\ 1 & -11 & -14\end{pmatrix}}$$



(a) $A=\begin{pmatrix}1 & 2 & -1\\ 2 & 3 & -1\\ 2 & -1 & 1\end{pmatrix}$

MATRIX OF MINORS $=\begin{pmatrix}2 & 4 & -8\\ 1 & 3 & -5\\ 1 & 1 & -1\end{pmatrix}$

MATRIX OF COFACTORS $=\begin{pmatrix}2 & -4 & -8\\ -1 & 3 & 5\\ 1 & -1 & -1\end{pmatrix}$

ADJUGATE MATRIX $=\begin{pmatrix}2 & -1 & 1\\ -4 & 3 & -1\\ -8 & 5 & -1\end{pmatrix}$

DET A $= 1\times2 + 2(-4) - 1(-8) = 2$

$\therefore A^{-1}=\frac{1}{2}\begin{pmatrix}2 & -1 & 1\\ -4 & 3 & -1\\ -8 & 5 & -1\end{pmatrix}$

(b) $B=\begin{pmatrix}-1 & -1 & 1\\ 6 & 2 & -5\\ 0 & -2 & 1\end{pmatrix}$

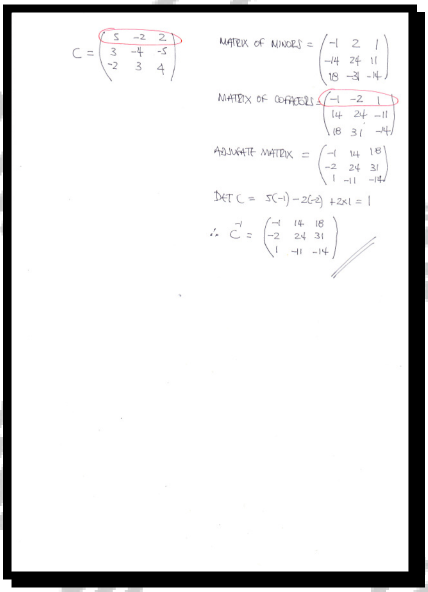
MATRIX OF MINORS $=\begin{pmatrix}-8 & 6 & -12\\ 1 & -1 & 2\\ 3 & -1 & 4\end{pmatrix}$

MATRIX OF COFACTORS $=\begin{pmatrix}-8 & -6 & -12\\ -1 & -1 & -2\\ 3 & 1 & 4\end{pmatrix}$

ADJUGATE MATRIX $=\begin{pmatrix}-8 & -1 & 3\\ -6 & -1 & 1\\ -12 & -2 & 4\end{pmatrix}$

DET B $= -1(-8) - 1(-6) + 1(-12) = 2$

$\therefore B^{-1}=\frac{1}{2}\begin{pmatrix}-8 & -1 & 3\\ -6 & -1 & 1\\ -12 & -2 & 4\end{pmatrix}$



$C=\begin{pmatrix}5 & -2 & 2\\ 3 & -4 & -5\\ -2 & 3 & 4\end{pmatrix}$

MATRIX OF MINORS $=\begin{pmatrix}-1 & 2 & 1\\ -14 & 24 & 11\\ 18 & -31 & -14\end{pmatrix}$

MATRIX OF COFACTORS $=\begin{pmatrix}-1 & -2 & 1\\ 14 & 24 & -11\\ 18 & 31 & -14\end{pmatrix}$

ADJUGATE MATRIX $=\begin{pmatrix}-1 & 14 & 18\\ -2 & 24 & 31\\ 1 & -11 & -14\end{pmatrix}$

DET C $= 5(-1) - 2(-2) + 2\times1 = 1$

$\therefore C^{-1}=\begin{pmatrix}-1 & 14 & 18\\ -2 & 24 & 31\\ 1 & -11 & -14\end{pmatrix}$

Question 10

The 3×3 matrix $\mathbf{A}$ is given below.

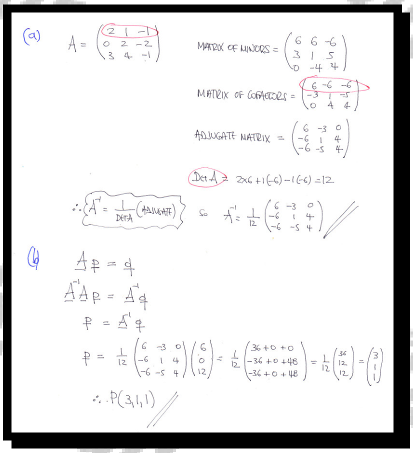
$$\mathbf{A}=\begin{pmatrix}2 & 1 & -1\\ 0 & 2 & -2\\ 3 & 4 & -1\end{pmatrix}.$$

a) Find the inverse of $\mathbf{A}$.

The point P has been mapped by $\mathbf{A}$ onto the point $Q(6,0,12)$.

b) Determine the coordinates of P.

$$\boxed{\mathbf{A}^{-1}=\frac{1}{12}\begin{pmatrix}6 & -3 & 0\\ -6 & 1 & 4\\ -6 & -5 & 4\end{pmatrix}},\ \boxed{P(3,1,1)}$$



(a) $\underline{A}=\begin{pmatrix}2 & 1 & -1\\ 0 & 2 & -2\\ 3 & 4 & -1\end{pmatrix}$

MATRIX OF MINORS $=\begin{pmatrix}6 & 6 & -6\\ 3 & 1 & 5\\ 0 & -4 & 4\end{pmatrix}$

MATRIX OF COFACTORS $=\begin{pmatrix}6 & -6 & -6\\ -3 & 1 & -5\\ 0 & 4 & 4\end{pmatrix}$

ADJUGATE MATRIX $=\begin{pmatrix}6 & -3 & 0\\ -6 & 1 & 4\\ -6 & -5 & 4\end{pmatrix}$

DET $\underline{A}$ $= 2\times6+1(-6)-1(-6)=12$

$\therefore\ \underline{A}^{-1}=\frac{1}{\text{DET}\underline{A}}(\text{ADJUGATE})$ So $\underline{A}^{-1}=\frac{1}{12}\begin{pmatrix}6 & -3 & 0\\ -6 & 1 & 4\\ -6 & -5 & 4\end{pmatrix}$ //

(b) $\underline{A}\,\underline{p}=\underline{q}$

$\underline{A}^{-1}\underline{A}\,\underline{p}=\underline{A}^{-1}\underline{q}$

$\underline{p}=\underline{A}^{-1}\underline{q}$

$\underline{p}=\frac{1}{12}\begin{pmatrix}6 & -3 & 0\\ -6 & 1 & 4\\ -6 & -5 & 4\end{pmatrix}\begin{pmatrix}6\\ 0\\ 12\end{pmatrix}=\frac{1}{12}\begin{pmatrix}36+0+0\\ -36+0+48\\ -36+0+48\end{pmatrix}=\frac{1}{12}\begin{pmatrix}36\\ 12\\ 12\end{pmatrix}=\begin{pmatrix}3\\ 1\\ 1\end{pmatrix}$

$\therefore\ P(3,1,1)$ //

Question 11

The 3×3 matrix $\mathbf{M}$ is given below.

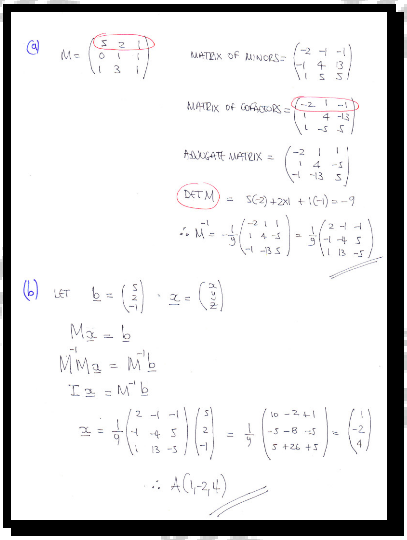
$$\mathbf{M}=\begin{pmatrix}5 & 2 & 1\\ 0 & 1 & 1\\ 1 & 3 & 1\end{pmatrix}.$$

a) Find the inverse of $\mathbf{M}$.

The point A has been transformed by $\mathbf{M}$ into the point $B(5,2,-1)$.

b) Determine the coordinates of A.

$$\boxed{\mathbf{M}^{-1}=\frac{1}{9}\begin{pmatrix}2 & -1 & -1\\ -1 & -4 & 5\\ 1 & 13 & -5\end{pmatrix}},\ \boxed{A(1,-2,4)}$$



(a) $M=\begin{pmatrix}5 & 2 & 1\\ 0 & 1 & 1\\ 1 & 3 & 1\end{pmatrix}$

MATRIX OF MINORS $=\begin{pmatrix}-2 & -1 & -1\\ -1 & 4 & 13\\ 1 & 5 & 5\end{pmatrix}$

MATRIX OF COFACTORS $=\begin{pmatrix}-2 & 1 & -1\\ 1 & 4 & -13\\ 1 & -5 & 5\end{pmatrix}$

ADJUGATE MATRIX $=\begin{pmatrix}-2 & 1 & 1\\ 1 & 4 & -5\\ -1 & -13 & 5\end{pmatrix}$

DET M $= 5(-2)+2\times1+1(-1)=-9$

$\therefore M^{-1}=-\frac{1}{9}\begin{pmatrix}-2 & 1 & 1\\ 1 & 4 & -5\\ -1 & -13 & 5\end{pmatrix}=\frac{1}{9}\begin{pmatrix}2 & -1 & -1\\ -1 & -4 & 5\\ 1 & 13 & -5\end{pmatrix}$

(b) LET $\underline{b}=\begin{pmatrix}5\\2\\-1\end{pmatrix}$. $\underline{x}=\begin{pmatrix}x\\y\\z\end{pmatrix}$

$M\underline{x}=\underline{b}$

$M^{-1}M\underline{a}=M^{-1}\underline{b}$

$I\underline{x}=M^{-1}\underline{b}$

$\underline{x}=\frac{1}{9}\begin{pmatrix}2 & -1 & -1\\ -1 & -4 & 5\\ 1 & 13 & -5\end{pmatrix}\begin{pmatrix}5\\2\\-1\end{pmatrix}=\frac{1}{9}\begin{pmatrix}10-2+1\\ -5-8-5\\ 5+26+5\end{pmatrix}=\begin{pmatrix}1\\-2\\4\end{pmatrix}$

$\therefore A(1,-2,4)$

Question 12

The 3×3 matrix $\mathbf{A}$ is given below.

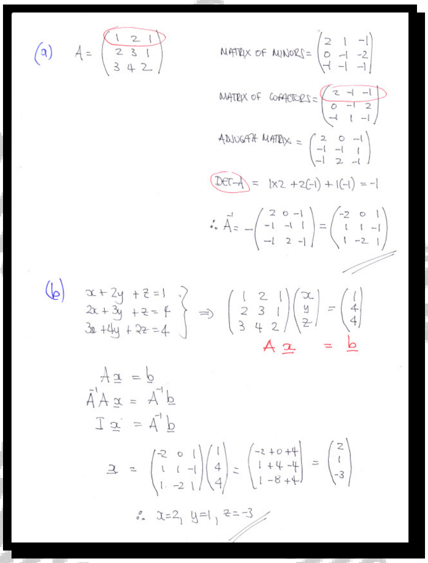
$$\mathbf{A}=\begin{pmatrix}1&2&1\\2&3&1\\3&4&2\end{pmatrix}.$$

a) Find the inverse of $\mathbf{A}$.

b) Hence, or otherwise, solve the system of equations

$$\begin{aligned}x+2y+z&=1\\2x+3y+z&=4\\3x+4y+2z&=4\end{aligned}$$

$$\boxed{\mathbf{A}^{-1}=\begin{pmatrix}-2&0&1\\1&1&-1\\1&-2&1\end{pmatrix}},\quad\boxed{x=2,\quad y=1,\quad z=-3}$$



(a) $A=\begin{pmatrix}1&2&1\\2&3&1\\3&4&2\end{pmatrix}$

MATRIX OF MINORS $=\begin{pmatrix}2&1&-1\\0&-1&-2\\-1&-1&-1\end{pmatrix}$

MATRIX OF COFACTORS $=\begin{pmatrix}2&-1&-1\\0&-1&2\\-1&1&-1\end{pmatrix}$

ADJUGATE MATRIX $=\begin{pmatrix}2&0&-1\\-1&-1&1\\-1&2&-1\end{pmatrix}$

DET A $=1\times2+2(-1)+1(-1)=-1$

$\therefore A^{-1}=-\begin{pmatrix}2&0&-1\\-1&-1&1\\-1&2&-1\end{pmatrix}=\begin{pmatrix}-2&0&1\\1&1&-1\\1&-2&1\end{pmatrix}$

(b) $\left.\begin{aligned}x+2y+z&=1\\2x+3y+z&=4\\3x+4y+2z&=4\end{aligned}\right\}\Rightarrow\begin{pmatrix}1&2&1\\2&3&1\\3&4&2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\4\\4\end{pmatrix}$

$A\underline{x}=\underline{b}$

$$\begin{aligned}A\underline{x}&=\underline{b}\\A^{-1}A\underline{x}&=A^{-1}\underline{b}\\I\underline{x}&=A^{-1}\underline{b}\end{aligned}$$

$\underline{x}=\begin{pmatrix}-2&0&1\\1&1&-1\\1&-2&1\end{pmatrix}\begin{pmatrix}1\\4\\4\end{pmatrix}=\begin{pmatrix}-2+0+4\\1+4-4\\1-8+4\end{pmatrix}=\begin{pmatrix}2\\1\\-3\end{pmatrix}$

$\therefore x=2,\ y=1,\ z=-3$

Question 13

The 3×3 matrix $\mathbf{M}$ is given below.

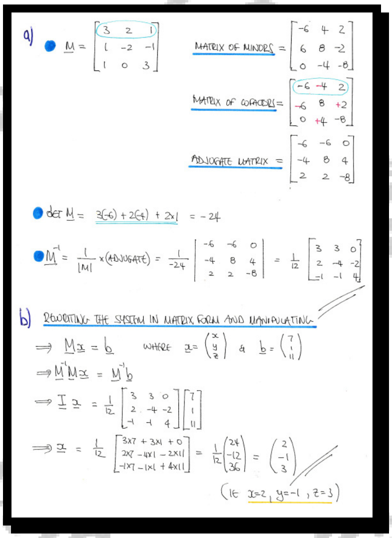
$$\mathbf{M}=\begin{pmatrix}3 & 2 & 1\\ 1 & -2 & -1\\ 1 & 0 & 3\end{pmatrix}.$$

a) Find the inverse of $\mathbf{M}$.

b) Hence, or otherwise, solve the following system of equations.

$$\begin{aligned}3x+2y+z&=7\\ x-2y-z&=1\\ x\quad\;\;+3z&=11\end{aligned}$$

$$\boxed{\mathbf{M}^{-1}=\frac{1}{12}\begin{pmatrix}3 & 3 & 0\\ 2 & -4 & -2\\ -1 & -1 & 4\end{pmatrix}},\quad \boxed{x=2,\ \ y=-1,\ \ z=3}$$



a)

- $\underline{M}=\begin{bmatrix}3 & 2 & 1\\ 1 & -2 & -1\\ 1 & 0 & 3\end{bmatrix}$ MATRIX OF MINORS $=\begin{bmatrix}-6 & 4 & 2\\ 6 & 8 & -2\\ 0 & -4 & -8\end{bmatrix}$

MATRIX OF COFACTORS $=\begin{bmatrix}-6 & -4 & 2\\ -6 & 8 & +2\\ 0 & +4 & -8\end{bmatrix}$

ADJUGATE MATRIX $=\begin{bmatrix}-6 & -6 & 0\\ -4 & 8 & 4\\ 2 & 2 & -8\end{bmatrix}$

- $\det \underline{M} = 3(-6)+2(-4)+2\times1 = -24$

- $\underline{M}^{-1} = \frac{1}{|M|}\times(\text{ADJUGATE}) = \frac{1}{-24}\begin{vmatrix}-6 & -6 & 0\\ -4 & 8 & 4\\ 2 & 2 & -8\end{vmatrix} = \frac{1}{12}\begin{bmatrix}3 & 3 & 0\\ 2 & -4 & -2\\ -1 & -1 & 4\end{bmatrix}$

b) REWRITING THE SYSTEM IN MATRIX FORM AND MANIPULATING

$\Rightarrow \underline{M}\underline{x} = \underline{b}$ WHERE $\underline{x}=\begin{pmatrix}x\\ y\\ z\end{pmatrix}$ & $\underline{b}=\begin{pmatrix}7\\ 1\\ 11\end{pmatrix}$

$\Rightarrow \underline{M}^{-1}\underline{M}\underline{x} = \underline{M}^{-1}\underline{b}$

$\Rightarrow \underline{I}\,\underline{x} = \frac{1}{12}\begin{bmatrix}3 & 3 & 0\\ 2 & -4 & -2\\ -1 & -1 & 4\end{bmatrix}\begin{bmatrix}7\\ 1\\ 11\end{bmatrix}$

$\Rightarrow \underline{x} = \frac{1}{12}\begin{bmatrix}3\times7+3\times1+0\\ 2\times7-4\times1-2\times11\\ -1\times7-1\times1+4\times11\end{bmatrix} = \frac{1}{12}\begin{pmatrix}24\\ -12\\ 36\end{pmatrix} = \begin{pmatrix}2\\ -1\\ 3\end{pmatrix}$

(IE $x=2,\ y=-1,\ z=3$)

Question 14

The 3×3 matrix $\mathbf{A}$ is given below.

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & -3 \\ 2 & 4 & 3 \\ -4 & 2 & -1 \end{pmatrix}$$

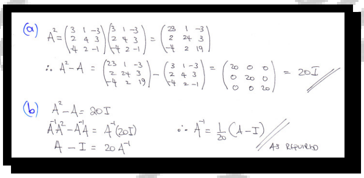
The matrix $\mathbf{A}$ is non singular.

a) Evaluate $\mathbf{A}^2 - \mathbf{A}$.

b) Show clearly that

$$\mathbf{A}^{-1} = \frac{1}{20}\left[\mathbf{A} - \mathbf{I}\right].$$

$\boxed{20\mathbf{I}}$



(a) $A^2 = \begin{pmatrix} 3 & 1 & -3 \\ 2 & 4 & 3 \\ -4 & 2 & -1 \end{pmatrix}\begin{pmatrix} 3 & 1 & -3 \\ 2 & 4 & 3 \\ -4 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 23 & 1 & -3 \\ 2 & 24 & 3 \\ -4 & 2 & 19 \end{pmatrix}$

$\therefore A^2 - A = \begin{pmatrix} 23 & 1 & -3 \\ 2 & 24 & 3 \\ -4 & 2 & 19 \end{pmatrix} - \begin{pmatrix} 3 & 1 & -3 \\ 2 & 4 & 3 \\ -4 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{pmatrix} = 20I$ //

(b) $A^2 - A = 20I$

$A^{-1}A^2 - A^{-1}A = A^{-1}(20I)$

$A - I = 20A^{-1}$

$\therefore A^{-1} = \frac{1}{20}(A - I)$ // AS REQUIRED

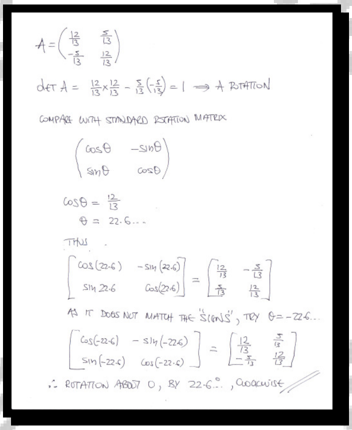
MATRIX TRANSFORMATIONS

Question 1

Describe fully the transformation given by the following 2×2 matrix.

$$\begin{pmatrix} \frac{12}{13} & \frac{5}{13} \\ -\frac{5}{13} & \frac{12}{13} \end{pmatrix}.$$

$\boxed{\text{rotation, anticlockwise, by } \arcsin\frac{5}{13}}$



$A = \begin{pmatrix} \frac{12}{13} & \frac{5}{13} \\ -\frac{5}{13} & \frac{12}{13} \end{pmatrix}$

$\det A = \frac{12}{13}\times\frac{12}{13} - \frac{5}{13}\left(-\frac{5}{13}\right) = 1 \Rightarrow$ A ROTATION

COMPARE WITH STANDARD ROTATION MATRIX

$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$\cos\theta = \frac{12}{13}$

$\theta = 22.6\ldots$

THUS

$\begin{bmatrix} \cos(22.6) & -\sin(22.6) \\ \sin 22.6 & \cos(22.6) \end{bmatrix} = \begin{bmatrix} \frac{12}{13} & -\frac{5}{13} \\ \frac{5}{13} & \frac{12}{13} \end{bmatrix}$

AS IT DOES NOT MATCH THE "SIGNS", TRY $\theta = -22.6\ldots$

$\begin{bmatrix} \cos(-22.6) & -\sin(-22.6) \\ \sin(-22.6) & \cos(-22.6) \end{bmatrix} = \begin{bmatrix} \frac{12}{13} & \frac{5}{13} \\ -\frac{5}{13} & \frac{12}{13} \end{bmatrix}$

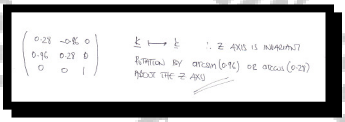
$\therefore$ ROTATION ABOUT O, BY $22.6^\circ\ldots$, CLOCKWISE

Question 2

Describe fully the transformation given by the following 3×3 matrix.

$$\begin{pmatrix} 0.28 & -0.96 & 0 \\ 0.96 & 0.28 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$\boxed{\text{rotation in the } z \text{ axis, anticlockwise, by } \arcsin\frac{24}{25}}$



$\begin{pmatrix} 0.28 & -0.96 & 0 \\ 0.96 & 0.28 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\underline{k} \longmapsto \underline{k}$ $\therefore$ Z AXIS IS INVARIANT

ROTATION BY $\arcsin(0.96)$ OR $\arccos(0.28)$ ABOUT THE Z AXIS

Question 3

A plane transformation maps the general point (x, y) to the general point (X, Y) by

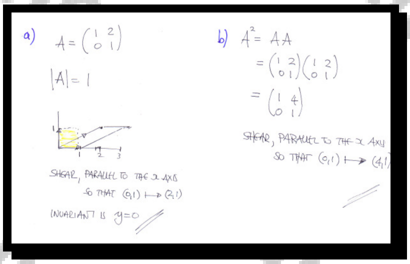
$$\begin{pmatrix} X \\ Y \end{pmatrix} = \mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix},$$

where $\mathbf{A}$ is the 2×2 matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

- **a)** Give a full geometrical description for the transformation represented by $\mathbf{A}$, stating the equation of the line of invariant points under this transformation
- **b)** Calculate $\mathbf{A}^2$ and describe geometrically the transformation it represents.

$\boxed{\text{shear parallel to } y = 0,\ (0,1) \mapsto (2,1)}$ $\boxed{\text{line of invariant points } y = 0}$,

$\boxed{\text{shear parallel to } y = 0,\ (0,1) \mapsto (4,1)}$



a) $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

$|A| = 1$

SHEAR, PARALLEL TO THE x AXIS

SO THAT $(0,1) \mapsto (2,1)$

INVARIANT IS $y=0$

b) $A^2 = AA$

$= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$

SHEAR, PARALLEL TO THE x AXI

SO THAT $(0,1) \mapsto (4,1$

Question 4

A plane transformation maps the points (x, y) to the points (X, Y) such that

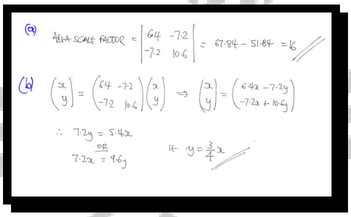
$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 6.4 & -7.2 \\ -7.2 & 10.6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

a) Find the **area** scale factor of the transformation.

The points which lie on a straight line through the origin remain invariant under this transformation.

b) Determine the equation of this straight line.

$\boxed{\text{SF} = 16}$, $\boxed{y = \frac{3}{4}x}$



(a) AREA SCALE FACTOR $= \begin{vmatrix} 6.4 & -7.2 \\ -7.2 & 10.6 \end{vmatrix} = 67.84 - 51.84 = 16$

(b) $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6.4 & -7.2 \\ -7.2 & 10.6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6.4x - 7.2y \\ -7.2x + 10.6y \end{pmatrix}$

$\therefore \; 7.2y = 5.4x$

OR

$7.2x = 9.6y$

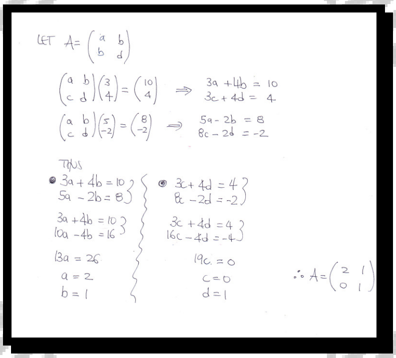
IE $y = \frac{3}{4}x$

Question 5

The transformation represented by the 2×2 matrix $\mathbf{A}$ maps the point $(3,4)$ onto the point $(10,4)$, and the point $(5,-2)$ onto the point $(8,-2)$.

Determine the elements of $\mathbf{A}$.

$$\boxed{\mathbf{A}=\begin{pmatrix}2 & 1\\ 0 & 1\end{pmatrix}}$$



LET $A=\begin{pmatrix}a & b\\ b & d\end{pmatrix}$

$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}3\\ 4\end{pmatrix}=\begin{pmatrix}10\\ 4\end{pmatrix} \Rightarrow \begin{array}{l}3a+4b=10\\ 3c+4d=4\end{array}$

$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}5\\ -2\end{pmatrix}=\begin{pmatrix}8\\ -2\end{pmatrix} \Rightarrow \begin{array}{l}5a-2b=8\\ 8c-2d=-2\end{array}$

THUS

- $\left.\begin{array}{l}3a+4b=10\\ 5a-2b=8\end{array}\right\}$
$\left.\begin{array}{l}3a+4b=10\\ 10a-4b=16\end{array}\right\}$
$13a=26$
$a=2$
$b=1$

- $\left.\begin{array}{l}3c+4d=4\\ 8c-2d=-2\end{array}\right\}$
$\left.\begin{array}{l}3c+4d=4\\ 16c-4d=-4\end{array}\right\}$
$19c=0$
$c=0$
$d=1$

$\therefore A=\begin{pmatrix}2 & 1\\ 0 & 1\end{pmatrix}$

Question 6

The 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ are given below

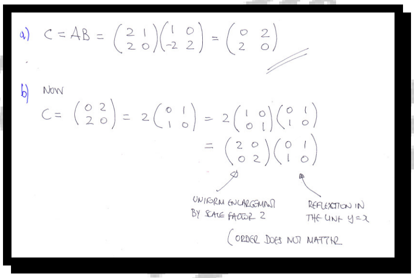
$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix}.$$

The matrix $\mathbf{C}$ represents the combined effect of the transformation represented by the $\mathbf{B}$, followed by the transformation represented by $\mathbf{A}$.

a) Determine the elements of $\mathbf{C}$.

b) Describe geometrically the transformation represented by $\mathbf{C}$.

$\boxed{\mathbf{C} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}}$, $\boxed{\text{enlargement by scale factor 2, reflection in the line } y = x\text{, in any order}}$



a) $C = AB = \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$

b) NOW

$C = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = 2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

UNIFORM ENLARGEMENT BY SCALE FACTOR 2

REFLECTION IN THE LINE $y = x$

(ORDER DOES NOT MATTER

Question 7

The 3×3 matrices $\mathbf{A}$ and $\mathbf{B}$ are given below.

$$\mathbf{A}=\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix} \quad\text{and}\quad \mathbf{B}=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}.$$

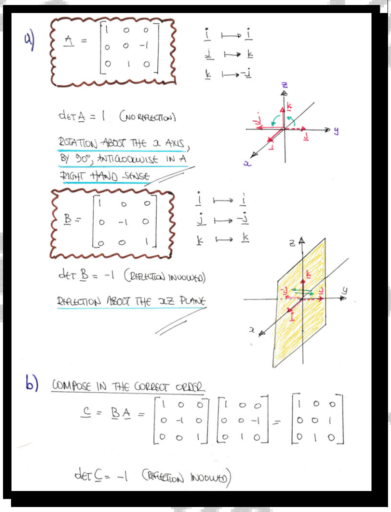
a) Describe geometrically the transformations given by each of the two matrices.

The matrix $\mathbf{C}$ is defined as the transformation defined by the matrix $\mathbf{A}$, followed by the transformation defined by the matrix $\mathbf{B}$.

b) Describe geometrically the transformation represented by $\mathbf{C}$.

$\boxed{\mathbf{A}\text{: rotation about } x \text{ axis, } 90^\circ \text{ anticlockwise}}$, $\boxed{\mathbf{B}\text{: reflection in the } xz \text{ plane}}$,

$\boxed{\mathbf{C}\text{: reflection in the plane } y=z}$



a) $\underline{A} = \begin{bmatrix}1&0&0\\0&0&-1\\0&1&0\end{bmatrix}$

$\underline{i} \longmapsto \underline{i}$
$\underline{j} \longmapsto \underline{k}$
$\underline{k} \longmapsto -\underline{j}$

$\det \underline{A} = 1$ (NO REFLECTION)

ROTATION ABOUT THE x AXIS, BY 90°, ANTICLOCKWISE IN A RIGHT HAND SENSE

$\underline{B} = \begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}$

$\underline{i} \longmapsto \underline{i}$
$\underline{j} \longmapsto -\underline{j}$
$\underline{k} \longmapsto \underline{k}$

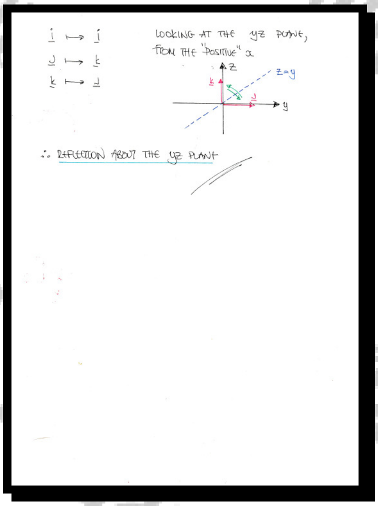
$\det \underline{B} = -1$ (REFLECTION INVOLVED)

REFLECTION ABOUT THE xz PLANE

b) COMPOSE IN THE CORRECT ORDER

$\underline{C} = \underline{B}\,\underline{A} = \begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&-1\\0&1&0\end{bmatrix} = \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$

$\det \underline{C} = -1$ (REFLECTION INVOLVED)



$\underline{i} \longmapsto \underline{i}$
$\underline{j} \longmapsto \underline{k}$
$\underline{k} \longmapsto \underline{j}$

LOOKING AT THE yz PLANE, FROM THE "POSITIVE" x

∴ REFLECTION ABOUT THE yz PLANE

Question 8

The 3×3 matrices $\mathbf{A}$ and $\mathbf{B}$ are given below.

$$\mathbf{A}=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\qquad\text{and}\qquad\mathbf{B}=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}.$$

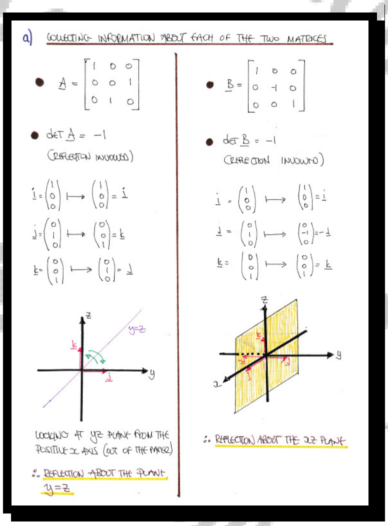
a) Describe geometrically the transformations given by each of the two matrices.

The matrix $\mathbf{C}$ is defined as the transformation defined by the matrix $\mathbf{A}$, followed by the transformation defined by the matrix $\mathbf{B}$.

b) Describe geometrically the transformation represented by $\mathbf{C}$.

$\boxed{\mathbf{A}:\text{reflection in the plane } y=z}$, $\boxed{\mathbf{B}:\text{reflection in the } xz \text{ plane}}$,

$\boxed{\mathbf{C}:\text{rotation in the } x \text{ axis, } 90°\text{, clockwise}}$



a) COLLECTING INFORMATION ABOUT EACH OF THE TWO MATRICES

- $\underline{A}=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$
- $\det \underline{A}=-1$
  (REFLECTION INVOLVED)

$\underline{i}=\begin{pmatrix}1\\0\\0\end{pmatrix}\longmapsto\begin{pmatrix}1\\0\\0\end{pmatrix}=\underline{i}$

$\underline{j}=\begin{pmatrix}0\\1\\0\end{pmatrix}\longmapsto\begin{pmatrix}0\\0\\1\end{pmatrix}=\underline{k}$

$\underline{k}=\begin{pmatrix}0\\0\\1\end{pmatrix}\longmapsto\begin{pmatrix}0\\1\\0\end{pmatrix}=\underline{j}$

LOOKING AT yz PLANE FROM THE POSITIVE x AXIS (OUT OF THE PAPER)

∴ REFLECTION ABOUT THE PLANE $y=z$

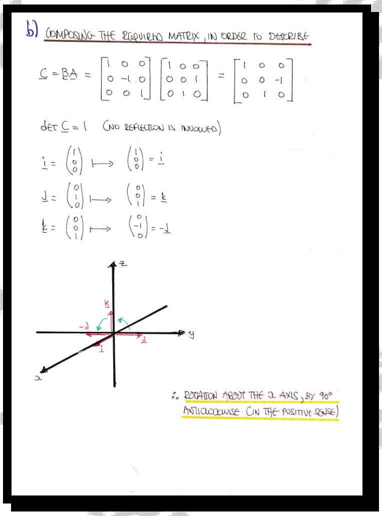
- $\underline{B}=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}$
- $\det \underline{B}=-1$
  (REFLECTION INVOLVED)

$\underline{i}=\begin{pmatrix}1\\0\\0\end{pmatrix}\longmapsto\begin{pmatrix}1\\0\\0\end{pmatrix}=\underline{i}$

$\underline{j}=\begin{pmatrix}0\\1\\0\end{pmatrix}\longmapsto\begin{pmatrix}0\\-1\\0\end{pmatrix}=-\underline{j}$

$\underline{k}=\begin{pmatrix}0\\0\\1\end{pmatrix}\longmapsto\begin{pmatrix}0\\0\\1\end{pmatrix}=\underline{k}$

∴ REFLECTION ABOUT THE xz PLANE



b) COMPOSING THE REQUIRED MATRIX, IN ORDER TO DESCRIBE

$\underline{C}=\underline{B}\underline{A}=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}=\begin{bmatrix}1&0&0\\0&0&-1\\0&1&0\end{bmatrix}$

$\det \underline{C}=1$ (NO REFLECTION IS INVOLVED)

$\underline{i}=\begin{pmatrix}1\\0\\0\end{pmatrix}\longmapsto\begin{pmatrix}1\\0\\0\end{pmatrix}=\underline{i}$

$\underline{j}=\begin{pmatrix}0\\1\\0\end{pmatrix}\longmapsto\begin{pmatrix}0\\0\\1\end{pmatrix}=\underline{k}$

$\underline{k}=\begin{pmatrix}0\\0\\1\end{pmatrix}\longmapsto\begin{pmatrix}0\\-1\\0\end{pmatrix}=-\underline{j}$

∴ ROTATION ABOUT THE x AXIS, BY 90° ANTICLOCKWISE (IN THE POSITIVE SENSE)

Question 9

The matrix $\mathbf{A}:\mathbb{R}^2 \mapsto \mathbb{R}^2$ and the matrix $\mathbf{A}:\mathbb{R}^2 \mapsto \mathbb{R}^2$ are defined as

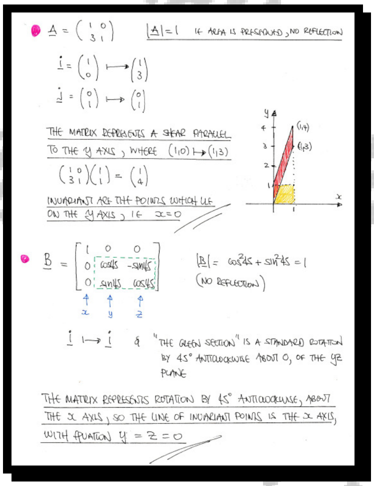
$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 45° & -\sin 45° \\ 0 & \sin 45° & \cos 45° \end{pmatrix}.$$

Describe geometrically the transformations given by each of these matrices.

State in each case the equation of the line of invariant points.

$\boxed{\mathbf{A}: \text{shear parallel to } y \text{ axis}, (1,0) \mapsto (3,1)}$,

$\boxed{\mathbf{B}: \text{rotation in the } x \text{ axis}, 45°, \text{anticlockwise}}$, $\boxed{\mathbf{A}: x = 0}$, $\boxed{\mathbf{B}: y = z = 0, \text{ i.e. } x \text{ axis}}$



● $\underline{A} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ $|\underline{A}| = 1$ IE AREA IS PRESERVED, NO REFLECTION

$\underline{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\underline{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

THE MATRIX REPRESENTS A SHEAR PARALLEL TO THE y AXIS, WHERE $(1,0) \mapsto (1,3)$

$\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

INVARIANT ARE THE POINTS WHICH LIE ON THE y AXIS, IE $x = 0$

● $\underline{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 45 & -\sin 45 \\ 0 & \sin 45 & \cos 45 \\ \uparrow & \uparrow & \uparrow \\ x & y & z \end{bmatrix}$ $|\underline{B}| = \cos^2 45 + \sin^2 45 = 1$ (NO REFLECTION)

$\underline{i} \longmapsto \underline{i}$ & "THE GREEN SECTION" IS A STANDARD ROTATION BY 45° ANTICLOCKWISE ABOUT O, OF THE yz PLANE

THE MATRIX REPRESENTS ROTATION BY 45° ANTICLOCKWISE, ABOUT THE x AXIS, SO THE LINE OF INVARIANT POINTS IS THE x AXIS, WITH EQUATION $y = z = 0$

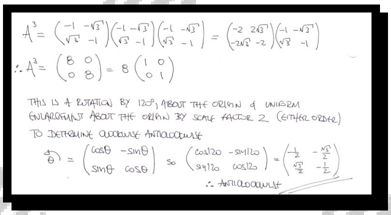
Question 10

The 2×2 matrix $\mathbf{A}$ is given below.

$$\mathbf{A}=\begin{pmatrix}-1 & \sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}.$$

Determine the elements of $\mathbf{A}^3$ and hence describe geometrically the transformation represented by $\mathbf{A}$.

$\boxed{\mathbf{A}^3=\begin{pmatrix}8 & 0\\ 0 & 8\end{pmatrix}}$, $\boxed{\text{rotation of } 120°\text{, anticlockwise \& enlargement of S.F. 2, both about the origin and in any order.}}$



$A^3=\begin{pmatrix}-1 & -\sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}\begin{pmatrix}-1 & -\sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}\begin{pmatrix}-1 & -\sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}=\begin{pmatrix}-2 & 2\sqrt{3}\\ -2\sqrt{3} & -2\end{pmatrix}\begin{pmatrix}-1 & -\sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}$

$\therefore A^3=\begin{pmatrix}8 & 0\\ 0 & 8\end{pmatrix}=8\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

THIS IS A ROTATION BY 120°, ABOUT THE ORIGIN & UNIFORM ENLARGEMENT ABOUT THE ORIGIN BY SCALE FACTOR 2 (EITHER ORDER)

TO DETERMINE CLOCKWISE ANTICLOCKWISE

$\overset{\circlearrowleft}{\theta}=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$ so $\begin{pmatrix}\cos120 & -\sin120\\ \sin120 & \cos120\end{pmatrix}=\begin{pmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}$

$\therefore$ ANTICLOCKWISE

Question 11

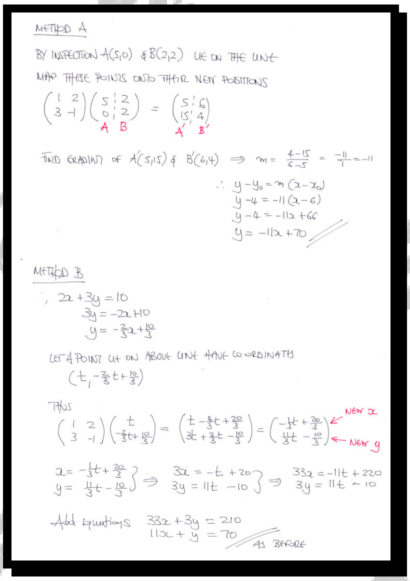
Find the image of the straight line with equation

$$2x + 3y = 10,$$

under the transformation represented by the 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}.$$

$$\boxed{11x + y = 70}$$



METHOD A

BY INSPECTION A(5,0) & B(2,2) LIE ON THE LINE

MAP THESE POINTS ONTO THEIR NEW POSITIONS

$$\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 15 & 4 \end{pmatrix}$$

A B A′ B′

FIND GRADIENT OF A′(5,15) & B′(6,4) $\Rightarrow m = \frac{4-15}{6-5} = \frac{-11}{1} = -11$

$$\therefore\ y - y_0 = m(x - x_0)$$
$$y - 4 = -11(x - 6)$$
$$y - 4 = -11x + 66$$
$$y = -11x + 70$$

METHOD B

$$2x + 3y = 10$$
$$3y = -2x + 10$$
$$y = -\tfrac{2}{3}x + \tfrac{10}{3}$$

LET A POINT LIE ON ABOVE LINE HAVE COORDINATES

$$\left(t, -\tfrac{2}{3}t + \tfrac{10}{3}\right)$$

THUS

$$\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} t \\ -\frac{2}{3}t + \frac{10}{3} \end{pmatrix} = \begin{pmatrix} t - \frac{4}{3}t + \frac{20}{3} \\ 3t + \frac{2}{3}t - \frac{10}{3} \end{pmatrix} = \begin{pmatrix} -\frac{1}{3}t + \frac{20}{3} \\ \frac{11}{3}t - \frac{10}{3} \end{pmatrix} \begin{matrix} \leftarrow \text{NEW } x \\ \leftarrow \text{NEW } y \end{matrix}$$

$$\left.\begin{matrix} x = -\frac{1}{3}t + \frac{20}{3} \\ y = \frac{11}{3}t - \frac{10}{3} \end{matrix}\right\} \Rightarrow \left.\begin{matrix} 3x = -t + 20 \\ 3y = 11t - 10 \end{matrix}\right\} \Rightarrow \begin{matrix} 33x = -11t + 220 \\ 3y = 11t - 10 \end{matrix}$$

Add equations $33x + 3y = 210$

$$11x + y = 70$$

AS BEFORE

Question 12

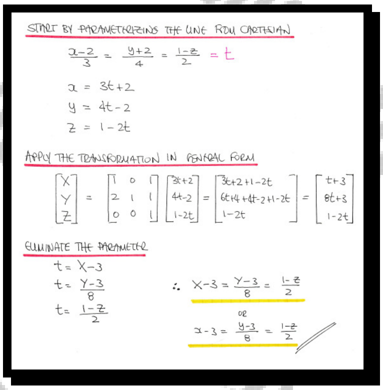
Find in Cartesian form the image of the straight line with equation

$$\frac{x-2}{3}=\frac{y+2}{4}=\frac{1-z}{2},$$

under the transformation represented by the 3×3 matrix $\mathbf{A}$, shown below.

$$\mathbf{A}=\begin{pmatrix}1&0&1\\2&1&1\\0&0&1\end{pmatrix}.$$

$$\boxed{x-3=\frac{y-3}{8}=\frac{1-z}{2}}$$



START BY PARAMETRIZING THE LINE FROM CARTESIAN

$$\frac{x-2}{3}=\frac{y+2}{4}=\frac{1-z}{2}=t$$

$x = 3t+2$

$y = 4t-2$

$z = 1-2t$

APPLY THE TRANSFORMATION IN GENERAL FORM

$$\begin{bmatrix}X\\Y\\Z\end{bmatrix}=\begin{bmatrix}1&0&1\\2&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}3t+2\\4t-2\\1-2t\end{bmatrix}=\begin{bmatrix}3t+2+1-2t\\6t+4+4t-2+1-2t\\1-2t\end{bmatrix}=\begin{bmatrix}t+3\\8t+3\\1-2t\end{bmatrix}$$

ELIMINATE THE PARAMETER

$t = X-3$

$t = \frac{Y-3}{8}$

$t = \frac{1-Z}{2}$

$$\therefore\ X-3=\frac{Y-3}{8}=\frac{1-Z}{2}$$

OR

$$x-3=\frac{y-3}{8}=\frac{1-z}{2}$$

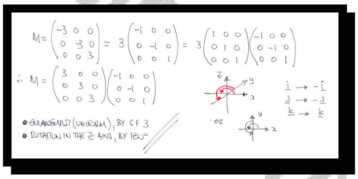
Question 13

$$\mathbf{M} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

The 3×3 matrix **M** above, describes two consecutive geometrical transformations of 3 dimensional space, which can be carried out in any order.

Describe geometrically each these two transformations.

$\boxed{\text{rotation about } z \text{ axis}, 180°}$, $\boxed{\text{uniform enlargement, S.F.} = 3}$



$$M = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{pmatrix} = 3\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\therefore M = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- ENLARGEMENT (UNIFORM), BY SF 3
- ROTATION IN THE Z AXIS, BY 180°

$\underline{i} \to -\underline{i}$

$\underline{j} \to -\underline{j}$

$\underline{k} \to \underline{k}$

Question 14

A plane transformation maps the general point (x, y) onto the general point (X, Y), by

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

a) Find the area scale factor of the transformation.

b) Determine the equation of the straight line of invariant points under this transformation.

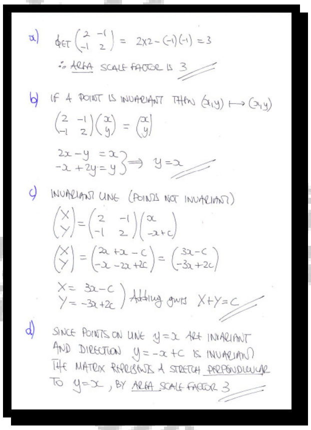
c) Show that all the straight lines with equation of the form

$$x + y = c,$$

where c is a constant, are invariant lines under this transformation.

d) Hence describe the transformation geometrically.

$\boxed{\text{SF} = 3}$, $\boxed{y = x}$, $\boxed{\text{stretch perpendicular to the line } y = x\text{, by area scale factor 3}}$



a) $\det\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = 2\times2 - (-1)(-1) = 3$

$\therefore$ AREA SCALE FACTOR IS 3

b) IF A POINT IS INVARIANT THEN $(x,y) \mapsto (x,y)$

$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\left.\begin{matrix} 2x - y = x \\ -x + 2y = y \end{matrix}\right\} \Longrightarrow y = x$

c) INVARIANT LINE (POINTS NOT INVARIANT)

$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ -x+c \end{pmatrix}$

$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 2x + x - c \\ -x - 2x + 2c \end{pmatrix} = \begin{pmatrix} 3x - c \\ -3x + 2c \end{pmatrix}$

$\left.\begin{matrix} X = 3x - c \\ Y = -3x + 2c \end{matrix}\right)$ Adding gives $X + Y = c$

d) SINCE POINTS ON LINE $y = x$ ARE INVARIANT AND DIRECTION $y = -x + c$ IS INVARIANT THE MATRIX REPRESENTS A STRETCH PERPENDICULAR TO $y = x$, BY AREA SCALE FACTOR 3

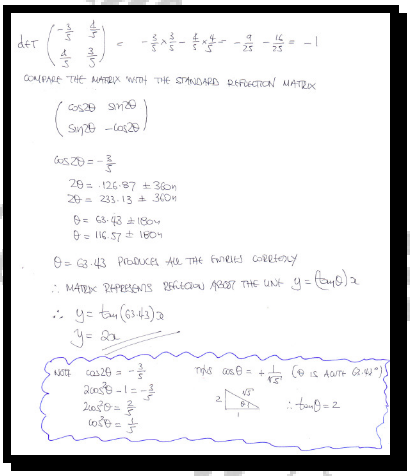
Question 15

Describe fully the transformation given by the following 2×2 matrix

$$\begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}.$$

The description must be supported by mathematical calculations.

$\boxed{\text{reflection in } y = 2x}$



$\det \begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix} = -\frac{3}{5} \times \frac{3}{5} - \frac{4}{5} \times \frac{4}{5} = -\frac{9}{25} - \frac{16}{25} = -1$

COMPARE THE MATRIX WITH THE STANDARD REFLECTION MATRIX

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

$\cos 2\theta = -\frac{3}{5}$

$2\theta = 126.87 \pm 360n$

$2\theta = 233.13 \pm 360n$

$\theta = 63.43 \pm 180n$

$\theta = 116.57 \pm 180n$

$\theta = 63.43$ PRODUCES ALL THE ENTRIES CORRECTLY

$\therefore$ MATRIX REPRESENTS REFLECTION ABOUT THE LINE $y = (\tan\theta)x$

$\therefore y = \tan(63.43)x$

$y = 2x$

NOTE $\cos 2\theta = -\frac{3}{5}$

$2\cos^2\theta - 1 = -\frac{3}{5}$

$2\cos^2\theta = \frac{2}{5}$

$\cos^2\theta = \frac{1}{5}$

THUS $\cos\theta = +\frac{1}{\sqrt{5}}$ (θ IS ACUTE $63.43°$)

$\therefore \tan\theta = 2$

Question 16

The matrices $\mathbf{A}$ and $\mathbf{B}$ are defined as

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & k \end{pmatrix},$$

where k is a scalar constant.

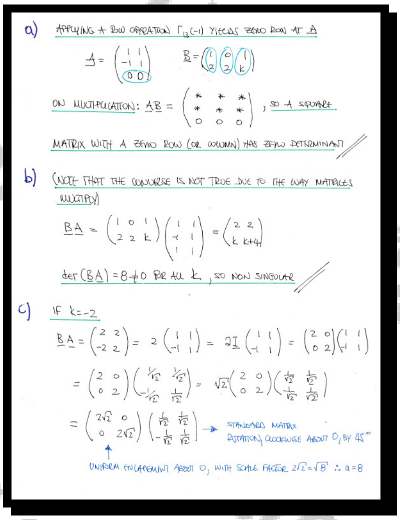
a) Without calculating $\mathbf{AB}$, show that $\mathbf{AB}$ is singular for all values of k.

b) Show that $\mathbf{BA}$ is non singular for all values of k.

When $k = -2$ the matrix $\mathbf{BA}$ represents a combination of a uniform enlargement with linear scale factor $\sqrt{a}$ and another transformation T.

c) Find the value of a and describe T geometrically.

$\boxed{a = 8}$, $\boxed{\text{rotation about } O\text{, clockwise, by } 45°}$



a) APPLYING A ROW OPERATION $r_{13}(-1)$ YIELDS ZERO ROW AT $\underline{A}$

$$\underline{A} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 0 \end{pmatrix} \qquad \underline{B} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & k \end{pmatrix}$$

ON MULTIPLICATION: $\underline{A}\underline{B} = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 0 \end{pmatrix}$, SO A SQUARE MATRIX WITH A ZERO ROW (OR COLUMN) HAS ZERO DETERMINANT

b) (NOTE THAT THE CONVERSE IS NOT TRUE DUE TO THE WAY MATRICES MULTIPLY)

$$\underline{B}\underline{A} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & k \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ k & k+4 \end{pmatrix}$$

$\det(\underline{B}\underline{A}) = 8 \neq 0$ FOR ALL k, SO NON SINGULAR

c) IF $k=-2$

$$\underline{B}\underline{A} = \begin{pmatrix} 2 & 2 \\ -2 & 2 \end{pmatrix} = 2\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = 2\underline{I}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & \frac{1}{\sqrt{2}} \end{pmatrix} = \sqrt{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 2\sqrt{2} & 0 \\ 0 & 2\sqrt{2} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

→ STANDARD MATRIX ROTATION, CLOCKWISE ABOUT O, BY 45°

↑ UNIFORM ENLARGEMENT ABOUT O, WITH SCALE FACTOR $2\sqrt{2} = \sqrt{8}$ $\therefore a = 8$

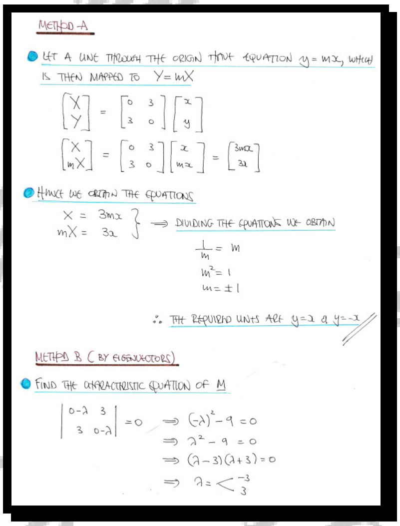
Question 17

The 2×2 matrix $\mathbf{M}$ is defined by

$$\mathbf{M} = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}.$$

Find, by calculation, the equations of the two lines which pass through the origin, that remain invariant under the transformation represented by $\mathbf{M}$.

$\boxed{y = \pm x}$



METHOD A

- LET A LINE THROUGH THE ORIGIN HAVE EQUATION $y = mx$, WHICH IS THEN MAPPED TO $Y = mX$

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} X \\ mX \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} x \\ mx \end{bmatrix} = \begin{bmatrix} 3mx \\ 3x \end{bmatrix}$$

- HENCE WE OBTAIN THE EQUATIONS

$$\left.\begin{aligned} X &= 3mx \\ mX &= 3x \end{aligned}\right\} \Longrightarrow \text{DIVIDING THE EQUATIONS WE OBTAIN}$$

$$\frac{1}{m} = m$$

$$m^2 = 1$$

$$m = \pm 1$$

$\therefore$ THE REQUIRED LINES ARE $y = x$ & $y = -x$ //

METHOD B (BY EIGENVECTORS)

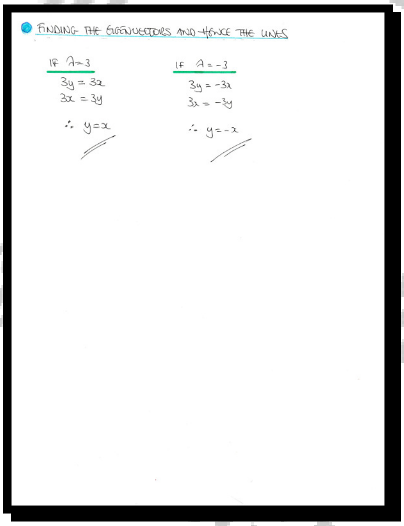
- FIND THE CHARACTERISTIC EQUATION OF $\underline{M}$

$$\begin{vmatrix} 0-\lambda & 3 \\ 3 & 0-\lambda \end{vmatrix} = 0 \Rightarrow (-\lambda)^2 - 9 = 0$$

$$\Rightarrow \lambda^2 - 9 = 0$$

$$\Rightarrow (\lambda-3)(\lambda+3) = 0$$

$$\Rightarrow \lambda = \begin{cases} -3 \\ 3 \end{cases}$$



- FINDING THE EIGENVECTORS AND HENCE THE LINES

IF $\lambda = 3$

$3y = 3x$

$3x = 3y$

$\therefore\ y = x$ //

IF $\lambda = -3$

$3y = -3x$

$3x = -3y$

$\therefore\ y = -x$ //

Question 18

The curve C has equation

$$5x^2 - 16xy + 13y^2 = 25 .$$

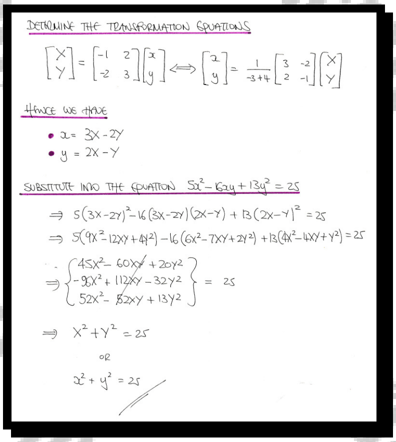
This curve is to be transformed by the 2×2 matrix $\mathbf{A}$, given below.

$$\mathbf{A} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}.$$

Show that, under this transformation matrix, the image of C is the circle with equation

$$x^2 + y^2 = 25 .$$

$\boxed{\text{proof}}$



DETERMINE THE TRANSFORMATION EQUATIONS

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \Longleftrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-3+4}\begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix}$$

HENCE WE HAVE

- $x = 3X - 2Y$
- $y = 2X - Y$

SUBSTITUTE INTO THE EQUATION $5x^2 - 16xy + 13y^2 = 25$

$$\Rightarrow 5(3X-2Y)^2 - 16(3X-2Y)(2X-Y) + 13(2X-Y)^2 = 25$$

$$\Rightarrow 5(9X^2 - 12XY + 4Y^2) - 16(6X^2 - 7XY + 2Y^2) + 13(4X^2 - 4XY + Y^2) = 25$$

$$\Rightarrow \left.\begin{cases} 45X^2 - \cancel{60XY} + 20Y^2 \\ -96X^2 + \cancel{112XY} - 32Y^2 \\ 52X^2 - \cancel{52XY} + 13Y^2 \end{cases}\right\} = 25$$

$$\Rightarrow X^2 + Y^2 = 25$$

OR

$$x^2 + y^2 = 25$$

Question 19

The 3×3 matrix $\mathbf{A}$ is defined by

$$\mathbf{A}=\begin{pmatrix}0&0&-1\\0&1&0\\1&0&0\end{pmatrix}.$$

a) Describe geometrically the transformation given by $\mathbf{A}$.

The 3×3 matrix $\mathbf{B}$ represents a rotation of 180° about the line $x=z$, $y=0$.

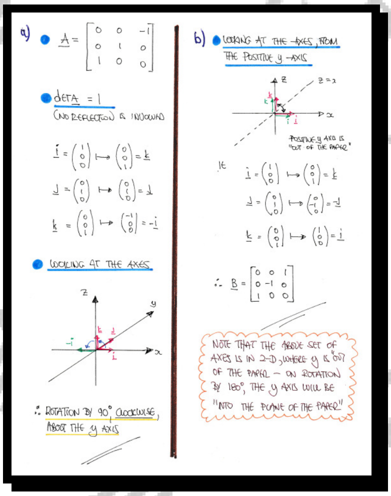
b) Determine the elements of $\mathbf{B}$.

The 3×3 matrix $\mathbf{C}$ is represents the transformation defined by $\mathbf{B}$, followed by the transformation defined by $\mathbf{A}$.

c) Describe geometrically the transformation represented by $\mathbf{C}$.

$\boxed{\mathbf{A}:\text{rotation about } y \text{ axis, } 90^\circ \text{ clockwise}}$, $\boxed{\mathbf{B}=\begin{pmatrix}0&0&1\\0&-1&0\\1&0&0\end{pmatrix}}$,

$\boxed{\mathbf{C}:\text{rotation about } z \text{ axis, } 180^\circ}$



a) ● $\underline{A}=\begin{bmatrix}0&0&-1\\0&1&0\\1&0&0\end{bmatrix}$

● $\det\underline{A}=1$

(NO REFLECTION) IS INVOLVED

$\underline{i}=\begin{pmatrix}1\\0\\0\end{pmatrix}\mapsto\begin{pmatrix}0\\0\\1\end{pmatrix}=\underline{k}$

$\underline{j}=\begin{pmatrix}0\\1\\0\end{pmatrix}\mapsto\begin{pmatrix}0\\1\\0\end{pmatrix}=\underline{j}$

$\underline{k}=\begin{pmatrix}0\\0\\1\end{pmatrix}\mapsto\begin{pmatrix}-1\\0\\0\end{pmatrix}=-\underline{i}$

● LOOKING AT THE AXES

∴ ROTATION BY 90°, CLOCKWISE, ABOUT THE y AXIS

b) ● LOOKING AT THE AXES, FROM THE POSITIVE y-AXIS

IE

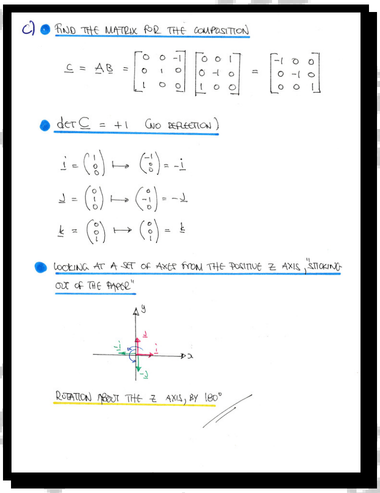
$\underline{i}=\begin{pmatrix}1\\0\\0\end{pmatrix}\mapsto\begin{pmatrix}0\\0\\1\end{pmatrix}=\underline{k}$

$\underline{j}=\begin{pmatrix}0\\1\\0\end{pmatrix}\mapsto\begin{pmatrix}0\\-1\\0\end{pmatrix}=-\underline{j}$

$\underline{k}=\begin{pmatrix}0\\0\\1\end{pmatrix}\mapsto\begin{pmatrix}1\\0\\0\end{pmatrix}=\underline{i}$

$\therefore\ \underline{B}=\begin{bmatrix}0&0&1\\0&-1&0\\1&0&0\end{bmatrix}$

NOTE THAT THE ABOVE SET OF AXES IS IN 2-D, WHERE y IS "OUT" OF THE PAPER – ON ROTATION BY 180°, THE y AXIS WILL BE "INTO THE PLANE OF THE PAPER"



c) ● FIND THE MATRIX FOR THE COMPOSITION

$\underline{C}=\underline{A}\,\underline{B}=\begin{bmatrix}0&0&-1\\0&1&0\\1&0&0\end{bmatrix}\begin{bmatrix}0&0&1\\0&-1&0\\1&0&0\end{bmatrix}=\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&1\end{bmatrix}$

● $\det\underline{C}=+1$ (NO REFLECTION)

$\underline{i}=\begin{pmatrix}1\\0\\0\end{pmatrix}\mapsto\begin{pmatrix}-1\\0\\0\end{pmatrix}=-\underline{i}$

$\underline{j}=\begin{pmatrix}0\\1\\0\end{pmatrix}\mapsto\begin{pmatrix}0\\-1\\0\end{pmatrix}=-\underline{j}$

$\underline{k}=\begin{pmatrix}0\\0\\1\end{pmatrix}\mapsto\begin{pmatrix}0\\0\\1\end{pmatrix}=\underline{k}$

● LOOKING AT A SET OF AXES FROM THE POSITIVE z AXIS, "STICKING OUT OF THE PAPER"

ROTATION ABOUT THE z AXIS, BY 180°

Question 20

The 3×3 matrix $\mathbf{R}$ is defined by

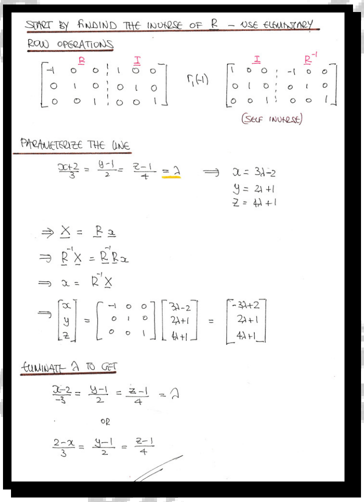
$$\mathbf{R} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The image of the straight line L, when transformed by $\mathbf{R}$, is the straight line with Cartesian equation

$$\frac{x+2}{3} = \frac{y-1}{2} = \frac{z-1}{4}.$$

Find a Cartesian equation for L.

$$\boxed{\frac{x-2}{-3} = \frac{y-1}{2} = \frac{z-1}{4}}$$



START BY FINDING THE INVERSE OF $\underline{R}$ – USE ELEMENTARY ROW OPERATIONS

$$\left[\begin{array}{ccc|ccc} \overset{}{-1} & \overset{\underline{R}}{0} & 0 & 1 & \overset{\underline{I}}{0} & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad r_1(-1) \quad \left[\begin{array}{ccc|ccc} 1 & \overset{\underline{I}}{0} & 0 & -1 & \overset{\underline{R}^{-1}}{0} & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

(SELF INVERSE)

PARAMETERIZE THE LINE

$$\frac{x+2}{3} = \frac{y-1}{2} = \frac{z-1}{4} = \lambda \implies \begin{array}{l} x = 3\lambda-2 \\ y = 2\lambda+1 \\ z = 4\lambda+1 \end{array}$$

$$\Rightarrow \underline{X} = \underline{R}\,\underline{x}$$

$$\Rightarrow \underline{R}^{-1}\underline{X} = \underline{R}^{-1}\underline{R}\,\underline{x}$$

$$\Rightarrow \underline{x} = \underline{R}^{-1}\underline{X}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3\lambda-2 \\ 2\lambda+1 \\ 4\lambda+1 \end{bmatrix} = \begin{bmatrix} -3\lambda+2 \\ 2\lambda+1 \\ 4\lambda+1 \end{bmatrix}$$

ELIMINATE λ TO GET

$$\frac{x-2}{-3} = \frac{y-1}{2} = \frac{z-1}{4} = \lambda$$

OR

$$\frac{2-x}{3} = \frac{y-1}{2} = \frac{z-1}{4}$$

Question 21

The 3×3 matrix $\mathbf{C}$ is defined by

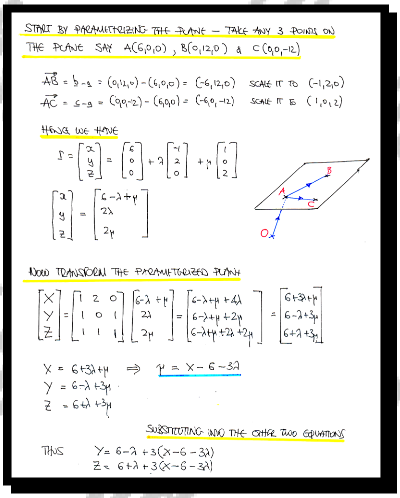
$$\mathbf{C} = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Find, in Cartesian form, the image of the plane with Cartesian equation

$$2x + y - z = 12$$

under the transformation defined by $\mathbf{C}$.

$$\boxed{3x + 4y - 5z = 12}$$



START BY PARAMETRIZING THE PLANE – TAKE ANY 3 POINTS ON THE PLANE SAY $A(6,0,0)$, $B(0,12,0)$ & $C(0,0,-12)$

$\overrightarrow{AB} = \underline{b} - \underline{a} = (0,12,0) - (6,0,0) = (-6,12,0)$ SCALE IT TO $(-1,2,0)$

$\overrightarrow{AC} = \underline{c} - \underline{a} = (0,0,-12) - (6,0,0) = (-6,0,-12)$ SCALE IT TO $(1,0,2)$

HENCE WE HAVE

$$\underline{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 - \lambda + \mu \\ 2\lambda \\ 2\mu \end{bmatrix}$$

NOW TRANSFORM THE PARAMETRIZED PLANE

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 6 - \lambda + \mu \\ 2\lambda \\ 2\mu \end{bmatrix} = \begin{bmatrix} 6 - \lambda + \mu + 4\lambda \\ 6 - \lambda + \mu + 2\mu \\ 6 - \lambda + \mu + 2\lambda + 2\mu \end{bmatrix} = \begin{bmatrix} 6 + 3\lambda + \mu \\ 6 - \lambda + 3\mu \\ 6 + \lambda + 3\mu \end{bmatrix}$$

$X = 6 + 3\lambda + \mu \implies \underline{\mu = X - 6 - 3\lambda}$

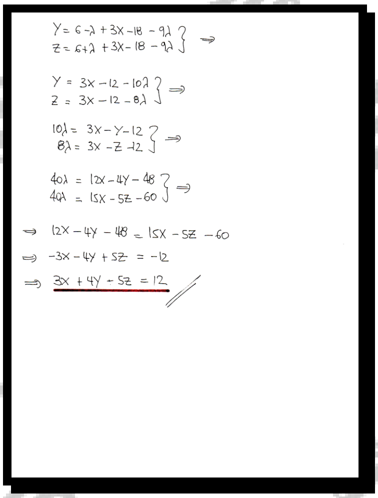
$Y = 6 - \lambda + 3\mu$

$Z = 6 + \lambda + 3\mu$

SUBSTITUTING INTO THE OTHER TWO EQUATIONS

THUS $\quad Y = 6 - \lambda + 3(X - 6 - 3\lambda)$

$\quad\quad\quad Z = 6 + \lambda + 3(X - 6 - 3\lambda)$



$$\left.\begin{aligned} Y &= 6 - \lambda + 3X - 18 - 9\lambda \\ Z &= 6 + \lambda + 3X - 18 - 9\lambda \end{aligned}\right\} \implies$$

$$\left.\begin{aligned} Y &= 3X - 12 - 10\lambda \\ Z &= 3X - 12 - 8\lambda \end{aligned}\right\} \implies$$

$$\left.\begin{aligned} 10\lambda &= 3X - Y - 12 \\ 8\lambda &= 3X - Z - 12 \end{aligned}\right\} \implies$$

$$\left.\begin{aligned} 40\lambda &= 12X - 4Y - 48 \\ 40\lambda &= 15X - 5Z - 60 \end{aligned}\right\} \implies$$

$\implies 12X - 4Y - 48 = 15X - 5Z - 60$

$\implies -3X - 4Y + 5Z = -12$

$\implies \underline{3X + 4Y - 5Z = 12}$ //

Question 22

A transformation $T:\mathbb{R}^2 \mapsto \mathbb{R}^2$ is represented by the 2×2 matrix $\mathbf{A}$ below.

$$\mathbf{A} = \begin{pmatrix} -3 & 8 \\ -1 & 3 \end{pmatrix}.$$

a) Find the determinant of $\mathbf{A}$ and explain its significance in sign and size.

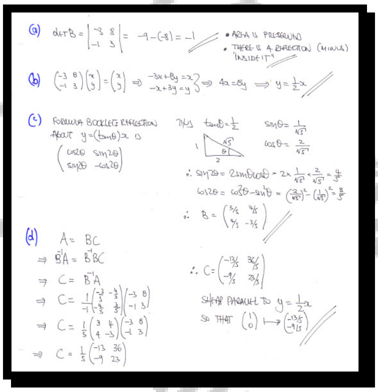
b) Find the equation of the line of the invariant points of $\mathbf{A}$.

c) Determine the entries of the 2×2 matrix $\mathbf{B}$ which represents a reflection about the line found in part **(b)**, giving all its entries as simple fractions.

The 2×2 matrix $\mathbf{A}$, consists of a shear represented by the matrix $\mathbf{C}$, followed by a reflection represented by the matrix $\mathbf{B}$.

d) Determine the elements of $\mathbf{C}$ and describe the shear.

$\boxed{\det \mathbf{A} = -1}$, $\boxed{y = \frac{1}{2}x}$, $\boxed{\mathbf{B} = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}}$, $\boxed{\mathbf{C} = \begin{pmatrix} -\frac{13}{5} & \frac{36}{5} \\ -\frac{9}{5} & \frac{23}{5} \end{pmatrix}}$



(a) $\det B = \begin{vmatrix} -3 & 8 \\ -1 & 3 \end{vmatrix} = -9-(-8) = -1$

- AREA IS PRESERVED
- THERE IS A REFLECTION (MINUS) "INSIDE IT"

(b) $\begin{pmatrix} -3 & 8 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \left.\begin{matrix} -3x+8y=x \\ -x+3y=y \end{matrix}\right\} \Rightarrow 4x=8y \Rightarrow y=\frac{1}{2}x$

(c) FORMULA BOOKLET: REFLECTION ABOUT $y=(\tan\theta)x$ IS $\begin{pmatrix} \cos2\theta & \sin2\theta \\ \sin2\theta & -\cos2\theta \end{pmatrix}$

THUS $\tan\theta = \frac{1}{2}$ $\quad \sin\theta = \frac{1}{\sqrt{5}}$ $\quad \cos\theta = \frac{2}{\sqrt{5}}$

$\therefore \sin2\theta = 2\sin\theta\cos\theta = 2\times\frac{1}{\sqrt{5}}\times\frac{2}{\sqrt{5}} = \frac{4}{5}$

$\cos2\theta = \cos^2\theta - \sin^2\theta = \left(\frac{2}{\sqrt{5}}\right)^2 - \left(\frac{1}{\sqrt{5}}\right)^2 = \frac{3}{5}$

$\therefore B = \begin{pmatrix} 3/5 & 4/5 \\ 4/5 & -3/5 \end{pmatrix}$

(d) $A = BC$

$\Rightarrow B^{-1}A = B^{-1}BC$

$\Rightarrow C = B^{-1}A$

$\Rightarrow C = \frac{1}{-1}\begin{pmatrix} -\frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix}\begin{pmatrix} -3 & 8 \\ -1 & 3 \end{pmatrix}$

$\Rightarrow C = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} -3 & 8 \\ -1 & 3 \end{pmatrix}$

$\Rightarrow C = \frac{1}{5}\begin{pmatrix} -13 & 36 \\ -9 & 23 \end{pmatrix}$

$\therefore C = \begin{pmatrix} -13/5 & 36/5 \\ -9/5 & 23/5 \end{pmatrix}$

SHEAR PARALLEL TO $y=\frac{1}{2}x$ SO THAT $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} -13/5 \\ -9/5 \end{pmatrix}$

Question 23

A transformation T, maps the general point (x, y) onto the general point (X, Y), by

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

a) Find the area scale factor of the transformation.

b) Determine the equation of the line of invariant points under this transformation.

c) Show that all the straight lines of the form

$$y = x + c,$$

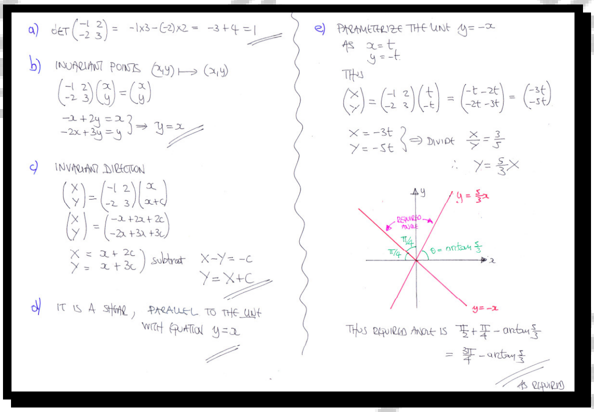
where c is a constant, are invariant lines under T.

d) Hence state the name of T.

e) Show that the acute angle formed by the straight line with equation $y = -x$ and its the image under T is

$$\frac{3\pi}{4} - \arctan\left(\frac{5}{3}\right).$$

$\boxed{\text{SF} = 1}$, $\boxed{y = x}$, $\boxed{\text{shear}}$



a) $\det\begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} = -1\times3-(-2)\times2 = -3+4 = 1$

b) INVARIANT POINTS $(x,y) \longmapsto (x,y)$

$\begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\left.\begin{array}{l} -x+2y = x \\ -2x+3y = y \end{array}\right\} \Rightarrow y = x$

c) INVARIANT DIRECTION

$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ x+c \end{pmatrix}$

$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} -x+2x+2c \\ -2x+3x+3c \end{pmatrix}$

$\left.\begin{array}{l} X = x+2c \\ Y = x+3c \end{array}\right)$ subtract $X-Y = -c$

$Y = X+C$

d) IT IS A SHEAR, PARALLEL TO THE LINE WITH EQUATION $y = x$

e) PARAMETERIZE THE LINE $y = -x$

AS $x = t$
$y = -t$

THUS

$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} t \\ -t \end{pmatrix} = \begin{pmatrix} -t-2t \\ -2t-3t \end{pmatrix} = \begin{pmatrix} -3t \\ -5t \end{pmatrix}$

$\left.\begin{array}{l} X = -3t \\ Y = -5t \end{array}\right\} \Rightarrow$ DIVIDE $\frac{X}{Y} = \frac{3}{5}$

$\therefore \ Y = \frac{5}{3}X$

THUS REQUIRED ANGLE IS $\frac{\pi}{2} + \frac{\pi}{4} - \arctan\frac{5}{3}$

$= \frac{3\pi}{4} - \arctan\frac{5}{3}$

AS REQUIRED

EIGENVALUES

&

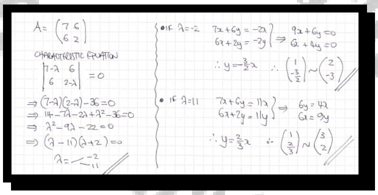
EIGENVECTORS

Question 1

Find the eigenvalues and the corresponding eigenvectors of the following 2×2 matrix.

$$\mathbf{A} = \begin{pmatrix} 7 & 6 \\ 6 & 2 \end{pmatrix}.$$

$$\boxed{\lambda = -2, \quad \mathbf{u} = \alpha \begin{pmatrix} 2 \\ -3 \end{pmatrix}}, \ \boxed{\lambda = 11, \quad \mathbf{u} = \beta \begin{pmatrix} 3 \\ 2 \end{pmatrix}}$$



$A = \begin{pmatrix} 7 & 6 \\ 6 & 2 \end{pmatrix}$

CHARACTERISTIC EQUATION

$\begin{vmatrix} 7-\lambda & 6 \\ 6 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow (7-\lambda)(2-\lambda) - 36 = 0$

$\Rightarrow 14 - 7\lambda - 2\lambda + \lambda^2 - 36 = 0$

$\Rightarrow \lambda^2 - 9\lambda - 22 = 0$

$\Rightarrow (\lambda - 11)(\lambda + 2) = 0$

$\lambda = \begin{matrix} -2 \\ 11 \end{matrix}$ //

- IF $\lambda = -2$ $\quad \left.\begin{matrix} 7x + 6y = -2x \\ 6x + 2y = -2y \end{matrix}\right\} \Rightarrow \begin{matrix} 9x + 6y = 0 \\ 6x + 4y = 0 \end{matrix}$

$\therefore y = -\frac{3}{2}x \quad \therefore \begin{pmatrix} 1 \\ -\frac{3}{2} \end{pmatrix} \sim \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ //

- IF $\lambda = 11$ $\quad \left.\begin{matrix} 7x + 6y = 11x \\ 6x + 2y = 11y \end{matrix}\right\} \Rightarrow \begin{matrix} 6y = 4x \\ 6x = 9y \end{matrix}$

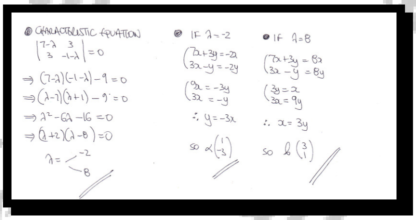
$\therefore y = \frac{2}{3}x \quad \therefore \begin{pmatrix} 1 \\ \frac{2}{3} \end{pmatrix} \sim \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ //

Question 2

Find the eigenvalues and the corresponding eigenvectors of the following 2×2 matrix.

$$\mathbf{C} = \begin{pmatrix} 7 & 3 \\ 3 & -1 \end{pmatrix}.$$

$$\boxed{\lambda = -2, \quad \mathbf{u} = \alpha \begin{pmatrix} 1 \\ -3 \end{pmatrix}}, \ \boxed{\lambda = 8, \quad \mathbf{u} = \beta \begin{pmatrix} 3 \\ 1 \end{pmatrix}}$$



- CHARACTERISTIC EQUATION

$\begin{vmatrix} 7-\lambda & 3 \\ 3 & -1-\lambda \end{vmatrix} = 0$

$\Rightarrow (7-\lambda)(-1-\lambda) - 9 = 0$

$\Rightarrow (\lambda-7)(\lambda+1) - 9 = 0$

$\Rightarrow \lambda^2 - 6\lambda - 16 = 0$

$\Rightarrow (\lambda+2)(\lambda-8) = 0$

$\lambda = \begin{matrix} -2 \\ 8 \end{matrix}$ //

- IF $\lambda = -2$

$\left(\begin{matrix} 7x + 3y = -2x \\ 3x - y = -2y \end{matrix}\right.$

$\left(\begin{matrix} 9x = -3y \\ 3x = -y \end{matrix}\right.$

$\therefore y = -3x$

so $\alpha \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ //

- IF $\lambda = 8$

$\left(\begin{matrix} 7x + 3y = 8x \\ 3x - y = 8y \end{matrix}\right.$

$\left(\begin{matrix} 3y = x \\ 3x = 9y \end{matrix}\right.$

$\therefore x = 3y$

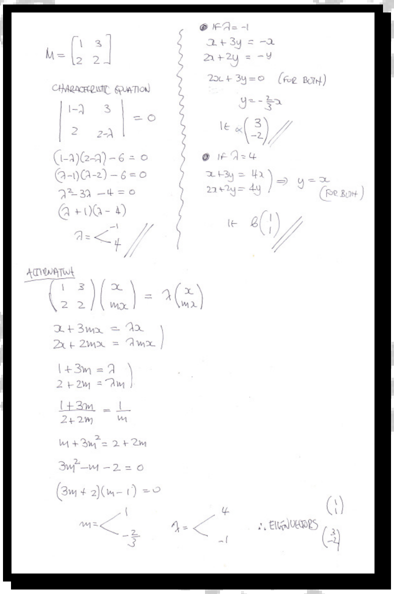
so $b \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ //

Question 3

Find the eigenvalues and the corresponding eigenvectors of the following 2×2 matrix.

$$\mathbf{M}=\begin{pmatrix}1 & 3\\ 2 & 2\end{pmatrix}.$$

$$\boxed{\lambda=-1,\quad \mathbf{u}=\alpha\begin{pmatrix}3\\-2\end{pmatrix}},\ \boxed{\lambda=4,\quad \mathbf{u}=\beta\begin{pmatrix}1\\1\end{pmatrix}}$$



$$M=\begin{bmatrix}1 & 3\\ 2 & 2\end{bmatrix}$$

CHARACTERISTIC EQUATION

$$\begin{vmatrix}1-\lambda & 3\\ 2 & 2-\lambda\end{vmatrix}=0$$

$$(1-\lambda)(2-\lambda)-6=0$$
$$(\lambda-1)(\lambda-2)-6=0$$
$$\lambda^2-3\lambda-4=0$$
$$(\lambda+1)(\lambda-4)$$
$$\lambda=\left\langle\begin{matrix}-1\\4\end{matrix}\right.\quad //$$

● IF $\lambda=-1$

$$x+3y=-x$$
$$2x+2y=-y$$

$2x+3y=0$ (FOR BOTH)

$$y=-\tfrac{2}{3}x$$

IE $\alpha\begin{pmatrix}3\\-2\end{pmatrix}$ //

● IF $\lambda=4$

$$\left.\begin{matrix}x+3y=4x\\2x+2y=4y\end{matrix}\right)\Rightarrow y=x$$

(FOR BOTH)

IE $\beta\begin{pmatrix}1\\1\end{pmatrix}$ //

ALTERNATIVE

$$\begin{pmatrix}1 & 3\\ 2 & 2\end{pmatrix}\begin{pmatrix}x\\mx\end{pmatrix}=\lambda\begin{pmatrix}x\\mx\end{pmatrix}$$

$$\left.\begin{matrix}x+3mx=\lambda x\\2x+2mx=\lambda mx\end{matrix}\right)$$

$$\left.\begin{matrix}1+3m=\lambda\\2+2m=\lambda m\end{matrix}\right)$$

$$\frac{1+3m}{2+2m}=\frac{1}{m}$$

$$m+3m^2=2+2m$$

$$3m^2-m-2=0$$

$$(3m+2)(m-1)=0$$

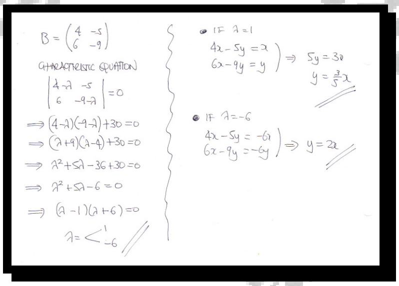
$$m=\left\langle\begin{matrix}1\\-\frac{2}{3}\end{matrix}\right.\qquad \lambda=\left\langle\begin{matrix}4\\-1\end{matrix}\right.\qquad \therefore \text{EIGENVECTORS}\ \begin{matrix}\begin{pmatrix}1\\1\end{pmatrix}\\[4pt]\begin{pmatrix}3\\-2\end{pmatrix}\end{matrix}$$

Question 4

Determine the eigenvalues and the corresponding equations of invariant lines of the following 2×2 matrix.

$$\mathbf{B} = \begin{pmatrix} 4 & -5 \\ 6 & -9 \end{pmatrix}.$$

$\boxed{\lambda = 1, y = \frac{3}{5}x}$, $\boxed{\lambda = -6, y = 2x}$



$B = \begin{pmatrix} 4 & -5 \\ 6 & -9 \end{pmatrix}$

CHARACTERISTIC EQUATION

$\begin{vmatrix} 4-\lambda & -5 \\ 6 & -9-\lambda \end{vmatrix} = 0$

$\Rightarrow (4-\lambda)(-9-\lambda)+30=0$

$\Rightarrow (\lambda+9)(\lambda-4)+30=0$

$\Rightarrow \lambda^2+5\lambda-36+30=0$

$\Rightarrow \lambda^2+5\lambda-6=0$

$\Rightarrow (\lambda-1)(\lambda+6)=0$

$\lambda = < \begin{matrix} 1 \\ -6 \end{matrix}$ //

- IF $\lambda = 1$

$\left.\begin{matrix} 4x-5y=x \\ 6x-9y=y \end{matrix}\right) \Rightarrow 5y=3x$

$y = \frac{3}{5}x$ //

- IF $\lambda = -6$

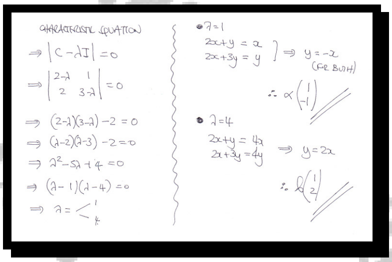
$\left.\begin{matrix} 4x-5y=-6x \\ 6x-9y=-6y \end{matrix}\right) \Rightarrow y=2x$ //

Question 5

Find the eigenvalues and the corresponding eigenvectors of the following 2×2 matrix.

$$\mathbf{C} = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}.$$

$\boxed{\lambda = 1, \quad \mathbf{u} = \alpha\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$, $\boxed{\lambda = 4, \quad \mathbf{u} = \beta\begin{pmatrix} 1 \\ 2 \end{pmatrix}}$



CHARACTERISTIC EQUATION

$\Rightarrow |C-\lambda I| = 0$

$\Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow (2-\lambda)(3-\lambda)-2=0$

$\Rightarrow (\lambda-2)(\lambda-3)-2=0$

$\Rightarrow \lambda^2-5\lambda+4=0$

$\Rightarrow (\lambda-1)(\lambda-4)=0$

$\Rightarrow \lambda = < \begin{matrix} 1 \\ 4 \end{matrix}$

- $\lambda = 1$

$\left.\begin{matrix} 2x+y=x \\ 2x+3y=y \end{matrix}\right] \Rightarrow y=-x$

(FOR BOTH)

$\therefore \alpha\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ //

- $\lambda = 4$

$\begin{matrix} 2x+y=4x \\ 2x+3y=4y \end{matrix} \Rightarrow y=2x$

$\therefore \beta\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ //

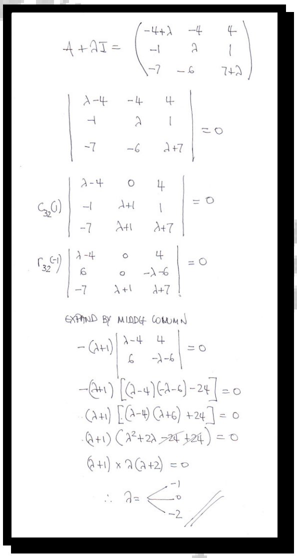
Question 6

The 3×3 matrix $\mathbf{A}$ is given below.

$$\mathbf{A} = \begin{pmatrix} -4 & -4 & 4 \\ -1 & 0 & 1 \\ -7 & -6 & 7 \end{pmatrix}. \quad .$$

Given that $\mathbf{I}$ is the 3×3 identity matrix, determine the values of the constant λ, so that $\mathbf{A} + \lambda\mathbf{I}$ is singular.

$$\boxed{\lambda = 0, -1, -2}$$



$$A + \lambda I = \begin{pmatrix} -4+\lambda & -4 & 4 \\ -1 & \lambda & 1 \\ -7 & -6 & 7+\lambda \end{pmatrix}$$

$$\begin{vmatrix} \lambda-4 & -4 & 4 \\ -1 & \lambda & 1 \\ -7 & -6 & \lambda+7 \end{vmatrix} = 0$$

$$C_{32}(1) \begin{vmatrix} \lambda-4 & 0 & 4 \\ -1 & \lambda+1 & 1 \\ -7 & \lambda+1 & \lambda+7 \end{vmatrix} = 0$$

$$r_{32}(-1) \begin{vmatrix} \lambda-4 & 0 & 4 \\ 6 & 0 & -\lambda-6 \\ -7 & \lambda+1 & \lambda+7 \end{vmatrix} = 0$$

EXPAND BY MIDDLE COLUMN

$$-(\lambda+1) \begin{vmatrix} \lambda-4 & 4 \\ 6 & -\lambda-6 \end{vmatrix} = 0$$

$$-(\lambda+1)\left[(\lambda-4)(-\lambda-6) - 24\right] = 0$$

$$(\lambda+1)\left[(\lambda-4)(\lambda+6) + 24\right] = 0$$

$$(\lambda+1)(\lambda^2+2\lambda \cancel{-24} \cancel{+24}) = 0$$

$$(\lambda+1) \times \lambda(\lambda+2) = 0$$

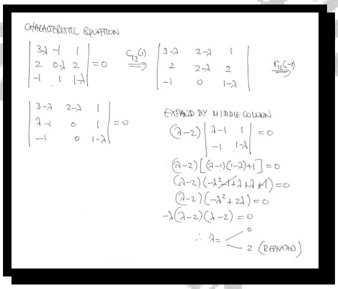
$$\therefore \lambda = \begin{cases} -1 \\ 0 \\ -2 \end{cases} //$$

Question 7

Find the eigenvalues and the corresponding eigenvectors of the following 3×3 matrix.

$$\mathbf{A}=\begin{pmatrix} 3 & -1 & 1 \\ 2 & 0 & 2 \\ -1 & 1 & 1 \end{pmatrix}.$$

$$\boxed{\lambda=0, \quad \lambda=2}$$



CHARACTERISTIC EQUATION

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ 2 & 0-\lambda & 2 \\ -1 & 1 & 1-\lambda \end{vmatrix}=0 \xRightarrow{C_{12}(1)} \begin{vmatrix} 3-\lambda & 2-\lambda & 1 \\ 2 & 2-\lambda & 2 \\ -1 & 0 & 1-\lambda \end{vmatrix} \xRightarrow{r_{12}(-1)}$$

$$\begin{vmatrix} 3-\lambda & 2-\lambda & 1 \\ \lambda-1 & 0 & 1 \\ -1 & 0 & 1-\lambda \end{vmatrix}=0$$

EXPAND BY MIDDLE COLUMN

$$(\lambda-2)\begin{vmatrix} \lambda-1 & 1 \\ -1 & 1-\lambda \end{vmatrix}=0$$

$$(\lambda-2)\left[(\lambda-1)(1-\lambda)+1\right]=0$$

$$(\lambda-2)(-\lambda^2\cancel{-1}+\lambda+\lambda\cancel{+1})=0$$

$$(\lambda-2)(-\lambda^2+2\lambda)=0$$

$$-\lambda(\lambda-2)(\lambda-2)=0$$

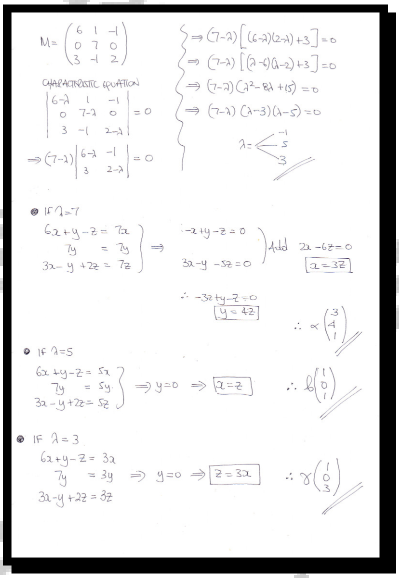
$$\therefore \lambda = \begin{cases} 0 \\ 2 \text{ (REPEATED)} \end{cases}$$

Question 8

Find the eigenvalues and the corresponding eigenvectors of the following 3×3 matrix.

$$\mathbf{M} = \begin{pmatrix} 6 & 1 & -1 \\ 0 & 7 & 0 \\ 3 & -1 & 2 \end{pmatrix}.$$

$$\boxed{\lambda_1 = 3, \quad \mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}}, \quad \boxed{\lambda_2 = 5, \quad \mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}, \quad \boxed{\lambda_3 = 7, \quad \mathbf{w} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}}$$



$$M = \begin{pmatrix} 6 & 1 & -1 \\ 0 & 7 & 0 \\ 3 & -1 & 2 \end{pmatrix}$$

CHARACTERISTIC EQUATION

$$\begin{vmatrix} 6-\lambda & 1 & -1 \\ 0 & 7-\lambda & 0 \\ 3 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (7-\lambda)\begin{vmatrix} 6-\lambda & -1 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (7-\lambda)\left[(6-\lambda)(2-\lambda)+3\right] = 0$$

$$\Rightarrow (7-\lambda)\left[(\lambda-6)(\lambda-2)+3\right] = 0$$

$$\Rightarrow (7-\lambda)(\lambda^2-8\lambda+15) = 0$$

$$\Rightarrow (7-\lambda)(\lambda-3)(\lambda-5) = 0$$

$$\lambda = \begin{cases} 7 \\ 5 \\ 3 \end{cases}$$

- IF $\lambda = 7$

$$\left.\begin{array}{r} 6x+y-z = 7x \\ 7y = 7y \\ 3x-y+2z = 7z \end{array}\right\} \Rightarrow \left.\begin{array}{r} -x+y-z=0 \\ 3x-y-5z=0 \end{array}\right) \text{Add } 2x-6z=0 \quad \boxed{x=3z}$$

$$\therefore -3z+y-z=0 \quad \boxed{y=4z}$$

$$\therefore \alpha\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$$

- IF $\lambda = 5$

$$\left.\begin{array}{r} 6x+y-z = 5x \\ 7y = 5y \\ 3x-y+2z = 5z \end{array}\right\} \Rightarrow y=0 \Rightarrow \boxed{x=z} \quad \therefore \beta\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

- IF $\lambda = 3$

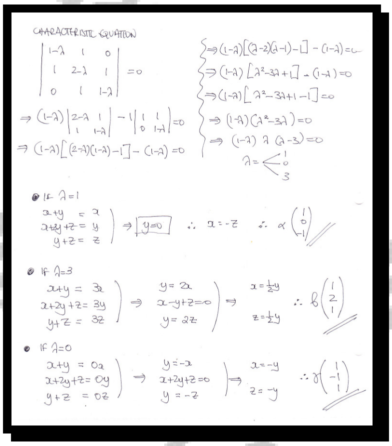
$$\begin{array}{r} 6x+y-z = 3x \\ 7y = 3y \\ 3x-y+2z = 3z \end{array} \Rightarrow y=0 \Rightarrow \boxed{z=3x} \quad \therefore \gamma\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

Question 9

Find the eigenvalues and the corresponding eigenvectors of the following 3×3 matrix.

$$\mathbf{A}=\begin{pmatrix}1&1&0\\1&2&1\\0&1&1\end{pmatrix}.$$

$$\boxed{\lambda_1=0,\quad \mathbf{u}=\begin{pmatrix}1\\-1\\1\end{pmatrix}},\ \boxed{\lambda_2=1,\quad \mathbf{v}=\begin{pmatrix}1\\0\\-1\end{pmatrix}},\ \boxed{\lambda_3=3,\quad \mathbf{w}=\begin{pmatrix}1\\2\\1\end{pmatrix}}$$



CHARACTERISTIC EQUATION

$$\begin{vmatrix}1-\lambda&1&0\\1&2-\lambda&1\\0&1&1-\lambda\end{vmatrix}=0$$

$$\Rightarrow (1-\lambda)\begin{vmatrix}2-\lambda&1\\1&1-\lambda\end{vmatrix}-1\begin{vmatrix}1&1\\0&1-\lambda\end{vmatrix}=0$$

$$\Rightarrow (1-\lambda)\left[(2-\lambda)(1-\lambda)-1\right]-(1-\lambda)=0$$

$$\Rightarrow (1-\lambda)\left[(\lambda-2)(\lambda-1)-1\right]-(1-\lambda)=0$$

$$\Rightarrow (1-\lambda)\left[\lambda^2-3\lambda+1\right]-(1-\lambda)=0$$

$$\Rightarrow (1-\lambda)\left[\lambda^2-3\lambda+1-1\right]=0$$

$$\Rightarrow (1-\lambda)(\lambda^2-3\lambda)=0$$

$$\Rightarrow (1-\lambda)\,\lambda\,(\lambda-3)=0$$

$$\lambda=\begin{cases}1\\0\\3\end{cases}$$

- IF $\lambda=1$

$$\left.\begin{aligned}x+y&=x\\x+2y+z&=y\\y+z&=z\end{aligned}\right\}\Rightarrow\boxed{y=0}\quad\therefore\ x=-z\quad\therefore\ \alpha\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

- IF $\lambda=3$

$$\left.\begin{aligned}x+y&=3x\\x+2y+z&=3y\\y+z&=3z\end{aligned}\right\}\Rightarrow\left.\begin{aligned}y&=2x\\x-y+z&=0\\y&=2z\end{aligned}\right\}\Rightarrow\begin{aligned}x&=\tfrac{1}{2}y\\z&=\tfrac{1}{2}y\end{aligned}\quad\therefore\ \beta\begin{pmatrix}1\\2\\1\end{pmatrix}$$

- IF $\lambda=0$

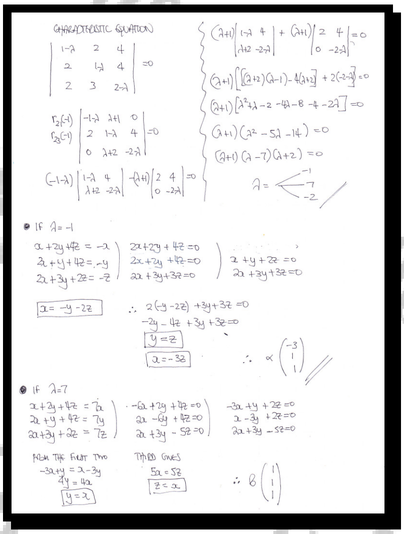
$$\left.\begin{aligned}x+y&=0x\\x+2y+z&=0y\\y+z&=0z\end{aligned}\right\}\Rightarrow\left.\begin{aligned}y&=-x\\x+2y+z&=0\\y&=-z\end{aligned}\right\}\Rightarrow\begin{aligned}x&=-y\\z&=-y\end{aligned}\quad\therefore\ \gamma\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

Question 10

Find the eigenvalues and the corresponding eigenvectors of the following 3×3 matrix.

$$\mathbf{A}=\begin{pmatrix}1 & 2 & 4\\ 2 & 1 & 4\\ 2 & 3 & 2\end{pmatrix}.$$

$$\boxed{\lambda_1=-1,\quad \mathbf{u}=\begin{pmatrix}-3\\1\\1\end{pmatrix}},\ \boxed{\lambda_2=-2,\quad \mathbf{v}=\begin{pmatrix}4\\4\\-5\end{pmatrix}},\ \boxed{\lambda_3=7,\quad \mathbf{w}=\begin{pmatrix}1\\1\\1\end{pmatrix}}$$



CHARACTERISTIC EQUATION

$$\begin{vmatrix}1-\lambda & 2 & 4\\ 2 & 1-\lambda & 4\\ 2 & 3 & 2-\lambda\end{vmatrix}=0$$

$$\begin{matrix}r_{21}(-1)\\ r_{23}(-1)\end{matrix}\begin{vmatrix}-1-\lambda & \lambda+1 & 0\\ 2 & 1-\lambda & 4\\ 0 & \lambda+2 & -2-\lambda\end{vmatrix}=0$$

$$(-1-\lambda)\begin{vmatrix}1-\lambda & 4\\ \lambda+2 & -2-\lambda\end{vmatrix}-(\lambda+1)\begin{vmatrix}2 & 4\\ 0 & -2-\lambda\end{vmatrix}=0$$

$$(\lambda+1)\begin{vmatrix}1-\lambda & 4\\ \lambda+2 & -2-\lambda\end{vmatrix}+(\lambda+1)\begin{vmatrix}2 & 4\\ 0 & -2-\lambda\end{vmatrix}=0$$

$$(\lambda+1)\Big[\big[(\lambda+2)(\lambda-1)-4(\lambda+2)\big]+2(-2-\lambda)\Big]=0$$

$$(\lambda+1)\left[\lambda^2+\lambda-2-4\lambda-8-4-2\lambda\right]=0$$

$$(\lambda+1)(\lambda^2-5\lambda-14)=0$$

$$(\lambda+1)(\lambda-7)(\lambda+2)=0$$

$$\lambda=\begin{cases}-1\\7\\-2\end{cases}$$

● IF $\lambda=-1$

$$\left.\begin{array}{l}x+2y+4z=-x\\2x+y+4z=-y\\2x+3y+2z=-z\end{array}\right)\quad\left.\begin{array}{l}2x+2y+4z=0\\2x+2y+4z=0\\2x+3y+3z=0\end{array}\right)\quad\begin{array}{l}x+y+2z=0\\2x+3y+3z=0\end{array}$$

$$\boxed{x=-y-2z}$$

$$\therefore\ 2(-y-2z)+3y+3z=0$$

$$-2y-4z+3y+3z=0$$

$$\boxed{y=z}$$

$$\boxed{x=-3z}$$

$$\therefore\ \alpha\begin{pmatrix}-3\\1\\1\end{pmatrix}$$

● IF $\lambda=7$

$$\left.\begin{array}{l}x+2y+4z=7x\\2x+y+4z=7y\\2x+3y+2z=7z\end{array}\right)\quad\left.\begin{array}{l}-6x+2y+4z=0\\2x-6y+4z=0\\2x+3y-5z=0\end{array}\right)\quad\begin{array}{l}-3x+y+2z=0\\x-3y+2z=0\\2x+3y-5z=0\end{array}$$

FROM THE FIRST TWO

$$-3x+y=x-3y$$

$$4y=4x$$

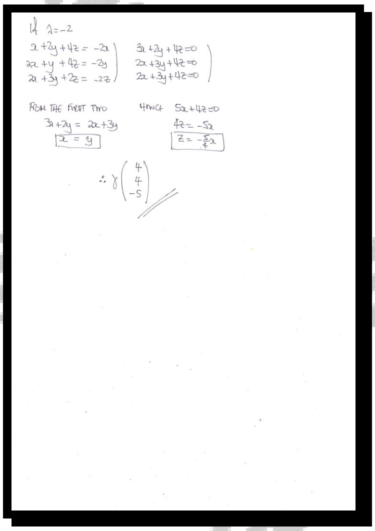
$$\boxed{y=x}$$

THIRD GIVES

$$5x=5z$$

$$\boxed{z=x}$$

$$\therefore\ \beta\begin{pmatrix}1\\1\\1\end{pmatrix}$$



IF $\lambda=-2$

$$\left.\begin{array}{l}x+2y+4z=-2x\\2x+y+4z=-2y\\2x+3y+2z=-2z\end{array}\right)\quad\left.\begin{array}{l}3x+2y+4z=0\\2x+3y+4z=0\\2x+3y+4z=0\end{array}\right)$$

FROM THE FIRST TWO

$$3x+2y=2x+3y$$

$$\boxed{x=y}$$

HENCE $5x+4z=0$

$$4z=-5x$$

$$\boxed{z=-\frac{5}{4}x}$$

$$\therefore\ \gamma\begin{pmatrix}4\\4\\-5\end{pmatrix}$$

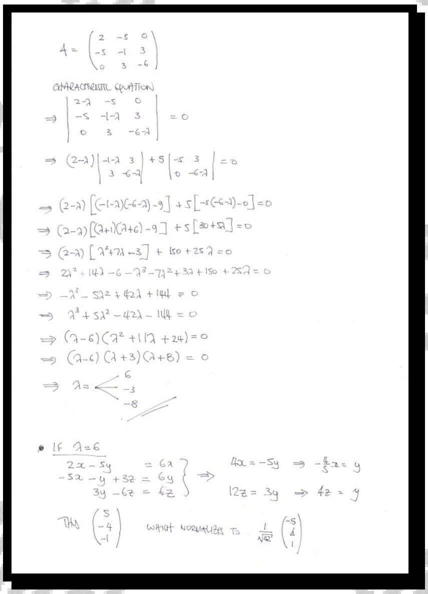
Question 11

The 3×3 matrix $\mathbf{A}$ is given below.

$$\mathbf{A}=\begin{pmatrix}2&-5&0\\-5&-1&3\\0&3&-6\end{pmatrix}.$$

Since $\mathbf{A}$ is symmetric, determine an orthogonal 3×3 matrix $\mathbf{P}$ and a diagonal 3×3 matrix $\mathbf{D}$ such that $\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P}=\mathbf{D}$.

$$\boxed{\mathbf{P}=\begin{pmatrix}-\frac{5}{\sqrt{42}}&\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{14}}\\\frac{4}{\sqrt{42}}&\frac{1}{\sqrt{3}}&\frac{2}{\sqrt{14}}\\\frac{1}{\sqrt{42}}&\frac{1}{\sqrt{3}}&-\frac{3}{\sqrt{14}}\end{pmatrix}},\ \boxed{\mathbf{D}=\begin{pmatrix}6&0&0\\0&-3&0\\0&0&-8\end{pmatrix}}$$



$$A=\begin{pmatrix}2&-5&0\\-5&-1&3\\0&3&-6\end{pmatrix}$$

CHARACTERISTIC EQUATION

$$\Rightarrow \begin{vmatrix}2-\lambda&-5&0\\-5&-1-\lambda&3\\0&3&-6-\lambda\end{vmatrix}=0$$

$$\Rightarrow (2-\lambda)\begin{vmatrix}-1-\lambda&3\\3&-6-\lambda\end{vmatrix}+5\begin{vmatrix}-5&3\\0&-6-\lambda\end{vmatrix}=0$$

$$\Rightarrow (2-\lambda)\left[(-1-\lambda)(-6-\lambda)-9\right]+5\left[-5(-6-\lambda)-0\right]=0$$

$$\Rightarrow (2-\lambda)\left[(\lambda+1)(\lambda+6)-9\right]+5\left[30+5\lambda\right]=0$$

$$\Rightarrow (2-\lambda)\left[\lambda^2+7\lambda-3\right]+150+25\lambda=0$$

$$\Rightarrow 2\lambda^2+14\lambda-6-\lambda^3-7\lambda^2+3\lambda+150+25\lambda=0$$

$$\Rightarrow -\lambda^3-5\lambda^2+42\lambda+144=0$$

$$\Rightarrow \lambda^3+5\lambda^2-42\lambda-144=0$$

$$\Rightarrow (\lambda-6)(\lambda^2+11\lambda+24)=0$$

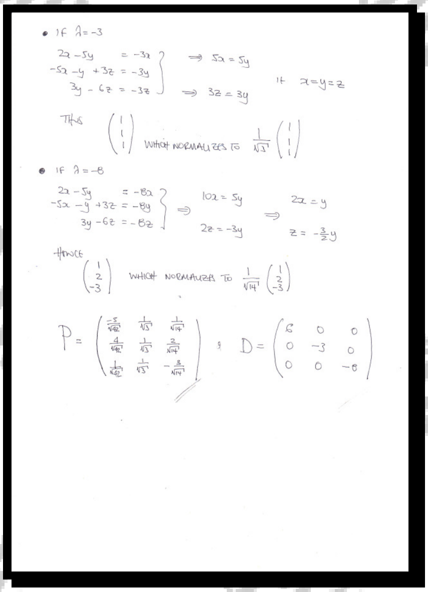
$$\Rightarrow (\lambda-6)(\lambda+3)(\lambda+8)=0$$

$$\Rightarrow \lambda=\begin{cases}6\\-3\\-8\end{cases}$$

- IF $\lambda=6$

$$\left.\begin{aligned}2x-5y\quad&=6x\\-5x-y+3z&=6y\\3y-6z&=6z\end{aligned}\right\}\Rightarrow\begin{aligned}4x=-5y&\Rightarrow -\tfrac{4}{5}x=y\\12z=3y&\Rightarrow 4z=y\end{aligned}$$

THUS $\begin{pmatrix}5\\-4\\-1\end{pmatrix}$ WHICH NORMALIZES TO $\frac{1}{\sqrt{42}}\begin{pmatrix}-5\\4\\1\end{pmatrix}$



- IF $\lambda=-3$

$$\left.\begin{aligned}2x-5y\quad&=-3x\\-5x-y+3z&=-3y\\3y-6z&=-3z\end{aligned}\right\}\begin{aligned}&\Rightarrow 5x=5y\\&\Rightarrow 3z=3y\end{aligned}\quad \text{it } x=y=z$$

THUS $\begin{pmatrix}1\\1\\1\end{pmatrix}$ WHICH NORMALIZES TO $\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$

- IF $\lambda=-8$

$$\left.\begin{aligned}2x-5y\quad&=-8x\\-5x-y+3z&=-8y\\3y-6z&=-8z\end{aligned}\right\}\Rightarrow\begin{aligned}10x&=5y\\2z&=-3y\end{aligned}\Rightarrow\begin{aligned}2x&=y\\z&=-\tfrac{3}{2}y\end{aligned}$$

HENCE

$\begin{pmatrix}1\\2\\-3\end{pmatrix}$ WHICH NORMALIZES TO $\frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\-3\end{pmatrix}$

$$P=\begin{pmatrix}\frac{-5}{\sqrt{42}}&\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{14}}\\\frac{4}{\sqrt{42}}&\frac{1}{\sqrt{3}}&\frac{2}{\sqrt{14}}\\\frac{1}{\sqrt{42}}&\frac{1}{\sqrt{3}}&-\frac{3}{\sqrt{14}}\end{pmatrix}\ ;\ D=\begin{pmatrix}6&0&0\\0&-3&0\\0&0&-8\end{pmatrix}$$

Question 12

The 3×3 matrix $\mathbf{A}$ is given below.

$$\mathbf{A}=\begin{pmatrix}1&0&4\\0&5&4\\4&4&3\end{pmatrix}.$$

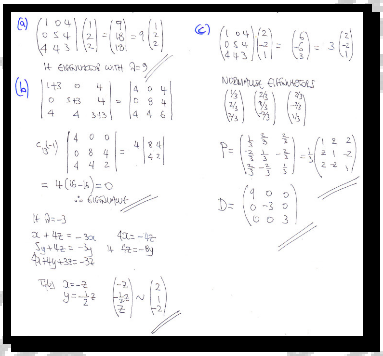
a) Verify that $\begin{pmatrix}1\\2\\2\end{pmatrix}$ is an eigenvector of $\mathbf{A}$ and state the corresponding eigenvalue.

b) Show that -3 is an eigenvalue of $\mathbf{A}$ and find the corresponding eigenvector.

c) Given further that $\begin{pmatrix}2\\-2\\1\end{pmatrix}$ is another eigenvector of $\mathbf{A}$, find the 3×3 matrices $\mathbf{P}$ and $\mathbf{D}$ such that

$$\mathbf{D}=\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P}$$

$$\boxed{\lambda=9},\ \boxed{\begin{pmatrix}2\\1\\-2\end{pmatrix}},\ \boxed{\mathbf{D}=\begin{pmatrix}9&0&0\\0&-3&0\\0&0&3\end{pmatrix}},\ \boxed{\mathbf{P}=\frac{1}{3}\begin{pmatrix}1&2&2\\2&1&-2\\2&-2&1\end{pmatrix}}$$



(a) $\begin{pmatrix}1&0&4\\0&5&4\\4&4&3\end{pmatrix}\begin{pmatrix}1\\2\\2\end{pmatrix}=\begin{pmatrix}9\\18\\18\end{pmatrix}=9\begin{pmatrix}1\\2\\2\end{pmatrix}$

It EIGENVECTOR WITH $\lambda=9$

(b) $\begin{vmatrix}1+3&0&4\\0&5+3&4\\4&4&3+3\end{vmatrix}=\begin{vmatrix}4&0&4\\0&8&4\\4&4&6\end{vmatrix}$

$C_{13}(-1)$ $\begin{vmatrix}4&0&0\\0&8&4\\4&4&2\end{vmatrix}=4\begin{vmatrix}8&4\\4&2\end{vmatrix}$

$=4(16-16)=0$

$\therefore$ EIGENVALUE

If $\lambda=-3$

$x+4z=-3x$ $\qquad 4x=-4z$

$5y+4z=-3y$ $\quad$ It $4z=-8y$

$4x+4y+3z=-3z$

Thus $x=-z$
$y=-\frac{1}{2}z$ $\qquad \begin{pmatrix}-z\\-\frac{1}{2}z\\z\end{pmatrix}\sim\begin{pmatrix}2\\1\\-2\end{pmatrix}$

(c) $\begin{pmatrix}1&0&4\\0&5&4\\4&4&3\end{pmatrix}\begin{pmatrix}2\\-2\\1\end{pmatrix}=\begin{pmatrix}6\\-6\\3\end{pmatrix}=3\begin{pmatrix}2\\-2\\1\end{pmatrix}$

NORMALISE EIGENVECTORS

$\begin{pmatrix}1/3\\2/3\\2/3\end{pmatrix}\ \begin{pmatrix}2/3\\1/3\\-2/3\end{pmatrix}\ \begin{pmatrix}2/3\\-2/3\\1/3\end{pmatrix}$

$P=\begin{pmatrix}\frac{1}{3}&\frac{2}{3}&\frac{2}{3}\\\frac{2}{3}&\frac{1}{3}&-\frac{2}{3}\\\frac{2}{3}&-\frac{2}{3}&\frac{1}{3}\end{pmatrix}=\frac{1}{3}\begin{pmatrix}1&2&2\\2&1&-2\\2&-2&1\end{pmatrix}$

$D=\begin{pmatrix}9&0&0\\0&-3&0\\0&0&3\end{pmatrix}$

Question 13

The 3×3 matrix $\mathbf{A}$ is given below.

$$\mathbf{A} = \begin{pmatrix} 4 & -1 & 1 \\ -1 & 6 & -1 \\ 1 & -1 & 4 \end{pmatrix}.$$

a) Show that $\lambda = 7$ is an eigenvalue of $\mathbf{A}$ and find the other two eigenvalues.

b) Find the eigenvector associated with the eigenvalue $\lambda = 7$.

The other two eigenvectors of $\mathbf{A}$ are

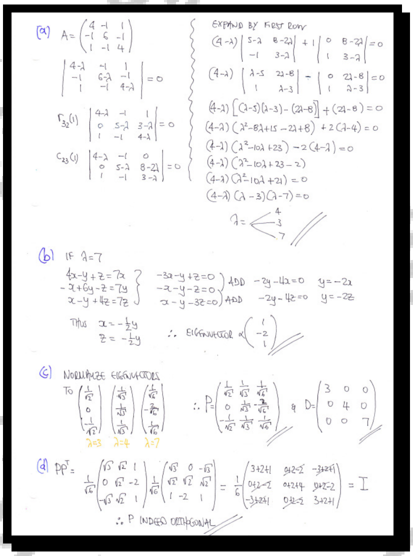
$$\mathbf{u} = \mathbf{i} - \mathbf{k} \qquad \text{and} \qquad \mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k},$$

where the eigenvalue of $\mathbf{v}$ is greater than the eigenvalue of $\mathbf{u}$.

c) Find a 3×3 matrix $\mathbf{P}$ and a diagonal 3×3 matrix $\mathbf{D}$ such that $\mathbf{D} = \mathbf{P}^{\mathrm{T}}\mathbf{AP}$.

d) Show that $\mathbf{P}$ is an orthogonal matrix.

$$\boxed{\lambda = 4, 3}, \quad \boxed{\alpha \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}, \quad \boxed{\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}}, \quad \boxed{\mathbf{D} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{pmatrix}}$$



(a) $A = \begin{pmatrix} 4 & -1 & 1 \\ -1 & 6 & -1 \\ 1 & -1 & 4 \end{pmatrix}$

$\begin{vmatrix} 4-\lambda & -1 & 1 \\ -1 & 6-\lambda & -1 \\ 1 & -1 & 4-\lambda \end{vmatrix} = 0$

$r_{32}(1)$ $\begin{vmatrix} 4-\lambda & -1 & 1 \\ 0 & 5-\lambda & 3-\lambda \\ 1 & -1 & 4-\lambda \end{vmatrix} = 0$

$C_{23}(1)$ $\begin{vmatrix} 4-\lambda & -1 & 0 \\ 0 & 5-\lambda & 8-2\lambda \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$

EXPAND BY FIRST ROW

$(4-\lambda)\begin{vmatrix} 5-\lambda & 8-2\lambda \\ -1 & 3-\lambda \end{vmatrix} + 1\begin{vmatrix} 0 & 8-2\lambda \\ 1 & 3-\lambda \end{vmatrix} = 0$

$(4-\lambda)\begin{vmatrix} \lambda-5 & 2\lambda-8 \\ 1 & \lambda-3 \end{vmatrix} - \begin{vmatrix} 0 & 2\lambda-8 \\ 1 & \lambda-3 \end{vmatrix} = 0$

$(4-\lambda)\left[(\lambda-5)(\lambda-3)-(2\lambda-8)\right] + (2\lambda-8) = 0$

$(4-\lambda)(\lambda^2-8\lambda+15-2\lambda+8) + 2(\lambda-4) = 0$

$(4-\lambda)(\lambda^2-10\lambda+23) - 2(4-\lambda) = 0$

$(4-\lambda)(\lambda^2-10\lambda+23-2)$

$(4-\lambda)(\lambda^2-10\lambda+21) = 0$

$(4-\lambda)(\lambda-3)(\lambda-7) = 0$

$\lambda = 4, 3, 7$

(b) IF $\lambda = 7$

$\left.\begin{aligned} 4x-y+z &= 7x \\ -x+6y-z &= 7y \\ x-y+4z &= 7z \end{aligned}\right\}$ $\begin{aligned} -3x-y+z &= 0 \\ -x-y-z &= 0 \\ x-y-3z &= 0 \end{aligned}$ ADD $-2y-4x=0$ $y=-2x$; ADD $-2y-4z=0$ $y=-2z$

Thus $x = -\frac{1}{2}y$, $z = -\frac{1}{2}y$ $\quad\therefore$ EIGENVECTOR $\alpha\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

(c) NORMALIZE EIGENVECTORS

To $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$ $\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$ $\begin{pmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$

$\lambda=3$ $\quad\lambda=4$ $\quad\lambda=7$

$\therefore P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}$ & $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{pmatrix}$

(d) $PP^{\mathrm{T}} = \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{3} & \sqrt{2} & 1 \\ 0 & \sqrt{2} & -2 \\ -\sqrt{3} & \sqrt{2} & 1 \end{pmatrix} \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{3} & 0 & -\sqrt{3} \\ \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1 & -2 & 1 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 3+2+1 & 0+2-2 & -3+2+1 \\ 0+2-2 & 0+2+4 & 0+2-2 \\ -3+2+1 & 0+2-2 & 3+2+1 \end{pmatrix} = I$

$\therefore$ P INDEED ORTHOGONAL

Question 14

The 2×2 matrix $\mathbf{A}$ is given below.

$$\mathbf{A}=\begin{pmatrix}7 & 6\\ 6 & 2\end{pmatrix}.$$

A straight line with equation $y=mx$, where m is a constant, remains invariant under the transformation represented by $\mathbf{A}$.

a) Show that

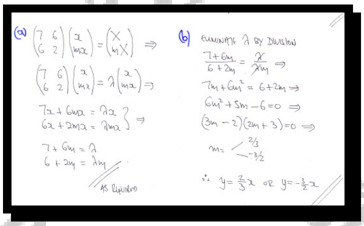
$$7+6m=\lambda$$

$$6+2m=\lambda m$$

where λ is a constant.

b) Hence find the two possible equations of this straight line.

$\boxed{y=\frac{2}{3}x}$, $\boxed{y=-\frac{3}{2}x}$



(a) $\begin{pmatrix}7 & 6\\ 6 & 2\end{pmatrix}\begin{pmatrix}x\\ mx\end{pmatrix}=\begin{pmatrix}X\\ mX\end{pmatrix} \Rightarrow$

$\begin{pmatrix}7 & 6\\ 6 & 2\end{pmatrix}\begin{pmatrix}x\\ mx\end{pmatrix}=\lambda\begin{pmatrix}x\\ mx\end{pmatrix} \Rightarrow$

$\left.\begin{array}{l}7x+6mx=\lambda x\\ 6x+2mx=\lambda mx\end{array}\right\} \Rightarrow$

$7+6m=\lambda$

$6+2m=\lambda m$

// AS REQUIRED

(b) ELIMINATE λ BY DIVISION

$\frac{7+6m}{6+2m}=\frac{\cancel{\lambda}}{\cancel{\lambda}m} \Rightarrow$

$7m+6m^2=6+2m \Rightarrow$

$6m^2+5m-6=0 \Rightarrow$

$(3m-2)(2m+3)=0 \Rightarrow$

$m= \begin{cases}2/3\\ -3/2\end{cases}$

$\therefore\ y=\frac{2}{3}x$ or $y=-\frac{3}{2}x$

Question 15

The 3×3 matrix $\mathbf{C}$ represents a geometric transformation $T:\mathbb{R}^3 \mapsto \mathbb{R}^3$.

$$\mathbf{C} = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$$

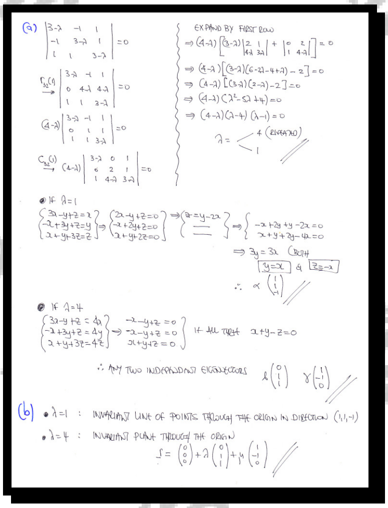
a) Find the eigenvalues and the corresponding eigenvectors of $\mathbf{C}$.

b) Describe the geometrical significance of the eigenvectors of $\mathbf{C}$ in relation to T.

$$\boxed{\lambda = 1, \quad \alpha\begin{pmatrix}1\\1\\-1\end{pmatrix}}, \ \boxed{\lambda = 4, \quad \beta\begin{pmatrix}0\\1\\1\end{pmatrix}, \gamma\begin{pmatrix}1\\-1\\0\end{pmatrix}},$$

$\boxed{\lambda = 1 \Leftrightarrow \text{invariant line of points through the origin}}$,

$\boxed{\lambda = 4 \Leftrightarrow \text{invariant plane through the origin}}$



(a) $\begin{vmatrix}3-\lambda & -1 & 1\\ -1 & 3-\lambda & 1\\ 1 & 1 & 3-\lambda\end{vmatrix}=0$

$\xrightarrow{r_{32}(1)} \begin{vmatrix}3-\lambda & -1 & 1\\ 0 & 4-\lambda & 4-\lambda\\ 1 & 1 & 3-\lambda\end{vmatrix}=0$

$(4-\lambda)\begin{vmatrix}3-\lambda & -1 & 1\\ 0 & 1 & 1\\ 1 & 1 & 3-\lambda\end{vmatrix}=0$

$\xrightarrow{C_{32}(1)} (4-\lambda)\begin{vmatrix}3-\lambda & 0 & 1\\ 0 & 2 & 1\\ 1 & 4-\lambda & 3-\lambda\end{vmatrix}=0$

EXPAND BY FIRST ROW

$\Rightarrow (4-\lambda)\left[(3-\lambda)\begin{vmatrix}2 & 1\\ 4-\lambda & 3-\lambda\end{vmatrix} + \begin{vmatrix}0 & 2\\ 1 & 4-\lambda\end{vmatrix}\right]=0$

$\Rightarrow (4-\lambda)[(3-\lambda)(6-2\lambda-4+\lambda)-2]=0$

$\Rightarrow (4-\lambda)[(3-\lambda)(2-\lambda)-2]=0$

$\Rightarrow (4-\lambda)(\lambda^2-5\lambda+4)=0$

$\Rightarrow (4-\lambda)(\lambda-4)(\lambda-1)=0$

$\lambda = \begin{cases} 4 \ \text{(REPEATED)} \\ 1 \end{cases}$ //

- IF $\lambda=1$

$\left.\begin{matrix}3x-y+z=x\\ -x+3y+z=y\\ x+y+3z=z\end{matrix}\right\} \Rightarrow \left(\begin{matrix}2x-y+z=0\\ -x+2y+z=0\\ x+y+2z=0\end{matrix}\right) \Rightarrow \left(\begin{matrix}z=y-2x\\ =\\ =\end{matrix}\right\} \Rightarrow \left\{\begin{matrix}-x+2y+y-2x=0\\ x+y+2y-4x=0\end{matrix}\right.$

$\Rightarrow 3y=3x$ (BOTH

$\boxed{y=x}$ & $\boxed{z=-x}$

$\therefore \alpha\begin{pmatrix}1\\1\\-1\end{pmatrix}$ //

- IF $\lambda=4$

$\left.\begin{matrix}3x-y+z=4x\\ -x+3y+z=4y\\ x+y+3z=4z\end{matrix}\right\} \Rightarrow \left.\begin{matrix}-x-y+z=0\\ -x-y+z=0\\ x+y+z=0\end{matrix}\right\}$ IT ALL THAT $x+y-z=0$

$\therefore$ ANY TWO INDEPENDENT EIGENVECTORS $\beta\begin{pmatrix}0\\1\\1\end{pmatrix}$ $\gamma\begin{pmatrix}1\\-1\\0\end{pmatrix}$ //

(b)
- $\lambda=1$: INVARIANT LINE OF POINTS THROUGH THE ORIGIN IN DIRECTION $(1,1,-1)$
- $\lambda=4$: INVARIANT PLANE THROUGH THE ORIGIN

$\underline{r} = \begin{pmatrix}0\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}0\\1\\1\end{pmatrix} + \mu\begin{pmatrix}1\\-1\\0\end{pmatrix}$ //

Question 16

The 2×2 matrix $\mathbf{A}$ is defined in terms of a constant k.

$$\mathbf{A} = \begin{pmatrix} 2 & 7 \\ 4 & k \end{pmatrix}$$

a) Given that $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$, find …

i. … the corresponding eigenvalue to the eigenvector.

ii. … the value of k

b) Find another eigenvector and the corresponding eigenvalue of $\mathbf{A}$.

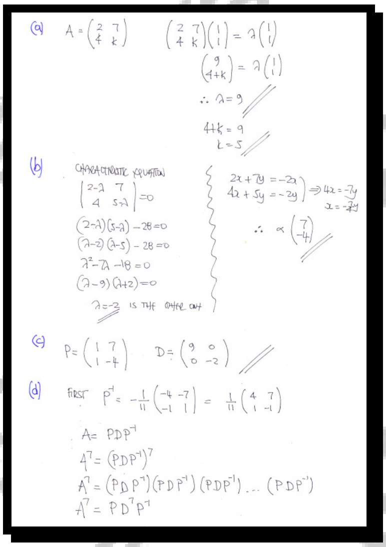
It is further given that $\mathbf{A}$ can be written as $\mathbf{A} = \mathbf{PDP}^{-1}$, where $\mathbf{D}$ is a 2×2 diagonal matrix and $\mathbf{P}$ is another 2×2 matrix.

c) Write down possible forms for the matrices $\mathbf{D}$ and $\mathbf{P}$.

d) Hence show clearly that

$$\mathbf{A}^7 = \begin{pmatrix} 1739180 & 3043789 \\ 1739308 & 3043661 \end{pmatrix}.$$

$\boxed{\lambda = 9}$, $\boxed{k = 5}$, $\boxed{\lambda = -2, \mathbf{u} = \begin{pmatrix} 7 \\ -4 \end{pmatrix}}$, $\boxed{\mathbf{D} = \begin{pmatrix} 9 & 0 \\ 0 & -2 \end{pmatrix}}$, $\boxed{\mathbf{P} = \begin{pmatrix} 1 & 7 \\ 1 & -4 \end{pmatrix}}$



(a) $A = \begin{pmatrix} 2 & 7 \\ 4 & k \end{pmatrix}$ $\begin{pmatrix} 2 & 7 \\ 4 & k \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lambda\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 9 \\ 4+k \end{pmatrix} = \lambda\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\therefore \lambda = 9$ //

$4+k = 9$

$k = 5$ //

(b) CHARACTERISTIC EQUATION

$\begin{vmatrix} 2-\lambda & 7 \\ 4 & 5-\lambda \end{vmatrix} = 0$

$(2-\lambda)(5-\lambda) - 28 = 0$

$(\lambda-2)(\lambda-5) - 28 = 0$

$\lambda^2 - 7\lambda - 18 = 0$

$(\lambda-9)(\lambda+2) = 0$

$\lambda = -2$ // IS THE OTHER ONE

$\left.\begin{matrix} 2x + 7y = -2x \\ 4x + 5y = -2y \end{matrix}\right\} \Rightarrow 4x = -7y$

$x = -\frac{7}{4}y$

$\therefore \propto \begin{pmatrix} 7 \\ -4 \end{pmatrix}$ //

(c) $P = \begin{pmatrix} 1 & 7 \\ 1 & -4 \end{pmatrix}$ $D = \begin{pmatrix} 9 & 0 \\ 0 & -2 \end{pmatrix}$ //

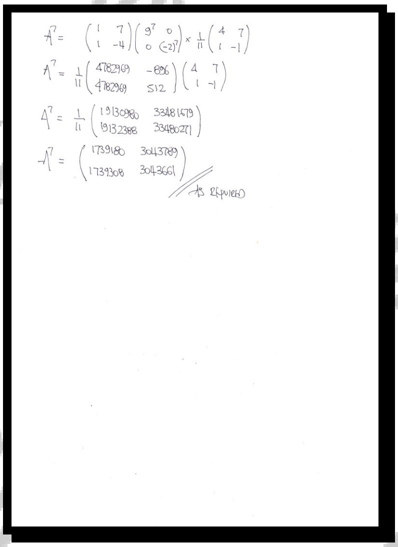
(d) FIRST $P^{-1} = -\frac{1}{11}\begin{pmatrix} -4 & -7 \\ -1 & 1 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 4 & 7 \\ 1 & -1 \end{pmatrix}$

$A = PDP^{-1}$

$A^7 = (PDP^{-1})^7$

$A^7 = (PDP^{-1})(PDP^{-1})(PDP^{-1}) \ldots (PDP^{-1})$

$A^7 = PD^7P^{-1}$



$A^7 = \begin{pmatrix} 1 & 7 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} 9^7 & 0 \\ 0 & (-2)^7 \end{pmatrix} \times \frac{1}{11}\begin{pmatrix} 4 & 7 \\ 1 & -1 \end{pmatrix}$

$A^7 = \frac{1}{11}\begin{pmatrix} 4782969 & -896 \\ 4782969 & 512 \end{pmatrix}\begin{pmatrix} 4 & 7 \\ 1 & -1 \end{pmatrix}$

$A^7 = \frac{1}{11}\begin{pmatrix} 19130980 & 33481679 \\ 19132388 & 33480271 \end{pmatrix}$

$A^7 = \begin{pmatrix} 1739180 & 3043789 \\ 1739308 & 3043661 \end{pmatrix}$ // AS REQUIRED

Question 17

The 3×3 matrix $\mathbf{A}$ is given below.

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 \\ 3 & -3 & 1 \\ 3 & -5 & 3 \end{pmatrix}.$$

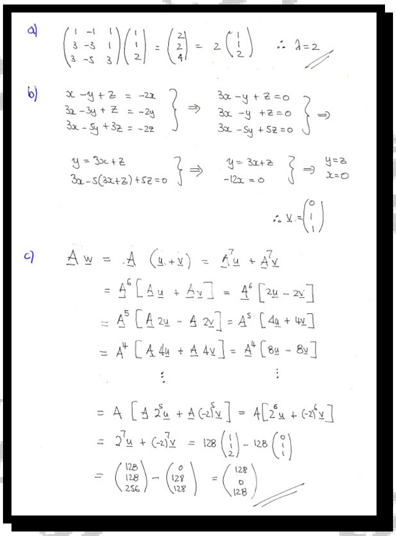
a) Given that $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$, find the corresponding eigenvalue.

b) Given that $\lambda = -2$ is an eigenvalue of $\mathbf{A}$, find a corresponding eigenvector $\mathbf{v}$.

The vector $\mathbf{w}$ is defined as $\mathbf{w} = \mathbf{u} + \mathbf{v}$.

c) Determine the vector $\mathbf{A}^7\mathbf{w}$.

$\boxed{\lambda = 2}$, $\boxed{\mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}$, $\boxed{\mathbf{A}^7\mathbf{w} = \begin{pmatrix} 128 \\ 0 \\ 128 \end{pmatrix}}$



a) $\begin{pmatrix} 1 & -1 & 1 \\ 3 & -3 & 1 \\ 3 & -5 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad \therefore \lambda = 2$

b) $\left.\begin{array}{l} x - y + z = -2x \\ 3x - 3y + z = -2y \\ 3x - 5y + 3z = -2z \end{array}\right\} \Rightarrow \left.\begin{array}{l} 3x - y + z = 0 \\ 3x - y + z = 0 \\ 3x - 5y + 5z = 0 \end{array}\right\} \Rightarrow$

$\left.\begin{array}{l} y = 3x + z \\ 3x - 5(3x+z) + 5z = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} y = 3x + z \\ -12x = 0 \end{array}\right\} \Rightarrow \begin{array}{l} y = z \\ x = 0 \end{array}$

$\therefore \underline{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

c) $\underline{A}\,\underline{w} = \underline{A}\,(\underline{u} + \underline{v}) = \underline{A}^7\underline{u} + \underline{A}^7\underline{v}$

$= \underline{A}^6\left[\underline{A}\,\underline{u} + \underline{A}\,\underline{v}\right] = \underline{A}^6\left[2\underline{u} - 2\underline{v}\right]$

$= \underline{A}^5\left[\underline{A}\,2\underline{u} - \underline{A}\,2\underline{v}\right] = \underline{A}^5\left[4\underline{u} + 4\underline{v}\right]$

$= A^4\left[\underline{A}\,4\underline{u} + \underline{A}\,4\underline{v}\right] = \underline{A}^4\left[8\underline{u} - 8\underline{v}\right]$

$\vdots \qquad \vdots$

$= A\left[\underline{A}\,2^5\underline{u} + \underline{A}\,(-2)^5\underline{v}\right] = A\left[2^6\underline{u} + (-2)^6\underline{v}\right]$

$= 2^7\underline{u} + (-2)^7\underline{v} = 128\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - 128\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 128 \\ 128 \\ 256 \end{pmatrix} - \begin{pmatrix} 0 \\ 128 \\ 128 \end{pmatrix} = \begin{pmatrix} 128 \\ 0 \\ 128 \end{pmatrix}$

Question 18

The 2×2 matrix $\mathbf{C}$ is defined as

$$\mathbf{C} = \begin{pmatrix} a & b+a \\ b-a & -a \end{pmatrix},$$

where a and b are constants.

- **a)** Determine the eigenvalues of $\mathbf{C}$ and their corresponding eigenvectors, giving the answers in terms of a and b where appropriate.

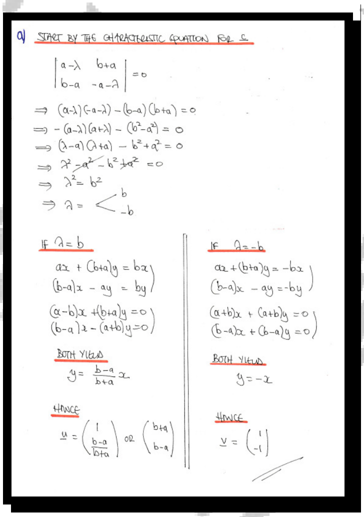
It is further given that $\mathbf{C} = \mathbf{PDP}^{-1}$, where $\mathbf{D}$ is a diagonal matrix and $\mathbf{P}$ is another 2×2 matrix.

- **b)** Write down the possible form of $\mathbf{D}$ and the possible form of $\mathbf{P}$ and hence show that

$$\mathbf{C}^9 = b^8\mathbf{C}.$$

$$\boxed{\lambda_1 = b, \quad \mathbf{u} = \begin{pmatrix} 1 \\ \dfrac{b-a}{b+a} \end{pmatrix} \text{ or } \mathbf{u} = \begin{pmatrix} b+a \\ b-a \end{pmatrix}}, \quad \boxed{\lambda = -b, \quad \mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}, \quad \boxed{\mathbf{D} = \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}},$$

$$\boxed{\mathbf{P} = \begin{pmatrix} b+a & 1 \\ b-a & -1 \end{pmatrix}}$$



a) START BY THE CHARACTERISTIC EQUATION FOR $\underline{C}$

$$\begin{vmatrix} a-\lambda & b+a \\ b-a & -a-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (a-\lambda)(-a-\lambda) - (b-a)(b+a) = 0$$
$$\Rightarrow -(a-\lambda)(a+\lambda) - (b^2-a^2) = 0$$
$$\Rightarrow (\lambda-a)(\lambda+a) - b^2 + a^2 = 0$$
$$\Rightarrow \lambda^2 \cancel{-a^2} - b^2 \cancel{+a^2} = 0$$
$$\Rightarrow \lambda^2 = b^2$$
$$\Rightarrow \lambda = \begin{cases} b \\ -b \end{cases}$$

IF $\lambda = b$

$$\left.\begin{aligned} ax + (b+a)y &= bx \\ (b-a)x - ay &= by \end{aligned}\right\}$$

$$\left.\begin{aligned} (a-b)x + (b+a)y &= 0 \\ (b-a)x - (a+b)y &= 0 \end{aligned}\right\}$$

BOTH YIELD

$$y = \frac{b-a}{b+a}x$$

HENCE

$$\underline{u} = \begin{pmatrix} 1 \\ \frac{b-a}{b+a} \end{pmatrix} \text{ OR } \begin{pmatrix} b+a \\ b-a \end{pmatrix}$$

IF $\lambda = -b$

$$\left.\begin{aligned} ax + (b+a)y &= -bx \\ (b-a)x - ay &= -by \end{aligned}\right\}$$

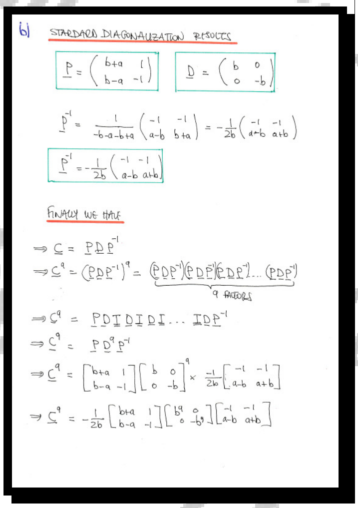
$$\left.\begin{aligned} (a+b)x + (a+b)y &= 0 \\ (b-a)x + (b-a)y &= 0 \end{aligned}\right\}$$

BOTH YIELD

$$y = -x$$

HENCE

$$\underline{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$



b) STANDARD DIAGONALIZATION RESULTS

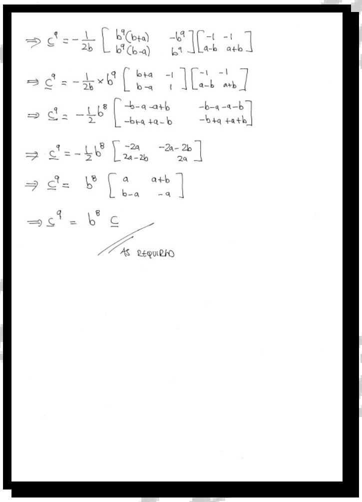
$$\boxed{\underline{P} = \begin{pmatrix} b+a & 1 \\ b-a & -1 \end{pmatrix}} \quad \boxed{\underline{D} = \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}}$$

$$\underline{P}^{-1} = \frac{1}{-b-a-b+a}\begin{pmatrix} -1 & -1 \\ a-b & b+a \end{pmatrix} = -\frac{1}{2b}\begin{pmatrix} -1 & -1 \\ a+b & a+b \end{pmatrix}$$

$$\boxed{\underline{P}^{-1} = -\frac{1}{2b}\begin{pmatrix} -1 & -1 \\ a-b & a+b \end{pmatrix}}$$

FINALLY WE HAVE

$$\Rightarrow \underline{C} = \underline{P}\,\underline{D}\,\underline{P}^{-1}$$
$$\Rightarrow \underline{C}^9 = (\underline{P}\,\underline{D}\,\underline{P}^{-1})^9 = \underbrace{(\underline{P}\,\underline{D}\,\underline{P}^{-1})(\underline{P}\,\underline{D}\,\underline{P}^{-1})(\underline{P}\,\underline{D}\,\underline{P}^{-1})\dots(\underline{P}\,\underline{D}\,\underline{P}^{-1})}_{9 \text{ FACTORS}}$$
$$\Rightarrow \underline{C}^9 = \underline{P}\,\underline{D}\,\underline{I}\,\underline{D}\,\underline{I}\,\underline{D}\,\underline{I}\dots\underline{I}\,\underline{D}\,\underline{P}^{-1}$$
$$\Rightarrow \underline{C}^9 = \underline{P}\,\underline{D}^9\,\underline{P}^{-1}$$
$$\Rightarrow \underline{C}^9 = \begin{bmatrix} b+a & 1 \\ b-a & -1 \end{bmatrix}\begin{bmatrix} b & 0 \\ 0 & -b \end{bmatrix}^9 \times \frac{-1}{2b}\begin{bmatrix} -1 & -1 \\ a-b & a+b \end{bmatrix}$$
$$\Rightarrow \underline{C}^9 = -\frac{1}{2b}\begin{bmatrix} b+a & 1 \\ b-a & -1 \end{bmatrix}\begin{bmatrix} b^9 & 0 \\ 0 & -b^9 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ a-b & a+b \end{bmatrix}$$



$$\Rightarrow \underline{C}^9 = -\frac{1}{2b}\begin{bmatrix} b^9(b+a) & -b^9 \\ b^9(b-a) & b^9 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ a-b & a+b \end{bmatrix}$$
$$\Rightarrow \underline{C}^9 = -\frac{1}{2b} \times b^9 \begin{bmatrix} b+a & -1 \\ b-a & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ a-b & a+b \end{bmatrix}$$
$$\Rightarrow \underline{C}^9 = -\frac{1}{2}b^8 \begin{bmatrix} -b-a-a+b & -b-a-a-b \\ -b+a+a-b & -b+a+a+b \end{bmatrix}$$
$$\Rightarrow \underline{C}^9 = -\frac{1}{2}b^8 \begin{bmatrix} -2a & -2a-2b \\ 2a-2b & 2a \end{bmatrix}$$
$$\Rightarrow \underline{C}^9 = b^8 \begin{bmatrix} a & a+b \\ b-a & -a \end{bmatrix}$$
$$\Rightarrow \underline{C}^9 = b^8\,\underline{C}$$

AS REQUIRED